Non-stretch mappings for a sharp estimate of the Beurling-Ahlfors operator *

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Abstract: In this paper we identify certain classes of non-stretch mappings that enjoy a sharp estimate of the Beurling-Ahlfors operator. We first make use of a property of subharmonic functions to prove that the Bañuelos-Wang conjecture and the Iwaniec conjecture are true for a class of mappings that satisfy a quasilinear conjugate Beltrami equation. By utilizing the principal solutions of Beltrami equations, we further explicitly construct some classes of non-stretch mappings for which the Bañuelos-Wang conjecture and the Iwaniec conjecture are true.

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1 Introduction

The *Beurling-Ahlfors operator* $\mathbf{T}$ is defined on $L^p(\mathbb{C})$, $1 < p < \infty$, by

$$\mathbf{T}f(z) = -\frac{1}{\pi}\mathrm{pv}\iint_{\mathbb{C}} \frac{f(\zeta)}{(z-\zeta)^2}dm(\zeta), \tag{1.1}$$

where pv means the Cauchy principal value and m is the Lebesgue measure in the plane $\mathbb{C}$. The Beurling-Ahlfors operator arises naturally in the study of the

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solutions of Beltrami equations [3, 5]. This operator and its multidimensional analogues are fundamental tools in several areas including quasiconformal mappings, partial differential equations, calculus of variations and differential geometry (see [3, 4, 5, 8, 18, 25, 30] and the references therein for more details).

For a function $f = u + iv : \mathbb{C} \to \mathbb{C}$, we denote its formal partial derivatives by

$$\bar{\partial} f = f_{\bar{z}} = \frac{1}{2}(f_x + i f_y) = \frac{1}{2}(u_x - v_y + i(u_y + v_x)),$$
$$\partial f = f_z = \frac{1}{2}(f_x - i f_y) = \frac{1}{2}(u_x + v_y + i(v_x - u_y)),$$

and write

$$Df = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}.$$

Let $\dot{W}^{1,p}(\mathbb{C}, \mathbb{C})$, $1 < p < \infty$, be the homogenous Sobolev space of complex-valued locally integrable functions in the plane whose distributional first derivatives are in $L^p(\mathbb{C})$. A function $f : \mathbb{C} \to \mathbb{C}$ is called *radial* if $f(re^{i\theta}) = g(r)$, while, f is said to be a *stretch mapping* if it is of the form $f(re^{i\theta}) = g(r)e^{i\theta}$, where $z = re^{i\theta}$, and g is a nonnegative locally Lipschitz function on $(0, \infty)$ with $g(0) = 0$ and $\lim_{r\to\infty} g(r) = 0$. Let S denote the set of all stretch mappings.

The Beurling-Ahlfors operator $\mathbf{T}$ is an isometric operator in $L^2(\mathbb{C})$ that sends $\bar{\partial} f$ to ∂f for $f \in \dot{W}^{1,p}(\mathbb{C}, \mathbb{C})$ (p52-53 in [3], or p94-96 in [5]). The Calderón-Zygmund lemma says that $\mathbf{T}$ has a finite L^p-norm bound C_p with $C_p \to 1$ as $p \to 2$ in $L^p(\mathbb{C})$ (p62-66 in [3]). In [26], Lehto showed that $||\mathbf{T}||_{L^p(\mathbb{C})} \geq p^* - 1$, $p^* = \max\{p, \frac{p}{p-1}\}$, by using a family of stretch mappings. Iwaniec [24] conjectured that $||\mathbf{T}||_{L^p(\mathbb{C})} = p^* - 1$. This conjecture is equivalent to the inequality

$$\iint_{\mathbb{C}} |\partial f|^p dm \leq (p^* - 1)^p \iint_{\mathbb{C}} |\bar{\partial} f|^p dm \tag{1.2}$$

for complex-valued functions $f \in \dot{W}^{1,p}(\mathbb{C}, \mathbb{C})$.

The *Bañuelos-Wang conjecture* is stated as follows [11]: For every function $f \in \dot{W}^{1,p}(\mathbb{C}, \mathbb{C})$, it is true that

$$\iint_{\mathbb{C}} \mathbf{B}_p(Df) dm \geq 0, \tag{1.3}$$

where the *Burkholder functional* $\mathbf{B}_p$ is given by

$$\mathbf{B}_p(Df) = ((p^* - 1)|\bar{\partial} f| - |\partial f|)(|\bar{\partial} f| + |\partial f|)^{p-1}. \tag{1.4}$$

The *Šverák conjecture* is as follows [31]: If $f \in \dot{W}^{1,2}(\mathbb{C}, \mathbb{C})$, then

$$\iint_{\mathbb{C}} \mathbf{S}(Df) dm \geq 0,$$

where the Šverák functional $\mathbf{S}$ is defined by

$$\mathbf{S}(Df) = \begin{cases} |\bar{\partial} f|^2 - |\partial f|^2, & \text{if } |\partial f| + |\bar{\partial} f| \leq 1, \\ 2|\bar{\partial} f| - 1, & \text{otherwise.} \end{cases}$$

The validity of the Šverák conjecture implies that of the Bañuelos-Wang conjecture (see Section 1 in [7] for a proof). By the *Burkholder inequality* (p16-17 in [13])

$$p(1-\frac{1}{p^*})^{p-1}((p^*-1)|\bar{\partial}f|-|\partial f|)(|\bar{\partial}f|+|\partial f|)^{p-1}\leq (p^*-1)^p|\bar{\partial}f|^p-|\partial f|^p, \quad (1.5)$$

the Bañuelos-Wang conjecture in turn implies the Iwaniec conjecture.

In 1952, Morrey [28] conjectured that the rank-one convexity of a functional $\mathbf{F}: M(m,n)\to\mathbb{R}$ does not imply its quasiconvexity when both m and n are at least 2, where $M(m,n)$ denotes the set of all $m\times n$ matrices with real entries. Due to the rank-one convexity of the Burkholder functional and the Šverák functional, the above three conjectures are also closely connected with the Morrey conjecture. One can see Section 5 in [7] or [32] for a precise statement of these relations.

Bañuelos and Wang [11] used martingale inequalities [13] to show that $||\mathbf{T}||_{L^p(\mathbb{C})}\leq 4(p^*-1)$. Utilizing an analytic approach with Bellman functions, Nazarov and Volberg [29] improved it and got $2(p^*-1)$. So far, the best result is $||\mathbf{T}||_{L^p(\mathbb{C})}\leq 1.575(p^*-1)$, obtained by Bañuelos and Janakiraman [9] by probabilistic techniques of Burkholder [13, 14]. One can refer to [12, 21] for its asymptotical estimates and see [19, 20] for the L^p-norm estimates of the powers $\mathbf{T}^n$.

On one hand, there have been efforts to decrease the constant C in the inequality

$$||\mathbf{T}f||_{L^p(\mathbb{C})}\leq C(p^*-1)||f||_{L^p(\mathbb{C})} \quad (1.6)$$

for all functions $f\in L^p(\mathbb{C})$, while, on the other hand, there were results establishing this inequality with $C=1$ but just for particular subclasses of $L^p(\mathbb{C})$.

Baernstein and Montgomery-Smith [7] showed that the Bañuelos-Wang conjecture holds for every stretch mapping $f\in S\cap\dot{W}^{1,p}(\mathbb{C},\mathbb{C})$ and consequently the Iwaniec conjecture is valid for this class of mappings. Recently, Volberg [32] extended the above result to complex radial functions.

Theorem A. *If a complex valued function f has an expression*

$$f(z)=f(|z|),\quad f\in C_0^\infty(\mathbb{C}),$$

then it follows

$$||\mathbf{T}f||_{L^p(\mathbb{C})}\leq (p^*-1)||f||_{L^p(\mathbb{C})}. \quad (1.7)$$

Let $\mathbb{H}$ be a separable Hilbert space over $\mathbb{R}$ with norm $|\cdot|$ and scalar product $<\cdot,\cdot>$, and $F:\mathbb{C}\to\mathbb{H}$ belong to $L^p(\mathbb{C})$. Bañuelos and Osękowski [10] used martingale inequalities to show that the inequality (1.7) holds for all radial functions F and the constant p^*-1 is the best possible for $1<p\leq 2$.

Let Ω be a simply-connected domain of $\mathbb{C}$. Recall that a *harmonic mapping* f defined on Ω is a solution of the conjugate Beltrami equation

$$\overline{f_{\bar{z}}}=af_z \quad (1.8)$$

in $W^{1,2}_{loc}(\Omega)$, where a is analytic and $|a|<1$ on Ω. We refer to [17, 22, 23] for the study of harmonic mappings. In [7], Baernstein and Montgomery-Smith proved the following

Theorem B. *If $f \in \dot{W}^{1,p}(\mathbb{C},\mathbb{C})$, $1 < p < \infty$, is harmonic on $\mathbb{C} \cup \infty \backslash \{|z| = 1\}$, then the inequality (1.3) holds.*

In this paper, we aim to give several new classes of complex-valued functions that validate the Bañuelos-Wang conjecture and the Iwaniec conjecture.

Firstly, we study the class of *logharmonic* mappings $f : \Omega \to \mathbb{C}$ which are solutions of the *quasilinear conjugate Beltrami equation*

$$\overline{f_{\bar{z}}} = a\frac{\bar{f}}{f}f_z \tag{1.9}$$

in $W^{1,2}_{loc}(\Omega)$, where a is an analytic function on Ω with $|a| < 1$. For two analytic functions h and g with $|g'h/gh'| < 1$ on Ω, $f = h\bar{g}$ satisfies (1.9) with $a = g'h/gh'$ almost everywhere. There are solutions of (1.9) which are not of the form $f = h\bar{g}$. For instance, $f(z) = z|z|^{2\alpha}$, $\Re\{\alpha\} > -1/2$, $f(1) = 1$, is a solution of (1.9) on $\mathbb{C}$ with $a = \bar{\alpha}/(1+\alpha)$. Denote by $\mathfrak{F}(a,\Omega)$ all nonconstant solutions in $W^{1,2}_{loc}(\Omega)$ satisfying (1.9) almost everywhere in Ω. Abdulhadi and Bshouty [2] obtained the representation theorem and boundary behaviors of functions in $\mathfrak{F}(a,\Omega)$. In [15], it is shown that a sense-preserving logharmonic mapping f in $C^2(\Omega)$ is ρ-harmonic with $\rho = \frac{1}{|f|^2}$, that is , it satisfies

$$f_{z\bar{z}} + (\log\rho)_\zeta \circ f f_z f_{\bar{z}} = 0 \tag{1.10}$$

almost everywhere in Ω, where $\zeta = f(z)$. See [1, 16] and the references therein for more properties about logharmonic mappings.

Let $\mathbb{D}$ be the unit disk of $\mathbb{C}$, and $\mathbb{D}^c$ the exterior of $\overline{\mathbb{D}}$. Set

$$\varphi(z) = \begin{cases} z, & z \in \overline{\mathbb{D}}, \\ 1/\bar{z}, & z \in \mathbb{D}^c. \end{cases}$$

Using the technique of subharmonic functions, we obtain

Theorem 1.1. *Suppose g is a locally univalent logharmonic mapping of the unit disk $\mathbb{D}$ in $W^{1,2}_{loc}(\mathbb{D})$. Let $f = g \circ \varphi$. If $f \in \dot{W}^{1,p}(\mathbb{C},\mathbb{C})$, then the Bañuelos-Wang conjecture and the Iwaniec conjecture are true for f.*

Secondly, we will use principal solutions to construct some classes of mappings validating the Bañuelos-Wang conjecture and the Iwaniec conjecture. Let μ be a measurable function satisfying $||\mu||_\infty \le 1$ on $\mathbb{C}$. A *principal solution* is a global $W^{1,2}_{loc}(\mathbb{C})$-solution of the *Beltrami equation*

$$f_{\bar{z}} = \mu f_z \tag{1.11}$$

with the asymptotic normalization

$$f(z) = z + b_1 z^{-1} + b_2 z^{-2} + \cdots, \quad \text{for } |z| \to \infty.$$

The function μ is called the *Beltrami coefficient* of (1.11). A series

$$\mu + \mu\mathbf{T}\mu + \mu\mathbf{T}\mu\mathbf{T}\mu + \mu\mathbf{T}\mu\mathbf{T}\mu\mathbf{T}\mu + \cdots$$

is called the *Neumann series*. When μ satisfies $||\mu||_\infty \leq k < 1$ and has a compact support, the Neumann series converges in $L^p(\mathbb{C})$ norm, where k is a constant (see p163 in [5]). If μ is degenerative, *i.e.*, $||\mu||_\infty$=1, the convergence of the Neumann series is not easy to be determined. For some particular classes of degenerative Beltrami coefficients μ, the convergence of their Neumann series can be determined if there exist explicit representations of $\mathbf{C}\mu$ and $\mathbf{T}\mu$ (see Lemma 3.1).

If the conjugate of a Beltrami coefficient μ is analytic, then we call it a *co-analytic Beltrami coefficient*. Let I be the identical mapping in this text. We show that if $f+I$ is a principal solution with a co-analytic Beltrami coefficient, then the Bañuelos-Wang conjecture and the Iwaniec conjecture are true for f (see Theorem 3.1).

Moreover, using the Parseval formula we give two classes of principal solutions $f+I$ with degenerative Beltrami coefficients that enable the corresponding mappings f validating the Bañuelos-Wang conjecture and the Iwaniec conjecture for $p=2$ and $p=4$ (see Example 3.2 and Theorem 3.2). We note that these mappings are not stretch or complex radial.

This rest of this paper is organized as follows. In Section 2, using the fact that the integral means of a subharmonic function are non-decreasing, we obtain the proof of Theorem 1.1. In Section 3, we use principle solutions to construct several classes of non-stretch mappings that validate the Bañuelos-Wang conjecture and the Iwaniec conjecture.

2 Proof of Theorem 1.1

Proof. By the assumption that $g \in W^{1,2}_{loc}(\mathbb{D})$ and $|a| < 1$, we have that, as a solution of (1.9), g is a locally quasiregular mapping of $\mathbb{D}$. Consequently, it is open and sense preserving. Denote by $\mathbb{Z}(g)$ the zero set of g. For any point $z_0 \in \mathbb{D}\backslash\mathbb{Z}(g)$, there exists a $r > 0$ such that $\log g$ is harmonic on $\mathbb{D}(z_0, r) = \{z||z - z_0| < r\}$ and thus $g \in C^\infty(\mathbb{D}(z_0, r))$. Hence, by (1.10) we have g is $\frac{1}{|g|^2}$-harmonic on $\mathbb{D}(z_0, r)$, that is, g satisfies

$$g g_{z\bar{z}} = g_z g_{\bar{z}}, \quad z \in \mathbb{D}(z_0, r). \tag{2.1}$$

Differentiating both sides of (2.1) in z, we obtain

$$g_{zz\bar{z}} = \frac{g_{zz} g_{\bar{z}}}{g}, \quad z \in \mathbb{D}(z_0, r).$$

The assumption of the locally univalence of g implies that $\log g$ is locally univalent on $\mathbb{D}(z_0, r)$. By the Lewy theorem [27], the harmonicity of $\log g$ on $\mathbb{D}(z_0, r)$ implies that the Jacobian $J_{\log g} > 0$ on $\mathbb{D}(z_0, r)$ and consequently $|g_z| > 0$ on $\mathbb{D}(z_0, r)$. Multiplying g_z to both sides of the above equality, we have

$$g_z g_{zz\bar{z}} = g_{zz} g_{z\bar{z}}, \quad z \in \mathbb{D}(z_0, r).$$

Direct computation shows that

$$\Delta \log |g_z| = 0 \tag{2.2}$$

holds for all $z \in \mathbb{D}\backslash\mathbb{Z}(g)$. This implies that $\log|g_z|$ is subharmonic on $\mathbb{D}$. The relation (1.9) and the subharmonicity of $\log|g_z|$ and $\log|a|$ show that $\log|g_{\bar z}|$ is also subharmonic on $\mathbb{D}$. Hence, the logarithms of both $|g_z|(|g_z|+|g_{\bar z}|)^{p-1}$ and $|g_{\bar z}|(|g_z|+|g_{\bar z}|)^{p-1}$ are subharmonic on $\mathbb{D}$. Thus, the functions themselves are subharmonic on $\mathbb{D}$.

Let $f = g\circ\varphi$ and $\zeta = \frac{1}{\bar z}$. For any $z \in \mathbb{D}^c$, it follows that

$$f_z = (g\circ\varphi)_z = (g(\frac{1}{\bar z}))_z = g_\zeta(\zeta)\zeta_z + g_{\bar\zeta}(\zeta)\bar\zeta_z = -\bar\zeta^2 g_{\bar\zeta}(\zeta), \tag{2.3}$$

and

$$f_{\bar z} = (g\circ\varphi)_{\bar z} = (g(\frac{1}{\bar z}))_{\bar z} = g_\zeta(\zeta)\zeta_{\bar z} + g_{\bar\zeta}(\zeta)\bar\zeta_{\bar z} = -\zeta^2 g_\zeta(\zeta). \tag{2.4}$$

For $z \in \mathbb{D}$, we have

$$f_z = (g\circ\varphi)_z = g_z, \quad f_{\bar z} = (g\circ\varphi)_{\bar z} = g_{\bar z}. \tag{2.5}$$

By the definition of $\mathbf{B}_p(Df)$ and the assumption that $f \in \dot W^{1,p}(\mathbb{C},\mathbb{C})$, we get from (2.3), (2.4) and (2.5) that

$$\begin{aligned}
&\iint_{\mathbb{C}} \mathbf{B}_p(Df)dm = \iint_{\mathbb{D}} \mathbf{B}_p(Df)dm + \iint_{\mathbb{D}^c} \mathbf{B}_p(Df)dm \\
&= \iint_{\mathbb{D}} [(p^*-1)|g_{\bar\zeta}| - |g_\zeta|](|g_\zeta|+|g_{\bar\zeta}|)^{p-1} dm(\zeta) \\
&+ \iint_{\mathbb{D}} [(p^*-1)|g_\zeta| - |g_{\bar\zeta}|](|g_\zeta|+|g_{\bar\zeta}|)^{p-1}|\zeta|^{2(p-2)} dm(\zeta) \\
&= \iint_{\mathbb{D}} [(p^*-1) - |\zeta|^{2(p-2)}]|g_{\bar\zeta}|(|g_\zeta|+|g_{\bar\zeta}|)^{p-1} r dr d\theta \\
&+ \iint_{\mathbb{D}} [(p^*-1)|\zeta|^{2(p-2)} - 1]|g_\zeta|(|g_\zeta|+|g_{\bar\zeta}|)^{p-1} r dr d\theta = I + II,
\end{aligned}$$

where $\zeta = re^{i\theta}$. It is clear that $I = II = 0$ when $p = 2$. If $2 < p < \infty$, then $I > 0$. Now we can also show that $II > 0$.

Write

$$I_1(r) = \frac{1}{2\pi}\int_0^{2\pi} |g_\zeta|(|g_\zeta|+|g_{\bar\zeta}|)^{p-1}d\theta, \quad I_2(r) = \frac{1}{2\pi}\int_0^{2\pi} |g_{\bar\zeta}|(|g_\zeta|+|g_{\bar\zeta}|)^{p-1}d\theta. \tag{2.6}$$

Then II can be written as

$$II = 2\pi\int_0^1 [(p-1)r^{2p-3} - r]I_1(r)dr.$$

Integration by parts gives

$$II = 2\pi\int_0^1 (\frac{r^2}{2} - \frac{r^{2p-2}}{2})dI_1(r). \tag{2.7}$$

When $2 < p < \infty$, the inequality $\frac{r^2}{2} - \frac{r^{2p-2}}{2} > 0$ holds for $0 < r < 1$. The subharmonic property of the integrand of $I_1(r)$ implies that $I_1(r)$ is non-decreasing for $0 < r < 1$, that is, $dI_1(r) \geq 0$ *a.e.*. Hence, $II > 0$.

When $1 < p < 2$, $II > 0$ is obvious and the inequality $I > 0$ can be deduced from the non-decreasing property of $I_2(r)$ on $(0, 1)$ and the technique that we use in the case $2 < p < \infty$. Thus, for $1 < p < \infty$, we have

$$\iint_{\mathbb{C}} \mathbf{B}_p(Df)dm \geq 0.$$

So, the Bañuelos-Wang conjecture is true for a mapping $f = g \circ \varphi \in \dot{W}^{1,p}(\mathbb{C}, \mathbb{C})$, when g satisfies the partial differential equation (1.9). As a consequence, the I-waniec conjecture is also true for this class of mappings. □

3 Non-stretch explicit examples constructed by principal solutions

The *Cauchy operator* is defined by

$$\mathbf{C}f(z) = -\frac{1}{\pi} \iint_{\mathbb{C}} \left(\frac{1}{\zeta - z} - \frac{\chi_{\mathbb{C}\setminus\mathbb{D}}}{\zeta}\right) f(\zeta)dm(\zeta), \tag{3.1}$$

for a function $f \in L^p(\mathbb{C})$, $p \geq 2$. For $f \in L^p(\mathbb{C})$, $p > 2$, $\mathbf{C}f$ is Hölder continuous with exponent $1 - 2/p$ (see Theorem 4.3.13 of [5] or [3]), while, for $f \in L^2(\mathbb{C})$, $\mathbf{C}f$ belongs to the space $VMO(\mathbb{C})$ (see Theorem 4.3.9 of [5]). When f is also compactly supported, the integral is going to be analytic near ∞ with the Laurent series

$$\mathbf{C}f(z) = \frac{1}{\pi} \iint_{\mathbb{C}} \frac{\chi_{\mathbb{C}\setminus\mathbb{D}}}{\zeta} f(\zeta)dm(\zeta) + \left(\frac{1}{\pi} \iint_{\mathbb{C}} f(\zeta)dm(\zeta)\right)\frac{1}{z} + \sum_{n=2}^{\infty} \frac{b_n}{z^n},$$

where b_n, $n \geq 2$, are constants. One can see Chapter 4 of [5] for more properties of the Cauchy operator. Before constructing explicit examples which are non-stretch, we need some lemmas. From the Green formula and a limit process, we have

Lemma A. *If $f \in L^p(\mathbb{C})$, $p \geq 2$, then the relations*

$$\partial \mathbf{C}f = \mathbf{T}f, \quad \bar{\partial}\mathbf{C}f = f. \tag{3.2}$$

hold in the distributional sense.

See p52-53 in [3] and p112 in [5] for a proof of Lemma A.

Let Ω be a bounded domain and χ_Ω the characteristic function of Ω. Let μ be a measurable function on $\mathbb{C}$ with $||\mu||_\infty \leq 1$. Then $\mu\chi_\Omega$ belongs to $L^p(\mathbb{C})$ for any $p \geq 2$ and thus $\mathbf{C}(\mu\chi_\Omega)$ and $\mathbf{T}(\mu\chi_\Omega)$ are well defined. Define

$$\mathbf{Q}f = \mu\chi_\Omega\mathbf{T}f$$

for $f \in L^p(\mathbb{C})$. Write

$$\mathbf{Q}^n f = \underbrace{\mathbf{Q} \circ \cdots \circ \mathbf{Q}}_{n}(f), \quad n \in \mathbb{N}^+.$$

By induction, $\mathbf{Q}^n(\mu\chi_\Omega)$ is well defined for all $n \in \mathbb{N}^+$. If the series $\sum_{n=1}^{\infty} \mathbf{Q}^n(\mu\chi_\Omega)$ converges and its sum h belongs to $L^p(\mathbb{C})$, $p \geq 2$, then

$$f = z + \mathbf{C}(\mu\chi_\Omega + h) \tag{3.3}$$

is a principal solution of the Beltrami equation

$$f_{\bar{z}} = \mu\chi_\Omega f_z.$$

Moreover, $f_z - 1 \in L^p(\mathbb{C})$, $p \geq 2$, and

$$f_z = 1 + \mathbf{T}(\mu\chi_\Omega + h), \quad f_{\bar{z}} = \mu\chi_\Omega + h.$$

Lemma 3.1. *Let $\mu = \bar{z}^n z^m$, where n and m are integers. Then the following relations hold. If $n \geq m$, then*

$$\mathbf{C}(\mu\chi_\mathbb{D})(z) = z^m \frac{\varphi(\bar{z})^{n+1}}{n+1} \tag{3.4}$$

and

$$\mathbf{T}(\mu\chi_\mathbb{D})(z) = \begin{cases} \frac{m}{n+1} z^{m-1}\bar{z}^{n+1}, & m \neq 0, z \in \mathbb{D}, \\ 0, & m = 0, z \in \mathbb{D}, \\ -\frac{n-m+1}{(n+1)z^{n-m+2}}, & \qquad z \in \mathbb{D}^c. \end{cases} \tag{3.5}$$

If $n = m - 1$, then

$$\mathbf{C}(\mu\chi_\mathbb{D})(z) = -\frac{1 - |z|^{2n+2}}{n+1}\chi_\mathbb{D} \tag{3.6}$$

and

$$\mathbf{T}(\mu\chi_\mathbb{D})(z) = z^n \bar{z}^{n+1}\chi_\mathbb{D}. \tag{3.7}$$

If $n \leq m - 2$, then

$$\mathbf{C}(\mu\chi_\mathbb{D})(z) = -\frac{z^{m-(n+1)}}{n+1}(1 - |z|^{2n+2})\chi_\mathbb{D} \tag{3.8}$$

and

$$\mathbf{T}(\mu\chi_\mathbb{D})(z) = (-\frac{m-(n+1)}{n+1} z^{m-(n+2)} + \frac{m}{n+1} z^{m-1}\bar{z}^{n+1})\chi_\mathbb{D}. \tag{3.9}$$

Proof. Let $\zeta = re^{i\theta}$. By the definition of the Cauchy operator, we have

$$\mathbf{C}(\mu\chi_\mathbb{D})(z) = -\frac{1}{\pi}\iint_\mathbb{C} \frac{\bar{\zeta}^n \zeta^m \chi_\mathbb{D}}{\zeta - z} dm(\zeta) = -2\int_0^1 r^{2n+1} I_z(r) dr, \tag{3.10}$$

where

$$I_z(r) = \frac{1}{2\pi i}\oint_{|\zeta|=r} \frac{1}{\zeta^{n-m+1}(\zeta - z)} d\zeta. \tag{3.11}$$

When $n \geq m$, we obtain

$$I_z(r) = -\frac{1}{z^{n-m+1}}\chi_{\mathbb{D}}. \tag{3.12}$$

Thus, it follows from (3.12) that

$$\mathbf{C}(\mu\chi_{\mathbb{D}})(z) = -2\int_0^1 r^{2n+1} I_z(r)dr = \frac{1}{(n+1)z^{n-m+1}}, \quad z \in \mathbb{D}^c$$

and

$$\mathbf{C}(\mu\chi_{\mathbb{D}})(z) = -2\int_0^{|z|} r^{2n+1} I_z(r)dr + \int_{|z|}^1 r^{2n+1} I_z(r)dr = \frac{1}{(n+1)} z^m \bar{z}^{n+1}, \quad z \in \overline{\mathbb{D}}.$$

By the first equality of (3.2) of Lemma A, one can get (3.5).

The proofs of the cases $n = m - 1$ and $n \leq m - 2$ can be obtained by the method used in the case $n \geq m$, we omit for simplicity. $\square$

Example 3.1. Let $\mu = z$. Then a principal solution of the Beltrami equation

$$f_{\bar{z}} = \mu\chi_{\mathbb{D}} f_z$$

is given by

$$f(z) = ze^{\varphi(\bar{z})} - 1 \tag{3.13}$$

Proof. Choose $m = 1, n = 0$ in Lemma 3.1. Then by the relation (3.7), we have

$$\mathbf{Q}(\mu\chi_{\mathbb{D}}) = z\bar{z}\chi_{\mathbb{D}}.$$

The relation (3.5) gives

$$\mathbf{Q}^2(\mu\chi_{\mathbb{D}}) = \frac{1}{2} z\bar{z}^2\chi_{\mathbb{D}}.$$

Hence, it follows from induction that

$$\mathbf{Q}^n(\mu\chi_{\mathbb{D}}) = \frac{1}{n!} z\bar{z}^n\chi_{\mathbb{D}}, \qquad n \in \mathbb{N}^+.$$

Set $\mathbf{Q}^0(\mu\chi_{\mathbb{D}}) = \mu\chi_{\mathbb{D}}$. By the convergence of the series $\sum_{n=0}^{\infty} \mathbf{Q}^n(\mu\chi_{\mathbb{D}})$ and the fact that its sum belongs to $L^p(\mathbb{C})$, $p \geq 2$, we have that $f = z + \mathbf{C}(\sum_{n=0}^{\infty} \mathbf{Q}^n(\mu\chi_{\mathbb{D}}))$ is a principal solution of the Beltrami equation $f_{\bar{z}} = z\chi_{\mathbb{D}} f_z$. Moreover, for $z \in \mathbb{D}$,

$$\begin{aligned} f(z) &= z + \mathbf{C}(\sum_{n=0}^{\infty} \mathbf{Q}^n(\mu\chi_{\mathbb{D}})) \\ &= z - (1 - |z|^2) + z(\frac{1}{2}\bar{z}^2 + \frac{1}{3\cdot 2!}\bar{z}^3 + \cdots + \frac{1}{(n+1)\cdot n!}\bar{z}^{n+1} + \cdots) \\ &= ze^{\bar{z}} - 1. \end{aligned}$$

Similarly, for $z \in \mathbb{D}^c$, we have $f(z) = ze^{\frac{1}{z}} - 1$. $\square$

Next, we will use principal solutions to construct several classes of mappings validating the Bañuelos-Wang conjecture and the Iwaniec conjecture.

Theorem 3.1. *Let I be the identical mapping and μ is co-analytic on $\mathbb{C}$. If $f+I$ is a principal solution with the Beltrami coefficient $\mu\chi_{\mathbb{D}}$, then*

$$\iint_{\mathbb{C}} \mathbf{B}_p(Df)dm \geq 0, \tag{3.14}$$

and the equality holds when $p=2$.

Proof. The assumption on μ implies that μ can be represented by a power series $\sum_{n=0}^{\infty} \overline{a_n}\bar{z}^n$. Owing to (3.5), we have that $\mu\chi_{\mathbb{D}}\mathbf{T}(\bar{z}^n\chi_{\mathbb{D}})=0$ for all $n\in\mathbb{N}^+$. Now the linearity of the Beurling-Ahlfors operator implies

$$\mathbf{Q}\mu\chi_{\mathbb{D}}(z)=0.$$

So,

$$\mathbf{Q}^n(\mu\chi_{\mathbb{D}})=0, \qquad n\in\mathbb{N}^+.$$

By the linearity of the Cauchy operator, we get

$$f+I=z+\mathbf{C}(\sum_{n=0}^{\infty}\mathbf{Q}^n(\mu\chi_{\mathbb{D}}))=z+\sum_{n=0}^{\infty}\mathbf{C}(\overline{a_n z^n}\chi_{\mathbb{D}}).$$

According to (3.4), we have

$$f(z)=\sum_{n=0}^{\infty}\mathbf{C}(\overline{a_n z^n}\chi_{\mathbb{D}})=\sum_{n=0}^{\infty}\overline{a_n}\frac{\varphi(\bar{z})^{n+1}}{n+1}. \tag{3.15}$$

Now we prove that f validates the Bañuelos-Wang conjecture.

$$\iint_{\mathbb{C}}\mathbf{B}_p(Df)dm=\iint_{\mathbb{D}}\mathbf{B}_p(Df)dm+\iint_{\mathbb{D}^c}\mathbf{B}_p(Df)dm=I+II.$$

By (3.15), we have

$$I=\iint_{\mathbb{D}}(p-1)|\sum_{n=0}^{\infty}a_n z^n|^p dxdy,$$

and

$$II=\iint_{\mathbb{D}^c}|\sum_{n=0}^{\infty}\overline{a_n}\frac{1}{z^{n+2}}|^p dxdy=\iint_{\mathbb{D}}|\sum_{n=0}^{\infty}a_n z^{n+2}|^p|z|^{-4}dm(z).$$

Let $z=re^{i\theta}$. Then,

$$\iint_{\mathbb{C}}\mathbf{B}_p(Df)dm=\iint_{\mathbb{D}}|\mu|^p((p-1)-r^{2(p-2)})rdrd\theta\geq 0,$$

and the equality holds when $p=2$. □

Generally, it is difficult to explicitly represent a principal solution for a given Beltrami coefficient. For some special classes of Beltrami coefficients, we can obtain their explicit principal solutions and use them to construct non-stretch examples validating the Bañuelos-Wang conjecture and the Iwaniec conjecture.

Example 3.2. Let $g(z) = f(z) - z + 1$, where $f(z)$ is given by Example 3.1. Then

$$\iint_{\mathbb{C}} \mathbf{B}_2(Dg)dm = 0, \quad \iint_{\mathbb{C}} \mathbf{B}_4(Dg)dm > 0.$$

Proof. By the equation (3.13), we get

$$g_z = \begin{cases} e^{\bar{z}} - 1, & |z| < 1, \\ e^{\frac{1}{z}} - \frac{1}{z}e^{\frac{1}{z}} - 1, & |z| > 1, \end{cases} \quad g_{\bar{z}} = \begin{cases} ze^{\bar{z}}, & |z| < 1, \\ 0, & |z| > 1. \end{cases} \tag{3.16}$$

It follows from the Parseval formula that

$$\iint_{\mathbb{D}} (|ze^z|^2 - |e^z - 1|^2)dm(z) = \pi \sum_{n=2}^{\infty} \frac{n-1}{(n!)^2} \tag{3.17}$$

and

$$\iint_{\mathbb{D}} \frac{|e^z - ze^z - 1|^2}{|z|^4} dm(z) = \pi \sum_{n=2}^{\infty} \frac{n-1}{(n!)^2}. \tag{3.18}$$

By the above two equations we have

$$\iint_{\mathbb{C}} \mathbf{B}_2(Dg)dm = \iint_{\mathbb{D}} (|ze^z|^2 - |e^z - 1|^2 - \frac{|e^z - ze^z - 1|^2}{|z|^4})dm(z) = 0.$$

From the power series $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ of e^z, it follows that

$$z^2e^{2z} = \sum_{n=2}^{\infty} \frac{2^{n-2}}{(n-2)!} z^n, \quad (e^z - 1)^2 = \sum_{n=2}^{\infty} \frac{2^n - 2}{n!} z^n, \tag{3.19}$$

$$(\frac{e^z - ze^z - 1}{z})^2 = \sum_{n=2}^{\infty} \frac{2^n(n-2) + 2}{n+2} \frac{z^n}{n!}. \tag{3.20}$$

Next we prove the second assertion of Example 3.2. By direct calculations, we have

$$\begin{aligned} \iint_{\mathbb{C}} \mathbf{B}_4(Dg)dm &= \iint_{\mathbb{C}} (3|g_{\bar{z}}|^4 - |g_z|^4 + 6|g_z|^2|g_{\bar{z}}|^2 + 8|g_z||g_{\bar{z}}|^3)dm \\ &\geq \iint_{\mathbb{C}} (3|g_{\bar{z}}|^4 - |g_z|^4)dm(z) = III - IV, \end{aligned}$$

where

$$III = \iint_{\mathbb{D}} [3|z^2e^{2z}|^2 - |(e^z - 1)^2|^2]dm(z), \quad IV = \iint_{\mathbb{D}} \frac{|(e^z - ze^z - 1)^2|^2}{|z|^4} dm(z).$$

Using the Parseval formula, we obtain from (3.19) and (3.20) that

$$
\begin{aligned}
III - IV &= \sum_{n=2}^{\infty} \frac{\pi}{[(n-2)!]^2(n+1)} \{ \frac{3 \cdot 2^{2n}}{16} - \frac{(2^n-2)^2(n+2)^2 + [2^n(n-2)+2]^2}{[(n-1)n(n+2)]^2} \} \\
&\geq \pi \{ \frac{31}{16} + \sum_{n=3}^{\infty} \frac{11}{144} \frac{4^n}{[(n-2)!]^2(n+1)} \} > 0.
\end{aligned}
$$

The proof of Example 3.2 is now complete. $\square$

Moreover, we can get a more general result as follows

Theorem 3.2. *Let I be the identical mapping and $\mu = \bar{z}^n z$ on $\mathbb{C}$, where $n \geq 1$. If $f + I$ is a principal solution of the Beltrami equation with the Beltrami coefficient $\mu \chi_{\mathbb{D}}$, then*

$$
\iint_{\mathbb{C}} \mathbf{B}_4(Df) dm > 0.
$$

Proof. By induction, we get from the equality (3.5) at Lemma 3.1 that

$$
\mathbf{Q}^k = \begin{cases} \frac{1}{k!} \frac{1}{(n+1)^k} \bar{z}^{k(n+1)}, & |z| \leq 1, \\ -\frac{kn+k+1}{k!(n+1)^k} \frac{1}{z^{kn+k+2}}, & |z| > 1, \end{cases} \tag{3.21}
$$

where $k \geq 1$. Hence, by the equality (3.4) of Lemma 3.1 we have

$$
\mathbf{C}(\mathbf{Q}^k(\mu \chi_{\mathbb{D}})) = \begin{cases} \frac{1}{(k+1)!} \frac{1}{(n+1)^{k+1}} \bar{z}^{k(n+1)} z, & |z| \leq 1, \\ \frac{1}{(k+1)!} \frac{1}{(n+1)^{k+1}} \frac{1}{z^{k(n+1)+k}}, & |z| > 1. \end{cases}
$$

Then the representation (3.3) gives

$$
f(z) = z e^{\frac{\varphi(\bar{z})^{n+1}}{n+1}} - z.
$$

Moreover, it follows

$$
f_z = \begin{cases} e^{\frac{\bar{z}^{n+1}}{n+1}} - 1, & |z| \leq 1, \\ e^{\frac{1}{(n+1)z^{n+1}}} - \frac{1}{z^{n+1}} e^{\frac{1}{(n+1)z^{n+1}}} - 1, & |z| > 1, \end{cases}
$$

and

$$
f_{\bar{z}} = z \bar{z}^n e^{\frac{\bar{z}^{n+1}}{n+1}} \chi_{\mathbb{D}}.
$$

Using change of variable, we have

$$
\begin{aligned}
\iint_{\mathbb{C}} \mathbf{B}_4(Df) dm &= \iint_{\mathbb{C}} (3|f_{\bar{z}}|^4 - |f_z|^4 + 6|f_z|^2|f_{\bar{z}}|^2 + 8|f_z||f_{\bar{z}}|^3) dm \\
&\geq \iint_{\mathbb{C}} (3|f_{\bar{z}}|^4 - |f_z|^4) dm = V - VI,
\end{aligned}
$$

where

$$
V = \iint_{\mathbb{D}} [3|z^{2(n+1)} e^{2\frac{z^{n+1}}{n+1}}|^2 - |(e^{\frac{z^{n+1}}{n+1}} - 1)^2|^2] dm(z),
$$

and

$$VI = \iint_{\mathbb{D}} \frac{|(e^{\frac{z^{n+1}}{n+1}} - z^{n+1}e^{\frac{z^{n+1}}{n+1}} - 1)^2|^2}{|z|^4} dm(z).$$

From the power series expansion $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$, it follows

$$(e^{\frac{z^{n+1}}{n+1}} - z^{n+1}e^{\frac{z^{n+1}}{n+1}} - 1)^2$$
$$= \sum_{k=2}^{\infty} \frac{1}{(k-2)!}(2^{k-2}(n+1)^2 - \frac{2^k-2}{k-1}((n+1) - \frac{1}{k})(\frac{z^{n+1}}{n+1})^k.$$

Utilizing the Parseval formula, we obtain, from (3.19) and the above relation, that

$$V - VI = 2\pi \sum_{k=2}^{\infty} \frac{1}{((k-2)!(n+1)^k)^2}\{(3 * 2^{2(k-2)}(n+1)^4 - \frac{(2^k-2)^2}{k^2(k-1)^2})$$
$$\frac{1}{2k(n+1)+2} - (2^{k-2}(n+1)^2 - \frac{2^k-2}{k-1}((n+1) - \frac{1}{k}))^2 \frac{1}{2k(n+1)-2}\}.$$

The assumptions that $n \geq 1$ and $k \geq 2$ imply that

$$2^{k-2}(n+1)^2 - \frac{2^k-2}{k-1}(n+1-\frac{1}{k}) > (\frac{2^k}{2} - \frac{2^k-2}{k-1})(n+1) \geq 0$$

and

$$2^{2(k-2)}(n+1)^4 - \frac{(2^k-2)^2}{k^2(k-1)^2} > 2^{2k} - \frac{2^{2k}}{4} = \frac{3}{4}2^{2k} > 0.$$

Thus, we have

$$\begin{aligned} V - VI >& 2\pi \sum_{k=2}^{\infty} \frac{1}{((k-2)!(n+1)^k)^2}\{\frac{3}{4}\frac{2^{2k}}{2k(n+1)+2} \\ &+ 2^{2(k-2)}(n+1)^4 \frac{k(n+1)-3}{2(k(n+1))^2-2)}\} > 0. \end{aligned}$$

Therefore, Theorem 3.2 follows. □

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