On backward shift algorithm for estimating poles of systems ⋆

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Abstract

In this paper, we present an algorithm of estimating poles of linear time-invariant systems by using the backward shift operator. It is proved that poles of rational functions, including zeros and multiplicities, are solutions of an algebra equation which can be obtained by taking backward shift operator to normalized reproductive kernels in the unit disc case. The algorithm is accordingly developed for frequency-domain identification. The robustness of this algorithm is proved. Some illustrative examples are presented to show the efficiency for systems with distinguished and multiple poles cases.

Key words: Backward shift operator, poles estimation, rational orthogonal basis, linear time-invariant systems, identification methods

1 Introduction

System identification is to build mathematical models which fit the measured data from discrete or continuous systems. An amount of methods have been developed for this problem, such as [5,6,26–28]. A classical guidebook for one getting to know this topic is [9]. For system identification of linear time-invariant (LTI) systems, the priori-knowledge of poles is important, especially for the methods that adopt rational orthogonal bases such as in[15–17,25]. In these methods, the estimated poles are used to construct rational orthogonal basis functions. A collection of these excellent results is [7].

In unit disc case, the general setting of a rational orthogonal basis is

$$\mathcal{B}_k(z)=\mathcal{B}_{\{a_1,.a_k\}}(z)\triangleq\frac{\sqrt{1-|a_k|^2}}{1-\overline{a}_k z}\prod_{l=1}^{k-1}\frac{z-a_l}{1-\overline{a}_l z},\quad (1)$$

where a_ks $(k=1,...)$ are in the unit disc, ($\overline{a}$ means conjugation of a). Many researchers work on choosing optimal n-poles $\{a_k\}_{k=1}^n$ in order to define the best rational orthogonal bases for a system. T. Oliveria e Silva derived the optimal pole conditions for Laguerre, Kautz and general orthogonal basis function models in [18–20], respectively. In [10–12], adaptive selection of poles is studied. Other attempts to estimate optimal pole positions of a Laguerre model are given in [23,3]. Generally speaking, the pole estimation of an LTI system, in practice, is not easy.

For a discrete LTI system which is causal and stable, let $\{x_k\},\{y_k\}$ be the input and output signals, respectively. There is a relation between $\{x_k\}$ and $\{y_k\}$ as

$$y_k=\{x_k\}*\{h_k\}=\sum_{l=0}^{+\infty}h_l x_{k-l},\quad (2)$$

where $\{h_k\}$ is the impulse response. With an operator q, $qx(k)=x(k+1)$, is drawn into, (2) can be represented as

$$y_k=\sum_{l=0}^{+\infty}h_l x_{k-l}=(\sum_{l=0}^{+\infty}h_l q^{-l})x_k.\quad (3)$$

The related function

$$G(z)=\sum_{l=0}^{+\infty}h_l z^{-l}\quad (4)$$

is the transfer function of the system. The values of the transfer function for z on the unit circle are called frequency responses. Under the stability and casuality as-

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sumption, $G(z)$ is a proper rational function with real coefficients. To identify $G(z)$ with general orthogonal bases, estimating poles for the basis functions plays a significant role.

As is well-known, the basis (1) can be obtained by the normalized reproducing kernels $e_a(z)$ with Gram-Schmidt process, where the normalized reproducing kernel at a, is given by

$$e_a(z) = \frac{\sqrt{1-|a|^2}}{1-\overline{a}z}.$$

For these kernels, there is a very good property when taking backward shift operator on them. Based on this property, we can estimate poles of e_a's instead of the orthogonal cases. In this paper, we are to locate poles of an LTI system based on a set of frequency domain measurements by using backward shift operator, which results in an algorithm, we call it *backward shift algorithm.*

This paper is arranged as follows. In section 2, we study each case of taking backward shift operator to the rational functions. After that, we introduce the backward shift algorithm in detail in section 3. Examples are given in section 4 to illustrate the proposed idea. Some conclusions are drawn in the last section.

2 Backward shift on rational functions

2.1 Backward shift operator

The backward shift operator, denoted by $\mathbf{S}$,

$$\mathbf{S}(f)(z) = \frac{f(z)-f(0)}{z}, \tag{5}$$

is the Banach space adjoint of the forward shift operator $\mathbf{F}(f)(z) = zf(z)$ in the Hardy-2 space in the unit disc, viz.,

$$\langle \mathbf{S}(f), g\rangle = \langle f, \mathbf{F}(g)\rangle, \ f, g \in H_2. \tag{6}$$

It is an important and interesting operator. Comprehensive studies in the operator and related topics can be found, for instance, in [1,14,4]. It is well known that a collection of countably many reproducing kernels of the Hardy space H_2, viz., conjugates of shifted Cauchy kernels, generates a backward shift invariant subspace.

For $0 \neq a \in \mathbb{D}$, the unit disc, we notice the kernel $e_a(z) = \frac{1}{1-\overline{a}z}$ (For convenience we will call $\overline{a}$ a pole of it, although we know precisely it is $\frac{1}{\overline{a}}$) is an eigenvector of $\mathbf{S}$, viz.,

$$\begin{aligned}\mathbf{S}(e_a)(z) &= \frac{e_a(z)-e_a(0)}{z} \\ &= \frac{\overline{a}}{1-\overline{a}z}.\end{aligned}$$

Therefore,

$$\begin{aligned}\mathbf{S}^2(e_a)(z) &= \mathbf{S}(\mathbf{S}(e_a))(z) \\ &= \frac{\overline{a}^2}{1-\overline{a}z},\end{aligned}$$

and, in general,

$$\mathbf{S}^n(e_a)(z) = \frac{\overline{a}^n}{1-\overline{a}z}. \tag{7}$$

An n-tuple $(a_1, ..., a_n)$ in the unit disc correspond to one of the following two n-tuples of partial fractions, being determined on whether some a_k's are zero. Denote by $b_1, ..., b_m$ all the distinguished ones among $a_1, ..., a_n$.

Case 1.

If none of the distinguished b_k's is zero, then it corresponds to

$$\frac{1}{1-\overline{b}_1 z}, ..., \frac{1}{(1-\overline{b}_1 z)^{l_1}}, ..., \frac{1}{1-\overline{b}_m z}, ..., \frac{1}{(1-\overline{b}_m z)^{l_m}},$$

where $l_1, ..., l_m$ are multiples of $b_1, ..., b_m$, respectively and $l_1 + ... + l_m = n$.

A rational function p/q, where p and q are co-prime polynomials, is a non-degenerate linear combination of the above linearly independent set of functions if and only if the degree of q is equal to n, and the degree of p is less than n.

Case 2.

If one of the distinguished b_k's is zero, say, $b_1 = 0$, with multiplicity l_1, then it corresponds to

$$1, ..., z^{l_1}, \frac{1}{1-\overline{b}_2 z}, ..., \frac{1}{(1-\overline{b}_2 z)^{l_2}}, ..., \frac{1}{(1-\overline{b}_m z)^{l_m}},$$

where $l_1 + ... + l_m = n$.

A rational function p/q, where p and q are co-prime polynomials, is a non-degenerate linear combination of the above linearly independent set of functions if and only if the degree of q is equal to $n - l_1$, and the degree of p is less than n.

These cases will be studied in detail in the following three subsections.

2.2 The distinguished non-zero poles case

In this subsection we treat the case where all b_k, $k = 1, ..., n$, are different with each other, that is, each multiplicity is 1. Assume that f is of the form

$$f(z)=\sum_{k=1}^{n}\frac{\lambda_k}{1-\bar{b}_k z}, \tag{8}$$

where λ_ks are non-zero, b_ks are non-zero and distinguished from each other. Applying, consecutively, the backward shift operator $\mathbf{S}$ to $f(z)$ n times, we have

$$\begin{cases} \mathbf{S}(f)(z)=\dfrac{\lambda_1\bar{b}_1}{1-\bar{b}_1 z}+\dfrac{\lambda_2\bar{b}_2}{1-\bar{b}_2 z}+...+\dfrac{\lambda_n\bar{b}_n}{1-\bar{b}_n z}, \\ \mathbf{S}^2(f)(z)=\dfrac{\lambda_1\bar{b}_1^2}{1-\bar{b}_1 z}+\dfrac{\lambda_2\bar{b}_2^2}{1-\bar{b}_2 z}+...+\dfrac{\lambda_n\bar{b}_n^2}{1-\bar{b}_n z}, \\ \vdots \\ \mathbf{S}^n(f)(z)=\dfrac{\lambda_1\bar{b}_1^n}{1-\bar{b}_1 z}+\dfrac{\lambda_2\bar{b}_2^n}{1-\bar{b}_2 z}+...+\dfrac{\lambda_n\bar{b}_n^n}{1-\bar{b}_n z}. \end{cases}$$

Since the b_ks are distinguished, $\{\frac{1}{1-\bar{b}_k z}\}_{k=1}^n$ is a linearly independent collection. There exists a unique non-zero sequence $\{\mu_k\}_{k=0}^n$ such that

$$\mu_0 f(z)+\mu_1\mathbf{S}(f)(z)+...+\mu_n\mathbf{S}^n(f)(z)=0. \tag{9}$$

Precisely,

$$\begin{cases} 0=(\mu_0+\mu_1\bar{b}_1+...+\mu_{n-1}\bar{b}_1^{n-1}+\mu_n\bar{b}_1^n)\dfrac{\lambda_1}{1-\bar{b}_1 z} \\ \quad+(\mu_0+\mu_1\bar{b}_2+...+\mu_{n-1}\bar{b}_2^{n-1}+\mu_n\bar{b}_2^n)\dfrac{\lambda_2}{1-\bar{b}_2 z} \\ \vdots \\ \quad+(\mu_0+\mu_1\bar{b}_n+...+\mu_{n-1}\bar{b}_n^{n-1}+\mu_n\bar{b}_n^n)\dfrac{\lambda_n}{1-\bar{b}_n z}. \end{cases}$$

The linear independence of $\{\frac{1}{1-\bar{b}_k z}\}_{k=1}^n$ implies

$$\begin{cases} \mu_0+\mu_1\bar{b}_1+\mu_2\bar{b}_1^2+...+\mu_{n-1}\bar{b}_1^{n-1}+\mu_n\bar{b}_1^n=0, \\ \mu_0+\mu_1\bar{b}_2+\mu_2\bar{b}_2^2+...+\mu_{n-1}\bar{b}_2^{n-1}+\mu_n\bar{b}_2^n=0, \\ \vdots \\ \mu_0+\mu_1\bar{b}_n+\mu_2\bar{b}_n^2+...+\mu_{n-1}\bar{b}_n^{n-1}+\mu_n\bar{b}_n^n=0, \end{cases}$$

Which means b_k, $k=1,...,n$, are the solutions of an algebraic equation

$$\mu_n x^n+\mu_{n-1}x^{n-1}+\mu_{n-2}x^{n-2}+...+\mu_1 x+\mu_0=0. \tag{10}$$

The equation (10) has n different roots, then $\mu_n\neq 0$, we may assume $\mu_n=1$.

2.3 The Multiple But Non-Zero Poles Case

In this subsection, we deal with rational function of order n with non-zero multiple poles. Without loss of generality, we assume $f(z)$ to be of the form

$$f(z)=\sum_{j=1}^{m}\frac{\lambda_j}{(1-\bar{b}_1 z)^j}+\sum_{k=m+1}^{n}\frac{\lambda_k}{1-\bar{b}_k z}, \tag{11}$$

where λ_k, for $k\geq m$, are all nonzero. First of all, by applying the backward shift operator to $\frac{1}{(1-\bar{b}_1 z)^m}$, we have

$$\mathbf{S}(\frac{1}{(1-\bar{b}_1 z)^m}) \\ =\bar{b}_1\left(\frac{1}{(1-\bar{b}_1 z)^m}+\frac{1}{(1-\bar{b}_1 z)^{m-1}}+...+\frac{1}{(1-\bar{b}_1 z)}\right). \tag{12}$$

Set $M(z)=\sum_{j=1}^m\frac{1}{(1-\bar{b}_1 z)^j}$, then

$$\mathbf{S}(M)(z) \\ =\bar{b}_1\frac{1}{(1-\bar{b}_1 z)}+\bar{b}_1\left(\frac{1}{(1-\bar{b}_1 z)^2}+\frac{1}{(1-\bar{b}_1 z)}\right) \\ +\bar{b}_1\left(\frac{1}{(1-\bar{b}_1 z)^m}+...\frac{1}{(1-\bar{b}_1 z)}\right) \\ =\bar{b}_1\left(\frac{1}{(1-\bar{b}_1 z)^m}+\frac{2}{(1-\bar{b}_1 z)^{m-1}}+...+\frac{m}{(1-\bar{b}_1 z)}\right). \tag{13}$$

Recursively,

$$\mathbf{S}^k(M)(z) \\ =\bar{b}_1^k\left(\frac{c_m^k}{(1-\bar{b}_1 z)^m}+\frac{c_{m-1}^k}{(1-\bar{b}_1 z)^{m-1}}+...+\frac{c_1^k}{(1-\bar{b}_1 z)}\right), \tag{14}$$

where c_j^k, $j=1,...,m; k=0,1,2,...$, are the coefficients of the order-jth term by taking kth backward shift operator. We can see the coefficients also satisfy the recursive rule,

$$c_j^{k+1}=\sum_{l=j}^{m}c_l^k, (j=1,2,...,m). \tag{15}$$

Obviously $c_m^0=c_{m-1}^0=...=c_1^0=1$, from (12) we know by taking backward shift operator to $\frac{1}{(1-\bar{b}_1 z)^m}$, it will generate m components in which the coefficients of the m-th order is 1, then there is $c_m^1=c_m^2...=c_m^k=c_m^{k+1}=...=1$. From (13), due to linear independence of $\{\frac{1}{(1-\bar{b}_1 z)^j}, j=1,...,m;\ \frac{1}{1-\bar{b}_{m+1}z},...,\frac{1}{1-\bar{b}_n z}\}$, there exists a unique sequence $\{\mu_k\}_{k=0}^n$, such that the same relation as equation (9) holds,

$$\mu_0 f(z) + \mu_1 \mathbf{S}(f)(z) ... + \mu_n \mathbf{S}^n(f)(z) = 0. \quad (16)$$

That results to

$$\begin{cases} \mu_0 + \mu_1 \bar{b}_1 + \mu_2 \bar{b}_1^2 + ... + +\mu_n \bar{b}_1^n = 0, \\ \mu_0 + \mu_1 \bar{b}_{m+1} ... + \mu_n \bar{b}_{m+1}^n = 0, \\ \vdots \\ \mu_0 + \mu_1 \bar{b}_n + \mu_2 \bar{b}_n^2 + ... + \bar{b}_n^n = 0, \end{cases} \quad (17)$$

and equations about $\bar{b}_1$,

$$\begin{cases} c_m^0 \mu_0 + c_m^1 \mu_1 \bar{b}_1 + ... c_m^n \mu_n \bar{b}_1^n = 0, \\ c_{m-1}^0 \mu_0 + c_{m-1}^1 \mu_1 \bar{b}_1 + ... + c_{m-1}^n \mu_n \bar{b}_1^n = 0, \\ \vdots \\ c_2^0 \mu_0 + c_2^1 \mu_1 \bar{b}_1 + ... c_2^n \mu_n \bar{b}_1^n = 0, \\ c_1^0 \mu_0 + c_1^1 \mu_1 \bar{b}_1 + ... c_1^n \mu_n \bar{b}_1^n = 0. \end{cases}$$

There are m equations of $\bar{b}_1$. By first eliminating μ_0 we have $m-1$ equations given by,

$$\begin{cases} c_{m-1}^0 \mu_1 + c_{m-1}^1 \mu_2 \bar{b}_1 + ... c_{m-1}^{n-1} \mu_n \bar{b}_1^{n-1} = 0, \\ c_{m-2}^0 \mu_1 + c_{m-2}^1 \mu_2 \bar{b}_1 + ... c_{m-2}^{n-1} \mu_n \bar{b}_1^{n-1} = 0, \\ \vdots \\ c_2^0 \mu_1 + c_2^1 \mu_2 \bar{b}_1 + ... c_2^{n-1} \mu_n \bar{b}_1^{n-1} = 0, \\ c_1^0 \mu_1 + c_1^1 \mu_2 \bar{b}_1 + ... c_1^{n-1} \mu_n \bar{b}_1^{n-1} = 0. \end{cases}$$

Repeating this process $(m-1)$ times, it comes

$$c_1^0 \mu_{m-1} + c_1^1 \mu_m \bar{b}_1 + ... c_1^{n-m+1} \mu_n \bar{b}_1^{n-m+1} = 0. \quad (18)$$

The following is a result about the coefficients c_1^k.

Lemma 1 *For any positive integer m, let $c_m^0 = c_{m-1}^0 = ... = c_1^0 = ... = c_m^k = c_m^{k+1} = ... = 1$, and c_j^k be given by the recursive formula (15), then we have*

$$c_1^k = \frac{(m+k-1)!}{(m-1)!k!}. \quad (19)$$

Proof. We use mathematical induction. First of all, $c_1^0 = \frac{(m-1)!}{(m-1)!0!} = 1$ and $c_1^1 = \frac{(m+1-1)!}{(m-1)!1!} = m$. Second, assume that the result holds for c_1^l, $1 < l \leq k$. For $l = k+1$, observing that c_l^k $(l = 1, ..., m)$ are the numbers of combinations, we have

$$\begin{aligned} c_1^k &= C_{m+k-1}^k, \\ c_2^k &= C_{m+k-2}^k, \\ &\vdots \\ c_m^k &= C_k^k. \end{aligned}$$

Then there holds

$$c_1^{k+1} = \sum_{l=1}^{m} c_l^k = \sum_{l=1}^{m} C_{m+k-l}^k.$$

Using the property of combination $C_m^n + C_m^{n+1} = C_{m+1}^{n+1}$ (for any positive integers m, n), we obtain

$$c_1^{k+1} = C_{m+k}^{k+1} = \frac{(m+k)!}{(m-1)!(k+1)!}. \quad (20)$$

The proof is complete.

The first equation in (17) implies $\bar{b}_1$ is a solution of an equation like (10). Equation (18) is the $(m-1)$th derivative of the first equation in (17) by lemma 1, then $\bar{b}_1$ is the $m-$multiple root. Therefore, the multiple poles are completely included in the solutions of equation (10). In the same way, we can assume $\mu_n = 1$.

2.4 The infinite poles case

In this subsection we will discuss the case in which polynomials are included. Set

$$f(z) = \sum_{k=0}^{m-1} c_k z^k + \sum_{l=m+1}^{n} \frac{\lambda_l}{1 - \bar{b}_l z}, \quad (21)$$

where c_{m-1} and λ_l $(l = m+1, ..., n)$ are nonzero. For the monomial z^{m-1}, when applying the backward shift operator to it, we have

$$\mathbf{S}(z^{m-1}) = \frac{z^{m-1} - 0}{z} = z^{m-2}. \quad (22)$$

After mth backward shift operator taken on it, it becomes zero, viz,

$$\mathbf{S}^m(z^{m-1}) = 0. \quad (23)$$

Then, for $f(z)$ given by (21), we have

$$\begin{cases} \mathbf{S}(f)(z) = \sum_{k=1}^{m-1} c_k z^{k-1} + \sum_{l=1}^{n} \frac{\lambda_k \bar{b}_k}{1-\bar{b}_k z}, \\ \vdots \\ \mathbf{S}^{m-1}(f)(z) = c_{m-1} + \sum_{l=1}^{n} \frac{\lambda_k \bar{b}_k^{m-1}}{1-\bar{b}_k z}, \\ \mathbf{S}^{m}(f)(z) = 0 + \sum_{l=1}^{n} \frac{\lambda_k \bar{b}_k^{m}}{1-\bar{b}_k z}, \\ \vdots \\ \mathbf{S}^{m+n}(f)(z) = 0 + \sum_{l=1}^{n} \frac{\lambda_k \bar{b}_k^{m+n}}{1-\bar{b}_k z}, \end{cases}$$

Similarly, because of the linear independence of $\{z^k\}$ and $\{\frac{1}{1-\bar{b}_l z}\}$, there exists a unique sequence $\{\mu_k\}_{k=0}^{n}$ such that

$$\mu_0 f(z) + \mu_1 \mathbf{S}(f)(z) + ...\mu_n \mathbf{S}^n(f)(z) = 0, \tag{24}$$

which gives

$$\begin{cases} \mu_0 c_0 + \mu_1 c_1 + \mu_2 c_2 + ... + \mu_{m-1} c_{m-1} = 0, \\ \mu_0 c_1 + \mu_1 c_2 + \mu_2 c_3 + ... + \mu_{m-2} c_{m-1} = 0, \\ \vdots \\ \mu_0 c_{m-3} + \mu_1 c_{m-2} + \mu_2 c_{m-1} = 0, \\ \mu_0 c_{m-2} + \mu_1 c_{m-1} = 0, \\ \mu_0 c_{m-1} = 0, \end{cases} \tag{25}$$

and

$$\begin{cases} \mu_0 + \mu_1 \bar{b}_1 + \mu_2 \bar{b}_1^2 + ...\mu_n \bar{b}_1^n = 0, \\ \vdots \\ \mu_0 + \mu_1 \bar{b}_n + \mu_2 \bar{b}_n^2 + ...\mu_n \bar{b}_n^n = 0. \end{cases} \tag{26}$$

Since $c_{m-1} \neq 0$, from equations (25) we can see $\mu_0 = \mu_1 = ... = \mu_{m-1} = 0$. While from equations (26), by the same way, we can see that $\{\bar{b}_k\}_{k=m+1}^{n}$ are solutions of equation

$$\mu_0 + \mu_1 x + ... + \mu_{n-1} x^{n-1} + \mu_n x^n = 0, \tag{27}$$

which can be simplified to

$$x^m(\mu_n x^{n-m} + \mu_{n-1} x^{n-m-1} ... + \mu_m) = 0. \tag{28}$$

It is easy to find that the truth $\{\bar{b}_k\}_{k=m+1}^{n}$ are exactly the solutions of

$$\mu_n x^{n-m} + \mu_{n-1} x^{n-1} ... + \mu_m = 0.$$

Without loss of generality, we can assume $\mu_n = 1$.

For the multiple $\{\bar{b}_k\}$ case, the discussion is similar to the above.

2.5 Sum-up

For any n-tuple $\{a_1, ..., a_n\}$ in $\mathbb{D}$, the corresponding finite orthogonal system

$$B_{\{a_1\}}, B_{\{a_1,a_2\}}, ..., B_{\{a_1,a_2,...,a_n\}},$$

where $B_{\{a_1,...,a_k\}}$ $(k = 1, ..., n)$ is defined by (1), is called an *n-Blaschke system*. It is proven in [22] that the *n-Blaschke system* may be obtained through Gram-Schmidt orthogonalization process from the reproductive kernels of order n,

$$e_{\{a_1\}}, e_{\{a_2\}}, ..., e_{\{a_n\}},$$

where if $a_k \neq 0$ has multiplicity l in the n-tuple $(a_1, ..., a_n)$, then

$$e_{\{a_k\}}(z) = \frac{1}{(1-\bar{a}_k z)^j} (j = 1, ..., l);$$

if $a_k = 0$ has multiplicity l, then

$$e_{\{a_k\}}(z) = z^{j-1} (j = 1, ..., l).$$

Any rational function consists of the kernels can be represented by linear combination of the corresponding finite orthogonal system. The studies in the above three subsections through backward shift operator perfectly correspond to the cases of the poles of the rational functions. Our conclusion is that:

Theorem 2 *Poles (including the zero and multiplicities) of rational functions are exactly solutions of an algebraic equation which can be derived through applying the backward shift operator.*

3 Backward shift algorithm

With Theorem 2, we can estimate poles of LTI systems based on frequency responses. Generally, the measured data are corrupted by noise and errors of measurement. These will lead to some errors in estimating coefficients $\{\mu_k\}_{k=0}^{n-1}$ and finally affect the accuracy of the estimated poles.

It is here assumed a set of measurements $\{E_k\}_{k=1}^{N}$ from a system $f(z)$, $f(z)$ be a proper rational function, is given by

$$E_k = f(e^{i\omega_k}) + v_k, \tag{29}$$

where $\omega_k = \frac{2\pi(k-1)}{N}$, $k = 1, ..., N$, and v_k is the corruption to $f(e^{i\omega_k})$. The corruption $\{v_k\}$ can be used to model a number of different error sources. It can be assumed to either be a bounded deterministic sequence, or a stochastic process with zero-mean and bounded covariance.

Without losing the accuracy, we consider the equation (9) in the form

$$\mu_0 f(z) + \mu_1 \mathbf{S}(f)(z) + ... + \mu_{n-1}\mathbf{S}^{n-1}(f)(z) = \mathbf{S}^n(f)(z). \tag{30}$$

The corresponding algebraic equation to (10) becomes

$$x^n - \mu_{n-1}x^{n-1} - \mu_{n-2}x^{n-2} - ... - \mu_1 x - \mu_0 = 0. \tag{31}$$

We call our algorithm *backward shift algorithm* given as follows.

Algorithm 1 *It has four steps:*

- *Generates data sets by taking backward shift operator to $\{E_k\}_{k=1}^N$ n times.*
- *Determine the coefficients μ_k's by relation (30).*
- *Find solutions of the algebraic equation (31) which gives the poles of the rational function.*
- *Find the composition coefficients by using a least-squares criterion.*

The following are processes in detail. Firstly, based on $\{f(e^{i\omega_k})\}_{k=1}^N$, $\{S(f)(e^{i\omega_k})\}_{k=1}^N$ can be obtained by

$$\mathbf{S}(f)(e^{i\omega_k}) = \frac{f(e^{i\omega_k}) - f(0)}{e^{i\omega_k}}. \tag{32}$$

With $f(0)$ approximated by

$$f(0) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\omega})d\omega \approx \frac{1}{N}\sum_{k=1}^N f(e^{i\omega_k}),$$

we denote

$$\widetilde{\mathbf{S}}(f)(e^{i\omega}) = \frac{f(e^{i\omega_k}) - \frac{1}{N}\sum_{k=1}^N f(e^{i\omega_k})}{e^{i\omega_k}}.$$

Repeating the process, we have

$$\begin{aligned}\mathbf{S}^{l-1}f(0) &= \frac{1}{2\pi}\int_0^{2\pi} \mathbf{S}^{l-1}f(e^{i\omega})d\omega \\ &\approx \frac{1}{N}\sum_{k=1}^N \widetilde{\mathbf{S}}^{l-1}f(e^{i\omega_k}),\end{aligned}$$

and

$$\begin{aligned}\widetilde{\mathbf{S}}^l(f)(e^{i\omega}) &= \frac{\widetilde{\mathbf{S}}^{l-1}(f)(e^{i\omega}) - \frac{1}{N}\sum_{k=1}^N \widetilde{\mathbf{S}}^{l-1}f(e^{i\omega_k})}{e^{i\omega}} \\ &\approx \mathbf{S}^l(f)(e^{i\omega}),\end{aligned} \tag{33}$$

The approximating data sets $\{\widetilde{\mathbf{S}}^2(f)(e^{i\omega_k})\}$, $\{\widetilde{\mathbf{S}}^3(f)(e^{i\omega_k})\}$, ..., $\{\widetilde{\mathbf{S}}^n(f)(e^{i\omega_k})\}$ are obtained. Therefore we obtain a linear system, denoted by

$$A\mu_N = b, \tag{34}$$

where $A = (A_{pq})_{N\times n}$, $A_{pq} = \widetilde{S}^{q-1}(f)(e^{i\omega_p})$, and μ, b are the columns given by

$$\begin{aligned}\mu &= [\mu_0 \ \ \mu_1 \ \ ... \ \ \mu_{n-1}]^T, \\ b &= [\widetilde{S}^n(f)(e^{i\omega_1}) \ \ \widetilde{S}^n(f)(e^{i\omega_2}) \ \ ... \ \ \widetilde{S}^n(f)(e^{i\omega_N})]^T,\end{aligned}$$

T being the transposition.

The problem (34) is a typical least-squares problem, since $f(z)$,$\mathbf{S}(f)(z)$,...,$\mathbf{S}^{n-1}(f)(z)$ are independent. To solve (34) is equivalent to solve

$$\mu_N = \arg\min_{\mu} \frac{1}{N}\sum_{k=1}^N |\sum_{l=1}^n \mu_{l-1}\widetilde{\mathbf{S}}^{l-1}(f)(e^{i\omega_k}) - \widetilde{\mathbf{S}}^n(f)(e^{i\omega_k})|^2, \tag{35}$$

whose normal equations are

$$R_N\mu_N = \frac{1}{N}A^T b,$$

where $R_N = \frac{1}{N}A^T A$,

$$(R_N)_{pq} = \frac{1}{N}\sum_{k=1}^N \widetilde{\mathbf{S}}^{p-1}(f)(e^{i\omega_k})\widetilde{\mathbf{S}}^{q-1}(f)(e^{i\omega_k}), \tag{36}$$

$p, q = 1, 2, ..., n.$

We have the following result.

Lemma 3 *For the matrix R_N defined by (36), there exists a matrix R^*, such that*

$$R^* = \lim_{N\to\infty} R_N. \tag{37}$$

Proof. It is easy to prove the result since $\mathbf{S}^{p-1}(f)(z)$ ($p = 1, ..., n$) are rational. First we can see

$$\begin{aligned}\lim_{N\to\infty}\widetilde{\mathbf{S}}(f)(e^{i\omega_k}) &= \lim_{N\to\infty}\frac{f(e^{i\omega_k}) - \frac{1}{N}\sum_{l=1}^N f(e^{i\omega_l})}{e^{i\omega_k}} \\ &= \frac{f(e^{i\omega_k}) - \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\omega})d\omega}{e^{i\omega_k}} \\ &= \frac{f(e^{i\omega_k}) - f(0)}{e^{i\omega_k}} \\ &= \mathbf{S}(f)(e^{i\omega_k}).\end{aligned} \tag{38}$$

Therefore, according to (38), we have

$$\begin{aligned}\lim_{N\to\infty}\widetilde{\mathbf{S}}^2(f)(e^{i\omega_k}) &= \frac{\mathbf{S}(f)(e^{i\omega_k})-\mathbf{S}(f)(0)}{e^{i\omega_k}}\\ &= \mathbf{S}^2(f)(e^{i\omega_k}).\end{aligned}$$

Likewise, uniformly in k,

$$\lim_{N\to\infty}\widetilde{\mathbf{S}}^{p-1}(f)(e^{i\omega_k}) = \mathbf{S}^{p-1}(f)(e^{i\omega_k}).$$

It means for each element of R_N,

$$\begin{aligned}&\lim_{N\to\infty}(R_N)_{pq}\\ &= \lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}\widetilde{\mathbf{S}}^{p-1}(f)(e^{i\omega_k})\widetilde{\mathbf{S}}^{q-1}(f)(e^{i\omega_k})\\ &= \frac{1}{2\pi}\int_0^{2\pi}\mathbf{S}^{p-1}(f)(e^{i\omega})\mathbf{S}^{q-1}(f)(e^{i\omega})d\omega.\end{aligned}$$

Accordingly, R_N converges when $N\to\infty$. Denote

$$\lim_{N\to\infty}R_N = R^*.$$

The proof is complete.

The rank of R^* is n, because $f, \mathbf{S}(f), ..., \mathbf{S}^{n-1}(f)$ are linear independent. Similarly, it can be proved there exists a b^* given by

$$b^* = \lim_{N\to\infty}\frac{1}{N}Ab.$$

From the proof of lemma 3, we notice if μ^* is the true solution of (30), then there holds

$$\lim_{N\to\infty}\mu_N = \mu^*. \tag{39}$$

In the noise case, assume the noise $\{v_k\}$ is bounded. Denote $\{\widetilde{\mathbf{S}}_d(f)(e^{i\omega_k})\}$, $\{\widetilde{\mathbf{S}}_d^2(f)(e^{i\omega_k})\}$, $\{\widetilde{\mathbf{S}}_d^3(f)(e^{i\omega_k})\}$,..., $\{\widetilde{\mathbf{S}}_d^n(f)(e^{i\omega_k})\}$ as the data sets obtained through the backward shift processes based on $\{E_k\}_{k=1}^N$, *i.e.* corrupted by noise. The following result states a property of this algorithm.

Theorem 4 *Suppose the noise $\{v_k\}$ is bounded by $\epsilon > 0$, that is, $|v_k| < \epsilon$. Let μ^* be the true solution of (30), $R^*\mu^* = b^*$, and μ'_N be the solution given by (35) with the sequence of data $\{\{\widetilde{\mathbf{S}}_d^l(f)(e^{i\omega_k})\}_{k=1}^N\}_{l=1}^n$ then there holds*

$$\lim_{\substack{\epsilon\to 0,\\ N\to\infty}}\mu'_N = \mu^*. \tag{40}$$

Proof. By substituting the measured data $\{E_k\}_{k=1}^N$ in the backward shift processes, the first data set is

$$\begin{aligned}&\widetilde{\mathbf{S}}_d(f)(e^{i\omega_k})\\ &= \frac{E_k - \frac{1}{N}\sum_{l=1}^N E_l}{e^{i\omega_k}}\\ &= \frac{f(e^{i\omega_k}) - \frac{1}{N}\sum_{l=1}^N f(e^{i\omega_l})}{e^{i\omega_k}} + \frac{v_k - \frac{1}{N}\sum_{l=1}^N v_l}{e^{i\omega_k}}\\ &= \widetilde{\mathbf{S}}(f)(e^{i\omega_k}) + \widetilde{\mathbf{S}}E(v_k),\end{aligned}$$

where $\widetilde{\mathbf{S}}E(v_k)$ stands for the error between $\widetilde{\mathbf{S}}_d(f)(e^{i\omega_k})$ and $\widetilde{\mathbf{S}}(f)(e^{i\omega_k})$. We can see

$$\lim_{\substack{\epsilon\to 0,\\ N\to\infty}}\widetilde{\mathbf{S}}E(v_k) = \lim_{\substack{\epsilon\to 0,\\ N\to\infty}}\frac{v_k - \frac{1}{N}\sum_{l=1}^N v_l}{e^{i\omega_k}} = 0. \tag{41}$$

The other data sets can be obtained from the following recursive formula, for $m = 2, 3, ..., n$,

$$\widetilde{\mathbf{S}}_d^m(f)(e^{i\omega_k}) = \widetilde{\mathbf{S}}^m(f)(e^{i\omega_k}) + \widetilde{\mathbf{S}}^m E(v_k), \tag{42}$$

where

$$\begin{aligned}\widetilde{\mathbf{S}}^m(f)(e^{i\omega_k}) &= \frac{\widetilde{\mathbf{S}}^{m-1}(f)(e^{i\omega_k}) - \frac{1}{N}\sum_{l=1}^N\widetilde{\mathbf{S}}^{m-1}(f)(e^{i\omega_l})}{e^{j\omega_k}},\\ \widetilde{\mathbf{S}}^m E(v_k) &= \frac{\widetilde{\mathbf{S}}^{m-1}E(v_k) - \frac{1}{N}\sum_{l=1}^N\widetilde{\mathbf{S}}^{m-1}E(v_l)}{e^{i\omega_k}}.\end{aligned} \tag{43}$$

The error between $\widetilde{\mathbf{S}}_d^m(f)(e^{i\omega_k})$ and $\widetilde{\mathbf{S}}^m(f)(e^{i\omega_k})$ is $\widetilde{\mathbf{S}}^m E(v_k)$. From (41), by mathematical induction, we also have

$$\lim_{\substack{\epsilon\to 0,\\ N\to\infty}}\widetilde{\mathbf{S}}^m E(v_k) = 0. \tag{44}$$

The equations for μ are

$$(A + \delta_A)\mu = b + \delta_b, \tag{45}$$

where $A_{pq} = \widetilde{\mathbf{S}}^{q-1}(f)(e^{i\omega_p})$, $\delta_{A_{pq}} = \widetilde{\mathbf{S}}^{q-1}E(v_p)(p, q = 1, 2, ..., n)$ and

$$\begin{aligned}b &= [\widetilde{\mathbf{S}}^n(f)(e^{i\omega_1})\ \ \widetilde{\mathbf{S}}^n(f)(e^{i\omega_2})\ \ ...\ \ \widetilde{\mathbf{S}}^n(f)(e^{i\omega_N})]^T,\\ \delta_b &= [\widetilde{\mathbf{S}}^n E(v_1)\ \ \widetilde{\mathbf{S}}^n E(v_2)\ \ ...\ \ \widetilde{\mathbf{S}}^n E(v_N)]^T.\end{aligned}$$

The normal equations become

$$(R_N + \delta_{R_N})\mu = \overline{b} + \delta_{\overline{b}}, \tag{46}$$

where

$$\begin{aligned} R_N &= \frac{1}{N}A^TA, \\ \delta_{R_N} &= \frac{1}{N}(A^t\delta_A + (\delta_A)^TA + (\delta_A)^T\delta_A), \\ \bar{b} &= \frac{1}{N}A^Tb, \\ \delta_{\bar{b}} &= \frac{1}{N}((\delta_A)^Tb + (\delta_A)^T\delta_b + A\delta_b). \end{aligned}$$

Since each element of δ_A and δ_b tends to 0 as $N \to \infty$ and $\epsilon \to 0$, besides each element of matrices A and b converges, then from (41) and (44), we can see that

$$\lim_{\substack{\epsilon\to 0,\\ N\to\infty}} \delta_{R_N} = 0, \quad \lim_{\substack{\epsilon\to 0,\\ N\to\infty}} \delta_{\bar{b}} = 0. \tag{47}$$

Let μ_N be the solution using data without noise and μ'_N be solution with noise. That is,

$$\begin{aligned} R_N\mu_N &= \bar{b}, \\ (R_N + \delta_{R_N})\mu'_N &= \bar{b} + \delta_{\bar{b}}. \end{aligned}$$

By perturbation theory [24], there is a result

$$\|\mu'_N - \mu_N\| \le \|\mu_N\| \frac{\kappa(R_N)}{1-\kappa(R_N)\frac{\|\delta_{R_N}\|}{\|R_N\|}} \left(\frac{\|\delta_{R_N}\|}{\|R_N\|} + \frac{\|\delta_{\bar{b}}\|}{\|\bar{b}\|} \right), \tag{48}$$

where $\kappa(R_N) = \|R_N\|\|R_N^{-1}\|$ is the condition number of R_N. We have the following error from μ'_N to μ^*

$$\begin{aligned} \|\mu'_N - \mu^*\| &\le \|\mu'_N - \mu_N\| + \|\mu_N - \mu^*\| \\ &\le \|\mu_N\| \frac{\kappa(R_N)}{1-\kappa(R_N)\frac{\|\delta_{R_N}\|}{\|R_N\|}} \left(\frac{\|\delta_{R_N}\|}{\|R_N\|} + \frac{\|\delta_{\bar{b}}\|}{\|\bar{b}\|} \right) \\ &\quad + \|\mu_N - \mu^*\|, \end{aligned} \tag{49}$$

From (48), it can be seen that

$$\lim_{\substack{\epsilon\to 0,\\ N\to\infty}} \|\mu'_N - \mu_N\| = 0; \tag{50}$$

On the other hand, from (39), there holds

$$\lim_{N\to\infty} \|\mu_N - \mu^*\| = 0. \tag{51}$$

Therefore, we have

$$\lim_{\substack{\epsilon\to 0,\\ N\to\infty}} \|\mu'_N - \mu^*\| = 0. \tag{52}$$

The proof is complete.

The inequality (49) also gives the error estimation in the matrix norm sense. Once μ'_N is obtained, the poles $\{\bar{a}_k\}$ can be acquired by solving equation (31) again. Using the least-squares criterion with $\{E_k\}_{k=1}^N$, the corresponding coefficients can be obtained.

Remark 5 *For order estimation of a rational function, by perturbation theory of matrix, there is a minimal perturbation that affects the rank of a matrix. Assume that the transfer function is of type (8) or (11), it is possible to determine the order through backward shift operator. We can repeat the shift processes, study the changes of A's rank in (34) and stop until A's rank does not change any more.*

Remark 6 *Observation and discussion show that if the transfer function $G(z)$ is not a rational function, then the proposed backward shift method for a prescribed n may have no relevance with the solution of the best approximation to $G(z)$ by rational functions of order not larger than n ([2]), although it does give the unique solution of the n-best approximation problem, that is $G(z)$ itself, when $G(z)$ is a rational function of order n.*

4 Numerical examples

In this section, we give some examples for illustrating the proposed backward shift algorithm, the first one is

$$f(z) = \frac{1}{1-0.5z} - \frac{3}{1+0.6z} + \frac{2}{1+0.7z} + \frac{3-5i}{6(1-(0.5+0.3i)z)} + \frac{3+5i}{6(1-(0.5-0.3i)z)},$$

in which each pole is different.

Table 1
$\{\bar{a}_k\}$ obtained in no noise case for example 1.

data number	$\bar{a}_1$	$\bar{a}_2$	$\bar{a}_3$
15	-0.7068	-0.5854	$0.4712 - 0.2740i$
20	-0.7005	-0.5992	$0.4975 - 0.2999i$
25	-0.7	-0.6	$0.4999 + 0.3i$
28	-0.7	-0.6	$0.5 + 0.3i$

Table 2
Continued $\{\bar{a}_k\}$ in no noise case for example 1.

data number	$\bar{a}_4$	$\bar{a}_5$
15	$0.4712 + 0.2740i$	0.2623
20	$0.4975 + 0.2999i$	0.4893
25	$0.4999 - 0.3i$	0.4997
28	$0.5 - 0.3i$	0.5

Table 3
$\{\bar{a}_k\}$ obtained for example 1 with noised data($m = 20$).

SNR	$\bar{a}_1$	$\bar{a}_2$	$\bar{a}_3$
10	$-0.8194 - 0.0971i$	$0.5130 - 0.3587i$	$-0.3527 + 0.1973i$
20	$-0.6620 - 0.0454i$	$-0.6411 + 0.0888i$	$0.4526 + 0.2298i$
30	$-0.7195 + 0.0086i$	$-0.5355 - 0.0331i$	$0.4767 - 0.2672i$
40	$-0.7208 + 0.0006i$	$-0.5775 - 0.0001i$	$0.4874 - 0.2837i$

Table 4
Continued $\{\overline{a}_k\}$ obtained for example 1 with noised data($m = 20$).

SNR	$\overline{a}_4$	$\overline{a}_5$
10	$0.4443 + 0.1699i$	$0.1049 + 0.3324i$
20	$0.4767 - 0.3016i$	$0.2041 - 0.2514i$
30	$0.4532 + 0.2650i$	$0.0802 + 0.1173i$
40	$0.4555 + 0.2930i$	$0.3667 + 0.0812i$

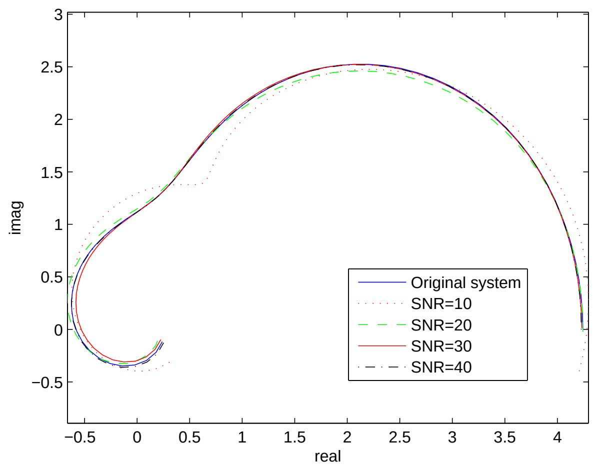

Fig. 1. Comparison of different noised data with SNR= 10,20,30,40, respectively, for example 1. And the data number m=20.

Table 1 and 2 show the results in no noise case, we can locate true poles of original system by using no more than 30 frequency responses for example 1. Table 3 and 4 are results by using the noised data (m=20) in each SNR (signal-to-noise ratio) level. Figure 1 shows frequency responses of approximating systems obtained by using noised data in different levels of SNR. We can see the approximations with order 5 is very good.

The second example is

$$f(z) = \frac{1}{1 - 0.9048z} + \frac{z^2 - 1}{(1 - 0.3679z)^2}, \qquad (53)$$

in which 0.3679 has multiplicity 2.

Table 5
$\{\overline{a}_k\}$ obtained in no noise case for example 2.

data number	$\overline{a}_1$	$\overline{a}_2$	$\overline{a}_3$
m=50	0.9040	$0.3655 + 0.0346i$	$0.3655 - 0.0346i$
m=80	0.9048	$0.3679 - 0.0017i$	$0.3679 + 0.0017i$
m=120	0.9048	0.3679	0.3679

Table 5 shows in the non-noise case, true solution can be gained. Table 6 are the results using the noised data. We can see the estimated poles are close to the original ones. In Figure 2, it shows the frequency responses of approximating systems obtained by using data added up with Gaussian noise in different levels of SNR.

Table 6
$\{\overline{a}_k\}$ obtained for example 2 with noised data.

SNR	$\overline{a}_1$	$\overline{a}_2$	$\overline{a}_3$
10	$0.8627 + 0.0008i$	$-0.4135 + 0.0591i$	$-0.0218 - 0.0374i$
20	0.8657	$-0.0898 - 0.1313i$	$-0.0842 + 0.1163i$
30	0.8800	$0.2316 + 0.2253i$	$0.2229 - 0.2176i$
40	0.8999	$0.3537 + 0.0866i$	$0.3502 - 0.0865i$

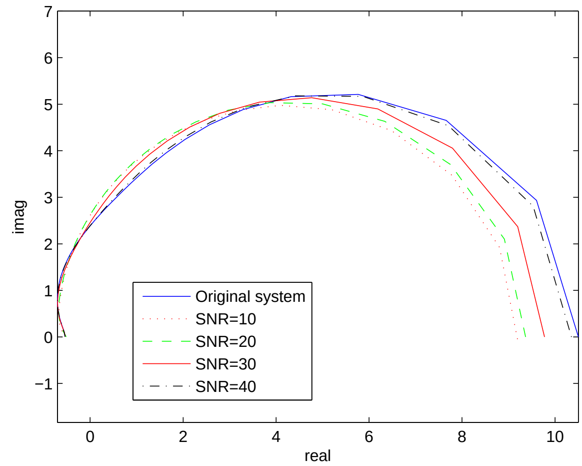

Fig. 2. Approximations of different noised data with SNR= 10,20,30,40, respectively, for example 2.

In the third example, we compare our algorithm with the existing algorithms given by [8] and [13]. We choose a simple second order system for comparison. Consider

$$g(z) = \frac{4}{z - 0.5} + \frac{5}{z - 0.2}, \qquad (54)$$

for algorithms in [8] and [13], it is to estimate poles 0.5 and 0.2 with measured data. The function $g(z)$ is analytic outside of the unit disc, by taking $z = 1/z$, then it is analytic in the unit disc and the proper part is

$$f(z) = \frac{4}{1 - 0.5z} + \frac{5}{1 - 0.2z}, \qquad (55)$$

we are to estimate the points 0.5 and 0.2 of $f(z)$ by the proposed method. The results are shown in the table 7 and 8 for both noise and no noise cases, respectively. All the data are equally spaced in the corresponding interval. In the noise case, the data are added up with Gaussian noise at $SNR = 20$. We can see our results are better.

Table 7
Results of a_1 and a_2 with no noise for example 3.

methods	data	a_1	a_2
Proposed	10	0.4999	0.1997
Proposed	15	0.5	0.2
Nara-Ando's [13]	10	$-0.9781 + 0.1564i$	$0.9646 - 0.0314i$
Kang-Lee's [8]	30	$-0.9638 - 1.0765i$	$-0.3846 + 0.8661i$

Table 8
Results of a_1 and a_2 with noise for example 3.

methods	data	a_1	a_2
Proposed	20	$0.4935 + 0.0196i$	$0.2080 + 0.0193i$
Nara-Ando's [13]	50	$0.8888 - 0.4782i$	$-0.6116 + 0.6413i$
Kang-Lee's [8]	50	$0.5384 + 0.6089i$	$0.0557 - 0.0774i$

5 Conclusions

This paper presents a new method for estimating poles of LTI systems by using backward shift operator. The numerical results show this algorithm is practical. The number of used data is not large. In fact, for the conjugate symmetry of the values on the unit circle of the transfer function, it only needs data that in the interval $(0, \pi)$, the rest on the other half circle can be denoted to be the conjugated value of the measured data, which decreases the cost of experiments. In each shift process, it computes a mean of errors, this will contribute to eliminate the large errors. From the illustrating examples, we can see this algorithm is convenient and efficient.

References

[1] A.B. Aleksandrov, Invariant subspace of the backward shift operator in the space H^p ($p \in (0,1)$), Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI), 92(1979), 7-29, 318, Investigations on linear operators and the theory of functions.

[2] L. Baratchart,& F. Seyfert, *An Lp analog of AAK theory for* $p \geq 2$, J. Funct. Anal., 191 (2002), no 1, 52-122.

[3] M. Casini, A. Garulli, & A. Vicino (2003), On worst-case approximation of feasible system sets via orthonomal basis functions, *IEEE Trans. Automat. Control,* no.1, 48, 96-101.

[4] J.A. Cima, & W.T. Ross, *The backward shift on the Hardy space*, American Mathematical Society, Providence, RI, 2000.

[5] G. Gu, & P.P. Khargonekar(1992). A class of algorithms for identification in H_∞, *Automatica*, no.2, 28, 299-312.

[6] A. Helmicki, C. Jacobson, & C. Nett(1991). Control oriented system identification:A worst case/deterministic approach in H_∞, *IEEE Trans. Automat. Control*, 36(10), 1163-1176.

[7] P.S.C. Heuberger, P.M.J. Van den Hof, & B. Wahlberg(Eds.)). *Modelling and Identification with Rational Orthogonal Basis Functions.* Springer, London, 2005.

[8] H. Kang, & H. Lee(2004). Identification of simple poles via boundary measurements and an application of EIT, *Inverse Problems*,vol.20, 1853C1863.

[9] L. Ljung(1999). *System Identification: Theory for the User, Second Edition.* Prentice-Hall, Inc., New Jersey.

[10] W. Mi, & T. Qian(2010), Frequency domain identification with adaptive rational orthogonal system , *Proceedings of IEEE International Conference on System Science and Engineering (ICSSE)*, 41-45.

[11] W. Mi, & T. Qian(2012), Frequency domain identification: an algorithm based on adaptive rational orthogonal system, Automatica 48 (2012), 1154-1162.

[12] W. Mi, T. Qian, & F. Wan(2012), A Fast adaptive model reduction method based on Takenaka-Mulmquist systems, *Systems & Control Letters*, 61, 223-230.

[13] T. Nara, & S. Ando(2010). Direct localization of poles of a meromorphic function from measurements on an incomplete boundary, *Inverse Problems*, vol.26, 015011.

[14] N.K. Nikol'skiĭ, *Treatise on the shift operator*, Springer-Verlag, Berlin, 1986.

[15] B. Ninness(1996). Frequency domain estimation using orthonormal bases. *In Proceedings of the 13th IFAC World Congress,* 381-386.

[16] B. Ninness, & F. Gustafsson(1997). A unifying construction of orthonormal bases for system identification. *IEEE Trans. Automat. Control,* 42, 515-512.

[17] B. Ninness, H. Hjalmarsson, & F. Gustafsson(1997). The fundamental role of general orthonormal bases in system identification. *IEEE Trans. Automat. Control,* 44, 1384-1406.

[18] T. Oliveria e Silval(1995), Optimality conditions for Laguerre lattice filters, *IEEE Signal Processing letters,* no.5, 2, 97-98.

[19] T. Oliveria e Silva1(1995), Optimality conditions For truncated Kautz networks with two periodically repeating complex conjugated poles, *IEEE Trans. Automat. Control,* no.2, 40, 342-346.

[20] T. Oliveria e Silva(1997), Stationarity conditions for L_2 error surface of generalized orthonormal basis functions lattice filter, *Signal Processing*, no.3, 56), 233-253.

[21] R. Pintelon, P. Guillaume, Y. Rolain, & J. Schoukens, H.V. Hamme(1994). Parametric identification of transfer functions in the frequency domain- a survey. *IEEE Trans. Automat. Control*, 39, 2245-2260.

[22] T. Qian, & E. Wegert(2013), Optimal approximation by Blaschke forms, *Complex Variables and Elliptic Equations,* Vol. 58, No. 1, 123C133.

[23] A.M. Sabatini(2000). A hybrid genetic algorithm for estimating the optimal time scale of linear systems approximations using Laguerre models, *IEEE Trans. Automat. Control,* no.5, 45, 1007-1011.

[24] G.W. Stewart, & J.G. Sun(1990), *Matrix perturbation theory*, Academic Press, London.

[25] D.K.de, Vries, & P.M.J. Van den Hof(1998). Frequency domain identification with generalized orthonomal basis functions. *IEEE Trans. Automat. Control,* v.43 No.5, pp. 656–669.

[26] B. Wahlberg,(1991). System identification using Laguerre models. *IEEE Trans. Automat. Control,* AC-36, 551-562.

[27] B. Wahlberg(1994). System identification using Kautz models. *IEEE Trans. Automat. Control,* AC-39, 1276-1282.

[28] B. Wahlberg, & P. Mäkilä(1996). On approximation of stable linear dynamical systems using Laguerre and Kautz functions. *Automatica,* 32, 693-708.