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New aspects of Beurling–Lax shift invariant subspaces

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ABSTRACT

In terms of forward and backward shift invariant subspaces, we characterize functions in Hardy spaces, or, analytic signals in the terminology of signal analysis, through multiplications between analytic and conjugate analytic signals. As applications, we give some necessary and sufficient conditions for solutions of the Bedrosian equation $H(fg) = f(Hg)$ when f or g is a bandlimited signal. We also solve the band preserving problem by means of the shift invariant subspace method, which establishes some necessary and sufficient conditions on the functions f that make fg have bandwidth within that of the function g.

1. Introduction

There are two mathematical analysis problems that have long been interested in physical practice and signal processing. The first is: under what conditions on f and g does the equality $H(fg) = fHg$ hold, where H is the Hilbert transformation defined by

$$H(g)(x) := \frac{1}{\pi} \lim_{\varepsilon \to 0^+} \int_{|t-x|>\varepsilon} \frac{g(t)}{x-t}\, dt. \tag{1.1}$$

The second is phrased as a band preserving problem: assume that $g \in L^2(\mathbb{R})$ is bandlimited, and has its band $[A, B]$, viz., $\operatorname{supp} \widehat{g} \subseteq [A, B]$. Then, under what conditions on f does the relation $\operatorname{supp} \widehat{fg} \subseteq [A, B]$ hold? Here the notation $\widehat{g}$ stands for the Fourier transform of g, defined by

$$\widehat{g}(\omega) = \int_{-\infty}^{\infty} g(x) e^{-i\omega x} dx, \tag{1.2}$$

and $\operatorname{supp} \widehat{g}$ is the support of $\widehat{g}$. The first problem amounts to generalize the classical results of Bedrosian given in [1]: For $f, g \in L^2(\mathbb{R})$, if $\operatorname{supp} \widehat{f} \subseteq [-A, A]$ and $\operatorname{supp} \widehat{g} \subseteq \mathbb{R} \setminus [-A, A]$ for a positive number A; or, alternatively, if $\operatorname{supp} \widehat{f} \subseteq [0, \infty)$ and $\operatorname{supp} \widehat{g} \subseteq [0, \infty)$, then $H(fg) = fHg$. The recent interest on the Bedrosian equation was motivated by the study of mono-components that, by the definition, are analytic signals with non-negative phase derivatives, the latter being defined as instantaneous frequencies [2–4]. Some interesting and useful results for the Bedrosian equation are established in [5–11]. About the second problem, if $g \in L^2(\mathbb{R})$ has its band $[A, B]$, by the Paley–Wiener theorem, it is known that $fg \in L^2(\mathbb{R})$ has the same band as that of g if and only if f is a quotient of two entire functions of the exponential type. A number of authors

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devoted to such construction of bandlimited signals (the band preserving problem) and then apply the results to the phase retrieval problem by the methodology in representing an entire function by Hardamard's formula in an infinite product [12–14]. However, almost all the existing constructive results of the band preserving problem are focused on $L^2(\mathbb{R})$ functions. Moreover, we do not know whether f can be totally described by the zero information of the Laplace transform of g.

In the paper, we will solve them by the method of shift invariant subspaces. An interesting thing will be found that the solutions of both problems are related to the conditions on analytic signals f and on conjugate analytic signals g such that their products fg are again analytic signals or conjugate analytic signals. In Section 2, we will characterize those conditions on f and g in terms of the forward and backward shift invariant subspaces. In Section 3, we will first make some summaries and self-appraisal to the existing results for the Bedrosian identity $H(fg) = fHg$. As applications, we will give a new characterization by using forward and backward shift invariant subspaces for the solutions of the Bedrosian equation $H(fg) = fHg$ when f or g is a bandlimited signal. Furthermore, we will establish necessary and sufficient conditions in terms of the same invariant subspaces on functions f that make the band of fg to be within that of g. It is shown that such functions f form the linear space spanned by rational functions whose zeros are those of the Laplace transform of g. The backward shift and forward shift approach to the problems is a new idea that has advantage over the Hardamard's formula method for its explicitness and computability.

2. Analytic signals as products of analytic and conjugate analytic signals

It is well-known that the Hilbert transform Hg given in (1.1) is well-defined for $g \in L^p(\mathbb{R}), 1 \leqslant p < \infty$ and $Hg \in L^p(\mathbb{R})$ for $g \in L^p(\mathbb{R}), 1 < p < \infty$. But for $g \in L^\infty(\mathbb{R}), Hg$ given in 1.1) may not exist. In order to also discuss Bedrosian identity $H(fg) = fHg$ for $g \in L^\infty(\mathbb{R})$, an immediate problem we encounter is to find an appropriate definition of the Hilbert transform on $L^\infty(\mathbb{R})$. Without loss of generality, for $g \in L^\infty(\mathbb{R})$, the Hilbert transformation is modified as [15]

$$(\widetilde{H}g)(x) = \frac{x+i}{\pi} \lim_{\epsilon \to 0+} \int_{|x-t|>\epsilon} \frac{g(t)}{(x-t)(t+i)}\, dt. \tag{2.3}$$

If $g \in L^p(\mathbb{R}), 1 \leqslant p < \infty$, the modified Hilbert transform $\widetilde{H}g$ given in (2.3) is essentially the same as the classical Hilbert transform given in (1.1) as $\widetilde{H}g = Hg + A_g$, where A_g is a constant of the form

$$A_g = \int_{-\infty}^{\infty} \frac{g(t)}{t+i} dt.$$

The Fourier transform given in (1.2) is well-defined for functions in $L^p(\mathbb{R})$ for $1 \leqslant p \leq 2$. But for $p > 2$, it can be shown that there are functions in $L^p(\mathbb{R})$ such that the Fourier transform given in (1.2) is not a well-defined function [16]. In order to generalize the study of band preserving problem from $L^2(\mathbb{R})$ spaces to $L^p(\mathbb{R})$ spaces, $1 \leqslant p \leqslant \infty$, the tempered distribution should be introduced. The tempered distribution as Fourier transformation is defined through

$$\langle \widehat{T}, \varphi \rangle = \langle T, \widehat{\varphi} \rangle, \quad \varphi \in \mathbb{S} \quad \text{the Schwartz class}. \tag{2.4}$$

The Fourier transform given in the distributional sense coincides with the traditional definition of Fourier transformation given in (1.2) for $1 \leqslant p \leq 2$. The support of a distribution T is defined as the complement of the largest open set on which T vanishes, where T is said to vanish on U if $\langle T, \phi \rangle = 0$ for any test function ϕ whose support is contained in U.

Let $\mathrm{L}^p_T(\mathbb{R}) := \{f \in L^p(\mathbb{R}) | Tf \in L^p(\mathbb{R})\}$ for $1 \leqslant p \leqslant \infty$, where T is the Hilbert transform interpreted as

$$Tf = \begin{cases} Hf, & 1 \leqslant p < \infty; \\ \widetilde{H}f, & p = \infty. \end{cases} \tag{2.5}$$

Obviously, when $1 < p < \infty$, the space $L^p_T(\mathbb{R})$ coincides with $L^p(\mathbb{R})$. For $p = 1$ and $p = \infty$, the spaces $L^p_T(\mathbb{R})$ are respectively proper subspaces of $L^p(\mathbb{R})$. For any function $f(x) \in L^p_T(\mathbb{R})$, we have the Hardy space decomposition

$$f(x) = \frac{1}{2}[f_+(x) + f_-(x)] = \frac{1}{2}[(f + iTf) + (f - iTf)]. \tag{2.6}$$

The signals $f_+ := f + iTf$ and $f_- := f - iTf$ are respectively called the *analytic signal* and the *conjugate analytic signal* associated with f. Let

$$\mathrm{H}^p(\mathbb{R}) := \{f : \quad f = u + iT(u) \quad \text{for certain function} \quad u \in L^p_T(\mathbb{R})\}$$

be the set of all analytic signals, and

$$\overline{\mathrm{H}^p(\mathbb{R})} := \{f : \quad f = u - iH(u) \quad \text{for certain function} \quad u \in L^p_T(\mathbb{R})\}$$

the set of all conjugate analytic signals. It is shown that the space $\mathrm{H}^p(\mathbb{R})$ is identical to the Banach subspace of $L^p(\mathbb{R})$ of the functions of non-negative spectrum [17], that is, $\mathrm{H}^p(\mathbb{R}) = \mathrm{FH}^p(\mathbb{R}^+) := \{f \in L^p(\mathbb{R}) | \operatorname{supp} \widehat{f} \subseteq [0, \infty)\}$, where $\mathbb{R}^+ := [0, \infty)$ and the support of $\widehat{f}$ is defined in the distributional sense. Alternately, the equivalent operation of (2.6) in the frequency domain is

$$\widehat{f}(\omega) = u(\omega)\widehat{f}(\omega) + u(-\omega)\widehat{f}(\omega),$$

where $u(\omega)$ is the step function given by

$$u(\omega) = \begin{cases} 1, & \omega \geqslant 0; \\ 0, & \omega < 0. \end{cases}$$

Moreover, there is a one-to-one correspondence between analytic signals in $\mathrm{H}^p(\mathbb{R})$ and functions in the Hardy space $\mathrm{H}^p(\mathbb{C}^+)$ [18], where $\mathrm{H}^p(\mathbb{C}^+)$ is the class of all holomorphic functions $F(z)$ on the upper half plane $\mathbb{C}^+ := \{z | z = x + iy, y > 0\}$ satisfying

$$\|F\|_{\mathrm{H}^p(\mathbb{C}^+)} := \begin{cases} \sup\limits_{y>0} \left\{ \int_{-\infty}^{+\infty} |F(x+iy)|^p \, dx \right\}^{1/p} < \infty, & \text{for} \quad 1 \leqslant p < \infty; \\ \sup\limits_{z\in\mathbb{C}^+} |F(z)| < \infty, & \text{for} \quad p = \infty. \end{cases}$$

For functions $f \in \mathrm{H}^p(\mathbb{R})$, we have [17]:

Lemma 2.1. *Let $1 \leqslant p \leqslant \infty$. Then $f \in \mathrm{H}^p(\mathbb{R})$ if and only if one of the following conditions holds:*

(1) $T(f) = -if$.
(2) $f \in \mathrm{FH}^p(\mathbb{R}^+)$.
(3) *f is the non-tangential boundary value of a function $F(z) \in H^p(\mathbb{C}^+)$, where, for $z \in \mathbb{C}^+$, $F(z)$ can be represented by*

$$F(z) = \begin{cases} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(t)}{t-z} dt, & 1 \leqslant p < \infty; \\ \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{z+i}{(t-z)(t+i)} f(t) dt, & p = \infty. \end{cases}$$

In what follows, let $1 \leqslant p, q, r \leqslant \infty$ such that $r^{-1} = p^{-1} + q^{-1}$. By the Hőlder inequality, it is easy to verify that the product of two analytic signals is an analytic signal and the product of two conjugate analytic signals is a conjugate analytic signal, as stated below.

Lemma 2.2. *If $f \in \mathrm{H}^p(\mathbb{R})$ and $g \in \mathrm{H}^q(\mathbb{R})$ are two nonzero analytic signals, then $fg \in \mathrm{H}^r(\mathbb{R})$. If $f \in \overline{\mathrm{H}^p(\mathbb{R})}$ and $g \in \overline{\mathrm{H}^q(\mathbb{R})}$ are nonzero conjugate analytic signals, then $fg \in \overline{\mathrm{H}^r(\mathbb{R})}$.*

Naturally, we would encounter the question in what circumstance the product of an analytic signal and a conjugate analytic signal is still analytic or conjugate analytic. The study of this problem is on its own interest, as well as of significant applications. Its solutions are related to characterizations of the solutions of the Bedrosian equation and of the band preserving problem. It is related to the well-known Nevanlinna decomposition theorem for $F \in \mathrm{H}^p(\mathbb{C}^+)$ [19]. Any function $F(z) \in \mathrm{H}^p(\mathbb{C}^+)$ can be decomposed as $F(z) = O_F(z)I_F(z),\ z \in \mathbb{C}^+$, where $O_F(z)$, the outer factor of F, is of the form

$$O_F(z) = \exp\left\{ \frac{i}{\pi} \int_{-\infty}^{\infty} \frac{1+tz}{z-t} \frac{\ln|F(t)|}{1+t^2} \, dt \right\}, \tag{2.7}$$

and $I_F(z)$ is the inner factor of F, that is, $I_F(z) \in H^\infty(\mathbb{C}^+)$ and $|I_F(x)| = 1$ for almost all $x \in \mathbb{R}$. Factorizing $I_F(z)$ canonically, we get $I_F(z) = e^{i(az+b)} B_F(z) S_F(z)$ for $z \in \mathbb{C}^+$, where a is a non-negative constant, b is a real constant, $B_F(z)$ is a Blaschke product and $S_F(z)$ is a singular inner function. Let E be the set of all different zeros of $F(z)$ on the upper half plane and $m(\lambda)$ the multiplicity of λ at $F(\lambda) = 0$. Then the Blaschke product B_F is of the form

$$B_F(z) = B_{E,m}(z) = \prod_{\lambda\in E} \left(\frac{|\lambda^2+1|}{\lambda^2+1} \cdot \frac{z-\lambda}{z-\bar{\lambda}} \right)^{m(\lambda)}, \tag{2.8}$$

where the pair (E, m) satisfies the Blaschke condition

$$\sum_{\lambda\in E} \frac{m(\lambda)\mathrm{Im}(\lambda)}{1+|\lambda|^2} < \infty.$$

In (2.8) we are with the convention that $|\lambda^2 + 1|/(\lambda^2 + 1)$ takes value 1 when $\lambda = i$. The singular inner function $S_F(z)$ is of the form

$$S_F(z) := \exp\left\{ \frac{i}{\pi} \int_{-\infty}^{\infty} \frac{1+zt}{t-z} \, d\mu(t) \right\},$$

where $\mu(t)$ is a real, bounded, increasing function with derivative $\mu'(t) = 0$ almost everywhere on $\mathbb{R}$. A fundamental property of Hardy space functions is that they have non-tangential boundary limits almost everywhere. Hence, for almost all $x \in \mathbb{R}$, the corresponding decomposition of nonzero function $f \in \mathrm{H}^p(\mathbb{R})$ is

$$f(x) = \lim_{y\to 0^+} F(x+iy) = O_f(x)I_f(x) = e^{i(ax+b)} O_f(x)B_f(x)S_f(x). \tag{2.9}$$

Note that, apart from the self-explanatory notation $F(x)$ as non-tangential boundary value of $F(z)$, we also denote the non-tangential boundary value function of $F \in \mathrm{H}^p(\mathbb{C}^+)$ by ∂F, and, for a function $f \in \mathrm{H}^p(\mathbb{R})$, denote the corresponding function in $\mathrm{H}^p(\mathbb{C}^+)$ by $\partial^{-1} f$.

Theorem 2.3. *Let $f \in \overline{\mathrm{H}^p(\mathbb{R})}$ and $g \in \mathrm{H}^q(\mathbb{R})$ be non-zero functions. Then the following statements are equivalent:*

(1) $fg \in \mathrm{H}^r(\mathbb{R})$.
(2) $\overline{f} \in \mathrm{H}^p(\mathbb{R}) \bigcap I_g \overline{\mathrm{H}^p(\mathbb{R})}$, *where I_g is the inner function of g.*
(3) $\frac{\overline{O_{\overline{f}}}}{O_{\overline{f}}} I_{\overline{f}} = \frac{\varphi_1}{\varphi_2}$ *and $g \in \varphi_1 \mathrm{H}^q(\mathbb{R})$, where $\varphi_1 = B_1 S_1$ and $\varphi_2 = B_2 S_2$ are two co-prime inner functions, that is, $B_1(z)$ and $B_2(z)$ have no common zeros, the singular measures μ_1 and μ_2 of S_1 and S_2 have no common minorant.*

Proof. We first assume (1) holds, that is, we have nonzero functions $f \in \overline{\mathrm{H}^p(\mathbb{R})}, g \in \mathrm{H}^q(\mathbb{R})$ and $fg \in \mathrm{H}^r(\mathbb{R})$. Then $\partial^{-1}(fg)(z) \in \mathrm{H}^r(\mathbb{C}^+)$ and the outer function of $\partial^{-1}(fg)(z)$ is represented by

$$O_{\partial^{-1}(fg)}(z) = \exp\left\{\frac{i}{\pi}\int_{-\infty}^{+\infty} \frac{1+zt}{(z-t)(1+t^2)} \ln|f(t)g(t)|dt\right\}.$$

By $\ln|fg| = \ln|g| + \ln|\overline{f}|$, we have $O_{\partial^{-1}(fg)} = O_{\partial^{-1}g} O_{\partial^{-1}\overline{f}}$. Hence, $O_{(fg)} = O_g O_{\overline{f}}$. By the factorization theorem, we therefore obtain

$$O_g I_g \overline{O_{\overline{f}} I_{\overline{f}}} = fg = O_{(fg)} I_{(fg)} = O_g O_{\overline{f}} I_{fg} = g \overline{I_g} O_{\overline{f}} I_{fg}. \tag{2.10}$$

From equation (2.10) and the assumptions g and f are nonzero functions, we have

$$f = \overline{I_g} O_{\overline{f}} I_{fg} \in \overline{I_g} \mathrm{H}^p(\mathbb{R}).$$

Hence, $\overline{f} \in \mathrm{H}^p(\mathbb{R}) \bigcap I_g \overline{\mathrm{H}^p(\mathbb{R})}$, proving that (1) implies (2). The relation (2.10) also shows

$$\frac{\overline{O_{\overline{f}}}}{O_{\overline{f}}} I_{\overline{f}} = \frac{I_g}{I_{fg}}, \tag{2.11}$$

that further implies

$$I_g \frac{\overline{O_{\overline{f}}}}{O_{\overline{f}}} \overline{I_{\overline{f}}} = I_{fg} \in \mathrm{H}^\infty(\mathbb{R}). \tag{2.12}$$

The relation (2.11) shows that $\frac{\overline{O_{\overline{f}}}}{O_{\overline{f}}} I_{\overline{f}}$ is the quotient of two inner functions I_f and I_{fg}. Eliminating the greatest common divisor of the two inner functions I_f and I_{fg}, there exist two co-prime inner functions $\varphi_1 = B_1 S_1$ and $\varphi_2 = B_2 S_2$ such that

$$\frac{\overline{O_{\overline{f}}}}{O_{\overline{f}}} I_{\overline{f}} = \frac{\varphi_1}{\varphi_2}. \tag{2.13}$$

From the fact that

$$I_g \frac{\varphi_2}{\varphi_1} = I_g \frac{\overline{O_{\overline{f}}}}{O_{\overline{f}}} \overline{I_{\overline{f}}} \in \mathrm{H}^\infty(\mathbb{R}),$$

we obtain that I_g must have a non-trivial divisor φ_1. Hence, we have $g = O_g I_g \in \varphi_1 \mathrm{H}^q(\mathbb{R})$ and $\frac{\overline{O_{\overline{f}}}}{O_{\overline{f}}} I_{\overline{f}} = \frac{\varphi_1}{\varphi_2}$, where φ_1 and φ_2 are two co-prime inner functions. This shows that (1) implies (3).

Now we show that (2) implies (1). If $\overline{f} \in \mathrm{H}^p(\mathbb{R}) \bigcap I_g \overline{\mathrm{H}^p(\mathbb{R})}$, then $\overline{f} \in \mathrm{H}^p(\mathbb{R})$ and $f = \overline{I_g} h$ for some $h \in \mathrm{H}^p(\mathbb{R})$. Since $g \in \mathrm{H}^q(\mathbb{R})$, we have $g = O_g I_g \in \mathrm{H}^q(\mathbb{R})$ and $O_g \in \mathrm{H}^q(\mathbb{R})$. Thus $fg = \overline{I_g} h O_g I_g = h O_g$. By the Hőlder inequality, it follows that $fg \in \mathrm{H}^r(\mathbb{R})$.

Now we show that (3) implies (1). If $g \in \varphi_1 \mathrm{H}^q(\mathbb{R})$ and $\frac{\overline{O_{\overline{f}}}}{O_{\overline{f}}} I_{\overline{f}} = \frac{\varphi_1}{\varphi_2}$, then there exists $h \in \mathrm{H}^q(\mathbb{R})$ such that $g = \varphi_1 h \in \mathrm{H}^q(\mathbb{R})$ and $\varphi_1 = \varphi_2 \frac{\overline{O_{\overline{f}}}}{O_{\overline{f}}} I_{\overline{f}}$. Since $\overline{f} \in \mathrm{H}^p(\mathbb{R})$, we have $f = \overline{\overline{f}} = \overline{O_{\overline{f}} I_{\overline{f}}}$. Therefore

$$fg = \varphi_1 h f = h \varphi_2 \frac{\overline{O_{\overline{f}}}}{O_{\overline{f}}} I_{\overline{f}} \overline{O_{\overline{f}} I_{\overline{f}}} = h \varphi_2 \overline{O_{\overline{f}}}.$$

By the Hőlder inequality, we conclude that $fg \in \mathrm{H}^r(\mathbb{R})$. The proof is complete. $\square$

The equivalence relation between (1) and (2) was first proved in [11]. Here we cite the proof again for the self-containing purpose, as well as for the convenience of proving their equivalence with (3). Symmetrically, or as a corollary of Theorem 2.3, we have an analogous result for the conjugate analytic case.

Corollary 2.4. *Let $f \in \overline{\mathrm{H}^p(\mathbb{R})}$ and $g \in \mathrm{H}^q(\mathbb{R})$ be non-zero functions. Then the following statements are equivalent:*

(1) $fg \in \overline{\mathrm{H}^r(\mathbb{R})}$.
(2) $g \in \mathrm{H}^q(\mathbb{R}) \cap I_{\bar f}\overline{\mathrm{H}^q(\mathbb{R})}$, *where $I_{\bar f}$ is the inner function of $\bar f$.*
(3) $\frac{O_g}{\overline{O_g}} I_g = \frac{\varphi_1}{\varphi_2}$ *and $\bar f \in \varphi_1 \mathrm{H}^p(\mathbb{R})$, where $\varphi_1 = B_1 S_1$ and $\varphi_2 = B_2 S_2$ are two co-prime inner functions.*

For a given nonzero function $g \in \mathrm{H}^q(\mathbb{R})$, we associate it with the spaces

$$X_g^p := \{\bar f : \ \bar f \in \mathrm{H}^p(\mathbb{R}) \quad \textit{and} \quad fg \in \overline{\mathrm{H}^r(\mathbb{R})}\}, \tag{2.14}$$

and

$$Y_g^p := \{\bar f : \ \bar f \in \mathrm{H}^p(\mathbb{R}) \quad \textit{and} \quad fg \in \mathrm{H}^r(\mathbb{R})\}. \tag{2.15}$$

We note that the induced spaces by g, X_g^p and Y_g^p, consist of Hardy space functions and hence are subspaces of $\mathrm{H}^p(\mathbb{R})$. By Corollary 2.4, we learn that if $\frac{O_g}{\overline{O_g}} I_g$ is not a quotient of two co-prime inner functions (for example, if $O_g(z)$ is an outer function with essential singularities in the lower half plane), then X_g^p is an empty set. If $\frac{O_g}{\overline{O_g}} I_g$ is the quotient of two co-prime inner functions φ_1 and φ_2, then X_g^p is $\varphi_1 \mathrm{H}^p(\mathbb{R})$. The space $\varphi_1 \mathrm{H}^p(\mathbb{R})$, usually referred as a forward shift invariant subspace, is invariant under the semigroup $\{S_a : a > 0\}$ of forward shift operators

$$(S_a h)(t) = e^{iat} h(t), \quad h \in H^p(\mathbb{R}).$$

By Theorem 2.3, the space Y_g^p is $\mathrm{H}^p(\mathbb{R}) \cap I_g \overline{\mathrm{H}^p(\mathbb{R})}$, where I_g is the inner function of g. When $g \in H^q(\mathbb{R})$ is an outer function, that is, $I_g = e^{ib}, b \in \mathbb{R}$ being trivial, the space Y_g^p is an empty set for $1 \leqslant p < \infty$. When $g \in H^q(\mathbb{R})$ is not an outer function, that is, I_g is not a constant, then the space Y_g^p is not empty, and is an invariant subspace of $\mathrm{H}^p(\mathbb{R})$ under the semigroup $\{S_a^* : a > 0\}$ of backward shift operators

$$(S_a^* h)(t) = e^{-iat} h(t), \quad h \in H^p(\mathbb{R}).$$

For $p^{-1} + p'^{-1} = 1$, the backward shift operator S_a^* is the adjoint of the forward shift operator S_a under the conjugate pairing between $\mathrm{H}^p(\mathbb{R})$ and $\mathrm{H}^{p'}(\mathbb{R})$,

$$\langle S_a f, g\rangle = \langle f, S_a^* g\rangle, \quad f \in \mathrm{H}^p(\mathbb{R}), \quad g \in \mathrm{H}^{p'}(\mathbb{R}).$$

The study of forward and backward shift invariant subspaces has a long history, some useful and interesting results can be found in [20–24]. Both forward shift invariant and backward shift invariant subspaces are expressible in terms of inner functions. The most classical characterization of forward shift invariant subspaces is referred to Beurling–Lax Theorem [20]. When the related inner function is just a Blaschke product, the corresponding backward shift invariant subspace can be explicitly characterized (also see [22]).

Lemma 2.5. *Let $1 \leqslant p \leqslant \infty$, and B_g be the Blaschke product given in (2.8). Then $\mathrm{H}^p(\mathbb{R}) \bigcap B_g \overline{\mathrm{H}^p(\mathbb{R})}$ is the closure in $L^p(\mathbb{R})$ of the sets of all rational functions of which each has its poles as a subset of those of B_g with multiplicities not exceeding those of the corresponding poles of B_g.*

By imposing the restriction that $\hat g$ be compactly supported, that is, g belongs to *bandlimited Hardy space* $\mathrm{FH}^q[A, B]$,

$$\mathrm{FH}^q[A,B] := \{g \in L^q(\mathbb{R}) : \ \mathrm{supp}\, \hat g \subseteq [A,B]\},$$

we will first give a characterization for the forward shift invariant subspace X_g^p described in Corollary 2.4 induced by $g \in \mathrm{FH}^q[A,B]$ for $0 \leqslant A < B$.

Corollary 2.6. *Let $g \in \mathrm{FH}^q[A,B]$ be a nonzero function. If $B \in \mathrm{supp}\, \hat g$ and $0 \leqslant A < B$. Then the forward invariant subspace X_g^p is $e^{iBx}\mathrm{H}^p(\mathbb{R})$.*

Proof. Let $h(x) := e^{iBx}\overline{g(x)}$. Then $\hat h(\omega) = \overline{\hat g(B - \omega)}$. Since $g \in \mathrm{FH}^q[A,B]$ is a nonzero function with $B \in \mathrm{supp}\, \hat g$, we have $h \in \mathrm{FH}^q[0, B - A] \subset \mathrm{H}^q(\mathbb{R})$ with $0 \in \mathrm{supp}\, \hat h$. By the factorization theorem, we have $O_h = O_g$ and

$$\frac{O_g(x)}{\overline{O_g(x)}} I_g(x) = \frac{O_g(x)}{\overline{g(x)}} = \frac{e^{iBx} O_g(x)}{e^{iBx}\overline{g(x)}} = \frac{e^{iBx} O_g(x)}{O_h(x) I_h(x)} = \frac{e^{iBx}}{I_h}. \tag{2.16}$$

Since $0 \in \mathrm{supp}\, \hat h$, we obtain that e^{iBx} and I_h are two co-prime inner functions. Hence, $\varphi_1(x) = e^{iBx}$ and $X_g^p = e^{iBx}\mathrm{H}^p(\mathbb{R})$. □

Let $g \in \mathrm{FH}^q[A,B]$ be a nonzero function, where $0 \leqslant A < B$. By the Paley–Wiener–Schwarz theorem [25], we learn that $g \in \mathrm{FH}^q[A,B]$ if and only if g is the non-tangential boundary limit of an entire function $G(z)$ of exponential type (not exceeding B) that belongs to $\mathrm{H}^q(\mathbb{C}^+)$, with the representation

$$G(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} g(x)\frac{e^{iB(z-x)} - e^{iA(z-x)}}{z-x}dx. \tag{2.17}$$

The following lemma is to show that the singular inner factor of $g \in \mathrm{FH}^q[A,B]$ is a constant in such case.

Lemma 2.7. *Let $g(x) \in \mathrm{FH}^q[A,B]$ be a nonzero function, where $0 \leqslant A < B$. Then the inner function of g is $e^{i(ax+b)}B_g(x)$, where $0 \leqslant a \leqslant B, b$ is a real constant, $B_g(x)$ is the boundary Blaschke product formed by the zeros of $G(z) := (\partial^{-1}g)(z)$ defined by (2.17) in the upper half plane and $|B_g(x)| = 1$ on the whole $\mathbb{R}$.*

Proof. The following proof is an adaptation of [19, Chapter II, Theorem 6.3]. Since the nonzero function $g \in \mathrm{FH}^q[A,B] \subset \mathrm{H}^p(\mathbb{R})$ for $0 \leqslant A < B$, $G(z) := (\partial^{-1}g) \in \mathrm{H}^q(\mathbb{C}^+)$ is an entire function of exponential type and $G(z) = e^{i(az+b)}O_G(z)B_G(z)S_G(z)$ for $z \in \mathbb{C}^+$, where $0 \leqslant a \leqslant B, b$ is a real constant, $O_G(z)$ is the outer function of $G(z), B_G(z)$ is the Blaschke product of $G(z)$ and $S_G(z)$ is the singular inner function of $G(z)$. In what follows, we will show that $B_G(z)$ is holomorphic across $\mathbb{R}$ and $S_G(z) \equiv 1$.

If the zeros of $B_G(z)$ had an accumulation point z_0 on $\mathbb{R}$, then the zeros of $G(z)$ would have an accumulation point z_0 on $\mathbb{R}$ such that $G(z_0) = 0$. This is impossible since $G(z)$ is non-trivial and holomorphic across $\mathbb{R}$. Consequently, $B_G(z)$ is holomorphic across $\mathbb{R}$ and $|B_G(x)| = 1$ on $\mathbb{R}$.

Below, we will show that $S_G(z) \equiv 1$. Let μ be the measure determining S_G. If μ had a point charge at x, then every derivative $G^{(n)}(z)$ of $G(z)$ satisfies $\lim_{y\to 0^+} G^{(n)}(x+iy) = 0$. This is impossible since $G(z)$ is analytic across $\mathbb{R}$ and $G(z)$ is a nonzero function. Thus $\mu(\{x\}) = 0$ for all $x \in \mathbb{R}$. Next suppose that $\mu(K) > 0$ for some compact subset K of $\mathbb{R}$. Since μ is singular, for μ-almost all x, we have $\lim_{y\to 0^+}|S_G(x+iy)| = 0$, and hence $\lim_{y\to 0+}|G(x+iy)| = 0$ (Theorem 6.2, [19]). Therefore, the holomorphic function $G(z)$ has zero values on infinitely many points of K. The compactness of K implies that G has to be a zero function, being contradictory with the assumption that $G(z)$ is nonzero. Therefore $\mu(K) = 0$. Hence, $\mathrm{supp}\mu = \emptyset$ and $S_G(z) \equiv 1$. This completes the proof. □

Denote by E the set of all different zeros of $G(z) := (\partial^{-1}g)(z)$ in the upper half plane and denote by $m(\lambda)$ the multiplicity of the zero λ of $G(z)$. With Lemma 2.5, we have a characterization for the backward shift invariant subspace Y_g^p when g is a band-limited signal.

Corollary 2.8. *Suppose that $g(x)$ is a nonzero function in $\mathrm{FH}^q[A,B], 0 \leqslant A < B$.Then the backward shift invariant subspace Y_g^p induced by g as given in (2.15) is identical with*

$$\mathrm{H}^p(\mathbb{R})\bigcap e^{iax}B_g\overline{\mathrm{H}^p(\mathbb{R})} = \mathrm{FH}^p[0,a]\bigoplus e^{iax}\mathcal{R}_{E,m},$$

where $0 \leqslant a \leqslant B$ and B_g is the Blaschke product formed by the points in E and the corresponding multiplicity function $m(\lambda)$, and $\mathcal{R}_{E,m}$ is the closed subspace

$$\mathcal{R}_{E,m} = \overline{\mathrm{span}}^p\left\{\frac{1}{(x-\overline{\lambda})^j},\quad \lambda \in E,\quad j = 1,\ldots,m(\lambda)\right\}$$

of the $L^p(\mathbb{R})$space.

Proof. By Lemma 2.7, we get that $Y_g^p = \mathrm{H}^p(\mathbb{R})\bigcap e^{iax}B_g(x)\overline{\mathrm{H}^p(\mathbb{R})}$, where $0 \leqslant a \leqslant B, B_g$ is the Blaschke product formed by the zeros together with their multiplicities of $G(z) := (\partial^{-1}g)(z)$ in the upper half plane.

Suppose that $f(x) \in \mathrm{H}^p(\mathbb{R})\bigcap e^{iax}B_g(x)\overline{\mathrm{H}^p(\mathbb{R})}$. Then $f(x) \in \mathrm{H}^p(\mathbb{R})$ and $f(x) \in e^{iax}B_g(x)\overline{\mathrm{H}^p(\mathbb{R})}$. Since $\mathrm{supp}\widehat{f} \subseteq [0,\infty)$, we have $\widehat{f}(\omega) = \widehat{f}(\omega)\chi_{[0,a]} + \widehat{f}(\omega)\chi_{[a,\infty]} = \widehat{f_1}(\omega) + \widehat{f_2}(\omega)$, where $\chi_{[a,b]}$ is the characteristic function of $[a,b]$. By the inverse Fourier transform in the distributional sense, f can be written as $f = f_1 + f_2$, where $f_1 \in \mathrm{FH}^p[0,a]$ and $f_2 \in \mathrm{FH}^p[a,\infty)$. Hence, we have $f_1(x)e^{-iax} \in \overline{\mathrm{H}^p(\mathbb{R})}$ and $f_2(x)e^{-iax} \in \mathrm{H}^p(\mathbb{R})$. We note that

$$f_1(x)e^{-iax} \in \overline{\mathrm{H}^p(\mathbb{R})} = B_g(x)\overline{B_g(x)\mathrm{H}^p(\mathbb{R})} \subset B_g(x)\overline{\mathrm{H}^p(\mathbb{R})},$$

and $f(x)e^{-iax} \in B_g(x)\overline{\mathrm{H}^p(\mathbb{R})}$. Therefore, $f_2(x)e^{-iax} = f(x)e^{-iax} - f_1(x)e^{-iax} \in B_g(x)\overline{\mathrm{H}^p(\mathbb{R})}$. Being combined with $f_2(x)e^{-iax} \in \mathrm{H}^p(\mathbb{R})$, it yields that $f_2(x)e^{-iax} \in \mathrm{H}^p(\mathbb{R})\bigcap B_g(x)\overline{\mathrm{H}^p(\mathbb{R})}$. By Lemma 2.5, it shows that $f_2(x) \in e^{iax}\mathcal{R}_{E,m}$.

Conversely, if $f(x) \in \mathrm{FH}^p[0,a]\bigoplus e^{iax}\mathcal{R}_{E,m}$, then $f(x) = f_1(x) + f_2(x)$, where $f_1(x) \in \mathrm{FH}^p[0,a]$ and $f_2(x) \in e^{iax}\mathcal{R}_{E,m}$. By Lemma 2.5, we obtain that $f_2(x) \in e^{iax}\left[\mathrm{H}^p(\mathbb{R})\bigcap B_g(x)\overline{\mathrm{H}^p(\mathbb{R})}\right] \subset \mathrm{H}^p(\mathbb{R})\bigcap e^{iax}B_g(x)\overline{\mathrm{H}^p(\mathbb{R})}$. Since $f_1(x) \in \mathrm{FH}^p[0,a]$, we obtain that $f_1(x)e^{-iax}B_g^{-1}(x) \in \overline{\mathrm{H}^p(\mathbb{R})}$. Hence, $f_1(x) \in \mathrm{H}^p(\mathbb{R})\bigcap e^{iax}B_g(x)\overline{\mathrm{H}^p(\mathbb{R})}$ and so $f(x) \in \mathrm{H}^p(\mathbb{R})\bigcap e^{iax}B_g(x)\overline{\mathrm{H}^p(\mathbb{R})}$. The proof is completed. □

Remark. In Corollary 2.8, if the point $0 \in \mathrm{supp}\widehat{g}$, then $a = 0$.

3. Solutions of the Bedrosian equation

The study about the identity $H(fg) = fHg$ has a long history. In 1963, E. Bedrosian found that: If $f, g \in L^2(\mathbb{R})$ satisfy (i) $\operatorname{supp}\widehat{f} \subseteq [0, \infty)$, $\operatorname{supp}\widehat{g} \subseteq [0, \infty)$; or, (ii) $\operatorname{supp}\widehat{f} \subseteq [-A, A]$, $\operatorname{supp}\widehat{g} \subseteq \mathbb{R} \setminus [-A, A]$ for a positive number A, then $H(fg) = fHg$. This result is known as the Bedrosian theorem and $H(fg) = fHg$ is called the Bedrosian identity. After that, some simple and useful sufficient conditions for the Bedrosian identity were given in the frequency domain in [5]. In 1986, some necessary and sufficient conditions for the Bedrosian identity $H(fg) = fHg$ were given for $f, g \in L^2(\mathbb{R})$ [6]. Recently, as an advent of the EMD (empirical mode decomposition) algorithm and the related transform in terms of the IMFs (intrinsic mode functions) as a product of EMD for non-stationary signal processing [4], the study of the Bedrosian identity received much attention again. Many new and interesting results in relation to the Bedrosian identity are reported in [17,8–11] and the references therein. For example, in [8], a new necessary and sufficient condition for the Bedrosian identity is presented in the frequency domain for $f \in L^2(\mathbb{R}), g \in L^2(\mathbb{R})$. In [17,9,11], the Bedrosian identity $H(fg) = fH(g)$ original for functions $f, g \in L^2(\mathbb{R})$ is extended to functions $f \in L^p_T(\mathbb{R})$, $g \in L^q_T(\mathbb{R})$ and $fg \in L^r_T(\mathbb{R})$, where $1 \leqslant p, r, q \leqslant \infty$ such that $r^{-1} = p^{-1} + q^{-1}$. In relation to the shift invariant subspaces characterizations of analytic signals given in Section 2, the necessary and sufficient conditions for the Bedrosian identity appeared in the mentioned references are summarized and generalized as follows:

Theorem 3.1. *Let $f \in L^p_T(\mathbb{R})$ and $g \in L^q_T(\mathbb{R})$ be nonzero functions, where $1 \leqslant p, r, q \leqslant \infty$ such that $r^{-1} = p^{-1} + q^{-1}$. Then the following assertions are equivalent:*

(1) $T(fg) = fTg$.
(2) $T(f_-g_+) = -if_-g_+$ *and* $T(f_+g_-) = if_+g_-$.
(3) $f_-g_+ \in \mathrm{H}^r(\mathbb{R})$ *and* $\overline{f_+g_-} \in \mathrm{H}^r(\mathbb{R})$.
(4) $f_-g_+ \in \mathrm{FH}^r(\mathbb{R}^+)$ *and* $\overline{f_+g_-} \in \mathrm{FH}^r(\mathbb{R}^+)$.
(5) $\overline{f_-} \in \mathrm{H}^p(\mathbb{R}) \cap I_{g_+}\overline{\mathrm{H}^p(\mathbb{R}^+)}$ *and* $f_+ \in \mathrm{H}^p(\mathbb{R}) \bigcap I_{\overline{g_-}}\overline{\mathrm{H}^p(\mathbb{R})}$ *if* g_+ *and* g_- *are nonzero functions.*
(6) $g_+ \in \varphi_1\mathrm{H}^q(\mathbb{R}), \overline{g_-} \in \varphi_1'\mathrm{H}^q(\mathbb{R})$ and $\frac{O_{\overline{f_-}}}{\overline{O_{\overline{f_-}}}} I_{\overline{f_-}} = \frac{\varphi_1}{\varphi_2}, \frac{O_{f_+}}{\overline{O_{f_+}}} I_{f_+} = \frac{\varphi_1'}{\varphi_2'}$, *where f_+ and f_- are nonzero functions, φ_1 and φ_2 is a pair of co-prime inner functions, φ_1' and φ_2' is also a pair of co-prime inner functions.*

Remark. In Theorem 3.1, the equivalences between (1), (2), (3), (4) were first given for $f, g \in L^2(\mathbb{R})$ in [6]. Recently, the equivalences between (1), (2), (3), (4) are generalized to $f \in L^p_T(\mathbb{R}), g \in L^q_T(\mathbb{R})$ [17,9]. The equivalence between (1) and (5) is given in [11]. The relation between (1) and (6) was first found in [10] for $f, g \in L^2(\mathbb{R})$. By Theorem 2.4, it is easy to verify that (1) is equivalent to (6) for the general indices.

Different necessary and sufficient conditions have different merits. For instance, by (4) of Theorem 3.1, it is easy to reproduce the classical sufficient condition for the Bedrosian identity, that is, if $f, g \in L^2(\mathbb{R})$ satisfy $\operatorname{supp}\widehat{f} \subset [-A, A]$, $\operatorname{supp}\widehat{g} \subset \mathbb{R} \setminus [-A, A]$ for a positive number A, then $H(fg) = fHg$. Moreover, by using the Titchmarsh convolution theorem, it is proved in [9] that if f is of low Fourier frequencies then it is necessary for g to have high Fourier frequencies to satisfy $T(fg) = fTg$ for $f \in L^p_T(\mathbb{R})$ and $g \in L^q_T(\mathbb{R})$. In this section, by (6) of Theorem 3.1, we re-produce the result by an alternative method.

Corollary 3.2. *Let $f \in \mathrm{FH}^p[A, B]$ and $g \in L^q_T(\mathbb{R})$ be nonzero functions. If the endpoints $A, B \in \operatorname{supp}\widehat{f}$, then $T(fg) = fTg$ if and only if $g_+ \in e^{-i\min\{0,A\}x}\mathrm{H}^q(\mathbb{R})$ and $\overline{g_-} \in e^{i\max\{0,B\}x}\mathrm{H}^q(\mathbb{R})$.*

Proof. If $f \in \mathrm{FH}^p[A, B]$ with $0 \leqslant A < B$ and the endpoints $A, B \in \operatorname{supp}\widehat{f}$, then $f_+ = \underline{f \in \mathrm{FH}^p}[A, B]$ with the point $B \in \operatorname{supp}\widehat{f_+}$, and $f_- \equiv 0$. By Theorem 3.1, it follows that $T(fg) = fTg$ if and only if $f_+g_- \in \overline{\mathrm{H}^r(\mathbb{R})}$. By Corollary 2.6, we obtain that $\overline{g_-} \in X^q_{f_+} = e^{iBx}\mathrm{H}^q(\mathbb{R})$. Hence, $H(fg) = gHg$ if and only if $g_+ \in \mathrm{H}^q(\mathbb{R})$ and $\overline{g_-} \in e^{iBx}\mathrm{H}^q(\mathbb{R})$.

The cases that $A < B \leq 0$ and $A < 0 < B$ can be treated in the same way. The proof is completed. □

In the Bedrosian equation the roles of f and g are not symmetric. Apart from the conditions concerning analyticity, almost all of the results concerning bandlimited functions impose the bandlimited condition to the function f. In contrast to this, by virtue of Corollary 2.8 and Theorem 3.1, we now give a sound characterization for the solutions of the Bedrosian equation for g being bandlimited.

Theorem 3.3. *Suppose that $g \in \mathrm{FH}^q[A, B]$ and $f \in \mathrm{L}^p_T(\mathbb{R})$ are nonzero functions, where the point $0 \in \operatorname{supp}\widehat{g}$. Then the following statements hold:*

(*i*) *If $0 = A < B$, then $T(fg) = fTg$ if and only if*

$$\overline{f_-} \in \overline{\operatorname{span}}^p \left\{ \frac{1}{(x - \overline{\lambda})^j}, \quad \lambda \in E_1, \quad j = 1, \ldots, m(\lambda) \right\};$$

(ii) If $A < B = 0$, then $T(fg) = fTg$ if and only if

$$f_+ \in \overline{\mathrm{span}}^p \left\{ \frac{1}{(x-\lambda)^j}, \quad \lambda \in E_2, \quad j = 1, \ldots, m(\lambda) \right\};$$

(iii) If $A < 0 < B$, then $T(fg) = fTg$ if and only if

$$f \in \overline{\mathrm{span}}^p \left\{ \frac{1}{(x-\lambda)^j}, \quad \lambda \in E_1 \cup E_2, \quad j = 1, \ldots, m(\lambda) \right\},$$

where E_1 and E_2 are respectively the sets of all different zeros of $G_+(z) := (\partial^{-1}g_+)(z)$ in the upper half plane and $G_-(z) := (\partial^{-1}g_-)(z)$ in the lower half plane, and $m(\lambda)$ denotes the multiplicity at λ.

Proof. Let $g \in FH^q[A,B]$ be a nonzero function with $0 = A < B$. Then $g = g_+ \in FH^q[A,B]$ is a nonzero function with the point $0 \in \mathrm{supp}\,\widehat{g_+}$, and $g_- \equiv 0$. By (5) of Theorem 3.1 and Corollary 2.8, we learn that $T(fg) = fTg$ if and only if

$$\overline{f_-} \in \overline{\mathrm{span}}^p \left\{ \frac{1}{(x-\overline{\lambda})^j}, \quad \lambda \in E_1, \quad j = 1, \ldots, m(\lambda) \right\},$$

where E_1 is the set of all different zeros of $G_+(z) := (\partial^{-1}g_+)(z)$ in the upper half plane.

Similarly, we can treat the case that $A < B = 0$ and $A < 0 < B$. The proof is finished. $\square$

Remark. : Moreover, if g is a real function in Theorem 3.3, then $g_+ = \overline{g_-}$, $G_2(z) = \overline{G_1(\overline{z})}$ and $E_2 = \overline{E_1} = \{\lambda | \overline{\lambda} \in E_1\}$.

4. An application in construction of bandlimited signals

Another question arising in physical practice is: Given a bandlimited function $g \in FH^q[A,B]$, find all the functions $f \in L^p(\mathbb{R})$ such that $fg \in FH^r[A,B]$. For a nonzero function $g \in FH^2[A,B]$, by the Paley–Wiener theorem, it is easily known that $fg \in FH^2[A,B]$ if and only if f is a quotient of two entire functions of the exponential type. But what is quotient of two entire functions of the exponential type? Can it be totally described by the zero information of the Laplace transform of g? In this section, we will make use of some knowledge presented in Section 2 to answer these questions. We will characterize, in terms of the zeros of the Laplace transform of $g \in FH^q[A,B]$, the functions $f \in L^p(\mathbb{R})$ that make fg have a bandlimit within that of g, viz., $fg \in FH^r[A,B]$.

Lemma 4.1. *Let $g \in FH^q[A,B]$ and $f \in L^p_H(\mathbb{R})$ be nonzero functions. Then $fg \in FH^r[A,B]$ if and only if $f_+g \in FH^r[A,B]$ and $f_-g \in FH^r[A,B]$.*

Proof. Since $f \in L^p_H(\mathbb{R})$, we have

$$fg = (1/2)g(f_+ + f_-) = (1/2)(gf_+ + gf_-),$$

where $f_+ \in H^p(\mathbb{R})$ and $\overline{f_-} \in H^p(\mathbb{R})$.

Let $h_1(x) := e^{-iAx}g(x)$ and $h_2(x) := e^{iBx}\overline{g(x)}$. Then $h_1 \in FH^q[0, B-A] \subset H^p(\mathbb{R})$ and $h_2 \in FH^q[0, B-A] \subset H^p(\mathbb{R})$ from the facts $\widehat{h_1}(\omega) = \widehat{g}(\omega + A)$ and $\widehat{h_2}(\omega) = \overline{\widehat{g}(B-\omega)}$. Since f_+ and $\overline{f_-} \in H^p(\mathbb{R})$, by Lemmas 2.1 and 2.2, we have $f_+h_1, \overline{f_-}h_2 \in H^r(\mathbb{R})$ and $\mathrm{supp}\,\widehat{f_+h_1} \subset [0,\infty), \mathrm{supp}\,\widehat{\overline{f_-}h_2} \subset [0,\infty)$. Hence, $\mathrm{supp}\,\widehat{f_+g} \subset [A,\infty)$ and $\mathrm{supp}\,\widehat{f_-g} \subset (-\infty, B]$ for $(\widehat{f_+g})(\omega) = (\widehat{h_1g})(\omega - A)$ and $(\widehat{f_-g})(\omega) = \overline{(\widehat{h_2\overline{f_-}})}(B-\omega)$. This gives that $fg \in FH^r[A,B]$ if and only if $fg_+ \in FH^r[A,B]$ and $f_-g \in FH^r[A,B]$. The proof is completed. $\square$

Below we will give necessary and sufficient conditions for functions f that make fg have the same bandlimit as g does.

Theorem 4.2. *Suppose that $g \in FH^q[A,B]$ and $f \in L^p_H(\mathbb{R})$ are nonzero functions. If the endpoints $A, B \in \mathrm{supp}\,\widehat{g}$, then $fg \in FH^r[A,B]$ if and only if*

$$f \in \overline{\mathrm{span}}^p \left\{ \frac{1}{(x-\lambda)^j}, \quad \lambda \in E, \quad j = 1, \ldots, m(\lambda) \right\},$$

where E is the set of all different zeros of $G(z) := (\partial^{-1}g)(z)$ given by (2.17) on $\mathbb{C} \setminus \mathbb{R}$ and $m(\lambda)$ denotes the order of λ.

Proof. Let $g_1(x) := e^{iBx}\overline{g(x)}$ and $g_2(x) := e^{-iAx}g(x)$. Then we have $g_1 \in FH^q[0, B-A] \subset H^p(\mathbb{R}), g_2 \in FH^q[0, B-A] \subset H^p(\mathbb{R})$, the point $0 \in \mathrm{supp}\,\widehat{g_1}$ and $0 \in \mathrm{supp}\,\widehat{g_2}$. Therefore, by Lemma 4.1 and Corollary 2.8, we obtain that $\overline{f_+}g_1 \in FH^r[0, B-A]$ and $f_-g_2 \in FH^r[0, B-A]$ if and only if $f_+ \in H^p(\mathbb{R}) \bigcap I_{g_1}(x)\overline{H^p(\mathbb{R})}$ and $\overline{f_-} \in H^p(\mathbb{R}) \bigcap I_{g_2}\overline{H^p(\mathbb{R})}$, where $I_{g_1}(x) := e^{ib_1}B_1(x)$ is the inner function of $g_1(x) := e^{iBx}\overline{g(x)}$, and $I_{g_2}(x) := e^{ib_2}B_2(x)$ is the inner function of $g_2(x) := e^{-iAx}g(x)$. By (2.17), we have $G_2(z) = e^{iBz}\overline{G(\bar{z})}$ and $G_1(z) = e^{iAz}G(z)$. Thus the zeros of $G_1(z)$ in the upper half plane are the conjugate of the zeros of $G(z)$ in the lower half plane. By Lemma 2.5, we obtain that

$$f \in \overline{\mathrm{span}}^{\mathrm{p}}\left\{\frac{1}{(\mathrm{x}-\lambda)^{\mathrm{j}}},\quad \lambda \in \mathrm{E},\quad \mathrm{j}=1,\ldots,\mathrm{m}(\lambda)\right\}.$$

The proof is finished. □

Theorem 4.2 gives a characterization of the solutions $f \in L^p(\mathbb{R}), 1 < p < \infty$, in terms of backward shift invariant subspaces. It, however, does not cover the cases $p = 1$ and $p = \infty$ due to the failure of the projectional Hardy spaces decomposition. Below we will treat the two exceptional cases by using an alternative approach.

Theorem 4.3. *Suppose that $g \in \mathrm{FH}^q[A,B]$ and $f \in L^p(\mathbb{R})$ are nonzero functions, $1 \leqslant p \leqslant \infty$. Then $fg \in \mathrm{FH}^r[A,B]$ if and only if*

$$f \in \overline{I_{g_1}}\mathrm{H}^p(\mathbb{R})\bigcap I_{g_2}\overline{\mathrm{H}^p(\mathbb{R})} = \overline{I_{g_1}}\left[\mathrm{H}^p(\mathbb{R})\bigcap I_{g_1}I_{g_2}\overline{H^p(\mathbb{R})}\right],$$

where $I_{g_1}(x) := e^{i(a_1x+b_1)}B_1(x)$ is the inner function of $g_1(x) := e^{-iAx}g(x)$ and $I_{g_2}(x) := e^{i(a_2x+b_2)}B_2(x)$ is the inner function of $g_2(x) := e^{iBx}\overline{g(x)}$.

Proof. Suppose that $g \in \mathrm{FH}^q[A,B]$ and $fg \in \mathrm{FH}^r[A,B]$. Let $g_1(x) := e^{-iAx}g(x)$ and $h_1(x) := f(x)g_1(x)$. Then $g_1 \in \mathrm{FH}^q[0,B-A]$ and $h_1 \in \mathrm{FH}^r[0,B-A]$. Hence, we have $g_1 = O_{g_1}I_{g_1}$ and $h_1 = O_{h_1}I_{h_1}$ by the decomposition theorem. From the facts that $g \in L^p(\mathbb{R})$ and $\ln|h_1| = \ln|fg_1| = \ln|f| + \ln|g_1|$, we have $O_f = \frac{O_{h_1}}{O_{g_1}} \in \mathrm{H}^p(\mathbb{R})$. Thus

$$f = \frac{h_1}{g_1} = \frac{O_{h_1}I_{h_1}}{O_{g_1}I_{g_1}} = \frac{O_fI_{h_1}}{I_{g_1}} \in \overline{I_{g_1}}\mathrm{H}^p(\mathbb{R}).$$

For $g_1 \in \mathrm{FH}^q[0,B-A]$ and $(\partial^{-1}g_1)(z) = e^{-iAz}G(z)$, by Lemma 2.7, we have $I_{g_1} = e^{i(a_1x+b_1)}B_1(x)$. On the other hand, letting $g_2(x) := e^{iBx}\overline{g(x)}$ and $h_2(x) := \overline{f(x)}g_2(x)$, there hold $g_2 \in \mathrm{FH}^p[0,B-A]$ and $h_2 \in \mathrm{FH}^r[0,B-A]$. Since $\ln|h_2| = \ln|h_1|$, it follows that $\ln|g_2| = \ln|g_1|$. By the decomposition theorem, we also have

$$\bar{f} = \frac{h_2}{g_2} = \frac{O_{h_2}I_{h_2}}{O_{g_2}I_{g_2}} = \frac{O_{h_1}I_{h_2}}{O_{g_1}I_{g_2}} = \frac{O_fI_{h_2}}{I_{g_2}} \in \overline{I_{g_2}}\mathrm{H}^p(\mathbb{R}).$$

For $g_2 \in \mathrm{FH}^q[0,B-A]$ and $(\partial^{-1}g_2)(z) = e^{iBz}\overline{G(\bar{z})}$, by Lemma 2.7, we have $I_{g_2} = e^{i(a_2x+b_2)}B_2(x)$. By combining with the above two facts, we have

$$f \in \overline{I_{g_1}}\mathrm{H}^p(\mathbb{R})\bigcap I_{g_2}\overline{\mathrm{H}^p(\mathbb{R})} = \overline{I_{g_1}}\left[\mathrm{H}^p(\mathbb{R})\bigcap I_{g_1}I_{g_2}\overline{H^p(\mathbb{R})}\right].$$

Conversely, if $f \in \overline{I_{g_1}}\mathrm{H}^p(\mathbb{R})\bigcap I_{g_2}\overline{\mathrm{H}^p(\mathbb{R})}$, then there exist $f_1, f_2 \in \mathrm{H}^p(\mathbb{R})$ such that $f = \overline{I_{g_1}}f_1$ and $\bar{f} = \overline{I_{g_2}}f_2$. Let $g_1(x) := e^{-iAx}g(x)$ and $g_2(x) := e^{iBx}\overline{g(x)}$. Since $g_1, g_2 \in \mathrm{FH}^q[0,B-A]$, we have $e^{-iAx}gf = O_{g_1}I_{g_1}\overline{I_{g_1}}f_1 = O_{g_1}f_1 \in \mathrm{H}^r(\mathbb{R})$ and $e^{iBx}\overline{g(x)f(x)} = O_{g_2}I_{g_2}\overline{I_{g_2}}f_2 = O_{g_2}f_2 \in \mathrm{H}^r(\mathbb{R})$. By Lemmas 2.1 and 2.2, we obtain that $\mathrm{supp}\,\widehat{fg} \subseteq [A,B]$ and $fg \in \mathrm{FH}^r[A,B]$. The proof is completed. □

Corollary 4.4. *Suppose $0 \not\equiv g \in \mathrm{FH}^q[A,B]$. If $G(z) := (\partial^{-1}g)(z)$ given by (2.17) has no zero points on $\mathbb{C}\setminus\mathbb{R}$, then there exists $f \in L^p(\mathbb{R})$ such that $fg \in \mathrm{FH}^r[A,B]$ if and only if*

$$f \in e^{i(A-a)x}\mathrm{H}^p(\mathbb{R})\bigcap e^{i(B-b)x}\overline{\mathrm{H}^p(\mathbb{R})} = \mathrm{FH}^p[A-a,B-b],$$

where $[a,b] := \mathrm{supp}\,\hat{g}$.

Remark. : From Corollary 4.4, we know that if the end points $A, B \in \mathrm{supp}\,\hat{g}$ and $G(z) := (\partial^{-1}g)(z)$ given by (2.17) has no zero points on $\mathbb{C}\setminus\mathbb{R}$, then there exists no nonzero function $f \in L^p(\mathbb{R}), 1 \leqslant p \leqslant \infty$, such that $fg \in \mathrm{FH}^r[A,B]$.

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