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Consecutive minimum phase expansion of physically realizable signals with applications

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In digital signal processing, it is a well know fact that a causal signal of finite energy is front loaded if and only if the corresponding analytic signal, or the physically realizable signal, is a minimum phase signal, or an outer function in the complex analysis terminology. Based on this fact, a series expansion method, called unwinding adaptive Fourier decomposition (AFD), to give rise to positive frequency representations with rapid convergence was proposed several years ago. It appears to be a promising positive frequency representation with great potential of applications. The corresponding algorithm, however, is complicated due to consecutive extractions of outer functions involving computation of Hilbert transforms. This paper is to propose a practical algorithm for unwinding AFD that does not depend on computation of Hilbert transform, but, instead, factorizes out the Blaschke product type of inner functions. The proposed method significantly improves applicability of unwinding AFD. As an application, we give the associated Dirac-type time-frequency distribution of physically realizable signals. Copyright © 2015 John Wiley & Sons, Ltd.

Keywords: unwinding AFD; Blaschke product; Dirac type time-frequency distribution

1. Introduction

In signal analysis functions in the Hardy spaces H^2 are identical, in various contexts, with analytic signals of finite energy, and in the periodic case, physically realizable signals. The close relation between functions in the Hardy spaces and signals of finite energy in general may be seen as follows ([1, 2]). For a real-valued signal of finite energy, denoted by f, we have the relations

$$f = 2\mathrm{Re}f^{+} - c_0, \tag{1}$$

where

$$f\left(e^{it}\right) = \sum_{k=-\infty}^{\infty} c_k e^{ikt}, \quad f^{+}\left(e^{it}\right) = \sum_{k=0}^{\infty} c_k e^{ikt},$$

where c_k's are the Fourier coefficients satisfying the relation

$$\|f\|^2 = \sum_{-\infty}^{\infty} |c_k|^2 < \infty.$$

In the aforementioned formulation, f^{+} is the related Hardy space function, or the analytic signal associated with f. The space of complex analytic functions in the unit disc

$$H^2(\mathbf{D}) = \left\{ f : \mathbf{D} \to \mathbf{C} \mid f(z) = \sum_{k=0}^{\infty} c_k z^k, \ \sum_{k=0}^{\infty} |c_k|^2 < \infty \right\}$$

is defined to be the Hardy H^2 space in the unit disc $\mathbf{D}$. The unit disc formulation corresponds to signals defined on compact intervals or periodic signals. Signals on the whole time range $(-\infty, \infty)$ are related to the Hardy space of the upper-half complex plane. In such case, a real-valued signal f of finite energy satisfies

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$$f = 2\mathrm{Re}f^{+}, \quad f^{+}(x) = \frac{1}{2\pi}\int_0^{\infty} e^{ix\xi}\hat{f}(\xi)d\xi, \hat{f}(\xi) = \int_{-\infty}^{\infty} e^{-ix\xi}f(x)dx, \quad \|f\|^2 = 2\|\hat{f}\|^2 < \infty.$$

The aforementioned relations show that the study of a real-valued signal, f, may be reduced to the study of its Hardy space projection f^{+}. This idea and the related theory and practice have been developed by a number of researchers ([1–11]).

Adaptive Fourier decompositions, or AFDs, amount to adaptively expanding functions in the Hardy spaces into linear combinations of the related Szegö kernels, or more precisely, the parameterized Szegö kernels or reproducing kernels of the underlying Hardy spaces, namely the Hardy H^2 spaces. With the concept AFD, besides the standard Szegö kernels, we also allow higher order Szegö kernels and other functions closely related to Szegö kernels, including partial fractions with multiple poles.

Because the work [11] and [12], AFDs, including what is lately called Core AFD and its variations ([10]), have become more sophisticated and have found their applications ([7, 13]).

The current paper concentrates on the unit disc context. The upper-half plane context has an analogous theory. The normalized Szegö kernel of the unit disc is

$$e_a(z) = \frac{\sqrt{1-|a|^2}}{1-\bar{a}z}.$$

We call it the parameterized Szegö kernel by the parameter $a \in \mathbf{D}$. It enjoys the reproducing property: For $f \in H^2(\mathbf{D})$,

$$\langle f, e_a\rangle = \sqrt{1-|a|^2}f(a).$$

The last formula is essentially the Cauchy formula.

Let $a_1, \ldots, a_n, \ldots$ be a sequence of complex numbers inside the unit disc where $a_n \neq a_m$ if $n \neq m$. Applying the Gram–Schmidt orthogonalization process to $e_{a_1}, \ldots, e_{a_n}, \ldots$, we obtain the associated orthonormal rational system, called Takenaka-Malmquist system or TM system in brief,

$$\{B_n\}_{k=0}^{\infty}, \quad B_n(z) = \frac{\sqrt{1-|a_n|^2}}{1-\bar{a}_n z}\prod_{k=1}^{n-1}\frac{z-a_k}{1-\bar{a}_k z}. \tag{2}$$

We note that the formula for B_n allows $a_n = a_m$ for $n \neq m$. Without the assumption $a_n \neq a_m, n \neq m$, the sequence $e_{a_1}, \ldots, e_{a_n}, \ldots$, should be replaced by a related sequence $E_{a_1}, \ldots, E_{a_n}, \ldots$, where multiples of a_n's are taken into account. The resulted $\{B_n\}_{n=1}^{\infty}$ by applying the G-S process to $E_{a_1}, \ldots, E_{a_n}, \ldots$ allowing multiple a_n's are of the same forms as those derived under the assumption $a_n \neq a_m, n \neq m$ ([11]). When all the a_n's are identical with 0 the system reduces to $\{z^{n-1}\}_{n=1}^{\infty}$, being the Fourier system for the Hardy space $H^2(\mathbf{D})$. This last remark gives a reason of the terminology adaptive Fourier decomposition by adaptive implementation of the rational system $\{B\}_{n=1}^{\infty}$.

The purpose of this article is to give a practical algorithm for unwinding AFD, that, in the engineering terminology, may be rephrased as consecutive minimum phase expansion of physically realizable signals. Before we get into the algorithm issue, we now first summarize the idea of Unwinding AFD from digital signal processing (DSP): it is a process of consecutive extractions of front loaded signals being incorporated into each step of Core AFD. We first recall the Nevanlinna Factorization Theorem.

From now on, we restrict ourselves to the Hardy H^2 space for the unit disc. For functions in Hardy spaces, the Nevanlinna Theorem asserts that if $f \in H^2(\mathbf{D})$, then $f(z) = O_f(z)I_f(z)$, where O_f and I_f are, respectively, the outer and inner function factors of f. The inner function part can be further decomposed as $I_f = B_f S_f$, where B_f and S_f are, respectively, called the Blaschke and singular inner function part of the inner function I_f. The aforementioned factorization decompositions are unique up to unimodular constant factors. The Outer, Blaschke, and singular inner functions in relation to f are given, respectively, by (6), (7), and (8).

We say a physically realizable signal of finite energy f is *front loaded*, if with the series expansion $f = \sum_{k=0}^{\infty} c_k z^k$, and for any other physically realizable signal of finite energy $g = \sum_{k=0}^{\infty} d_k z^k$, there always holds

$$\sum_{k=0}^{n}|c_k|^2 \geq \sum_{k=0}^{n}|d_k|^2, \quad \text{for any positive integer } n.$$

In DSP, it is a well known fact that a physically realizable function of finite energy f is front loaded if and only if f itself is an outer function, or, equivalently, a minimum phase signal.

Within each step of the theoretical unwinding AFD algorithm, extracting out the outer function part of a function $f \in H^2(\mathbf{D})$ requires computing a Hilbert transform. The cost of computing Hilbert transform is high, but the accuracy is low, which prevents unwinding AFD from practical use. There have been a number of competitive algorithms for computation of Hilbert transform that are all insufficient to treat the complexity of Unwinding AFD. Due to the super effectiveness of unwinding AFD (see [10]), however, a deep study on the algorithm is necessary. This article presents an algorithm that, with a low-computation cost, significantly lifts up the convergence speed of unwinding AFD series. The writing plan of the paper is as follows. In Section 2, we give a revision on the theoretical algorithm of unwinding AFD. In Section 3, we introduce our new and practical algorithm with examples for comparison. In Section 4, we introduce the Dirac-type time-frequency distribution based on the positive frequency decomposition of signals obtained from Unwinding AFD. In the final section, Section 5, we draw conclusions.

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2. A revision on unwinding AFD

Because of the relation (1), we can restrict ourselves to merely the Hardy spaces, and, precisely, $H^2(\mathbf{D})$. Let $f_1 = f \in H^2(\mathbf{D})$. By using the Nevanlinna's factorization theorem ([14]), we have the decomposition $f_1 = O_1 I_1$, where O_1 and I_1 are, respectively, the outer and inner functions associated with f_1. Note that in the factorization, we ignore the difference between two functions, deferring only by a multiplicative unimodular constant. For any $a_1 \in \mathbf{D}$, we have the identity

$$f(z) = I_1(z)\langle O_1, e_{a_1}\rangle e_{a_1}(z) + I_1(z)\left(\frac{O_1(z) - \langle O_1, e_{a_1}\rangle e_{a_1}(z)}{\frac{z-a_1}{1-\bar{a}_1 z}}\right)\frac{z-a_1}{1-\bar{a}_1 z}.$$

The reasoning of doing such factorization and expansion is referred to a concept called 'energy delay' in DSP; see, for instance, [12]. Denote

$$f_2(z) = \frac{O_1(z) - \langle O_1, e_{a_1}\rangle e_{a_1}(z)}{\frac{z-a_1}{1-\bar{a}_1 z}},$$

and apply the Nevanlinna factorization to f_2 and get $f_2 = I_2 O_2$. Next, to f_2, we repeat what is performed to f_1, and so on. We obtain, after n steps,

$$f(z) = \sum_{k=1}^{n}\langle O_k, e_{a_k}\rangle I^{(k)}(z)B_k(z) + I_n(z)\prod_{k=1}^{n}\frac{z-a_k}{1-\bar{a}_k z}f_{n+1}, \tag{3}$$

where

$$I^{(k)} = I_1 \cdots I_k,\ f_k = I_k O_k,\ f_{k+1}(z) = \frac{O_k(z) - \langle O_k, e_{a_k}\rangle e_{a_k}(z)}{\frac{z-a_k}{1-\bar{a}_k z}}. \tag{4}$$

The aforementioned equation is an identity for any $a_1, \ldots, a_n$ in the unit disc. What makes it to be unwinding AFD is that the selection of each a_k satisfies the maximal selection principle

$$a_k = \max\arg\{|\langle Q_k, e_a\rangle| \mid a \in \mathbf{D}\}.$$

In [11], it is proved that such $a_k \in \mathbf{D}$ exists, and under maximal selections of a_k's, there holds

$$f(z) = \sum_{k=1}^{\infty}\langle O_k, e_{a_k}\rangle I^{(k)}(z)B_k(z). \tag{5}$$

We note that the decomposition is orthonormal. Based on the analogous property of a TM system, the unit-norm property of $I^{(k)}(z)B_k(z)$ is obvious. Now, we show the orthogonality property. For any positive integers k, l, we have, by invoking Cauchy's Theorem,

$$\left\langle I^{(k+l)}B_{k+l}, I^{(k)}B_k\right\rangle = \frac{1}{2\pi}\int_0^{2\pi} I^{(k+1)}\left(e^{it}\right)\cdots I^{(k+l)}\left(e^{it}\right)\cdot B_{k+l}\left(e^{it}\right)\bar{B}_k\left(e^{it}\right)dt = 0.$$

The nonlinear optimal selections of the a_k's in core AFD already induce considerable complexity in its algorithm, what is added in the Unwinding AFD case is the factorization that induces much more complexity. Performing the Nevanlinna factorization to any function $f \in H^2(\mathbf{D})$, we have, more precisely, $f = O_f I_f$, where O_f is the outer function and I_f is the inner function of f, but further, $I_f = B_f S_f$, where B_f is the Blaschke product part, and S_f is the singular inner function part of f. They are given, respectively, by the following formulas: for $z \in \mathbf{D}$,

$$O_f(z) = Ce^{\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}\log|f(e^{it})|dt}, \quad |C| = 1; \tag{6}$$

and

$$B_f(z) = z^m\prod_{k=m+1}^{\infty}\frac{-|b_k|}{b_k}\frac{z-b_k}{1-\bar{b}_k z}, \tag{7}$$

where $b_k, k = 1, 2, \ldots$, be all the zeros of f, and $b_1 = \cdots = b_m = 0$, and $b_{m+l} \neq 0, l = 1, 2, \cdots$; and,

$$S_f(z) = e^{-\int_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}d\mu(t)}, \tag{8}$$

where $d\mu(t)$ is a positive regular Borel measure singular to Lebesgue measure. We note that in order to obtain unwinding AFD, we need to compute both the outer and inner functions, and what we need, in fact, are the boundary limits of those functions.

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3. Computation of inner and outer functions

The natural method to obtain O_f and I_f is to first work out O_f by using the formula (6). We eventually need the boundary limit function $O_f(e^{it})$. Computing the boundary limit is equivalent to computing the Hilbert transform of $\log|f(e^{it})|$. Once we achieve in getting O_f we use the formula $I_f = f/O_f$ to get the inner function, including its boundary limit. An algorithm for such computation costs a lot and cannot be accurate, because of the singularity of the kernel of Hilbert transformation in general. This motivates an alternative strategy. We proceed with first working out the inner function part, and then the outer function part. To do so, we introduce the mild assumption that the Hardy space function under consideration may be analytically extended across the unit circle. The Hardy space theory then implies that f has a trivial and, in fact, constant singular inner function, and B_f only has finitely many zeros as the 'winding' part. In such case, $I_f = B_f$ is easy to be obtained (7). Accordingly, and, accordingly, $O_f = f/B_f$. In practice, we are given a discrete set of data on the boundary. By analytic interpolation methods, we can find, in fact, a family of Hardy space functions that satisfy the discrete data boundary condition, and being analytically extendable across the boundary of the unit disc. If, theoretically, the involved Hardy space signal is not analytically extendable across the unit circle, then the introduced method only unwinds finite Blaschke products parts and leaves infinite and singular inner function parts as they are. In such case, the results of approximation by Hardy space functions of finitely many zeros guarantee applicability of the algorithm ([15]).

3.1. A practical method for finding zeros

From now on, we assume that $f \in H^2(\mathbf{D})$ is analytically extendable across the unit circle. In such case, $S_f = 1$. In practice, it is often the case. If not, we, in fact, seek for an approximating function of such property. Under such assumption, we now claim that f has a finite number of zeros on $\overline{\mathbf{D}}$. If not, the zeros of f have an accumulation point on $\overline{\mathbf{D}}$. The theorem of isolate zeros in complex analysis implies that $f \equiv 0$, which is the case of triviality.

Suppose that f has N_1 zeros in $\mathbf{D}$ and N_2 zeros on $\partial\mathbf{D}$. Then,

$$N = N_1 + \frac{1}{2}N_2 = \frac{1}{2\pi i}\int_{|z|=1} \frac{f'(z)}{f(z)}dz. \tag{9}$$

Because f has a finite number of zeros in $\mathbf{D}$, we have

$$N_1 = \lim_{\delta\to 0^+} \frac{1}{2\pi i}\int_{|z|=1-\delta} \frac{f'(z)}{f(z)}dz. \tag{10}$$

It is obvious that there exists $\delta > 0$ such that all zeros in $\mathbf{D}$ belong to $\mathbf{D}_{1-\delta} = \{z \in \mathbf{C} \mid |z| < 1-\delta\}$. We have that $f(b) = 0$ implies

$$b = \arg\min_{z\in\mathbf{D}_{1-\delta}} |f(z)|. \tag{11}$$

because of these simple facts, we can find the N_1 zeros by the following steps.
Procedure:

1. Determine N by (9);
2. Set $g_1 = f$ and apply (11) to g_1, then obtain b_1 satisfying $g_1(b_1) = 0$;
3. Set $g_k = g_{k-1}\frac{1-\bar{b}_{k-1}z}{z-b_{k-1}}$ and apply (11) to g_k, then obtain b_k satisfying $g_k(b_k) = 0, 2 \le k \le N$;
4. The procedure ceases if $|g_k(b_k)| > \epsilon_0 > 0$, where ϵ_0 is a given constant. Otherwise, repeat step 3.

The procedure shows that in the process of AFD we factorize as many as possible Möbius transforms in each step in order to speed up the convergence.

4. Time frequency analysis of the decomposition

In this section, we concentrate on the time-frequency distribution induced by the proposed method. Recently, a novel development of Dirac-type time-frequency distribution based on mono-component decomposition of signals is made in [7]. Here, we present the time-frequency distribution of the proposed method in spirit of [7].

First, we recall the definition of periodic mono-component signals [12]. A signal of finite energy $s(e^{it}) = \rho(t)e^{i\theta(t)}$ on $[0, 2\pi]$ is a mono-component, if there holds the function equation

$$\mathcal{H}s = -is,$$

and suitably defined $\theta'(t) \ge 0$, where $\mathcal{H}(f)(e^{ix}) := \frac{1}{2\pi}p.v\int_0^{2\pi} f(e^{it})\cot\left(\frac{x-t}{2}\right)dt$ is the circular Hilbert transform. Note that a signal of finite energy is in the Hardy space $H^2(\mathbf{D})$ if and only if $Hs = -is$ on the boundary.

The Blaschke product of f with the assumed properties in the previous section

$$B_f\left(e^{it}\right) = e^{imt}\prod_{k=m+1}^{N} \frac{-|b_k|}{b_k}\frac{e^{it}-b_k}{1-\bar{b}_k e^{it}}$$

is a mono-component signal.

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In fact, $B_f(e^{it}) = |B_f(e^{it})|e^{i\theta(t)} = e^{i\theta(t)}$, it is easy to calculate

$$\theta'(t) = \sum_{k=1}^{N} \frac{1-|b_k|^2}{1-2|b_k|\cos(t-\theta_{b_k})+|b_k|^2},$$

where $b_k = |b_k|e^{i\theta_{b_k}}$. We also have

$$0 < \frac{1-|b_k|}{1+|b_k|} \le \frac{1-|b_k|^2}{1-2|b_k|\cos(t-\theta_{b_k})+|b_k|^2} \le \frac{1+|b_k|}{1-|b_k|},$$

therefore, $\theta'(t) > 0$.

For a decomposition like $f = \sum_{k=1}^{\infty}\langle O_k, e_{a_k}\rangle I^{(k)} B_{\{a_1,\ldots,a_k\}}$, if we let $a_1 = 0$, then all $\{B_{\{0,a_2,\ldots,a_k\}}(e^{it})\}_{k=1}^{\infty}$ are mono-components. Let $\langle O_k, e_{a_k}\rangle I^{(k)}(e^{it})B_{\{a_1,\ldots,a_k\}}(e^{it}) = \rho_k(t)e^{i\theta_k(t)}$. Simple computation on the Szegö kernel gives

$$\rho_k(t) = |\langle O_k, e_{a_k}\rangle| \frac{\sqrt{1-|a_k|^2}}{\sqrt{1-2|a_k|\cos(t-\theta_{a_k})+|a_k|^2}}$$

and

$$\theta_k'(t) = \frac{|a_k|\cos(t-\theta_{a_k})-|a_k|^2}{1-2|a_k|\cos(t-\theta_{a_k})+|a_k|^2} + \sum_{l=1}^{k-1}\frac{1-|a_l|^2}{1-2|a_l|\cos(t-\theta_{a_l})+|a_l|^2} + \sum_{j=1}^{k}\sum_{l=1}^{N_j}\frac{1-|b_l^{(j)}|^2}{1-2|b_l^{(j)}|\cos\left(t-\theta_{b_l^{(j)}}\right)+|b_l^{(j)}|^2},$$

where N_j denotes the number of zeros of f_j as defined in (4).

4.1. *TTFD and CTTFD*

As in [7], the *transient time frequency distribution* (TTFD) of a mono-component signal $s(t) = \rho(t)e^{i\theta(t)}$ is defined by

$$P(t,\xi) = \rho^2(t)\delta_M(\xi - \theta'(t)), \quad (t,\xi) \in \mathbb{R} \times \left[-\frac{1}{2M}, +\infty\right)$$

where

$$\delta_M(\xi - \theta'(t)) = \begin{cases} M, & \text{if} \quad \xi \in \left[\theta'(t) - \frac{1}{2M}, \theta'(t) + \frac{1}{2M}\right], \\ 0, & \text{if} \quad \xi \notin \left[\theta'(t) - \frac{1}{2M}, \theta'(t) + \frac{1}{2M}\right]. \end{cases}$$

Table I. Example 1.

k	1	2	3	4	5	6	7	8	9	10
a_k	0.84 − 0.44i	−0.03 + 0.80i	−0.12 − 0.83i	−0.65 − 0.07i	0.50 + 0.06i	−0.65 − 0.64i	−0.77 + 0.38i	−0.34 + 0.46i	−0.45 + 0.22i	0.02 − 0.77i
b_k	0.92 − 0.17i	−0.30 + 0.07i	−0.17 + 0.81i	0.07 − 0.90i	0.80i	−0.06 − 0.10i	−0.28 − 0.22i	−0.03 + 0.01i	0.41 − 0.65*i*	−0.10 + 0.16i
c_k	−0.58 + 0.03i	−0.19 − 0.22i	0.59 − 0.48i	−0.02 + 0.82i	−0.08 − 0.06i	−0.84 − 0.81i	0.98 + 1.71i	0.24 + 0.91i	0.57 + 1.49i	0.22 − 0.03i
k	11	12	13	14	15	16	17	18	19	20
a_k	−0.62 − 0.01i	0.60 + 0.34i	−0.70 + 0.06i	−0.36 + 0.38i	0.29 − 0.67i	−0.87 + 0.32i	0.63 + 0.02i	0.05 − 0.47i	−0.07 + 0.53i	−0.36−0.15i
b_k	0.63 + 0.69i	−0.53 + 0.05i	0.43 + 0.63i	0.59 + 0.47i	0.48 + 0.09i	0.68i	0.48 − 0.60i	0.68 + 0.10i	0.11 + 0.69i	−0.06 − 0.08i
c_k	0.81 + 1.92i	1.5100 + 0.92i	−0.43 − 1.81i	0.54 − 1.27i	0.68 + 0.58i	−0.01 − 0.12i	1.91 − 0.14i	−0.98 + 1.07i	−0.49 + 1.70i	0.08 + 0.39i

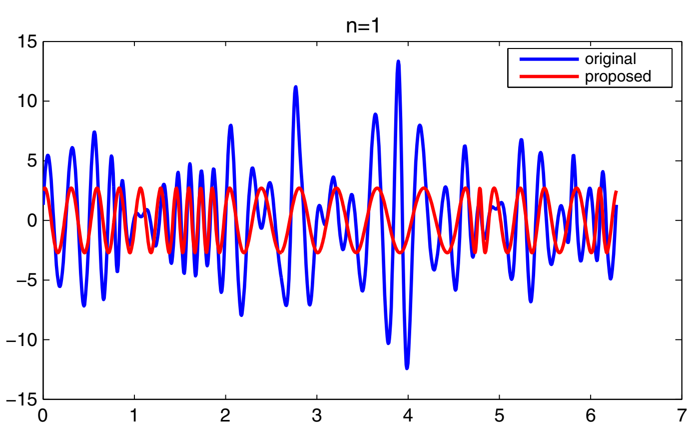

Figure 1. Example 1. $n = 1$ (unwinding adaptive Fourier decomposition).

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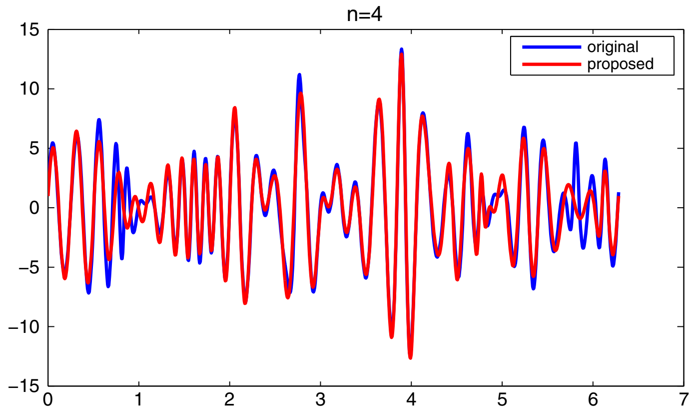

Figure 2. Example 1. $n = 4$ (unwinding adaptive Fourier decomposition).

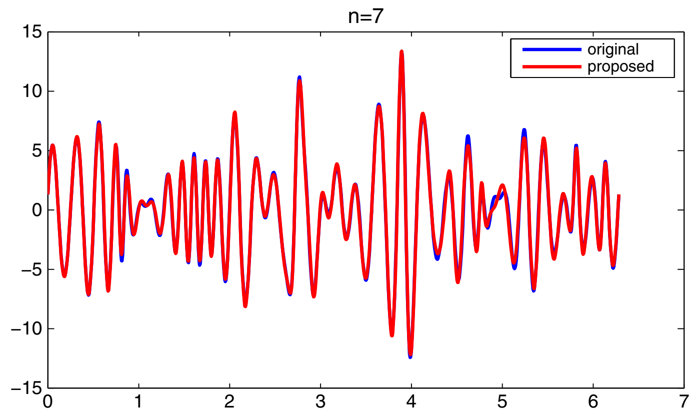

Figure 3. Example 1. $n = 7$ (unwinding adaptive Fourier decomposition).

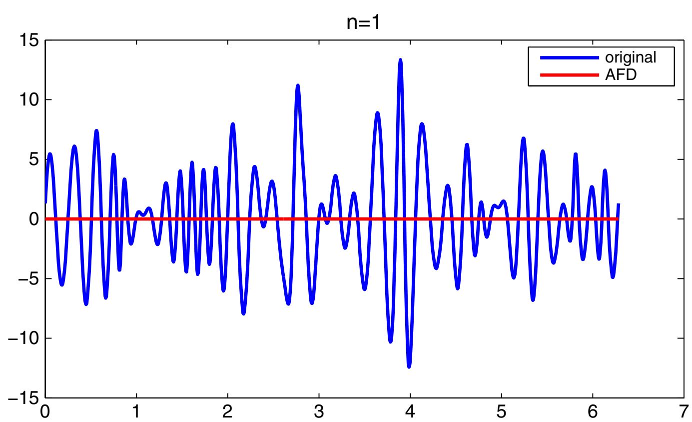

Figure 4. Example 1. $n = 1$ (adaptive Fourier decomposition).

where M is a large enough positive number to be determined in practice. When M goes to infinity, δ_M becomes the distributional Dirac function. It is convenience and applicable that we make M to be a finite number.

If $s(e^{it})$ is a multi-component signal, then $s(e^{it})$ can be decomposed into a sum of mono-component signals, say, for instance, in the Unwinding AFD way:

$$s\left(e^{it}\right) = \sum_{k=1}^{\infty} s_k(t) = \sum_{k=1}^{\infty} \langle O_k, e_{a_k} \rangle I^{(k)}\left(e^{it}\right) B_{\{0,a_2,\ldots,a_k\}}\left(e^{it}\right) = \sum_{k=1}^{\infty} \rho_k(t) e^{i\theta_k(t)}.$$

Then, the corresponding *composing TTFD* is defined through

$$P(t,\xi) = \sum_{k=1}^{\infty} P_k(t,\xi) = \sum_{k=1}^{\infty} \rho_k^2(t)\delta_M\left(\xi - \theta_k'(t)\right),$$

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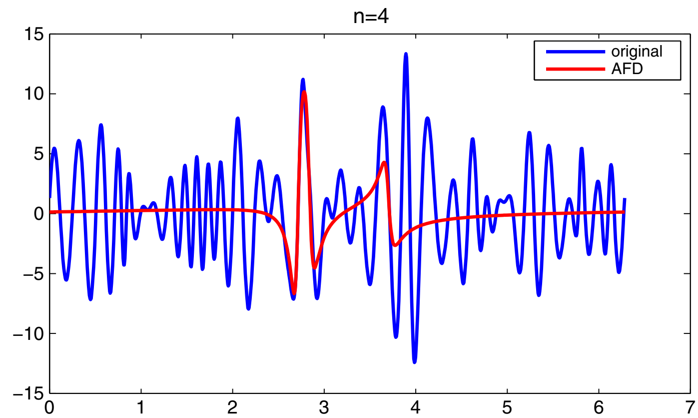

Figure 5. Example 1. $n = 4$ (adaptive Fourier decomposition).

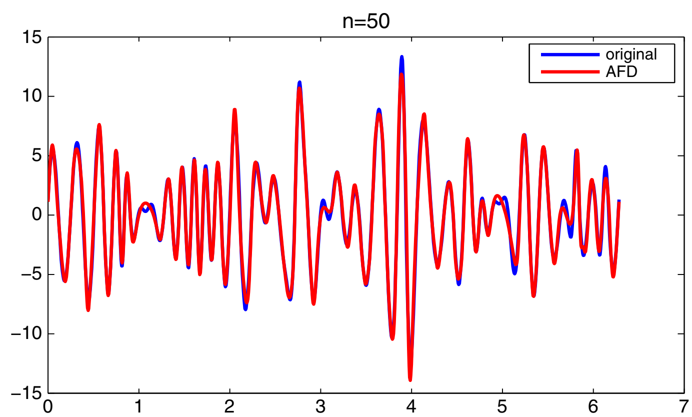

Figure 6. Example 1. $n = 50$ (adaptive Fourier decomposition).

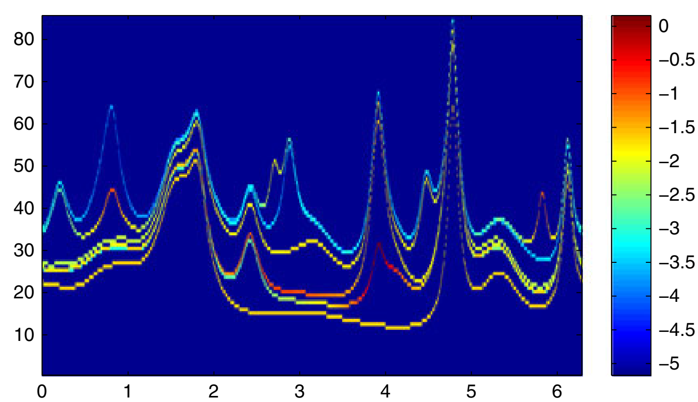

Figure 7. Example 1. Transient time frequency given by unwinding adaptive Fourier decomposition.

where $P_k(t, \xi)$ is the TTFD of $s_k(t)$.
For more details about TTFD and composing TTFD, see [7].

5. Experiments

Here, we give comparisons of the proposed method and AFD. Although AFD gives a fast approximation of signals in energy sense, it does not perform well for signals with high frequency. We will show that the proposed method has the advantage in approximating high-frequency signals.

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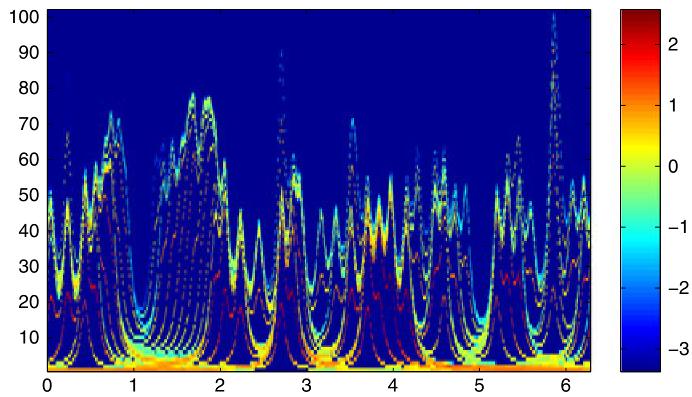

Figure 8. Example 1. Transient time frequency given by adaptive Fourier decomposition.

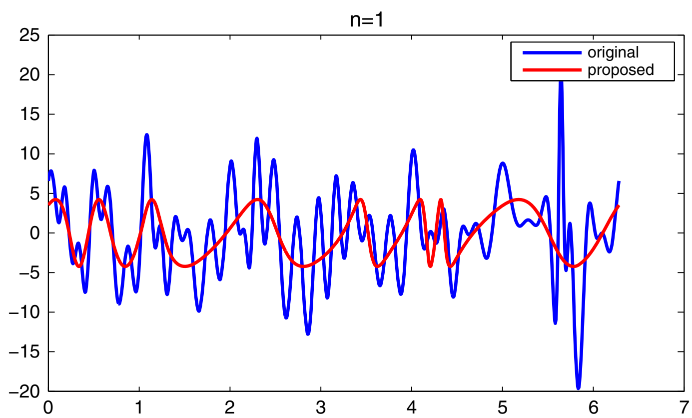

Figure 9. Example 2. $n = 1$ (unwinding adaptive Fourier decomposition).

Table II. Example 2.

k	1	2	3	4	5	6	7	8	9	10
a_k	$0.76 - 0.57i$	$0.34 + 0.16i$	$-0.55 + 0.69i$	$0.12 - 0.04i$	$0.74 - 0.43i$	$-0.30 + 0.54i$	$0.64 + 0.06i$	$-0.14 - 0.76i$	$-0.36 - 0.71i$	$0.59 - 0.50i$
b_k	$-0.13 - 0.08i$	$0.25 + 0.69i$	$0.75 - 0.51i$	$0.57 + 0.17i$	$-0.59 - 0.20i$	$-0.04 - 0.23i$	$-0.13 - 0.25i$	$-0.40 + 0.50i$	$0.78i$	$-0.80 - 0.01i$
c_k	$0.64 + 0.41i$	$-0.23 + 0.83i$	$0.69 + 0.36i$	$0.26 - 1.02i$	$-2.98 - 0.69i$	$-0.30 - 0.76i$	$0.79 - 0.59i$	$1.05 + 0.36i$	$-0.32 + 0.34i$	$0.52 - 0.23i$
k	11	12	13	14	15	16	17	18	19	20
a_k	$-0.06 - 0.63i$	$0.66 + 0.33i$	$0.25 + 0.74i$	$-0.37 - 0.27i$	$0.45i$	$0.67 + 0.33i$	$0.27 + 0.78i$	$-0.22 - 0.62i$	$0.83 - 0.20i$	$-0.38 - 0.06i$
b_k	$-0.33 - 0.02i$	$-0.44 + 0.68i$	$0.46 + 0.39i$	$0.56 + 0.28i$	$0.60 + 0.31i$	$-0.19 - 0.29i$	$-0.54 + 0.03i$	$0.91 + 0.12i$	$-0.63 + 0.02i$	$-0.38 - 0.03i$
c_k	$0.24 - 1.00i$	$0.92 - 0.33i$	$1.96 + 0.60i$	$-0.91 - 0.13i$	$2.10 + 1.44i$	$0.21 - 0.82i$	$-0.80 + 0.29i$	$-0.81 + 1.15i$	$0.12 - 0.86i$	$0.14 - 0.06i$

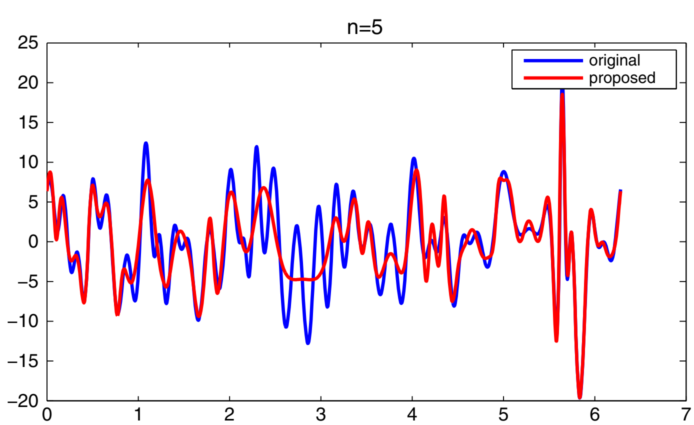

Figure 10. Example 2. $n = 5$ (unwinding adaptive Fourier decomposition).

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5.1. Example 1

Denote by $B_n = B_{\{a_1,\dots,a_k\}}$ the modified Blaschke product as given in (1.2). The signal is given by the samples of the following function

$$g_1\left(e^{it}\right) = B_{\{b_1,\dots,b_{20}\}}\left(e^{it}\right)\sum_{j=1}^{20} c_k B_{\{a_1,\dots,a_k\}}\left(e^{it}\right),$$

where $t \in [0, 2\pi)$. The parameters (a_k, b_k and c_k) of Example 1 are given in Table I.

Specifically, the samples of g_1 are from $t_j = \frac{2\pi(j-1)}{1024}, j = 1, 2, \dots, 1024$.

From Figures 1–3 and Figures 5–7, we know that the proposed method can give a good approximation to g_1 with $n = 7$, while AFD gives an approximation in a similar level with $n = 50$. Comparing Figure 4 and Figure 8, we see that TFD given by the proposed method is more reasonable than the one given by AFD.

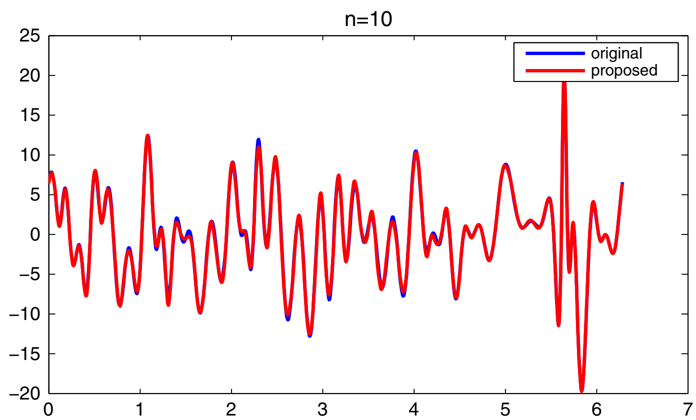

Figure 11. Example 2. $n = 10$ (unwinding adaptive Fourier decomposition).

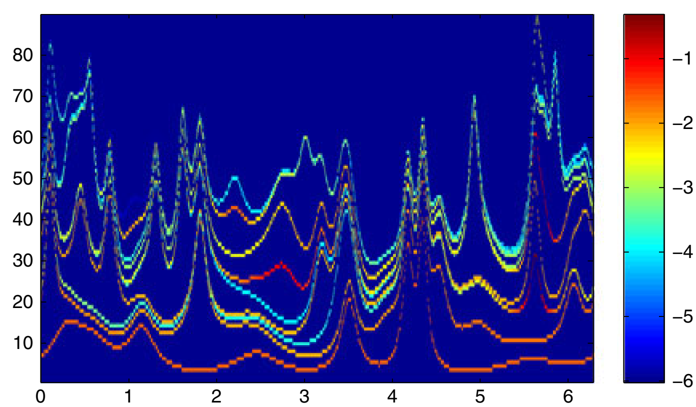

Figure 12. Example 2. Transient time frequency given by unwinding adaptive Fourier decomposition.

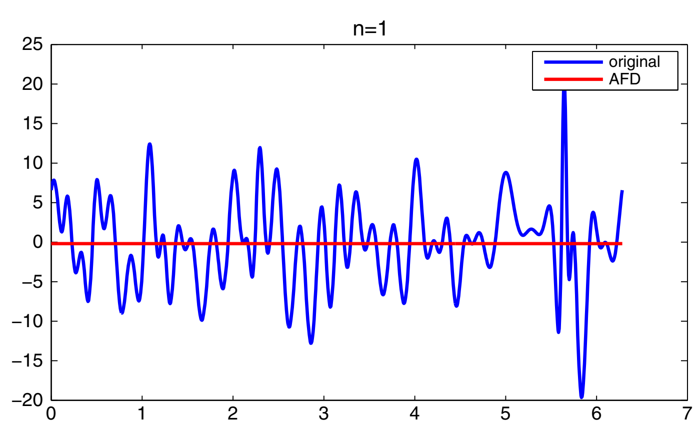

Figure 13. Example 2. $n = 1$ (adaptive Fourier decomposition).

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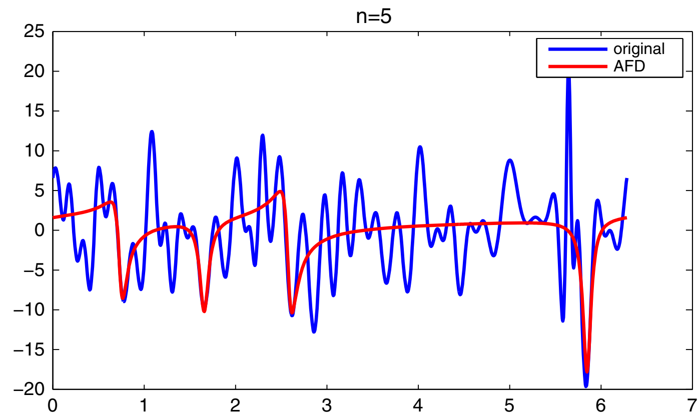

Figure 14. Example 2. $n = 5$ (adaptive Fourier decomposition).

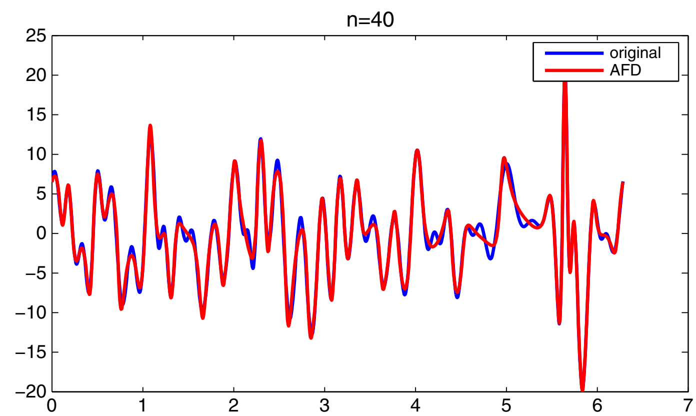

Figure 15. Example 2. $n = 40$ (adaptive Fourier decomposition).

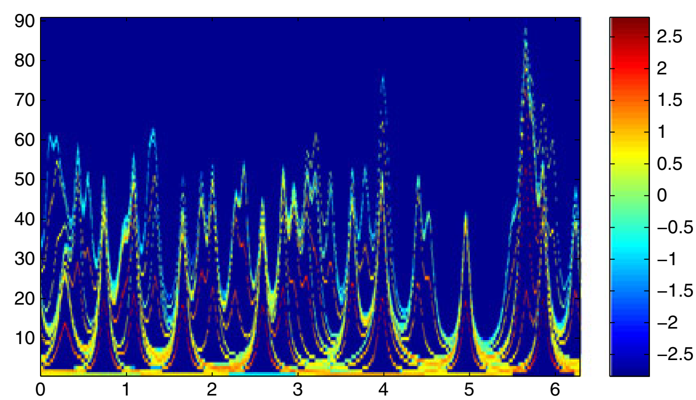

Figure 16. Example 2. Transient time frequency given by adaptive Fourier decomposition.

5.2. *Example 2*

Here, we consider that the example is of the following form, which is with slight difference of Example 1

$$g_2\left(e^{it}\right) = \left(B_{\{b_1,\ldots,b_{20}\}}\left(e^{it}\right) + 1\right)\sum_{j=1}^{20} c_k B_{\{a_1,\ldots,a_k\}}\left(e^{it}\right),$$

where $t \in [0, 2\pi)$. The parameters (a_k, b_k and c_k) of Example 2 are given in Table II.

Similarly, the samples of g_2 are from the values of g_2 at points $t_j = \frac{2\pi(j-1)}{1024}, j = 1, 2, \ldots, 1024$.
Unlike g_1, g_2 can not be first factorized out $(B_{\{b_1,\ldots,b_{20}\}} + 1)$ by the proposed method. However, the results show that performances of the proposed method are still better than the AFDs of the same order. We can easily conclude this by comparing Figures 9–12 and Figures 13–16.

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6. Conclusion

In this paper, a new and effective algorithm for Unwinding AFD is presented and tested through theoretical examples. We then use the new algorithm and the resulted positive frequency representation to give a Dirac type time-frequency distribution of signals.

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