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澳門科技大學
UNIVERSIDADE DE CIÊNCIA E TECNOLOGIA DE MACAU
MACAU UNIVERSITY OF SCIENCE AND TECHNOLOGY

澳門四高校聯合入學考試（語言科及數學科）

Joint Admission Examination for Macao Four Higher Education Institutions (Languages and Mathematics)

2023 年試題及參考答案
2023 Examination Paper and Suggested Answer

數學附加卷 Mathematics Supplementary Paper

注意事項：

1. 考生獲發文件如下:

   1.1 本考卷包括封面共 22 版

   1.2 草稿紙一張

2. 請於本考卷封面填寫聯考編號、考場、樓宇、考室及座號。

3. 本考卷共有五條解答題，每題二十分，任擇三題作答。全卷滿分為六十分。

4. 必須在考卷內提供的橫間頁內作答，寫在其他地方的答案將不會獲評分。

5. 必須將解題步驟清楚寫出。只當答案和所有步驟正確而清楚地表示出來，考生方可獲得滿分。

6. 本考卷的圖形並非按比例繪畫。

7. 考試中不可使用任何形式的計算機。

8. 請用藍色或黑色原子筆作答。

9. 考試完畢，考生須交回本考卷及草稿紙。

Instructions:

1. Each candidate is provided with the following documents:

   1.1 Question paper including cover page – 22 pages

   1.2 One sheet of draft paper

2. Fill in your JAE No., campus, building, room and seat no. on the front page of the examination paper.

3. There are 5 questions in this paper, each carries 20 marks. Answer any 3 questions. Full mark of this paper is 60.

4. Put your answers in the lined pages provided. Answers put elsewhere will not be marked.

5. Show all your steps in getting to the answer. Full credits will be given only if the answer and all the steps are correct and clearly shown.

6. The diagrams in this examination paper are not drawn to scale.

7. Calculators of any kind are not allowed in the examination.

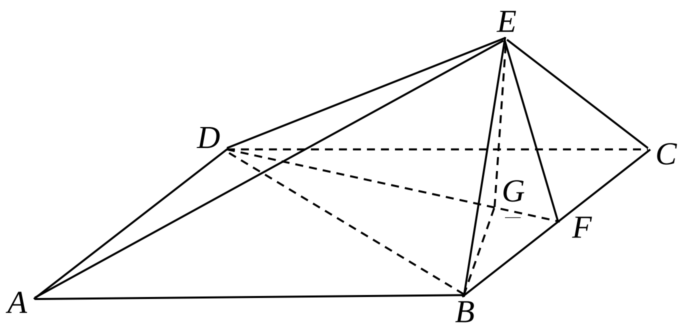
8. Answer the questions with a blue or black ball pen.

9. Candidates must return the question paper and draft paper at the end of the examination.

任擇三題作答，每題二十分。請把答案寫在緊接每條題目之後的 3 頁橫間頁內。

Answer any 3 questions, each carries 20 marks. Write down the answers on the 3 lined pages following each question.

1.



如上圖，*E-ABCD* 是四棱錐，其底 *ABCD* 為菱形，$|AB| = 2\sqrt{2}$ 及 $\angle DAB = \frac{\pi}{3}$。

$|EB| = |EC|$，$\angle BEC = \frac{\pi}{2}$，*F* 為 *BC* 的中點，*G* 為 *E* 在 *DF* 的垂足。$|DE| = \sqrt{5}$。

(a) 證明 $\cos(\angle DFE) = \frac{\sqrt{3}}{4}$。[提示: 求 $|EF|$ 及 $|DF|$。] (8 分)

(b) 證明 *EG* 垂直平面 *ABCD*。[提示: 證明 ΔEGB是直角三角形。] (7 分)

(c) 求 $|AE|$。 (5 分)

In the above figure, *E-ABCD* is a pyramid, its base *ABCD* is a rhombus, $|AB| = 2\sqrt{2}$ and $\angle DAB = \frac{\pi}{3}$. $|EB| = |EC|$, $\angle BEC = \frac{\pi}{2}$, *F* is the midpoint of *BC*, *G* is the foot of perpendicular from *E* to *DF*. $|DE| = \sqrt{5}$.

(a) Show that $\cos(\angle DFE) = \frac{\sqrt{3}}{4}$**.** [Hint. Find $|EF|$ and $|DF|$.] (8 marks)

(b) Show that *EG* is perpendicular to plane *ABCD*.

[Hint. Show that ΔEGB is a right-angled triangle.] (7 marks)

(c) Find $|AE|$. (5 marks)

2. (a) 已知函數 $f(x) = x^3 - 12x + 6$。

(i) 求 $f'(x)$ 及 $f''(x)$。 (2 分)

(ii) 求 $f(x)$ 的局部極大值和局部極小值。 (3 分)

(iii) 求曲線 $y = f(x)$ 的拐點。 (2 分)

(iv) 運用 (i) – (iii) 的結果，繪出曲線 $y = f(x)$。 (3 分)

(b) 已知直線 $L: y = x + 4$ 是曲線 $C: y = x^3 + 3x^2 + x$ 在點 A 的一條切線。

(i) 求點 A。 (4 分)

(ii) 求由直線 L 与曲線 C 所包圍的區域的面積。 (6 分)

(a) Given function $f(x) = x^3 - 12x + 6$.

(i) Find $f'(x)$ and $f''(x)$. (2 marks)

(ii) Find the local maximum and local minimum values of $f(x)$. (3 marks)

(iii) Find the inflection point(s) of the curve $y = f(x)$. (2 marks)

(iv) Using the results in (i) – (iii), sketch the curve $y = f(x)$. (3 marks)

(b) Given that the line $L: y = x + 4$ is a tangent line of the curve $C: y = x^3 + 3x^2 + x$ at point A.

(i) Find the point A. (4 marks)

(ii) Find the area of the region bounded by the line L and the curve C. (6 marks)

3. 已知橢圓 E: $\frac{x^2}{9}+\frac{y^2}{4}=1$ 及 E 以外的一點 $A(h,k)$。設 L_1 和 L_2 是過點 A 且與 E 相切的兩條直線。

(a) (i) 證明過點 A 直線 $y=mx+(k-mh)$ 與 E 相切當且僅當

$$9m^2-(k-mh)^2+4=0。$$ (5 分)

(ii) 設 m_1 和 m_2 是 (a)(i)中以 m 為未知量的二次方程的兩個根，證明

$$m_1+m_2=\frac{2hk}{h^2-9} 及 m_1m_2=\frac{k^2-4}{h^2-9}。$$ (3 分)

(b) 若 L_1 與 L_2 互相垂直，求點 A 的軌跡。 (6 分)

(c) 若 $(h,k)=(5,4)$，求 L_1 與 L_2 的夾角，答案以 $\tan^{-1}$ 表示。 (6 分)

Given ellipse E: $\frac{x^2}{9}+\frac{y^2}{4}=1$ and a point $A(h,k)$ outside E. Suppose L_1 and L_2 are two lines passing through A and tangent to E.

(a) (i) Show that the straight line $y=mx+(k-mh)$ that passes through A is tangent to the ellipse E if and only if

$$9m^2-(k-mh)^2+4=0.$$ (5 marks)

(ii) Let m_1 and m_2 be the two roots of the quadratic equation in unknown m in (a)(i). Show that $m_1+m_2=\frac{2hk}{h^2-9}$ and $m_1m_2=\frac{k^2-4}{h^2-9}$. (3 marks)

(b) If L_1 and L_2 are perpendicular, find the locus of A. (6 marks)

(c) If $(h,k)=(5,4)$, find the angle between L_1 and L_2. Express your answer in terms of $\tan^{-1}$. (6 marks)

4. 設 $i=\sqrt{-1}$。

(a) 設 $w=x+yi$，其中 x 和 y 為實數。若 w 滿足方程 $z-3\bar{z}+|z|=-1+16i$，
求 w 的值。 (8 分)

(b) (i) 用棣美弗定理，證明

$$\cos 3\theta=\cos^3\theta-3\cos\theta\sin^2\theta \quad \text{and} \quad \sin 3\theta=3\cos^2\theta\sin\theta-\sin^3\theta$$。

推導出 $\tan 3\theta=\frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}$。 (5 分)

(ii) 用 (i) 的結果，解方程 $x^3-3\sqrt{3}x^2-3x+\sqrt{3}=0$，答案以 tan 表示。 (7 分)

Let $i=\sqrt{-1}$.

(a) Let $w=x+yi$, where x and y are real numbers. If w satisfies the equation $z-3\bar{z}+|z|=-1+16i$, find the value of w. (8 marks)

(b) (i) Using De Moivre's theorem, show that

$$\cos 3\theta=\cos^3\theta-3\cos\theta\sin^2\theta \quad \text{and} \quad \sin 3\theta=3\cos^2\theta\sin\theta-\sin^3\theta.$$

Deduce that $\tan 3\theta=\frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}$. (5 marks)

(ii) Using the result in (i), solve the equation $x^3-3\sqrt{3}x^2-3x+\sqrt{3}=0$.
Express your answer in terms of tan. (7 marks)

5. (a) (i) 已知公式

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

及

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B \text{。}$$

推導出

$\sin x - \sin y = 2\cos\frac{x+y}{2}\sin\frac{x-y}{2}$ 及 $\cos x - \cos y = -2\sin\frac{x+y}{2}\sin\frac{x-y}{2}$ 。 (4 分)

(ii) 證明恆等式 $\begin{vmatrix} 1 & 1 & 1 \\ \sin 2\theta & \sin 4\theta & \sin 8\theta \\ \cos 2\theta & \cos 4\theta & \cos 8\theta \end{vmatrix} = -4\sin\theta\sin 2\theta\sin 3\theta$ 。 (6 分)

(b) 設 $0 \le \theta \le \frac{\pi}{2}$。若以下以 x、y 和 z 為未知量的方程組

$$\begin{cases} x + y + z = 6 \\ (\sin 2\theta)x + (\sin 4\theta)y + (\sin 8\theta)z = \sqrt{3} \\ (\cos 2\theta)x + (\cos 4\theta)y + (\cos 8\theta)z = -3 \end{cases}$$

有多於一個解，求此方程組的通解。 (10 分)

(a) (i) Given the formulas

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

and

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B.$$

Deduce that

$\sin x - \sin y = 2\cos\frac{x+y}{2}\sin\frac{x-y}{2}$ and $\cos x - \cos y = -2\sin\frac{x+y}{2}\sin\frac{x-y}{2}$. (4 marks)

(ii) Prove the identity $\begin{vmatrix} 1 & 1 & 1 \\ \sin 2\theta & \sin 4\theta & \sin 8\theta \\ \cos 2\theta & \cos 4\theta & \cos 8\theta \end{vmatrix} = -4\sin\theta\sin 2\theta\sin 3\theta$. (6 marks)

(b) Let $0 \le \theta \le \frac{\pi}{2}$. Suppose the following system of equations with unknowns x, y and z

$$\begin{cases} x + y + z = 6 \\ (\sin 2\theta)x + (\sin 4\theta)y + (\sin 8\theta)z = \sqrt{3} \\ (\cos 2\theta)x + (\cos 4\theta)y + (\cos 8\theta)z = -3 \end{cases}$$

has more than one solution. Find the general solution of this system of equations. (10 marks)

參考答案：

1. (a) 由 F 為 BC 的中點和 $|EB|=|EC|$，得 $EF\perp BC$ 和 $\angle EBF=\frac{1}{2}(\pi-\angle BEC)=\frac{\pi}{4}$。
所以，$|EF|=|BF|\tan(\angle EBF)=\left(\frac{1}{2}|BC|\right)\cdot 1=\frac{1}{2}|AB|=\sqrt{2}$。
由 $\angle DAB=\frac{\pi}{3}$ 和 $|AD|=|AB|$，得 ΔDAB 是等邊三角形，故 $|DB|=|AB|=2\sqrt{2}$。
由 F 為 BC 的中點和 $|DB|=|AB|=|DC|$，得 $DF\perp BC$。
所以，$|DF|^2=|DB|^2-|BF|^2=8-2=6$。
因此，$\cos(\angle DEF)=\frac{|DF|^2+|EF|^2-|DE|^2}{2|DF||EF|}=\frac{6+2-5}{2\sqrt{6}\sqrt{2}}=\frac{\sqrt{3}}{4}$。

(b) 在 ΔEFG 中，$|EG|=|EF|\sin(\angle DFE)=|EF|\sqrt{1-\cos^2(\angle DFE)}=\sqrt{2}\sqrt{1-\left(\frac{\sqrt{3}}{4}\right)^2}=\frac{\sqrt{26}}{4}$。
在 ΔBFG 中，$|BG|=\sqrt{|BF|^2+|FG|^2}=\sqrt{2+|EF|^2\cos^2(\angle DFE)}=\sqrt{2+2(\frac{3}{16})}=\frac{\sqrt{38}}{4}$。
在 ΔEFB 中，$|BE|=\sqrt{|EF|^2+|BF|^2}=\sqrt{2+2}=2$。
因 $|EG|^2+|BG|^2=4=|BE|^2$，故 $\angle EGB=\frac{\pi}{2}$。
由 $EG\perp DF$ 及 $EG\perp GB$，得 $EG\perp ABCD$。

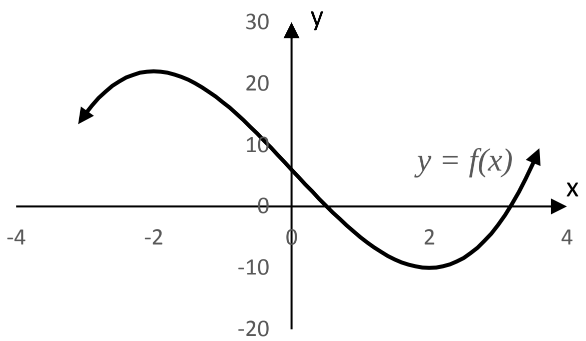
(c) 由 F 為 BC 的中點和 $|DB|=|DC|(=|BC|)$，得 $\angle BDF=\frac{1}{2}\angle BDC=\frac{\pi}{6}$。
連 AG。因 $\angle ADG=\angle ADB+\angle BDF=\frac{\pi}{3}+\frac{\pi}{6}=\frac{\pi}{2}$，故 $|AG|^2=|AD|^2+|DG|^2$。
因 $|DG|=|DF|-|FG|=\sqrt{6}-|EF|\cos(\angle DFE)=\sqrt{6}-\sqrt{2}\left(\frac{\sqrt{3}}{4}\right)=\frac{3\sqrt{6}}{4}$，
故 $|AG|^2=|AD|^2+|DG|^2=8+\frac{54}{16}=\frac{91}{8}$。
所以，$|AE|=\sqrt{|AG|^2+|EG|^2}=\sqrt{\frac{91}{8}+\frac{13}{8}}=\sqrt{13}$。

2. (a)(i) $f'(x)=3x^2-12$， $f''(x)=6x$。

(ii) $f'(x)=0\Leftrightarrow x=-2$ 或 $x=2$。
當 $x<-2$，$f'(x)>0$，故 $f(x)$ 是遞增的。
當 $-2<x<2$， $f'(x)<0$ ，故 $f(x)$ 是遞減的。
當 $2<x$，$f'(x)>0$ ，故 $f(x)$ 是遞增的。
因此， $f(-2)=22$ 是一局部極大值， $f(2)=-10$ 是一局部極小值。

(iii) $f''(x)=0\Leftrightarrow x=0$。當 $x<0$， $f''(x)<0$; 當 $x>0$， $f''(x)>0$。
因此，$(0,6)$ 是曲線 $y=f(x)$ 的拐點。

(iv)



(b)(i) 直線 L 的斜率是 1。在點 A，有 $1=\frac{dy}{dx}=3x^2+6x+1$，解此方程，得 $x=0$ 或 $x=-2$。

當 $x=0$，在直線 L 上的點是 (0,4)，但在曲線 C 上的點是 (0,0)，故 $x=0$ 不合。

當 $x=-2$, $(-2,2)$ 是在直線 L 和在曲線 C 之上。因此，點 A 是 $(-2,2)$。

(ii) 解 $\begin{cases} y=x^3+3x^2+x \\ y=x+4 \end{cases}$，(由(i)，$x=-2$ 是一個解) 得 $x=-2$ 或 $x=1$。

當 $-2<x<1$，直線 $y=x+4$ 在曲線 $y=x^3+3x^2+x$ 之上。

因此，所求面積為

$$\int_{-2}^{1}(x+4)-(x^3+3x^2+x)\,dx = \int_{-2}^{1}4-x^3-3x^2\,dx$$
$$=\left[4x-\frac{1}{4}x^4-x^3\right]_{-2}^{1}$$
$$=\frac{27}{4}\text{。}$$

3. (a)(i) 由 $\begin{cases} \frac{x^2}{9}+\frac{y^2}{4}=1 \\ y=mx+(k-mh) \end{cases}$，得 $4x^2+9[mx+(k-mh)]^2=36$，即

$$(4+9\mathrm{m}^2)x^2+18m(k-mh)x+9(k-mh)^2-36=0\text{。} \qquad (1)$$

因直線$y=mx+(k-mh)$ 與 E 相切，(1) 有重根，其判別式為 0，即

$[18m(k-mh)]^2-4(4+9\mathrm{m}^2)[9(k-mh)^2-36]=0$。由此，得

$$9m^2-(k-mh)^2+4=0\text{。} \qquad (2)$$

(ii) 由 (2) 得

$$(9-h^2)m^2+2hkm+(4-k^2)=0\text{。} \qquad (3)$$

因 m_1 和 m_2 是 (3) 的根，故 $m_1+m_2=\frac{2hk}{h^2-9}$ 及 $m_1m_2=\frac{k^2-4}{h^2-9}$。

(b)當 L_1 和 L_2 不是一對垂直和水平線時，

$$m_1m_2=-1 \Rightarrow \frac{k^2-4}{h^2-9}=-1 \Rightarrow h^2+k^2=13, h\neq\pm3, k\neq\pm2\text{。}$$

當 L_1 和 L_2 是一對垂直和水平線時，容易得出 $(h,k)=(\pm3,\pm2)$。

故 A 的軌跡是 $x^2+y^2=13$。

(c) 對 $i = 1,2,$ 設 $m_i = \tan\theta_i$ 為 L_i 的斜率，其中 $-\frac{\pi}{2} < \theta_i < \frac{\pi}{2}$。

由 $(h,k) = (5,4)$，得 $m_1 + m_2 = \frac{5}{2}$ 及 $m_1 m_2 = \frac{3}{4}$。

設 L_1 與 L_2 的夾角為 α， $0 < \alpha < \frac{\pi}{2}$，則

$$\tan\alpha = |\tan(\theta_1 - \theta_2)| = \left|\frac{\tan\theta_1 - \tan\theta_2}{1+\tan\theta_1\tan\theta_2}\right| = \left|\frac{m_1 - m_2}{1+m_1 m_2}\right| = \frac{\sqrt{(m_1+m_2)^2 - 4m_1 m_2}}{|1+m_1 m_2|}$$
$$= \frac{2\sqrt{13}}{7} 。$$

因此，L_1 與 L_2 的夾角是 $\tan^{-1}\frac{2\sqrt{13}}{7}$。

4. (a)
$$\begin{aligned} z - 3\bar{z} + |z| = -1 + 16i &\Rightarrow (x+yi) - 3(x-yi) + \sqrt{x^2+y^2} = -1 + 16i \\ &\Rightarrow -2x + \sqrt{x^2+y^2} + 4yi = -1 + 16i \\ &\Rightarrow \begin{cases} -2x + \sqrt{x^2+y^2} = -1 \\ 4y = 16 \end{cases} 。\end{aligned}$$

由第二條方程得 $y = 4$。對 $y = 4$，解第一條方程，有

$$\begin{aligned} -2x + \sqrt{x^2+16} = -1 &\Rightarrow \sqrt{x^2+16} = 2x - 1 \\ &\Rightarrow x^2 + 16 = (2x-1)^2 \text{ 及 } x \geq 1/2 \\ &\Rightarrow 3x^2 - 4x - 15 = 0 \text{ 及 } x \geq 1/2 \\ &\Rightarrow (x-3)(3x+5) = 0 \text{ 及 } x \geq 1/2 \\ &\Rightarrow x = 3 \end{aligned}$$

故 $w = 3 + 4i$。

(b) (i)
$$\begin{aligned} \cos 3\theta + i\sin 3\theta &= (\cos\theta + i\sin\theta)^3 \\ &= \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta \\ &= (\cos^3\theta - 3\cos\theta\sin^2\theta) + i(3\cos^2\theta\sin\theta - \sin^3\theta) 。\end{aligned}$$

比較實部及虛部，得

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta \text{ 及 } \sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta 。$$

由此，

$$\tan 3\theta = \frac{\sin 3\theta}{\cos 3\theta} = \frac{(3\cos^2\theta\sin\theta - \sin^3\theta)/\cos^3\theta}{(\cos^3\theta - 3\cos\theta\sin^2\theta)/\cos^3\theta} = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta} 。$$

(ii) 設 $x = \tan\theta$，$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$。則

$$\begin{aligned} & x^3 - 3\sqrt{3}x^2 - 3x + \sqrt{3} = 0 \\ \Rightarrow\ & \tan^3\theta - 3\sqrt{3}\tan^2\theta - 3\tan\theta + \sqrt{3} = 0 \\ \Rightarrow\ & \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta} = \sqrt{3} \\ \Rightarrow\ & \tan 3\theta = \sqrt{3} \\ \Rightarrow\ & 3\theta = \frac{\pi}{3} + n\pi,\ n \text{ 是整數} \\ \Rightarrow\ & \theta = \frac{\pi}{9} + \frac{n\pi}{3},\ n \text{ 是整數。} \end{aligned}$$

因 $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$，故方程的根為 $\tan(-\frac{2\pi}{9})$，$\tan\frac{\pi}{9}$ 及 $\tan\frac{4\pi}{9}$。

[註: 若設 $0 \leq \theta < \frac{\pi}{2}$ 或 $\frac{\pi}{2} < \theta < \pi$，則 θ 的值為 $\frac{\pi}{9}$，$\frac{4\pi}{9}$ 及 $\frac{7\pi}{9}$。因 $\tan(-\frac{2\pi}{9}) = \tan\frac{7\pi}{9}$，兩組答案是相同的。]

5. (a)(i) $\sin x - \sin y$

$= \sin\left(\frac{x+y}{2} + \frac{x-y}{2}\right) - \sin\left(\frac{x+y}{2} - \frac{x-y}{2}\right)$

$= \left(\sin\frac{x+y}{2}\cos\frac{x-y}{2} + \cos\frac{x+y}{2}\sin\frac{x-y}{2}\right) - \left(\sin\frac{x+y}{2}\cos\frac{x-y}{2} - \cos\frac{x+y}{2}\sin\frac{x-y}{2}\right)$

$= 2\cos\frac{x+y}{2}\sin\frac{x-y}{2}$

$\cos x - \cos y$

$= \cos\left(\frac{x+y}{2} + \frac{x-y}{2}\right) - \cos\left(\frac{x+y}{2} - \frac{x-y}{2}\right)$

$= \left(\cos\frac{x+y}{2}\cos\frac{x-y}{2} - \sin\frac{x+y}{2}\sin\frac{x-y}{2}\right) - \left(\cos\frac{x+y}{2}\cos\frac{x-y}{2} + \sin\frac{x+y}{2}\sin\frac{x-y}{2}\right)$

$= -2\sin\frac{x+y}{2}\sin\frac{x-y}{2}$

(a)(ii)

$$\begin{vmatrix} 1 & 1 & 1 \\ \sin 2\theta & \sin 4\theta & \sin 8\theta \\ \cos 2\theta & \cos 4\theta & \cos 8\theta \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ \sin 2\theta & \sin 4\theta - \sin 2\theta & \sin 8\theta - \sin 2\theta \\ \cos 2\theta & \cos 4\theta - \cos 2\theta & \cos 8\theta - \cos 2\theta \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ \sin 2\theta & 2\cos 3\theta\sin\theta & 2\cos 5\theta\sin 3\theta \\ \cos 2\theta & -2\sin 3\theta\sin\theta & -2\sin 5\theta\sin 3\theta \end{vmatrix}$$

$$= 4\sin\theta\sin 3\theta \begin{vmatrix} 1 & 0 & 0 \\ \sin 2\theta & \cos 3\theta & \cos 5\theta \\ \cos 2\theta & -\sin 3\theta & -\sin 5\theta \end{vmatrix}$$

$$= 4\sin\theta\sin 3\theta\,(-\sin 5\theta\cos 3\theta + \cos 5\theta\sin 3\theta)$$
$$= 4\sin\theta\sin 3\theta\,[-\sin(5\theta - 3\theta)]$$
$$= -4\sin\theta\sin 2\theta\sin 3\theta$$

(b) 因方程組有多於一解，用(a)(ii) 的結果，有 $-4\sin\theta\sin 2\theta\sin 3\theta = 0$。

因 $0 \le \theta \le \frac{\pi}{2}$，故 $\theta = 0$ 或 $\theta = \frac{\pi}{2}$ 或 $\theta = \frac{\pi}{3}$。

若 $\theta = 0$ 或 $\theta = \frac{\pi}{2}$，第二條方程變成 $0x + 0y + 0z = \sqrt{3}$，從而可知方程組沒有解。

故 $\theta \ne 0$ 及 $\theta \ne \frac{\pi}{2}$。

當 $\theta = \frac{\pi}{3}$，方程組變成 $\begin{cases} x + y + z = 6 \\ \frac{\sqrt{3}}{2}x - \frac{\sqrt{3}}{2}y + \frac{\sqrt{3}}{2}z = \sqrt{3} \\ -\frac{1}{2}x - \frac{1}{2}y - \frac{1}{2}z = -3 \end{cases}$。

解 $\begin{cases} x + y + z = 6 \\ \frac{\sqrt{3}}{2}x - \frac{\sqrt{3}}{2}y + \frac{\sqrt{3}}{2}z = \sqrt{3} \end{cases}$，得 $x = 4 - t$，$y = 2$，$z = t$，$t \in \mathbb{R}$。

Suggested Answers:

1. (a) From F is the midpoint of BC and $|EB| = |EC|$, we get $EF \perp BC$ and $\angle EBF = \frac{1}{2}(\pi - \angle BEC) = \frac{\pi}{4}$.
Then, $|EF| = |BF|\tan(\angle EBF) = \left(\frac{1}{2}|BC|\right) \cdot 1 = \frac{1}{2}|AB| = \sqrt{2}$.
From $\angle DAB = \frac{\pi}{3}$ and $|AD| = |AB|$, we get ΔDAB is equilateral. So, $|DB| = |AB| = 2\sqrt{2}$.
From F is the midpoint of BC and $|DB| = |AB| = |DC|$, we get $DF \perp BC$.
Then, $|DF|^2 = |DB|^2 - |BF|^2 = 8 - 2 = 6$.
Hence, $\cos(\angle DEF) = \frac{|DF|^2+|EF|^2-|DE|^2}{2|DF||EF|} = \frac{6+2-5}{2\sqrt{6}\sqrt{2}} = \frac{\sqrt{3}}{4}$.

(b) In ΔEFG, $|EG| = |EF|\sin(\angle DFE) = |EF|\sqrt{1-\cos^2(\angle DFE)} = \sqrt{2}\sqrt{1-\left(\frac{\sqrt{3}}{4}\right)^2} = \frac{\sqrt{26}}{4}$.
In ΔBFG, $|BG| = \sqrt{|BF|^2+|FG|^2} = \sqrt{2+|EF|^2\cos^2(\angle DFE)} = \sqrt{2+2(\frac{3}{16})} = \frac{\sqrt{38}}{4}$.
In ΔEFB, $|BE| = \sqrt{|EF|^2+|BF|^2} = \sqrt{2+2} = 2$.
Since $|EG|^2 + |BG|^2 = 4 = |BE|^2$, we have $\angle EGB = \frac{\pi}{2}$.
From $EG \perp DF$ and $EG \perp GB$, we get $EG \perp ABCD$.

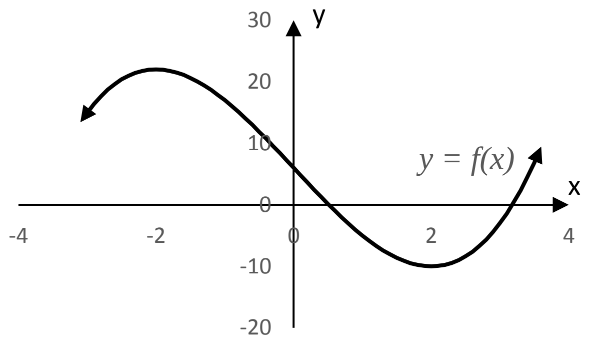
(c) From F is the midpoint of BC and $|DB| = |DC|(= |BC|)$, we get $\angle BDF = \frac{1}{2}\angle BDC = \frac{\pi}{6}$.
Join A and G. Since $\angle ADG = \angle ADB + \angle BDF = \frac{\pi}{3} + \frac{\pi}{6} = \frac{\pi}{2}$, we have $|AG|^2 = |AD|^2 + |DG|^2$.
Since $|DG| = |DF| - |FG| = \sqrt{6} - |EF|\cos(\angle DFE) = \sqrt{6} - \sqrt{2}\left(\frac{\sqrt{3}}{4}\right) = \frac{3\sqrt{6}}{4}$,
we have $|AG|^2 = |AD|^2 + |DG|^2 = 8 + \frac{54}{16} = \frac{91}{8}$.
Hence, $|AE| = \sqrt{|AG|^2+|EG|^2} = \sqrt{\frac{91}{8}+\frac{13}{8}} = \sqrt{13}$.

2. (a)(i) $f'(x) = 3x^2 - 12$, $f''(x) = 6x$.

(ii) $f'(x) = 0 \Leftrightarrow x = -2$ or $x = 2$.
When $x < -2$, $f'(x) > 0$, and thus $f(x)$ is increasing.
When $-2 < x < 2$, $f'(x) < 0$, and thus $f(x)$ is decreasing.
When $2 < x$, $f'(x) > 0$, and thus $f(x)$ is increasing.
Hence, $f(-2) = 22$ is a local maximum value and $f(2) = -10$ is a local minimum value.

(iii) $f''(x) = 0 \Leftrightarrow x = 0$. When $x < 0$, $f''(x) < 0$; when $x > 0$, $f''(x) > 0$.
Hence, $(0,6)$ is an inflection point of the curve $y = f(x)$.

(iv)



(b)(i) The slope of line L is 1. At A, $1 = \frac{dy}{dx} = 3x^2 + 6x + 1$. Solving, $x = 0$ or $x = -2$.
When $x = 0$, the point on line L is $(0,4)$ but the point on the curve C is $(0,0)$.
Thus, $x = 0$ is rejected.
When $x = -2$, $(-2,2)$ is on line L and on curve C. Hence, A is $(-2,2)$.

(ii) Solving $\begin{cases} y = x^3 + 3x^2 + x \\ y = x + 4 \end{cases}$, (from (i), $x = -2$ is a solution) we get $x = -2$ or $x = 1$.
When $-2 < x < 1$, the line $y = x + 4$ is above the curve $y = x^3 + 3x^2 + x$.
Hence, the area is

$$\int_{-2}^{1} (x+4) - (x^3 + 3x^2 + x)\,dx = \int_{-2}^{1} 4 - x^3 - 3x^2\,dx$$
$$= \left[4x - \frac{1}{4}x^4 - x^3\right]_{-2}^{1}$$
$$= \frac{27}{4}.$$

3. (a)(i) From $\begin{cases} \frac{x^2}{9} + \frac{y^2}{4} = 1 \\ y = mx + (k - mh) \end{cases}$, we get $4x^2 + 9[mx + (k - mh)]^2 = 36$, that is,

$$(4 + 9\mathrm{m}^2)x^2 + 18m(k - mh)x + 9(k - mh)^2 - 36 = 0. \qquad (1)$$

Since the line $y = mx + (k - mh)$ is tangent to E, (1) has a double root and so its discriminant is 0. That is, $[18m(k - mh)]^2 - 4(4 + 9\mathrm{m}^2)[9(k - mh)^2 - 36] = 0$. Hence, we get

$$9m^2 - (k - mh)^2 + 4 = 0. \qquad (2)$$

(ii) From (2) we get

$$(9 - h^2)m^2 + 2hkm + (4 - k^2) = 0. \qquad (3)$$

Since m_1 and m_2 are roots of (3), we have $m_1 + m_2 = \frac{2hk}{h^2-9}$ and $m_1 m_2 = \frac{k^2-4}{h^2-9}$.

(b) When L_1 and L_2 are not a pair of horizontal and vertical lines,

$$m_1 m_2 = -1 \Rightarrow \frac{k^2-4}{h^2-9} = -1 \Rightarrow h^2 + k^2 = 13, h \neq \pm 3, k \neq \pm 2.$$

When L_1 and L_2 form a pair of horizontal and vertical lines, readily, $(h,k) = (\pm 3, \pm 2)$.
Hence, the locus of A is $x^2 + y^2 = 13$.

(c) For $i = 1,2$, let $m_i = \tan\theta_i$ be the slope of L_i , where $-\frac{\pi}{2} < \theta_i < \frac{\pi}{2}$.

As $(h,k) = (5,4)$, we have $m_1 + m_2 = \frac{5}{2}$ and $m_1 m_2 = \frac{3}{4}$.

Suppose the angle between L_1 and L_2 is α, where $0 < \alpha < \frac{\pi}{2}$. Then,

$$\tan\alpha = |\tan(\theta_1 - \theta_2)| = \left|\frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1 \tan\theta_2}\right| = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \frac{\sqrt{(m_1 + m_2)^2 - 4m_1 m_2}}{|1 + m_1 m_2|} = \frac{2\sqrt{13}}{7}.$$

Hence, the angle between L_1 and L_2 is $\tan^{-1}\frac{2\sqrt{13}}{7}$.

4. (a)

$$\begin{aligned} z - 3\bar{z} + |z| = -1 + 16i &\Rightarrow (x + yi) - 3(x - yi) + \sqrt{x^2 + y^2} = -1 + 16i \\ &\Rightarrow -2x + \sqrt{x^2 + y^2} + 4yi = -1 + 16i \\ &\Rightarrow \begin{cases} -2x + \sqrt{x^2 + y^2} = -1 \\ 4y = 16 \end{cases}. \end{aligned}$$

From the second equation, $y = 4$. Solving the first equation with $y = 4$, we have

$$\begin{aligned} -2x + \sqrt{x^2 + 16} = -1 &\Rightarrow \sqrt{x^2 + 16} = 2x - 1 \\ &\Rightarrow x^2 + 16 = (2x - 1)^2 \quad \text{and} \quad x \geq 1/2 \\ &\Rightarrow 3x^2 - 4x - 15 = 0 \quad \text{and} \quad x \geq 1/2 \\ &\Rightarrow (x - 3)(3x + 5) = 0 \quad \text{and} \quad x \geq 1/2 \\ &\Rightarrow x = 3 \end{aligned}$$

Hence, $w = 3 + 4i$.

(b) (i)

$$\begin{aligned} \cos 3\theta + i\sin 3\theta &= (\cos\theta + i\sin\theta)^3 \\ &= \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta \\ &= (\cos^3\theta - 3\cos\theta\sin^2\theta) + i(3\cos^2\theta\sin\theta - \sin^3\theta). \end{aligned}$$

Comparing the real and imaginary parts, we get

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta \quad \text{and} \quad \sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta.$$

Hence,

$$\tan 3\theta = \frac{\sin 3\theta}{\cos 3\theta} = \frac{(3\cos^2\theta\sin\theta - \sin^3\theta)/\cos^3\theta}{(\cos^3\theta - 3\cos\theta\sin^2\theta)/\cos^3\theta} = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}.$$

(ii) Let $x = \tan\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then,

$$\begin{aligned} & x^3 - 3\sqrt{3}x^2 - 3x + \sqrt{3} = 0 \\ \Rightarrow\ & \tan^3\theta - 3\sqrt{3}\tan^2\theta - 3\tan\theta + \sqrt{3} = 0 \\ \Rightarrow\ & \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta} = \sqrt{3} \\ \Rightarrow\ & \tan 3\theta = \sqrt{3} \\ \Rightarrow\ & 3\theta = \frac{\pi}{3} + n\pi,\ n \text{ is an integer} \\ \Rightarrow\ & \theta = \frac{\pi}{9} + \frac{n\pi}{3},\ n \text{ is an integer.} \end{aligned}$$

Since $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, the roots of the equation are $\tan(-\frac{2\pi}{9})$, $\tan\frac{\pi}{9}$ and $\tan\frac{4\pi}{9}$.

[Remark: If we let $0 \leq \theta < \frac{\pi}{2}$ or $\frac{\pi}{2} < \theta < \pi$, then the values of θ are $\frac{\pi}{9}$, $\frac{4\pi}{9}$ and $\frac{7\pi}{9}$.

The answers are the same because $\tan(-\frac{2\pi}{9}) = \tan\frac{7\pi}{9}$.]

5. (a)(i)

$$\sin x-\sin y$$
$$=\sin\left(\frac{x+y}{2}+\frac{x-y}{2}\right)-\sin\left(\frac{x+y}{2}-\frac{x-y}{2}\right)$$
$$=\left(\sin\frac{x+y}{2}\cos\frac{x-y}{2}+\cos\frac{x+y}{2}\sin\frac{x-y}{2}\right)-\left(\sin\frac{x+y}{2}\cos\frac{x-y}{2}-\cos\frac{x+y}{2}\sin\frac{x-y}{2}\right)$$
$$=2\cos\frac{x+y}{2}\sin\frac{x-y}{2}.$$

$$\cos x-\cos y$$
$$=\cos\left(\frac{x+y}{2}+\frac{x-y}{2}\right)-\cos\left(\frac{x+y}{2}-\frac{x-y}{2}\right)$$
$$=\left(\cos\frac{x+y}{2}\cos\frac{x-y}{2}-\sin\frac{x+y}{2}\sin\frac{x-y}{2}\right)-\left(\cos\frac{x+y}{2}\cos\frac{x-y}{2}+\sin\frac{x+y}{2}\sin\frac{x-y}{2}\right)$$
$$=-2\sin\frac{x+y}{2}\sin\frac{x-y}{2}.$$

(a)(ii)

$$\begin{vmatrix}1&1&1\\\sin 2\theta&\sin 4\theta&\sin 8\theta\\\cos 2\theta&\cos 4\theta&\cos 8\theta\end{vmatrix}=\begin{vmatrix}1&0&0\\\sin 2\theta&\sin 4\theta-\sin 2\theta&\sin 8\theta-\sin 2\theta\\\cos 2\theta&\cos 4\theta-\cos 2\theta&\cos 8\theta-\cos 2\theta\end{vmatrix}$$

$$=\begin{vmatrix}1&0&0\\\sin 2\theta&2\cos 3\theta\sin\theta&2\cos 5\theta\sin 3\theta\\\cos 2\theta&-2\sin 3\theta\sin\theta&-2\sin 5\theta\sin 3\theta\end{vmatrix}$$

$$=4\sin\theta\sin 3\theta\begin{vmatrix}1&0&0\\\sin 2\theta&\cos 3\theta&\cos 5\theta\\\cos 2\theta&-\sin 3\theta&-\sin 5\theta\end{vmatrix}$$

$$=4\sin\theta\sin 3\theta\,(-\sin 5\theta\cos 3\theta+\cos 5\theta\sin 3\theta)$$
$$=4\sin\theta\sin 3\theta\,[-\sin(5\theta-3\theta)]$$
$$=-4\sin\theta\sin 2\theta\sin 3\theta$$

(b) Since the system of equations has more than one solution, using the result in (a)(ii), we have $-4\sin\theta\sin 2\theta\sin 3\theta=0$.

As $0\le\theta\le\frac{\pi}{2}$, we have $\theta=0$ or $\theta=\frac{\pi}{2}$ or $\theta=\frac{\pi}{3}$.

If $\theta=0$ or $\theta=\frac{\pi}{2}$, the second equation becomes $0x+0y+0z=\sqrt{3}$, from which we know that the system of equations has no solution. Hence, $\theta\neq 0$ and $\theta\neq\frac{\pi}{2}$.

When $\theta=\frac{\pi}{3}$, the system of equations becomes $\begin{cases}x+y+z=6\\\frac{\sqrt{3}}{2}x-\frac{\sqrt{3}}{2}y+\frac{\sqrt{3}}{2}z=\sqrt{3}\\-\frac{1}{2}x-\frac{1}{2}y-\frac{1}{2}z=-3\end{cases}$.

Solving $\begin{cases}x+y+z=6\\\frac{\sqrt{3}}{2}x-\frac{\sqrt{3}}{2}y+\frac{\sqrt{3}}{2}z=\sqrt{3}\end{cases}$, we get $x=4-t,\ y=2,\ z=t,\ t\in\mathbb{R}$.