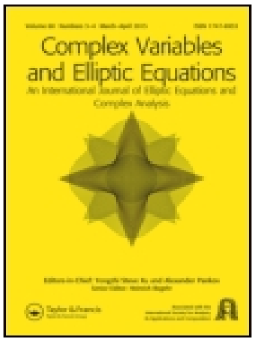

Taylor & Francis
Taylor & Francis Group

Complex Variables and Elliptic Equations
An International Journal

ISSN: 1747-6933 (Print) 1747-6941 (Online) Journal homepage: http://www.tandfonline.com/loi/gcov20

Stronger uncertainty principles for hypercomplex signals

Yan Yang, Pei Dang & Tao Qian

To cite this article: Yan Yang, Pei Dang & Tao Qian (2015) Stronger uncertainty principles for hypercomplex signals, Complex Variables and Elliptic Equations, 60:12, 1696-1711, DOI: 10.1080/17476933.2015.1041938

To link to this article: http://dx.doi.org/10.1080/17476933.2015.1041938

Complex Variables and Elliptic Equations, 2015
Vol. 60, No. 12, 1696–1711, http://dx.doi.org/10.1080/17476933.2015.1041938

Stronger uncertainty principles for hypercomplex signals

Yan Yang[a], Pei Dang[b*] and Tao Qian[c]

[a] *School of Mathematics and Computational Science, Sun Yat-Sen University, Guangzhou, China;* [b] *Faculty of Information Technology, Macau University of Science and Technology, Macao, China;* [c] *Faculty of Science and Technology, Department of Mathematics, University of Macau, Macao, China*

Communicated by J. DU

(*Received 7 March 2015; accepted 13 April 2015*)

In this paper, for real para-vector-valued signals, we obtain stronger uncertainty principles in terms of covariance and absolute covariance based on Fourier transform in both directional and the spatial cases. We provide certain conditions that give rise to the equal relation between the two uncertainty principles. Examples are presented to verify the results.

Keywords: uncertainty principle in higher dimensions; Fourier transform; covariance

AMS Subject Classifications: 46F10; 30G35

1. Introduction

Uncertainty principle in time–frequency planes plays an important role in signal processing [1–11] and in physics [12–21]. The classical form of uncertainty principle states that for a given signal of unit energy $f(t)$ with Fourier transform

$$\hat{f}(\omega) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-\mathbf{i}\omega t} \mathrm{d}t,$$

the product of spreads of the signal in the time domain and the frequency domain is bounded by a lower bound

$$\sigma_t^2 \sigma_\omega^2 \geq \frac{1}{4},$$

where σ_t^2 and σ_ω^2 are the duration and bandwidth of a signal $f(t)$, defined, respectively, by

$$\sigma_t^2 := \int_{-\infty}^{\infty} (t - \langle t \rangle)^2 |f(t)|^2 \mathrm{d}t$$

and

$$\sigma_\omega^2 := \int_{-\infty}^{\infty} (\omega - \langle \omega \rangle)^2 |\hat{f}(\omega)|^2 \mathrm{d}\omega,$$

*Corresponding author. Email: pdang@must.edu.mo

respectively. Here,

$$\langle t \rangle := \int_{-\infty}^{\infty} t|f(t)|^2 \mathrm{d}t$$

is the mean time and

$$\langle \omega \rangle := \int_{-\infty}^{\infty} \omega|\hat{f}(\omega)|^2 \mathrm{d}\omega$$

is the mean frequency.

If $f(t)$ is expressed in the polar form $f(t) = |f(t)|e^{\mathbf{i}\theta(t)} = \rho(t)e^{\mathbf{i}\theta(t)}$, then a stronger version of uncertainty principle [13] is

$$\sigma_t \sigma_\omega \geq \left| -\frac{1}{2} + \mathbf{i}\mathrm{Cov}_{\mathrm{t}\omega} \right| = \frac{1}{2}\sqrt{1 + 4\mathrm{Cov}_{\mathrm{t}\omega}^2}, \tag{1.1}$$

where $\mathrm{Cov}_{\mathrm{t}\omega}$ is the covariance of the signal defined by

$$\mathrm{Cov}_{t\omega} := \int_{-\infty}^{\infty} (t - \langle t \rangle)(\theta'(t) - \langle \omega \rangle)\rho^2(t)\mathrm{d}t.$$

The covariance is a measurement of the relation between instantaneous frequency, $\theta'(t)$, and time t.

Recently, in [22], Dang et al. strengthen the result of (1.1), they obtained

$$\sigma_t \sigma_\omega \geq \left| -\frac{1}{2} + \mathbf{i}\mathrm{COV}_{\mathrm{t}\omega} \right| = \frac{1}{2}\sqrt{1 + 4\mathrm{COV}_{\mathrm{t}\omega}^2}, \tag{1.2}$$

where $\mathrm{COV}_{t\omega}$ is the absolute covariance of a signal defined by

$$\mathrm{COV}_{t\omega} := \int_{-\infty}^{\infty} |(t - \langle t \rangle)(\theta'(t) - \langle \omega \rangle)|\rho^2(t)\mathrm{d}t.$$

Due to the trivial inequality $\int_{-\infty}^{\infty}(t - \langle t \rangle)(\theta'(t) - \langle \omega \rangle)\rho^2(t)\mathrm{d}t \leq \int_{-\infty}^{\infty} |(t - \langle t \rangle)(\theta'(t) - \langle \omega \rangle)|\rho^2(t)\mathrm{d}t$, (1.2) is stronger than (1.1).

Without loss of generality, we let $\langle t \rangle = 0$ and $\langle \omega \rangle = 0$. The essence of uncertainty principle will not be affected.

For the importance of uncertainty principle, there are many efforts to extend it to various types of functions and integral transformations. Recently, researchers discussed the uncertainty relations for fractional Fourier transform [6,10,23] and linear canonical transform.[9,20,24,25] A stronger uncertainty principle in LCT involving the phase derivative of the signal was discussed in [26].

While in higher dimensional spaces, how to describe the uncertainty principle? In Clifford algebra, Hitzer et al. [27–30] investigated a directional uncertainty principle for the Clifford–Fourier transform, which describes how the variances (in arbitrary but fixed directions) of a multi-vector-valued function and its Clifford–Fourier transform are related. Using the scalar-valued phase derivative of hypercomplex signals,[31] two uncertainty principles, of which one is for scalar-valued hypercomplex signals and the other is for axial form hypercomplex signals, for Fourier transforms were studied in [32]. In [33], we prove the classical uncertainty principles without covariance using the LCT of hypercomplex signal. To our knowledge, a work on the investigation of the stronger uncertainty relations with covariance of hypercomplex signal is not carried out yet.

In the present work, we study the real para-vector-valued signals. Using the polar form of it, the stronger uncertainty principles with covariance and absolute covariance for the real para-vector-valued signal are established. These uncertainty principles prescribe a larger bound on the product of the effective widths of real para-vector-valued signals in the time and frequency domains. Examples are given to verify the results.

The article is organized as follows. Section 2 gives a brief introduction to some general definitions and basic properties of Clifford analysis. In Section 3, some important properties about Fourier transforms are recalled. They are necessary to prove the uncertainty principles. The latest results about the Heisenberg uncertainty principle with absolute covariance and covariance are generalized for the real para-vector-valued signals in Section 4. At last, we give some examples to verity the results in Section 5.

2. Clifford algebra

The theory of Clifford algebras is intimately connected with the theory of quadratic forms and orthogonal transformations. They generalize the real numbers, complex numbers, quaternions and several other hypercomplex number systems.[34,35] Clifford algebras have important applications in a variety of fields including geometry, theoretical physics and digital image processing. They are named after the English geometer William Kingdon Clifford.

Most of the basic knowledge and notation in relation to Clifford algebra hereby are referred to [36] and [37].

Let $\mathbf{e}_1, \ldots, \mathbf{e}_m$ be *basic elements* satisfying $\mathbf{e}_i\mathbf{e}_j + \mathbf{e}_j\mathbf{e}_i = -2\delta_{ij}$, where $\delta_{ij} = 1$ if $i = j$, and $\delta_{ij} = 0$ otherwise, $i, j = 1, 2, \ldots, m$. Let

$$\mathbf{R}_1^m = \left\{x_0 + \underline{x} \mid x_0 \in \mathbf{R}, \underline{x} \in \mathbf{R}^m\right\},$$

where

$$\mathbf{R}^m = \left\{\underline{x} \mid \underline{x} = x_1\mathbf{e}_1 + \cdots + x_m\mathbf{e}_m, x_j \in \mathbf{R}, j = 1, 2, \ldots, m\right\}$$

be identical with the usual Euclidean space $\mathbf{R}^m$.

An element in $\mathbf{R}^m$ is called a *real vector* and an element in $\mathbf{R}_1^m$ is called a *real para-vector*. The multiplication of two real para-vectors $x_0 + \underline{x} = \sum_{j=0}^m x_j e_j$ and $y_0 + \underline{y} = \sum_{j=0}^m y_j e_j$ is given by

$$(x_0 + \underline{x})(y_0 + \underline{y}) = (x_0y_0 + \underline{x} \cdot \underline{y}) + (x_0\underline{y} + y_0\underline{x}) + (\underline{x} \wedge \underline{y})$$

with

$$\underline{x} \cdot \underline{y} = -\sum_{j=1}^m x_j y_j = \frac{1}{2}(\underline{x}\,\underline{y} + \underline{y}\,\underline{x}) = -\langle \underline{x}, \underline{y}\rangle$$

$$\underline{x} \wedge \underline{y} = \sum_{i<j} e_{ij}(x_i y_j - x_j y_i) = \frac{1}{2}(\underline{x}\,\underline{y} - \underline{y}\,\underline{x}).$$

There are three parts altogether, a *scalar part* $x_0y_0 + \underline{x} \cdot \underline{y}$, a vector part $x_0\underline{y} + y_0\underline{x}$ and a *bi-vector part* $\underline{x} \wedge \underline{y}$, respectively. We denote the scalar part of $(x_0 + \underline{x})(y_0 + \underline{y})$ by $\mathrm{Sc}[(x_0 + \underline{x})(y_0 + \underline{y})]$.

The real (complex) Clifford algebra generated by $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m$, denoted by $Cl_{0,m}$, is the associative algebra over the real (complex) field $\mathbf{R}$ ($\mathbf{C}$). A general element in $Cl_{0,m}$, therefore, is of the form

$$x = \sum_S x_S \mathbf{e}_S,$$

$x_s \in \mathbf{R}$ ($\mathbf{C}$) and $\mathbf{e}_S = \mathbf{e}_{i_1}\mathbf{e}_{i_2}\ldots\mathbf{e}_{i_l}$, and S runs over all the ordered subsets of $\{1, 2, \ldots, m\}$, namely

$$S = \{1 \le i_1 < i_2 < \cdots < i_l \le m\}, \quad 1 \le l \le m.$$

The *conjugation* of $\mathbf{e}_S$ is defined by $\overline{\mathbf{e}}_S := \overline{\mathbf{e}}_{il}\ldots\overline{\mathbf{e}}_{i1}$, here $\overline{\mathbf{e}}_j = -\mathbf{e}_j$. Especially, we have $\overline{\mathbf{e}_i\mathbf{e}_j} = -\mathbf{e}_i\mathbf{e}_j$. So the Clifford conjugates of a vector $\underline{x}$ and a bi-vector $\underline{x} \wedge \underline{y}$ are $\overline{\underline{x}} = -\underline{x}$ and $\overline{\underline{x} \wedge \underline{y}} = -\underline{x} \wedge \underline{y}$, respectively.

The natural *inner product* between x and y in $Cl_{0,m}$, denoted by $\langle x, y\rangle$, is the complex number $\sum_S x_S\overline{y_S}$, where $x = \sum_S x_S\mathbf{e}_S, x_S \in \mathbf{C}$ and $y = \sum_S y_S\mathbf{e}_S, y_S \in \mathbf{C}$. The norm associated with this inner product is

$$|x| = \langle x, x\rangle^{\frac{1}{2}} = \left(\sum_S |x_S|^2\right)^{\frac{1}{2}}.$$

For $p = 1$ and 2, the Clifford-valued modules $L^p(\mathbf{R}^m; Cl_{0,m})$ are defined by

$$L^p(\mathbf{R}^m; Cl_{0,m}) := \Big\{ f : \mathbf{R}^m \to Cl_{0,m} \mid \|f\|^p_{L^p(\mathbf{R}^m;Cl_{0,m})} = \int_{\mathbf{R}^m} |f(\underline{x})|^p \mathrm{d}\underline{x} < \infty \Big\}.$$

For two Clifford-valued signals $f, g \in L^2(\mathbf{R}^m; Cl_{0,m})$ can be equipped with a Hermitian inner product,

$$\langle f, g\rangle_{L^2(\mathbf{R}^m;Cl_{0,m})} := \mathrm{Sc}\left[\int_{\mathbf{R}^m} f(\underline{x})\overline{g(\underline{x})}\mathrm{d}\underline{x}\right] \tag{2.1}$$

whose associated norm is

$$\|f\|_{L^2(\mathbf{R}^m;Cl_{0,m})} := \left(\int_{\mathbf{R}^m} |f(\underline{x})|^2\mathrm{d}\underline{x}\right)^{1/2}.$$

In this paper, we study the signals which are defined in $\mathbf{R}^m$ taking values in $\mathbf{R}^m_1$. That is

$$f(\underline{x}) : \mathbf{R}^m \longrightarrow \mathbf{R}^m_1,$$
$$f(\underline{x}) = f_0(\underline{x}) + f_1(\underline{x})\mathbf{e}_1 + f_2(\underline{x})\mathbf{e}_2 + \cdots + f_m(\underline{x})\mathbf{e}_m,$$

where $f_i(\underline{x}), i = 1, 2, \ldots, m$ is real-valued functions.

For any signal $f(\underline{x}) \in (\mathbf{R}^m, \mathbf{R}^m_1)$, we have the polar form [31]:

$$\begin{aligned} f(\underline{x}) &= f_0(\underline{x}) + f_1(\underline{x})\mathbf{e}_1 + f_2(\underline{x})\mathbf{e}_2 + \cdots + f_m(\underline{x})\mathbf{e}_m \\ &= |f(\underline{x})|e^{\underline{u}(\underline{x})\theta(\underline{x})} \\ &= \rho(\underline{x})e^{\underline{u}(\underline{x})\theta(\underline{x})}, \end{aligned}$$

with *amplitude*

$$\rho(\underline{x}) := |f(\underline{x})| = \sqrt{f_0^2(\underline{x}) + f_1^2(\underline{x}) + \cdots + f_m^2(\underline{x})}$$

and *orientation*

$$\underline{u}(x) := \frac{f_1(\underline{x})\mathbf{e}_1 + f_2(\underline{x})\mathbf{e}_2 + \cdots + f_m(\underline{x})\mathbf{e}_m}{\sqrt{f_1^2(\underline{x}) + f_2^2(\underline{x}) + \cdots + f_m^2(\underline{x})}}$$

belongs to the unit sphere $S^{m-1} := \{\underline{x} \in \mathbf{R}^m \mid |\underline{x}|^2 = 1\}$ of m-dimensional Euclidean space $\mathbf{R}^m$. The *phase angle* is

$$\theta(\underline{x}) := \arctan \frac{\sqrt{f_1^2(\underline{x}) + f_2^2(\underline{x}) + \cdot + f_m^2(\underline{x})}}{f_0(\underline{x})} \in [0, \pi]$$

and the *phase vector* is $e^{\underline{u}(\underline{x})\theta(\underline{x})}$.

3. Fourier transform of hypercomplex signals

If $f \in L^1(\mathbf{R}^m, Cl_{0,m})$, the *Fourier transform* of f is defined by

$$F\{f\}(\underline{\xi}) := \frac{1}{\sqrt{(2\pi)^m}} \int_{\mathbf{R}^m} e^{-\mathbf{i}\langle \underline{x}, \underline{\xi} \rangle} f(\underline{x}) \mathrm{d}\underline{x} \tag{3.1}$$

where $\langle \underline{x}, \underline{\xi} \rangle := x_1\xi_1 + \cdots + x_m\xi_m$ is the usual inner product in Euclidean space $\mathbf{R}^m$ and the *inverse Fourier transform* by

$$f(\underline{x}) = \frac{1}{\sqrt{(2\pi)^m}} \int_{\mathbf{R}^m} e^{\mathbf{i}\langle \underline{x}, \underline{\xi} \rangle} F\{f\}(\underline{\xi}) \mathrm{d}\underline{\xi}.$$

Note If $f \in L^1(\mathbf{R}^m, \mathbf{R}_1^m)$, then $F\{f\}$ is a complex para-vector valued.

Let $f(\underline{x}), g(\underline{x}) \in L^2(\mathbf{R}^m, Cl_{0,m})$, for

$$\langle f(\underline{x}), g(\underline{x}) \rangle = \mathrm{Sc} \int_{\mathbf{R}^m} f(\underline{x}) \overline{g(\underline{x})} \mathrm{d}\underline{x},$$

the well-known Plancherel Theorem holds

$$\mathrm{Sc}\left[\int_{\mathbf{R}^m} f(\underline{x}) \overline{g(\underline{x})} \mathrm{d}\underline{x}\right] = \mathrm{Sc}\left[\int_{\mathbf{R}^m} F\{f\}(\underline{\xi}) \overline{F\{g\}(\underline{\xi})} \mathrm{d}\underline{\xi}\right].$$

In particular, for $f = g \in L^2(\mathbf{R}^m, \mathbf{R}_1^m)$, the Parseval Theorem is obtained:

$$\int_{\mathbf{R}^m} |f(\underline{x})|^2 \mathrm{d}\underline{x} = \int_{\mathbf{R}^m} |F\{f\}(\underline{\xi})|^2 \mathrm{d}\underline{\xi}. \tag{3.2}$$

Next, we prove the following partial derivative properties.

LEMMA 3.1 *Let $f(\underline{x})$ be a real para-vector-valued signal. If $f(\underline{x})$ and $\frac{\partial f(\underline{x})}{\partial x_k} \in L^1(\mathbf{R}^m)$ for $k = 1, \ldots, m$, then*

$$F\left\{\frac{\partial}{\partial x_k} f(\underline{x})\right\}(\underline{\xi}) = \mathbf{i}\xi_k F\{f\}(\underline{\xi}). \tag{3.3}$$

Proof Applying the integration by parts and complex value $-\mathbf{i}u_k$ can be commutative with any para-vector-valued signals, we obtain

$$F\left\{\frac{\partial}{\partial x_k}f(\underline{x})\right\}(\underline{\xi}) = \frac{1}{\sqrt{(2\pi)^m}}\int_{\mathbf{R}^m}\left[\frac{\partial}{\partial x_k}f(\underline{x})\right]e^{-\mathbf{i}\langle\underline{x},\underline{\xi}\rangle}\mathrm{d}\underline{x}$$
$$= -\frac{1}{\sqrt{(2\pi)^m}}\int_{\mathbf{R}^m}(-\mathbf{i}\xi_k)f(\underline{x})e^{-\mathbf{i}\langle\underline{x},\underline{\xi}\rangle}\mathrm{d}\underline{x}$$
$$= \mathbf{i}\xi_k F\{f\}(\underline{\xi}).$$

□

LEMMA 3.2 *Let $f(\underline{x})$ be a real para-vector-valued signal. If $\frac{\partial f}{\partial x_k} \in L^2(\mathbf{R}^m)$ for $k = 1, \ldots, m$, then*

$$\int_{\mathbf{R}^m}\xi_k^2|F\{f\}(\underline{\xi})|^2\mathrm{d}\underline{\xi} = \int_{\mathbf{R}^m}\left|\frac{\partial}{\partial x_k}f(\underline{x})\right|^2\mathrm{d}\underline{x}. \tag{3.4}$$

Proof Applying (3.3) in Lemma 3.1 and Parseval Theorem of Fourier transform, we obtain

$$\int_{\mathbf{R}^m}\xi_k^2|F\{f\}(\underline{\xi})|^2\mathrm{d}\underline{\xi} = \int_{\mathbf{R}^m}\left|\mathbf{i}\xi_k F\{f(\underline{x})\}(\underline{\xi})\right|^2\mathrm{d}\underline{\xi}$$
$$= \int_{\mathbf{R}^m}\left|F\left\{\frac{\partial}{\partial x_k}f(\underline{x})\right\}(\underline{\xi})\right|^2\mathrm{d}\underline{\xi}$$
$$= \int_{\mathbf{R}^m}\left|\frac{\partial}{\partial x_k}f(\underline{x})\right|^2\mathrm{d}\underline{x}.$$

□

4. Uncertainty principles

In the following, we explicitly prove and generalize the latest result about the stronger uncertainty principle with absolute covariance to real para-vector-valued signals. We also give sufficient and necessary conditions such that they minimize the uncertainty product.

Before this, we need the following propositions.

PROPOSITION 4.1 *For any real para-vector-valued signal $f(\underline{x}) = A(f)(\underline{x})e^{\underline{u}(\underline{x})\theta(\underline{x})}$, if $\frac{\partial}{\partial x_k}\underline{u}(\underline{x})$ exists for $k = 1, 2, \ldots, m$, then*

$$\mathrm{Sc}\left[\left(\frac{\partial}{\partial x_k}\underline{u}(\underline{x})\right)\underline{u}(\underline{x})\right] = 0. \tag{4.1}$$

Proof For

$$\underline{u}(\underline{x}) = \frac{f_1(\underline{x})\mathbf{e}_1 + f_2(\underline{x})\mathbf{e}_2 + \cdots + f_m(\underline{x})\mathbf{e}_m}{\sqrt{f_1^2(\underline{x}) + f_2^2(\underline{x}) + \cdots + f_m^2(\underline{x})}},$$

we have $\underline{u}^2(\underline{x}) = -1$. by calculation directly, we have

$$\begin{aligned}0 &= \frac{\partial}{\partial x_k}\underline{u}^2(\underline{x})\\ &= \left(\frac{\partial}{\partial x_k}\underline{u}(\underline{x})\right)\underline{u}(\underline{x}) + \underline{u}(x)\left(\frac{\partial}{\partial x_k}\underline{u}(\underline{x})\right)\\ &= \left(\frac{\partial}{\partial x_k}\underline{u}(\underline{x})\right)\underline{u}(\underline{x}) + \overline{\left(\frac{\partial}{\partial x_k}\underline{u}(\underline{x})\right)\underline{u}(\underline{x})}\\ &= 2\mathrm{Sc}\left[\left(\frac{\partial}{\partial x_k}\underline{u}(\underline{x})\right)\underline{u}(\underline{x})\right].\end{aligned}$$

This completes the proof. □

PROPOSITION 4.2 *For any real para-vector-valued signal* $f(\underline{x}) = \rho(\underline{x})e^{\underline{u}(\underline{x})\theta(\underline{x})}$, *if* $\frac{\partial}{\partial x_k}e^{\underline{u}(\underline{x})\theta(\underline{x})}$ *exists for* $k = 1, 2, \ldots, m$, *then*

$$\mathrm{Sc}\left[\left(\frac{\partial}{\partial x_k}e^{\underline{u}(\underline{x})\theta(\underline{x})}\right)e^{-\underline{u}(\underline{x})\theta(\underline{x})}\right] = 0. \tag{4.2}$$

Proof Applying the generalized Euler formula, we have

$$e^{\underline{u}(\underline{x})\theta(\underline{x})} = \cos\theta(\underline{x}) + \underline{u}(\underline{x})\sin\theta(\underline{x}).$$

By calculation directly, we have

$$\begin{aligned}&\frac{\partial}{\partial x_k}e^{\underline{u}(\underline{x})\theta(\underline{x})}\\ &\quad = -\sin\theta(\underline{x})\frac{\partial\theta(\underline{x})}{\partial x_k} + \frac{\partial\underline{u}(x)}{\partial x_k}\sin\theta(\underline{x}) + \underline{u}(\underline{x})\cos\theta(\underline{x})\frac{\partial\theta(\underline{x})}{\partial x_k}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\left(\frac{\partial}{\partial x_k}e^{\underline{u}(\underline{x})\theta(\underline{x})}\right)e^{-\underline{u}(\underline{x})\theta(\underline{x})}\\ &= \left(-\sin\theta(\underline{x})\frac{\partial\theta(\underline{x})}{\partial x_k} + \frac{\partial\underline{u}(x)}{\partial x_k}\sin\theta(\underline{x}) + \underline{u}(\underline{x})\cos\theta(\underline{x})\frac{\partial\theta(\underline{x})}{\partial x_k}\right)\left(\cos\theta(\underline{x}) - \underline{u}(\underline{x})\sin\theta(\underline{x})\right)\\ &= \frac{\partial\theta(\underline{x})}{\partial x_k}\underline{u}(\underline{x}) + \frac{\partial\underline{u}(\underline{x})}{\partial x_k}\sin\theta(\underline{x})\cos\theta(\underline{x}) - \frac{\partial\underline{u}(\underline{x})}{\partial x_k}\underline{u}(\underline{x})\sin^2\theta(\underline{x}).\end{aligned} \tag{4.3}$$

Clearly, the scalar part of the multiplication of $\left(\frac{\partial}{\partial x_k}e^{\underline{u}(\underline{x})\theta(\underline{x})}\right)$ and $e^{-\underline{u}(\underline{x})\theta(\underline{x})}$ is decided by the scalar part of

$$-\frac{\partial\underline{u}(\underline{x})}{\partial x_k}\underline{u}(\underline{x})\sin^2\theta(\underline{x}).$$

Due to Proposition 4.1, we have

$$\mathrm{Sc}\left[\left(\frac{\partial}{\partial x_k}e^{\underline{u}(\underline{x})\theta(\underline{x})}\right)e^{-\underline{u}(\underline{x})\theta(\underline{x})}\right] = 0.$$

This completes the proof. □

Remark 4.1 In one-dimensional cases, for signal $f(x) = \rho(x)e^{\mathbf{i}\theta(x)}$, it is easy to see that

$$\left(\frac{\partial}{\partial x}e^{\mathbf{i}\theta(x)}\right)e^{-\mathbf{i}\theta(x)} = \mathbf{i}\theta'(x).$$

Remark 4.2 From formulas (4.2) and (4.3), we find that the multiplication of $\left(\frac{\partial}{\partial x_k}e^{\underline{u}(\underline{x})\theta(\underline{x})}\right)$ and $e^{-\underline{u}(\underline{x})\theta(\underline{x})}$ has two parts: the vector part and the bi-vector part. Therefore, we have

$$\left(\frac{\partial}{\partial x_k}e^{\underline{u}(\underline{x})\theta(\underline{x})}\right)e^{-\underline{u}(\underline{x})\theta(\underline{x})} + \overline{\left(\frac{\partial}{\partial x_k}e^{\underline{u}(\underline{x})\theta(\underline{x})}\right)e^{-\underline{u}(\underline{x})\theta(\underline{x})}} = 0.$$

PROPOSITION 4.3 *For any real para-vector-valued signal $f(\underline{x}) = \rho(\underline{x})e^{\underline{u}(\underline{x})\theta(\underline{x})}$, if $\frac{\partial}{\partial x_k}f(\underline{x})$ exists for $k = 1, 2, \ldots, m$, then*

$$\left|\frac{\partial}{\partial x_k}f(\underline{x})\right|^2 = \left[\frac{\partial}{\partial x_k}\rho(\underline{x})\right]^2 + \rho^2(\underline{x})\left|\left(\frac{\partial}{\partial x_k}e^{\underline{u}(\underline{x})\theta(\underline{x})}\right)\left(e^{-\underline{u}(\underline{x})\theta(\underline{x})}\right)\right|^2. \tag{4.4}$$

Proof For $f(\underline{x}) = \rho(\underline{x})e^{\underline{u}(\underline{x})\theta(\underline{x})}$, we have

$$\begin{aligned}\frac{\partial}{\partial x_k}f(\underline{x}) &= \frac{\partial}{\partial x_k}\left[\rho(\underline{x})e^{\underline{u}(\underline{x})\theta(\underline{x})}\right]\\ &= \left(\frac{\partial}{\partial x_k}\rho(\underline{x})\right)e^{\underline{u}(\underline{x})\theta(\underline{x})} + \rho(\underline{x})\frac{\partial}{\partial x_k}\left(e^{\underline{u}(\underline{x})\theta(\underline{x})}\right).\end{aligned}$$

Therefore,

$$\begin{aligned}&\left\|\frac{\partial}{\partial x_k}f(\underline{x})\right\|^2 = \frac{\partial}{\partial x_k}f(\underline{x})\overline{\frac{\partial}{\partial x_k}f(\underline{x})}\\ &= \left[\left(\frac{\partial}{\partial x_k}\rho\right)e^{\underline{u}\theta} + \rho\frac{\partial}{\partial x_k}\left(e^{\underline{u}\theta}\right)\right]\left[\left(\frac{\partial}{\partial x_k}\rho\right)e^{-\underline{u}\theta} + \rho\overline{\frac{\partial}{\partial x_k}\left(e^{\underline{u}\theta}\right)}\right]\\ &= \left(\frac{\partial}{\partial x_k}\rho\right)^2 + \rho^2\frac{\partial}{\partial x_k}\left(e^{\underline{u}\theta}\right)\overline{\frac{\partial}{\partial x_k}\left(e^{\underline{u}\theta}\right)} + \rho\left(\frac{\partial}{\partial x_k}\rho\right)\left[\frac{\partial}{\partial x_k}\left(e^{\underline{u}\theta}\right)e^{-\underline{u}\theta} + \overline{\frac{\partial}{\partial x_k}\left(e^{\underline{u}\theta}\right)e^{-\underline{u}\theta}}\right].\end{aligned}$$

By Remark 4.2, we have

$$\frac{\partial}{\partial x_k}\left(e^{\underline{u}\theta}\right)e^{-\underline{u}\theta} + \overline{\frac{\partial}{\partial x_k}\left(e^{\underline{u}\theta}\right)e^{-\underline{u}\theta}} = 0.$$

While,

$$\begin{aligned}\rho^2\frac{\partial}{\partial x_k}\left(e^{\underline{u}\theta}\right)\overline{\frac{\partial}{\partial x_k}\left(e^{\underline{u}\theta}\right)} &= \rho^2\frac{\partial}{\partial x_k}\left(e^{\underline{u}\theta}\right)e^{-\underline{u}\theta}\overline{\frac{\partial}{\partial x_k}\left(e^{\underline{u}\theta}\right)e^{-\underline{u}\theta}}\\ &= \rho^2\left|\frac{\partial}{\partial x_k}\left(e^{\underline{u}\theta}\right)e^{-\underline{u}\theta}\right|^2.\end{aligned}$$

This completes the proof. □

Clearly, using (3.4) and (4.4), we have

THEOREM 4.1 *For any real para-vector-valued signal* $f(\underline{x}) = \rho(\underline{x})e^{\underline{u}(\underline{x})\theta(\underline{x})}$*, if* $f \in L^1 \bigcap L^2(\mathbf{R}^m)$*, and for* $k = 1, 2, \ldots, m$*,* $\frac{\partial}{\partial x_k} f$ *exist and are also in* $L^2(\mathbf{R}^m, \mathbf{R}_1^m)$*, then*

$$\int_{\mathbf{R}^m} \xi_k^2 |F\{f\}(\underline{\xi})|^2 \mathrm{d}\underline{\xi} = \int_{\mathbf{R}^m} \left[\frac{\partial}{\partial x_k}\rho(\underline{x})\right]^2 \mathrm{d}\underline{x} + \int_{\mathbf{R}^m} \rho^2(\underline{x}) \left|\left(\frac{\partial}{\partial x_k} e^{\underline{u}(\underline{x})\theta(\underline{x})}\right)\left(e^{-\underline{u}(\underline{x})\theta(\underline{x})}\right)\right|^2 \mathrm{d}\underline{x}. \tag{4.5}$$

Remark 4.3 (4.5) is an effective formula to compute $\int_{\mathbf{R}^2} \xi_k^2 |F\{f\}(\underline{\xi})|^2 \mathrm{d}\underline{\xi}$. Using this formula, we can avoid computing the Fourier transform of $f(\underline{x})$.

Due to Remark 4.1, in classical cases, we have [13]:

$$\sigma_\omega^2 = \int_{-\infty}^{\infty} \rho'^2(x)\mathrm{d}x + \int_{-\infty}^{\infty} \rho^2(x)\theta'^2(x)\mathrm{d}x.$$

THEOREM 4.2 (Uncertainty Principle in spatial case) *Let* $f(\underline{x})$ *be a real para-vector-valued signal with* $\|f(\underline{x})\|_{L^2} = 1$*. If* $x_k f$ *and* $\frac{\partial f}{\partial x_k} \in L^2(\mathbf{R}^m)$ *for* $k = 1, \ldots, m$*, then*

$$\left(\int_{\mathbf{R}^m} x_k^2 |f(\underline{x})|^2 \mathrm{d}\underline{x}\right)\left(\int_{\mathbf{R}^m} \xi_k^2 |F\{f\}(\underline{u})|^2 \mathrm{d}\underline{u}\right) \geq \frac{1}{4} + \mathrm{COV}_{x_k\xi_k}^2, \tag{4.6}$$

where the absolute covariance of every variable is defined by

$$\mathrm{COV}_{x_k\xi_k} := \int_{\mathbf{R}^m} \left| x_k \left(\frac{\partial}{\partial x_k} e^{\underline{u}(\underline{x})\theta(\underline{x})}\right) e^{-\underline{u}(\underline{x})\theta(\underline{x})}\right| \rho^2(\underline{x})\mathrm{d}\underline{x}.$$

The equality (4.6) *holds if and only if*

$$f(\underline{x}) = e^{-\frac{\alpha_1}{2}x_1^2 - \cdots - \frac{\alpha_m}{2}x_m^2} e^{\underline{u}(\underline{x})\theta(\underline{x})}$$

and

$$\left|\left(\frac{\partial}{\partial x_k} e^{\underline{u}(\underline{x})\theta(\underline{x})}\right) e^{-\underline{u}(\underline{x})\theta(\underline{x})}\right| = \beta_k |x_k|.$$

Here $\alpha_k > 0$ *and* $\beta_k > 0$ *for* $k = 1, \ldots, m$.

Proof Applying formula (4.5), we have

$$\begin{aligned}
&\left(\int_{\mathbf{R}^m} x_k^2 |f(\underline{x})|^2 \mathrm{d}\underline{x}\right)\left(\int_{\mathbf{R}^m} \xi_k^2 |F\{f\}(\underline{\xi})|^2 \mathrm{d}\underline{\xi}\right) \\
&= \left(\int_{\mathbf{R}^m} x_k^2 \rho^2 \mathrm{d}\underline{x}\right)\left(\int_{\mathbf{R}^m} \left[\frac{\partial}{\partial x_k}\rho(\underline{x})\right]^2 \mathrm{d}\underline{x} + \int_{\mathbf{R}^m} \rho^2 \left|\left(\frac{\partial}{\partial x_k} e^{\underline{u}(\underline{x})\theta(\underline{x})}\right)\left(e^{-\underline{u}(\underline{x})\theta(\underline{x})}\right)\right|^2 \mathrm{d}\underline{x}\right) \\
&= \left(\int_{\mathbf{R}^m} x_k^2 \rho^2 \mathrm{d}\underline{x}\right)\left(\int_{\mathbf{R}^m} \left[\frac{\partial}{\partial x_k}\rho(\underline{x})\right]^2 \mathrm{d}\underline{x}\right) \\
&+ \left(\int_{\mathbf{R}^m} x_k^2 \rho^2 \mathrm{d}\underline{x}\right)\left(\int_{\mathbf{R}^2} \rho^2(\underline{x}) \left|\left(\frac{\partial}{\partial x_k} e^{\underline{u}(\underline{x})\theta(\underline{x})}\right)\left(e^{-\underline{u}(\underline{x})\theta(\underline{x})}\right)\right|^2 \mathrm{d}\underline{x}\right)
\end{aligned} \tag{4.7}$$

Using Hölder inequality, we have

$$\begin{aligned}
&\left(\int_{\mathbf{R}^m} x_k^2\rho^2 \mathrm{d}\underline{x}\right)\left(\int_{\mathbf{R}^m}\left[\frac{\partial}{\partial x_k}\rho(\underline{x})\right]^2\mathrm{d}\underline{x}\right)\\
&\quad\geq\left(\int_{\mathbf{R}^m}\left|x_k\rho\left[\frac{\partial}{\partial x_k}\rho(\underline{x})\right]\right|\mathrm{d}\underline{x}\right)^2\\
&\quad\geq\left|\int_{\mathbf{R}^m}x_k\rho\left[\frac{\partial}{\partial x_k}\rho(\underline{x})\right]\mathrm{d}\underline{x}\right|^2\\
&\quad=\left|\int_{\mathbf{R}^m}\frac{1}{2}\frac{\partial}{\partial x_k}\left(\rho^2x_k\right)\mathrm{d}\underline{x}-\int_{\mathbf{R}^2}\frac{1}{2}\rho^2\mathrm{d}\underline{x}\right|^2\\
&\quad=\frac{1}{4}.
\end{aligned}\tag{4.8}$$

The first term of (4.8) is a perfect differential and integrates to zero. The second term gives one half since we assume the signal is unit energy.

Similarly, we have

$$\begin{aligned}
&\left(\int_{\mathbf{R}^m} x_k^2\rho^2 \mathrm{d}\underline{x}\right)\left(\int_{\mathbf{R}^m}\rho^2(\underline{x})\left|\left(\frac{\partial}{\partial x_k}e^{\underline{u}(\underline{x})\theta(\underline{x})}\right)\left(e^{-\underline{u}(\underline{x})\theta(\underline{x})}\right)\right|^2\mathrm{d}\underline{x}\right)\\
&\quad\geq\left(\int_{\mathbf{R}^m}\left|x_k\left(\frac{\partial}{\partial x_k}e^{\underline{u}(\underline{x})\theta(\underline{x})}\right)\left(e^{-\underline{u}(\underline{x})\theta(\underline{x})}\right)\right|\rho^2\mathrm{d}\underline{x}\right)^2\\
&\quad=\mathrm{COV}^2_{x_k\xi_k}.
\end{aligned}\tag{4.9}$$

connecting (4.7)–(4.9), the inequality (4.6) holds.

Next, we deduce the conditions under which the equation holds in (4.6). The equation in (4.8) holds if and only if $\frac{\partial}{\partial x_k}\rho(\underline{x}) = \pm\alpha_k x_k\rho(\underline{x})$, where $\alpha_k > 0$. That is $\rho(\underline{x}) = e^{\pm\frac{\alpha_k}{2}x_k^2}$. For $f(\underline{x}) \in L^2(\mathbf{R}^m)$, then we choose $\rho(\underline{x}) = e^{-\frac{\alpha_k}{2}x_k^2}$.

Clearly, the equation holds in (4.9) if and only if

$$\left|\left(\frac{\partial}{\partial x_k}e^{\underline{u}(\underline{x})\theta(\underline{x})}\right)e^{-\underline{u}(\underline{x})\theta(\underline{x})}\right| = \beta_k|x_k|,\ \ \beta_k > 0.$$

This completes the proof. □

Corollary 4.1 *Let $f(\underline{x})$ be a real para-vector-valued signal with $\|f(\underline{x})\|_{L^2} = 1$. If $x_k f$ and $\frac{\partial f}{\partial x_k} \in L^2(\mathbf{R}^m)$ for $k = 1,\ldots,m$,, then*

$$\left(\int_{\mathbf{R}^m} x_k^2|f(\underline{x})|^2\mathrm{d}\underline{x}\right)\left(\int_{\mathbf{R}^m}\xi_k^2|F\{f\}(\underline{u})|^2\mathrm{d}\underline{u}\right) \geq \frac{1}{4} + \mathrm{Cov}^2_{x_k\xi_k},\tag{4.10}$$

where the covariance for every variable is defined by

$$\mathrm{Cov}_{x_k\xi_k} := \int_{\mathbf{R}^m} x_k\left(\frac{\partial}{\partial x_k}e^{\underline{u}(\underline{x})\theta(\underline{x})}\right)e^{-\underline{u}(\underline{x})\theta(\underline{x})}\rho^2(\underline{x})\mathrm{d}\underline{x}.$$

Theorem 4.3 (Uncertainty Principle in directional case) *Let $f(\underline{x}) = |f(\underline{x})|e^{\underline{u}(\underline{x})\theta(\underline{x})}$ be a real para-vector-valued signal with $\|f\|_{L^2} = 1$. If $x_k f(\underline{x})$, $\frac{\partial}{\partial x_k}f(\underline{x}) \in L^2(\mathbf{R}^m)$, for*

$k = 1, 2, \ldots, m$, then

$$\left(\int_{\mathbf{R}^m} |\underline{x}|^2 |f(\underline{x})|^2 \mathrm{d}\underline{x}\right)\left(\int_{\mathbf{R}^m} |\underline{\xi}|^2 |F\{f\}(\underline{\xi})|^2 \mathrm{d}\underline{\xi}\right) \geq \frac{m^2}{4} + \mathrm{COV}^2_{\underline{x}\underline{\xi}}, \tag{4.11}$$

where the absolute covariance in directional case is

$$\mathrm{COV}_{\underline{x}\underline{\xi}} := \sum_{k=1}^{m} \mathrm{COV}_{x_k \xi_k}.$$

The equality (4.11) holds if and only if $f(\underline{x}) = e^{-\frac{\alpha}{2}|\underline{x}|^2} e^{\underline{u}(\underline{x})\theta(\underline{x})}$ and $\left|\left(\frac{\partial}{\partial x_k} e^{\underline{u}(\underline{x})\theta(\underline{x})}\right) e^{-\underline{u}(\underline{x})\theta(\underline{x})}\right| = \beta |x_k|$. Here $\alpha > 0$ and $\beta > 0$.

Proof Applying (3.4) and (4.4), we have

$$\begin{aligned}
&\left(\int_{\mathbf{R}^m} |\underline{x}|^2 |f(\underline{x})|^2 \mathrm{d}\underline{x}\right)\left(\int_{\mathbf{R}^m} |\underline{\xi}|^2 |F\{f\}(\underline{\xi})|^2 \mathrm{d}\underline{\xi}\right) \\
&\quad = \left(\int_{\mathbf{R}^m} \sum_{k=1}^{m} x_k^2 \rho^2 \mathrm{d}\underline{x}\right)\left(\int_{\mathbf{R}^m} \sum_{k=1}^{m} \xi_k^2 |F\{f\}(\underline{\xi})|^2 \mathrm{d}\underline{\xi}\right) \\
&\quad = \left(\int_{\mathbf{R}^m} \sum_{k=1}^{m} x_k^2 \rho^2 \mathrm{d}\underline{x}\right)\left(\int_{\mathbf{R}^m} \sum_{k=1}^{m} \left|\frac{\partial}{\partial x_k} f(\underline{x})\right|^2 \mathrm{d}\underline{x}\right) \\
&\quad = \left(\int_{\mathbf{R}^m} \sum_{k=1}^{m} x_k^2 \rho^2 \mathrm{d}\underline{x}\right)\left(\int_{\mathbf{R}^m} \sum_{k=1}^{m} \left[\frac{\partial}{\partial x_k} \rho(\underline{x})\right]^2 + \rho^2(\underline{x}) \left|\left(\frac{\partial}{\partial x_k} e^{\underline{u}\theta}\right)\left(e^{-\underline{u}\theta}\right)\right|^2 \mathrm{d}\underline{x}\right) \\
&\quad = \left(\int_{\mathbf{R}^m} \sum_{k=1}^{m} x_k^2 \rho^2 \mathrm{d}\underline{x}\right)\left(\int_{\mathbf{R}^m} \sum_{k=1}^{m} \left[\frac{\partial}{\partial x_k} \rho(\underline{x})\right]^2 \mathrm{d}\underline{x}\right) \\
&\qquad + \left(\int_{\mathbf{R}^m} \sum_{k=1}^{m} x_k^2 \rho^2 \mathrm{d}\underline{x}\right)\left(\int_{\mathbf{R}^m} \sum_{k=1}^{m} \rho^2(\underline{x}) \left|\left(\frac{\partial}{\partial x_k} e^{\underline{u}(\underline{x})\theta(\underline{x})}\right)\left(e^{-\underline{u}(\underline{x})\theta(\underline{x})}\right)\right|^2 \mathrm{d}\underline{x}\right).
\end{aligned}$$

Applying the Schwarz inequality of continuous and discrete cases, we have

$$\begin{aligned}
&\left(\int_{\mathbf{R}^m} \sum_{k=1}^{m} x_k^2 \rho^2 \mathrm{d}\underline{x}\right)\left(\int_{\mathbf{R}^m} \sum_{k=1}^{m} \left[\frac{\partial}{\partial x_k} \rho(\underline{x})\right]^2 \mathrm{d}\underline{x}\right) \\
&\quad \geq \left|\int_{\mathbf{R}^m} \left(\sum_{k=1}^{m} x_k^2 \rho^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^{m} \left(\frac{\partial}{\partial x_k} \rho\right)^2\right)^{\frac{1}{2}} \mathrm{d}\underline{x}\right|^2 \\
&\quad \geq \left|\int_{\mathbf{R}^m} \sum_{k=1}^{m} \left(\frac{\partial}{\partial x_k} \rho\right) x_k \rho \mathrm{d}\underline{x}\right|^2 \\
&\quad = \frac{m^2}{4}.
\end{aligned}$$

(4.8) is used in the last step. Similarly, we have

$$\left(\int_{\mathbf{R}^m}\sum_{k=1}^{m}x_k^2\rho^2\mathrm{d}\underline{x}\right)\left(\int_{\mathbf{R}^m}\sum_{k=1}^{m}\rho^2(\underline{x})\left|\left(\frac{\partial}{\partial x_k}e^{\underline{u}(\underline{x})\theta(\underline{x})}\right)\left(e^{-\underline{u}(\underline{x})\theta(\underline{x})}\right)\right|^2\mathrm{d}\underline{x}\right)$$
$$\geq\left(\sum_{k=1}^{m}\int_{\mathbf{R}^m}\left|x_k\left(\frac{\partial}{\partial x_k}e^{\underline{u}\theta}\right)e^{-\underline{u}\theta}\right|\rho^2\mathrm{d}\underline{x}\right)^2.$$

Similarly, like Theorem 4.2, the equality (4.11) holds if and only if $f(\underline{x})=e^{-\frac{\alpha}{2}|\underline{x}|^2}e^{\underline{u}(\underline{x})\theta(\underline{x})}$ and $|(\frac{\partial}{\partial x_k}e^{\underline{u}(\underline{x})\theta(\underline{x})})e^{-\underline{u}(\underline{x})\theta(\underline{x})}|=\beta|x_k|$. Here $\alpha>0$ and $\beta>0$. This completes the proof. □

Corollary 4.2 *Let $f(\underline{x})=|f(\underline{x})|e^{\underline{u}(\underline{x})\theta(\underline{x})}$ be a real para-vector-valued signal with $\|f\|_{L^2}=1$. If $x_kf(\underline{x})$, $\frac{\partial}{\partial x_k}f(\underline{x})\in L^2(\mathbf{R}^m)$, for $k=1,2,\ldots,m$ then*

$$\left(\int_{\mathbf{R}^m}|\underline{x}|^2|f(\underline{x})|^2\mathrm{d}\underline{x}\right)\left(\int_{\mathbf{R}^m}|\underline{\xi}|^2|F\{f\}(\underline{\xi})|^2\mathrm{d}\underline{\xi}\right)$$
$$\geq\frac{m^2}{4}+\left|\mathrm{Cov}_{\underline{x}\underline{\xi}}\right|^2,$$

where the covariance in directional case is

$$\mathrm{Cov}_{\underline{x}\underline{\xi}}:=\sum_{k=1}^{m}\mathrm{Cov}_{x_k\xi_k}.$$

5. Example

Example 5.1 Consider a real para-vector-valued signal of unit energy

$$f(\underline{x})=\left(\frac{\alpha}{\pi}\right)^{\frac{m}{4}}e^{\frac{-\alpha|\underline{x}|^2}{2}}e^{\underline{u}\frac{|\underline{x}|^2}{2}},$$

where α is a positive real number and $\underline{u}\in S^m$ is a vector-valued constant.

Computing directly, we have

$$\int_{\mathbf{R}^m}x_k^2|f(\underline{x})|^2\mathrm{d}\underline{x}=\left(\frac{\alpha}{\pi}\right)^{\frac{m}{2}}\int_{\mathbf{R}^m}x_k^2e^{-\alpha|\underline{x}|^2}\mathrm{d}\underline{x}$$
$$=\frac{1}{2\alpha},$$

and

$$\int_{\mathbf{R}^m}\xi_k^2|F\{f\}(\underline{\xi})|^2\mathrm{d}\underline{\xi}=\int_{\mathbf{R}^m}\left(\frac{\partial}{\partial x_k}\rho\right)^2\mathrm{d}\underline{x}+\int_{\mathbf{R}^2}\rho^2\left|\left(\frac{\partial}{\partial x_k}e^{\underline{u}\theta}\right)e^{-\underline{u}\theta}\right|^2\mathrm{d}\underline{x}$$
$$=\left(\frac{\alpha}{\pi}\right)^{\frac{m}{2}}\alpha^2\int_{\mathbf{R}^m}x_k^2e^{-\alpha|\underline{x}|^2}\mathrm{d}\underline{x}+\left(\frac{\alpha}{\pi}\right)^{\frac{m}{2}}\int_{\mathbf{R}^m}x_k^2e^{-\alpha|\underline{x}|^2}\mathrm{d}\underline{x}$$
$$=\frac{\alpha}{2}+\frac{1}{2\alpha}.$$

It is easy to see that $\text{Cov}_{x_k\xi_k} = \frac{\underline{u}}{2\alpha}$, $\text{COV}_{x_k\xi_k} = \frac{1}{2\alpha}$, $k = 1, 2, \ldots, m$. Then

$$\begin{aligned}&\left(\int_{\mathbf{R}^m} x_k^2 |f(\underline{x})|^2 \mathrm{d}\underline{x}\right)\left(\int_{\mathbf{R}^m} \xi_k^2 |F\{f\}(\underline{\xi})|^2 \mathrm{d}\underline{\xi}\right)\\ &= \frac{1}{4} + \frac{1}{4\alpha^2}\\ &= \frac{1}{4} + \left(\frac{1}{2\alpha}\right)^2 = \frac{1}{4} + \text{COV}^2_{x_1\xi_1}\\ &= \frac{1}{4} + |\frac{\underline{u}}{2\alpha}|^2 = \frac{1}{4} + |\text{Cov}_{x_1\xi_1}|^2\end{aligned}$$

and

$$\begin{aligned}&\left(\int_{\mathbf{R}^m} |\underline{x}|^2 |f(\underline{x})|^2 \mathrm{d}\underline{x}\right)\left(\int_{\mathbf{R}^m} |\underline{\xi}|^2 |F\{f\}(\underline{\xi})|^2 \mathrm{d}\underline{\xi}\right)\\ &= \frac{m^2}{4} + \frac{m^2}{4\alpha^2}\\ &= \frac{m^2}{4} + \left(\frac{m}{2\alpha}\right)^2 = \frac{m^2}{4} + \text{COV}^2_{\underline{x}\underline{\xi}}\\ &= \frac{m^2}{4} + |\frac{m\underline{u}}{2\alpha}|^2 = \frac{m^2}{4} + \left|\text{Cov}_{\underline{x}\underline{\xi}}\right|^2.\end{aligned}$$

Note that, in this case, the stronger forms of uncertainty principle of Theorems 4.2 and 4.3 become equalities. In fact, $|(\frac{\partial}{\partial x_k} e^{\underline{u}(\underline{x})\theta(\underline{x})})e^{-\underline{u}(\underline{x})\theta(\underline{x})}| = |\underline{u}x_k|$, which satisfies the conditions as given in (4.6) and (4.11).

Example 5.2 Consider a real para-vector-valued signal of unit energy

$$f(\underline{x}) = \left(\frac{\alpha}{\pi}\right)^{\frac{m}{4}} e^{\frac{-\alpha|\underline{x}|^2}{2}} e^{\beta_1 x_1 \mathbf{e}_1},$$

where α is a positive real number and $\beta_1 \in \mathbf{R}$.

By Example 5.1, we have

$$\begin{aligned}\int_{\mathbf{R}^m} x_k^2 |f(\underline{x})|^2 \mathrm{d}\underline{x} &= \left(\frac{\alpha}{\pi}\right)^{\frac{m}{2}} \int_{\mathbf{R}^m} x_k^2 e^{-\alpha|\underline{x}|^2} \mathrm{d}\underline{x}\\ &= \frac{1}{2\alpha}.\end{aligned} \tag{5.1}$$

By direct calculation, we have

$$\begin{aligned}\int_{\mathbf{R}^m} \xi_1^2 |F\{f\}(\underline{\xi})|^2 \mathrm{d}\underline{\xi} &= \int_{\mathbf{R}^m} \left(\frac{\partial}{\partial x_1}\rho\right)^2 \mathrm{d}\underline{x} + \int_{\mathbf{R}^m} \rho^2 \left|\left(\frac{\partial}{\partial x_1} e^{\underline{u}\theta}\right) e^{-\underline{u}\theta}\right|^2 \mathrm{d}\underline{x}\\ &= \left(\frac{\alpha}{\pi}\right)^{\frac{m}{2}} \alpha^2 \int_{\mathbf{R}^m} x_1^2 e^{-\alpha|\underline{x}|^2} \mathrm{d}\underline{x} + \left(\frac{\alpha}{\pi}\right)^{\frac{m}{2}} \beta_1^2 \int_{\mathbf{R}^m} e^{-\alpha|\underline{x}|^2} \mathrm{d}\underline{x}\\ &= \frac{\alpha}{2} + \beta_1^2\end{aligned} \tag{5.2}$$

and for $k = 2, \ldots, m$, we have

$$\int_{\mathbf{R}^m} \xi_k^2 |F\{f\}(\underline{\xi})|^2 \mathrm{d}\underline{\xi} = \int_{\mathbf{R}^m} \left(\frac{\partial}{\partial x_k}\rho\right)^2 \mathrm{d}\underline{x} + \int_{\mathbf{R}^m} \rho^2 \left|\left(\frac{\partial}{\partial x_k} e^{\underline{u}\theta}\right) e^{-\underline{u}\theta}\right|^2 \mathrm{d}\underline{x}$$
$$= \left(\frac{\alpha}{\pi}\right)^{\frac{m}{2}} \alpha^2 \int_{\mathbf{R}^m} x_k^2 e^{-\alpha|\underline{x}|^2} \mathrm{d}\underline{x}$$
$$= \frac{\alpha}{2}. \tag{5.3}$$

Clearly, we have $\mathrm{Cov}_{x_k\xi_k} = 0$, for $k = 1, 2, \ldots, m$, and $\mathrm{COV}_{x_1\xi_1} = \frac{|\beta_1|}{\sqrt{\pi\alpha}}$, $\mathrm{COV}_{x_k\xi_k} = 0$ for $k = 2, \ldots, m$.

Therefore, we have

$$\left(\int_{\mathbf{R}^m} x_1^2 |f(\underline{x})|^2 \mathrm{d}\underline{x}\right)\left(\int_{\mathbf{R}^m} \xi_1^2 |F\{f\}(\underline{\xi})|^2 \mathrm{d}\underline{\xi}\right)$$
$$= \frac{1}{4} + \frac{\beta_1^2}{2\alpha}$$
$$> \frac{1}{4} + \frac{\beta_1^2}{\pi\alpha} = \frac{1}{4} + \mathrm{COV}^2_{x_1\xi_1}$$
$$> \frac{1}{4} = \frac{1}{4} + |\mathrm{Cov}_{x_1\xi_1}|^2 \tag{5.4}$$

and for $k = 2, \ldots, m$

$$\left(\int_{\mathbf{R}^m} x_k^2 |f(\underline{x})|^2 \mathrm{d}\underline{x}\right)\left(\int_{\mathbf{R}^m} \xi_k^2 |F\{f\}(\underline{\xi})|^2 \mathrm{d}\underline{\xi}\right)$$
$$= \frac{1}{4}$$
$$= \frac{1}{4} + \mathrm{COV}^2_{x_k\xi_k}$$
$$= \frac{1}{4} + |\mathrm{Cov}_{x_k\xi_k}|^2. \tag{5.5}$$

Expressions (5.4) and (5.5) verify Theorem 4.2.

Applying (5.1)–(5.3), we have

$$\int_{\mathbf{R}^m} |\underline{x}|^2 |f(\underline{x})|^2 \mathrm{d}\underline{x} = \frac{m}{2\alpha},$$
$$\int_{\mathbf{R}^m} |\underline{\xi}|^2 |F\{f\}(\underline{\xi})|^2 \mathrm{d}\underline{\xi} = \frac{m\alpha}{2} + \beta_1^2.$$

Then

$$\left(\int_{\mathbf{R}^m} |\underline{x}|^2 |f(\underline{x})|^2 \mathrm{d}\underline{x}\right)\left(\int_{\mathbf{R}^m} |\underline{\xi}|^2 |F\{f\}(\underline{\xi})|^2 \mathrm{d}\underline{\xi}\right)$$
$$= \frac{m^2}{4} + \frac{m}{2\alpha}\beta_1^2$$
$$> \frac{m^2}{4} + \frac{\beta_1^2}{\pi\alpha} = \frac{m^2}{4} + \mathrm{COV}^2_{\underline{x}\underline{\xi}}$$
$$> \frac{m^2}{4} = \frac{m^2}{4} + \left|\mathrm{Cov}_{\underline{x}\underline{\xi}}\right|^2. \tag{5.6}$$

Here $\mathrm{Cov}_{\underline{x\xi}} = 0$ and $\mathrm{COV}_{\underline{x\xi}} = \frac{|\beta_1|}{\sqrt{\pi\alpha}}$. (5.6) verifies Theorem 4.3.

Disclosure statement

No potential conflict of interest was reported by the authors.

Funding

This work was supported by Macao Science and Technology Development Fund, MSAR. Ref. 018/2014/A1; Macao Government FDCT 098/2012/A3; and University of Macau Multi-Year Research [grant number (MYRG) MYRG116(Y1-L3)-FST13-QT].

References

[1] Da ZX. Modern signal processing. 2nd ed. Beijing: Tsinghua University Press; 2002. p. 362.
[2] Dembo A, Cover TM. Information theoretic inequalities. IEEE Trans. Inform. Theory. 1991;37:1501–1508.
[3] Hardy G, Littlewood JE, Polya G. Inequalities. 2nd ed. London: Cambridge University Press; 1952.
[4] Heinig H, Smith M. Extensions of the Heisenberg-Weyl inequality. Int. J. Math. Math. Sci. 1986;9:185–192.
[5] Loughlin PJ, Cohen L. The uncertainty principle: global, local, or both? IEEE Trans. Signal Porcess. 2004;52:1218–1227.
[6] Mustard D. Uncertainty principle invariant under fractional Fourier transform. J. Aust. Math. Soc. Ser. B. 1991;33:180–191.
[7] Majernik V, Eva M, Shpyrko S. Uncertainty relations expressed by Shannon-like entropies. CEJP. 2003;3:393–420.
[8] Ozaktas HM, Aytur O. Fractional Fourier domains. Signal Process. 1995;46:119–124.
[9] Stern A. Sampling of compact signals in offset linear canonical transform domains. Signal Image Video Process. 2007;1:259–367.
[10] Shinde S, Gadre VM. An uncertainty principle for real signals in the fractional Fourier transform domain. IEEE Trans. Signal Process. 2001;49:2545–2548.
[11] Xu GL, Wang XT, Xu XG. Three uncertainty relations for real signals associated with linear canonical transform. IET Signal Process. 2009;3:85–92.
[12] Aytur O, Ozaktas HM. Non-orthogonal domains in phase space of quantum optics and their relation to fractional Fourier transform. Opt. Commun. 1995;120:166–170.
[13] Cohen L. Time-frequency analysis: theory and applications. Upper Saddle River (NJ): Prentice Hall; 1995.
[14] Iwo BB. Entropic uncertainty relations in quantum mechanics. Accardi L, Von Waldenfels W, editors. Quantum probability and applications II, Vol. 1136, Lecture notes in mathematics. Berlin: Springer; 1985. p. 90.
[15] Iwo BB. Formulation of the uncertainty relations in terms of the Rényi entropies. Phys. Rev. A. 2006;74:052101.
[16] Iwo BB. Rényi entropy and the uncertainty relations. Adenier G, Fuchs CA, Yu A, editors. Foundations of probability and physics, Khrennikov, AIP Conference Proceedings 889. Melville: American Institute of Physics; 2007. p. 52–62
[17] Maassen H. A discrete entropic uncertainty relation, quantum probability and applications. Lecture notes in mathematics. Berlin: Springer; 1990. p. 263–266.

[18] Maassen H, Uffink JBM. Generalized entropic uncertainty relations. Phys. Rev. Lett. 1988;60:1103–1106.
[19] Rényi A. On measures of information and entropy. Proc. Fouth Berkeley Symp. Math. Stat. Probab. 1960;1: 547–561.
[20] Stern A. Uncertainty principles in linear canonical transform domains and some of their implications in optics. J. Opt. Soc. Am. A. 2008;25:647–652.
[21] Wódkiewicz K. Operational approach to phase-space measurements in quantum mechanics. Phys. Rev. Lett. 1984;52:1064–1067.
[22] Dang P, Deng GT, Qian T. Uncertainty principle involving phase derivative. J. Funct. Anal. 2013;265:2239–2266.
[23] Ozaktas HM, Zalevsky Z, Kutay MA. The fractional Fourier transform with applications in optics and signal processing. Chichester: Wiley; 2001.
[24] Kou KI, Xu RH, Zhang YH. Paley-Wiener theorems and uncertainty principles for the windowned linear canonical transform. Math. Methods Appl. Sci. 2012;35:2122–2132.
[25] Sharma KK, Joshi SD. Uncertainty principles for real signals in linear canonical transform domains. IEEE Trans. Signal Process. 2008;56:2677–2683.
[26] Dang P, Deng GT, Qian T. A tighter uncertainty principle for linear cannonical transform in terms of phase derivative. IEEE Trans. Signal Process. 2013;61:5153–5164.
[27] Hitzer E, Mawardi B. Uncertainty principle for the Clifford geometric algebra $Cl_{n,0}, n = 3(mod 4)$ based on Clifford Fourier transform. In: Qian T, Vai MI, Xu Y, editors. The Springer (SCI) Book Series 'Applied and Numerical Harmonic Analysis'. Switzerland: 2007. p. 47–56.
[28] Hitzer E, Mawardi B. Clifford Fourier transform on multivector fields and uncertainty principles for dimensions $n = 2(mod 4)$ and $n = 3(mod 4)$. Adv. Appl. Clifford Algebras. 2008. doi:10.1007/s00006-008-0098-3. Online First, 25 May.
[29] Mawardi B, Hitzer E. Clifford Fourier transformation and uncertainty principle for the Clifford geometric algebra $Cl_{3,0}$ Adv. Appl. Clifford Algebras. 2006;16:41–61.
[30] Mawardi B, Hitzer E. Clifford algebra $Cl_{3,0}$-valued wavelet transformation, Clifford wavelet uncertainty inequality and Clifford Gabor wavelets. Int. J. Wavelet Multiresolution Inf. Process. 2007;5:997–1019.
[31] Yang Y, Qian T, Sommen F. Phase derivative of monogenic functions in higher dimensional spaces. Complex Anal. Operator Theory. 2012;6:987–1010.
[32] Yang Y, Dang P, Qian T. Space-frequency analysis in higher dimensions and applications. Annali di Matematica 2014;1–16. doi:10.1007/s1023-014-0406-6
[33] Yang Y, Kou KI. Uncertainty principles for hypercomplex signals in the linear canonical transform domains. Signal Process. 2014;95:67–75.
[34] Clifford WK. Preliminary sketch of bi-quaternions. Proc. London Math. Soc. 1873;4:381–395.
[35] Clifford WK. Mathematical papers. London: Macmillan; 1882.
[36] Brackx F, Delanghe R, Sommen F. Clifford analysis. Boston (MA): Pitman; 1982.
[37] Delanghe R, Sommen F, Soucek V. Clifford algebra and spinor valued functions. Dordrecht: Kluwer; 1992.
[38] Korn P. Some uncertainty principle for time-frequency transforms for the Cohen class. IEEE Trans. Signal Process. 2005;53:523–527.
[39] Ozaktas HM, Kutay MA, Zalevsky Z. The fractional Fourier transform with applications in optics and signal processing. New York (NY): Wiley; 2000.