L^p POLYHARMONIC DIRICHLET PROBLEMS IN REGULAR DOMAINS IV: THE UPPER-HALF SPACE

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Abstract. In this article, we consider a class of Dirichlet problems with L^p boundary data for polyharmonic functions in the upper-half space. By introducing a sequence of new kernel functions for the upper-half space, called higher order Poisson kernels, integral representation solutions of the problems are provided.

1. Introduction

In recent years, there has been a great deal of studies on integral representations of polyanalytic, metaanalytic, polyharmonic and metaharmonic functions in various types of planar or higher dimensional domains [2–13, 15–21, 24, 25]. The aim is to find integral representation solutions of some BVPs (boundary value problems) of certain partial differential equations with various types of boundary data, including the Hölder continuous, continuous, L^p, Hardy, Besov, Sobolev types, and so on. The BVP types include Dirichlet, Neumann, Schwarz, Robin and some mixed problems in regular domains (in the unit disc: [2, 3, 5, 9–11]; and in the upper-half plane: [4,6,8,12,15]) and in irregular domains (C^1 domians [7] and Lipschitz domains: [6, 21,24]), as well as in Riemann manifolds [19,20]. Among other things, polyharmonic Dirichlet problems (for short, PHD problems) arouse considerable interest.

The objective of this article is to solve the PHD problems with L^p data in the upper-half space, $\mathbb{R}^{n+1}_+$

$$\begin{cases} \Delta^m u = 0 \text{ in } \mathbb{R}^{n+1}_+ \\ \Delta^j u = f_j \text{ on } \partial\mathbb{R}^{n+1}_+ = \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $n \geq 2$ is a natural number, $\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times \mathbb{R}_+ = \{x = (\underline{x}, y) : \underline{x} \in \mathbb{R}^n,\ y \in \mathbb{R},\ y > 0\}$, $\underline{x} = (x_1, \ldots, x_n)$, $\Delta \equiv \Delta_{n+1} := \sum_{k=1}^n \frac{\partial^2}{\partial x_k^2} + \frac{\partial^2}{\partial y^2}$, $f_j \in L^p(\mathbb{R}^n)$, $m \in \mathbb{N}$, $0 \leq j < m$, and $p \geq 1$. By introducing a sequence of new kernel functions, we will give integral representation solutions of the PHD problems (1.1). The kernel functions can be regarded as higher order Poisson kernels for the upper-half space (see next section). To the authors' knowledge, this result for integral representations of the solutions of the BVPs with L^p boundary data for polyharmonic equations is completely new. The existing results on such PHD problems ([2–6, 8–10, 15–18, 21, 24, 25] and references therein) only deal with the existence and uniqueness under suitable assumptions (for example, boundedness of non-tangential maximal

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boundary data) as well as estimates of the solutions, but do not present a complete and coherent integral representation theory.

2. Higher order Poisson kernels

Definition 2.1. Let D be a simply connected (bounded or unbounded) domain in $\mathbb{R}^{n+1}$ with smooth boundary ∂D and $k \in \mathbb{N} \cup \{\infty\}$, $C^k(D)$ denotes the set of the functions that have continuous partial derivatives of order k in D. If f is a continuous function defined on $D \times \partial D$ satisfying $f(\cdot, v) \in C^k(D)$ for any fixed $v \in \partial D$ and $f(x, \cdot) \in C(\partial D)$ for any fixed $x \in D$, then f is said to be $C^k \times C$ on $D \times \partial D$ and written as $f \in (C^k \times C)(D \times \partial D)$.

Definition 2.2. A sequence of real-valued functions of two variables $\{G_m(\cdot, \cdot)\}_{m=1}^{\infty}$ defined on $\mathbb{R}^{n+1}_+ \times \mathbb{R}^n$ is called a sequence of higher order Poisson kernels, and, precisely, $G_m(\cdot, \cdot)$ is the mth order Poisson kernel, if they satisfy the following conditions.
1. For all $m \in \mathbb{N}$, $G_m \in (C^\infty \times C)(\mathbb{R}^{n+1}_+ \times \mathbb{R}^n)$, the non-tangential boundary value

$$\lim_{\substack{x \to (u,0) \\ x \in \mathbb{R}^{n+1}_+,\, u \in \mathbb{R}^n}} G_m(x, v) = G_m((u, 0), v)$$

exists for all $u \in \mathbb{R}^n$ and $u \neq v \in \mathbb{R}^n$; $G_m(\cdot, u)$ can be continuously extended to $\overline{\mathbb{R}^{n+1}_+} \setminus \{(u, 0)\}$ for any fixed $u \in \mathbb{R}^n$;
2. $G_1(e_{n+1}, v) = \frac{2}{\omega_n} \frac{1}{(|v|^2+1)^{\frac{n+1}{2}}}$. where ω_n is the surface area of the unit ball in $\mathbb{R}^{n+1}$ and equals to $\frac{2\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})}$, $e_{n+1} = (0, \ldots, 0, 1) \in \mathbb{R}^{n+1}_+$, and for $m \in \mathbb{N}$,

$$|G_m(x, v)| \leq M \frac{y}{(1 + |v|^2)^{\frac{n}{2}}}$$

for any $(x, v) \in D_c \times \{v \in \mathbb{R}^n : |v| > T\}$, where D_c is any compact subset of $\overline{\mathbb{R}^{n+1}_+}$, T is a sufficiently large positive real number and M denotes some positive constant depending only on D_c and T.
3. $\Delta G_1(x, v) = 0$ and $\Delta G_m(x, v) = G_{m-1}(x, v)$ for $m > 1$.
4. $\lim_{x \to (u,0),\, x \in \mathbb{R}^{n+1}_+,\, u \in \mathbb{R}^n} \int_{\mathbb{R}^n} G_1(x, v)\gamma(v)dv = \gamma(u)$, a.e., for any $\gamma \in L^p(\mathbb{R}^n)$, $p \geq 1$;
5. $\lim_{x \to (u,0),\, x \in \mathbb{R}^{n+1}_+,\, u \in \mathbb{R}^n} \int_{\mathbb{R}^n} G_m(x, v)\gamma(v)dv = 0$ for any $\gamma \in L^p(\mathbb{R}^n)$, $p \geq 1$, $m \geq 2$,

where all limits are non-tangential [22].

We note that the Poisson kernel for the upper-half space $\mathbb{R}^{n+1}_+$ is [22]

$$P_y(\underline{x}) = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \frac{y}{(|\underline{x}|^2 + y^2)^{\frac{n+1}{2}}} \tag{2.1}$$

where $\underline{x} \in \mathbb{R}^n$, $y > 0$, and $|\underline{x}| = \sqrt{x_1^2 + \cdots + x_n^2}$.

Set

$$D_1(x, v) = P_y(\underline{x} - v) = c_n \frac{y}{(|\underline{x} - v|^2 + y^2)^{\frac{n+1}{2}}}, \tag{2.2}$$

where $x = (\underline{x}, y) \in \mathbb{R}^{n+1}_+$, in which $\underline{x} \in \mathbb{R}^n$ and $y > 0$; $v \in \mathbb{R}^n$, and

$$c_n = \frac{2}{\omega_n} = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}}. \tag{2.3}$$

Lemma 2.3. *Let $x = (\underline{x}, y) \in \mathbb{R}^{n+1}_+$, $\underline{x} \in \mathbb{R}^n$ and $y > 0$, then for any $s \in \mathbb{R}$,*

$$\Delta\left(y|x|^s\right) = s(s+n+1)y|x|^{s-2} \tag{2.4}$$

and

$$\Delta\left(y|x|^s \log|x|\right) = s(s+n+1)y|x|^{s-2}\log|x| + (2s+n+1)y|x|^{s-2}, \tag{2.5}$$

where $\Delta = \sum_{k=1}^n \frac{\partial^2}{\partial x_k^2} + \frac{\partial^2}{\partial y^2}$ and $|x| = \sqrt{x_1^2 + \cdots + x_n^2 + y^2}$.

Proof. For $1 \le k \le n$, we have

$$\frac{\partial}{\partial x_k}\left(y|x|^s\right) = syx_k|x|^{s-2}, \quad \frac{\partial}{\partial x_k}\left(y|x|^s\log|x|\right) = yx_k|x|^{s-2}(s\log|x|+1);$$

and

$$\begin{aligned} \frac{\partial^2}{\partial x_k^2}\left(y|x|^s\right) &= \frac{\partial}{\partial x_k}\left(syx_k|x|^{s-2}\right) \\ &= sy|x|^{s-2} + s(s-2)yx_k^2|x|^{s-4}, \end{aligned} \tag{2.6}$$

$$\begin{aligned} \frac{\partial^2}{\partial x_k^2}\left(y|x|^s\log|x|\right) &= \frac{\partial}{\partial x_k}\left(yx_k|x|^{s-2}(s\log|x|+1)\right) \\ &= y|x|^{s-2}(s\log|x|+1) + yx_k^2|x|^{s-4}[s(s-2)\log|x| + 2s - 2]. \end{aligned} \tag{2.7}$$

On the other hand,

$$\frac{\partial}{\partial y}\left(y|x|^s\right) = |x|^s + sy^2|x|^{s-2}, \quad \frac{\partial}{\partial y}\left(y|x|^s\log|x|\right) = |x|^s\log|x| + y^2|x|^{s-2}(s\log|x|+1);$$

and

$$\begin{aligned} \frac{\partial^2}{\partial y^2}\left(y|x|^s\right) &= \frac{\partial}{\partial y}\left(|x|^s + sy^2|x|^{s-2}\right) \\ &= 3sy|x|^{s-2} + s(s-2)y^3|x|^{s-4}, \end{aligned} \tag{2.8}$$

$$\begin{aligned} \frac{\partial^2}{\partial y^2}\left(y|x|^s\log|x|\right) &= \frac{\partial}{\partial y}\left[|x|^s\log|x| + y^2|x|^{s-2}(s\log|x|+1)\right] \\ &= 3y|x|^{s-2}(s\log|x|+1) + y^3|x|^{s-4}[s(s-2)\log|x| + 2s - 2]. \end{aligned} \tag{2.9}$$

Therefore, from (2.6)-(2.9), by direct calculations,

$$\begin{aligned} \Delta\left(y|x|^s\right) &= \left[\sum_{k=1}^n \frac{\partial^2}{\partial x_k^2} + \frac{\partial^2}{\partial y^2}\right]\left(y|x|^s\right) \\ &= s(s+n+1)y|x|^{s-2} \end{aligned}$$

and

$$\begin{aligned} \Delta\left(y|x|^s\log|x|\right) &= \left[\sum_{k=1}^n \frac{\partial^2}{\partial x_k^2} + \frac{\partial^2}{\partial y^2}\right]\left(y|x|^s\log|x|\right) \\ &= s(s+n+1)y|x|^{s-2}\log|x| + (2s+n+1)y|x|^{s-2}. \end{aligned}$$

□

Denote

$$\alpha_s = s(s+n+1) \tag{2.10}$$

for any $s \in \mathbb{R}$. Thus, when $s \neq 0$, we can rewrite (2.4) and (2.5) as follows:

$$\Delta\left(\frac{1}{\alpha_s} y|x|^s\right) = y|x|^{s-2} \tag{2.11}$$

and

$$\Delta\left(\frac{1}{\alpha_s} y|x|^s \log|x|\right) = y|x|^{s-2}\log|x| + \left(\frac{1}{s} + \frac{1}{s+n+1}\right) y|x|^{s-2}. \tag{2.12}$$

Moreover, we also have

$$\Delta\left(\frac{1}{n+1} y \log|x|\right) = y|x|^{-2}. \tag{2.13}$$

Lemma 2.4. *Let $x = (\underline{x}, y) \in \mathbb{R}^{n+1}_+$, $\underline{x} \in \mathbb{R}^n$ and $y > 0$, and $v \in \mathbb{R}^n$. For $m \in \mathbb{N}$ and $m \geq 2$, define*

$$D_m(x,v) = \frac{c_n}{\beta_1\beta_2\cdots\beta_{m-1}} y\left(|\underline{x}-v|^2 + y^2\right)^{m-1-\frac{n+1}{2}} \tag{2.14}$$

if n is even, and

$$D_m(x,v) = \begin{cases} \frac{c_n}{\beta_1\beta_2\cdots\beta_{m-1}} y\left(|\underline{x}-v|^2+y^2\right)^{m-1-\frac{n+1}{2}}, \quad m \leq \frac{n+1}{2}, \\ \frac{c_n}{(n+1)\beta_1\beta_2\cdots\beta_{\frac{n+1}{2}-1}\alpha_2\alpha_4\cdots\alpha_{2m-n-3}} y\left(|\underline{x}-v|^2+y^2\right)^{m-1-\frac{n+1}{2}} \\ \times\left[\log\sqrt{\frac{|\underline{x}-v|^2+y^2}{1+|v|^2}} - \sum_{t=1}^{m-\frac{n+3}{2}}\left(\frac{1}{t}+\frac{1}{t+n+1}\right)\right], \ m \geq \frac{n+3}{2} \end{cases} \tag{2.15}$$

if n is odd, where $\beta_k = \alpha_{2k-n-1}$, $k = 1, 2, \ldots, m-1$, α_s is given by (2.10) and c_n is given by (2.3). Then

$$\Delta D_1(x,v) = 0 \text{ and } \Delta D_m(x,v) = D_{m-1}(x,v), \ m \geq 2, \tag{2.16}$$

where D_1 is given by (2.2).

Proof. By straightforward calculations, it immediately follows from (2.11)-(2.13). □

In what follows, we need to introduce ultraspherical polynomials [1,23], $P_l^{(\lambda)}$ and $Q_l^{(\lambda)}$, which can be respectively defined by the generating functions

$$(1 - 2r\xi + r^2)^{-\lambda} = \sum_{l=0}^{\infty} P_l^{(\lambda)}(\xi) r^l \tag{2.17}$$

and

$$(1 - 2r\xi + r^2)^{-\lambda}\log(1 - 2r\xi + r^2) = \sum_{l=0}^{\infty} Q_l^{(\lambda)}(\xi) r^l, \tag{2.18}$$

where $\lambda \neq 0$, $0 \leq |r| < 1$ and $|\xi| \leq 1$. $P_l^{(\lambda)}$ and $Q_l^{(\lambda)}$ have the following explicit expressions:

$$P_l^{(\lambda)}(\xi) = \frac{1}{l!}\left\{\frac{d^l}{dr^l}\left[(1-2r\xi+r^2)^{-\lambda}\right]\right\}_{r=0} = \sum_{j=0}^{[\frac{l}{2}]}(-1)^j\frac{\Gamma(l-j+\lambda)}{\Gamma(\lambda)j!(l-2j)!}(2\xi)^{l-2j} \tag{2.19}$$

and

$$Q_l^{(\lambda)}(\xi) = -\frac{d}{d\lambda}\left[P_l^{(\lambda)}(\xi)\right] = \sum_{j=0}^{[\frac{l}{2}]}\sum_{k=0}^{l-j-1}(-1)^{j+1}\frac{\Gamma(l-j+\lambda)}{(\lambda+k)\Gamma(\lambda)j!(l-2j)!}(2\xi)^{l-2j}, \tag{2.20}$$

where $[\frac{l}{2}]$ denotes the integer part of $\frac{l}{2}$. If necessary, for some special values of λ, say $\lambda = \lambda_0$, the above expressions may be extended and interpreted as limits for $\lambda \to \lambda_0$ (for example, λ_0 is a non-positive integer). Some other properties of the ultraspherical polynomials can be also found in [1, 23].

For sufficiently large $|v| \geq |x|$ and any real numbers $\lambda \neq 0$ and $s > 0$,

$$\begin{aligned}(|\underline{x}-v|^2+y^2)^{-\lambda} &= (|v|^2 - 2\underline{x}\cdot v + |x|^2)^{-\lambda} \\ &= |v|^{-2\lambda}\left[1-2\frac{|x|}{|v|}\left(\frac{\underline{x}}{|x|}\cdot\frac{v}{|v|}\right)+\frac{|x|^2}{|v|^2}\right]^{-\lambda} \\ &= |v|^{-2\lambda}\sum_{l=0}^{\infty}P_l^{(\lambda)}(\underline{x}\cdot v_{S^n}/|x|)\left(\frac{|x|}{|v|}\right)^l \\ &= \sum_{l=0}^{\infty}|x|^l P_l^{(\lambda)}(\underline{x}\cdot v_{S^n}/|x|)|v|^{-(l+2\lambda)}\end{aligned} \tag{2.21}$$

and

$$\begin{aligned}\frac{1}{|v|^s} &= \left[\frac{1}{\sqrt{1+|v|^2}}\frac{1}{\sqrt{1-\frac{1}{1+|v|^2}}}\right]^s \\ &= \frac{1}{(1+|v|^2)^{\frac{s}{2}}}\sum_{\mu=0}^{\infty}\binom{\mu+\frac{s}{2}-1}{\mu}\frac{1}{(1+|v|^2)^{\mu}} \\ &= \sum_{\mu=0}^{\infty}\binom{\mu+\frac{s}{2}-1}{\mu}\frac{1}{(1+|v|^2)^{\mu+\frac{s}{2}}},\end{aligned} \tag{2.22}$$

where $v = |v|v_{S^n}$. Therefore

$$\begin{aligned}(|\underline{x}-v|^2+y^2)^{-\lambda} &= \sum_{l=0}^{[-2\lambda]}|x|^l P_l^{(\lambda)}(\underline{x}\cdot v_{S^n}/|x|)|v|^{-(l+2\lambda)} \\ &+ \sum_{l=[-2\lambda]+1}^{\infty}|x|^l P_l^{(\lambda)}(\underline{x}\cdot v_{S^n}/|x|)\end{aligned} \tag{2.23}$$

$$\times \sum_{\mu=0}^{\infty} \binom{\mu+\frac{l}{2}+\lambda-1}{\mu} \frac{1}{(1+|v|^2)^{\mu+\frac{l}{2}+\lambda}}.$$

Similarly, we have

$$(|\underline{x}-v|^2+y^2)^{-\lambda} \log \frac{|\underline{x}-v|^2+y^2}{1+|v|^2} \tag{2.24}$$

$$=(|v|^2-2\underline{x}\cdot v+|x|^2)^{-\lambda} \log \frac{|v|^2-2\underline{x}\cdot v+|x|^2}{1+|v|^2}$$

$$=|v|^{-2\lambda}\left[1-2\frac{|x|}{|v|}\left(\frac{\underline{x}}{|x|}\cdot\frac{v}{|v|}\right)+\frac{|x|^2}{|v|^2}\right]^{-\lambda}\left\{\log\left[1-2\frac{|x|}{|v|}\left(\frac{\underline{x}}{|x|}\cdot\frac{v}{|v|}\right)+\frac{|x|^2}{|v|^2}\right]\right.$$

$$\left. -\log\left[1+\frac{1}{|v|^2}\right]\right\}$$

$$=|v|^{-2\lambda}\sum_{l=0}^{\infty} Q_l^{(\lambda)}(\underline{x}\cdot v_{S^n}/|x|)\left(\frac{|x|}{|v|}\right)^l - |v|^{-2\lambda}\sum_{l=0}^{\infty} P_l^{(\lambda)}(\underline{x}\cdot v_{S^n}/|x|)\left(\frac{|x|}{|v|}\right)^l$$

$$\times \sum_{k=1}^{\infty}(-1)^k \frac{1}{k}\frac{1}{|v|^{2k}}$$

$$=\sum_{l=0}^{\infty}|x|^l Q_l^{(\lambda)}(\underline{x}\cdot v_{S^n}/|x|)|v|^{-(l+2\lambda)}$$

$$-\sum_{l=2}^{\infty}\sum_{s=1}^{\left[\frac{l}{2}\right]}(-1)^s\frac{1}{s}|x|^{l-2s}P_{l-2s}^{(\lambda)}(\underline{x}\cdot v_{S^n}/|x|)|v|^{-(l+2\lambda)}.$$

Definition 2.5. Let f be a continuous function defined in $\mathbb{R}^n$ that can be expanded as

$$f(\zeta)=\sum_{k=-\infty}^{m} c_k(\zeta)|\zeta|^k \tag{2.25}$$

for sufficiently large $|\zeta|$, where integer $m \geq -n$ and coefficient functions $c_k(\zeta)$ are continuous in $\mathbb{R}^n$. Denote

$$\text{S.P.}[f](\zeta)=\sum_{k=0}^{m} c_k(\zeta)|\zeta|^k+\sum_{k=1}^{n-1}\sum_{\mu=0}^{\left[\frac{k-1}{2}\right]}\binom{\frac{k}{2}-1}{\mu} c_{2\mu-k}(\zeta)\frac{1}{(1+|\zeta|^2)^{\frac{k}{2}}} \tag{2.26}$$

and

$$\text{I.P.}[f](\zeta)=\sum_{k=n}^{\infty}\sum_{\mu=0}^{\left[\frac{k-1}{2}\right]}\binom{\frac{k}{2}-1}{\mu} c_{2\mu-k}(\zeta)\frac{1}{(1+|\zeta|^2)^{\frac{k}{2}}} \tag{2.27}$$

for sufficiently large $|\zeta|$. If I.P.$[f]$ is L^p integrable in the complement of a sufficiently large ball centered at the origin in $\mathbb{R}^n$ for $p>1$, then S.P.$[f]$ is called the singular part of f and I.P.$[f]$ is called the integrable part of f at infinity in the L^p sense, $p>1$.

We immediately have

Proposition 2.6. *Let f be defined as in Definition 2.5, then for sufficiently large $|\zeta|$,*

$$f(\zeta) = \text{S.P.}[f](\zeta) + \text{I.P.}[f](\zeta). \tag{2.28}$$

Proof. Due to (2.25) and (2.22), for sufficiently large $|\zeta|$,

$$\begin{aligned} f(\zeta) =& \sum_{s=0}^{m} c_s(\zeta)|\zeta|^s + \sum_{s=1}^{\infty} c_{-s}(\zeta)\frac{1}{|\zeta|^s} \\ =& \sum_{s=0}^{m} c_s(\zeta)|\zeta|^s + \sum_{s=1}^{\infty} c_{-s}(\zeta)\left[\sum_{\mu=0}^{\infty}\binom{\mu+\frac{s}{2}-1}{\mu}\frac{1}{(1+|\zeta|^2)^{\mu+\frac{s}{2}}}\right] \\ =& \sum_{s=0}^{m} c_s(\zeta)|\zeta|^s + \sum_{k=1}^{\infty}\sum_{\mu=0}^{\left[\frac{k-1}{2}\right]}\binom{\frac{k}{2}-1}{\mu} c_{2\mu-k}(\zeta)\frac{1}{(1+|\zeta|^2)^{\frac{k}{2}}} \\ =& \text{S.P.}[f](\zeta) + \text{I.P.}[f](\zeta). \end{aligned} \tag{2.29}$$

□

Theorem 2.7. *Let*

$$G_m(x,v) = D_m(x,v) - \text{S.P.}[D_m](x,v), \tag{2.30}$$

where

(2.31)

$$\begin{aligned} \text{S.P.}[D_m](x,v) =& \frac{c_n}{\beta_1\beta_2\cdots\beta_{m-1}} y\Big[\sum_{l=0}^{2m-n-3}|x|^l P_l^{(\frac{n+3}{2}-m)}(\underline{x}\cdot v_{S^n}/|x|)|v|^{2m-n-3-l} \\ &+ \sum_{k=2m-n-2}^{2m-4}\sum_{\mu=0}^{\left[\frac{k}{2}\right]}\binom{\frac{k}{2}-m+\frac{n+1}{2}}{\mu}|x|^{k-2\mu}P_{k-2\mu}^{(\frac{n+3}{2}-m)}(\underline{x}\cdot v_{S^n}/|x|) \\ &\times\frac{1}{(1+|v|^2)^{\frac{k}{2}-m+\frac{n+3}{2}}}\Big] \end{aligned}$$

for any m and even n, or any odd n with $m \leq \frac{n+1}{2}$; and

(2.32)

$$\begin{aligned} \text{S.P.}[D_m](x,v) =& \frac{c_n}{(n+1)\beta_1\beta_2\cdots\beta_{\frac{n+1}{2}-1}\alpha_2\alpha_4\cdots\alpha_{2m-n-3}} y \\ &\times\Big\{\Big[\sum_{l=0}^{2m-n-3}|x|^l Q_l^{(\frac{n+3}{2}-m)}(\underline{x}\cdot v_{S^n}/|x|)|v|^{2m-n-3-l} \\ &+ \sum_{k=2m-n-2}^{2m-4}\sum_{\mu=0}^{\left[\frac{k}{2}\right]}\binom{\frac{k}{2}-m+\frac{n+1}{2}}{\mu}|x|^{k-2\mu}Q_{k-2\mu}^{(\frac{n+3}{2}-m)}(\underline{x}\cdot v_{S^n}/|x|) \\ &\times\frac{1}{(1+|v|^2)^{\frac{k}{2}-m+\frac{n+3}{2}}}\Big] \end{aligned}$$

$$-\frac{1}{2}\Big[\sum_{l=2}^{2m-n-3}\sum_{s=1}^{\left[\frac{l}{2}\right]}(-1)^s\frac{1}{s}|x|^{l-2s}P_{l-2s}^{(\frac{n+3}{2}-m)}(\underline{x}\cdot v_{S^n}/|x|)|v|^{2m-n-3-l}$$

$$+\sum_{k=2m-n-2}^{2m-4}\sum_{\mu=0}^{\left[\frac{k}{2}\right]}\sum_{s=1}^{\left[\frac{k}{2}-\mu\right]}(-1)^s\frac{1}{s}\binom{\frac{k}{2}-m+\frac{n+1}{2}}{\mu}|x|^{k-2\mu-2s}$$

$$\times P_{k-2\mu-2s}^{(\frac{n+3}{2}-m)}(\underline{x}\cdot v_{S^n}/|x|)\frac{1}{(1+|v|^2)^{\frac{k}{2}-m+\frac{n+3}{2}}}\Big]$$

$$-\sum_{t=1}^{m-\frac{n+3}{2}}\left(\frac{1}{t}+\frac{1}{t+n+1}\right)\Big[\sum_{l=0}^{2m-n-3}|x|^lP_l^{(\frac{n+3}{2}-m)}(\underline{x}\cdot v_{S^n}/|x|)$$

$$\times|v|^{2m-n-3-l}+\sum_{k=2m-n-2}^{2m-4}\sum_{\mu=0}^{\left[\frac{k}{2}\right]}\binom{\frac{k}{2}-m+\frac{n+1}{2}}{\mu}|x|^{k-2\mu}$$

$$\times P_{k-2\mu}^{(\frac{n+3}{2}-m)}(\underline{x}\cdot v_{S^n}/|x|)\frac{1}{(1+|v|^2)^{\frac{k}{2}-m+\frac{n+3}{2}}}\Big]\Big\}$$

for any odd n with $m\geq\frac{n+3}{2}$, where α_s, β_s are given as in Lemma 2.4, c_n is given by (2.3), and the generalized ultraspherical polynomials $P^{(\frac{n+3}{2}-m)}$, $Q^{(\frac{n+3}{2}-m)}$ are defined by (2.17) and (2.18). Then $\{G_m(x,v)\}_{m=1}^{\infty}$ is a sequence of higher order Poisson kernels defined in Definition 2.2.

Proof. By using the definition of the singular part, S.P.$[\cdot]$ and the relations (2.21), (2.22) and (2.24), performing similar calculations as for getting (2.23) and (2.29), we get (2.31) and (2.32). Note the explicit expressions (2.31) and (2.32), it immediately follows that for any $m\in\mathbb{N}$, $G_m\in(C^\infty\times C)(\mathbb{R}_+^{n+1}\times\mathbb{R}^n)$, the non-tangential boundary value

$$\lim_{\substack{x\to(u,0)\\ x\in\mathbb{R}_+^{n+1},\,u\in\mathbb{R}^n}}G_m(x,v)=G_m((u,0),v)$$

exists for all $u\in\mathbb{R}^n$ and $u\neq v\in\mathbb{R}^n$. Further more, $G_m(\cdot,u)$ can be continuously extended to $\overline{\mathbb{R}_+^{n+1}}\setminus\{(u,0)\}$ for any fixed $u\in\mathbb{R}^n$, i.e., the property 1 in Definition 2.2 holds.

Note that

$$D_1(x,v)=c_n\frac{y}{(|\underline{x}-v|^2+y^2)^{\frac{n+1}{2}}}.$$

So by the definition of the singular part,

$$\text{S.P.}[D_1](x,v)\equiv 0. \tag{2.33}$$

Therefore

$$G_1(x,v)=D_1(x,v)=P_y(\underline{x}-v). \tag{2.34}$$

Then $G_1(e_{n+1},v)=\frac{2}{\omega_n}\frac{1}{(|v|^2+1)^{\frac{n+1}{2}}}$ and $\lim_{x\to(u,0),\,x\in\mathbb{R}_+^{n+1},\,u\in\mathbb{R}^n}\int_{\mathbb{R}^n}G_1(x,v)\gamma(v)dv=\gamma(u)$, a.e., for any $\gamma\in L^p(\mathbb{R}^n)$, $p\geq 1$. Moreover, by the definition, for sufficiently large $|v| > |x|$,

(2.35)

$$\text{I.P.}[D_m](x,v) = \begin{cases} A_{m,n} y C_{m,n}(x,v)\frac{1}{|v|^n}, \ n\, even\, and\, any\, m, or\, n\, odd\, and\, m \le \frac{n+1}{2}, \\ B_{m,n} y \widetilde{C}_{m,n}(x,v)\frac{1}{|v|^n}, \ n\, odd\, and\, m \ge \frac{n+3}{2}, \end{cases}$$

where $A_{m,n}$ and $B_{m,n}$ are positive constants depending only on m and n,

$$C_{m,n}(x,v) = |x|^{2m-3}\left\{\frac{d^{2m-3}}{dr^{2m-3}}\left[(1-2r(\underline{x}\cdot v_{S^n}/|x|)+r^2)^{m-\frac{n+3}{2}}\right]\right\}_{r=\theta} \tag{2.36}$$

and

(2.37)

$$\begin{aligned}\widetilde{C}_{m,n}(x,v) =& |x|^{2m-3}\Big\{\frac{d^{2m-3}}{dr^{2m-3}}\Big[(1-2r(\underline{x}\cdot v_{S^n}/|x|)+r^2)^{m-\frac{n+3}{2}} \\ &\times\left[\frac{1}{2}\log\frac{1-2r(\underline{x}\cdot v_{S^n}/|x|)+r^2}{1+r^2} - \sum_{t=1}^{m-\frac{n+3}{2}}\left(\frac{1}{t}+\frac{1}{t+n+1}\right)\right]\Big]\Big\}_{r=\vartheta}\end{aligned}$$

with $0<\theta,\vartheta<1$. Therefore, for any compact subset D_c of $\overline{\mathbb{R}^{n+1}_+}$ and $x=(\underline{x},y)\in D_c$, by the continuity of $C_{m,n}$ and $\widetilde{C}_{m,n}$, we have

$$|G_m(x,v)| = |\text{I.P.}[D_m](x,v)| \le M\frac{y}{(1+|v|^2)^{\frac{n}{2}}}, \tag{2.38}$$

where $(x,v)\in D_c\times\{v\in\mathbb{R}^n : |v|>T\}$, T is a sufficiently large positive real number and M is a positive constant depending only on D_c and T. Thus the properties 2 and 4 in Definition 2.2 are established.

From (2.31) and (2.32), we can simply denote

$$\begin{aligned}\text{S.P.}[D_m](x,v) =& C_m y\Big[\sum_{l=0}^{2m-n-3} c_{m,l}(x,v)|v|^l \\ &+ \sum_{k=2m-n-2}^{2m-4} c_{m,-k}(x,v)\frac{1}{(1+|v|^2)^{\frac{k}{2}-m+\frac{n+3}{2}}}\Big],\end{aligned} \tag{2.39}$$

where C_m is a constant depending only on m, n, and the coefficient functions $c_{m,l}$ and $c_{m,-k}$ can be explicitly expressed by the ultraspherical polynomials $P^{(\frac{n+3}{2}-m)}$ and $Q^{(\frac{n+3}{2}-m)}$. Therefore,

$$\begin{aligned}\Delta\Big[\text{S.P.}[D_m](x,v)\Big] =& C_m\Big[\sum_{l=0}^{2m-n-3}\Delta[yc_{m,l}(x,v)]|v|^l \\ &+ \sum_{k=2m-n-2}^{2m-4}\Delta[yc_{m,-k}(x,v)]\frac{1}{(1+|v|^2)^{\frac{k}{2}-m+\frac{n+3}{2}}}\Big].\end{aligned} \tag{2.40}$$

By Lemma 2.4, we have

$$\Delta G_m - G_{m-1} = \text{S.P.}[D_{m-1}] - \Delta\Big[\text{S.P.}[D_m]\Big] \tag{2.41}$$

for any $m\ge 2$. Due to (2.38) and (2.39),

$$\Delta G_m = G_{m-1}$$

for any $m \geq 2$. By taking into account $\Delta G_1 = 0$, the property 3 in Definition 2.2 follows.

Finally, we show that the property 5 holds. Denote by $\vee_\alpha(u)$ the cone in $\mathbb{R}^{n+1}_+$ with vertex $(u, 0)$ and aperture $\alpha > 0$, viz.

$$\vee_\alpha(u) = \{(\underline{x}, y) \in \mathbb{R}^{n+1}_+ : |\underline{x} - u| < \alpha y\}. \tag{2.42}$$

In what follows, we often use the truncated cone

$$\triangledown_{\alpha,\eta}(u) = \{(\underline{x}, y) \in \mathbb{R}^{n+1}_+ : |\underline{x} - u| < \alpha y,\ 0 \leq y \leq \eta\}. \tag{2.43}$$

Case 1: $2 \leq m \leq \frac{n+1}{2}$. Take a splitting,

$$\begin{aligned}\int_{\mathbb{R}^n} G_m(x,v)\gamma(v)dv &= \int_{|v-u|<\delta} G_m(x,v)\gamma(v)dv + \int_{\delta\leq|v-u|\leq T} G_m(x,v)\gamma(v)dv \\ &\quad + \int_{|v-u|>T} G_m(x,v)\gamma(v)dv \\ &\triangleq \mathrm{I} + \mathrm{II} + \mathrm{III},\end{aligned} \tag{2.44}$$

where u is any fixed point in $\mathbb{R}^n$, $\delta, T > 0$, δ is sufficiently small while T is sufficiently large, $x \in \triangledown_{\alpha,\eta}(u)$, $0 < \eta < \min\{\delta, \frac{1}{2}\}$, and $\gamma \in L^p(\mathbb{R}^n)$, $p \geq 1$. By the property 1, $y^{-1}G_m(x,v)$ is continuous on the compact set $\triangledown_{\alpha,\eta}(u) \times \{v \in \mathbb{R}^n : \delta \leq |v-u| \leq T\}$. Therefore,

$$\mathrm{II} \to 0 \text{ as } x \to (u,0), x \in \triangledown_{\alpha,\eta}(u). \tag{2.45}$$

By the property 2, for sufficiently large T, $x \in \triangledown_{\alpha,\eta}(u)$ and $|v-u| > T$, we have

$$|G_m(x,v)| \leq M\frac{y}{(1+|v|^2)^{\frac{n}{2}}},$$

where M is a constant depending only on δ and T. So

$$|G_m(x,v)\gamma(v)| \leq M\frac{y}{(1+|v|^2)^{\frac{n}{2}}}|\gamma(v)|. \tag{2.46}$$

The RHS of the above inequality belongs to $L^1(\mathbb{R}^n)$, because $\frac{1}{(1+|v|^2)^{\frac{n}{2}}} \in L^q(\mathbb{R}^n) \cap C_0(\mathbb{R}^n)$ and $\gamma \in L^p(\mathbb{R}^n)$ for any $p \geq 1$ and $q > 1$, where $C_0(\mathbb{R}^n)$ is the set of all functions defined on $\mathbb{R}^n$ vanishing at infinity. Since by (2.46), $G_m(x,v)\gamma(v) \to 0$ as $x \to (u,0)$ for any $x \in \triangledown_{\alpha,\eta}(u)$ and $|v-u| > T$, by (2.46) and Lebesgue's dominated convergence theorem,

$$\mathrm{III} \to 0 \text{ as } x \to (u,0), x \in \triangledown_{\alpha,\eta}(u). \tag{2.47}$$

Write that

$$\begin{aligned}\mathrm{I} &= \int_{|v-u|<\delta} D_m(x,v)\gamma(v)dv - \int_{|v-u|<\delta} \mathrm{S.P.}[D_m](x,v)\gamma(v)dv \\ &\triangleq \mathrm{I}_1 - \mathrm{I}_2.\end{aligned} \tag{2.48}$$

Similarly to (2.45), by taking into account $y^{-1}\mathrm{S.P.}[D_m](x,v) \in C(\triangledown_{\alpha,\eta}(u) \times \{v \in \mathbb{R}^n : |v-u| \leq \delta\})$,

$$\mathrm{I}_2 \to 0 \text{ as } x \to (u,0), x \in \triangledown_{\alpha,\eta}(u). \tag{2.49}$$

For $x \in \bigtriangledown_{\alpha,\eta}(u)$ and $|v-u|<\delta<\frac{1}{2}$,

$$\begin{aligned} D_m(x,v) &= c_m \frac{y}{(|\underline{x}-v|^2+y^2)^{\frac{n+3}{2}-m}} \\ &\le c_m \frac{y}{\Big[||v-u|-|\underline{x}-u||^2+y^2\Big]^{\frac{n+3}{2}-m}} \\ &= c_m \frac{y}{\Big[|v-u|^2+|\underline{x}-u|(1-2|v-u|)+y^2\Big]^{\frac{n+3}{2}-m}} \\ &\le c_m \frac{y}{|v-u|^{(n+3)-2m}}, \end{aligned} \tag{2.50}$$

where $c_m = \frac{c_n}{\beta_1\beta_2\cdots\beta_{m-1}}$. Therefore,

$$\mathrm{I}_1 \le c_m y \int_{|v-u|<\delta} \frac{1}{|v-u|^{(n+3)-2m}} \gamma(v)dv \tag{2.51}$$

$$= c_m y \int_{|v'|<\delta} |v'|^{m-2}\gamma(u+v')dv'. \tag{2.52}$$

So

$$\mathrm{I}_1 \to 0 \text{ as } x \to (u,0), x \in \bigtriangledown_{\alpha,\eta}(u). \tag{2.53}$$

Therefore, in this case, by (2.44), (2.45), (2.47)-(2.49), (2.53),

$$\lim_{x\to(u,0),\, x\in\mathbb{R}^{n+1}_+,\, u\in\mathbb{R}^n} \int_{\mathbb{R}^n} G_m(x,v)\gamma(t)dv = 0$$

for any $\gamma \in L^p(\mathbb{R}^n)$, $p \ge 1$.

Case 2: $m \ge \frac{n+3}{2}$. For sufficiently large $T>0$, we can split

$$\begin{aligned} \int_{\mathbb{R}^n} G_m(x,v)\gamma(v)dv &= \int_{|v-u|\le T} G_m(x,v)\gamma(v)dv + \int_{|v-u|>T} G_m(x,v)\gamma(v)dv \\ &\triangleq \mathrm{J}_1 + \mathrm{J}_2, \end{aligned} \tag{2.54}$$

where

$$\begin{aligned} \mathrm{J}_1 &= \int_{|v-u|\le T} G_m(x,v)\gamma(v)dv \\ &= \int_{|v-u|\le T} D_m(x,v)\gamma(v)dv - \int_{|v-u|\le T} \mathrm{S.P.}[D_m](x,v)\gamma(v)dv \\ &\triangleq \mathrm{J}_{11} - \mathrm{J}_{12}. \end{aligned} \tag{2.55}$$

Similarly to (2.47) and (2.49), we have

$$\mathrm{J}_2 \to 0 \text{ as } x \to (u,0), x \in \bigtriangledown_{\alpha,\eta}(u) \tag{2.56}$$

and

$$\mathrm{J}_{12} \to 0 \text{ as } x \to (u,0), x \in \bigtriangledown_{\alpha,\eta}(u). \tag{2.57}$$

Since $m \ge \frac{n+3}{2}$, by (2.14) and (2.15), $y^{-1}D_m(x,v) \in C(\bigtriangledown_{\alpha,\eta}(u) \times \{v \in \mathbb{R}^n : |v-u| \le T\})$. Similarly to (2.53),

$$\mathrm{J}_{11} \to 0 \text{ as } x \to (u,0), x \in \bigtriangledown_{\alpha,\eta}(u). \tag{2.58}$$

By (2.56)-(2.58), we have

$$\lim_{x\to(u,0),\, x\in\mathbb{R}^{n+1}_+,\, u\in\mathbb{R}^n} \int_{\mathbb{R}^n} G_m(x,v)\gamma(v)dv = 0$$

for any $\gamma \in L^p(\mathbb{R}^n)$, $p \geq 1$.

We thus conclude the property 5 of Definition 2.2. The proof is complete. $\square$

3. Polyharmonic Dirichlet problems in the upper-half space

In this section, we solve the PHD problem (1.1), viz.,

$$\begin{cases} \Delta^m u = 0 \text{ in } \mathbb{R}^{n+1}_+ \\ \Delta^j u = f_j \text{ on } \partial\mathbb{R}^{n+1}_+ = \mathbb{R}^n, \end{cases} \tag{3.1}$$

where $n \geq 2$, $\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times \mathbb{R}_+ = \{x = (\underline{x}, y) : \underline{x} \in \mathbb{R}^n,\, y \in \mathbb{R},\, y > 0\}$, $\Delta \equiv \Delta_{n+1} := \sum_{k=1}^n \frac{\partial^2}{\partial x_k^2} + \frac{\partial^2}{\partial y^2}$, $f_j \in L^p(\mathbb{R}^n)$, $m \in \mathbb{N}$, $0 \leq j < m$, and $p \geq 1$.

To do so, firstly as a special case extension of Theorem 2.27 in [14], we establish

Lemma 3.1. *Let D be a simply connected unbounded domain in $\mathbb{R}^{n+1}$ with smooth unbounded boundary $\partial D \subset \mathbb{R}^n$. If $f \in (C^1 \times C)(D \times \partial D)$ and there exist $g_0, g_1 \in L^p(\partial D)$, $p \geq 1$ such that*

$$|f(x,v)| \leq M_0 \frac{g_0(v)}{(1+|v|^2)^{\frac{n}{2}}} \tag{3.2}$$

and

$$|\frac{\partial}{\partial x_j} f(x,v)| \leq M_1 \frac{g_1(v)}{(1+|v|^2)^{\frac{n}{2}}} \tag{3.3}$$

hold for any $(x,v) \in D_c \times \{v \in \partial D : |v| > T\}$ and $j = 1, 2, \ldots, n+1$, where D_c is a compact subset of D, T is a sufficiently large positive real number and M_0, M_1 are positive constants depending only on D_c and T, then

$$\frac{\partial}{\partial x_j}\left(\int_{\partial D} f(x,v)dv\right) = \int_{\partial D} \frac{\partial f}{\partial x_j}(x,v)dv \tag{3.4}$$

for any $1 \leq j \leq n+1$.

Proof. Fix $X = (x_1, x_2, \ldots, x_{n+1}) \in D$ and $j \in \{1, 2, \ldots, n+1\}$, take $X_l = X + t_l e_j$ with $\lim_{l\to+\infty} t_l = 0$, and $e_j = (0, \ldots, 1, \ldots, 0) \in \mathbb{R}^{n+1}$ whose the jth element is 1 and the other ones are zero. Denote

$$\begin{aligned} D_l(X,v) &= \frac{f(X_l,v) - f(X,v)}{t_l} \\ &= \frac{\partial}{\partial x_j} f(X + \theta t_l e_j, v), \end{aligned} \tag{3.5}$$

where $0 < \theta < 1$, then by (3.3),

$$|D_l(X,v)| \leq M_1 \frac{g_1(v)}{(1+|v|^2)^{\frac{n}{2}}} \tag{3.6}$$

uniformly in $\{v \in \partial D : |v| > T\}$ whenever $X_l \in \{Y : |Y - X| \leq R\} \subset D$ for some $R > 0$ and sufficiently large $T > 0$. Since $f \in (C^1 \times C)(D \times \partial D)$ and

$$\lim_{l \to +\infty} D_l(X, v) = \frac{\partial f}{\partial x_j}(X, v), \ v \in \partial D, \tag{3.7}$$

by (3.2), (3.6), the continuity of f on the compact set $\{Y : |Y - X| \leq R\} \times \{v \in \partial D : |v| \leq T\}$, and Lebesgue's dominated convergence theorem,

$$\begin{aligned}
\lim_{l \to +\infty} \int_{\partial D} D_l(X, v) dv &= \lim_{l \to +\infty} \left[\int_{|v| \leq T, v \in \partial D} D_l(X, v) dv + \int_{|v| > T, v \in \partial D} D_l(X, v) dv \right] \\
&= \int_{|v| \leq T, v \in \partial D} \frac{\partial f}{\partial x_j}(X, v) dv + \int_{|v| > T, v \in \partial D} \frac{\partial f}{\partial x_j}(X, v) dv \\
&= \int_{\partial D} \frac{\partial f}{\partial x_j}(X, v) dv,
\end{aligned}$$

i.e.,

$$\lim_{l \to +\infty} \frac{\int_{\partial D} f(X_l, v) dv - \int_{\partial D} f(X, v) dv}{t_l} = \int_{\partial D} \frac{\partial f}{\partial x_j}(X, v) dv. \tag{3.8}$$

Since X and the sequence X_l are arbitrarily chosen, then

$$\frac{\partial}{\partial x_j} \left(\int_{\partial D} f(X, v) dv \right) = \int_{\partial D} \frac{\partial f}{\partial x_j}(X, v) dv$$

for any $1 \leq j \leq n + 1$ and $X \in D$. □

As an immediate consequence, we have

Corollary 3.2. *Let D be a simply connected unbounded domain in $\mathbb{R}^{n+1}$ with smooth unbounded boundary $\partial D \subset \mathbb{R}^n$. If $f \in (C^2 \times C)(D \times \partial D)$ and there exist $g_0, g_1, g_2 \in L^p(\partial D)$, $p \geq 1$ such that*

$$|f(x, v)| \leq M_0 \frac{g_0(v)}{(1 + |v|^2)^{\frac{n}{2}}}, \tag{3.9}$$

$$|\frac{\partial}{\partial x_j} f(x, v)| \leq M_1 \frac{g_1(v)}{(1 + |v|^2)^{\frac{n}{2}}} \tag{3.10}$$

and

$$|\frac{\partial^2}{\partial x_j^2} f(x, v)| \leq M_2 \frac{g_2(v)}{(1 + |v|^2)^{\frac{n}{2}}} \tag{3.11}$$

hold for any $(x, v) \in D_c \times \{v \in \partial D : |v| > T\}$ and $j = 1, 2, \ldots, n + 1$, where D_c is any compact subset of D, T is a sufficiently large positive real number and M_0, M_1, M_2 are positive constants depending only on D_c and T, then

$$\Delta \left(\int_{\partial D} f(x, v) dv \right) = \int_{\partial D} \Delta f(x, v) dv. \tag{3.12}$$

From the above corollary, we can obtain the following theorem concerning differentiability of integrals of higher order Poisson kernels.

Theorem 3.3. *Let* $\{ G_m(x,v) \}_{m=1}^{\infty}$ *be the sequence of higher order Poisson kernels as in Theorem 2.7, then for any* $m > 1$ *and* $\gamma \in L^p(\mathbb{R}^n)$*,* $p \geq 1$*,*

$$\Delta\left(\int_{\mathbb{R}^n} G_m(x,v)\gamma(v)dv\right) = \int_{\mathbb{R}^n} G_{m-1}(x,v)\gamma(v)dv. \tag{3.13}$$

Proof. From the property 1 in Definition 2.2, we know that $G_m \in (C^2 \times C)(\mathbb{R}_+^{n+1} \times \mathbb{R}^n)$. For sufficiently large $T > 0$,

$$\begin{aligned} G_m(x,v) =& D_m(x,v) - \text{S.P.}[D_m](x,v) = \text{I.P.}[D_m](x,v) \\ =& \sum_{k=2m-3}^{\infty} c_{m,-k}(x,v)\frac{1}{(1+|v|^2)^{\frac{k}{2}-m+\frac{n+3}{2}}} \end{aligned} \tag{3.14}$$

for any $(x,v) \in \{x \in \mathbb{R}_+^{n+1} : |x| \leq \frac{T}{2}\} \times \{v \in \mathbb{R}^n : |v| > T\}$, where $c_{m,-k}$ can be explicitly expressed by the ultraspherical polynomials $P^{(\frac{n+3}{2}-m)}$ and $Q^{(\frac{n+3}{2}-m)}$. So by the property 2 in Definition 2.2, i.e., (2.38) and arguments similar to (2.38), we obtain

$$|G_m(x,v)| \leq M_0\frac{1}{(1+|v|^2)^{\frac{n}{2}}}, \tag{3.15}$$

$$|\frac{\partial}{\partial x_j}G_m(x,v)| \leq M_1\frac{1}{(1+|v|^2)^{\frac{n}{2}}} \tag{3.16}$$

and

$$|\frac{\partial^2}{\partial x_j^2}G_m(x,v)| \leq M_2\frac{1}{(1+|v|^2)^{\frac{n}{2}}} \tag{3.17}$$

for any $m \geq 2$ and $(x,v) \in D_c \times \{v \in \mathbb{R}^n : |v| > T\}$, where D_c is any compact subset of $\overline{\mathbb{R}_+^{n+1}}$, T is a sufficiently large positive real number and M_0, M_1, M_2 are positive constants depending only on D_c and T. Therefore, by Corollary 3.2, for any $m > 1$,

$$\Delta\left(\int_{\mathbb{R}^n} G_m(x,v)\gamma(v)dv\right) = \int_{\mathbb{R}^n} G_{m-1}(x,v)\gamma(v)dv. \qquad \square$$

Now we can give the main result for polyharmonic Dirichlet problems in the upper-half space as follows.

Theorem 3.4. *Let* $\{ G_m(x,v) \}_{m=1}^{\infty}$ *be the sequence of higher order Poisson kernels on* $\mathbb{R}_+^{n+1} \times \mathbb{R}^n$*, given by (2.30), then for any* $m \geq 1$*, the PHD problem (1.1) is solvable and its general solution is given by*

$$u(x) = \sum_{j=1}^{m}\int_{\mathbb{R}^n} G_j(x,v)f_{j-1}(v)dv + u_h(x), \tag{3.18}$$

where $u_h(x)$ *denotes the general solution of the accompanying homogeneous PHD problem*

$$\begin{cases} \Delta^n u = 0 \text{ in } \mathbb{R}_+^{n+1}, \\ \Delta^j u = 0 \text{ on } \partial\mathbb{R}_+^{n+1} = \mathbb{R}^n. \end{cases} \tag{3.19}$$

Proof. Note the inductive property 3 of higher order Poisson kernels stated as in Definition 2.2, and let the polyharmonic operators Δ^l, $1 \le l \le m-1$, act on the two sides of (3.18); by Theorem 3.3, we have

$$\Delta^l u(x) = \sum_{j=l+1}^{m} \int_{\mathbb{R}^n} G_{j-l}(x,v) f_{j-1}(v) dv + \Delta^l u_h(x). \tag{3.20}$$

Thus, since $\Delta^l u_h = 0$ on $\mathbb{R}^n$,

$$\Delta^l u(s) = f_l(s),\ s \in \mathbb{R}^n,\ 0 \le l \le m-1 \tag{3.21}$$

follows from the property 5 of higher order Poisson kernels and the nice property of G_1, i.e.,

$$\lim_{\substack{x \to (s,0) \\ x \in \mathbb{R}^{n+1}_+, s \in \mathbb{R}^n}} \int_{\mathbb{R}^n} G_1(x,v)\gamma(v) dv = \gamma(s) \tag{3.22}$$

for any $\gamma \in L^p(\mathbb{R}^n)$, $p \ge 1$. Similarly, letting the polyharmonic operators Δ^n act on the two sides of (3.18), we have $\Delta^n u(x) = 0$ for any $x \in \mathbb{R}^{n+1}_+$. Thus (3.18) is a solution of the PHD problem (1.1).

Denote

$$u^*(x) = \sum_{j=1}^{m} \int_{\mathbb{R}^n} G_j(x,v) f_{j-1}(v) dv. \tag{3.23}$$

The above argument shows that u^* is a special solution of the PHD problem (1.1). Since u_h is the general solution of the accompanying homogenous PHD problem (3.19), then it is immediate from linear algebra that (3.18) is the general solution of the PHD problem (1.1). □

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