Singular integrals associated to the Laplacian on the affine group $ax+b$

G. I. Gaudry*, T. Qian**, and P. Sjögren*

Abstract. We consider singular integral operators of the form (a) $Z_1 L^{-1} Z_2$, (b) $Z_1 Z_2 L^{-1}$, and (c) $L^{-1} Z_1 Z_2$, where Z_1 and Z_2 are nonzero right-invariant vector fields, and L is the L^2-closure of a canonical Laplacian. The operators (a) are shown to be bounded on L^p for all $p \in (1, \infty)$ and of weak type (1, 1), whereas all of the operators in (b) and (c) are not of weak type (p, p) for any $p \in [1, \infty)$.

0. Introduction

The affine group of the line is the set $\{(b, a)\colon\ a \in \mathbf{R}^+, b \in \mathbf{R}\}$ with group product $(b, a) \cdot (d, c) = (b+ad, ac)$. By taking the logarithm of the second coordinate, one can identify this group with $G = \{(s, t)\colon\ s, t \in \mathbf{R}\}$, with group product $(u, v) \cdot (s, t) = (u+se^v, v+t)$. There is a natural left-invariant Riemannian metric on G. This is obtained by transferring the hyperbolic metric from the upper half-plane to G. We call this transferred metric the "hyperbolic metric" on G. If we set

$$z = \cosh t + \tfrac{1}{2} s^2 e^{-t},$$

then $\operatorname{arccosh} z$ is the distance from $(0, 0)$ to the point (s, t) with respect to the hyperbolic metric.

The left-invariant and the right-invariant Haar measures dm and dn on G are given by

$$dm(s, t) = e^{-t}\, ds\, dt, \quad \text{and} \quad dn(s, t) = ds\, dt,$$

respectively. The modular function is $\Delta(s, t) = e^{-t}$.

The group G has (up to unitary equivalence) just two infinite-dimensional irreducible unitary representations σ^+ and σ^-, defined as follows (cf. [H 1]): Both

* Research supported by the Australian Research Council.

** Research carried out as a National Research Fellow.

are realised on $L^2(\mathbf{R})$ and, for f in $L^2(\mathbf{R})$,

$$[\sigma^{\pm}(s,t)f](x) = \exp(\pm ise^x)f(x+t).$$

Let X, Y be the basis for the Lie algebra of G defined by

$$\exp sX = (s,0), \quad \exp tY = (0,t).$$

Still using X and Y to denote the right-invariant differential operators induced by X and Y, respectively, we have

$$Xf(s,t) = \partial_s f(s,t), \tag{1}$$

and

$$Yf(s,t) = s\partial_s f(s,t) + \partial_t f(s,t). \tag{2}$$

For $f\in C_c^\infty(\mathbf{R})$, the derived representations are

$$[d\sigma^{\pm}(X)f](x) = \frac{d}{ds}\exp(\pm ise^x)f(x)\Big|_{s=0} = \pm ie^x f(x),$$

$$[d\sigma^{\pm}(Y)f](x) = \frac{d}{dt}f(x+t)\Big|_{t=0} = \frac{d}{dx}f(x).$$

We shall consider the following right-invariant operator on $C_c^\infty(G)$,

$$L_0 = -X^2 - Y^2,$$

which is positive, formally self-adjoint, and has a closure L on $\mathrm{L}^2(G, dm)$. Let Z_1 and Z_2 be any two nonzero vectors in the span of X and Y. We shall study the following operators:

$$Z_1L^{-1}Z_2, \quad Z_1Z_2L^{-1}, \quad L^{-1}Z_1Z_2.$$

They act on functions carried by G. The precise meaning of these operators will be given later. It is not hard to show that the Riesz transforms $Z_1L^{-1/2}$ and $L^{-1/2}Z_2$ are bounded on $\mathrm{L}^2(G, dm)$ — see Lemma 4. Consequently, the operator $Z_1L^{-1}Z_2$ is also bounded on $\mathrm{L}^2(G, dm)$. This simple argument does not apply to the other operators. Indeed, they are unbounded on every L^p space, $1\leq p<\infty$.

Our main results are the following.

Theorem 1. *The operator $Z_1L^{-1}Z_2$ is bounded from $\mathrm{L}^p(G, dm)$ to $\mathrm{L}^p(G, dm)$, $1<p<\infty$, and bounded from $\mathrm{L}^1(G, dm)$ to weak $\mathrm{L}^1(G, dm)$.*

Theorem 2. *The operators $Z_1Z_2L^{-1}$ and $L^{-1}Z_1Z_2$ are not of weak type (p,p) for any $p\in[1,\infty)$.*

The procedure we use to prove Theorem 1 and Theorem 2 is to split the kernels into their local parts and their parts at infinity. We prove that the local parts of the kernels in all cases give rise to bounded operators on $L^p(G, dm)$, $1<p<\infty$, and from $L^1(G, dm)$ to weak $L^1(G, dm)$. To study the parts of the kernels at infinity, we further split each of them into two pieces that correspond to $t>0$ and $t<0$. In the following, C denotes a positive, finite constant which may vary from line to line, and may depend on parameters according to the context.

Results similar to some of ours, but for solvable groups of polynomial growth, have been obtained by G. Alexopoulos [A]. Positive results about the boundedness of Riesz transforms on groups of polynomial growth have been obtained by L. Saloff-Coste [S-C]. Boundedness of the Riesz transforms on unimodular, non-amenable groups has been proved by Lohoué [L 1]. We point out that the local behaviour of the kernels treated in this paper could also be handled by using the results or methods of [B], [F—S], [L 2] and [L 3]. The results of Fefferman and Sánchez-Calle imply local boundedness of second-order derivatives of the fundamental solution of subelliptic operators on manifolds, and therefore apply to our case.

The authors would like to thank R. Burns and M. Cowling for helpful conversations on this topic. Special thanks are due to G. Mauceri and Chun Li for the impetus they gave to our study of singular integrals on the affine group.

1. Preliminaries on operator theory

In this section we shall work on $L^2=L^2(G, dm)$. By L_0 we denote the right-invariant differential operator $-X^2-Y^2$ with domain $C_c^\infty(G)$, which is dense in L^2. By the general theory of Laplacian operators on Lie groups (see, for example, [H 2]), L_0 has a self-adjoint closure, denoted by L. We write the spectral resolution of L as

$$L = \int_{-\infty}^{+\infty} \lambda \, dE_\lambda,$$

where E_λ is a projection-valued measure. The domain of L is

$$D(L) = \left\{f\colon \int \lambda^2 d\langle E_\lambda f, f\rangle < \infty\right\}.$$

For the relevant spectral theory, see for instance [R—S].

Lemma 1. *The operator L is positive and one-to-one.*

Proof. Since L_0 is positive, the closure L is also positive. To see that L is one-to-one, we study the derived representation of L_0:

$$A_0 = d\sigma^\pm(L) = -\frac{d}{dx^2}+e^{2x},$$

where A_0 is a Sturm—Liouville operator in $L^2(\mathbf{R})$ with dense domain $C_c^\infty(\mathbf{R})$. Then A_0 has a closure, denoted by A, which is one-to-one, since

$$\langle A_0 f, f\rangle = \left\langle \frac{df}{dx}, \frac{df}{dx}\right\rangle + \langle e^{(\cdot)} f, e^{(\cdot)} f\rangle \geqq \|e^{(\cdot)} f\|^2.$$

By the Plancherel theorem for the affine group ([Kh], Theorem 2), L itself is therefore one-to-one. □

It follows from Lemma 1 that E_λ is carried by the open half-line $(0, +\infty)$. For any real α, we can thus define the power L^α as a self-adjoint operator by the formula

$$L^\alpha = \int_0^\infty \lambda^\alpha \, dE_\lambda.$$

The domain

$$D(L^\alpha) = \left\{f\colon \int \lambda^{2\alpha} d\langle E_\lambda f, f\rangle < \infty\right\}$$

is dense. The range $R(L^\alpha)$ is seen to equal $D(L^{-\alpha})$; in particular, it is also dense. One has

$$D(L^\alpha) \subseteqq D(L^\beta) \tag{3}$$

for $\alpha<\beta<0$ and for $0<\beta<\alpha$. For any $\alpha, \beta\in\mathbf{R}$, there is an inclusion

$$L^\alpha L^\beta \subseteqq L^{\alpha+\beta}, \tag{4}$$

in the sense that $L^\alpha L^\beta$ coincides with $L^{\alpha+\beta}$ on $\{f\in D(L^\beta)\colon L^\beta f\in D(L^\alpha)\}$, which is a dense subspace of $D(L^{\alpha+\beta})$. In particular, $L^{-\alpha}$ is the inverse of L^α.

We now recall the following results of Hulanicki [H 1]. The spectrum of L is contained in $[0, +\infty)$, and, for any $\lambda\notin[0, +\infty)$,

$$(\lambda - L)^{-1} = k_\lambda *, \tag{5}$$

where $k_\lambda *$ denotes the left-convolution operator with kernel k_λ:

$$k_\lambda * f(x) = \int k_\lambda(y) f(y^{-1}x)\, dm(y) = \int k_\lambda(xy) f(y^{-1})\, dm(y),$$

and

$$k_\lambda(s, t) = e^{t/2}\mathfrak{Q}_{-\nu-1/2}\left(\cosh t + \tfrac{1}{2} s^2 e^{-t}\right) = e^{t/2}\mathfrak{Q}_{-\nu-1/2}(z).$$

Here $\nu=\sqrt{-\lambda}$, with $\operatorname{Re}(\nu)<0$, and $\mathfrak{Q}_\nu$ is the Legendre function of the second kind with the parameter $\mu=0$.

It was proved in [H 1] that

$$k_\lambda\in L^1, \quad \lambda\notin\mathbf{R},$$

When $\lambda=0$ and ν is taken to be 0, the expression k_0 makes sense; it is not in L^1, however.

We shall need a certain number of results concerning the functions $\mathfrak{Q}_\tau$, particularly their asymptotics. First, the formula in [MOS], p. 196:

$$\mathfrak{Q}_\tau(z) = -\gamma - \psi(\tau+1) - \tfrac{1}{2}\log\left(\tfrac{1}{2}z - \tfrac{1}{2}\right) + O(|z-1|) \quad \text{as} \quad z \to 1+0, \tag{6}$$

where γ is Euler's constant, and $\psi(x)=\Gamma'(x)/\Gamma(x)$. We also have the asymptotic formulas, as $z\searrow 1$, that can be obtained from [MOS], pp. 174, 196:

$$\mathfrak{Q}^{(m)}_{-1/2}(z) = (-1)^m \frac{(m-1)!}{2}(z-1)^{-m}+O\left((z-1)^{-m+1}\right), \quad m = 1, 2, \ldots, \tag{7}$$

and the asymptotic formula at ∞ that can be obtained from [MOS], pp. 174, 153:

$$\mathfrak{Q}^{(m)}_{-1/2}(z) = \gamma_m z^{-1/2-m}+O(z^{-5/2-m}), \tag{8}$$

where

$$\gamma_0 = 2C_1, \quad \gamma_1 = -C_1, \quad \text{and} \quad \gamma_2 = \tfrac{3}{2}C_1. \tag{9}$$

We shall also need the formula, valid for $\operatorname{Re}(\tau)>-1$,

$$\mathfrak{Q}_\tau(z) = 2^{-\tau-1}\int_{-1}^{1}(z-u)^{-\tau-1}(1-u^2)^\tau\, du, \tag{10}$$

which can be obtained from the fourth formula on p. 186 of [MOS] by making the change of variable $u=-\cos t$.

Lemma 2. *The kernel k_0 is locally integrable; $L(C_c^\infty(G))$ is dense; and, for any $\varphi\in C_c^\infty(G)$, we have $k_0*(-L)\varphi=\varphi$.*

Proof. The function

$$k_0(s,t) = e^{t/2}\mathfrak{Q}_{-1/2}\left(\cosh t+\tfrac{1}{2}s^2e^{-t}\right)$$

has singularities at $(0,0)$ and ∞. The singularity at $(0,0)$ is integrable by (6).

The density of $L(C_c^\infty(G))$ follows from Lemma 1. To prove the equality in the lemma, taking a test function φ and using (5), we have

$$\begin{aligned} k_0*(-L)\varphi-\varphi &= \left(k_0*(-L)\varphi-k_0*(\lambda-L)\varphi\right)+\left(k_0*(\lambda-L)\varphi-k_\lambda*(\lambda-L)\varphi\right)\\ &= -\lambda k_0*\varphi+(k_0-k_\lambda)*(\lambda-L)\varphi. \end{aligned}$$

Hence,

$$\begin{aligned} \left|(k_0*(-L)\varphi-\varphi)\right| &\leq \limsup_{\lambda\to 0}\left(|\lambda|\,|k_0*\varphi|+|(k_0-k_\lambda)*\psi_\lambda|\right)\\ &= \limsup_{\lambda\to 0}|(k_0-k_\lambda)*\psi_\lambda| \end{aligned}$$

where $\psi_\lambda=(\lambda-L)\varphi$ are uniformly bounded in $C_c^\infty(G)$ for λ small. From the formula (10), one can see that k_0 and k_λ are uniformly controlled by a locally integrable function as $\lambda\to 0-$. Now we can use the Lebesgue dominated convergence theorem and conclude that the last expression is zero a.e. The lemma has now been proved. □

By considering the operator X to be distribution-valued, we define the operator $XL^{-1/2}$ to have domain

$$D(XL^{-1/2}) = \{f \in D(L^{-1/2}) \colon XL^{-1/2} f \in \mathrm{L}^2\}.$$

Lemma 3. *The domain of $XL^{-1/2}$ is dense in $\mathrm{L}^2(G, dm)$.*

Proof. By the general properties of fractional powers, $C_c^\infty(G) \subseteqq D(L) \subseteqq D(L^{1/2})$; so it suffices to show that $L^{1/2}(C_c^\infty(G))$ is dense in $\mathrm{L}^2(G, dm)$.

The subspace $L^{1/2}(D(L))$ is dense, since it contains all functions which can be written $f = \int_\varepsilon^R dE_\lambda f$ for $\varepsilon > 0$, $R < \infty$. Thus we need show only that $L^{1/2}(C_c^\infty(G))$ is dense in $L^{1/2}(D(L))$.

To this end, we let $f \in D(L) \subseteqq D(L^{1/2})$. Since L is the closure of L_0, we can find a sequence $\{f_j\}$ in $C_c^\infty(G)$ such that $f_j \to f$ and $Lf_j \to Lf$. This implies that $L^{1/2} f_j \to L^{1/2} f$, since

$$\|L^{1/2}(f_j - f)\|_2^2 = \langle L(f_j - f), f_j - f \rangle \leqq \|L(f_j - f)\|_2 \|f_j - f\|_2. \quad \square$$

The same argument and conclusion are valid for the operator $YL^{-1/2}$.

The following lemma is the main result about $\mathrm{L}^2(G, dm)$-boundedness. Cf. also [B—R] and [Str], Corollary 2.6.

Lemma 4. *The operators $XL^{-1/2}$, $YL^{-1/2}$, $L^{-1/2}X$, and $L^{-1/2}Y$ have bounded extensions on $\mathrm{L}^2(G, dm)$.*

Proof. In fact, for $\varphi \in C_c^\infty(G)$,

$$\begin{aligned} \langle X\varphi, X\varphi \rangle + \langle Y\varphi, Y\varphi \rangle &= \langle -X^2 \varphi, \varphi \rangle + \langle -Y^2 \varphi, \varphi \rangle \\ &= \langle L\varphi, \varphi \rangle \\ &= \langle L^{1/2}\varphi, L^{1/2}\varphi \rangle. \end{aligned}$$

Now take $g \in L^{1/2}(C_c^\infty(G))$ and let $\varphi = L^{-1/2} g$ in the above inequality. We get

$$\|XL^{-1/2} g\|^2 + \|YL^{-1/2} g\|^2 = \|g\|^2.$$

Since $L^{1/2}(C_c^\infty(G))$ is dense, as shown in the proof of Lemma 3, we therefore deduce the desired boundedness of $XL^{-1/2}$ and $YL^{-1/2}$. Since $L^{-1/2}$ is self-adjoint and X and Y are skew-adjoint, we conclude the desired boundedness of $L^{-1/2}X$ and $L^{-1/2}Y$. $\square$

Corollary 1. *The operators $XL^{-1}X$, $YL^{-1}Y$, $XL^{-1}Y$ and $YL^{-1}X$ are bounded on $\mathrm{L}^2(G, dm)$.*

Proof. This is because $XL^{-1}X = XL^{-1/2}L^{-1/2}X$, etc. $\square$

2. Kernel expressions and local behaviour of the kernels

We call an operator an *essentially principal operator* if, by subtracting a constant multiple of the Dirac operator at the origin, it becomes a principal value left-convolution operator. (In computing a "principal value" we integrate over the complement of the Euclidean ball $\{(s,t)\colon |(s,t)|<\varepsilon\}$ of radius ε about $(0,0)$ and then let $\varepsilon\to 0$.) If A is an essentially principal operator, we denote the associated convolution kernel by $\operatorname{Ker}(A)$.

Lemma 5. *All of the operators of the type $Z_1L^{-1}Z_2$, $Z_1Z_2L^{-1}$, and $L^{-1}Z_1Z_2$ are essentially principal operators. Furthermore, we have*

$$
\begin{aligned}
\operatorname{Ker}(XL^{-1}X) &= e^t\partial_s^2 k_0(s,t) = e^{t/2}\mathfrak{Q}'_{-1/2}(z)+s^2e^{-t/2}\mathfrak{Q}''_{-1/2}(z)\\
\operatorname{Ker}(YL^{-1}Y) &= (s\partial_s\partial_t - s\partial_s+\partial_t^2-\partial_t)k_0(s,t)\\
&= e^{t/2}\Big[-\tfrac{1}{4}\mathfrak{Q}_{-1/2}(z)+(\cosh t-s^2e^{-t})\mathfrak{Q}'_{-1/2}(z)\\
&\qquad +(\sinh^2 t-\tfrac{1}{4}s^4e^{-2t})\mathfrak{Q}''_{-1/2}(z)\Big]\\
\operatorname{Ker}(XL^{-1}Y) &= (\partial_s\partial_t-\partial_s)k_0(s,t)\\
&= -\tfrac{3}{2}se^{-t/2}\mathfrak{Q}'_{-1/2}(z)+se^{-t/2}\big(\sinh t-\tfrac{1}{2}s^2e^{-t}\big)\mathfrak{Q}''_{-1/2}(z)\\
\operatorname{Ker}(YL^{-1}X) &= e^t(s\partial_s^2+\partial_s\partial_t+\partial_s)k_0(s,t)\\
&= \tfrac{3}{2}se^{t/2}\mathfrak{Q}'_{-1/2}(z)+se^{t/2}\big(\sinh t+\tfrac{1}{2}s^2e^{-t}\big)\mathfrak{Q}''_{-1/2}(z)\\
\operatorname{Ker}(X^2L^{-1}) &= \partial_s^2k_0(s,t)\\
&= e^{-t}\operatorname{Ker}(XL^{-1}X)\\
\operatorname{Ker}(XYL^{-1}) &= (s\partial_s^2+\partial_s\partial_t+\partial_s)k_0(s,t)\\
&= e^{-t}\operatorname{Ker}(YL^{-1}X)\\
\operatorname{Ker}(YXL^{-1}) &= (s\partial_s^2+\partial_s\partial_t)k_0(s,t)\\
&= \tfrac{1}{2}se^{-t/2}\mathfrak{Q}'_{-1/2}(z)+se^{-t/2}\big(\sinh t+\tfrac{1}{2}s^2e^{-t}\big)\mathfrak{Q}''_{-1/2}(z)\\
\operatorname{Ker}(L^{-1}X^2) &= e^{2t}\partial_s^2k_0(s,t)\\
&= e^t\operatorname{Ker}(XL^{-1}X)\\
\operatorname{Ker}(L^{-1}XY) &= e^t\partial_s\partial_tk_0(s,t)\\
&= -\tfrac{1}{2}se^{t/2}\mathfrak{Q}'_{-1/2}(z)+se^{t/2}\big(\sinh t-\tfrac{1}{2}s^2e^{-t}\big)\mathfrak{Q}''_{-1/2}(z)\\
\operatorname{Ker}(L^{-1}YX) &= e^t(\partial_t\partial_s-\partial_s)k_0(s,t)\\
&= e^t\operatorname{Ker}(XL^{-1}Y).
\end{aligned}
$$

The formulas for $L^{-1}Y^2$ and Y^2L^{-1} can be obtained from the corresponding formulas involving X^2.

Proof. Let $\psi\in C_c^\infty(G)$. In the computations that follow, we write $x=(u, v)$, $y=(s, t)$. The formula (7) shows that Xk_0 and Yk_0 are locally integrable. It follows that

$$XL^{-1}\psi(x) = \frac{d}{dh}\bigg|_{h=0}\int k_0(\exp(hX)xy)\psi(y^{-1})\,dm(y) \tag{11}$$
$$= \int Xk_0(xy)\psi(y^{-1})\,dm(y) = Xk_0*\psi(x).$$

Similarly,

$$YL^{-1}\psi = Yk_0*\psi. \tag{12}$$

Consider now the formula for $XL^{-1}X$. If $\varphi\in C_c^\infty(G)$, we have from (11) that

$$XL^{-1}X\varphi(x) = Xk_0*X\varphi(x) = \int Xk_0(xy)\,X\varphi(y^{-1})\,dm(y).$$

If $y=(s, t)$, then

$$\frac{d}{dh}\bigg|_{h=0}\varphi(\exp(hX)y^{-1}) = -e^t\frac{\partial}{\partial s}(\varphi(-e^{-t}s, -t)).$$

Hence

$$XL^{-1}X\varphi(x) = -\lim_{\varepsilon\to 0}\int_{|xy|>\varepsilon} Xk_0(xy)\frac{\partial}{\partial s}(\varphi(-e^{-t}s, -t))\,ds\,dt \tag{13}$$
$$= \lim_{\varepsilon\to 0}\int_{|(s,t)|>\varepsilon}\frac{\partial}{\partial s}(Xk_0)(y)\,\varphi(y^{-1}x)\,ds\,dt$$
$$+\lim_{\varepsilon\to 0}\int_{-\varepsilon}^{\varepsilon} Xk_0(\sqrt{\varepsilon^2-t^2}, t)\{\varphi(e^{-t}(u-\sqrt{\varepsilon^2-t^2}), v-t)$$
$$-\varphi(e^{-t}(u+\sqrt{\varepsilon^2-t^2}), v-t)\}\,dt,$$

by integration by parts. Now from the asymptotic formula (7), we see that

$$Xk_0(s, t) = -\frac{se^{-t/2}}{s^2+t^2}+O(1)$$

as $|(s, t)|\to 0$. So the last limit in (13) is equal to

$$-\lim_{\varepsilon\to 0}\int_{-\varepsilon}^{\varepsilon}\frac{\sqrt{\varepsilon^2-t^2}}{\varepsilon^2}e^{-t/2}\{\varphi(e^{-t}(u-\sqrt{\varepsilon^2-t^2}), v-t)-\varphi(e^{-t}(u+\sqrt{\varepsilon^2-t^2}), v-t)\}\,dt$$

$$= -\lim_{\varepsilon\to 0}\int_{-1}^{1}\sqrt{1-a^2}\,e^{-\varepsilon a/2}\{\varphi(e^{-\varepsilon a}(u-\varepsilon\sqrt{1-a^2}), v-a\varepsilon)$$
$$-\varphi(e^{-\varepsilon a}(u+\varepsilon\sqrt{1-a^2}), v-a\varepsilon)\}\,da,$$

which is $C\varphi(x)$, where C is a constant. Thus we have

$$(XL^{-1}X)\varphi(x) = \lim_{\varepsilon\to 0}\int_{|y|>\varepsilon} e^t\left(\frac{\partial^2}{\partial s^2}k_0\right)(y)\varphi(y^{-1}x)\,dm(y)+C\varphi(x).$$

This establishes the first formula for $XL^{-1}X$. The second follows from the expression for k_0.

Consider next the kernel of $XL^{-1}Y$. Keeping (2) in mind, we see that

$$XL^{-1}Y\varphi(x) = -\int Xk_0(s,t)\frac{\partial}{\partial t}\{\varphi((-e^{-t}s,-t)(u,v))\}e^{-t}\,ds\,dt$$

$$= -\lim_{\varepsilon\to 0}\int_{|(s,t)|>\varepsilon} e^{-t}Xk_0(s,t)\frac{\partial}{\partial t}\{\varphi((-e^{-t}s,-t)(u,v))\}\,ds\,dt.$$

A calculation using integration by parts now shows that

(14)

$$XL^{-1}Y\varphi(x) = \lim_{\varepsilon\to 0}\int_{|(s,t)|>\varepsilon}\left[-Xk_0(s,t)+\frac{\partial}{\partial t}Xk_0(s,t)\right]\varphi((-e^{-t}s,-t)(u,v))e^{-t}\,ds\,dt$$

$$+\lim_{\varepsilon\to 0}\int_{-\varepsilon}^{\varepsilon} ds\left[e^{-\sqrt{\varepsilon^2-s^2}}Xk_0\left(s,\sqrt{\varepsilon^2-s^2}\right)\varphi\left(\left(-e^{-\sqrt{\varepsilon^2-s^2}}s,-\sqrt{\varepsilon^2-s^2}\right)(u,v)\right)\right.$$

$$\left.-e^{\sqrt{\varepsilon^2-s^2}}Xk_0\left(s,-\sqrt{\varepsilon^2-s^2}\right)\varphi\left(\left(-e^{\sqrt{\varepsilon^2-s^2}}s,\sqrt{\varepsilon^2-s^2}\right)(u,v)\right)\right].$$

Since the terms $\varphi\left(\left(-e^{\pm\sqrt{\varepsilon^2-s^2}}s,\pm\sqrt{\varepsilon^2-s^2}\right)(u,v)\right)$ tend uniformly to $\varphi(x)$ as $\varepsilon\to 0$, it remains to consider

$$\lim_{\varepsilon\to 0}\int_{-\varepsilon}^{\varepsilon} ds\left[e^{-\sqrt{\varepsilon^2-s^2}}Xk_0\left(s,\sqrt{\varepsilon^2-s^2}\right)-e^{\sqrt{\varepsilon^2-s^2}}Xk_0\left(s,-\sqrt{\varepsilon^2-s^2}\right)\right]$$

$$= \lim_{\varepsilon\to 0}\int_{-1}^{1}\varepsilon\,da\left[e^{-\varepsilon\sqrt{1-a^2}}Xk_0\left(\varepsilon a,\varepsilon\sqrt{1-a^2}\right)-e^{\varepsilon\sqrt{1-a^2}}Xk_0\left(\varepsilon a,-\varepsilon\sqrt{1-a^2}\right)\right].$$

From (7), we see that

$$Xk_0\left(\varepsilon a,\varepsilon\sqrt{1-a^2}\right) = -\frac{\varepsilon a}{\varepsilon^2}+O(1)$$

and

$$Xk_0\left(\varepsilon a,-\varepsilon\sqrt{1-a^2}\right) = -\frac{\varepsilon a}{\varepsilon^2}+O(1).$$

Hence

$$\lim_{\varepsilon\to 0}\int_{-1}^{1}\varepsilon\,da\left[e^{-\varepsilon\sqrt{1-a^2}}Xk_0\left(\varepsilon a,\varepsilon\sqrt{1-a^2}\right)-e^{\varepsilon\sqrt{1-a^2}}Xk_0\left(\varepsilon a,-\varepsilon\sqrt{1-a^2}\right)\right]=0.$$

From (14), we have

$$XL^{-1}Y\varphi(x) = \text{p.v.}\int \left(\frac{\partial^2}{\partial s\,\partial t} - \frac{\partial}{\partial s}\right) k_0(y)\,\varphi(y^{-1}x)\,dm(y).$$

This establishes the formula for $\operatorname{Ker}(XL^{-1}Y)$.

The other formulas are proved using similar calculations. In treating cases such as $L^{-1}XY$ and $L^{-1}X^2$ one needs to note that, if $\psi \in C_c^\infty(G)$, then

$$L^{-1}X\psi = (e^t Xk_0) * \psi$$

and

$$L^{-1}Y\psi = \left(\frac{\partial k_0}{\partial t} - k_0\right) * \psi. \quad \square$$

We say that the local part of the kernel K gives rise to a bounded operator from $L^p(G, dm)$ to $L^p(G, dm)$ ($L^1(G, dm)$ to weak $L^1(G, dm)$), if for any $\psi \in C_c^\infty(G)$ that is 1 on a neighbourhood of e, the operator

$$\varphi \mapsto \text{p.v.}\int (\psi K)(y)\,\varphi(y^{-1}x)\,dm(y)$$

is bounded from $L^p(G, dm)$ to $L^p(G, dm)$ ($L^1(G, dm)$ to weak $L^1(G, dm)$), where the operator norm depends also on the "cut-off" function ψ.

Lemma 6. *The local parts of all of the kernels given in Lemma 5 give rise to bounded operators from* $L^2(G, dm)$ *to* $L^2(G, dm)$.

Proof. Our plan is as follows. Notice that the kernel expressions given in Lemma 5 are all linear combinations of terms of the form $s^m e^{\pm nt}\partial_s^p \partial_t^q k_0(s, t)$, where $m=0, 1, n, p, q=0, 1, 2, m+q \leqq 2, 0<p+q \leqq 2$; so it is enough to show that

(i) $\partial_s k_0$, $\partial_t k_0$, $s\partial_s^2 k_0$, and $s\partial_s\partial_t k_0$ are locally integrable;

(ii) the local parts of the kernels $\partial_s^2 k_0$, $\partial_t^2 k_0$ and $\partial_s\partial_t k_0$ generate bounded operators on $L^2(G, dm)$.

Here we have ignored the factors $e^{\pm nt}$, since $e^{\pm nt}-1=O(t)$; with this latter factor the terms in (ii) become integrable. By using the asymptotic formulas (6) and (7), a simple calculation gives assertion (i). We now establish (ii).

Ignoring the factor e^t, the local behaviour of $\partial_s^2 k_0$ is the same as the local behaviour of $\operatorname{Ker}(XL^{-1}X)$. Let ψ be a cut-off function: $\psi \in C_c^\infty(G)$, with $\psi \geqq 0$, $\psi(x)=1$ on a neighbourhood of e. If we show that $(\psi-1)\operatorname{Ker}(XL^{-1}X)$ is integrable, then by Corollary 1 we can conclude that $\psi\operatorname{Ker}(XL^{-1}X)$ gives an $L^2(G, dm)$-bounded operator and so does $\psi\partial_s^2 k_0$. In fact, when z is large, formula (8) gives

$$|\partial_s^2 k_0(s, t)| \leqq C(e^{-t/2} z^{-3/2} + s^2 e^{-3t/2} z^{-5/2}).$$

Since

$$z^{-1} \leqq \begin{cases} 2e^{-t} & \text{for } |s| \leqq e^t,\ t > 0 \\ \dfrac{2e^t}{1+s^2} & \text{otherwise,} \end{cases}$$

we have

$$|\partial_s^2 k_0(s,t)| \leqq C \begin{cases} e^{-2t} & \text{for } |s| \leqq e^t,\ t > 0 \\ \dfrac{e^t}{1+|s|^3} & \text{otherwise.} \end{cases}$$

It is easy to verify, by using the estimate above, that $\mathrm{Ker}(XL^{-1}X)$ is integrable at ∞.

Now we prove that the local part of $\partial_t\partial_s k_0$ determines a bounded operator on $L^2(G, dm)$. As $\partial_s k_0$ is locally integrable, we can compare the local part of $\partial_t\partial_s k_0$ with the local part of $\mathrm{Ker}(XL^{-1}Y)$; using Corollary 1 along with an argument like the above reduces the problem to showing that the latter kernel is integrable at ∞. In fact,

$$\mathrm{Ker}(XL^{-1}Y) = -se^{-t/2}\left[z\mathfrak{Q}''_{-1/2}(z)+\tfrac{3}{2}\mathfrak{Q}'_{-1/2}(z)\right]+se^{t/2}\mathfrak{Q}''_{-1/2}(z).$$

By formula (8), the first term of the above sum behaves like $se^{-t/2}z^{-7/2}$ and the second term behaves like $se^{t/2}z^{-5/2}$. Both are integrable at ∞.

Similarly, we can show that the local part of $\mathrm{Ker}(YL^{-1}Y)$ generates a bounded operator on $L^2(G, dm)$, and this simultaneously gives the same property for the kernel $\partial_t^2 k_0$. In fact,

$$\mathrm{Ker}(YL^{-1}Y) = e^{t/2}\left[(z^2-1)\mathfrak{Q}''_{-1/2}(z)+z\mathfrak{Q}'_{-1/2}(z)-\tfrac{1}{4}\mathfrak{Q}_{-1/2}(z)\right]$$
$$-s^2e^{-t/2}\left[z\mathfrak{Q}''_{-1/2}(z)+\tfrac{3}{2}\mathfrak{Q}'_{-1/2}(z)\right].$$

By using formula (8), we see that the leading terms in the above two parts cancel, leaving the terms $e^{t/2}z^{-5/2}$ and $s^2e^{-t/2}z^{-7/2}$, both of which are integrable at ∞. □

Definition. Let A be an operator of the form $Af = \text{p.v. } K*f$, defined on $C_c(X)$, where X is either $\mathbf{R}^n$ or G. We say that A is *locally bounded from* $L^p(X)$ *to* $L^p(X)$ if, for each compact set $S \subseteqq X$, A is bounded from $C^\infty_{c,S} = \{f \in C_c^\infty\colon \operatorname{supp} f \subseteqq S\}$ into $L^p(X)$ with respect to the L^p-norm, with a bound that possibly depends on S. We define in a similar way what is meant by saying that A is *locally bounded from $L^1(X)$ to weak $L^1(X)$*.

Lemma 7. *Let A be an operator of the form $Af = \text{p.v. } K*f$ that is locally bounded from $L^2(G, dm)$ to $L^2(G, dm)$, and whose kernel K has compact support. Let L be* *the kernel on* $\mathbf{R}^2\times\mathbf{R}^2$ *such that* $L(x,y)=K(xy^{-1})\in C^\infty$ *for* $x\neq y$. *If* $L(x,y)$ *satisfies the standard Calderón—Zygmund estimate:*

$$(C-Z)\qquad |L(x,y)|+|x-y|\big(|\nabla_x L(x,y)|+|\nabla_y L(x,y)|\big)\leqq C|x-y|^{-2},$$

for all $x\neq y$, *then* A *is bounded from* $\mathrm{L}^1(G,dm)$ *to weak* $\mathrm{L}^1(G,dm)$.

In proving Lemma 7, we shall need the following result.

Lemma 8. *Denote by* H_ϱ, $\varrho>0$, *the hyperbolic ball of radius* ϱ *centred at* e. *Given* ε, $\delta>0$, *there exist a sequence* $t_1, t_2, \ldots$ *of points of* G *and positive integers* n *and* s *such that*

(i) $G=\bigcup_i H_\varepsilon t_i$.

(ii) *Each point* $y\in G$ *belongs to at most* n *of the sets* $H_\varepsilon t_i$.

(iii) *Each* $y\in G$ *belongs to at most* s *of the sets* $H_{\varepsilon+\delta}t_i$.

Remarks on Lemma 8. This result, with conditions (i) and (ii) only stated explicitly, can be found in [P], p. 66. Condition (iii) is actually a consequence of (ii). If a point g lies in k of the sets $\{H_{\varepsilon+\delta}t_i\}$, we may as well suppose these are $H_{\varepsilon+\delta}t_1, \ldots, H_{\varepsilon+\delta}t_k$. Then $t_i\in H_{\varepsilon+\delta}g$, and so $H_\varepsilon t_i\subseteq H_{2\varepsilon+\delta}g$, $i=1,\ldots,k$. It follows from (ii) that the sets $H_\varepsilon t_i$ $(i=1,\ldots,k)$ overlap at most n times at any point. Let m_r denote right Haar measure. Then

$$\sum_{i=1}^k \chi_{H_\varepsilon t_i}\leqq n\chi_{H_{2\varepsilon+\delta}g}$$

and so

$$km_r(H_\varepsilon)\leqq nm_r(H_{2\varepsilon+\delta}).$$

The number k is therefore bounded above. (This argument can be found in [B—R].)

Proof of Lemma 7. Observe that

$$\begin{aligned}Af(x) &= \int K(xy^{-1})\Delta(y^{-1})f(y)\,dm(y)\\ &= \int_{\mathbf{R}^2} L(x,y)f(y)\,dy.\end{aligned}$$

Since A is locally bounded from $\mathrm{L}^2(G,dm)$ to $\mathrm{L}^2(G,dm)$, and its kernel has compact support, A is also locally bounded from $\mathrm{L}^2(\mathbf{R}^2)$ to $\mathrm{L}^2(\mathbf{R}^2)$. Choose $\tau>0$ large enough so that

$$\text{(15)}\qquad \operatorname{supp}K\cdot H_1\subseteq H_\tau.$$

The local boundedness of the operator A implies that

$$\text{(16)}\qquad \|Af\|_{\mathrm{L}^2(\mathbf{R}^2)}\leqq C\|f\|_{\mathrm{L}^2(\mathbf{R}^2)}$$

for all $f\in\mathrm{L}^2(\mathbf{R}^2)$ such that $\operatorname{supp}f\subseteq H_\tau$.

Now choose $h \in C_c^\infty(G)$ so that $h=1$ on H_1, $0 \leqq h \leqq 1$, and $h=0$ outside H_τ. Let

$$(17) \qquad A_h f(x) = \text{p.v.} \int_{\mathbf{R}^2} L(x, y) h(y) f(y)\, dy.$$

Clearly,

$$(18) \qquad A_h f = Af \quad \text{if} \quad \operatorname{supp} f \subseteqq H_1.$$

It follows from (16) that

$$\|A_h f\|_{L^2(\mathbf{R}^2)} \leqq C \|f\|_{L^2(\mathbf{R}^2)}$$

and so A_h is a bounded operator from $L^2(\mathbf{R}^2)$ to $L^2(\mathbf{R}^2)$. Its kernel $L(x, y)h(y)$ satisfies the standard Calderón—Zygmund estimates on $\mathbf{R}^2$ by (C—Z). It follows that A_h is of weak type (1, 1) (cf. [S], pp. 30—34, for instance). Since the kernel of A_h has compact support, and Lebesgue measure and Haar measure are locally equivalent, it follows from (18) that

$$(19) \qquad m(\{x\colon |Af(x)| > \lambda\}) \leqq \frac{C}{\lambda} \|f\|_{L^1(G, dm)}$$

for all $f \in L^1(G, dm)$ such that $\operatorname{supp} f \subseteqq H_1$.

To complete the proof of Lemma 7, take $\varepsilon=1$ in Lemma 8. Choose a sequence $\{\psi_i\}$ of measurable functions such that (a) $\psi_i > 0$ on $H_1 t_i$; (b) $\psi_i = 0$ off $H_1 t_i$; (c) $\sum_i \psi_i(x) = 1$ everywhere. For instance, take ψ_i to be $\chi_{H_1 t_i} / \sum_j \chi_{H_1 t_j}$. Let $\tau_x f$ denote the right translate of f by amount x: $\tau_x f(y) = f(yx)$. Then $\tau_{x^{-1}} A \tau_x = A$ for all x. Notice that for any $\varphi \in C_c^\infty(G)$, $\operatorname{supp} \tau_{t_i}(\psi_i \varphi) \subseteqq H_1$, and $\operatorname{supp}(A\tau_{t_i}(\psi_i\varphi)) \subseteqq H_\tau$ from (15). Using Lemma 8, (18), and (19), we have that, for all $\lambda > 0$,

$$\begin{aligned}
m(\{y \in G\colon |A\varphi(y)| > \lambda\}) &= m\big(\{y \in G\colon \big|A\big(\textstyle\sum_i \psi_i\big)\varphi(y)\big| > \lambda\}\big) \\
&\leqq m\big(\{y \in G\colon \textstyle\sum_i |A(\psi_i \varphi)(y)| > \lambda\}\big) \\
&\leqq \textstyle\sum_i m\big(\{y \in G\colon |A(\psi_i \varphi)(y)| > \lambda/s\}\big) \\
&= \textstyle\sum_i m\big(\{y \in G\colon |\tau_{t_i^{-1}} A \tau_{t_i}(\psi_i \varphi)(y)| > \lambda/s\}\big) \\
&= \textstyle\sum_i \Delta(t_i)\, m\big(\{y \in G\colon |A\tau_{t_i}(\psi_i \varphi)(y)| > \lambda/s\}\big) \\
&\leqq C\frac{s}{\lambda} \textstyle\sum_i \Delta(t_i) \|\tau_{t_i}(\psi_i \varphi)\|_{L^1(G, m)} \\
&= C\frac{s}{\lambda} \textstyle\sum_i \|(\psi_i \varphi)\|_{L^1(G, m)} \\
&\leqq C\frac{s}{\lambda} \|\varphi\|_{L^1(G, dm)}.
\end{aligned}$$

This completes the proof of Lemma 7. □

Lemma 9. *The local parts of all the kernels given in Lemma 5 give rise to bounded operators from* $L^p(G, dm)$ *to* $L^p(G, dm)$, $1<p<\infty$, *and from* $L^1(G, dm)$ *to weak* $L^1(G, dm)$.

Proof. By K we denote any one of the mentioned kernels, and let $L(x, y)=K(xy^{-1})$. Once we have proved that $L(x, y)$ satisfies the standard Calderón—Zygmund estimate, we can invoke Lemma 7. Actually we need to deal only with the terms indicated in (ii) in the proof of Lemma 6. Noticing that, for x close to y, $|xy^{-1}|\approx|x-y|$, and $z-1\approx\frac{1}{2}(s^2+t^2)$, for z close to 1, and using formulas (6) and (7), it is easy to check the required estimates for these terms. We omit the details. □

Corollary 2. *The local parts of the kernels of* $Z_1L^{-1}Z_2$, $Z_1Z_2L^{-1}$ *and* $L^{-1}Z_1Z_2$ *generate bounded operators from* $L^p(G, dm)$ *to* $L^p(G, dm)$, $1<p<\infty$, *and from* $L^1(G, dm)$ *to weak* $L^1(G, dm)$.

Lemma 10. *The kernels of the operators* $Z_1L^{-1}Z_2$ *are all integrable at infinity.*

Proof. It clearly suffices to consider the cases where Z_1 and Z_2 are either of the basis elements X and Y. The integrability of the kernels $XL^{-1}X$, $YL^{-1}Y$, and $XL^{-1}Y$ was established in the course of proving Lemma 6. To see that the kernel of $YL^{-1}X$ is integrable, observe that

$$\operatorname{Ker}(YL^{-1}X) = se^{t/2}\left(\tfrac{3}{2}\mathfrak{Q}'_{-1/2}(z)+z\mathfrak{Q}''_{-1/2}(z)\right)-se^{-t/2}\mathfrak{Q}''_{-1/2}(z).$$

By formulas (8) and (9), the first term in the last expression is $O(se^{t/2}z^{-7/2})$, and the second term is $O(se^{-t/2}z^{-5/2})$. Both terms are integrable at ∞ and so therefore is $\operatorname{Ker}(YL^{-1}X)$. □

Theorem 1 is now immediate from Corollary 2 and Lemma 10.

3. Global behaviour of the kernels

We have just shown that the kernels of the operators $Z_1L^{-1}Z_2$ are all integrable at infinity. In this section, we deal with the unboundedness from $L^p(G, dm)$ to weak $L^p(G, dm)$ of the parts at infinity of the kernels of $Z_1Z_2L^{-1}$ and $L^{-1}Z_1Z_2$. Note that, by Corollary 2, this will establish Theorem 2.

We begin with some basic formulas, and information about the asymptotic behaviour of the kernels. In the first lemma, the notation r_a, $(a>0)$ is used for the standard a-dilate of the function r on $(-\infty, \infty)$: $r_a(u)=a^{-1}r(u/a)$.

Lemma 11. *Let* $W(s,t)=r(s)e^t\chi_{(-\infty,0]}(t)$, *where* r *is an arbitrary integrable function. If* f *is an integrable function on* G, *supported in the region* $t>0$, *then*

$$W*f(s,t) = r_{e^{-t}} *_{\mathbf{R}} \check{h}(se^{-t})$$

for $t<0$, *where* $h(u)=\int_{-\infty}^{\infty} f(-e^{-v}u, -v)\,dv$ *and* $\check{h}(u)=h(-u)$.

Proof. We have, for $t<0$,

$$\begin{aligned} W*f(s,t) &= \int_{-\infty}^{\infty}\int_{t+v<0} e^{t+v} r(s+e^t u) f(-e^{-v}u, -v) e^{-v}\,du\,dv \\ &= \int_{-\infty}^{\infty} e^t r(s+e^t u)\,du \int_{v<-t} f(-e^{-v}u, -v)\,dv \\ &= \int_{-\infty}^{\infty} r_{e^{-t}}(se^{-t}+u) \int_{-\infty}^{\infty} f(-e^{-v}u, -v)\,dv \\ &= r_{e^{-t}} * \check{h}(se^{-t}). \end{aligned}$$

Lemma 12. *The kernel of the operator* $Z_1Z_2L^{-1}$ *is integrable at infinity in the region* $t>0$. *In the neighbourhood of infinity in the region* $t<0$, *the kernel is the sum of an integrable function and a function of the form* $W(s,t)=r(s)e^t\chi_{(-\infty,0]}(t)$, *where*

$$r(s) = A\frac{2s^2-1}{(1+s^2)^{5/2}} + B\frac{s}{(1+s^2)^{5/2}} + C\frac{s}{(1+s^2)^{3/2}},$$

A, B and C are constants, and $r\not\equiv 0$.

Proof. Let $Z_i=a_iX+b_iY$, where $a_i, b_i\in\mathbf{C}$, and $|a_i|+|b_i|\neq 0$, $i=1,2$. Since $(X^2+Y^2)L^{-1}=I$, the identity, we reduce the question to the consideration of the operator $(aX^2+bXY+cYX)L^{-1}$, where $a=a_1a_2-b_1b_2$, $b=a_1b_2$, $c=a_2b_1$, and a, b, and c are not all zero. It follows from Lemma 5 that

$$\operatorname{Ker}(YXL^{-1}) = \operatorname{Ker}(XYL^{-1}) - se^{-t/2}\mathfrak{Q}'_{-1/2}(z)$$

and, by using (8) and (9), that

$$\begin{aligned} \operatorname{Ker}(XYL^{-1}) &= \tfrac{3}{2}se^{-t/2}\mathfrak{Q}'_{-1/2}(z) + se^{-t/2}(z-e^{-t})\mathfrak{Q}''_{-1/2}(z) \\ &= se^{-t/2}\left(\tfrac{3}{2}\mathfrak{Q}'_{-1/2}(z) + z\mathfrak{Q}''_{-1/2}(z)\right) - se^{-3t/2}\mathfrak{Q}''_{-1/2}(z) \\ &= -se^{-3t/2}\mathfrak{Q}''_{-1/2}(z) + \text{integrable term.} \end{aligned}$$

So we may assume that, at infinity, the kernel is the sum of an integrable term and a term of the form

$$(20)\quad a\left(e^{-t/2}\mathfrak{Q}'_{-1/2}(z) + s^2e^{-3t/2}\mathfrak{Q}''_{-1/2}(z)\right) - (b+c)\left(se^{-3t/2}\mathfrak{Q}''_{-1/2}(z)\right) - cse^{-t/2}\mathfrak{Q}'_{-1/2}(z).$$

Since every term in the last expression is integrable in the region $t\geqq 0$, the first part of the lemma is proved.

Consider the region $t<0$. Setting aside terms that are integrable as $t\to-\infty$, we see that (20) is a constant multiple of

$$\big(ar_1(s)-3(b+c)r_2(s)+cr_3(s)\big)e^t,$$

where

$$r_1=\frac{2s^2-1}{(1+s^2)^{5/2}},\quad r_2=\frac{s}{(1+s^2)^{5/2}},\quad r_3=\frac{s}{(1+s^2)^{3/2}}.$$

Notice that, from the assumptions on a_i and b_i,

$$r=ar_1-3(b+c)r_2+cr_3\not\equiv 0. \tag{21}$$

It is clear that this last function is of the required form. □

The following lemma implies the remaining part of Theorem 2.

Lemma 13. *The operators $L^{-1}Z_1Z_2$ and $Z_1Z_2L^{-1}$ are not bounded from $\mathrm{L}^p(G, dm)$ to weak $\mathrm{L}^p(G, dm)$ for any $p\in[1,\infty)$.*

Proof. We deal first with $L^{-1}Z_1Z_2$ and $p=1$. It suffices to show that the part of the kernel at infinity is not in weak $\mathrm{L}^1(G, dm)$. As in the proof of Lemma 12, we reduce the problem to considering the operator $L^{-1}(aX^2+bXY+cYX)$, where a, b and c are complex numbers and are not all zero. Using Lemma 5, (8), and (9), and setting aside terms that are integrable at ∞, we have that $\operatorname{Ker}(L^{-1}Z_1Z_2)(s,t)$ behaves like

$$a\Big(-e^{3t/2}z^{-3/2}+\tfrac{3}{2}s^2e^{t/2}z^{-5/2}\Big)+\Big(\tfrac{1}{2}b+\tfrac{3}{2}c\Big)se^{t/2}z^{-3/2}$$
$$+\tfrac{3}{2}(b+c)se^{t/2}(e^t-z)z^{-5/2}=M(s,t).$$

If $a\neq 0$, then in the region $t>0$ and $|s|<\varepsilon_1e^t$, where ε_1 is positive and small, we have

$$|M(s,t)|\geqq C,$$

and, for any given small λ,

$$m\big(\{(s,t)\colon |M(s,t)|>\lambda\}\big)\geqq C\iint_{t>0,\,|s|<\varepsilon_1e^t}e^{-t}\,ds\,dt=\infty.$$

If $a=0$ and $\frac{1}{2}b+\frac{3}{2}c\neq 0$, then in the region $t>0$, $\varepsilon_2e^t<|s|<e^t$, where ε_2 is close to 1, we have

$$|M(s,t)|\geqq C$$

for all sufficiently large t, say $t\geqq t_0$. So,

$$m\big(\{(s,t)\colon |M(s,t)|>\lambda\}\big)\geqq C\int_{t_0}^{\infty}\int_{e^t>|s|>\varepsilon_2e^t}e^{-t}\,ds\,dt=\infty.$$

If $a=0$ and $\frac{1}{2}b+\frac{3}{2}c=0$, then $b+c$ cannot be zero. Otherwise a, b and c would all be 0. In this case we consider the region: $t>0$ and $\frac{1}{2}\varepsilon_1 e^t<|s|<\varepsilon_1 e^t$, with ε_1 positive and small. We then have

$$|M(s,t)| \geqq C\left|(b+c)\varepsilon_1\left(\tfrac{1}{4}-\tfrac{1}{2}\varepsilon_1^3\right)\right|.$$

So,

$$m\big(\{(s,t)\colon |M(s,t)| > \lambda\}\big) \geqq C\iint_{t>0,\frac{1}{2}\varepsilon_1 e^t<|s|<\varepsilon_1 e^t} e^{-t}\,ds\,dt = \infty.$$

Now we conclude that $\operatorname{Ker}(L^{-1}Z_1Z_2)$ is not in weak $L^1(G, dm)$, and so $L^{-1}Z_1Z_2$ is not bounded from $L^1(G, dm)$ to weak $L^1(G, dm)$.

The case $p>1$ is established by noting that the argument just given holds without change for the function $L^{-1}Z_1Z_2\chi_U$ in place of $\operatorname{Ker}(L^{-1}Z_1Z_2)$, if U is a sufficiently small neighbourhood of the identity.

We now deal with the operator $Z_1Z_2L^{-1}$, starting with the case $p>1$. We may assume that the kernel is a function W of the form given in Lemma 12. Fix a nonzero function $h\in C_c(\mathbf{R})$. For $T>0$, let $\psi_T=T^{-1}\chi_{[-T,0]}$ and $f_T(u,v)=h(-e^{-v}u)\psi_T(-v)$. Then,

$$\int_{-\infty}^{\infty} f_T\big((u,v)^{-1}\big)\,dv = h(u)$$

for all T. So by Lemma 11, $W*f_T$ is independent of T in the region where $t<0$. Since $\|f_T\|_{L^p(G,dm)}\to 0$ as $T\to\infty$, this completes the proof of the case $p>1$.

The proof of the case $p=1$ reduces to showing that there is no constant C such that

$$m\big(\{(s,t)\colon |W*f(s,t)| > \lambda\}\big) \leqq \frac{C}{\lambda}\|f\|_{L^1(G,dm)} \tag{22}$$

for all $\lambda>0$, and all $f\in C_c^\infty(G)$, where W and r are as in Lemma 12.

Note that if $f\geqq 0$ is supported in the set $\{(s,t)\colon t>0\}$, then the corresponding function h in Lemma 11 satisfies

$$\|h\|_{L^1(\mathbf{R})} = \|f\|_{L^1(G,dm)}.$$

On the other hand, if $h\geqq 0$, $h\in L^1(\mathbf{R})$, is given, we can find a corresponding f: take the function f_T above with $T=1$, for instance. It is therefore enough to disprove the inequality

$$m\big(\{(s,t)\colon t<0, |r_{e^{-t}}*\check{h}(se^{-t})| > \lambda\}\big) \leqq \frac{C}{\lambda}\|h\|_{L^1(\mathbf{R})}. \tag{23}$$

Let μ be the measure in the upper half-plane given by $d\mu=dx\,dy/y$. Making the change of variables $x=e^{-t}s$, $y=e^{-t}$, one finds that (23) is equivalent to the inequality

$$\mu\big(\{(x,y)\colon y>1, |r_y*h(x)| > \lambda\}\big) \leqq \frac{C}{\lambda}\|h\|_{L^1(\mathbf{R})}. \tag{24}$$

But (24) is self-improving, in the sense that, if it were true, it would also hold without the condition $y>1$ in the left-hand side. To see this, apply (24) with h replaced by its dilate h_b and λ by λ/b, for some $b>0$. Observe that

$$r_y * h_b = (r_{y/b} * h)_b,$$

so that the inequality $|r_y * h_b(x)| > \lambda/b$ is equivalent to $|r_{y/b} * h(x/b)| > \lambda$. This gives

$$\text{(25)}\quad \mu\big(\{(x,y)\colon y>1,\, |r_y * h_b(x)| > \lambda/b\}\big) = \int_1^\infty \frac{dy}{y}\, b \int_{\{x:\, |r_{y/b} * h(x)| > \lambda\}} dx$$
$$= b\mu\big(\{(x,y)\colon y > b^{-1},\, |r_y * h(x)| > \lambda\}\big).$$

The inequality (24) now implies that the last expression in (25) is dominated by $C(b/\lambda)\|h\|_1$. This means that if (24) holds, it also holds with the condition $y>1$ replaced by the condition $y>b^{-1}$ with the same constant C. Then one simply lets $b\to\infty$ to get the self-improving property.

It therefore suffices to prove the following lemma in order to complete the proof of Lemma 13.

Lemma 14. *There is no constant C such that*

$$\text{(26)}\qquad \mu\big(\{(x,y)\colon y>0,\, |r_y * h(x)| > \lambda\}\big) \leqq \frac{C}{\lambda}\|h\|_{L^1(\mathbf{R})}$$

for all $h\in L^1(\mathbf{R}^1)$.

Proof. The function r is in C^∞ and satisfies:

$$r(s) = O(|s|^{-2}),\quad r'(s) = O(|s|^{-3}),\quad \int r(s)\,ds = 0.$$

Take a function $\varphi\in C^\infty$ with the same properties as r and such that

$$\int r(s)\,\varphi(-s)\,ds \neq 0.$$

One can e.g. take $\varphi=r$. Fix large natural numbers q and p. For a large natural number N, we shall consider

$$\text{(27)}\qquad h_N(x) = \sum_{n=n_0}^{N} \sum_{k=1}^{2^{qn}/p} \pm\varphi(2^{qn}x - pk).$$

For convenience, we let p be a power of 2, and let n_0 be the smallest value of n for which $2^{qn}\geqq p$. The signs in (27) will be chosen later.

Claim 1. *One can choose q, p and $\lambda>0$ so that for all sign choices and all large N,*

$$\mu\big(\{(x,y)\colon y>0,\, |r_y * h_N(x)| > \lambda\}\big) \geqq CN,$$

with $C>0$ independent of the signs and of N.

Claim 2. *The signs $\pm$ in (27) can be chosen so that $\|h_N\|_{L^1(\mathbf{R})} = O(\sqrt{N})$, as $N\to\infty$.*

These would clearly disprove (26).

Proof of Claim 1. Since $r*\varphi(0)\neq 0$, there exists a $\delta>0$ such that

$$|r_u*\varphi(x)| > \delta,$$

for (x, u) in a neighbourhood U of $(0, 1)$ in $\mathbf{R}^2_+$. We can take $U\subseteqq[-1, 1]\times[\frac{1}{2}, 2]$. Write

$$\varphi_{nk}(x) = \varphi(2^{qn}x-pk),$$

so that

$$\varphi_{nk}(x) = 2^{-qn}\varphi_{2^{-qn}}(x-pk2^{qn}).$$

Then

$$r_y*\varphi_{nk}(x) = r_{2^{qn}y}*\varphi(2^{qn}x-pk). \tag{28}$$

Thus $|r_y*\varphi_{nk}(x)|>\delta$ if $(2^{qn}x-pk, 2^{qn}y)\in U$, which is equivalent to $(x, y)\in U_{nk}$ for some set

$$U_{nk}\subseteqq[(pk-1)2^{-qn}, (pk+1)2^{-qn}]\times[2^{-qn-1}, 2^{-qn+1}].$$

The interval $[(pk-1)2^{-qn}, (pk+1)2^{-qn}]$ appearing here should be considered as the essential part of the support of φ_{nk}. Clearly $\mu(U_{nk})=2^{-qn}\mu(U)$. Now take $N>n_0$ and let $(x, y)\in U_{\nu\varkappa}$ for some integers ν, $\varkappa$ with $n_0\leqq\nu\leqq N$ and $1\leqq\varkappa\leqq 2^{\nu q}/p$. In the sum

$$\sum_{n=n_0}^{N}\sum_{k=1}^{2^{qn}/p}\pm r_y*\varphi_{nk}(x) = r_y*h_N(x). \tag{29}$$

the term with $n=\nu$, $k=\varkappa$ is greater than δ in absolute value. We shall see that the other terms are much smaller, and start by deriving an inequality for $r_\gamma*\varphi$, $\gamma>0$. Note that

$$r_\gamma*\varphi(z) = \int r_\gamma(s)\big(\varphi(z-s)-\varphi(z)\big)\,ds, \tag{30}$$

since $\int r(s)\,ds=0$.

Assume first that $\gamma\leqq 2$. Applying the Mean Value Theorem to the part of the integral over the region $|s|<(1+|z|)/2$, we get

(31)

$$\begin{aligned}\Big|\int_{|s|<(1+|z|)/2} r_\gamma(s)\big(\varphi(z-s)-\varphi(z)\big)\,ds\Big| &\leqq \int_{|s|<(1+|z|)/2}|r_\gamma(s)|\,|s|\,ds \sup_{|\xi|>(|z|-1)/2}|\varphi'(\xi)|\\ &\leqq C(1+|z|)^{-3}\gamma\int_{|s|<(1+|z|)/2\gamma}|r(s)|\,|s|\,ds\\ &\leqq C\frac{\gamma}{1+|z|^3}\log\Big(1+\frac{1+|z|}{\gamma}\Big)\\ &\leqq C\frac{\gamma\log(1+\gamma^{-1})}{1+|z|^2}.\end{aligned}$$

The remaining part of the integral in (30) is at most

$$\begin{aligned}(32)\qquad &\int_{|s|>(1+|z|)/2}|r_\gamma(s)|\,|\varphi(z-s)|\,ds+\int_{|s|>(1+|z|)/2}|r_\gamma(s)|\,ds\,|\varphi(z)|\\ &\leqq \int|\varphi(\xi)|\,d\xi \sup_{|s|>(1+|z|)/2}|r_\gamma(s)|+\frac{C}{1+|z|^2}\int_{|s|>(1+|z|)/2\gamma}|r(s)|\,ds\\ &\leqq C\left(\frac{\gamma}{1+|z|^2}+\frac{\gamma}{1+|z|^3}\right).\end{aligned}$$

We conclude from (31) and (32) that, if $\gamma\leqq 2$, then

$$(33)\qquad |r_\gamma*\varphi(z)|\leqq\frac{C\gamma^\varepsilon}{1+|z|^2}$$

for any fixed $\varepsilon<1$.

Next if $\gamma\geqq\frac{1}{2}$, we write $r_\gamma*\varphi=(r*\varphi_{1/\gamma})_\gamma$. Since φ and r satisfy the same conditions, we can apply (33) with r and φ interchanged, and replace γ by $\frac{1}{\gamma}$. This gives

$$(34)\qquad |r_\gamma*\varphi(z)|\leqq\frac{C\gamma^{1-\varepsilon}}{\gamma^2+|z|^2}\quad\text{for all}\quad \gamma>\frac{1}{2}.$$

In the double sum in (29), consider first the terms with $n=\nu, k\neq\varkappa$. Since $\frac{1}{2}\leqq 2^{q\nu}y\leqq 2$, (28) and (34) yield

$$\sum\nolimits_{k\neq\varkappa}|r_y*\varphi_{\nu k}(x)|\leqq\sum\nolimits_{k\neq\varkappa}\frac{C}{1+|2^{q\nu}x-pk|^2}.$$

Since $(x,y)\in U_{\nu\varkappa}$, we have $|2^{q\nu}x-p\varkappa|<1$, so that

$$\sum\nolimits_{k\neq\varkappa}|r_y*\varphi_{\nu k}(x)|\leqq C\sum\nolimits_{k\neq\varkappa}\frac{1}{p^2|k-\varkappa|^2}\leqq Cp^{-2}\leqq\frac{\delta}{6},$$

if p is large enough. Take now those terms in (29) with $n<\nu$. Then $2^{qn}y\leqq 2^{qn}2^{-q\nu+1}\leqq 2$, and we can apply (33) to get

$$\begin{aligned}\sum\nolimits_{n=n_0}^{\nu-1}\sum\nolimits_{k=1}^{2^{qn}/p}|r_y*\varphi_{nk}(x)| &\leqq \sum\nolimits_{n<\nu}\sum\nolimits_k\frac{C(2^{qn}y)^\varepsilon}{1+|2^{qn}x-pk|^2}\\ &\leqq C\sum\nolimits_{n<\nu}(2^{qn}2^{-q\nu})^\varepsilon\sum\nolimits_j\frac{1}{1+p^2j^2}\\ &\leqq C\sum\nolimits_{n<\nu}2^{\varepsilon q(n-\nu)}\leqq\frac{\delta}{6},\end{aligned}$$

if q is large enough. In the remaining case $n>\nu$, we apply (34) since $2^{qn}y>\frac{1}{2}$. The result is

$$\sum_{n=\nu+1}^{N}\sum_{k=1}^{2^{qn}/p}|r_y*\varphi_{nk}(x)| \leqq \sum_{n>\nu}\sum_k \frac{C(2^{qn}y)^{1-\varepsilon}}{(2^{qn}y)^2+|2^{qn}x-pk|^2}$$

$$\leqq C\sum_{n>\nu}(2^{qn}y)^{1-\varepsilon}\int\frac{dt}{(2^{qn}y)^2+t^2}$$

$$\leqq C\sum_{n>\nu}(2^{qn}y)^{1-\varepsilon}(2^{qn}y)^{-1}$$

$$\leqq C\sum_{n>\nu}2^{-\varepsilon q(n-\nu)}\leqq\frac{\delta}{6},$$

for large q. Altogether, this means that if $(x, y)\in U_{\nu\varkappa}$, the sum of the terms in (29) with $(n, k)\neq(\nu, \varkappa)$ is no larger than $\delta/2$ for large p, q. It follows that $|r_y*h_N(x)|>\delta/2$ in each $U_{\nu\varkappa}$. Since

$$\sum_{\nu<N}\sum_{\varkappa=1}^{2^{q\nu}/p}\mu(U_{\nu\varkappa})\sim N,$$

Claim 1 follows with $\lambda=\delta/2$.

Proof of Claim 2. On the set of all sign choices in (27), consider the probability measure which makes the signs into independent Bernoulli variables. Denote by $\mathbf{E}$ the corresponding expectation. Then

$$\mathbf{E}|h_N| = \mathbf{E}\left|\sum_{n,k}\pm\varphi_{n,k}(x)\right| \tag{35}$$

$$\leqq C\left(\sum_{n,k}|\varphi_{nk}(x)|^2\right)^{1/2}$$

$$= C\left(\sum_{n=n_0}^{N}\sum_{k=1}^{2^{qn}/p}|\varphi(2^{qn}x-pk)|^2\right)^{1/2}$$

because of Khinchin's inequality. Now $\varphi(u)=O(|u|^{-2})$ as $|u|\to+\infty$. So

$$|\varphi(2^{qn}x-pk)|^2\leqq\frac{C}{1+|2^{qn}x-pk|^4} \tag{36}$$

for all $n\leqq N$, $k\leqq 2^{qn}/p$. From (35) and (36), it follows that

$$\int_{|x|\leqq 2}\mathbf{E}|h_N(x)|\,dx\leqq C\sqrt{N} \tag{37}$$

for all N.

If $|x|>2$, then $|2^{qn}x-pk|\geqq 2^{qn}|x|/2$, and so, by (35) and (36) again,

$$\text{(38)}\qquad \int_{|x|>2} \mathbf{E}\,|h_N(x)|\,dx \leqq C\int_{|x|>2}\sqrt{N}\sup_{n\leqq N}\left(\sum_{k=1}^{2^{qn}/p}\frac{1}{1+|2^{qn}x-pk|^4}\right)^{1/2}dx$$

$$\leqq C\sqrt{N}\int_{|x|>2}\sup_{n\leqq N}\left(\frac{2^{qn}/p}{2^{4qn}x^4}\right)^{1/2}dx$$

$$\leqq C\sqrt{N}\int_{|x|>2}\frac{dx}{x^2}.$$

Taking (37) and (38) into account, we see that

$$\mathbf{E}\int|h_N(x)|\,dx = C\sqrt{N}$$

for all N. Thus there exists, for each N, a choice of signs for which

$$\int|h_N|\,dx \leqq C\sqrt{N}.$$

This proves Claim 2.

References

A. ALEXOPOULOS, G., Inégalités de Harnack paraboliques et transformées de Riesz sur les groupes de Lie résolubles à croissance polynômiale du volume, *C. R. Acad. Sci. Paris Sér. I Math.* **309** (1989), 661—662.

B. BAKRY, D. Etude des transformées de Riesz dans les variétés riemanniennes à courbure de Ricci minorée, *Séminaire de Probabilités XXI*, 137—172, Lecture Notes in Math. **1247**, Springer-Verlag, Berlin—New York, 1987.

B—R. BURNS, R. J. and ROBINSON, D. W., A priori inequalities on Lie groups, *Preprint.*

F—S. FEFFERMAN, C. and SÁNCHEZ-CALLE, A. Fundamental solutions for second order subelliptic operators, *Ann. of Math.* **124** (1986), 247—272.

H 1. HULANICKI, A., On the spectrum of the Laplacian on the affine group of the real line, *Studia Math.* **54** (1976), 199—204.

H 2. HULANICKI, A., Subalgebra of $L_1(G)$ associated with Laplacian on a Lie group, *Colloq. Math.* **31** (1974), 269—297.

Kh. KHALIL, I., Sur l'analyse harmonique du groupe affine de la droite, *Studia Math.* **51** (1974), 140—166.

L 1. LOHOUÉ, N. Transformées de Riesz et fonctions de Littlewood—Paley sur les groupes non moyennables, *C. R. Acad. Sci. Paris, Sér. I*, **306** (1988), 327—330.

L 2. LOHOUÉ, N. Inégalités de Sobolev pour les formes différentielles sur une variété riemannienne, *C. R. Acad. Sci. Paris, Sér. I*, **301** (1985), 277—280.

L 3. LOHOUÉ, N. Comparaison des champs de vecteurs et des puissances du Laplacien sur une variété riemannienne à courbure non positive, *J. Funct. Anal.* **61** (1985), 164—201.

MOS. MAGNUS, W., OBERHETTINGER, F. and SONI, R. P., *Formulas and theorems for the special functions of mathematical physics*, 3rd Edition, Springer-Verlag, Berlin—Heidelberg—New York, 1966.

P. PIERS, J.-P., *Amenable locally compact groups*, John Wiley and Sons, New York, 1984.

R—S. REED, M. and SIMON, B., *Methods of modern mathematical physics, I. Functional analysis*, Academic Press, New York, 1972.

S—C. SALOFF-COSTE, L., Analyse sur les groupes de Lie à croissance polynomiale, *Ark. Mat.* **28** (1990), 315—331.

S. STEIN, E. M., *Singular integrals and differentiability properties of functions*, Princeton University Press, Princeton 1970.

Str. STRICHARTZ, R. S., Analysis of the Laplacian on the complete Riemannian manifold, *J. Funct. Anal.* **52** (1983), 48—79.

Received July 1, 1991

G. I. Gaudry
School of Information Science and Technology
The Flinders University of South Australia
PO Box 2100
Adelaide. S.A. 5001
Australia

T. Qian
Department of Mathematics
University of New England
Armidale. NSW 2350
Australia

P. Sjögren
Department of Mathematics
Chalmers University of Technology
and
University of Göteborg
S-412 96 Göteborg
Sweden