J. Math. Anal. Appl. 416 (2014) 659–671

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Holomorphic approximation of L_2-functions on the unit sphere in $\mathbb{R}^3$

Nele De Schepper [a,*], Tao Qian [b], Frank Sommen [a], Jinxun Wang [b]

[a] *Clifford Research Group, Department of Mathematical Analysis, Faculty of Engineering and Architecture, Ghent University, Belgium*
[b] *Department of Mathematics, Faculty of Science and Technology, University of Macau, Taipa, Macao, China*

ARTICLE INFO

Article history:
Received 8 September 2013
Available online 6 March 2014
Submitted by M. Peloso

Keywords:
Quaternionic analysis
Orthogonal polynomials
Holomorphic signals
Holomorphic polynomials

ABSTRACT

In this paper we construct an embedding of holomorphic functions in two complex variables into the unit ball in $\mathbb{R}^3$. This leads to a closed subspace of the L_2-functions on the unit sphere spanned by quaternionic polynomials for which we construct orthonormal bases and study the related convergence properties.

1. Introduction

The most widely used approximation for L_2-functions on the unit sphere in $\mathbb{R}^m$ is the approximation by spherical harmonics. Specifically, a spherical harmonic of degree k is a k-homogeneous polynomial $S_k(\underline{x})$ which is harmonic, i.e. $\Delta_{\underline{x}} S_k(\underline{x}) = 0$, $\Delta_{\underline{x}} = \sum_{j=1}^m \partial_{x_j}^2$ being the Laplacian, and every $f \in L_2(S^{m-1})$ admits an orthogonal decomposition in spherical harmonics of the form $f(\underline{\omega}) = \sum_{k=0}^\infty S_k(\underline{\omega})$, $\underline{\omega} = \frac{\underline{x}}{|\underline{x}|} \in S^{m-1}$.

Clifford analysis forms a refinement of harmonic analysis. It starts with the construction of a Clifford algebra with generators $e_1, \ldots, e_m$ and relations $e_j^2 = -1$, $e_j e_k = -e_k e_j$, $j \neq k$ and leads to the Dirac operator $\partial_{\underline{x}} = \sum_{j=1}^m e_j \partial_{x_j}$ for which $\partial_{\underline{x}}^2 = -\Delta_{\underline{x}}$; solutions of $\partial_{\underline{x}} f = 0$ are called monogenic functions. Spherical monogenics of degree k are then defined as k-homogeneous polynomials $P_k(\underline{x})$ which are monogenic: $\partial_{\underline{x}} P_k(\underline{x}) = 0$ and the series $\sum_{k=0}^\infty P_k(\underline{\omega}) \in L_2(S^{m-1})$ form the closed subspace of monogenic signals $ML_2(S^{m-1})$. For any scalar function $f \in L_2(S^{m-1})$ there exists a monogenic signal $\sum_{k=0}^\infty P_k(\underline{\omega}) = g$ for which $[g]_0 = f$, $[\ .\ ]_0$ denoting the scalar part. The monogenic function theory has a lot of interesting properties; we refer to the extended literature containing the books [2,4,5,7] etc.

* Corresponding author. Fax: +32 (0)9 264 49 87.
E-mail addresses: nds@cage.ugent.be (N. De Schepper), fsttq@umac.mo (T. Qian), fs@cage.ugent.be (F. Sommen), wjxpyh@gmail.com (J. Wang).

http://dx.doi.org/10.1016/j.jmaa.2014.02.065
0022-247X/

One disadvantage is however the fact that the product of monogenic functions is no longer monogenic. This leads us to the idea of constructing an embedding of holomorphic functions of several complex variables into the unit ball in Euclidean space. In $\mathbb{R}^3$, it works as follows: consider Clifford generators e_1, e_2 with defining relations $e_1^2 = e_2^2 = -1$ and $e_1 e_2 = -e_2 e_1$. Then the "holomorphic correspondence" is the map from the holomorphic functions in two complex variables $g(z_1, z_2) = \sum_{p,q=0}^{\infty} z_1^p z_2^q C_{p,q}$ into the corresponding series in $\mathbb{R}^3$:

$$g(z_1, z_2) \longmapsto \sum_{p,q=0}^{\infty} (x_0 + \underline{x})^p (x_1 - e_1 e_2 x_2)^q C_{p,q}$$

with $\underline{x} = x_1 e_1 + x_2 e_2$. This embedding leads to a closed subspace of $L_2(S^2)$ called the space of holomorphic signals on S^2.

In general, holomorphic signals are of special interest, because they are closed under the operations of addition, subtraction, multiplication and division. Moreover, when defined, the composition of two holomorphic signals is also still a holomorphic signal. Moreover, every holomorphic signal admits an amplitude-phase representation such that the amplitude of the product of holomorphic signals is the product of the amplitudes of the holomorphic signals, and the phase of the product of holomorphic signals is the sum of the phases of the holomorphic signals. By using the above defined holomorphic correspondence, functions on the unit sphere are transformed into holomorphic signals so that techniques and advantages of holomorphic signals become available.

The first task in this paper (see Section 3) is to construct the Gram–Schmidt orthonormal basis that corresponds to the holomorphic polynomials $(x_0 + \underline{x})^p (x_1 - e_1 e_2 x_2)^q$. In Section 4 we also study convergence properties for these holomorphic bases and for q fixed the series converges for all $x_0 + \underline{x}$ with $x_0^2 + |\underline{x}|^2 < 1$. In case both p and q are variable, we are still able to prove convergence for $\sqrt{x_0^2 + |\underline{x}|^2} < \sqrt{2} - 1$.

Although beyond the scope of this paper, we would like to remark that the holomorphic correspondence can be generalized to a mapping from functions in m complex variables $g(z_1, \ldots, z_m) = \sum_{q_1,\ldots,q_m=0}^{\infty} z_1^{q_1} \cdots z_m^{q_m} C_{q_1,\ldots,q_m}$ to functions in Euclidean space $\mathbb{R}^{m+1}$ of the form

$$g(z_1, \ldots, z_m) \longmapsto \sum_{q_1,\ldots,q_m=0}^{\infty} (x_0 + \underline{x})^{q_1} (x_1 - e_1 \underline{x}_1)^{q_2} \cdots (x_{m-1} - e_{m-1} e_m x_m)^{q_m} C_{q_1,\ldots,q_m}$$

whereby $\underline{x} = \sum_{j=1}^{m} x_j e_j$, $\underline{x}_1 = \sum_{j=2}^{m} x_j e_j$, $\underline{x}_2 = \sum_{j=3}^{m} x_j e_j$ etc.

This then leads to the holomorphic signals on S^m and several problems related to it, including orthonormal bases on S^m, convergence properties and the study of the pull-back of the L_2-inner product on S^m to the space of holomorphic functions on for instance the polydisk $B_1 \times \cdots \times B_m$. In particular, the direct image of a holomorphic function in the polydisk surely leads to a converging series in the unit ball. This and various other problems form the basis for extended future research.

2. Preliminaries

We will work in the algebra $\mathbb{H}$ of quaternions. Let e_1, e_2 be two imaginary units of $\mathbb{H}$, satisfying the multiplication rules $e_1^2 = e_2^2 = -1$ and $e_1 e_2 = -e_2 e_1$. The conjugation in $\mathbb{H}$ is determined by $\overline{e_1} = -e_1$ and $\overline{e_2} = -e_2$. For any $x = x_0 + x_1 e_1 + x_2 e_2 \in \mathbb{R}^3$ we also write $x = x_0 + \underline{x}$, where $\underline{x} = x_1 e_1 + x_2 e_2$.

Let S^2 be the unit sphere in $\mathbb{R}^3$. The space $L_2(S^2)$ consists of all functions defined on S^2, taking values in $\mathbb{H}$, and being square integrable on S^2 with respect to the surface area element dS. The inner product on $L_2(S^2)$ is defined by

$$\langle f, g \rangle := \frac{1}{4\pi} \int_{\xi \in S^2} \overline{f(\xi)} g(\xi)\, dS, \quad f, g \in L_2\big(S^2\big),$$

which leads to an induced norm given by

$$\|f\| := \sqrt{\langle f, f\rangle}, \quad f \in L_2\big(S^2\big).$$

It is known that the set of polynomials is dense in $L_2(S^2)$. Moreover, we observe that

$$L_2\big(S^2\big) = \overline{\operatorname{span}}\big\{x_0^a \underline{x}^b (x_1 - e_1e_2x_2)^c\colon\ a, b, c \in \mathbb{N}\big\},$$

where the span denotes the right $\mathbb{H}$-linear span.

Since $2x_0 = x + \bar{x}$, $2\underline{x} = x - \bar{x}$, and on the unit sphere $\bar{x} = x^{-1}$, we have

$$\begin{aligned} L_2\big(S^2\big) &= \overline{\operatorname{span}}\big\{x^a \bar{x}^b (x_1 - e_1e_2x_2)^c\colon\ a, b, c \in \mathbb{N}\big\} \\ &= \overline{\operatorname{span}}\big\{x^p (x_1 - e_1e_2x_2)^q\colon\ p \in \mathbb{Z},\ q \in \mathbb{N}\big\}. \end{aligned}$$

In this paper we will restrict ourselves to half of the above generating set, we consider namely the set of holomorphic polynomials

$$\mathcal{HP} = \big\{x^p (x_1 - e_1e_2x_2)^q\colon\ p, q \in \mathbb{N}\big\}.$$

The closed formula for the orthonormalization of $\mathcal{HP}$ will be given. These kind of holomorphic polynomials are in fact closely related to the spherical monogenics. To see this, let us first recall some definitions.

A quaternion-valued function f, defined in an open set $\Omega \subset \mathbb{R}^3$, is called left monogenic in Ω if it is in Ω a null solution of D, i.e. $Df = 0$, where the differential operator $D := \partial_{x_0} + \partial_{\underline{x}} = (\partial/\partial x_0) + e_1(\partial/\partial x_1) + e_2(\partial/\partial x_2)$ is the so-called Cauchy–Riemann operator. A left monogenic polynomial of degree k is called a left inner spherical monogenic of degree k. The collection of all such monogenic polynomials is denoted by $\mathcal{M}_k$.

A useful tool to construct a monogenic function from a given smooth function $\mathbb{R}^2 \supset \Omega \ni \underline{x} \to f(\underline{x}) \in \mathbb{H}$ is the Cauchy–Kowalevski (CK) extension (see [2,10]), given by

$$\mathrm{CK}(f)(x) = e^{-x_0\partial_{\underline{x}}} f = \sum_{n=0}^{\infty} \frac{(-x_0)^n}{n!} \partial_{\underline{x}}^n f(\underline{x}).$$

Since for any $p, q \in \mathbb{N}$

$$\begin{aligned} \partial_{\underline{x}}\big(\underline{x}^{2p}(x_1 - e_1e_2x_2)^q\big) &= -2p\underline{x}^{2p-1}(x_1 - e_1e_2x_2)^q, \\ \partial_{\underline{x}}\big(\underline{x}^{2p+1}(x_1 - e_1e_2x_2)^q\big) &= -2(p+q+1)\underline{x}^{2p}(x_1 - e_1e_2x_2)^q, \end{aligned}$$

we arrive at

$$\begin{aligned} \mathrm{CK}\big(\underline{x}^p(x_1 - e_1e_2x_2)^q\big) &= \sum_{j=0}^{p} C_{p,j}^q \underline{x}^{p-j} x_0^j (x_1 - e_1e_2x_2)^q \\ &= \big(A_{p,q}\big(x_0, |\underline{x}|\big) + \underline{x}B_{p,q}\big(x_0, |\underline{x}|\big)\big)(x_1 - e_1e_2x_2)^q, \end{aligned}$$

where $C_{p,j}^q$, $A_{p,q}$ and $B_{p,q}$ are real-valued, and $x \to A_{p,q}(x_0, |\underline{x}|) + \underline{x}B_{p,q}(x_0, |\underline{x}|)$ is an axial monogenic function (see [3,11]) when $q = 0$. From Proposition 3.1 (see the next section) we can obtain that

$$\mathcal{B}_k = \big\{\mathrm{CK}\big(\underline{x}^p(x_1 - e_1e_2x_2)^q\big)\colon\ p, q \in \mathbb{N},\ p + q = k\big\}$$

is an orthogonal basis of $\mathcal{M}_k$ with respect to the inner product on $L_2(S^2)$. Indeed, let us take two different functions b_1 and b_2 from $\mathcal{B}_k$:

$$b_1 = \mathrm{CK}\big(\underline{x}^{p_1}(x_1 - e_1e_2x_2)^{q_1}\big), \qquad b_2 = \mathrm{CK}\big(\underline{x}^{p_2}(x_1 - e_1e_2x_2)^{q_2}\big)$$

with $q_1 \neq q_2$ ($p_1+q_1 = p_2+q_2 = k$). Each b_u, $u = 1, 2$, is a linear combination of $x_0^j\underline{x}^{p_u-j}(x_1-e_1e_2x_2)^{q_u}$ with $p_u+q_u = k$, which can be rewritten as a linear combination of $x_0^i x^j(x_1-e_1e_2x_2)^{q_u}$ with $i+j = p_u = k-q_u$. By direct verification (see also the proof of Proposition 3.1), it easily follows that

$$\big\langle x_0^{i_1}x^{j_1}(x_1 - e_1e_2x_2)^{q_1}, x_0^{i_2}x^{j_2}(x_1 - e_1e_2x_2)^{q_2}\big\rangle = 0$$

whenever $q_1 \neq q_2$, which proves the orthogonality of the basis β_k.

This construction should be compared with the results in [1,4].

The study of the projection operator from $L_2(S^2)$ to $\overline{\operatorname{span}}\mathcal{HP}$ and the related approximation problems will be our next research objectives.

3. Orthonormalization of $\mathcal{HP}$

For any $p, q \in \mathbb{N}$, let

$$\alpha_{p,q}(x) = x^p(x_1 - e_1e_2x_2)^q = (x_0 + \underline{x})^p(x_1 - e_1e_2x_2)^q.$$

The aim of this section is to orthogonalize this sequence.

We start with the following lemma, a result which can also be found in [6], p. 372, 3.631, formula 8.

Lemma 3.1. *For $m \in \mathbb{N}$ and $\gamma > -1$, we have*

$$\int_0^\pi \cos(m\theta)(\sin\theta)^\gamma\, d\theta = \frac{2^{-\gamma}\pi\cos(\frac{m\pi}{2})\Gamma(\gamma+1)}{\Gamma(1-\frac{m}{2}+\frac{\gamma}{2})\Gamma(1+\frac{m}{2}+\frac{\gamma}{2})}.$$

Proof. We first prove the result for m odd. As

$$\int_0^\pi \cos\big((2m+1)\theta\big)(\sin\theta)^\gamma\, d\theta = \int_0^\pi \cos\big((2m+1)(\pi-t)\big)\big(\sin(\pi-t)\big)^\gamma\, dt$$

$$= -\int_0^\pi \cos\big((2m+1)t\big)(\sin t)^\gamma\, dt,$$

we obtain that

$$\int_0^\pi \cos\big((2m+1)\theta\big)(\sin\theta)^\gamma\, d\theta = 0.$$

We now prove by induction the case m even, i.e. for all $\gamma > -1$

$$\int_0^\pi \cos(2m\theta)(\sin\theta)^\gamma\, d\theta = \frac{2^{-\gamma}\pi(-1)^m\Gamma(\gamma+1)}{\Gamma(1-m+\frac{\gamma}{2})\Gamma(1+m+\frac{\gamma}{2})}.$$

When $m=0$, we have that (see e.g. [9], p. 8, last formula with $\beta=0$)

$$\int_0^\pi \cos(2m\theta)(\sin\theta)^\gamma\,d\theta = \int_0^\pi (\sin\theta)^\gamma\,d\theta = \frac{2^{-\gamma}\pi\Gamma(\gamma+1)}{(\Gamma(1+\frac{\gamma}{2}))^2}.$$

Assume that the identity holds for $m=k$, we now prove it for $m=k+1$. Applying partial integration, we find

$$\begin{aligned}\int_0^\pi \cos(2k\theta)(\sin\theta)^{\gamma+2}\,d\theta &= \frac{1}{2k}\int_0^\pi (\sin\theta)^{\gamma+2}\,d\sin(2k\theta)\\ &= -\frac{\gamma+2}{2k}\int_0^\pi \sin(2k\theta)\cos\theta(\sin\theta)^{\gamma+1}\,d\theta,\end{aligned}$$

from which it follows that

$$-2\int_0^\pi \sin(2k\theta)\cos\theta(\sin\theta)^{\gamma+1}\,d\theta = \frac{4k}{\gamma+2}\int_0^\pi \cos(2k\theta)(\sin\theta)^{\gamma+2}\,d\theta.$$

Hence, when $m=k+1$, we obtain consecutively

$$\begin{aligned}&\int_0^\pi \cos(2m\theta)(\sin\theta)^\gamma\,d\theta\\ &\quad= \int_0^\pi \big(\cos(2k\theta)\cos(2\theta)-\sin(2k\theta)\sin(2\theta)\big)(\sin\theta)^\gamma\,d\theta\\ &\quad= \int_0^\pi \big(\cos(2k\theta)\big(1-2(\sin\theta)^2\big)-2\sin(2k\theta)\sin\theta\cos\theta\big)(\sin\theta)^\gamma\,d\theta\\ &\quad= \int_0^\pi \cos(2k\theta)(\sin\theta)^\gamma\,d\theta - 2\int_0^\pi \cos(2k\theta)(\sin\theta)^{\gamma+2}\,d\theta - 2\int_0^\pi \sin(2k\theta)\cos\theta(\sin\theta)^{\gamma+1}\,d\theta\\ &\quad= \int_0^\pi \cos(2k\theta)(\sin\theta)^\gamma\,d\theta + \left(\frac{4k}{\gamma+2}-2\right)\int_0^\pi \cos(2k\theta)(\sin\theta)^{\gamma+2}\,d\theta\\ &\quad= \frac{2^{-\gamma}\pi(-1)^k\Gamma(\gamma+1)}{\Gamma(1-k+\frac{\gamma}{2})\Gamma(1+k+\frac{\gamma}{2})} + \left(\frac{4k}{\gamma+2}-2\right)\frac{2^{-\gamma-2}\pi(-1)^k\Gamma(\gamma+3)}{\Gamma(2-k+\frac{\gamma}{2})\Gamma(2+k+\frac{\gamma}{2})}\\ &\quad= \frac{2^{-\gamma}\pi(-1)^{k+1}\Gamma(\gamma+1)}{\Gamma(-k+\frac{\gamma}{2})\Gamma(2+k+\frac{\gamma}{2})},\end{aligned}$$

where in the second last line we have used the induction hypothesis. □

The above result leads to the following proposition.

Proposition 3.1. *For $p_1, p_2, q_1, q_2 \in \mathbb{N}$:*

$$\langle \alpha_{p_1,q_1}, \alpha_{p_2,q_2} \rangle \in \mathbb{R},$$

and when $q_1 \neq q_2$, or $p_1 - p_2$ is odd (i.e. p_1 and p_2 have different parity), we have

$$\langle \alpha_{p_1,q_1}, \alpha_{p_2,q_2} \rangle = 0.$$

Proof. Assume $p_1 \geqslant p_2$. Using spherical coordinates, set $x_0 = \cos\theta$, $x_1 = \sin\theta\cos\beta$ and $x_2 = \sin\theta\sin\beta$ with $0 \leqslant \theta \leqslant \pi$, $0 \leqslant \beta < 2\pi$, we get $dS = \sin\theta\, d\theta\, d\beta$, and

$$\begin{aligned}
&\overline{\alpha_{p_1,q_1}(x)}\alpha_{p_2,q_2}(x) \\
&\quad = (x_1 + e_1e_2x_2)^{q_1}(x_0 - \underline{x})^{p_1}(x_0 + \underline{x})^{p_2}(x_1 - e_1e_2x_2)^{q_2} \\
&\quad = (\sin\theta)^{q_1+q_2}\big(\cos(q_1\beta) + e_1e_2\sin(q_1\beta)\big)\left(\cos\big((p_1-p_2)\theta\big) - \frac{\underline{x}}{|\underline{x}|}\sin\big((p_1-p_2)\theta\big)\right) \\
&\qquad \cdot \big(\cos(q_2\beta) - e_1e_2\sin(q_2\beta)\big) \\
&\quad = (\sin\theta)^{q_1+q_2}\cos\big((p_1-p_2)\theta\big)\big(\cos\big((q_1-q_2)\beta\big) + e_1e_2\sin\big((q_1-q_2)\beta\big)\big) \\
&\qquad - (\sin\theta)^{q_1+q_2}\sin\big((p_1-p_2)\theta\big)\big(\cos(q_1\beta) + e_1e_2\sin(q_1\beta)\big)\frac{\underline{x}}{|\underline{x}|} \\
&\qquad \cdot \big(\cos(q_2\beta) - e_1e_2\sin(q_2\beta)\big) \\
&\quad = (\sin\theta)^{q_1+q_2}\cos\big((p_1-p_2)\theta\big)\big(\cos\big((q_1-q_2)\beta\big) + e_1e_2\sin\big((q_1-q_2)\beta\big)\big) \\
&\qquad - (\sin\theta)^{q_1+q_2}\sin\big((p_1-p_2)\theta\big)\big(e_1\cos\big((q_1+q_2+1)\beta\big) + e_2\sin\big((q_1+q_2+1)\beta\big)\big).
\end{aligned}$$

We thus obtain

$$\begin{aligned}
\langle \alpha_{p_1,q_1}, \alpha_{p_2,q_2} \rangle &= \frac{1}{2}\delta_{q_1,q_2}\int_0^\pi \cos\big((p_1-p_2)\theta\big)(\sin\theta)^{2q_1+1}\, d\theta \\
&= \frac{2^{-2q_1-2}\pi\Gamma(2q_1+2)\cos(\frac{p_1-p_2}{2}\pi)}{\Gamma(\frac{3}{2} - \frac{p_1-p_2}{2} + q_1)\Gamma(\frac{3}{2} + \frac{p_1-p_2}{2} + q_1)}\delta_{q_1,q_2},
\end{aligned} \tag{1}$$

where in the last line we have used the previous lemma. □

So, the orthogonalization of $\mathcal{HP} = \{\alpha_{p,q}\colon\ p, q \in \mathbb{N}\}$ is equivalent to the separate orthogonalization of $\{\alpha_{2s,q}\colon\ s \in \mathbb{N}\}$ and $\{\alpha_{2s+1,q}\colon\ s \in \mathbb{N}\}$ for each fixed $q \in \mathbb{N}$.

Since by (1) it follows that

$$\langle \alpha_{2s_1,q}, \alpha_{2s_2,q} \rangle = \frac{(-1)^{s_1-s_2}2^{-2q-2}\pi\Gamma(2q+2)}{\Gamma(\frac{3}{2} - s_1 + s_2 + q)\Gamma(\frac{3}{2} + s_1 - s_2 + q)} = \langle \alpha_{2s_1+1,q}, \alpha_{2s_2+1,q} \rangle, \tag{2}$$

and

$$\|\alpha_{2s,q}\|^2 = \frac{2^{-2q-2}\pi\Gamma(2q+2)}{(\Gamma(\frac{3}{2}+q))^2} = \|\alpha_{2s+1,q}\|^2,$$

we just need to consider the orthogonalization of $\{\alpha_{2s,q}\colon\ s \in \mathbb{N}\}$ with q being fixed (see also Remark 3.1).

For convenience, we now change notations. Let $\alpha_1(x) = (x_1 - e_1e_2x_2)^q, \alpha_2(x) = x^2(x_1 - e_1e_2x_2)^q, \ldots, \alpha_n(x) = x^{2n-2}(x_1 - e_1e_2x_2)^q, \ldots$. Hence we put $\alpha_{2s,q}(x) = \alpha_{s+1}(x)$, $s = 0, 1, 2, \ldots$. Then according to the Gram–Schmidt orthogonalization process, the sequence $\{\alpha_n\}_{n=1}^{\infty}$ can be orthogonalized by setting

$$\begin{cases} \beta_1 := \alpha_1, \\ \beta_n := \alpha_n - \sum_{i=1}^{n-1} \frac{\langle \alpha_n, \beta_i \rangle}{\langle \beta_i, \beta_i \rangle} \beta_i, \quad n \geqslant 2. \end{cases} \tag{3}$$

Thus $\{B_n\} := \{\frac{\beta_n}{\|\beta_n\|}\}$ becomes an orthonormal polynomial system.

Remark 3.1. The orthogonalization of $\{\alpha_{2s+1,q}\colon s \in \mathbb{N}\}$ with q fixed, is then given by $\{x\beta_n\}$, since $\langle xf, xg\rangle = \langle f, g\rangle$ as $\bar{x}x = 1$ for $x \in S^2$.

By (2) and straightforward calculations, one obtains

$$\begin{aligned}
&\beta_1 = \alpha_1, \\
&\|\beta_1\|^2 = \frac{2^{-2q-2}\pi\Gamma(2q+2)}{(\Gamma(\frac{3}{2}+q))^2}, \\
&\frac{\langle \alpha_2, \beta_1 \rangle}{\langle \beta_1, \beta_1 \rangle} = -\frac{2q+1}{2q+3}, \\
&\beta_2 = \alpha_2 + \frac{2q+1}{2q+3}\alpha_1, \\
&\|\beta_2\|^2 = \frac{2^{-2q-2}\pi\Gamma(2q+3)}{(\Gamma(\frac{5}{2}+q))^2}, \\
&\frac{\langle \alpha_3, \beta_2 \rangle}{\langle \beta_2, \beta_2 \rangle} = -\frac{2(2q+1)}{(2q+5)}, \\
&\beta_3 = \alpha_3 + \frac{2(2q+1)}{2q+5}\alpha_2 + \frac{2q+1}{2q+5}\alpha_1, \\
&\|\beta_3\|^2 = \frac{2^{-2q-1}\pi\Gamma(2q+4)}{(\Gamma(\frac{7}{2}+q))^2}, \\
&\frac{\langle \alpha_4, \beta_2 \rangle}{\langle \beta_2, \beta_2 \rangle} = \frac{3(2q-1)(2q+1)}{(2q+5)(2q+7)}, \\
&\frac{\langle \alpha_4, \beta_3 \rangle}{\langle \beta_3, \beta_3 \rangle} = -\frac{3(2q+1)}{2q+7}, \\
&\text{etc.}
\end{aligned}$$

In fact, we have the following result.

Theorem 3.1. *Let $\{\beta_n\}_{n=1}^{\infty}$ be defined through (3), then for any $n \in \mathbb{N}$, we have*

$$\begin{aligned}
\beta_n = \alpha_n &+ \binom{n-1}{1}\frac{2q+1}{2q+2n-1}\alpha_{n-1} + \binom{n-1}{2}\frac{(2q+1)(2q+3)}{(2q+2n-1)(2q+2n-3)}\alpha_{n-2} \\
&+ \cdots + \binom{n-1}{k}\frac{(2q+1)(2q+3)\cdots(2q+2k-1)}{(2q+2n-1)(2q+2n-3)\cdots(2q+2n-2k+1)}\alpha_{n-k} \\
&+ \cdots + \binom{n-1}{n-2}\frac{(2q+1)(2q+3)}{(2q+2n-1)(2q+2n-3)}\alpha_2 + \frac{2q+1}{2q+2n-1}\alpha_1
\end{aligned}$$

$$= \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{\Gamma(\frac{1}{2}+k+q)\Gamma(\frac{1}{2}-k+n+q)}{\Gamma(\frac{1}{2}+q)\Gamma(\frac{1}{2}+n+q)} \alpha_{n-k}$$

$$= \sum_{k=1}^{n} \binom{n-1}{k-1} \frac{\Gamma(\frac{1}{2}+k+q)\Gamma(\frac{1}{2}-k+n+q)}{\Gamma(\frac{1}{2}+q)\Gamma(\frac{1}{2}+n+q)} \alpha_{k},$$

where in the last line we have executed the substitution $k' = n - k$ *and used* $\binom{n-1}{n-k} = \binom{n-1}{k-1}$, *and*

$$\|\beta_n\|^2 = \frac{(n-1)!2^{-2q-2}\pi\Gamma(2q+n+1)}{(\Gamma(\frac{1}{2}+n+q))^2}.$$

Moreover, for any $i, j \in \mathbb{N}$ *with* $i > j$, *it holds that*

$$\frac{\langle\alpha_i, \beta_j\rangle}{\langle\beta_j, \beta_j\rangle} = (-1)^{i-j} \binom{i-1}{i-j} \frac{(2q+1)(2q-1)\cdots(2q-2i+2j+3)}{(2q+2i-1)(2q+2i-3)\cdots(2q+2j+1)}$$

$$= (-1)^{i-j} \binom{i-1}{i-j} \frac{\Gamma(\frac{3}{2}+q)\Gamma(\frac{1}{2}+j+q)}{\Gamma(\frac{1}{2}+i+q)\Gamma(\frac{3}{2}-i+j+q)}.$$

To prove these results, we need the following lemmas.

Lemma 3.2. *For all non-negative integers* $s < j$, *we have*

$$\sum_{k=0}^{j} (-1)^k \binom{j}{k} k^s = 0.$$

This lemma has a close connection with the Stirling numbers of the second kind (see e.g. [8]), and it is well-known (see e.g. [6], p. 4, 0.154, formula 3).

Lemma 3.3. *For any positive integer* j, *we have*

$$\sum_{k=1}^{j} (-1)^k \binom{j-1}{k-1} \Gamma\left(\frac{1}{2}+k+q\right) \Gamma\left(\frac{1}{2}-k+j+q\right) \frac{\Gamma(\frac{3}{2}-i+j+q)}{\Gamma(\frac{3}{2}-i+k+q)} \frac{\Gamma(\frac{1}{2}+i+q)}{\Gamma(\frac{3}{2}+i-k+q)}$$

$$= (-1)^j \frac{\Gamma(2q+j+1)\Gamma(\frac{3}{2}+q)\Gamma(\frac{1}{2}+q)}{\Gamma(2q+2)} (i-1)(i-2)\cdots(i-j+1). \tag{4}$$

Proof. We observe that

$$\frac{\Gamma(\frac{3}{2}-i+j+q)}{\Gamma(\frac{3}{2}-i+k+q)} \frac{\Gamma(\frac{1}{2}+i+q)}{\Gamma(\frac{3}{2}+i-k+q)}$$

$$= \left(\frac{3}{2}-i+(j-1)+q\right)\left(\frac{3}{2}-i+(j-2)+q\right)\cdots\left(\frac{3}{2}-i+k+q\right)$$

$$\cdot \left(\frac{1}{2}+(i-1)+q\right)\left(\frac{1}{2}+(i-2)+q\right)\cdots\left(\frac{3}{2}+i-k+q\right)$$

is a polynomial in i of degree $(j-k)+(k-1) = j-1$, and so is the case of the right hand side of (4). Thus, it suffices to show that: (a) The left hand side of (4) has roots $i = 1, 2, \ldots, j-1$. (b) The coefficients of i^{j-1} in both sides are equal, namely,

$$\sum_{k=1}^{j}\binom{j-1}{k-1}\Gamma\left(\frac{1}{2}+k+q\right)\Gamma\left(\frac{1}{2}-k+j+q\right)=\frac{\Gamma(2q+j+1)\Gamma(\frac{3}{2}+q)\Gamma(\frac{1}{2}+q)}{\Gamma(2q+2)}. \tag{5}$$

When i is a positive integer and less than j, we can see that

$$\begin{aligned}&\frac{\Gamma(\frac{1}{2}+k+q)\Gamma(\frac{1}{2}-k+j+q)}{\Gamma(\frac{3}{2}-i+k+q)\Gamma(\frac{3}{2}+i-k+q)}\\ &\quad=\left(\frac{1}{2}+(k-1)+q\right)\left(\frac{1}{2}+(k-2)+q\right)\cdots\left(\frac{3}{2}-i+k+q\right)\\ &\qquad\cdot\left(\frac{1}{2}-k+(j-1)+q\right)\left(\frac{1}{2}-k+(j-2)+q\right)\cdots\left(\frac{3}{2}+i-k+q\right)\end{aligned}$$

is a polynomial in k of degree $(i-1)+(j-i-1)=j-2$. Hence (a) follows immediately by Lemma 3.2. Now we prove (5) by induction. The case $j=1$ is clear. Suppose that (5) is true for some certain j, then for the next integer $j+1$, we get

$$\begin{aligned}&\sum_{k=1}^{j+1}\binom{j}{k-1}\Gamma\left(\frac{1}{2}+k+q\right)\Gamma\left(\frac{1}{2}-k+(j+1)+q\right)\\ &\quad=\sum_{k=1}^{j+1}\left[\binom{j-1}{k-1}+\binom{j-1}{k-2}\right]\Gamma\left(\frac{1}{2}+k+q\right)\Gamma\left(\frac{1}{2}-k+(j+1)+q\right)\\ &\quad=\sum_{k=1}^{j}\binom{j-1}{k-1}\left(\frac{1}{2}-k+j+q\right)\Gamma\left(\frac{1}{2}+k+q\right)\Gamma\left(\frac{1}{2}-k+j+q\right)\\ &\qquad+\sum_{k=1}^{j}\binom{j-1}{k-1}\left(\frac{1}{2}+k+q\right)\Gamma\left(\frac{1}{2}+k+q\right)\Gamma\left(\frac{1}{2}-k+j+q\right)\\ &\quad=(1+j+2q)\sum_{k=1}^{j}\binom{j-1}{k-1}\Gamma\left(\frac{1}{2}+k+q\right)\Gamma\left(\frac{1}{2}-k+j+q\right)\\ &\quad=(1+j+2q)\frac{\Gamma(2q+j+1)\Gamma(\frac{3}{2}+q)\Gamma(\frac{1}{2}+q)}{\Gamma(2q+2)}\\ &\quad=\frac{\Gamma(2q+(j+1)+1)\Gamma(\frac{3}{2}+q)\Gamma(\frac{1}{2}+q)}{\Gamma(2q+2)},\end{aligned}$$

where in the second last line we have used the induction hypothesis. □

Proof of Theorem 3.1. For $n=1,2,\ldots,$ let

$$\beta_n=\sum_{k=1}^{n}\binom{n-1}{k-1}\frac{\Gamma(\frac{1}{2}+k+q)\Gamma(\frac{1}{2}-k+n+q)}{\Gamma(\frac{1}{2}+q)\Gamma(\frac{1}{2}+n+q)}\alpha_k,$$

where $\alpha_k=x^{2k-2}(x_1-e_1e_2x_2)^q$ as before. Then from (2) and (4), we obtain

$$\begin{aligned}\langle\alpha_i,\beta_j\rangle&=\sum_{k=1}^{j}\binom{j-1}{k-1}\frac{\Gamma(\frac{1}{2}+k+q)\Gamma(\frac{1}{2}-k+j+q)}{\Gamma(\frac{1}{2}+q)\Gamma(\frac{1}{2}+j+q)}\frac{(-1)^{i-k}\pi 2^{-2q-2}\Gamma(2q+2)}{\Gamma(\frac{3}{2}-i+k+q)\Gamma(\frac{3}{2}+i-k+q)}\\ &=(-1)^{i-j}(i-1)(i-2)\cdots(i-j+1)\frac{\pi 2^{-2q-2}\Gamma(\frac{3}{2}+q)\Gamma(2q+j+1)}{\Gamma(\frac{1}{2}+i+q)\Gamma(\frac{1}{2}+j+q)\Gamma(\frac{3}{2}-i+j+q)},\end{aligned} \tag{6}$$

which implies that $\langle \alpha_i, \beta_j \rangle = 0$ for $i < j$. Since β_i is a linear combination of $\alpha_1, \ldots, \alpha_i$, we conclude that $\langle \beta_i, \beta_j \rangle = 0$ for $i < j$, and hence it is true for all $i \neq j$, since $\langle \beta_i, \beta_j \rangle = \overline{\langle \beta_j, \beta_i \rangle}$. We also note that

$$\|\beta_j\|^2 = \langle \beta_j, \beta_j \rangle = \langle \alpha_j, \beta_j \rangle = \frac{(j-1)! 2^{-2q-2} \pi \Gamma(2q+j+1)}{(\Gamma(\frac{1}{2}+j+q))^2},$$

where we have used (6).

Therefore, for $i \geqslant j$, we have

$$\frac{\langle \alpha_i, \beta_j \rangle}{\langle \beta_j, \beta_j \rangle} = (-1)^{i-j} \binom{i-1}{j-1} \frac{\Gamma(\frac{3}{2}+q)\Gamma(\frac{1}{2}+j+q)}{\Gamma(\frac{1}{2}+i+q)\Gamma(\frac{3}{2}-i+j+q)}.$$

Moreover, it is clear that

$$\alpha_n = \sum_{i=1}^{n} \left\langle \alpha_n, \frac{\beta_i}{\|\beta_i\|} \right\rangle \frac{\beta_i}{\|\beta_i\|} = \sum_{i=1}^{n} \frac{\langle \alpha_n, \beta_i \rangle}{\langle \beta_i, \beta_i \rangle} \beta_i = \beta_n + \sum_{i=1}^{n-1} \frac{\langle \alpha_n, \beta_i \rangle}{\langle \beta_i, \beta_i \rangle} \beta_i.$$

Consequently,

$$\beta_n = \alpha_n - \sum_{i=1}^{n-1} \frac{\langle \alpha_n, \beta_i \rangle}{\langle \beta_i, \beta_i \rangle} \beta_i,$$

which means that $\{\beta_n\}_{n=1}^{\infty}$ is exactly the outcome of the Gram–Schmidt orthogonalization process of (3). □

4. Pointwise convergence of the series

For any positive integer n and $q \in \mathbb{N}$, we let

$$\beta_{n,q}(x) = \beta_n(x) = \sum_{k=1}^{n} \binom{n-1}{k-1} \frac{\Gamma(\frac{1}{2}+k+q)\Gamma(\frac{1}{2}-k+n+q)}{\Gamma(\frac{1}{2}+q)\Gamma(\frac{1}{2}+n+q)} x^{2k-2}(x_1 - e_1 e_2 x_2)^q,$$

then from the previous section we know that

$$\bigcup_{q=0}^{\infty} \bigcup_{n=1}^{\infty} \left(\{\beta_{n,q}(x)\} \cup \{x\beta_{n,q}(x)\} \right)$$

consists an orthogonal system on the unit sphere S^2 in $\mathbb{R}^3$.

We have the following result.

Proposition 4.1. *Suppose $\{\lambda_n\} \in l^2$ (i.e. $\sum_{n=1}^{\infty} |\lambda_n|^2 < \infty$), then for each $q \in \mathbb{N}$,*

$$\sum_{n=1}^{\infty} \lambda_n \frac{\beta_{n,q}(x)}{\|\beta_{n,q}\|}$$

is convergent in the open unit ball $\{x\colon\ |x| < 1\}$.

Proof. Changing the order of summation, we obtain

$$\sum_{n=1}^{\infty}\lambda_n\frac{\beta_{n,q}(x)}{\|\beta_{n,q}\|}=\sum_{n=1}^{\infty}\lambda_n\sum_{k=1}^{n}\binom{n-1}{k-1}\frac{2^{q+1}\Gamma(\frac{1}{2}+k+q)\Gamma(\frac{1}{2}-k+n+q)}{\sqrt{\pi}\Gamma(\frac{1}{2}+q)\sqrt{(n-1)!\Gamma(2q+n+1)}}x^{2k-2}(x_1-e_1e_2x_2)^q$$
$$=C_q\sum_{k=1}^{\infty}\frac{\Gamma(\frac{1}{2}+k+q)}{(k-1)!}\left(\sum_{n=k}^{\infty}\frac{\Gamma(\frac{1}{2}-k+n+q)}{(n-k)!}\sqrt{\frac{(n-1)!}{\Gamma(2q+n+1)}}\lambda_n\right)x^{2k-2}(x_1-e_1e_2x_2)^q,$$

where $C_q=\frac{2^{q+1}}{\sqrt{\pi}\Gamma(\frac{1}{2}+q)}$. Observe that

$$\left|\sum_{n=k}^{\infty}\frac{\Gamma(\frac{1}{2}-k+n+q)}{(n-k)!}\sqrt{\frac{(n-1)!}{\Gamma(2q+n+1)}}\lambda_n\right|=\left|\sum_{n=0}^{\infty}\frac{\Gamma(\frac{1}{2}+n+q)}{n!}\sqrt{\frac{(n+k-1)!}{\Gamma(2q+n+k+1)}}\lambda_{n+k}\right|$$
$$\leqslant\sum_{n=0}^{\infty}\frac{\Gamma(\frac{1}{2}+n+q)}{n!}\sqrt{\frac{n!}{\Gamma(2q+n+2)}}|\lambda_{n+k}|,$$

since $\frac{(n+k-1)!}{\Gamma(2q+n+k+1)}$ is decreasing in k. Using Cauchy–Schwarz inequality, we find

$$\left|\sum_{n=k}^{\infty}\frac{\Gamma(\frac{1}{2}-k+n+q)}{(n-k)!}\sqrt{\frac{(n-1)!}{\Gamma(2q+n+1)}}\lambda_n\right|\leqslant\sqrt{\sum_{n=0}^{\infty}\frac{(\Gamma(\frac{1}{2}+n+q))^2}{n!\Gamma(2q+n+2)}}\sqrt{\sum_{n=1}^{\infty}|\lambda_n|^2}$$
$$=\sqrt{\lim_{m\to\infty}\sum_{n=0}^{m}\frac{(\Gamma(\frac{1}{2}+n+q))^2}{n!\Gamma(2q+n+2)}}\sqrt{\sum_{n=1}^{\infty}|\lambda_n|^2}$$
$$=\sqrt{\lim_{m\to\infty}\frac{4(\Gamma(\frac{3}{2}+m+q))^2}{(1+2q)^2m!\Gamma(2q+m+2)}}\sqrt{\sum_{n=1}^{\infty}|\lambda_n|^2}$$
$$=\frac{2}{1+2q}\sqrt{\sum_{n=1}^{\infty}|\lambda_n|^2},$$

where we have made use of the following identity

$$\sum_{n=0}^{m}\frac{(\Gamma(\frac{1}{2}+n+q))^2}{n!\Gamma(2q+n+2)}=\frac{4(\Gamma(\frac{3}{2}+m+q))^2}{(1+2q)^2m!\Gamma(2q+m+2)},$$

which can be proved by induction on m, and $\lim_{m\to\infty}\frac{(\Gamma(\frac{3}{2}+m+q))^2}{m!\Gamma(2q+m+2)}=1$. Moreover, we note that

$$\frac{\Gamma(\frac{1}{2}+k+q)}{(k-1)!}\sim k^{q+\frac{1}{2}}\quad(k\to\infty),$$

so the series is always convergent in $|x|<1$. □

Considering also the summation over q, we obtain the following convergence result.

Proposition 4.2. *Let $C = \sum_{q=0}^{\infty}\sum_{n=1}^{\infty}|\lambda_{n,q}|^2 < \infty$, then*

$$\sum_{q=0}^{\infty}\sum_{n=1}^{\infty}\lambda_{n,q}\frac{\beta_{n,q}(x)}{\|\beta_{n,q}\|}$$

is convergent when $|x| < \sqrt{2}-1$.

Proof. Again changing the order of summation, we get

$$\sum_{q=0}^{\infty}\sum_{n=1}^{\infty}\lambda_{n,q}\frac{\beta_{n,q}(x)}{\|\beta_{n,q}\|} = \sum_{q=0}^{\infty}\sum_{k=1}^{\infty}\left(\sum_{n=k}^{\infty}\binom{n-1}{k-1}\frac{2^{q+1}\Gamma(\frac{1}{2}+k+q)\Gamma(\frac{1}{2}-k+n+q)}{\sqrt{\pi}\Gamma(\frac{1}{2}+q)\sqrt{(n-1)!\Gamma(2q+n+1)}}\lambda_{n,q}\right)x^{2k-2}(x_1-e_1e_2x_2)^q.$$

So, similar to the above proof, we find

$$\begin{aligned}\left|\sum_{q=0}^{\infty}\sum_{n=1}^{\infty}\lambda_{n,q}\frac{\beta_{n,q}(x)}{\|\beta_{n,q}\|}\right| &\leqslant \sqrt{C}\sum_{q=0}^{\infty}\sum_{k=1}^{\infty}\left(\frac{2^{q+1}\Gamma(\frac{1}{2}+k+q)}{\sqrt{\pi}\Gamma(\frac{1}{2}+q)(k-1)!}\frac{2}{1+2q}|x|^{2k-2}\right)|x_1-e_1e_2x_2|^q \\ &= 2\sqrt{\frac{C}{\pi}}\sum_{q=0}^{\infty}\frac{1}{\Gamma(\frac{3}{2}+q)}\sum_{k=1}^{\infty}\left(\frac{\Gamma(\frac{1}{2}+k+q)}{(k-1)!}|x|^{2k-2}\right)\bigl(2|x_1-e_1e_2x_2|\bigr)^q. \qquad (7)\end{aligned}$$

Applying Maclaurin series, we have that

$$\begin{aligned}\bigl(1-|x|^2\bigr)^{-(q+\frac{3}{2})} &= \sum_{i=0}^{\infty}\frac{\Gamma(q+\frac{3}{2}+i)}{\Gamma(q+\frac{3}{2})}\frac{|x|^{2i}}{i!} \\ &= \sum_{k=1}^{\infty}\frac{\Gamma(q+\frac{1}{2}+k)}{\Gamma(q+\frac{3}{2})}\frac{|x|^{2k-2}}{(k-1)!}.\end{aligned}$$

Hence, (7) becomes

$$2\sqrt{\frac{C}{\pi}}\sum_{q=0}^{\infty}\bigl(1-|x|^2\bigr)^{-\frac{3}{2}-q}\bigl(2|x_1-e_1e_2x_2|\bigr)^q \leqslant 2\sqrt{\frac{C}{\pi}}\bigl(1-|x|^2\bigr)^{-\frac{3}{2}}\sum_{q=0}^{\infty}\left(\frac{2|x|}{1-|x|^2}\right)^q,$$

which converges if $|x| < 1$ and $\frac{2|x|}{1-|x|^2} < 1$, hence if $|x| < \sqrt{2}-1$. □

Acknowledgments

This work was supported by Macao FDCT 098/2012/A3, research grant of the University of Macau No. UL017/08-Y4/MAT/QT01/FST, and Multi-Year Research Grant of the University of Macau No. MYRG116(Y1-L3)-FST13-QT.

References

[1] S. Bock, K. Gürlebeck, R. Lávička, V. Souček, Gelfand–Tsetlin bases for spherical monogenics in dimension 3, Rev. Mat. Iberoam. 28 (2012) 1165–1192.
[2] F. Brackx, R. Delanghe, F. Sommen, Clifford Analysis, Res. Notes Math., vol. 76, Pitman (Advanced Publishing Program), Boston, MA, 1982.
[3] A.K. Common, F. Sommen, Axial monogenic functions from holomorphic functions, J. Math. Anal. Appl. 179 (1993) 610–629.
[4] R. Delanghe, F. Sommen, V. Souček, Clifford Algebra and Spinor-Valued Functions, Math. Appl., vol. 53, Kluwer Academic Publishers Group, Dordrecht, 1992.

[5] J. Gilbert, M. Murray, Clifford Algebras and Dirac Operators in Harmonic Analysis, Cambridge University Press, Cambridge, 1991.
[6] I.S. Gradshteyn, I.M. Ryzhik, Table of Integrals, Series and Products, Academic Press, New York–London–Toronto–Sydney–San Francisco, 1980.
[7] K. Gürlebeck, W. Sprössig, Quaternionic and Clifford Calculus for Physicists and Engineers, Math. Methods Pract., Wiley, Chichester, 1997.
[8] C. Jordan, Calculus of Finite Differences, third edition, Chelsea Publishing Company, New York, 1965.
[9] W. Magnus, F. Oberhettinger, R.P. Soni, Formulas and Theorems for the Special Functions of Mathematical Physics, Springer-Verlag, Berlin–Heidelberg–New York, 1966.
[10] F. Sommen, A product and an exponential function in hypercomplex function theory, Appl. Anal. 12 (1981) 13–26.
[11] F. Sommen, Special functions in Clifford analysis and axial symmetry, J. Math. Anal. Appl. 130 (1988) 110–133.