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UNIVERSIDADE DE CIÊNCIA E TECNOLOGIA DE MACAU
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澳門四高校聯合入學考試(語言科及數學科)

Joint Admission Examination for Macao Four Higher Education Institutions (Languages and Mathematics)

2022 試題及參考答案
2022 Examination Paper and Suggested Answer

數學正卷 Mathematics Standard Paper

第一部份 選擇題。請選出每題之*最佳答案*。

1. 在下列集合中，表示空集的是

A. $\{0\}$　　B. $\{x : \sin x + \cos x = 3\}$　　C. $\{x : x^2 - 1 = 0, x \in \mathbb{R}\}$

D. $\{\varnothing\}$　　E. $\{(x, y) : x^2 + y^2 = 0, x \in \mathbb{R}, y \in \mathbb{R}\}$

2. 若 $a > b > 0$ 及 $m > 0$，下列哪一個不等式是正確的？

A. $\frac{b}{a} > \frac{b+m}{a+m}$　　B. $\frac{a}{b} < \frac{a-m}{b-m}$　　C. $\frac{b}{a} < \frac{b+m}{a+m}$

D. $\frac{a}{b} > \frac{a-m}{b-m}$　　E. 以上皆非

3. 若多項式 $x^3 + 3x^2 + ax + b$ 分別被 $x-2$ 及 $x+1$ 除，餘數相等。求 a 之值。

A. 4　　B. −3　　C. −4　　D. 9　　E. −6

4. $\sqrt{7 - 4\sqrt{3}} =$

A. $\sqrt{3} - \sqrt{2}$　　B. $2 - \sqrt{3}$　　C. $\sqrt{3} - 2$

D. $2 - \sqrt{6}$　　E. $2 - 2\sqrt{3}$

5. 若方程 $px^2 - 2(p+3)x + p - 1 = 0$ 有實根，求 p 的取值範圍。

A. $0 \le p \le \frac{5}{7}$　　B. $p \ge -\frac{5}{7}$　　C. $-\frac{5}{7} \le p \le 1$

D. $p \ge -\frac{9}{7}$　　E. 以上皆非

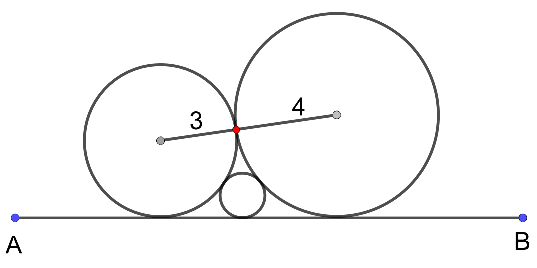
6. 已知 $2^a = 5^b = \sqrt{10}$，則 $\frac{1}{a} + \frac{1}{b}$ 之值是多少？

A. 2　　B. 1　　C. $\sqrt{2}$　　D. $\frac{\sqrt{2}}{2}$　　E. $\frac{1}{2}$

7. 右圖中三個圓及線段 AB 互切。兩個大圓半徑分別為 3 單位及 4 單位。求小圓的半徑。

A. $84 - 48\sqrt{3}$　　B. $\sqrt{2} - 1$　　C. $2\sqrt{2} - 2$

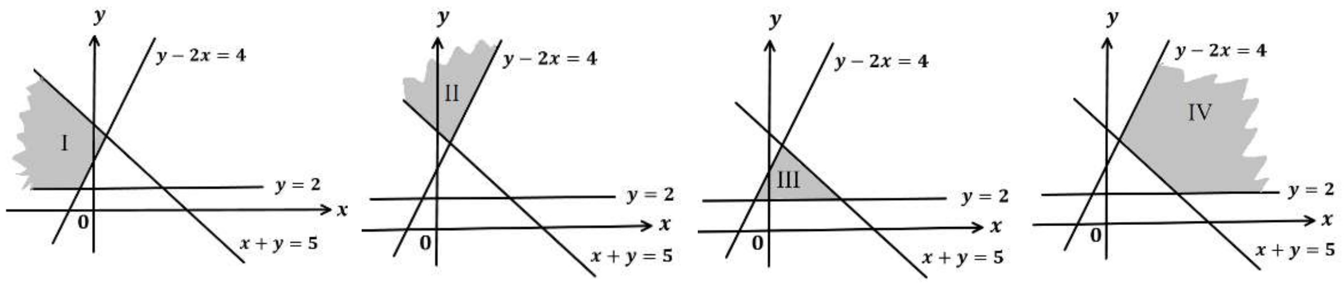
D. $6 - 4\sqrt{2}$　　E. $42 - 24\sqrt{3}$



8. 約翰在一次遊戲中勝出的概率是 $\frac{1}{4}$。他在連續三次遊戲中勝出至少一次的概率是多少？

A. $\frac{1}{64}$　　B. $\frac{27}{64}$　　C. $\frac{35}{64}$　　D. $\frac{37}{64}$　　E. $\frac{43}{64}$

9. 下圖中，哪一區域是不等式組 $\begin{cases} y-2x\le 4 \\ x+y\ge 5 \\ y-2\ge 0 \end{cases}$ 的解集？



A. I　　B. II　　C. III　　D. IV　　E. 以上皆非

10. 已知直線 L 穿過點 $(1,2)$，並與直線 $2x-3y=4$ 垂直。直線 L 和 y 軸相交於

A. $\left(0,\frac{7}{2}\right)$　　B. $\left(0,\frac{5}{2}\right)$　　C. $(0,3)$　　D. $\left(0,-\frac{1}{2}\right)$　　E. $\left(0,\frac{3}{2}\right)$

11. 已知某室內表演場所觀眾席有 6400 張座椅，場內每行都有座椅 32 張。為了配合現行防疫社交距離措施，同一行最多可有連續 4 張座椅被佔用。根據以上要求，每場表演的入座人數最多為

A. 5000　　B. 5200　　C. 5400　　D. 5600　　E. 5800

12. 已知方程式 $3x^2-8x+m=0$ 的兩個實根為 x_1 和 x_2。若 $\frac{1}{x_1}$ 和 $\frac{1}{x_2}$ 的算術平均數為 2，則 m 之值是多少？

A. -2　　B. -1　　C. 4　　D. 1　　E. 2

13. 從 1, 2, 3, 4, 5 中先後選取兩個不同數，分別作為一個兩位數的十位和個位，此兩位數小於 40 的概率為

A. $\frac{1}{5}$　　B. $\frac{2}{5}$　　C. $\frac{3}{5}$　　D. $\frac{4}{5}$　　E. 1

14. 方程 $6^x-2^x=2^{x+1}-6^{x+1}+2^{x+2}$ 的解為

A. $x=-1$　　B. $x=-\frac{1}{2}$　　C. $x=0$　　D. $x=\frac{1}{2}$　　E. $x=1$

15. 彼得先面向正東方走了 n 公里，然後他向右轉了 150 度並走了 3 公里，結果他離出發點恰好為 $\sqrt{3}$ 公里。求 n 之值。

A. $3\sqrt{3}$　　B. $\frac{\sqrt{3}}{3}$　　C. $\frac{1}{3}$　　D. 3　　E. $2\sqrt{3}$ 或 $\sqrt{3}$

第二部份 解答題。

1. 已知二次函數 $f(x)=ax^2+bx+c$ 的圖像 C 通過點 $(5,0)$，對稱軸為 $x=2$，且 $f(x)$ 有最小值 -9。

(a) 求 a、b 和 c 之值。 (3 分)

(b) 將圖像 C 向左平移 3 個單位，再向上平移 3 個單位，求所得圖像的函數表達式。 (2 分)

(c) 設函數 $g(x)=f(3\sin x)$，求函數 $g(x)$ 的最大值和最小值。 (3 分)

2. 設 $\{a_n\}_{n\geq 1}$ 是首項為 1 的等比數列。數列 $\{b_n\}_{n\geq 1}$ 滿足 $b_n=\dfrac{na_n}{4}$。已知 $a_1, 4a_2, 16a_3$ 成等差數列。

(a) 求 $\{a_n\}_{n\geq 1}$ 和 $\{b_n\}_{n\geq 1}$ 的通項公式。 (3 分)

(b) 求 $\{a_n\}_{n\geq 1}$ 的前 n 項和 S_n 及 $\{b_n\}_{n\geq 1}$ 的前 n 項和 T_n。 (5 分)

3. 已知橢圓 $C:\dfrac{x^2}{b^2}+\dfrac{y^2}{a^2}=1$ $(a>b>0)$ 的離心率為 $\dfrac{\sqrt{2}}{2}$，且點 $(2\sqrt{2},4)$ 在 C 上。

(a) 求 C 的方程。 (3 分)

(b) 設直線 $L: y=k_1x+b$ 不通過原點 O 且不平行於坐標軸。直線 L 與 C 有兩個交點 A 和 B，線段AB 的中點為 M，直線 OM 的斜率為 k_2。證明 $k_1k_2=-2$。 (5 分)

4. (a) 把 $\sqrt{3}\cos\theta-\sin\theta$ 以 $r\cos(\theta+\beta)$ 形式表示，並求 $\sqrt{3}\cos\theta-\sin\theta$ 的取值範圍。 (2 分)

(b) 求 $\sqrt{3}\cos\theta-\sin\theta=-1$ 在 $0\leq\theta\leq 2\pi$ 的解。答案要以弧度表示。 (3 分)

(c) 如果 $\sqrt{3}\cos\theta-\sin\theta=-\dfrac{1}{2}$，且 $0\leq\theta\leq\pi$，求 $\sin\left(\dfrac{\theta}{2}+\dfrac{\pi}{12}\right)$。 (3 分)

5. 用數學歸納法證明對任意正整數 n，$3^{4n+2}+5^{2n+1}$ 能被 14 整除。 (8 分)

參考答案

第一部份 選擇題。

題目編號	最佳答案
1	B
2	C
3	E
4	B
5	D
6	A
7	A
8	D
9	D
10	A
11	B
12	E
13	C
14	C
15	E

(第二部份答案由下頁開始)

<u>第二部份</u> 解答題。

1. (a) 由題意知

$$\begin{cases} 25a+5b+c=0 \\ b=-4a \\ 4a+2b+c=-9 \end{cases},$$

解得

$$a=1,\ b=-4,\ c=-5\ 。$$

因此所求函數解析式為

$$f(x)=x^2-4x-5\ 。$$

(b) $y=(x+1)^2-6=x^2+2x-5$ 。

(c) $g(x)=(3\sin x)^2-4(3\sin x)-5=(3\sin x-2)^2-9$ 。

由於 $-3\le 3\sin x\le 3$ ，$2\in[-3,3]$ ，函數 $g(x)$ 在 $3\sin x=-3$ 處取得最大值，即

$$\text{Max } g(x)=f(-3)=16\ 。$$

函數 $g(x)$ 在 $3\sin x=2$ 處取得最小值, 即

$$\text{Min } g(x)=f(2)=-9\ 。$$

2. (a) 設 $a_n=a_1q^{n-1}$ 。由 $a_1, 4a_2, 16a_3$ 成等差數列知 $16a_3+a_1=8a_2$ ，

即 $16a_1q^2+a_1=8a_1q$ 。由於 $a_1=1$ ，可得 $q=\frac{1}{4}$ 。因此

$$a_n=\left(\frac{1}{4}\right)^{n-1},\ b_n=\frac{n}{4^n}\ 。$$

(b) 由等比數列前 n 項和公式得，

$$S_n=\frac{4}{3}\left(1-\frac{1}{4^n}\right)\ 。$$

由

$$T_n=\frac{1}{4}+\frac{2}{4^2}+\frac{3}{4^3}+\frac{4}{4^4}+\cdots+\frac{n-1}{4^{n-1}}+\frac{n}{4^n}\ , \qquad (1)$$

我們有

$$4T_n=1+\frac{2}{4}+\frac{3}{4^2}+\frac{4}{4^3}+\cdots+\frac{n}{4^{n-1}}\ 。 \qquad (2)$$

(2) 式減 (1) 式得

$$3T_n=1+\frac{1}{4}+\frac{1}{4^2}+\frac{1}{4^3}+\cdots+\frac{1}{4^{n-1}}-\frac{n}{4^n}=\frac{4}{3}\left(1-\frac{1}{4^n}\right)-\frac{n}{4^n}\ 。$$

因此

$$T_n=\frac{4}{9}\left(1-\frac{1}{4^n}\right)-\frac{n}{3\cdot 4^n}\ 。$$

3. (a) 由題意知 a 和 b 滿足

$$\begin{cases} \dfrac{\sqrt{a^2-b^2}}{a}=\dfrac{\sqrt{2}}{2} \\ \dfrac{8}{b^2}+\dfrac{16}{a^2}=1 \end{cases},$$

解得 $a^2=32$ 及 $b^2=16$。因此所求橢圓方程為

$$\frac{x^2}{16}+\frac{y^2}{32}=1 \text{。}$$

(b) 聯立直線 $y=k_1x+b$ 及橢圓 $\frac{x^2}{16}+\frac{y^2}{32}=1$ 得

$$\frac{x^2}{16}+\frac{(k_1x+b)^2}{32}=1 \text{。}$$

整理得

$$(2+k_1^2)x^2+2k_1bx+b^2-32=0 \text{。}$$

線段 AB 的中點 $M(x_0,y_0)$ 的坐標為

$$x_0=\frac{-k_1b}{2+k_1^2} \text{，} y_0=\frac{2b}{2+k_1^2} \text{。}$$

於是我們得到直線 OM 的斜率為

$$k_2=\frac{-2}{k_1} \text{。}$$

因此，$k_1k_2=-2$。

4. (a) $\sqrt{3}\cos\theta-\sin\theta=2(\frac{\sqrt{3}}{2}\cos\theta-\frac{1}{2}\sin\theta)=2(\cos\frac{\pi}{6}\cos\theta-\sin\frac{\pi}{6}\sin\theta)=2\cos(\theta+\frac{\pi}{6})$。

由此知 $-2\le\sqrt{3}\cos\theta-\sin\theta\le 2$。

(b) 由 (a) 可知 $\sqrt{3}\cos\theta-\sin\theta=2\cos(\theta+\frac{\pi}{6})$。

由 $\sqrt{3}\cos\theta-\sin\theta=-1$，得 $\cos(\theta+\frac{\pi}{6})=-\frac{1}{2}$。

因為 $0\le\theta\le 2\pi$，我們有 $\theta+\frac{\pi}{6}=\frac{2}{3}\pi$ 或 $\frac{4}{3}\pi$，即 $\theta=\frac{\pi}{2}$ 或 $\frac{7}{6}\pi$。

(c) 因為 $\sqrt{3}\cos\theta-\sin\theta=-\frac{1}{2}$，所以 $2\cos(\theta+\frac{\pi}{6})=-\frac{1}{2}$，即 $\cos(\theta+\frac{\pi}{6})=-\frac{1}{4}$。

由二倍角公式得 $1-2\sin^2(\frac{\theta}{2}+\frac{\pi}{12})=-\frac{1}{4}$，因此 $\sin^2(\frac{\theta}{2}+\frac{\pi}{12})=\frac{5}{8}$。由於 $0\le\theta\le\pi$，從而有 $0\le\frac{\theta}{2}+\frac{\pi}{12}<\pi$，因此 $\sin(\frac{\theta}{2}+\frac{\pi}{12})=\sqrt{\frac{5}{8}}=\frac{\sqrt{10}}{4}$。

5. 證明：設 $S(n)$ 表示命題“$3^{4n+2}+5^{2n+1}$ 能被 14 整除”。

(1) 當 $n=1$ 時，原式 $=3^{4+2}+5^{2+1}=3^6+5^3=729+125=854$ ，$854\div 14=16$ ，$S(1)$ 成立。

(2) 假設 $n=k(k\in\mathbb{Z}^+)$ 時，$S(n)$ 成立，即 $3^{4k+2}+5^{2k+1}$ 能被14整除。

當 $n=k+1(k\in\mathbb{Z}^+)$ 時，

$$\begin{aligned}&3^{4(k+1)+2}+5^{2(k+1)+1}\\=&3^{4k+4+2}+5^{2k+2+1}\\=&3^4\cdot 3^{4k+2}+5^2\cdot 5^{2k+1}\\=&81\cdot 3^{4k+2}+25\cdot 5^{2k+1}\\=&56\cdot 3^{4k+2}+25\cdot 3^{4k+2}+25\cdot 5^{2k+1}\\=&56\cdot 3^{4k+2}+25\cdot(3^{4k+2}+5^{2k+1}),\end{aligned}$$

上式中 $56\cdot 3^{4k+2}$ 可以被14整除，且 $25\cdot(3^{4k+2}+5^{2k+1})$ 根據假設也可以被 14 整除，$S(k+1)$ 也成立。

由 (1)，(2) 及數學歸納法，對任意正整數 n，$3^{4n+2}+5^{2n+1}$ 能被 14 整除。

<u>Part I</u> Multiple choice questions. Choose the *best answer* for each question.

1. Which of the following is an empty set?

A. $\{0\}$ B. $\{x : \sin x + \cos x = 3\}$ C. $\{x : x^2 - 1 = 0, x \in \mathbb{R}\}$

D. $\{\varnothing\}$ E. $\{(x, y) : x^2 + y^2 = 0, x \in \mathbb{R}, y \in \mathbb{R}\}$

2. If $a > b > 0$ and $m > 0$, which of the following inequalities is true?

A. $\frac{b}{a} > \frac{b+m}{a+m}$ B. $\frac{a}{b} < \frac{a-m}{b-m}$ C. $\frac{b}{a} < \frac{b+m}{a+m}$

D. $\frac{a}{b} > \frac{a-m}{b-m}$ E. none of the above

3. If the polynomial $x^3 + 3x^2 + ax + b$ is divided by $x-2$ and $x+1$ respectively, the remainders are equal. Find the value of a.

A. 4 B. –3 C. –4 D. 9 E. –6

4. $\sqrt{7-4\sqrt{3}} =$

A. $\sqrt{3}-\sqrt{2}$ B. $2-\sqrt{3}$ C. $\sqrt{3}-2$

D. $2-\sqrt{6}$ E. $2-2\sqrt{3}$

5. If the equation $px^2 - 2(p+3)x + p - 1 = 0$ has real roots, find the range of p.

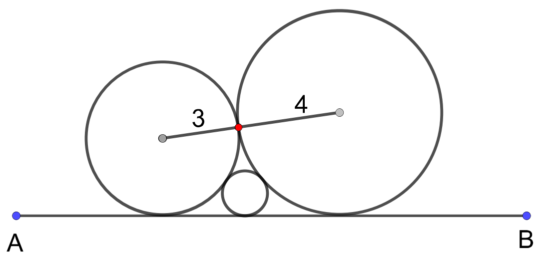
A. $0 \le p \le \frac{5}{7}$ B. $p \ge -\frac{5}{7}$ C. $-\frac{5}{7} \le p \le 1$

D. $p \ge -\frac{9}{7}$ E. none of the above

6. Given $2^a = 5^b = \sqrt{10}$, what is the value of $\frac{1}{a} + \frac{1}{b}$?

A. 2 B. 1 C. $\sqrt{2}$ D. $\frac{\sqrt{2}}{2}$ E. $\frac{1}{2}$

7. In the right figure, the three circles and the line segment AB touch each other. The two larger circles have radii 3 units and 4 units, respectively. Find the radius of the smallest circle.



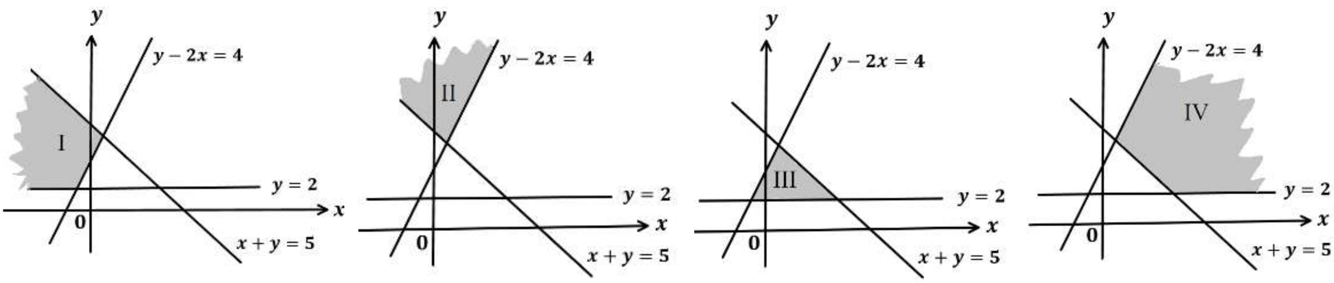
A. $84-48\sqrt{3}$ B. $\sqrt{2}-1$ C. $2\sqrt{2}-2$

D. $6-4\sqrt{2}$ E. $42-24\sqrt{3}$

8. John has probability $\frac{1}{4}$ of winning a game. What is the probability that he wins at least one game in three consecutive games?

A. $\frac{1}{64}$ B. $\frac{27}{64}$ C. $\frac{35}{64}$ D. $\frac{37}{64}$ E. $\frac{43}{64}$

9. In the below figures, which of the regions is the solution set to the system of inequalities $\begin{cases} y-2x \le 4 \\ x+y \ge 5 \\ y-2 \ge 0 \end{cases}$?



A. I B. II C. III D. IV E. none of the above

10. Suppose that the line L passes through the point (1, 2), and is perpendicular to the line $2x-3y=4$. L and the y-axis will intersect at

A. $\left(0,\frac{7}{2}\right)$ B. $\left(0,\frac{5}{2}\right)$ C. (0, 3) D. $\left(0,-\frac{1}{2}\right)$ E. $\left(0,\frac{3}{2}\right)$

11. There are 6400 chairs in an indoor stadium. Each row of the venue has 32 chairs. Current social distancing measure requires that no more than 4 consecutive chairs could be occupied in the same row. To comply with this requirement, the maximum number of chairs that could be occupied in each performance is

A. 5000 B. 5200 C. 5400 D. 5600 E. 5800

12. Suppose the two real roots of the equation $3x^2-8x+m=0$ are x_1 and x_2. If the arithmetic mean of $\frac{1}{x_1}$ and $\frac{1}{x_2}$ is 2, what is the value of m?

A. −2 B. −1 C. 4 D. 1 E. 2

13. Two different numbers are picked up from 1, 2, 3, 4, and 5 sequentially to form the tens and unit digits of a two-digit number. The probability that the two-digit number is less than 40 is

A. $\frac{1}{5}$ B. $\frac{2}{5}$ C. $\frac{3}{5}$ D. $\frac{4}{5}$ E. 1

14. The solution of the equation $6^x-2^x=2^{x+1}-6^{x+1}+2^{x+2}$ is

A. $x=-1$ B. $x=-\frac{1}{2}$ C. $x=0$ D. $x=\frac{1}{2}$ E. $x=1$

15. Peter first faced east and walked for n kilometers, and then he turned 150 degrees to the right and walked 3 kilometers. Now he was $\sqrt{3}$ kilometers away from the starting point. Find the value of n.

A. $3\sqrt{3}$ B. $\frac{\sqrt{3}}{3}$ C. $\frac{1}{3}$ D. 3 E. $2\sqrt{3}$ or $\sqrt{3}$

Part II **Problem-solving questions.**

1. Given that the graph *C* of the quadratic function $f(x) = ax^2 + bx + c$ passes through the point (5, 0). Its axis of symmetry is $x = 2$, and the minimum value of $f(x)$ is −9.
   (a) Find the values of *a*, *b* and *c*. (3 marks)
   (b) Find the expression of the function after shifting the graph *C* to the left by 3 units, and then shifting it up by 3 units. (2 marks)
   (c) Let $g(x) = f(3\sin x)$. Find the maximum and minimum values of $g(x)$. (3 marks)

2. Let $\{a_n\}_{n\geq 1}$ be a geometric sequence with 1 as the first term, and sequence $\{b_n\}_{n\geq 1}$ is given as $b_n = \frac{na_n}{4}$. Suppose that a_1, $4a_2$, $16a_3$ form an arithmetic sequence.
   (a) Find the general terms for $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$. (3 marks)
   (b) Find S_n, sum of the first *n* terms for $\{a_n\}_{n\geq 1}$, and T_n, sum of the first *n* terms for $\{b_n\}_{n\geq 1}$. (5 marks)

3. Suppose the eccentricity of the ellipse $C: \frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \ (a > b > 0)$ is $\frac{\sqrt{2}}{2}$, and $(2\sqrt{2}, 4)$ is a point lying on *C*.
   (a) Find the equation of *C*. (3 marks)
   (b) Let $L: y = k_1 x + b$ be one line which does not pass through the origin *O* and is not parallel to the coordinate axes. There are two intersections *A* and *B* for the line *L* and the ellipse *C*. The midpoint of the line segment *AB* is *M*, and the slope of the line *OM* is k_2. Show that $k_1 k_2 = -2$. (5 marks)

4. (a) Express $\sqrt{3}\cos\theta - \sin\theta$ in the form of $r\cos(\theta + \beta)$, and find the range of $\sqrt{3}\cos\theta - \sin\theta$. (2 marks)
   (b) Solve $\sqrt{3}\cos\theta - \sin\theta = -1$ for $0 \leq \theta \leq 2\pi$. Answer is to be presented in radians. (3 marks)
   (c) If $\sqrt{3}\cos\theta - \sin\theta = -\frac{1}{2}$, and $0 \leq \theta \leq \pi$, find $\sin\left(\frac{\theta}{2} + \frac{\pi}{12}\right)$. (3 marks)

5. Prove by mathematical induction that $3^{4n+2} + 5^{2n+1}$ is divisible by 14 for any positive integer *n*. (8 marks)

Suggested Answer

<u>Part I</u> Multiple choice questions.

Question Number	Best Answer
1	B
2	C
3	E
4	B
5	D
6	A
7	A
8	D
9	D
10	A
11	B
12	E
13	C
14	C
15	E

(Answers for Part II start from next page)

<u>Part II</u> Problem-solving questions.

1. (a) The coefficients of the quadratic function satisfy the following system

$$\begin{cases} 25a+5b+c=0 \\ b=-4a \\ 4a+2b+c=-9 \end{cases}.$$

Solving the equation system yields

$$a=1,\ b=-4,\ c=-5.$$

The function is

$$f(x)=x^2-4x-5.$$

(b) $y=(x+1)^2-6=x^2+2x-5$.

(c) We have $g(x)=(3\sin x)^2-4(3\sin x)-5=(3\sin x-2)^2-9$.

Since $-3\le 3\sin x\le 3$, $2\in[-3,3]$, the function obtains its maximum value at $3\sin x=-3$, that is,

$$\text{Max } g(x) = f(-3)=16.$$

The function attains its minimum value at $3\sin x=2$, that is,

$$\text{Min } g(x) = f(2)=-9.$$

2. (a) Let $a_n=a_1q^{n-1}$. As $a_1, 4a_2, 16a_3$ are in arithmetic progression, so $16a_3+a_1=8a_2$, i.e.,

$$16a_1q^2+a_1=8a_1q.$$

Since $a_1=1$, we have $q=\frac{1}{4}$. Thus

$$a_n=\left(\frac{1}{4}\right)^{n-1},\ b_n=\frac{n}{4^n}.$$

(b) From the geometric sum formula, we get $S_n=\frac{4}{3}\left(1-\frac{1}{4^n}\right)$.

Since

$$T_n=\frac{1}{4}+\frac{2}{4^2}+\frac{3}{4^3}+\frac{4}{4^4}+\cdots+\frac{n-1}{4^{n-1}}+\frac{n}{4^n}, \quad (1)$$

we have

$$4T_n=1+\frac{2}{4}+\frac{3}{4^2}+\frac{4}{4^3}+\cdots+\frac{n}{4^{n-1}}. \quad (2)$$

Subtracting (1) from (2) gives

$$3T_n=1+\frac{1}{4}+\frac{1}{4^2}+\frac{1}{4^3}+\cdots+\frac{1}{4^{n-1}}-\frac{n}{4^n}=\frac{4}{3}\left(1-\frac{1}{4^n}\right)-\frac{n}{4^n}.$$

Hence

$$T_n=\frac{4}{9}\left(1-\frac{1}{4^n}\right)-\frac{n}{3\cdot 4^n}.$$

3. (a) a and b satisfy the system

$$\begin{cases} \dfrac{\sqrt{a^2-b^2}}{a}=\dfrac{\sqrt{2}}{2} \\ \dfrac{8}{b^2}+\dfrac{16}{a^2}=1 \end{cases}.$$

This implies that $a^2=32$, $b^2=16$. Thus the equation is

$$\frac{x^2}{16}+\frac{y^2}{32}=1.$$

(b) Combining the equations of the line $y=k_1x+b$ and the ellipse $\frac{x^2}{16}+\frac{y^2}{32}=1$, we get

$$\frac{x^2}{16}+\frac{(k_1x+b)^2}{32}=1.$$

That is, $(2+k_1^2)x^2+2k_1bx+b^2-32=0$. Then the coordinates of M are

$$x_0=\frac{-k_1b}{2+k_1^2},\ y_0=\frac{2b}{2+k_1^2}.$$

Therefore, the slope of the line segment OM is

$$k_2=\frac{-2}{k_1}.$$

Consequently, $k_1k_2=-2$.

4. (a) $\sqrt{3}\cos\theta-\sin\theta=2(\frac{\sqrt{3}}{2}\cos\theta-\frac{1}{2}\sin\theta)=2(\cos\frac{\pi}{6}\cos\theta-\sin\frac{\pi}{6}\sin\theta)=2\cos(\theta+\frac{\pi}{6})$.

It follows from the above that $-2\le\sqrt{3}\cos\theta-\sin\theta\le 2$.

(b) From the result of (a), we get $\sqrt{3}\cos\theta-\sin\theta=2\cos(\theta+\frac{\pi}{6})$.

Since $\sqrt{3}\cos\theta-\sin\theta=-1$, so $\cos(\theta+\frac{\pi}{6})=-\frac{1}{2}$.

Since $0\le\theta\le 2\pi$, we have $\theta+\frac{\pi}{6}=\frac{2}{3}\pi$ or $\frac{4}{3}\pi$, that is $\theta=\frac{\pi}{2}$ or $\frac{7}{6}\pi$.

(c) Since $\sqrt{3}\cos\theta-\sin\theta=-\frac{1}{2}$, therefore $2\cos(\theta+\frac{\pi}{6})=-\frac{1}{2}$, that is $\cos(\theta+\frac{\pi}{6})=-\frac{1}{4}$.

By the double-angle formula, we have $1-2\sin^2(\frac{\theta}{2}+\frac{\pi}{12})=-\frac{1}{4}$, so $\sin^2(\frac{\theta}{2}+\frac{\pi}{12})=\frac{5}{8}$. And since $0\le\theta\le\pi$, we get $0\le\frac{\theta}{2}+\frac{\pi}{12}$.

Therefore

$$\sin(\frac{\theta}{2}+\frac{\pi}{12})=\sqrt{\frac{5}{8}}=\frac{\sqrt{10}}{4}.$$

5. Proof: Let $S(n)$ be the statement "$3^{4n+2}+5^{2n+1}$ is divisible by 14".

(1). When $n=1$,

$$3^{4+2}+5^{2+1}=3^6+5^3=729+125=854,\ 854\div 14=16.$$

Therefore, $S(1)$ is true.

(2). Assume that $S(n)$ is true for $n=k(k\in\mathbb{Z}^+)$, i.e., $3^{4k+2}+5^{2k+1}$ is divisible by 14, where $k\in\mathbb{Z}^+$.

So when $n=k+1(k\in\mathbb{Z}^+)$,

$$\begin{aligned}&3^{4(k+1)+2}+5^{2(k+1)+1}\\=&3^{4k+4+2}+5^{2k+2+1}\\=&3^4\cdot 3^{4k+2}+5^2\cdot 5^{2k+1}\\=&81\cdot 3^{4k+2}+25\cdot 5^{2k+1}\\=&56\cdot 3^{4k+2}+25\cdot 3^{4k+2}+25\cdot 5^{2k+1}\\=&56\cdot 3^{4k+2}+25\cdot(3^{4k+2}+5^{2k+1}),\end{aligned}$$

Where $56\cdot 3^{4k+2}$ is divisible by 14, and according to the assumption, $25\cdot(3^{4k+2}+5^{2k+1})$ is also divisible by 14.

In other words, $S(k+1)$ is also true.

By (1), (2) and the Principle of Mathematical Induction, the statement is true for all positive integers n.