A CLASS OF UNBOUNDED FOURIER MULTIPLIERS ON THE UNIT COMPLEX BALL

PENGTAO LI, JIANHAO LV, AND TAO QIAN

Abstract. In this paper, we introduce a class of Fourier multiplier operators M_b on $n-$complex unit sphere, where the symbol $b \in H^s(S_\omega)$. We obtained the Sobolev boundedness of M_b. Our result implies that the operators M_b take a role of fractional differential operators on $\partial\mathbb{B}$.

1. Introduction

In this paper, we introduce a class of unbounded holomorphic Fourier multipliers M_b on $n-$complex unit sphere. We further study the boundedness of M_b on Sobolev spaces. Our results generalize the theory of Fourier multipliers on Lipschitz curves in $\mathbb{C}$ to $n-$complex unit sphere $\mathbb{B}_n$. We refer the reader to Gaudry-Qian-Wang [3], McIntosh-Qian [8], and Qian [9, 10] for further information on multipliers on Lipschitz curves.

Our motivation originates from the following example on the unit sphere in $\mathbb{C}^n$. The explicit formula of the Cauchy-Szegö kernel

$$H(z,\overline{\xi}) = \frac{1}{\omega_{2n-1}}\frac{1}{(1-z\overline{\xi'})^n}.$$

Let $\{p_k^\nu\}$ denote the orthonormal system in the space of holomorphic functions in $\mathbb{B}_n$. The following result is well-known.

$$H(z,\overline{\xi}) = \sum_{k=0}^{\infty}\sum_{\nu=1}^{N_k} p_\nu^k(z)\overline{p_\nu^k(\xi)}, z \in \mathbb{B}_n, \xi \in \partial\mathbb{B}_n \tag{1.1}$$

See Theorem 2.1 and (2.4) below for details. Formally, (1.1) can be seen as the special case of (1.2) below. Let S_ω be the sector defined as

$$S_\omega = \Big\{z \in \mathbb{C} :\ z \neq 0 \text{ and } |\arg z| < \omega\Big\}.$$

Assume that

(1) b is holomorphic on S_ω;

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(2) b is bounded near the origin;
(3) $|b(z)| \leq C|z|^s$ for $|z| > 1$.

We consider the function:

$$H_b(z, \overline{\xi}) = \sum_{k=1}^{\infty} b(k) \sum_{\nu=1}^{N_k} p_\nu^k(z) \overline{p_\nu^k(\xi)}. \tag{1.2}$$

If $b(z) \equiv 1$, then (1.2) becomes (1.1). For $s = 0$, Cowling-Qian [1] introduced a class of bounded holomorphic multipliers on $L^2(\partial\mathbb{B}_n)$. In this paper, we consider the case $s \neq 0$. For this case, b is unbounded on $\{z : |z| > 1\}$. We prove that if $b \in H^s(S_\omega)$, then

$$|H_b(z, \overline{\xi})| = \frac{C_{\mu'}}{\delta(\nu, \mu')|1 - z\overline{\xi'}|^{n+s}}.$$

See Theorem 3.4.

In Section 4, we introduce a class of Fourier multipliers M_b with $b \in H^s(S_\omega)$, $s \neq 0$. Unlike the ones of Cowling-Qian [1], our multipliers b are unbounded on S_ω. Take $b(k) = k^s$. Plancherel's theorem implies that M_b is not bounded on $L^2(\partial\mathbb{B}_n)$. Hence for such M_b, we need to consider their boundedness on some function spaces with higher regularity. Let $r, s \in [0, \infty)$. We prove that if $b \in H^s(S_\omega)$, M_b is bounded from Sobolev space $W^{p,r+s}(\partial\mathbb{B}_n)$ to Sobolev space $W^{p,r}(\partial\mathbb{B}_n)$, $1 < p < \infty$. Our result implies that the operators M_b take a role of fractional differential operators on $\partial\mathbb{B}_n$. See Theorem 4.5.

The rest of this paper is organized as follows. In Section 2, we state some basic preliminaries and notations which will be used in the sequel. In Section 3, we estimate the kernels generated by holomorphic multipliers $b \in H^s(S_\omega)$. The Sobolev boundedness of the operators M_b is given in Section 4.

Notations: $\mathsf{U} \approx \mathsf{V}$ represents that there is a constant $c > 0$ such that $c^{-1}\mathsf{V} \leq \mathsf{U} \leq c\mathsf{V}$ whose right inequality is also written as $\mathsf{U} \lesssim \mathsf{V}$. Similarly, one writes $\mathsf{V} \gtrsim \mathsf{U}$ for $\mathsf{V} \geq c\mathsf{U}$.

2. Preliminaries and notations

In this section we state some preliminaries and notations and refer the reader to Gong [4], Hua [5] and Rudin [13] for further information. We use z as a general element of $\mathbb{C}^n$, i.e. $z = (z_1, \cdots, z_n)$, $z_i \in \mathbb{C}$, $i = 1, 2, \cdots, n$, $n \geq 2$. Denote $\overline{z} = [\overline{z_1}, \cdots \overline{z_n}]$. The notation z is considered to be a row vector. Denote by $\mathbb{B}_n$ the open unit ball $\{z \in \mathbb{C}^n : |z| < 1\}$, where $|z| = \Big(\sum_{i=1}^{n} |z_i|^2\Big)^{1/2}$. The unit sphere in $\mathbb{C}^n$ is denoted by

$$\partial\mathbb{B}_n = \mathbb{S}^{2n-1} = \Big\{z \in \mathbb{C}^n : |z| = 1\Big\}.$$

The open ball centered at z with radius r will be denoted by $B(z, r)$. A general element on $\partial\mathbb{B}_n$ is usually denoted by ξ. The constant ω_{2n-1} involved in the Cauchy-Szegö kernel is the surface area of $\partial\mathbb{B}_n$ and is equal to $\frac{2\pi^n}{\Gamma(n)}$. For $z, w \in \mathbb{C}^n$, we use the notation $zw' = \sum_{k=1}^{n} z_k w_k$. The theory developed in this paper is relevant to the radial Dirac operator

$$D=\sum_{k=1}^{n} z_k \frac{\partial}{\partial z_k}.$$

Now we state some basis knowledge of basis functions in the space of holomorphic function in $\mathbb{B}_n$ and some relevant function spaces on $\partial\mathbb{B}_n$. We refer to Hua [5] for details. Let k be a nonnegative integer. We consider the colum vector $z^{[k]}$ with components

$$\sqrt{\frac{k!}{k_1!\cdots k_n!}}z_1^{k_1}\cdots z_n^{k_n},\ k_1+\cdots k_n=k.$$

The dimension of $z^{[k]}$ is

$$N_k=\frac{1}{k!}n(n+1)\cdot(n+k-1)=C^k_{n+k-1}.$$

Let dz and $d\sigma(\xi)$ be the Lebesgue volume element of $\mathbb{C}^n$ and the Lebesgue area element of $\partial\mathbb{B}_n$, respectively. Define

$$\begin{cases} H_1^k=\displaystyle\int_{\mathbb{B}_n}\overline{z^{[k]\prime}}\cdot z^{[k]}dz,\\ H_2^k=\displaystyle\int_{\partial\mathbb{B}_n}\overline{\xi^{[k]\prime}}\cdot \xi^{[k]}d\sigma(\xi).\end{cases}$$

It is easy to prove that H_1^k and H_2^k are positive definite Hermitian matrices of order N_k. There exists a matrix Γ such that

$$\begin{cases}\overline{\Gamma'}\cdot H_1^k\cdot\Gamma=\Lambda,\\ \overline{\Gamma'}\cdot H_2^k\cdot\Gamma=I,\end{cases}\tag{2.1}$$

where $\Lambda=[\beta_1^k,\cdots,\beta_n^k]$ is a diagonal matrix and I is the identity matrix. Set

$$\begin{cases} z_{[k]}=z^{[k]}\cdot\Gamma;\\ \xi_{[k]}=\xi^{[k]}\cdot\Gamma.\end{cases}$$

Denote by $p_\nu^k(z)$ the components of the vectors $z_{[k]}$. From (2.1), we can see that

$$\int_{\mathbb{B}_n} p_\nu^k(z)\overline{p_\mu^k(z)}dz=\delta_{\nu\mu}\cdot\delta_{kl}\cdot\beta_\nu^k,\tag{2.2}$$

$$\int_{\partial\mathbb{B}_n} p_\nu^k(\xi)\overline{p_\mu^k(\xi)}d\sigma(\xi)=\delta_{\nu\mu}\cdot\delta_{kl}.\tag{2.3}$$

The following theorem is well known.

Theorem 2.1. *The system of functions*

$$\left\{(\beta_\nu^k)^{-\frac{1}{2}}p_\nu^k,\ k=0,1,2,\cdots,\nu=1,2,\cdots,N_k\right\}$$

is a complete orthonormal system in the space of holomorphic functions in $\mathbb{B}_n$. The system $\{p_\nu^k\}$ is orthonormal, but not complete in the space of continuous functions on $\partial\mathbb{B}_n$.

The explicit formula of the Cauchy-Szegö kernel

$$H(z,\overline{\xi}) = \frac{1}{\omega_{2n-1}}\frac{1}{(1-z\overline{\xi'})^n}$$

on $\partial\mathbb{B}_n$ was first deduced in Hua [5] by using the system $\{p_v^k\}$ and the relation

$$H(z,\overline{\xi}) = \sum_{k=0}^{\infty}\sum_{v=1}^{N_k} p_v^k(z)\overline{p_v^k(\xi)}, z\in\mathbb{B}_n, \xi\in\mathbb{B}_n. \tag{2.4}$$

For $z,\omega\in\mathbb{B}_n\bigcup\partial\mathbb{B}_n$, the nonisotropic distance $d(z,\omega)$ is defined as

$$d(z,\omega) = \left|1 - z\bar{\omega}'\right|^{1/2}.$$

It can be easily shown that $d(\cdot,\cdot)$ is a metric on $\partial\mathbb{B}_n$. For $\xi\in\partial\mathbb{B}_n$ and $\varepsilon > 0$, we define the ball corresponding to $d(\cdot,\cdot)$ as

$$S(\xi,\varepsilon) = \left\{\eta\in\partial\mathbb{B}_n,\ d(\xi,\eta)\le\varepsilon\right\}.$$

The complement set of $S(\xi,\varepsilon)$ in $\partial\mathbb{B}_n$ is denoted by $S^c(\xi,\varepsilon)$.

Set

$$\mathcal{A} = \left\{f:\ f \text{ is holomorphic in } B(0,1+\delta) \text{ for some } \delta>0\right\}.$$

If $f\in\mathcal{A}$, then

$$f(z) = \sum_{k=0}^{\infty}\sum_{v=0}^{N_k} c_{kv}p_v^k(z),$$

where c_{kv} are the Fourier coefficients of f:

$$c_{kv} = \int_{\partial\mathbb{B}_n}\overline{p_v^k(\xi)}f(\xi)d\sigma(\xi),$$

and for any positive integer l, the series

$$\sum_{k=0}^{\infty}k^l\sum_{v=0}^{N_k}c_{kv}p_v^k(z)$$

is uniformally and absolutely convergent in any compact ball contained in $B(0,1+\delta)$ in which f is defined.

Denote by $\mathcal{U}$ the unitary group of $\mathbb{C}^n$ consisting of all unitary operators on the Hilbert space $\mathbb{C}^n$ under the complex inner product $\langle z,w\rangle = z\overline{w'}$. These are the linear operators U that preserve inner products:

$$\langle Uz,\ Uw\rangle = \langle z,\ w\rangle.$$

Clearly, $\mathcal{U}$ is a compact subset of $O(2n)$. It is easy to verify that $\mathcal{A}$ is invariant under $U\in\mathcal{U}$. If $f\in\mathcal{A}$, then f is defined by its values on $\partial\mathbb{B}_n$. In Section 3, we treat $f|_{\partial\mathbb{B}_n}$ as identical to $f\in\mathcal{A}$.

3. The kernel generated by holomorphic multipliers

Set

$$\begin{aligned} S_\omega &= \Big\{z \in \mathbb{C} \mid z \neq 0 \text{ and } |\arg z| < \omega\Big\}, \\ S_\omega(\pi) &= \Big\{z \in \mathbb{C} \mid z \neq 0, |\mathrm{Re}(z)| \leq \pi \text{ and } |\arg(\pm z)| < \omega\Big\}, \\ W_\omega(\pi) &= \Big\{z \in \mathbb{C} \mid z \neq 0, |\mathrm{Re}(z)| \leq \pi \text{ and } \mathrm{Im}(z) > 0\Big\} \bigcup S_\omega(\pi), \\ H_\omega &= \Big\{z \in \mathbb{C} \mid z = e^{i\omega}, \omega \in W_\omega(\pi)\Big\}. \end{aligned}$$

The following function space is relevant:

Definition 3.1. Let $-1 < s < \infty$. $H^s(S_\omega)$ is defined as the set of all holomorphic functions in S_ω such that

(1) b is bounded for $|z| \leq 1$;
(2) $|b(z)| \leq C_\mu |z|^s, z \in S_\mu, 0 < \mu < \omega$.

Remark 3.2. The classes $H^s(S_\omega)$ are generalizations of $H^\infty(S_\omega)$ which is introduced by A. McIntosh and his collaborators. We refer to Li-McIntosh-Semmes [6], McIntosh [7], McIntosh-Qian [8], Qian [12] and the reference therein for further information on $H^\infty(S_\omega)$.

Let

$$\varphi_b(z) = \sum_{k=1}^{\infty} b(k) z^k.$$

Lemma 3.3. *Let $b \in H^s(S_\omega)$, $-1 < s < \infty$. Then φ_b can be holomorphically extended to H_ω. Moreover, for $0 < \mu < \mu' < \omega$ and $l = 0, 1, 2, \ldots,$*

$$\left|\left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)^l \varphi_b(z)\right| \lesssim \frac{C_{\mu'} l!}{\delta^l(\mu,\mu')\,|1-z|^{l+1+s}},\ z \in H_\mu,$$

where $\delta(\mu,\mu') = \min\{\frac{1}{2}, \tan(\mu,\mu'\}$; $C_{\mu'}$ are the constants in Definition 3.1.

Proof. Let

$$V_\omega = \Big\{z \in \mathbb{C} : \mathrm{Im}(z) > 0\Big\} \bigcup S_\omega \bigcup (-S_\omega),$$
$$W_\omega = V_\omega \cap \Big\{z \in \mathbb{C} :\ -\pi \leq \mathrm{Re} z \leq \pi\Big\}$$

and ρ_θ is the ray $r\exp(i\theta)$, $0 < r < \infty$, where θ is chosen so that $\rho_\theta \subsetneq S_\omega$. Define

$$\Psi_b(z) = \frac{1}{2\pi}\int_{\rho(\theta)} \exp(i\xi z) b(\xi)\mathrm{d}\xi,\ z \in V_\omega,$$

where $\exp(iz\xi)$ is exponentially decaying as $\xi \to \infty$ along ρ_θ. Then we get

$$\begin{aligned} \Big||z|^{1+s}\,\Psi_b(z)\Big| &= \left|\frac{1}{2\pi}\int_{\rho(\theta)} \exp(i\xi z)\,|z|^{1+s}\, b(\xi)\mathrm{d}z\right| \\ &\lesssim \frac{C_{\mu'}}{2\pi}\int_0^\infty \exp(-r|z|\sin(\theta+\arg z))(r|z|)^s \mathrm{d}(r|z|)^s \\ &\lesssim C_{\mu'}, \end{aligned} \tag{3.1}$$

which implies $|\Psi_b(z)| \lesssim 1/|z|^{1+s}$. Define

$$\psi_b(z) = 2\pi \sum_{n=-\infty}^{\infty} \Psi_b(z + 2n\pi), \quad z \in \bigcup_{n=-\infty}^{\infty} (2n\pi + W_\omega).$$

It is easy to see that ψ_b is holomorphically and 2π -periodically defined in the described region, and $|\psi_b(z)| \lesssim 1/|z|^{1+s}$. Let

$$\varphi_b(z) \;=\; \psi_b\left(\frac{\log z}{i}\right).$$

For $z \in \exp(iS_\omega)$, we write $z = e^{iu}$, where $u \in S_\omega$. Then $\sin\frac{|u|}{2} \lesssim \frac{|u|}{2}$. This implies that $2 - 2\cos|u| \lesssim |u|^2$ and $|1 - e^{i|u|}| \lesssim |u|$. Therefore, (3.1) gives

$$\begin{aligned} |\varphi_b(z)| &\lesssim \frac{C_{\mu'}}{|\log z|^{1+s}} \lesssim \frac{C_{\mu'}}{|\log|z||^{1+s}} \\ &\lesssim \frac{C_{\mu'}}{|1-z|^{1+s}}. \end{aligned}$$

Take the ball

$$B(z,r) = \left\{\xi : |z - \xi| < \delta(\mu,\mu')|1 - z|\right\}.$$

Applying Cauchy's integral formula, we obtain

$$\varphi_b^{(l)}(z) = \frac{l!}{2\pi i}\int_{\partial B(z,r)} \frac{\varphi(\eta)}{(\eta - z)^{1+l}} d\eta.$$

For any $\eta \in \partial B(z,r)$, we have $|\eta - z| \geq (1 - \delta(\mu,\mu'))|1 - z|$. Then we have

$$\begin{aligned} \left|\varphi_b^{(l)}(z)\right| &\lesssim \frac{l!\|b\|_{H^s(S_\omega^c)}}{\delta^l(\mu,\mu')|1-z|^l}\left|\int_{\partial B(z,r)} \frac{1}{|1-\eta|^{1+s}} d\eta\right| \\ &\lesssim \frac{l!}{\delta^l(\mu,\mu')|1-z|^{l+1+s}}. \end{aligned}$$

□

Theorem 3.4. *Let $b \in H^s(S_\omega)$ and*

$$H_b(z,\bar{\xi}) = \sum_{k=1}^{\infty} b(k) \sum_{v=1}^{N_k} p_v^k(z)\overline{p_v^k(\xi)}, \quad z \in \mathbb{B}_n,\ \xi \in \partial\mathbb{B}_n.$$

Then

$$H_b(z,\bar{\xi}) \;=\; \frac{1}{(n-1)!\omega_{2n-1}}(r^{n-1}\varphi_b(r))^{(n-1)}\Big|_{r=z\bar{\xi}'}$$

is holomorphically defined for $z \in \mathbb{B}_n, \xi \in \partial\mathbb{B}_n$ such that $z\bar{\xi}' \in H_\omega$, where φ_b is the function defined in Lemma 3.3, Moreover, for $0 < \mu < \mu' < \omega$ and $l = 0, 1, 2, \ldots$,

$$\left|D_z^l H_b(z,\bar{\xi})\right| \lesssim \frac{C_{\mu'} l!}{\delta^l(\mu,\mu')\left|1 - z\bar{\xi}'\right|^{n+l+s}}, \quad z\bar{\xi}' \in H_\mu,$$

where $\delta(\mu,\mu') = \min\{1/2, \tan(\mu' - \mu)\}$, $C_{\mu'}$ are the constant in the definition of the function space $H^s(S_\omega)$.

Proof. Recall that

$$\begin{cases} \varphi_b(z) = \sum_{k=1}^{\infty} b(k) z^k; \\ r^{n-1}\varphi_b(r) = \sum_{k=1}^{\infty} b(k) r^{n+k-1}. \end{cases}$$

Then we have

$$\begin{aligned} \frac{1}{(n-1)!}\left(r^{n-1}\varphi_b(r)\right)^{(n-1)} &= \frac{1}{(n-1)!}\sum_{k=1}^{\infty} b(k)(n+k-1)(n+k-2)\dots(k+1)r^k \\ &= \sum_{k=1}^{\infty} b(k) r^k \frac{(n+k-1)!}{(n-1)!k!} \\ &= \sum_{k=1}^{\infty} \frac{(n+k-1)(n+k-2)(n+1)n}{k!} b(k) r^k, \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{1}{(n-1)!}\left(r^{n-1}\varphi_b(r)\right)^{(n-1)}\Big|_{r=z\bar{\xi}'} &= \sum_{k=1}^{\infty} b(k)\frac{(n+k-1)(n+k-2)(n+1)n}{k!}(z\bar{\xi}')^k \\ &= \omega_{2n-1}\sum_{k=1}^{\infty} b(k)\sum_{v=1}^{N_k} p_v^k(z)\overline{p_v^k(\xi)} \\ &= \omega_{2n-1} H_b(z,\bar{\xi}). \end{aligned}$$

□

By [10, Theorem 3], we could obtain the following result.

Theorem 3.5. *Let s be an negative integer. If $b \in H^s(S_{\omega,\pm})$,*

$$H_b(z,\xi) = \sum_{k=1}^{\infty} b(k)\sum_{v=1}^{N_k} p_v^k(z) p_\mu^l(\xi),\ z \in \mathbb{B},\ \xi \in \partial\mathbb{B}_n,$$

then

$$\left|D_z^l H_b(z,\bar{\xi})\right| \lesssim \frac{C_\mu l!\left[|\ln|1-z\bar{\xi}'||+1\right]}{\delta^l(\mu,\mu')|1-z\bar{\xi}'|^{n+l+s}}.$$

Proof. The proof is similar to Theorem 3.4. we omit it. □

4. Sobolev spaces and unbounded Fourier multipliers

4.1. **Integral representation of multipliers.** Given $b \in H^s(S_\omega)$. We define an Fourier multiplier operator $M_b : \mathcal{A} \to \mathcal{A}$ by

$$M_b(f)(\xi) = \sum_{k=1}^{\infty} b(k)\sum_{v=0}^{N_k} c_{kv} p_v^k(\xi),\ \xi \in \partial\mathbb{B}_n,$$

where $\{c_{kv}\}$ are the Fourier coefficients of the test function $f \in \mathcal{A}$.

For the above operator M_b, a Plemelj type formula holds.

Theorem 4.1. *Let $b \in H^s(S_\omega)$, $s > 0$. Take $b_1(z) = z^{-s_1} b(z)$, where $s_1 = [s] + 1$. Operator M_b has a singular integral expression. For $f \in \mathcal{A}$,*

$$\begin{aligned} M_b(f)(\xi) &= \lim_{\varepsilon\to 0}\Big[\int_{S^c(\xi,\varepsilon)} H_{b_1}(\xi,\overline{\eta}) D_\eta^{s_1} f(\eta) d\sigma(\eta) \\ &\quad + (D_z^{s_1} f)(\xi)\int_{S^c(\xi,\varepsilon)} H_{b_1}(\xi,\overline{\eta}) d\sigma(\eta)\Big], \end{aligned}$$

where $\int_{S(\xi,\varepsilon)} H_{b_1}(\xi,\overline{\eta}) d\sigma(\eta)$ is a bounded function of $\xi \in \partial\mathbb{B}_n$ and ε.

Proof. Let

$$M_b(f)(\rho\xi) = \sum_{k=1}^{\infty} b(k) \sum_{v=1}^{N_k} c_{kv} p_v^k(\rho\xi), \quad \xi \in \partial\mathbb{B}_n,$$

where

$$c_{kv} = \int_{\partial B} \overline{p_v^k(\eta)} f(\eta) \mathrm{d}\sigma(\eta).$$

We can see that

$$\begin{aligned} D_z z^{[l]} &= \sqrt{\frac{l!}{l_1! l_2! \cdots l_n!}} \sum_{k=1}^{n} z_k \frac{\partial}{\partial z_k}\left(z_1^{l_1} z_2^{l_2} \cdots z_n^{l_n}\right) \\ &= \sqrt{\frac{l!}{l_1! l_2! \cdots l_n!}} \sum_{k=1}^{n} z_k l_k z_1^{l_1} z_2^{l_2} \cdots z_{k-1}^{l_{k-1}} z_k^{l_k-1} z_{k+1}^{l_{k+1}} \cdots z_n^{l_n} \\ &= \sqrt{\frac{l!}{l_1! l_2! \cdots l_n!}} \Big(\sum_{k=1}^{n} l_k\Big) z_1^{l_1} z_2^{l_2} \cdots z_n^{l_n} \\ &= l z^{[l]}, \end{aligned}$$

which implies that $D_z p_v^k = k p_v^k$. Then we have

$$\begin{aligned} M_b(f)(\rho\xi) &= \sum_{k=1}^{\infty} b(k) \sum_{v=1}^{N_k} \int_{\partial B} p_v^k(\rho\xi) \overline{p_v^k(\eta)} f(\eta) \mathrm{d}\sigma(\eta) \\ &= \sum_{k=1}^{\infty} b(k) \frac{1}{k^{s_1}} \sum_{v=1}^{N_k} \int_{\partial B} p_v^k(\rho\xi) k^{s_1} \overline{p_v^k(\eta)} f(\eta) \mathrm{d}\sigma(\eta) \\ &= \sum_{k=1}^{\infty} b(k) \frac{1}{k^{s_1}} \sum_{v=1}^{N_k} \int_{\partial B} p_v^k(\rho\xi) D_\eta^{s_1} \overline{p_v^k(\eta)} f(\eta) \mathrm{d}\sigma(\eta). \end{aligned}$$

By integration by parts,

$$\begin{aligned} M_b(f)(\rho\xi) &= \sum_{k=1}^{\infty} b(k) \frac{1}{k^{s_1}} \sum_{v=1}^{N_k} \int_{\partial B} p_v^k(\rho\xi) \overline{p_v^k(\eta)} (D_\eta^{s_1} f)(\eta) \mathrm{d}\sigma(\eta) \\ &= \sum_{k=1}^{\infty} b_1(k) \sum_{v=1}^{N_k} \int_{\partial B} p_v^k(\rho\xi) \overline{p_v^k(\eta)} (D_\eta^{s_1} f)(\eta) \mathrm{d}\sigma(\eta). \end{aligned}$$

For any $\varepsilon > 0$, we have

$$\begin{aligned} M_b(f)(\rho\xi) &= \int_{S^c(\xi,\varepsilon)} H_{b_1}(\rho\xi,\bar{\eta}) D_\eta^{s_1} f(\eta) \mathrm{d}\sigma(\eta) \\ &+ \int_{S(\xi,\varepsilon)} H_{b_1}(\rho\xi,\bar{\eta})(-D_\xi^{s_1} f(\xi) + D_\eta^{s_1} f(\eta)) \mathrm{d}\sigma(\eta) \\ &+ D_\xi^{s_1} f(\xi) \int_{S(\xi,\varepsilon)} H_{b_1}(\rho\xi,\bar{\eta}) \mathrm{d}\sigma(\eta) \\ &=: I_1(\rho,\varepsilon) + I_2(\rho,\varepsilon) + D_\xi^{s_1} f(\xi) I_3(\rho,\varepsilon), \end{aligned}$$

where

$$\begin{aligned} I_1(\rho,\varepsilon) &= \int_{S^c(\xi,\varepsilon)} H_{b_1}(\rho\xi,\bar{\eta}) D_\eta^{s_1} f(\eta) \mathrm{d}\sigma(\eta), \\ I_2(\rho,\varepsilon) &= \int_{S(\xi,\varepsilon)} H_{b_1}(\rho\xi,\bar{\eta})(-D_\xi^{s_1} f(\xi) + D_\eta^{s_1} f(\eta)) \mathrm{d}\sigma(\eta), \\ I_3(\rho,\varepsilon) &= \int_{S(\xi,\varepsilon)} H_{b_1}(\rho\xi,\bar{\eta}) \mathrm{d}\sigma(\eta). \end{aligned}$$

For $\rho \to 1-0$, we have

$$\begin{aligned} \lim_{\rho\to 1-0} I_1(\rho,\varepsilon) &= \lim_{\rho\to 1-0} \int_{S^c(\xi,\varepsilon)} H_{b_1}(\rho\xi,\bar{\eta}) D_\eta^{s_1} f(\eta) \mathrm{d}\sigma(\eta) \\ &= \int_{S^c(\xi,\varepsilon)} H_{b_1}(\xi,\bar{\eta}) D_\eta^{s_1} f(\eta) \mathrm{d}\sigma(\eta). \end{aligned}$$

Now we consider $I_2(\rho,\varepsilon)$. Let $\xi = [1,0,\ldots,0]$. For $\eta \in \partial\mathbb{B}_n$, write

$$\begin{cases} \eta_1 = re^{i\theta}, \eta_2 = v_2, \eta_3 = v_3, \ldots, \eta_n = v_n; \\ v = [v_2, v_3, \ldots, v_n]. \end{cases}$$

For such $\eta \in \partial\mathbb{B}_n$, $v\bar{v}' = 1 - r^2$. Without loss of generality, assume $\xi = 1$. We get

$$\left|1 - \xi\bar{\eta}'\right|^{1/2} = \left|1 - re^{i\theta}\right|^{1/2} = [(1 - r\cos\theta)^2 + (r\sin\theta)^2]^{1/4} \leq \varepsilon,$$

which implies that

$$\cos\theta \geq \frac{1 + r^2 - \varepsilon^4}{2r}.$$

The above estimate implies

$$S(\xi,\varepsilon) = \left\{\eta \mid v\bar{v}' = 1 - r^2, \cos\theta \geq \frac{1 + r^2 - \varepsilon^4}{2r}\right\}.$$

Since

$$\frac{1 + r^2 - \varepsilon^4}{2r} \leq \cos\theta \leq 1,$$

we obtain $1 - r \leq \varepsilon^2$ and then

$$v\bar{v}' = 1 - r^2 \leq 1 - (1 - \varepsilon^2)^2 = 2\varepsilon^2 - \varepsilon^4.$$

Denote

$$a = a(r,\varepsilon) = \arccos\left(\frac{1+r^2-\varepsilon^4}{2r}\right).$$

Since $(1-r)^2 \le \varepsilon^4$ and $1-y = O(\arccos^2 y)$, we get $a = O(\varepsilon^2)$. It is easy to see

$$\begin{aligned} |\xi-\eta|^2 &= |1-re^{i\theta}|^2 + \sum_{k=2}^{n} |v_k|^2 \\ &= (1+r^2-2r\cos\theta) + (1-r^2) \\ &= 2-2r\cos\theta \end{aligned}$$

and

$$\begin{aligned} d^4(\xi,\eta) &= 1+r^2-2r\cos\theta \\ &= (2-2r\cos\theta)-(1-r^2) \\ &= |\xi-\eta|^2-(1+r)(1-r), \end{aligned}$$

that is, $d^2(\xi,\eta) \le |\xi-\eta|$. Because

$$d^2(\xi,\eta) = [1+r^2-2r\cos\theta]^{1/2} \ge 1-r,$$

then we have $1-r \le d^2(\xi,\eta)$, so

$$|\xi-\eta|^2 \le d^4(\xi,\eta) + (1+r)d^2(\xi,\eta).$$

Since $d^2(\xi,\eta) \le 2$, then

$$|\xi-\eta|^2 \le 2d^2(\xi,\eta) + 2d^2(\xi,\eta) = 4d^2(\xi,\eta),$$

that is

$$|\xi-\eta| \le 2d(\xi,\eta).$$

Since $f \in \mathcal{A}$, we have

$$|f(\xi)-f(\eta)| \lesssim |\xi-\eta| \lesssim d(\xi,\eta).$$

For $\rho \in (0,1)$

$$\begin{aligned} |I_2(\rho,\varepsilon)| &\lesssim \int_{S(\xi,\varepsilon)} \left|H_{b_1}(\rho\xi,\bar{\eta})\right| |f(\xi)-f(\eta)| \,\mathrm{d}\sigma(\eta) \\ &\lesssim \int_{S(\xi,\varepsilon)} \frac{d(\xi,\eta)}{|1-\xi\bar{\eta}'|^n} \mathrm{d}\sigma(\eta) \\ &\lesssim \int_{v\bar{v}'\le 2\varepsilon^2-\varepsilon^4} \int_{-a}^{a} \frac{1}{\left|1-re^{i\theta}\right|^{n-1/2}} \mathrm{d}\theta \mathrm{d}v. \end{aligned}$$

For $n = 2$,

$$\begin{aligned}
\frac{1}{2a}\int_{-a}^{a}\frac{1}{\left|1-re^{i\theta}\right|^{2-1/2}}\mathrm{d}\theta &\leq \left(\frac{1}{2a}\int_{-a}^{a}\frac{1}{\left|1-re^{i\theta}\right|^{2}}\mathrm{d}\theta\right)^{3/4}\\
&\leq \left(\frac{1}{2a}\int_{-\pi}^{\pi}\frac{1}{\left|1-re^{i\theta}\right|^{2}}\mathrm{d}\theta\right)^{3/4}\\
&\leq \left(\frac{1}{2a}\right)^{3/4}\frac{1}{(1-r^2)^{3/4}}.
\end{aligned}$$

Then we get

$$\begin{aligned}
|I_2(\rho,\varepsilon)| &\lesssim \int_{v\bar{v}'\leq 2\varepsilon^2-\varepsilon^4} a^{1/4}\frac{1}{(1-r^2)^{3/4}}\mathrm{d}v\\
&\lesssim \varepsilon^{1/2}\int_{v\bar{v}'\leq 2\varepsilon^2-\varepsilon^4}\frac{1}{(v\bar{v}')^{3/4}}\mathrm{d}v\\
&= \varepsilon^{1/2}\int_0^{\sqrt{2\varepsilon^2-\varepsilon^4}}\frac{t}{t^{3/2}}\mathrm{d}t\\
&\lesssim \varepsilon\to 0
\end{aligned}$$

For $n > 2$, we have

$$\begin{aligned}
\int_{-a}^{a}\frac{1}{\left|1-re^{i\theta}\right|^{n-1/2}}\mathrm{d}\theta &\lesssim \int_{-a}^{a}\frac{\left|1-r^2\right|^{n-1/2-2}}{\left|1-re^{i\theta}\right|^{n-1/2}}\frac{1}{\left|1-r^2\right|^{n-1/2-2}}\mathrm{d}\theta\\
&\lesssim \frac{1}{\left|1-r^2\right|^{n-1/2-1}}\int_{-\pi}^{\pi}\frac{1}{\left|1-re^{i\theta}\right|^{2}}\mathrm{d}\theta\\
&\lesssim \frac{1}{\left|1-r^2\right|^{n-1/2-1}},
\end{aligned}$$

then we get

$$|I_2(\rho,\varepsilon)| \lesssim \int_0^{\sqrt{2\varepsilon^2-\varepsilon^4}} t^{2n-3}\frac{1}{t^{2n-3}}\mathrm{d}t \lesssim \sqrt{2\varepsilon^2}\to 0.$$

Now we prove if $\rho\to 1-0$, $I_3(\rho,\varepsilon)$ has a limit uniformly bounded for ε near 0. Integrating as before, we have

$$\begin{aligned}
I_3(\rho,\varepsilon) &= \int_{S(\xi,\varepsilon)} H_{b_1}(\rho\xi,\bar{\eta})\mathrm{d}\sigma(\eta)\\
&= \int_{v\bar{v}'\leq 2\varepsilon^2-\varepsilon^4}\int_{-a}^{a}\left(t^{n-1}\varphi_{b_1}(t)\right)^{(n-1)}\Big|_{t=\rho re^{i\theta}}\mathrm{d}\theta\mathrm{d}v.
\end{aligned}$$

Let $s=\rho re^{i\theta}$. Then $\mathrm{d}s = is\mathrm{d}\theta$. We get

$$I_3(\rho,\varepsilon) = -i\int_{v\bar{v}'\leq 2\varepsilon^2-\varepsilon^4}\int_{\rho re^{-ia}}^{\rho re^{ia}}\left(s^{n-1}\varphi_{b_1}(s)\right)^{(n-1)}\mathrm{d}s\mathrm{d}v.$$

By integration by parts, the inside integral with respect to the variable t becomes

$$\begin{aligned}
&\int_{-a}^{a} \left(t^{n-1}\varphi_{b_1}(t)\right)^{(n-1)}\Big|_{t=\rho re^{i\theta}} \mathrm{d}\theta \\
&= \left[\sum_{k=1}^{n-1}(k-1)!\frac{\left(t^{n-1}\varphi_{b_1}(t)\right)^{(n-k-1)}}{t^k}\right]\Bigg|_{\rho re^{-ia}}^{\rho re^{ia}} + (n-1)!\int_{\rho re^{-ia}}^{\rho re^{ia}} \frac{\varphi_{b_1}(t)}{t}\mathrm{d}t \\
&= \sum_{k=1}^{n-1}[J_k(t)]_{\rho re^{-ia}}^{\rho re^{ia}} + L(r,a).
\end{aligned}$$

We first estimate J_k,

$$\begin{aligned}
&\int_{v\bar{v}'\leq 2\varepsilon^2-\varepsilon^4} J_k\left(\rho re^{\pm ia}\right)\mathrm{d}v \\
&\lesssim \int_{v\bar{v}'\leq 2\varepsilon^2-\varepsilon^4} (k-1)!\frac{\left(\rho re^{\pm ia}\right)^k}{\left(\rho re^{\pm ia}\right)^k}\frac{1}{\left|1-\rho re^{\pm ia}\right|^{n-k}}\mathrm{d}v \\
&\lesssim \int_{v\bar{v}'\leq 2\varepsilon^2-\varepsilon^4} \frac{1}{\left|1-\rho re^{\pm ia}\right|^{n-k}}\mathrm{d}v.
\end{aligned}$$

Because $\left|1-\rho re^{\pm ia}\right|^2 = 1+\rho^2r^2-2\rho r\cos a$,

$$\begin{aligned}
\left|1-\rho re^{\pm ia}\right|^2 - \left|1-re^{\pm ia}\right|^2 &= \rho^2r^2-2\rho r\cos a-(r^2-2r\cos a) \\
&= r^2(\rho^2-1)+2r\cos a(1-\rho).
\end{aligned}$$

Since $\cos a = \frac{1+r^2-\varepsilon^4}{2r}$, we have

$$\begin{aligned}
\left|1-\rho re^{\pm ia}\right|^2 - \left|1-re^{\pm ia}\right|^2 &= r^2(\rho^2-1)+(1+r^2-\varepsilon^4)(1-\rho) \\
&= (1-\rho)[1+r^2-\varepsilon^4-(1+\rho)r^2] \\
&= (1-\rho)(1-\rho r^2-\varepsilon^4) > 0.
\end{aligned}$$

So

$$\left|1-\rho re^{\pm ia}\right| \geq \left|1-re^{\pm ia}\right| = \varepsilon^2.$$

For k, when $\varepsilon \to 0$, we obtain

$$\begin{aligned}
\int_{v\bar{v}'\leq 2\varepsilon^2-\varepsilon^4} J_k\left(\rho re^{\pm ia}\right)\mathrm{d}v &\lesssim \frac{1}{\varepsilon^{2n-2k}}\int_{v\bar{v}'\leq 2\varepsilon^2-\varepsilon^4}\mathrm{d}v \\
&\lesssim \frac{1}{\varepsilon^{2n-2k}}\int_0^{\sqrt{2\varepsilon^2-\varepsilon^4}} t^{2n-3}\mathrm{d}t \\
&\lesssim \frac{\varepsilon^{2n-2}}{\varepsilon^{2n-2k}} \lesssim 1.
\end{aligned}$$

On the other hand

$$\begin{aligned}(n-1)!\int_{\rho re^{-ia}}^{\rho re^{ia}}\frac{\varphi_{b_1}(t)}{t}\mathrm{d}t &= i(n-1)!\int_{-a}^{a}\varphi_{b_1}(t)\Big|_{t=\rho re^{i\theta}}\mathrm{d}\theta \\ &\lesssim 1,\ (\text{ when } \rho\to 0)\end{aligned}$$

that implies

$$\int_{v\bar{v}'\leq 2\varepsilon^2-\varepsilon^4} L(\rho r, a)\mathrm{d}v.$$

□

4.2. **Sobolev spaces on $\partial\mathbb{B}_n$ via Fourier mulitpliers.** Sobolev spaces on the n-complex unit sphere $\partial\mathbb{B}_n$ are defined as follows. We define the fractional integral operator $\mathcal{I}^s$ on $\partial\mathbb{B}_n$ as follows. Let

$$f(z)=\sum_{k=0}^{\infty}\sum_{v=0}^{N_k}c_{kv}p_v^k(z).$$

For $-\infty<s<\infty$, the operator $\mathcal{I}^s$ is defined by

$$\mathcal{I}^s f(z)=\sum_{k=0}^{\infty}\sum_{v=0}^{N_k}k^s c_{kv}p_v^k(z).$$

For $s\in\mathbb{Z}_+$, we can see that the operators $\mathcal{I}^s$ become the ordinary differential operators with higher orders.

Theorem 4.2. *Let $s\in\mathbb{Z}_+$. $D_z^s=\mathcal{I}^s$ on $L^2(\partial\mathbb{B}_n)$.*

Proof. Without loss of generality, we assume that $f\in\mathcal{A}$. Then

$$f(z)=\sum_{k=0}^{\infty}\sum_{v=0}^{N_k}c_{kv}p_v^k(z),$$

where c_{kv} are the Fourier coefficients of f:

$$c_{kv}=\int_{\partial\mathbb{B}_n}\overline{p_v^k(\xi)}f(\xi)d\sigma(\xi).$$

So

$$\begin{aligned}D_z^s f(z) &= \sum_{k=0}^{\infty}\sum_{v=0}^{N_k}\int_{\partial\mathbb{B}_n}\overline{p_v^k(\xi)}f(\xi)d\sigma(\xi)D_z^s(p_v^k)(z)\\ &= \sum_{k=0}^{\infty}k^s\sum_{v=0}^{N_k}\int_{\partial\mathbb{B}_n}\overline{p_v^k(\xi)}f(\xi)d\sigma(\xi)p_v^k(z).\end{aligned}$$

□

Definition 4.3. Let $s\in[0,+\infty)$. The Sobolev norm $\|\cdot\|_{W^{2,s}(\partial\mathbb{B}_n)}$ on $\partial\mathbb{B}_n$ is defined as

$$\|f\|_{W^{2,s}(\partial\mathbb{B}_n)}=:\|\mathcal{I}^s f\|_2<\infty.$$

The Sobolev spaces on $\partial\mathbb{B}_n$ is defined as the closure of $\mathcal{A}$ under the norm $\|\cdot\|_{W^{2,s}(\partial\mathbb{B}_n)}$, that is $W^{2,s}(\partial\mathbb{B}_n)=\overline{\mathcal{A}}^{\|\cdot\|_{W^{2,s}(\partial\mathbb{B}_n)}}$.

Remark 4.4. By the Plancherel theorem, $f \in W^{2,s}(\partial\mathbb{B}_n)$ if and only if

$$\Big(\sum_{k=1}^{\infty} k^{2s} \sum_{v=0}^{N_k} |c_{kv}|^2\Big)^{1/2} < \infty.$$

Now we consider the Sobolev boundedness of M_b.

Theorem 4.5. *Given $r, s \in [0, +\infty)$ and $b \in H^s(S_\omega)$. The Fourier multiplier operator M_b is bounded from $W^{2,r+s}(\partial\mathbb{B}_n)$ to $W^{2,r}(\partial\mathbb{B}_n)$.*

Proof. Write

$$\mathcal{I}^s f(z) = \sum_{k=0}^{\infty} \sum_{v=0}^{N_k} c^s_{kv} p^k_v(z).$$

By the orthogonality of $\{p^k_v\}$, we can see that $c^s_{kv} = k^s c_{kv}$. Let $b(z) = z^{-s} b(z)$. Because $b \in H^s(S_\omega)$, we can see that $b_1 \in H^\infty(S_\omega)$. This implies that

$$\begin{aligned}
\mathcal{I}^r(M_b(f))(\xi) &= \sum_{k=1}^{\infty} b(k) k^r \sum_{v=0}^{N_k} c_{kv} p^k_v(\xi) \\
&= \sum_{k=1}^{\infty} b_1(k) k^{r+s} \sum_{v=0}^{N_k} c_{kv} p^k_v(\xi) \\
&= M_{b_1}(\mathcal{I}^{r+s} f)(\xi).
\end{aligned}$$

Finally, by [1, Theorem 3], we can see that

$$\begin{aligned}
\|M_b(f)\|_{W^{2,r}} &= \|\mathcal{I}^r(M_b(f))\|_2 \\
&= \|M_{b_1}(\mathcal{I}^{r+s} f)\|_2 \\
&\le C\|\mathcal{I}^{r+s} f\|_2.
\end{aligned}$$

This completes the proof of Theorem 4.5.

□

5. Conflict of Interests

The authors declare that they have no conflict of interest in this submitted manuscript.

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