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Extra-strong uncertainty principles in relation to phase derivative for signals in Euclidean spaces

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ABSTRACT

This paper devotes to studying uncertainty principles of Heisenberg type for signals defined on $\mathbf{R}^n$ taking values in a Clifford algebra. For real-para-vector-valued signals possessing all first-order partial derivatives we obtain two uncertainty principles of which both correspond to the strongest form of the Heisenberg type uncertainty principles for the one-dimensional space. The lower-bounds of the new uncertainty principles are in terms of a scalar-valued phase derivative. Through Hardy spaces decomposition we also obtain two forms of uncertainty principles for real-valued signals of finite energy with the first order Sobolev type smoothness.

1. Introduction

In history both W. Heisenberg and N. Wiener promoted the concept uncertainty principle in relation to physics [13,16]. In quantum mechanics the Heisenberg uncertainty principle addresses a fundamental problem: the values of a pair of canonically conjugate observables such as the position and the momentum of particles cannot be both determined precisely in any quantum state. In the language of harmonic analysis, the uncertainty principle says that a nonzero function and its Fourier transform cannot both be sharply localized. It is Gabor's fundamental work [15] that really brings uncertainty principle to the sight of signal analysts. The *classical uncertainty principle* [15] can be represented by the inequality

$$\sigma_{t,s}^2 \sigma_{\omega,s}^2 \geq \frac{1}{4}, \tag{1.1}$$

where $\sigma_{t,s}^2$ and $\sigma_{\omega,s}^2$ are respectively the duration and bandwidth of a signal $s(t) \in L^2(\mathbf{R})$ with $\|s\|_2 = 1$, defined as

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$$\sigma_{t,s}^2 \triangleq \int_{-\infty}^{\infty} (t - \langle t \rangle_s)^2 |s(t)|^2 dt \tag{1.2}$$

and

$$\sigma_{\omega,s}^2 \triangleq \int_{-\infty}^{\infty} (\omega - \langle \omega \rangle_s)^2 |\hat{s}(\omega)|^2 d\omega, \tag{1.3}$$

where $\langle t \rangle_s$ and $\langle \omega \rangle_s$ are the means of time t and Fourier frequency ω, respectively, defined by

$$\langle t \rangle_s \triangleq \int_{-\infty}^{\infty} t|s(t)|^2 dt, \tag{1.4}$$

and

$$\langle \omega \rangle_s \triangleq \int_{-\infty}^{\infty} \omega |\hat{s}(\omega)|^2 d\omega, \tag{1.5}$$

where $\hat{s}(\omega)$ is the Fourier transformation of $s(t)$, defined by

$$\hat{s}(\omega) \triangleq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-it\omega} s(t) dt. \tag{1.6}$$

For a signal expressed as $s(t) = |s(t)|e^{i\varphi(t)}$ a stronger result is available [8,7], referred to as the *strong uncertainty principle*:

$$\sigma_{t,s}^2 \sigma_{\omega,s}^2 \geq \frac{1}{4} + \mathrm{Cov}_s^2, \tag{1.7}$$

where

$$\mathrm{Cov}_s = \int_{-\infty}^{\infty} t\varphi'(t)|s(t)|^2 dt - \langle t \rangle_s \langle \omega \rangle_s = \int_{-\infty}^{\infty} (t - \langle t \rangle_s)(\varphi'(t) - \langle \omega \rangle_s)|s(t)|^2 dt$$

is the *covariance* of the signal.

The recent paper [9] strengthens the result (1.7) through proving a larger lower-bound, called *extra-strong uncertainty principle*:

$$\sigma_{t,s}^2 \sigma_{\omega,s}^2 \geq \frac{1}{4} + \mathrm{COV}_s^2, \tag{1.8}$$

where COV_s is the *absolute covariance* of the signal, defined as

$$\mathrm{COV}_s = \int_{-\infty}^{\infty} |(t - \langle t \rangle_s)(\varphi'(t) - \langle \omega \rangle_s)||s(t)|^2 dt. \tag{1.9}$$

By definition, COV_s is obviously larger than Cov_s. In our view, all the progresses of the classical uncertainty principle correspond to the three mentioned forms, viz., (1.1), (1.7) and (1.8).

Recently some researchers studied uncertainty principles for signals in higher dimensional Euclidean spaces with the Clifford algebra setting [1,2,18,17,21,22,24], as well as the quaternionic setting [3,14,23]. For higher dimensional signals, uncertainty principles expose, as an important aspect, how the variance of a multivariate vector-valued function and the variance of its Clifford Fourier transform, or its quaternionic Fourier transform, if appropriate, are related. In the present study, we work with the Clifford algebra setting. The literature [1,17,18] obtain a "directional" uncertainty principle for multivariate and Clifford-valued functions $f : \mathbf{R}^m \to Cl_{m,0}$, $m = 2, 3 \pmod 4$. The obtained directional uncertainty principle essentially corresponds to the primary step (1.1). The work [21] obtains two uncertainty principles for signals in higher dimensions of which one is for real-scalar-valued signals, and the other for axial-form signals. In order to state the obtained results in [21] and those in the present paper we need to recall a number of definitions. We first note that the improvements of the classical uncertainty principle (1.1) are all dependent on a phase derivative concept. In signal analysis *phase derivative* is well accepted as *instantaneous frequency* of a signal, the latter being of central importance in signals analysis. This paper further promotes the role of phase derivative in uncertainty principles for higher dimensional spaces.

Let $\mathbf{e}_1, \ldots, \mathbf{e}_m$ denote the basic elements spanning the Euclidean space $\mathbf{R}^m$ of which each is like the complex imaginary element $\mathbf{i}$ with the property $\mathbf{e}_k^2 = -1$, but with the anti-commutativity $\mathbf{e}_k\mathbf{e}_l = -\mathbf{e}_l\mathbf{e}_k$, $k \neq l$. The corresponding real- and complex-Clifford algebras both are of the linear dimension 2^m whose elements are of the form $x = x_0 + x'$, where x_0 and x' are the scalar and non-scalar parts of x, denoted, respectively, by $x_0 = \mathrm{Sc}\{x\}$ and $x' = \mathrm{Nsc}\{x\}$ (for more details see Section 2). We will be working on functions defined on $\mathbf{R}^m$ or $\mathbf{R}_1^m = \{x = x_0 + \underline{x} : \ x_0 \in \mathbf{R}, \ \underline{x} \in \mathbf{R}^m\}$, and taking values in the complex Clifford algebra. The elements in $\mathbf{R}^m$ and $\mathbf{R}_1^m$ are, respectively, called *vectors* and *para-vectors*. To define the phase and amplitude and their derivatives of a para-vector-valued function $f(\underline{x})$ defined on a region $\Omega \subset \mathbf{R}^n$, we represent $f(\underline{x})$ in the polar coordinate form [22]:

$$\begin{aligned} f(\underline{x}) &= f_0(\underline{x}) + f_1(\underline{x})\mathbf{e}_1 + f_2(\underline{x})\mathbf{e}_2 + \cdots + f_m(\underline{x})\mathbf{e}_m \\ &= \rho(\underline{x})[\frac{f_0(\underline{x})}{\rho(\underline{x})} + \frac{u(\underline{x})}{\rho(\underline{x})}] \\ &= \rho(\underline{x})[\frac{f_0(\underline{x})}{\rho(\underline{x})} + \frac{u(\underline{x})}{|u(\underline{x})|}\frac{|u(\underline{x})|}{\rho(\underline{x})}] \\ &= \rho(\underline{x})[\cos\theta(\underline{x}) + \frac{u(\underline{x})}{|u(\underline{x})|}\sin\theta(\underline{x})] \\ &= \rho(\underline{x})e^{\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}, \end{aligned} \tag{1.10}$$

where $u(\underline{x}) = f_1(\underline{x})\mathbf{e}_1 + f_2(\underline{x})\mathbf{e}_2 + \cdots + f_m(\underline{x})\mathbf{e}_m$, $\rho(\underline{x}) = \sqrt{f_0^2(\underline{x}) + f_1^2(\underline{x}) + f_2^2(\underline{x}) + \cdots + f_m^2(\underline{x})}$ is called the *amplitude*, and $\theta(\underline{x}) = \arctan\frac{|u(\underline{x})|}{f_0(\underline{x})}$ is called the *phase angle*. We, in particular, note that $\left(\frac{u(\underline{x})}{|u(\underline{x})|}\right)^2 = -1$, that partially justifies the above introduced phase concept. Below we adopt the phase derivative definitions given in [24] and define amplitude derivatives of $f(\underline{x})$ in the consistent way.

In higher dimensions, to define phase derivatives, the following first-order partial differential operator, called *Dirac differential operator* will be used, that reduces to the operator $\frac{1}{i}\frac{d}{dt}$ in the one-dimensional case:

$$\underline{D} = \frac{\partial}{\partial x_1}\mathbf{e}_1 + \cdots + \frac{\partial}{\partial x_m}\mathbf{e}_m.$$

Definition 1.1. Let $f(\underline{x}) = \rho(\underline{x})e^{\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})} \in L^2(\mathbf{R}^m)$ be real-para-vector-valued, and all the first order partial derivatives of $f(\underline{x})$ exist. Then there are two alternative ways to define the phase derivative of f, namely

$$\theta_1'(\underline{x}) = \mathrm{Sc}\left\{[\underline{D}f(\underline{x})][f(\underline{x})]^{-1}\right\} \tag{1.11}$$

or

$$\theta_2'(\underline{x}) = \mathrm{Sc}\{[\underline{D}\theta(\underline{x})]\frac{u(\underline{x})}{|u(\underline{x})|}\}. \tag{1.12}$$

Two alternative amplitude derivatives are accordingly

$$\rho_1'(\underline{x}) \triangleq \underline{D}\rho(\underline{x}) \tag{1.13}$$

and

$$\rho_2'(\underline{x}) \triangleq \rho(\underline{x})\mathrm{Nsc}\left\{[\underline{D}f(\underline{x})][f(\underline{x})]^{-1}\right\}. \tag{1.14}$$

The above phase derivatives $\theta_1'(\underline{x})$, $\theta_2'(\underline{x})$ are first studied in [24]. For $m = 1$, θ_1' and θ_2' coincide with each other, both being identical with the classical phase derivative, and so do ρ_1' and ρ_2'. In higher dimensions the two above defined amplitude derivatives lead to two definitions of variance of the frequency, viz., $\mathrm{var}_{\underline{\xi}}$ and $\mathrm{var}^*_{\underline{\xi}}$, as given in Definition 1.2 (for more details see Remark 4.5).

With the Dirac operator, like the Cauchy–Riemann operator in the one complex variable case, one can define Clifford holomorphic functions in the higher dimensional spaces, being analogous with the complex holomorphic functions of one complex variable, called *monogenic functions*. In particular, the Hardy spaces constituted by the para-vector-valued monogenic functions in the upper-half space coincide with the conjugate harmonic systems in the sense of Stein and Weiss [20]. The Hardy spaces employed in this theory are not restricted to only contain para-vector-valued monogenic functions.

Definition 1.2. Let $f(\underline{x}) = \rho(\underline{x})e^{\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})} \in L^2(\mathbf{R}^m)$ be real-para-vector-valued with $\|f\|_2 = 1$. Assume that all the first order partial derivatives of $f(\underline{x})$ exist. Then the *mean of time* is given by

$$\langle\underline{x}\rangle = \int_{\mathbf{R}^m} \mathbf{i}\underline{x}|f(\underline{x})|^2 d\underline{x}, \tag{1.15}$$

the *variance of* $\underline{x}$ is

$$\mathrm{var}_{\underline{x}} = \int_{\mathbf{R}^m} (\mathbf{i}\underline{x} - \langle\underline{x}\rangle)^2 |f(\underline{x})|^2 d\underline{x}, \tag{1.16}$$

the *mean of frequency* is

$$\langle\underline{\xi}\rangle = \int_{\mathbf{R}^m} |\mathbf{i}\underline{\xi}||\hat{f^+}(\underline{\xi})|^2 d\underline{\xi} - \int_{\mathbf{R}^m} |\mathbf{i}\underline{\xi}||\hat{f^-}(\underline{\xi})|^2 d\underline{\xi}, \tag{1.17}$$

the *mean of* $\underline{\xi}^2$ is defined by

$$\langle\underline{\xi}^2\rangle = \int_{\mathbf{R}^m} |\mathbf{i}\underline{\xi}|^2 |\hat{f}(\underline{\xi})|^2 d\underline{\xi},$$

the *variance of frequency* is alternatively defined by one of the following two formulas:

$$\mathrm{var}_{\underline{\xi}} = \int_{\mathbf{R}^m} [|\mathbf{i}\underline{\xi}| - \langle\underline{\xi}\rangle]^2 |\hat{f^+}(\underline{\xi})|^2 d\underline{\xi} + \int_{\mathbf{R}^m} [-|\mathbf{i}\underline{\xi}| - \langle\underline{\xi}\rangle]^2 |\hat{f^-}(\underline{\xi})|^2 d\underline{\xi}, \tag{1.18}$$

or

$$\mathrm{var}^*_{\underline{\xi}} \triangleq \int_{\mathbf{R}^m} [\theta_1'(\underline{x}) - \langle \underline{\xi} \rangle]^2 |f(\underline{x})|^2 d\underline{x} + \int_{\mathbf{R}^m} |\rho_1'(\underline{x})|^2 d\underline{x}. \tag{1.19}$$

The *covariance* is defined by

$$\mathrm{Cov} = \int_{\mathbf{R}^m} (\mathbf{i}\underline{x} - \langle \underline{x} \rangle)[\theta_1'(\underline{x}) - \langle \underline{\xi} \rangle]|f(\underline{x})|^2 d\underline{x}, \tag{1.20}$$

and the *absolute covariance* is defined by

$$\mathrm{COV} = \int_{\mathbf{R}^m} |\mathbf{i}\underline{x} - \langle \underline{x} \rangle||\theta_1'(\underline{x}) - \langle \underline{\xi} \rangle||f(\underline{x})|^2 d\underline{x}, \tag{1.21}$$

where $\hat{f}(\underline{\xi})$, $\hat{f^+}(\underline{\xi})$ and $\hat{f^-}(\underline{\xi})$ are Fourier transform of $f(\underline{x})$, $f^+(\underline{x})$ and $f^-(\underline{x})$, respectively, and f^+ and f^- are the upper- and lower-Hardy H^2-spaces projections of f, with the relation $f = f^+ + f^-$.

We will, in the main part of this paper, give justifications and detailed explanations for the above introduced definitions. Now we are ready to review some existing results. For real-scalar-valued signals $f(\underline{x}) \in L^2(\mathbf{R}^m)$ the work [21] obtains

$$\mathrm{var}_{\underline{x}} \mathrm{var}_{\underline{\xi}} \geq \frac{m^2}{4}; \tag{1.22}$$

while for signals of the axial-form $f(\underline{x}) = U(|\underline{x}|) + \frac{\underline{x}}{|\underline{x}|} V(|\underline{x}|) \in L^2(\mathbf{R}^m)$ where $U(|\underline{x}|)$ and $V(|\underline{x}|)$ are scalar-valued, [21] obtains

$$\mathrm{var}_{\underline{x}} \mathrm{var}_{\underline{\xi}} \geq [-\frac{m}{2} + (m-1) \int_{\mathbf{R}^m} V^2 d\underline{x}]^2 + \mathrm{Cov}^2. \tag{1.23}$$

It is easy to see that when $m = 1$, the lower bounds of (1.22) and (1.23) reduce, respectively, to those of (1.1) and (1.7). With this observation we say that (1.22) and (1.23) correspond to the weaker types of uncertainty principles. Moreover, (1.23) is restricted to only signals of the axial-form. The work [22] gives an uncertainty principle for real-para-vector-valued signal $f(\underline{x}) = \rho(\underline{x}) e^{\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}$, read as

$$\left(\int_{\mathbf{R}^m} |\underline{x}|^2 |f(\underline{x})|^2 d\underline{x} \right) \left(\int_{\mathbf{R}^m} |\underline{\xi}|^2 |\hat{f}(\underline{\xi})|^2 d\underline{\xi} \right) \geq \frac{m^2}{4} + \mathrm{COV}^2_{\underline{x}\underline{\xi}}, \tag{1.24}$$

where the *absolute covariance* $\mathrm{COV}_{\underline{x}\underline{\xi}}$ is defined by

$$\mathrm{COV}_{\underline{x}\underline{\xi}} := \sum_{k=1}^{m} \int_{\mathbf{R}^m} \left| x_k \mathrm{NSc} \left\{ \left(\frac{\partial}{\partial x_k} e^{\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})} \right) e^{-\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})} \right\} \right| |f(\underline{x})|^2 d\underline{x}, \tag{1.25}$$

where $\rho(\underline{x})$, $u(\underline{x})$ and $\theta(\underline{x})$ are defined in (1.10).

The uncertainty principle given by (1.24) does reduce to the strongest form (1.8) for $m = 1$. It, however, for $m > 1$, does not use the global scalar-valued phase derivatives given in Definition 1.1. It would be very interesting to establish uncertainty principles in terms of the scalar-valued phase derivatives and compare their sharpness. In the present paper through a scalar-valued phase derivative related COV, we achieve to obtain two incomparable uncertainty principles, namely

$$\mathrm{var}_{\underline{x}}\mathrm{var}^*_{\underline{\xi}} \geq \frac{m^2}{4} + \mathrm{COV}^2 \tag{1.26}$$

and

$$\mathrm{var}_{\underline{x}}\mathrm{var}_{\underline{\xi}} \geq \frac{1}{4}|\mathbf{i}m + M|^2 + \mathrm{COV}^2. \tag{1.27}$$

The uncertainty principles (1.26) and (1.27), for real-para-vector-valued signals, are under the assumption that all the first order classical partial derivatives exist. Both those uncertainty principles essentially correspond to (1.8). As in the classical case both those results and the proofs are naturally related to phase and amplitude derivatives of the signals.

If we do not assume existence of the first order partial derivatives, we cannot directly define the phase and amplitude derivatives. Without such smoothness assumption we, instead, use the Hardy spaces decomposition. As example, we consider real-scalar-valued signals. For those signals, under a Sobolev type smoothness condition (weak smoothness), two types of uncertainty principles, corresponding to (1.26) and (1.27), respectively, are derived.

The paper is organized as follows. In Section 2, we recall some basic knowledge of Clifford algebras. In Section 3, we define and analyze phase and amplitude derivatives of real-para-vector-valued signals in the Clifford algebra setting. Section 4 is devoted to studying the means and variances of time and frequency in the Clifford algebra setting. Section 5 discusses uncertainty principles for real-para-vector-valued signals with all the first order partial derivatives. We deduce two different types of uncertainty principles, both corresponding to the strongest form in the one-dimensional case, (1.8). In Section 6 we treat non-smooth real scalar-valued signals by using the Hardy spaces decomposition method.

2. Preliminaries

We now recall some basic knowledge of Clifford algebra (see [6,12]). Let $\mathbf{e}_1, \ldots, \mathbf{e}_m$ be *basic elements* satisfying $\mathbf{e}_i\mathbf{e}_j + \mathbf{e}_j\mathbf{e}_i = -2\delta_{ij}$, where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise, $i, j = 1, 2, \ldots, m$. Let

$$\mathbf{R}^m_1 = \{x_0 + \underline{x},\ \underline{x} \in \mathbf{R}^m\},$$

where

$$\mathbf{R}^m = \{\underline{x} = x_1\mathbf{e}_1 + \ldots + x_m\mathbf{e}_m : x_j \in \mathbf{R},\ j = 1, 2, \ldots, m\}$$

is identical with the usual m-dimensional Euclidean space.

An element in $\mathbf{R}^m$ is called a *vector*. The real (complex) Clifford algebra generated by $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m$, denoted by $\mathbf{R}_m$ ($\mathbf{C}_m$), is the non-commutative algebra generated by $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m$, over the real (complex) field $\mathbf{R}$ ($\mathbf{C}$). A general element in $\mathbf{R}_m$ is of the form $x = \sum_T x_T\mathbf{e}_T$, where $x_T \in \mathbf{R}$, and $\mathbf{e}_T = \mathbf{e}_{i_1}\mathbf{e}_{i_2}\cdots\mathbf{e}_{i_l}$, being called *induced products*, where T runs over all the ordered subsets of $\{1, \ldots, m\}$, namely

$$T = \{1 \leq i_1 < \cdots < i_k \leq m\},\quad 1 \leq k \leq m.$$

When $T = \emptyset$, we set $\mathbf{e}_\emptyset = \mathbf{e}_0 = 1$. We denote $|T| = l$ where l is the number of the indices involved. A general Clifford number x may be decomposed into

$$x = \sum_{l=0}^{m} x^{(l)},\quad x^{(l)} = \sum_{|T|=l} x_T\mathbf{e}_T.$$

A Clifford number of the form $x^{(l)}$ is called a Clifford number of l-form. A Clifford number of 2-form is also called a *bi-vector*.

The multiplication of two vectors $\underline{x} = \sum_{j=1}^m x_j \mathbf{e}_j$ and $\underline{y} = \sum_{j=1}^m y_j \mathbf{e}_j$ is given by

$$\underline{xy} = \underline{x} \cdot \underline{y} + \underline{x} \wedge \underline{y}$$

where $\underline{x} \cdot \underline{y}$ is a scalar, denoted by $\mathrm{Sc}(\underline{xy})$ and given by

$$\underline{x} \cdot \underline{y} = -\sum_{j=1}^m x_j y_j = \frac{1}{2}(\underline{xy} + \underline{yx}) = -\langle \underline{x}, \underline{y} \rangle,$$

and $\underline{x} \wedge \underline{y}$ is the non-scalar part of $\underline{xy}$, denoted by $\mathrm{NSc}(\underline{xy})$ and given by

$$\underline{x} \wedge \underline{y} = \sum_{i<j} e_{ij}(x_i y_j - x_j y_i) = \frac{1}{2}(\underline{xy} - \underline{yx}),$$

being a bi-vector, denoted by $\mathrm{Bi}(\underline{xy})$.

The Clifford conjugation and reversion of $\mathbf{e}_T = \mathbf{e}_{i_1} \cdots \mathbf{e}_{i_l}$ are $\bar{\mathbf{e}}_T = \bar{\mathbf{e}}_{i_l} \cdots \bar{\mathbf{e}}_{i_1}$, $\bar{\mathbf{e}}_j = -\mathbf{e}_j$ and $\tilde{\mathbf{e}}_T = \mathbf{e}_{i_l} \cdots \mathbf{e}_{i_1}$. So the Clifford conjugation of a vector $\underline{x}$ is $\overline{\underline{x}} = -\underline{x}$.

It is easy to verify that $0 \neq \underline{x} \in \mathbf{R}^m$ implies

$$\underline{x}^{-1} = \frac{\overline{\underline{x}}}{|\underline{x}|^2}.$$

The natural inner product between $x = \Sigma_T x_T \mathbf{e}_T$ and $y = \sum_T y_T \mathbf{e}_T$ in $\mathbf{C}_m$, denoted by $\langle x, y \rangle$, is the complex number $\Sigma_T x_T \overline{y}_T$. The norm associated with this inner product is

$$|x| = \langle x, x \rangle^{\frac{1}{2}} = \left(\Sigma_T |x_T|^2\right)^{\frac{1}{2}}.$$

Let $f(x) = \Sigma_T f_T(x) \mathbf{e}_T$, where $x = x_0 + \underline{x} \in \mathbf{R}_1^m$, and f_T are complex-valued functions. We will use the *homogeneous Dirac operator* $\underline{D}$ and the *non-homogeneous Dirac operator* D, where

$$\underline{D} = \frac{\partial}{\partial x_1} \mathbf{e}_1 + \cdots + \frac{\partial}{\partial x_m} \mathbf{e}_m,$$

and

$$D = \frac{\partial}{\partial x_0} + \underline{D}.$$

We define the "left" and "right" role of the operators $\underline{D}$ and D, respectively, by

$$\underline{D} f = \sum_{i=1}^m \sum_T \frac{\partial f_T}{\partial x_i} \mathbf{e}_i \mathbf{e}_T, \ Df = \sum_{i=0}^m \sum_T \frac{\partial f_T}{\partial x_i} \mathbf{e}_i \mathbf{e}_T$$

and

$$f \underline{D} = \sum_{i=1}^m \sum_T \frac{\partial f_T}{\partial x_i} \mathbf{e}_T \mathbf{e}_i, \ fD = \sum_{i=0}^m \sum_T \frac{\partial f_T}{\partial x_i} \mathbf{e}_T \mathbf{e}_i.$$

If f has all continuous first order partial derivatives and $Df = 0$ in a (connected and open) domain $\Omega \subseteq \mathbf{R}_1^m$, then we say that f is *left-monogenic* in Ω; and, if $fD = 0$ in $\Omega \subseteq \mathbf{R}_1^m$, we say that f is *right-monogenic* in Ω. If f is both left- and right-monogenic, then we say that f is *monogenic*.

We call

$$E(x) = \frac{\overline{x}}{|x|^{m+1}}$$

the *Cauchy kernel* in $\mathbf{R}_1^m$. It is easy to verify that $E(x)$ is a monogenic function in $\mathbf{R}_1^m \backslash \{0\}$.

3. Derivatives of phase and amplitude of signals on $\mathbf{R}^m$

Phase derivative is well accepted as instantaneous frequency of a signal, the latter playing an important role in signal analysis ([4] and [5]). In the studies of uncertainty principles of one-dimensional signals, the phase and amplitude derivatives are involved [8,11,10]. In the present paper we will consider uncertainty principle for signals on $\mathbf{R}^m$. To develop a comprehensive theory in higher dimensions we need to formulate appropriate phase and amplitude derivative concepts for multivariate functions. We first have a revision on the one-dimensional case.

Let $s(t) = \rho(t)e^{i\varphi(t)}$ be a signal defined on $\mathbf{R}$, $\rho(t) = |s(t)|$. Assume that the classical derivatives of $s(t)$, $\rho(t)$ and $\varphi(t)$ all exist. Taking the derivative with respect to t and dividing $s(t)$ on both sides, we obtain

$$\varphi'(t) = \mathrm{Im}\left[\frac{s'(t)}{s(t)}\right], \tag{3.28}$$

and

$$\rho'(t) = \rho(t)\mathrm{Re}\left[\frac{s'(t)}{s(t)}\right], \tag{3.29}$$

where $\mathrm{Im}\left[\frac{s'(t)}{s(t)}\right]$ and $\mathrm{Re}\left[\frac{s'(t)}{s(t)}\right]$ denote, respectively, the imaginary and the real parts of $\frac{s'(t)}{s(t)}$.

In the following, we assume that signals $f(\underline{x})$ are defined in $\mathbf{R}^m$ taking values in $\mathbf{R}_1^m$, that is,

$$f(\underline{x}) = f_0(\underline{x}) + f_1(\underline{x})\mathbf{e}_1 + f_2(\underline{x})\mathbf{e}_2 + \cdots + f_m(\underline{x})\mathbf{e}_m,$$

where $f_i(\underline{x})$, $i = 0, 1, 2, \cdots, m$, are real-scalar-valued and have all the classical first order partial derivatives.

Similarly with one-dimensional case we define the *Fourier transform* of $f \in L^1(\mathbf{R}^m)$ by

$$\hat{f}(\underline{\xi}) = \frac{1}{(2\pi)^{\frac{m}{2}}} \int_{\mathbf{R}^m} e^{-\mathbf{i}\langle \underline{x}, \underline{\xi}\rangle} f(\underline{x})d\underline{x}. \tag{3.30}$$

If $\hat{f}$ is also in $L^1(\mathbf{R}^m)$, then the inversion formula holds, that is

$$f(\underline{x}) = \frac{1}{(2\pi)^{\frac{m}{2}}} \int_{\mathbf{R}^m} e^{\mathbf{i}\langle \underline{x}, \underline{\xi}\rangle} \hat{f}(\underline{\xi})d\underline{\xi}. \tag{3.31}$$

There holds the Plancherel Theorem

$$\|\hat{f}\|_2^2 = \|f\|_2^2, \qquad f \in L^1(\mathbf{R}^m) \cap L^2(\mathbf{R}^m).$$

Through a density argument, both the Fourier transformation and its inverse can be extended to $L^2(\mathbf{R}^m)$ in which the Plancherel Theorem remains to hold. When we use the formulas (3.30) and (3.31) for $L^2(\mathbf{R}^m)$ functions, we keep in mind that the convergence of the integrals is in the L^2-sense.

The *Hilbert transform* of $f(\underline{x})$ is defined as any of the following equivalent forms

$$\begin{aligned} Hf(\underline{x}) &= \frac{1}{(2\pi)^{\frac{m}{2}}}\int\limits_{\mathbf{R}^m} \frac{\underline{D}e^{\mathbf{i}\langle \underline{x},\underline{\xi}\rangle}}{|\underline{\xi}|}\hat{f}(\underline{\xi})d\underline{\xi} \\ &= \frac{1}{(2\pi)^{\frac{m}{2}}}\int\limits_{\mathbf{R}^m} \frac{\mathbf{i}\underline{\xi}}{|\underline{\xi}|}e^{\mathbf{i}\langle \underline{x},\underline{\xi}\rangle}\hat{f}(\underline{\xi})d\underline{\xi} \\ &= \frac{2}{\sigma_m}\lim_{\varepsilon\to 0^+}\int\limits_{|\underline{x}-\underline{t}|>\varepsilon} \frac{\overline{\underline{x}-\underline{t}}}{|\underline{x}-\underline{t}|^{m+1}}f(\underline{t})d\underline{t} \\ &= -\sum_{j=1}^{m}\mathbf{e}_j R_j(f)(\underline{x}), \end{aligned}$$

where $R_j(f)(\underline{x}) = \frac{2}{\sigma_m}\lim_{\varepsilon\to 0^+}\int_{|\underline{x}-\underline{t}|>\varepsilon}\frac{x_j-t_j}{|\underline{x}-\underline{t}|^{m+1}}f(\underline{t})d\underline{t}$ is the jth-Riesz transform of f [20], $\sigma_m = 2\pi^{\frac{m+1}{2}}/\Gamma(\frac{m+1}{2})$ is the surface area of the unit sphere of $\mathbf{R}^{m+1}$.

For $f(\underline{x}) \in L^2(\mathbf{R}^m)$, we have the decomposition $f(\underline{x}) = f^+(\underline{x}) + f^-(\underline{x}), \hat{f^{\pm}}(\underline{\xi}) = \chi_{\pm}(\underline{\xi})\hat{f}(\underline{\xi})$, where $f^{\pm}(x) \in H_2^{\pm}(\mathbf{R}^m)$, and (see [19])

$$\begin{aligned} f^{\pm}(\underline{x}) &= \frac{1}{2}[f(\underline{x}) \pm Hf(\underline{x})] \\ &= \frac{1}{(2\pi)^{\frac{m}{2}}}\int\limits_{\mathbf{R}^m} e^{\mathbf{i}\langle \underline{x},\underline{\xi}\rangle}\chi_{\pm}(\underline{\xi})\hat{f}(\underline{\xi})d\underline{\xi} \\ &= \lim_{x_0\to 0^{\pm}} f^{\pm}(x), \end{aligned} \tag{3.32}$$

where

$$f^{\pm}(x) = \frac{1}{(2\pi)^{\frac{m}{2}}}\int\limits_{\mathbf{R}^m} e^{\mp x_0|\underline{\xi}|}e^{i\langle \underline{x},\underline{\xi}\rangle}\chi_{\pm}(\underline{\xi})\hat{f}(\underline{\xi})d\underline{\xi}, \quad x = x_0 + \underline{x} \in \mathbf{R}_1^{m\pm},$$

and $\chi_{\pm}(\underline{\xi}) = \frac{1}{2}(1 \pm \mathbf{i}\frac{\underline{\xi}}{|\underline{\xi}|})$. It is straightforward to verify that $\chi_{\pm}^2(\underline{\xi}) = \chi_{\pm}(\underline{\xi})$, $\chi_+(\underline{\xi}) + \chi_-(\underline{\xi}) = 1$, $\chi_+(\underline{\xi})\chi_-(\underline{\xi}) = \chi_-(\underline{\xi})\chi_+(\underline{\xi}) = 0$ and $|\underline{\xi}|\chi_{\pm}(\underline{\xi}) = \pm\mathbf{i}\underline{\xi}\chi_{\pm}(\underline{\xi})$. $H_2^+(\mathbf{R}^m)$ and $H_2^-(\mathbf{R}^m)$ are the non-tangential boundary values of the Hardy spaces functions on $\mathbf{R}_1^{m+}$ and $\mathbf{R}_1^{m-}$, the latter being the upper- and the lower-half $(m+1)$-dimensional Euclidean spaces (see [19]).

Remark 3.1. When $m = 1$, $\underline{\xi} = x_1\mathbf{e}_1$, and

$$\chi_{\pm}(\underline{\xi}) = \frac{1}{2}(1 \pm \mathbf{i}\frac{\underline{\xi}}{|\underline{\xi}|}) = \frac{1}{2}(1 \pm \mathbf{i}\frac{x_1\mathbf{e}_1}{|x_1\mathbf{e}_1|}) = \frac{1}{2}[1 \pm \mathbf{i}\mathrm{sgn}(x_1)\mathbf{e}_1].$$

Let $\mathbf{e}_1 = -\mathbf{i}$, then

$$\chi_{\pm}(\underline{\xi}) = \frac{1}{2}[1 \pm \mathbf{i}\mathrm{sgn}(x_1)\mathbf{e}_1] = \frac{1}{2}[1 \pm \mathrm{sgn}(x_1)] = \chi_{\pm}(x_1).$$

Based on this, when $m = 1$,

$$\underline{D} = \frac{\partial}{\partial x_1}\mathbf{e}_1 = \frac{1}{\mathbf{i}}\frac{\partial}{\partial x_1}.$$

In the introduction part, we represent $f(\underline{x})$ in the polar coordinate form

$$f(\underline{x}) = f_0(\underline{x}) + f_1(\underline{x})\mathbf{e}_1 + f_2(\underline{x})\mathbf{e}_2 + \cdots + f_m(\underline{x})\mathbf{e}_m = \rho(\underline{x})e^{\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})},$$

and define phase and amplitude derivatives. In fact, the phase derivatives $\theta_1'(\underline{x})$, $\theta_2'(\underline{x})$ correspond to, respectively, the right- and left-hand side of (3.28). We explain this through computations.

On the one hand, since

$$f(\underline{x}) = \rho(\underline{x})e^{\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})},$$

we have

$$f(\underline{x})\overline{f(\underline{x})} = |f(\underline{x})|^2 = \rho^2(\underline{x}), \quad \text{and} \quad [f(\underline{x})]^{-1} = \frac{\overline{f(\underline{x})}}{f(\underline{x})\overline{f(\underline{x})}} = \frac{1}{\rho(\underline{x})}e^{-\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}.$$

On the other hand,

$$\underline{D}f(\underline{x}) = \underline{D}[\rho(\underline{x})e^{\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}] = [\underline{D}\rho(\underline{x})]e^{\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})} + \rho(\underline{x})[\underline{D}e^{\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}].$$

Hence,

$$\begin{aligned} &[\underline{D}f(\underline{x})][f(\underline{x})]^{-1} \\ &= \left\{[\underline{D}\rho(\underline{x})]e^{\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})} + \rho(\underline{x})[\underline{D}e^{\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}]\right\}[\frac{1}{\rho(\underline{x})}e^{-\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}] \\ &= \frac{[\underline{D}\rho(\underline{x})]}{\rho(\underline{x})} + [\underline{D}e^{\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}]e^{-\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}. \end{aligned} \tag{3.33}$$

We further obtain

$$\begin{aligned} &[\underline{D}e^{\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}]e^{-\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})} \\ &= \left\{\underline{D}[\cos\theta(\underline{x}) + \frac{u(\underline{x})}{|u(\underline{x})|}\sin\theta(\underline{x})]\right\}e^{-\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})} \\ &= \left\{-\sin\theta(\underline{x})[\underline{D}\theta(\underline{x})] + \sin\theta(\underline{x})[\underline{D}\frac{u(\underline{x})}{|u(\underline{x})|}] + \cos\theta(\underline{x})[\underline{D}\theta(\underline{x})]\frac{u(\underline{x})}{|u(\underline{x})|}\right\}e^{-\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})} \\ &= \left\{\sin\theta(\underline{x})[\underline{D}\theta(\underline{x})]\frac{u(\underline{x})}{|u(\underline{x})|}\frac{u(\underline{x})}{|u(\underline{x})|} + \sin\theta(\underline{x})[\underline{D}\frac{u(\underline{x})}{|u(\underline{x})|}]\right. \\ &\quad \left.+ \cos\theta(\underline{x})[\underline{D}\theta(\underline{x})]\frac{u(\underline{x})}{|u(\underline{x})|}\right\}e^{-\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})} \\ &= \left\{[\underline{D}\theta(\underline{x})]\frac{u(\underline{x})}{|u(\underline{x})|}[\cos\theta(\underline{x}) + \sin\theta(\underline{x})\frac{u(\underline{x})}{|u(\underline{x})|}] + \sin\theta(\underline{x})[\underline{D}\frac{u(\underline{x})}{|u(\underline{x})|}]\right\}e^{-\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})} \\ &= \left\{[\underline{D}\theta(\underline{x})]\frac{u(\underline{x})}{|u(\underline{x})|}e^{\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})} + \sin\theta(\underline{x})[\underline{D}\frac{u(\underline{x})}{|u(\underline{x})|}]\right\}e^{-\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})} \\ &= [\underline{D}\theta(\underline{x})]\frac{u(\underline{x})}{|u(\underline{x})|} + \sin\theta(\underline{x})[\underline{D}\frac{u(\underline{x})}{|u(\underline{x})|}]e^{-\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}. \end{aligned}$$

Therefore,

$$[\underline{D}f(\underline{x})][f(\underline{x})]^{-1} = \frac{[\underline{D}\rho(\underline{x})]}{\rho(\underline{x})} + [\underline{D}\theta(\underline{x})]\frac{u(\underline{x})}{|u(\underline{x})|} + \sin\theta(\underline{x})[\underline{D}\frac{u(\underline{x})}{|u(\underline{x})|}]e^{-\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}. \tag{3.34}$$

If we want to define the phase derivation like the right-hand side of (3.28), then we have

$$\theta_1'(\underline{x}) \triangleq \mathrm{Sc}\left\{[\underline{D}f(\underline{x})][f(\underline{x})]^{-1}\right\}.$$

If we want to define it in accordance with the left-hand side of (3.28), which is a direct application of the spherical Dirac operator to the phase function, we should have

$$\theta_2'(\underline{x}) \triangleq \mathrm{Sc}\{[\underline{D}\theta(\underline{x})]\frac{u(\underline{x})}{|u(\underline{x})|}\}.$$

According to (3.33) and (3.34), taking into account the fact that $\frac{[\underline{D}\rho(\underline{x})]}{\rho(\underline{x})}$ is a pure vector, we have, explicitly,

$$\begin{aligned}\theta_1'(\underline{x}) &= \mathrm{Sc}\left\{[\underline{D}f(\underline{x})][f(\underline{x})]^{-1}\right\}\\ &= \mathrm{Sc}\left\{[\underline{D}\theta(\underline{x})]\frac{u(\underline{x})}{|u(\underline{x})|}\right\} + \mathrm{Sc}\left\{\sin\theta(\underline{x})[\underline{D}\frac{u(\underline{x})}{|u(\underline{x})|}]e^{-\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}\right\}\\ &= \theta_2'(\underline{x}) + \mathrm{Sc}\left\{\sin\theta(\underline{x})[\underline{D}\frac{u(\underline{x})}{|u(\underline{x})|}]e^{-\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}\right\}. \end{aligned} \tag{3.35}$$

Therefore, the difference between the two phase derivatives is $\mathrm{Sc}\left\{\sin\theta(\underline{x})[\underline{D}\frac{u(\underline{x})}{|u(\underline{x})|}]e^{-\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}\right\}$. Similarly, based on the left-hand and right-hand sides of (3.29), we can define *amplitude derivative* as

$$\rho_1'(\underline{x}) \triangleq \underline{D}\rho(\underline{x}), \tag{3.36}$$

or

$$\rho_2'(\underline{x}) \triangleq \rho(\underline{x})\mathrm{Nsc}\left\{[\underline{D}f(\underline{x})][f(\underline{x})]^{-1}\right\}. \tag{3.37}$$

Based on (3.33), we have

$$\underline{D}\rho(\underline{x}) = \rho(\underline{x})\left\{[\underline{D}f(\underline{x})][f(\underline{x})]^{-1} - [\underline{D}e^{\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}]e^{-\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}\right\}. \tag{3.38}$$

Since $\underline{D}\rho(\underline{x})$ is vector-valued, (3.36) can be further represented as

$$\begin{aligned}\rho_1'(\underline{x}) &= \underline{D}\rho(\underline{x})\\ &= \rho(\underline{x})\left\{\mathrm{Nsc}\{[\underline{D}f(\underline{x})][f(\underline{x})]^{-1}\} - \mathrm{Nsc}\{[\underline{D}e^{\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}]e^{-\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}\}\right\}\\ &= \rho_2'(\underline{x}) - \rho(\underline{x})\mathrm{Nsc}\{[\underline{D}e^{\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}]e^{-\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}\}. \end{aligned} \tag{3.39}$$

In the one-dimensional case, corresponding to the homogeneous case $m = 1$, under mild conditions $\underline{D}\frac{u(\underline{x})}{|u(\underline{x})|} = \frac{1}{\mathbf{i}}\frac{d}{dx_1}[\frac{1}{\mathbf{i}}\mathrm{sgn}(f_1)] = 0$. Then $\mathrm{Sc}\left\{\sin\theta(\underline{x})[\underline{D}\frac{u(\underline{x})}{|u(\underline{x})|}]e^{-\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}\right\}$ and $\mathrm{Nsc}\{[\underline{D}e^{\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}]e^{-\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}\}$ reduce to zero. As a consequence, $\theta_1'(\underline{x}) = \theta_2'(\underline{x})$, and $\rho_1'(\underline{x}) = \rho_2'(\underline{x})$.

4. Mean and variance of time and frequency for signals on $\mathbf{R}^m$

In this section, for a real-para-vector-valued signal $f(\underline{x})$ and its Clifford Fourier transform $\hat{f}(\underline{\xi})$, we are to define means of $\underline{x}$ and frequency $\underline{\xi}$. Since, for one-dimensional signal $s(t)$, $t \in \mathbf{R}$, the mean of t is called the *mean of time*, in higher dimensions we also phrase the corresponding quantities as the mean of time instead of the mean of the space variable $\underline{x}$. We first review the related knowledge for the one-dimensional case in subsection *4.1*. Then in subsection *4.2* we give definitions of means and variances of time and frequency for signals on $\mathbf{R}^m$ with the Clifford algebra setting.

4.1. Means and variances of time and frequency for signals on $\mathbf{R}$

The revision on the one-dimensional case is methodological. It would hint which would be the reasonable ways to define the counterpart concepts in higher dimensions. The definitions of means and variances of time and frequency for signals on $\mathbf{R}$ are given in the introduction part. In [8], $\langle\omega\rangle_s$ and $\sigma^2_{\omega,s}$ are represented in the time domain. The representations of $\langle\omega\rangle_s$ and $\sigma^2_{\omega,s}$ in the time domain give reasons for the means, as well as the phase and amplitude derivatives. Here we enclose the proof of the results.

Theorem 4.1. *Let $s(t) = \rho(t)e^{i\varphi(t)} \in L^2(\mathbf{R})$, $\rho(t) = |s(t)|$ with $\|s\|_2 = 1$. Assume that the classical derivatives $\rho'(t)$, $\varphi'(t)$ and $s'(t)$ all exist and are Lebesgue measurable, and $s'(t)$ is in $L^2(\mathbf{R})$. Then there hold*

$$\langle\omega\rangle_s = \int_{-\infty}^{\infty} \varphi'(t)|s(t)|^2 dt, \tag{4.40}$$

and

$$\sigma^2_{\omega,s} = \int_{-\infty}^{\infty} [\rho'(t)]^2 dt + \int_{-\infty}^{\infty} [\varphi'(t) - \langle\omega\rangle_s]^2 |s(t)|^2 dt. \tag{4.41}$$

Proof. Since $s, s' \in L^2(\mathbf{R})$, then $\hat{s}(\omega),\ \omega\hat{s}(\omega) \in L^2(\mathbf{R})$. Hölder's inequality implies $\omega|\hat{s}(\omega)|^2, \omega^2|\hat{s}(\omega)|^2 \in L^1(\mathbf{R})$, and hence $\langle\omega\rangle_s$ and $\sigma^2_{\omega,s}$ are well defined. Then

$$\begin{aligned}
\langle\omega\rangle_s &= \int_{-\infty}^{\infty} \omega|\hat{s}(\omega)|^2 d\omega \\
&= -\int_{-\infty}^{\infty} is'(t)\overline{s(t)} dt \\
&= -i \int_L \frac{s'(t)}{s(t)} |s(t)|^2 dt \\
&= \int_L \mathrm{Im}\left\{\frac{s'(t)}{s(t)}\right\} |s(t)|^2 dt \\
&= \int_{-\infty}^{\infty} \varphi'(t)|s(t)|^2 dt,
\end{aligned}$$

and

$$\begin{aligned}
\sigma_{\omega,s}^2 &= \int_{-\infty}^{\infty} [\omega - \langle\omega\rangle_s]^2 |\hat{s}(\omega)|^2 d\omega \\
&= \int_{-\infty}^{\infty} [-is'(t) - \langle\omega\rangle_s s(t)]\overline{[-is'(t) - \langle\omega\rangle_s s(t)]} dt \\
&= \int_L |\frac{s'(t)}{s(t)} - i\langle\omega\rangle_s|^2 |s(t)|^2 dt \\
&= \int_L \mathrm{Re}^2 \left\{\frac{s'(t)}{s(t)}\right\} |s(t)|^2 dt + \int_{-\infty}^{\infty} \left\{\mathrm{Im}[\frac{s'(t)}{s(t)}] - \langle\omega\rangle_s\right\}^2 |s(t)|^2 dt \\
&= \int_{-\infty}^{\infty} [\rho'(t)]^2 \, dt + \int_{-\infty}^{\infty} [\varphi'(t) - \langle\omega\rangle_s]^2 \, |s(t)|^2 dt,
\end{aligned}$$

where $L = \{t \in \mathbf{R} | s(t) \neq 0\}$. □

4.2. Means and variances of time and frequency for signals on $\mathbf{R}^m$

Now we discuss the means and variances of time and frequency for signals on $\mathbf{R}^m$. The paper [21] proposes some definitions of means and variances of time and frequency for real-valued signals on $\mathbf{R}^m$, that essentially correspond to (1.4), (1.5), (1.2) and (1.3). In this study, we also use the similar method to define mean and variance of time and frequency. Since the signals we consider are para-vector-valued, our definitions of mean and variance of frequency have some differences from those in [21]. We have gave our definition of mean and variance of time and frequency in the introduction part. In the following, we give some comments about our definition.

Remark 4.2. When $m = 1$, the mean and variance of time for $f(\underline{x}) \in L^2(\mathbf{R}^m)$, that is, (1.15) and (1.16), are reduced to

$$\langle\underline{x}\rangle = \int_{\mathbf{R}^1} \mathbf{i}x_1\mathbf{e}_1 |f(\underline{x})|^2 d\underline{x} = \int_{\mathbf{R}^1} x_1 |f(\underline{x})|^2 d\underline{x},$$

and

$$\mathrm{var}_{\underline{x}} = \int_{\mathbf{R}^1} (\mathbf{i}x_1\mathbf{e}_1 - \langle\underline{x}\rangle)^2 |f(\underline{x})|^2 d\underline{x} = \int_{\mathbf{R}^1} (x_1 - \langle\underline{x}\rangle)^2 |f(\underline{x})|^2 d\underline{x}.$$

Those coincide with (1.4) and (1.2), respectively, in the one-dimensional case.

In the following remark we verify that $\mathrm{var}_{\underline{x}}$ is a real number.

Remark 4.3. It is easy to see that the mean of time

$$\langle\underline{x}\rangle = \int_{\mathbf{R}^m} \mathbf{i}\underline{x} |f(\underline{x})|^2 d\underline{x},$$

is complex-vector-valued number and can be written in another form

$$\langle \underline{x} \rangle = \mathbf{i}\underline{a},$$

where $\underline{a}$ is real-vector-valued number. Then

$$\mathrm{var}_{\underline{x}} = \int\limits_{\mathbf{R}^m} (\mathbf{i}\underline{x} - \langle \underline{x} \rangle)^2 |f(\underline{x})|^2 d\underline{x} = \int\limits_{\mathbf{R}^m} |\underline{x} - \underline{a}|^2 |f(\underline{x})|^2 d\underline{x},$$

where we used $(\underline{x} - \underline{a})^2 = -|\underline{x} - \underline{a}|^2$.

Remark 4.4. When $m = 1$, the mean and variance of frequency defined in (1.17) and (1.18) are reduced to

$$\begin{aligned}\langle \underline{\xi} \rangle &= \int\limits_{\mathbf{R}^1} |\mathbf{i}\xi_1 \mathbf{e}_1| |\hat{f^+}(\underline{\xi})|^2 d\underline{\xi} - \int\limits_{\mathbf{R}^1} |\mathbf{i}\xi_1 \mathbf{e}_1| |\hat{f^-}(\underline{\xi})|^2 d\underline{\xi} \\ &= \int\limits_0^\infty \xi_1 |\hat{f}(\underline{\xi})|^2 d\underline{\xi} + \int\limits_{-\infty}^0 \xi_1 |\hat{f}(\underline{\xi})|^2 d\underline{\xi} \\ &= \int\limits_{-\infty}^\infty \xi_1 |\hat{f}(\underline{\xi})|^2 d\underline{\xi},\end{aligned}$$

and

$$\begin{aligned}\mathrm{var}_{\underline{\xi}} &= \int\limits_{\mathbf{R}^1} [|\mathbf{i}\xi_1 \mathbf{e}_1| - \langle \underline{\xi} \rangle]^2 |\chi_+(\xi_1)\hat{f}(\underline{\xi})|^2 d\underline{\xi} + \int\limits_{\mathbf{R}^1} [-|\mathbf{i}\xi_1 \mathbf{e}_1| - \langle \underline{\xi} \rangle]^2 |\chi_-(\xi_1)\hat{f}(\underline{\xi})|^2 d\underline{\xi} \\ &= \int\limits_0^\infty [\xi_1 - \langle \underline{\xi} \rangle]^2 |\hat{f}(\underline{\xi})|^2 d\underline{\xi} + \int\limits_{-\infty}^0 [\xi_1 - \langle \underline{\xi} \rangle]^2 |\hat{f}(\underline{\xi})|^2 d\underline{\xi} \\ &= \int\limits_{-\infty}^\infty (\xi_1 - \langle \underline{\xi} \rangle)^2 |\hat{f}(\underline{\xi})|^2 d\underline{\xi}.\end{aligned}$$

Those, respectively, coincide with (1.5) and (1.3).

Remark 4.5. In Definition 1.2, we have two versions for variance of frequency. Those two versions both are inspired by the one-dimensional case, that is (4.41). As Remark 4.4 indicates, $\mathrm{var}_{\underline{\xi}}$ corresponds to (4.41). Theorem 4.7 represents $\mathrm{var}_{\underline{\xi}}$ in the time domain, that is

$$\mathrm{var}_{\underline{\xi}} = \int\limits_{\mathbf{R}^m} [\theta_1'(\underline{x}) - \langle \underline{\xi} \rangle]^2 |f(\underline{x})|^2 d\underline{x} + \int\limits_{\mathbf{R}^m} |\rho_2'(\underline{x})|^2 d\underline{x}, \tag{4.42}$$

that gives support to use $\rho_2'(\underline{x})$. Replacing $\rho_2'(\underline{x})$ with $\rho_1'(\underline{x})$ in (4.42) we have an alternative version of variance, that is,

$$\int\limits_{\mathbf{R}^m} [\theta_1'(\underline{x}) - \langle \underline{\xi} \rangle]^2 |f(\underline{x})|^2 d\underline{x} + \int\limits_{\mathbf{R}^m} |\rho_1'(\underline{x})|^2 d\underline{x},$$

that is just the definition of $\mathrm{var}^*_{\underline{\xi}}$. When $m = 1$, $\sigma^2_{\omega,s}$, $\mathrm{var}_{\underline{\xi}}$ and $\mathrm{var}^*_{\underline{\xi}}$ all coincide. When $m \geq 2$ they are, however, not. In the following we will establish an uncertainty principle for each of the two formulations $\mathrm{var}_{\underline{\xi}}$ and $\mathrm{var}^*_{\underline{\xi}}$.

In the following two theorems, we represent $\langle \underline{\xi} \rangle$, $\langle \underline{\xi}^2 \rangle$ and $\mathrm{var}_{\underline{\xi}}$ in time domain.

Theorem 4.6. *Let $f(\underline{x}) = \rho(\underline{x}) e^{\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})} \in L^2(\mathbf{R}^m)$ be real-para-vector-valued and $\underline{D}f(\underline{x}) \in L^2(\mathbf{R}^m)$ with $\|f\|_2 = 1$. Then there holds*

$$\langle \underline{\xi} \rangle = \int_{\mathbf{R}^m} \theta_1'(\underline{x}) |f(\underline{x})|^2 d\underline{x}, \tag{4.43}$$

where θ_1' is given in Definition 1.1.

Proof. Since $f(\underline{x}) \in L^2(\mathbf{R}^m)$ and $\underline{D}f(\underline{x}) \in L^2(\mathbf{R}^m)$, $\langle \underline{\xi} \rangle$ is well defined. Since $f(\underline{x}) \in L^2(\mathbf{R}^m)$ is real-para-vector-valued, we have

$$[f(\underline{x})]^{-1} = \frac{\overline{f(\underline{x})}}{|f(\underline{x})|^2},$$

$\hat{f}(\underline{\xi})$ is a complex para-vector-valued function, $\chi_+(\underline{\xi})\hat{f}(\underline{\xi})$ and $\chi_-(\underline{\xi})\hat{f}(\underline{\xi})$ are complex-valued and are sums of some 0-form, 1-form and 2-form. There holds the relation

$$|\chi_\pm(\underline{\xi})\hat{f}(\underline{\xi})|^2 = \mathrm{Sc}\{\chi_\pm(\underline{\xi})\hat{f}(\underline{\xi})\overline{[\chi_\pm(\underline{\xi})\hat{f}(\underline{\xi})]}\}.$$

In what follows the property $|\underline{\xi}|\chi_\pm(\underline{\xi}) = \pm\mathbf{i}\underline{\xi}\chi_\pm(\underline{\xi})$ will be used. The property can be obtained directly by using the expression $\chi_\pm(\underline{\xi}) = \frac{1}{2}(1 \pm \mathbf{i}\frac{\underline{\xi}}{|\underline{\xi}|})$.

Then we have

$$\begin{aligned}
\langle \underline{\xi} \rangle &= \int_{\mathbf{R}^m} |\mathbf{i}\underline{\xi}||\hat{f^+}(\underline{\xi})|^2 d\underline{\xi} - \int_{\mathbf{R}^m} |\mathbf{i}\underline{\xi}||\hat{f^-}(\underline{\xi})|^2 d\underline{\xi} \\
&= \mathrm{Sc}\left\{ \int_{\mathbf{R}^m} |\underline{\xi}|\chi_+(\underline{\xi})\hat{f}(\underline{\xi})\overline{\chi_+(\underline{\xi})\hat{f}(\underline{\xi})} d\underline{\xi} - \int_{\mathbf{R}^m} |\underline{\xi}|\chi_-(\underline{\xi})\hat{f}(\underline{\xi})\overline{\chi_-(\underline{\xi})\hat{f}(\underline{\xi})} d\underline{\xi} \right\} \\
&= \mathrm{Sc}\left\{ \int_{\mathbf{R}^m} \mathbf{i}\underline{\xi}\chi_+(\underline{\xi})\hat{f}(\underline{\xi})\overline{\chi_+(\underline{\xi})\hat{f}(\underline{\xi})} d\underline{\xi} + \int_{\mathbf{R}^m} \mathbf{i}\underline{\xi}\chi_-(\underline{\xi})\hat{f}(\underline{\xi})\overline{\chi_-(\underline{\xi})\hat{f}(\underline{\xi})} d\underline{\xi} \right\} \\
&= \mathrm{Sc}\left\{ \int_{\mathbf{R}^m} \mathbf{i}\underline{\xi}[\chi_+(\underline{\xi})\hat{f}(\underline{\xi}) + \chi_-(\underline{\xi})\hat{f}(\underline{\xi})]\overline{[\chi_+(\underline{\xi})\hat{f}(\underline{\xi}) + \chi_-(\underline{\xi})\hat{f}(\underline{\xi})]} d\underline{\xi} \right\} \\
&= \mathrm{Sc}\left\{ \int_{\mathbf{R}^m} \mathbf{i}\underline{\xi}\hat{f}(\underline{\xi})\overline{\hat{f}(\underline{\xi})} d\underline{\xi} \right\} \\
&= \mathrm{Sc}\left\{ \int_{\mathbf{R}^m} \underline{D}f(\underline{x})\overline{f(\underline{x})} d\underline{x} \right\} \\
&= \mathrm{Sc}\left\{ \int_{\mathbf{R}^m} [\underline{D}f(\underline{x})][f(\underline{x})]^{-1} f(\underline{x})\overline{f(\underline{x})} d\underline{x} \right\} \\
&= \mathrm{Sc}\left\{ \int_{\mathbf{R}^m} [\underline{D}f(\underline{x})][f(\underline{x})]^{-1} |f(\underline{x})|^2 d\underline{x} \right\}
\end{aligned}$$

$$= \int_{\mathbf{R}^m} \mathrm{Sc}\left\{[\underline{D}f(\underline{x})][f(\underline{x})]^{-1}\right\}|f(\underline{x})|^2 d\underline{x}$$

$$= \int_{\mathbf{R}^m} \theta_1'(\underline{x})|f(\underline{x})|^2 d\underline{x}, \tag{4.44}$$

where the equalities

$$\mathrm{Sc}\left\{\int_{\mathbf{R}^m} \mathbf{i}\underline{\xi}\{\chi_+(\underline{\xi})\hat{f}(\underline{\xi})\overline{[\chi_-(\underline{\xi})\hat{f}(\underline{\xi})]}\}d\underline{\xi}\right\} = 0, \tag{4.45}$$

and

$$\mathrm{Sc}\left\{\int_{\mathbf{R}^m} \mathbf{i}\underline{\xi}\{\chi_-(\underline{\xi})\hat{f}(\underline{\xi})\overline{[\chi_+(\underline{\xi})\hat{f}(\underline{\xi})]}\}d\underline{\xi}\right\} = 0, \tag{4.46}$$

guarantee the fourth equal relation of (4.44).

We only prove (4.45). The proof of (4.46) is similar with the proof of (4.45).

$$\begin{aligned}
&\mathrm{Sc}\left\{\int_{\mathbf{R}^m} \mathbf{i}\underline{\xi}\{\chi_+(\underline{\xi})\hat{f}(\underline{\xi})\overline{[\chi_-(\underline{\xi})\hat{f}(\underline{\xi})]}\}d\underline{\xi}\right\}\\
&= \mathrm{Sc}\left\{\int_{\mathbf{R}^m} |\underline{\xi}|\chi_+(\underline{\xi})\hat{f}(\underline{\xi})\overline{[\chi_-(\underline{\xi})\hat{f}(\underline{\xi})]}d\underline{\xi}\right\}\\
&= \int_{\mathbf{R}^m} |\underline{\xi}|\mathrm{Sc}\left\{\chi_+(\underline{\xi})\hat{f}(\underline{\xi})\overline{[\chi_-(\underline{\xi})\hat{f}(\underline{\xi})]}\right\}d\underline{\xi}\\
&= \int_{\mathbf{R}^m} |\underline{\xi}|\mathrm{Sc}\left\{\overline{[\chi_-(\underline{\xi})\hat{f}(\underline{\xi})]}\chi_+(\underline{\xi})\hat{f}(\underline{\xi})\right\}d\underline{\xi}\\
&= \mathrm{Sc}\left\{\int_{\mathbf{R}^m} |\underline{\xi}|\overline{[\hat{f}(\underline{\xi})]}\,\overline{\chi_-(\underline{\xi})}\chi_+(\underline{\xi})\hat{f}(\underline{\xi})d\underline{\xi}\right\}\\
&= 0,
\end{aligned}$$

where we used the property $\overline{\chi_-(\underline{\xi})}\chi_+(\underline{\xi}) = \chi_-(\underline{\xi})\chi_+(\underline{\xi}) = 0$. The proof is complete. □

Theorem 4.7. *Let* $f(\underline{x}) = \rho(\underline{x})e^{\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})} \in L^2(\mathbf{R}^m)$ *be real-para-vector-valued and* $\underline{D}f(\underline{x}) \in L^2(\mathbf{R}^m)$ *with* $\|f\|_2 = 1$*. Then there hold*

$$\langle\underline{\xi}^2\rangle = \int_{\mathbf{R}^m} [\theta_1'(\underline{x})]^2|f(\underline{x})|^2 d\underline{x} + \int_{\mathbf{R}^m} |\rho_2'(\underline{x})|^2 d\underline{x}, \tag{4.47}$$

and

$$\mathrm{var}_{\underline{\xi}} = \int_{\mathbf{R}^m} [\theta_1'(\underline{x}) - \langle\underline{\xi}\rangle]^2|f(\underline{x})|^2 d\underline{x} + \int_{\mathbf{R}^m} |\rho_2'(\underline{x})|^2 d\underline{x}, \tag{4.48}$$

where θ_1' *and* ρ_2' *are given in Definition 1.1.*

Proof. Since $f(\underline{x}) \in L^2(\mathbf{R}^m)$ and $\underline{D}f(\underline{x}) \in L^2(\mathbf{R}^m)$, $\langle \underline{\xi}^2 \rangle$ and $\mathrm{var}_{\underline{\xi}}$ are well defined. Since $\hat{f}(\underline{\xi})$ is complex-para-vector-valued, we have

$$|\mathbf{i}\underline{\xi}|^2|\hat{f}(\underline{\xi})|^2 = |\mathbf{i}\underline{\xi}\hat{f}(\underline{\xi})|^2 = \mathrm{Sc}\{\mathbf{i}\underline{\xi}\hat{f}(\underline{\xi})\overline{[\mathbf{i}\underline{\xi}\hat{f}(\underline{\xi})]}\}.$$

Since $f(\underline{x})$ is a real-para-vector-valued function, we have that $\underline{D}f(\underline{x})$ is a sum of some 0-forms, 1-forms and 2-forms. Thus $[\underline{D}f(\underline{x})][f(\underline{x})]^{-1}$ is a sum of some 0-forms, 1-forms, 2-forms and 3-forms. By direct computation, we have

$$\begin{aligned}
&\mathrm{Sc}\left\{[\underline{D}f(\underline{x})][f(\underline{x})]^{-1}\overline{[\underline{D}f(\underline{x})][f(\underline{x})]^{-1}}\right\} \\
&= |[\underline{D}f(\underline{x})][f(\underline{x})]^{-1}|^2 \\
&= \mathrm{Sc}^2\left\{[\underline{D}f(\underline{x})][f(\underline{x})]^{-1}\right\} + |\mathrm{NSc}\left\{[\underline{D}f(\underline{x})][f(\underline{x})]^{-1}\right\}|^2.
\end{aligned}$$

Then

$$\begin{aligned}
\langle \underline{\xi}^2 \rangle &= \int_{\mathbf{R}^m} |\mathbf{i}\underline{\xi}|^2|\hat{f}(\underline{\xi})|^2 d\underline{\xi} \\
&= \int_{\mathbf{R}^m} |\mathbf{i}\underline{\xi}\hat{f}(\underline{\xi})|^2 d\underline{\xi} \\
&= \mathrm{Sc}\left\{\int_{\mathbf{R}^m} \mathbf{i}\underline{\xi}\hat{f}(\underline{\xi})\overline{\mathbf{i}\underline{\xi}\hat{f}(\underline{\xi})} d\underline{\xi}\right\} \\
&= \mathrm{Sc}\left\{\int_{\mathbf{R}^m} \underline{D}f(\underline{x})\overline{\underline{D}f(\underline{x})} d\underline{x}\right\} \\
&= \mathrm{Sc}\left\{\int_{\mathbf{R}^m} [\underline{D}f(\underline{x})][f(\underline{x})]^{-1}|f(\underline{x})|^2\overline{[\underline{D}f(\underline{x})][f(\underline{x})]^{-1}} d\underline{x}\right\} \\
&= \mathrm{Sc}\left\{\int_{\mathbf{R}^m} [\underline{D}f(\underline{x})][f(\underline{x})]^{-1}\overline{[\underline{D}f(\underline{x})][f(\underline{x})]^{-1}}|f(\underline{x})|^2 d\underline{x}\right\} \\
&= \int_{\mathbf{R}^m} \mathrm{Sc}\left\{[\underline{D}f(\underline{x})][f(\underline{x})]^{-1}\overline{[\underline{D}f(\underline{x})][f(\underline{x})]^{-1}}\right\}|f(\underline{x})|^2 d\underline{x} \\
&= \int_{\mathbf{R}^m} \left\{\mathrm{Sc}^2\left\{[\underline{D}f(\underline{x})][f(\underline{x})]^{-1}\right\} + |\mathrm{NSc}\left\{[\underline{D}f(\underline{x})][f(\underline{x})]^{-1}\right\}|^2\right\}|f(\underline{x})|^2 d\underline{x} \\
&= \int_{\mathbf{R}^m} \mathrm{Sc}^2\left\{[\underline{D}f(\underline{x})][f(\underline{x})]^{-1}\right\}|f(\underline{x})|^2 d\underline{x} + \int_{\mathbf{R}^m} |\mathrm{NSc}\left\{[\underline{D}f(\underline{x})][f(\underline{x})]^{-1}\right\}|^2|f(\underline{x})|^2 d\underline{x},
\end{aligned}$$

and we further have

$$\begin{aligned}
\mathrm{var}_{\underline{\xi}} &= \int_{\mathbf{R}^m} [|\mathbf{i}\underline{\xi}| - \langle\underline{\xi}\rangle]^2|\hat{f^+}(\underline{\xi})|^2 d\underline{\xi} + \int_{\mathbf{R}^m} [-|\mathbf{i}\underline{\xi}| - \langle\underline{\xi}\rangle]^2|\hat{f^-}(\underline{\xi})|^2 d\underline{\xi} \\
&= \int_{\mathbf{R}^m} [|\mathbf{i}\underline{\xi}|^2 - 2\langle\underline{\xi}\rangle|\mathbf{i}\underline{\xi}| + \langle\underline{\xi}\rangle^2]|\hat{f^+}(\underline{\xi})|^2 d\underline{\xi} + \int_{\mathbf{R}^m} [|\mathbf{i}\underline{\xi}|^2 + 2\langle\underline{\xi}\rangle|\mathbf{i}\underline{\xi}| + \langle\underline{\xi}\rangle^2]|\hat{f^-}(\underline{\xi})|^2 d\underline{\xi}
\end{aligned}$$

$$= \int\limits_{\mathbf{R}^m} |\mathbf{i}\underline{\xi}|^2 |\hat{f^+}(\underline{\xi})|^2 d\underline{\xi} + \int\limits_{\mathbf{R}^m} |\mathbf{i}\underline{\xi}|^2 |\hat{f^-}(\underline{\xi})|^2 d\underline{\xi}$$

$$- 2\langle \underline{\xi} \rangle [\int\limits_{\mathbf{R}^m} |\mathbf{i}\underline{\xi}||\hat{f^+}(\underline{\xi})|^2 d\underline{\xi} - \int\limits_{\mathbf{R}^m} |\mathbf{i}\underline{\xi}||\hat{f^-}(\underline{\xi})|^2 d\underline{\xi}]$$

$$+ \langle \underline{\xi} \rangle^2 [\int\limits_{\mathbf{R}^m} |\hat{f^+}(\underline{\xi})|^2 d\underline{\xi} + \int\limits_{\mathbf{R}^m} |\hat{f^-}(\underline{\xi})|^2 d\underline{\xi}]$$

$$= \int\limits_{\mathbf{R}^m} |\mathbf{i}\underline{\xi}|^2 |\hat{f}(\underline{\xi})|^2 d\underline{\xi} - 2\langle \underline{\xi} \rangle^2 + \langle \underline{\xi} \rangle^2 \int\limits_{\mathbf{R}^m} |\hat{f}(\underline{\xi})|^2 d\underline{\xi}$$

$$= \int\limits_{\mathbf{R}^m} \mathrm{Sc}^2 \left\{ [\underline{D} f(\underline{x})][f(\underline{x})]^{-1} \right\} |f(\underline{x})|^2 d\underline{x}$$

$$+ \int\limits_{\mathbf{R}^m} |\mathrm{NSc} \left\{ [\underline{D} f(\underline{x})][f(\underline{x})]^{-1} \right\} |^2 |f(\underline{x})|^2 d\underline{x}$$

$$- 2\langle \underline{\xi} \rangle \int\limits_{\mathbf{R}^m} \mathrm{Sc} \left\{ [\underline{D} f(\underline{x})][f(\underline{x})]^{-1} \right\} |f(\underline{x})|^2 d\underline{x} + \langle \underline{\xi} \rangle^2 \int\limits_{\mathbf{R}^m} |f(\underline{x})|^2 d\underline{x}$$

$$= \int\limits_{\mathbf{R}^m} \left\{ \mathrm{Sc} \left\{ [\underline{D} f(\underline{x})][f(\underline{x})]^{-1} \right\} - \langle \underline{\xi} \rangle \right\}^2 |f(\underline{x})|^2 d\underline{x}$$

$$+ \int\limits_{\mathbf{R}^m} |\mathrm{NSc} \left\{ [\underline{D} f(\underline{x})][f(\underline{x})]^{-1} \right\} |^2 |f(\underline{x})|^2 d\underline{x}$$

$$= \int\limits_{\mathbf{R}^m} [\theta_1'(\underline{x}) - \langle \underline{\xi} \rangle]^2 |f(\underline{x})|^2 d\underline{x} + \int\limits_{\mathbf{R}^m} |\rho_2'(\underline{x})|^2 d\underline{x},$$

where we used

$$\int\limits_{\mathbf{R}^m} |\mathbf{i}\underline{\xi}|^2 |\hat{f^+}(\underline{\xi})|^2 d\underline{\xi} + \int\limits_{\mathbf{R}^m} |\mathbf{i}\underline{\xi}|^2 |\hat{f^-}(\underline{\xi})|^2 d\underline{\xi} = \int\limits_{\mathbf{R}^m} |\mathbf{i}\underline{\xi}|^2 |\hat{f}(\underline{\xi})|^2 d\underline{\xi} \tag{4.49}$$

and

$$\int\limits_{\mathbf{R}^m} |\hat{f^+}(\underline{\xi})|^2 d\underline{\xi} + \int\limits_{\mathbf{R}^m} |\hat{f^-}(\underline{\xi})|^2 d\underline{\xi} = \int\limits_{\mathbf{R}^m} |\hat{f}(\underline{\xi})|^2 d\underline{\xi}. \tag{4.50}$$

Now we prove (4.49).

$$\int\limits_{\mathbf{R}^m} |\mathbf{i}\underline{\xi}|^2 |\hat{f^+}(\underline{\xi})|^2 d\underline{\xi} + \int\limits_{\mathbf{R}^m} |\mathbf{i}\underline{\xi}|^2 |\hat{f^-}(\underline{\xi})|^2 d\underline{\xi}$$

$$= \mathrm{Sc} \left\{ \int\limits_{\mathbf{R}^m} |\mathbf{i}\underline{\xi}|^2 [\overline{\hat{f^+}(\underline{\xi})} \hat{f^+}(\underline{\xi}) + \overline{\hat{f^-}(\underline{\xi})} \hat{f^-}(\underline{\xi})] d\underline{\xi} \right\}$$

$$= \mathrm{Sc} \Big\{ \int\limits_{\mathbf{R}^m} |\mathbf{i}\underline{\xi}|^2 \{ \overline{[\chi_+(\underline{\xi}) \hat{f}(\underline{\xi})]} [\chi_+(\underline{\xi}) \hat{f}(\underline{\xi})] + \overline{[\chi_+(\underline{\xi}) \hat{f}(\underline{\xi})]} [\chi_-(\underline{\xi}) \hat{f}(\underline{\xi})]$$

$$+ \overline{[\chi_-(\underline{\xi}) \hat{f}(\underline{\xi})]} [\chi_+(\underline{\xi}) \hat{f}(\underline{\xi})] + \overline{[\chi_-(\underline{\xi}) \hat{f}(\underline{\xi})]} [\chi_-(\underline{\xi}) \hat{f}(\underline{\xi})] \} d\underline{\xi} \Big\}$$

$$= \mathrm{Sc}\Big\{ \int_{\mathbf{R}^m} |\mathbf{i}\underline{\xi}|^2 \overline{[\chi_+(\underline{\xi})\hat{f}(\underline{\xi}) + \chi_-(\underline{\xi})\hat{f}(\underline{\xi})]}[\chi_+(\underline{\xi})\hat{f}(\underline{\xi}) + \chi_-(\underline{\xi})\hat{f}(\underline{\xi})]d\underline{\xi}\Big\}$$

$$= \mathrm{Sc}\Big\{ \int_{\mathbf{R}^m} |\mathbf{i}\underline{\xi}|^2 \overline{[\hat{f}(\underline{\xi})]}\hat{f}(\underline{\xi})d\underline{\xi}\Big\}$$

$$= \int_{\mathbf{R}^m} |\mathbf{i}\underline{\xi}|^2 |\hat{f}(\underline{\xi})|^2 d\underline{\xi}.$$

The proof of (4.50) is similar with that of (4.49), that is

$$\int_{\mathbf{R}^m} |\hat{f^+}(\underline{\xi})|^2 d\underline{\xi} + \int_{\mathbf{R}^m} |\hat{f^-}(\underline{\xi})|^2 d\underline{\xi}$$

$$= \mathrm{Sc}\left\{ \int_{\mathbf{R}^m} [\overline{\hat{f^+}(\underline{\xi})}\hat{f^+}(\underline{\xi}) + \overline{\hat{f^-}(\underline{\xi})}\hat{f^-}(\underline{\xi})]d\underline{\xi} \right\}$$

$$= \mathrm{Sc}\Big\{ \int_{\mathbf{R}^m} \{\overline{[\chi_+(\underline{\xi})\hat{f}(\underline{\xi})]}[\chi_+(\underline{\xi})\hat{f}(\underline{\xi})] + \overline{[\chi_+(\underline{\xi})\hat{f}(\underline{\xi})]}[\chi_-(\underline{\xi})\hat{f}(\underline{\xi})]$$

$$+ \overline{[\chi_-(\underline{\xi})\hat{f}(\underline{\xi})]}[\chi_+(\underline{\xi})\hat{f}(\underline{\xi})] + \overline{[\chi_-(\underline{\xi})\hat{f}(\underline{\xi})]}[\chi_-(\underline{\xi})\hat{f}(\underline{\xi})]\}d\underline{\xi}\Big\}$$

$$= \mathrm{Sc}\Big\{ \int_{\mathbf{R}^m} \overline{[\chi_+(\underline{\xi})\hat{f}(\underline{\xi}) + \chi_-(\underline{\xi})\hat{f}(\underline{\xi})]}[\chi_+(\underline{\xi})\hat{f}(\underline{\xi}) + \chi_-(\underline{\xi})\hat{f}(\underline{\xi})]d\underline{\xi}\Big\}$$

$$= \mathrm{Sc}\Big\{ \int_{\mathbf{R}^m} \overline{[\hat{f}(\underline{\xi})]}\hat{f}(\underline{\xi})d\underline{\xi}\Big\}$$

$$= \int_{\mathbf{R}^m} |\hat{f}(\underline{\xi})|^2 d\underline{\xi}.$$

In the proofs of (4.49) and (4.50) we used

$$\overline{[\chi_+(\underline{\xi})\hat{f}(\underline{\xi})]}[\chi_-(\underline{\xi})\hat{f}(\underline{\xi})] = \overline{\hat{f}(\underline{\xi})}\,\overline{\chi_+(\underline{\xi})}\chi_-(\underline{\xi})\hat{f}(\underline{\xi}) = 0$$

and

$$\overline{[\chi_-(\underline{\xi})\hat{f}(\underline{\xi})]}[\chi_+(\underline{\xi})\hat{f}(\underline{\xi})] = \overline{\hat{f}(\underline{\xi})}\,\overline{\chi_-(\underline{\xi})}\chi_+(\underline{\xi})\hat{f}(\underline{\xi}) = 0.$$

The proof is complete. □

Remark 4.8. The reference [21] not only gives the definitions of the mean and variance of time and frequency for real-valued signals, as noted in the start of subsection 4.2, but also represents the mean and variance of frequency in the time domain. However, the method used in [21] is only valid for real-scalar-valued signals. Theorem 4.6 and Theorem 4.7 effectively represent the mean and variance of frequency in the time domain for real-para-vector-valued signals.

5. Uncertainty principle for real-para-vector-valued signals

The following is an application of Minkowski's inequality.

Lemma 5.1. *Let $g(\underline{x}) = g_0(\underline{x}) + \sum_{|T|=1}^{m} g_T(\underline{x})e_T \in L^1(\mathbf{R}^m; \mathbf{R}_m)$. Then there holds*

$$\int_{\mathbf{R}^m} |g(\underline{x})|d\underline{x} \geq |\int_{\mathbf{R}^m} g(\underline{x})d\underline{x}|. \tag{5.51}$$

By Lemma 5.1 and the Hölder inequality, we immediately have the following lemma.

Lemma 5.2. *Let $f(\underline{x}) = \rho(\underline{x})e^{\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})} \in L^2(\mathbf{R}^m)$ be real-para-vector-valued, and $\underline{x}f(\underline{x}), \underline{D}f(\underline{x}) \in L^2(\mathbf{R}^m)$ with $\|f\|_2 = 1$. Then*

$$\begin{aligned}
&\mathrm{var}_{\underline{x}} \int_{\mathbf{R}^m} |\rho_1'(\underline{x})|^2 d\underline{x} \\
&= \int_{\mathbf{R}^m} (\mathbf{i}\underline{x} - \langle \underline{x} \rangle)^2 |f(\underline{x})|^2 d\underline{x} \\
&\quad \cdot \int_{\mathbf{R}^m} |\mathrm{Nsc}\{[\underline{D}f(\underline{x})][f(\underline{x})]^{-1}\} - \mathrm{Nsc}\{[\underline{D}e^{\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}]e^{-\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}\}|^2 |f(\underline{x})|^2 d\underline{x} \\
&\geq |\int_{\mathbf{R}^m} \left\{\mathrm{Nsc}\{[\underline{D}f(\underline{x})][f(\underline{x})]^{-1}\} - \mathrm{Nsc}\{[\underline{D}e^{\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}]e^{-\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}\}\right\} (\mathbf{i}\underline{x} - \langle \underline{x} \rangle) |f(\underline{x})|^2 d\underline{x}|^2.
\end{aligned}$$

Lemma 5.3. *Let $f(\underline{x}) = \rho(\underline{x})e^{\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})} \in L^2(\mathbf{R}^m)$ be real-para-vector-valued. Assume that all the first order partial derivatives of $f(\underline{x})$ exist. Then*

$$\begin{aligned}
&\mathrm{Nsc}\{[\underline{D}f(\underline{x})]\overline{f(\underline{x})}\}\mathbf{i}\underline{x} \\
&= \frac{1}{2}[\underline{D}|f(\underline{x})|^2 \mathbf{i}\underline{x}] - \frac{1}{2}|f(\underline{x})|^2[\underline{D}\mathbf{i}\underline{x}] + |f(\underline{x})|^2 \mathrm{NSc}\{[\underline{D}e^{\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}]e^{-\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}\}\mathbf{i}\underline{x}.
\end{aligned}$$

Proof. As we indicate in Theorem 4.7, $\underline{D}f(\underline{x})$ is a sum of some 0-forms, 1-forms and 2-forms, then by direct calculation we know that $[\underline{D}f(\underline{x})]\overline{f(\underline{x})}$ is real Clifford-valued and a sum of some 0-forms, 1-forms, 2-forms and 3-forms. We will use the notation $\mathrm{Tri}[h(\underline{x})]$ to denote the 3-form part of $h(\underline{x})$ for a Clifford-valued function $h(\underline{x})$.

From

$$\underline{D}f(\underline{x}) = [\underline{D}\rho(\underline{x})]e^{\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})} + \rho(\underline{x})[\underline{D}e^{\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}] \quad \text{and} \quad \overline{f(\underline{x})} = \rho(\underline{x})e^{-\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})},$$

we have

$$[\underline{D}f(\underline{x})]\overline{f(\underline{x})} = \rho(\underline{x})\underline{D}\rho(\underline{x}) + |f(\underline{x})|^2[\underline{D}e^{\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}]e^{-\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}. \tag{5.52}$$

It is easy to see

$$f(\underline{x})\overline{[\underline{D}f(\underline{x})]} = \overline{[\underline{D}f(\underline{x})]\overline{f(\underline{x})}} = -\rho(\underline{x})\underline{D}\rho(\underline{x}) + |f(\underline{x})|^2\overline{[\underline{D}e^{\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}]e^{-\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}}.$$

Then we have

$$\begin{aligned}
&[\underline{D}f(\underline{x})]\overline{f(\underline{x})} - f(\underline{x})\overline{[\underline{D}f(\underline{x})]} \\
&= 2\rho(\underline{x})\underline{D}\rho(\underline{x}) + |f(\underline{x})|^2\{[\underline{D}e^{\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}]e^{-\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})} - \overline{[\underline{D}e^{\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}]e^{-\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}}\}
\end{aligned}$$

$$= 2\rho(\underline{x})\underline{D}\rho(\underline{x}) + $$
$$|f(\underline{x})|^2 \left\{2\mathrm{NSc}\{[\underline{D}e^{\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}]e^{-\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}\} - 2\mathrm{Tri}\{[\underline{D}e^{\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}]e^{-\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}\}\right\},$$

and

$$\begin{aligned}
&[\underline{D}f(\underline{x})\overline{f(\underline{x})} - f(\underline{x})\overline{\underline{D}f(\underline{x})}]\mathbf{i}\underline{x} \\
&= 2\rho(\underline{x})[\underline{D}\rho(\underline{x})]\mathbf{i}\underline{x} + 2|f(\underline{x})|^2\mathrm{NSc}\{[\underline{D}e^{\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}]e^{-\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}\}\mathbf{i}\underline{x} \\
&\quad - 2|f(\underline{x})|^2\mathrm{Tri}\{[\underline{D}e^{\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}]e^{-\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}\}\mathbf{i}\underline{x} \\
&= \underline{D}[|f(\underline{x})|^2\mathbf{i}\underline{x}] - |f(\underline{x})|^2[\underline{D}\mathbf{i}\underline{x}] + 2|f(\underline{x})|^2\mathrm{NSc}\{[\underline{D}e^{\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}]e^{-\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}\}\mathbf{i}\underline{x} \\
&\quad - 2|f(\underline{x})|^2\mathrm{Tri}\{[\underline{D}e^{\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}]e^{-\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}\}\mathbf{i}\underline{x}.
\end{aligned}$$

Since $\rho(\underline{x})[\underline{D}\rho(\underline{x})]$ is vector-valued, by (5.52) we have

$$\mathrm{Tri}\{[\underline{D}f(\underline{x})]\overline{f(\underline{x})}\} = |f(\underline{x})|^2\mathrm{Tri}\{[\underline{D}e^{\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}]e^{-\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}\}.$$

Hence

$$\begin{aligned}
&\mathrm{Nsc}\{[\underline{D}f(\underline{x})]\overline{f(\underline{x})}\}\mathbf{i}\underline{x} \\
&= \frac{1}{2}\left\{[\underline{D}f(\underline{x})]\overline{f(\underline{x})} - f(\underline{x})\overline{[\underline{D}f(\underline{x})]}\right\}\mathbf{i}\underline{x} + \mathrm{Tri}\{[\underline{D}f(\underline{x})]\overline{f(\underline{x})}\}\mathbf{i}\underline{x} \\
&= \frac{1}{2}\underline{D}[|f(\underline{x})|^2\mathbf{i}\underline{x}] - \frac{1}{2}|f(\underline{x})|^2[\underline{D}\mathbf{i}\underline{x}] + |f(\underline{x})|^2\mathrm{NSc}\{[\underline{D}e^{\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}]e^{-\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}\}\mathbf{i}\underline{x} \\
&\quad - |f(\underline{x})|^2\mathrm{Tri}\{[\underline{D}e^{\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}]e^{-\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}\}\mathbf{i}\underline{x} + \mathrm{Tri}\{[\underline{D}f(\underline{x})]\overline{f(\underline{x})}\}\mathbf{i}\underline{x} \\
&= \frac{1}{2}\underline{D}[|f(\underline{x})|^2\mathbf{i}\underline{x}] - \frac{1}{2}|f(\underline{x})|^2[\underline{D}\mathbf{i}\underline{x}] + |f(\underline{x})|^2\mathrm{NSc}\{[\underline{D}e^{\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}]e^{-\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}\}\mathbf{i}\underline{x}. \qquad \square
\end{aligned}$$

Now we study uncertainty principle by adopting $\mathrm{var}^*_{\underline{\xi}}$ as variance of frequency. The following is one of our main results.

Theorem 5.4. *Let $f(\underline{x}) = \rho(\underline{x})e^{\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})} \in L^2(\mathbf{R}^m)$ be real-para-vector-valued, $\underline{x}f(\underline{x}), \underline{D}f(\underline{x}) \in L^2(\mathbf{R}^m)$ and $\|f\|_2 = 1$. Then there holds*

$$\mathrm{var}_{\underline{x}}\mathrm{var}^*_{\underline{\xi}} \geq \frac{m^2}{4} + \mathrm{COV}^2, \tag{5.53}$$

where $\mathrm{var}_{\underline{x}}, \mathrm{var}^*_{\underline{\xi}}$ and COV are defined in Definition 1.2.

Proof. To prove inequality (5.53), due to (1.19), we need to prove the following two inequalities:

$$\mathrm{var}_{\underline{x}}\int_{\mathbf{R}^m} |\rho_1'(\underline{x})|^2 d\underline{x} \geq \frac{m^2}{4}, \tag{5.54}$$

and

$$\mathrm{var}_{\underline{x}}\int_{\mathbf{R}^m} [\theta_1'(\underline{x}) - \langle\underline{\xi}\rangle]^2|f(\underline{x})|^2 d\underline{x} \geq \mathrm{COV}^2. \tag{5.55}$$

Now we prove the inequality (5.54). By using Lemma 5.2, we have

$$\begin{aligned}
&\mathrm{var}_{\underline{x}} \int_{\mathbf{R}^m} |\rho_1'(\underline{x})|^2 d\underline{x} \\
&= \int_{\mathbf{R}^m} (\mathbf{i}\underline{x} - \langle \underline{x}\rangle)^2 |f(\underline{x})|^2 d\underline{x} \\
&\quad \cdot \int_{\mathbf{R}^m} |\mathrm{Nsc}\{[\underline{D}f(\underline{x})][f(\underline{x})]^{-1}\} - \mathrm{Nsc}\{[\underline{D}e^{\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}]e^{-\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}\}|^2 |f(\underline{x})|^2 d\underline{x} \\
&\geq |\int_{\mathbf{R}^m} \left\{\mathrm{Nsc}\{[\underline{D}f(\underline{x})][f(\underline{x})]^{-1}\} - \mathrm{Nsc}\{[\underline{D}e^{\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}]e^{-\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}\}\right\} (\mathbf{i}\underline{x} - \langle \underline{x}\rangle)|f(\underline{x})|^2 d\underline{x}|^2 \\
&= |\int_{\mathbf{R}^m} \mathrm{Nsc}\{[\underline{D}f(\underline{x})][f(\underline{x})]^{-1}|f(\underline{x})|^2\}(\mathbf{i}\underline{x} - \langle \underline{x}\rangle) d\underline{x} \\
&\quad - \int_{\mathbf{R}^m} \mathrm{Nsc}\{[\underline{D}e^{\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}]e^{-\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}\}(\mathbf{i}\underline{x} - \langle \underline{x}\rangle)|f(\underline{x})|^2 d\underline{x}|^2 \\
&= |\int_{\mathbf{R}^m} \mathrm{Nsc}\{[\underline{D}f(\underline{x})]\overline{f(\underline{x})}\}\mathbf{i}\underline{x} d\underline{x} \\
&\quad - \int_{\mathbf{R}^m} \mathrm{Nsc}\{[\underline{D}e^{\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}]e^{-\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}\}\mathbf{i}\underline{x}|f(\underline{x})|^2 d\underline{x}|^2 \\
&= |\int_{\mathbf{R}^m} \left\{\frac{1}{2}\underline{D}[|f(\underline{x})|^2\mathbf{i}\underline{x}] - \frac{1}{2}|f(\underline{x})|^2[\underline{D}\mathbf{i}\underline{x}] + |f(\underline{x})|^2 \mathrm{NSc}\{[\underline{D}e^{\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}]e^{-\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}\}\mathbf{i}\underline{x}\right\} d\underline{x} \\
&\quad - \int_{\mathbf{R}^m} \mathrm{Nsc}\{[\underline{D}e^{\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}]e^{-\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}\}\mathbf{i}\underline{x}|f(\underline{x})|^2 d\underline{x}|^2 \\
&= |\int_{\mathbf{R}^m} \left\{\frac{1}{2}\underline{D}[|f(\underline{x})|^2\mathbf{i}\underline{x}] - \frac{1}{2}|f(\underline{x})|^2[\underline{D}\mathbf{i}\underline{x}]\right\} d\underline{x}|^2 \\
&= \frac{m^2}{4}.
\end{aligned} \tag{5.56}$$

The third equality in (5.56) is a consequence of the relation

$$\int_{\mathbf{R}^m} \left\{\mathrm{Nsc}\{[\underline{D}f(\underline{x})][f(\underline{x})]^{-1}\} - \mathrm{Nsc}\{[\underline{D}e^{\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}]e^{-\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}\}\right\} \langle \underline{x}\rangle |f(\underline{x})|^2 d\underline{x} = 0. \tag{5.57}$$

It is to be proved below. The fourth equality in (5.56) is obtained by using Lemma 5.3. The last equality in (5.56) follows from the following two relations:

$$\int_{\mathbf{R}^m} \underline{D}[|f(\underline{x})|^2\mathbf{i}\underline{x}]d\underline{x} = 0 \quad \text{and} \quad \underline{D}(\mathbf{i}\underline{x}) = -\mathbf{i}m.$$

Next we show (5.57). We recall, by invoking (3.38),

$$\int_{\mathbf{R}^m} \left\{\mathrm{Nsc}\{[\underline{D}f(\underline{x})][f(\underline{x})]^{-1}\} - \mathrm{Nsc}\{[\underline{D}e^{\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}]e^{-\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}\}\right\} \langle \underline{x}\rangle |f(\underline{x})|^2 d\underline{x}$$

$$= \int_{\mathbf{R}^m} \frac{\underline{D}\rho(\underline{x})}{\rho(\underline{x})} |f(\underline{x})|^2 d\underline{x} \langle \underline{x} \rangle$$
$$= \int_{\mathbf{R}^m} [\underline{D}\rho(\underline{x})]\rho(\underline{x}) d\underline{x} \langle \underline{x} \rangle$$
$$= \frac{1}{2} \int_{\mathbf{R}^m} [\underline{D}\rho^2(\underline{x})] d\underline{x} \langle \underline{x} \rangle$$
$$= 0.$$

Finally, we prove (5.55) through Hölder's inequality:

$$\begin{aligned}
&\mathrm{var}_{\underline{x}} \int_{\mathbf{R}^m} [\theta_1'(\underline{x}) - \langle \underline{\xi} \rangle]^2 |f(\underline{x})|^2 d\underline{x} \\
&= \int_{\mathbf{R}^m} (\mathbf{i}\underline{x} - \langle \underline{x} \rangle)^2 |f(\underline{x})|^2 d\underline{x} \int_{\mathbf{R}^m} \{\mathrm{Sc}\{[\underline{D}f(\underline{x})][f(\underline{x})]^{-1}\} - \langle \underline{\xi} \rangle\}^2 |f(\underline{x})|^2 d\underline{x} \\
&\geq \{ \int_{\mathbf{R}^m} |\mathbf{i}\underline{x} - \langle \underline{x} \rangle||\mathrm{Sc}\{[\underline{D}f(\underline{x})][f(\underline{x})]^{-1}\} - \langle \underline{\xi} \rangle||f(\underline{x})|^2 d\underline{x}\}^2 \\
&= \mathrm{COV}^2. \qquad \square
\end{aligned}$$

By using $\mathrm{var}_{\underline{\xi}}$ as variance of frequency, we obtain, alternatively,

Theorem 5.5. *Let $f(\underline{x}) = \rho(\underline{x}) e^{\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})} \in L^2(\mathbf{R}^m)$ be real-para-vector-valued, $\underline{x}f(\underline{x}), \underline{D}f(\underline{x}) \in L^2(\mathbf{R}^m)$ and $\|f\|_2 = 1$. Then there holds*

$$\mathrm{var}_{\underline{x}} \mathrm{var}_{\underline{\xi}} \geq \frac{1}{4} |\mathbf{i}m + M|^2 + \mathrm{COV}^2, \tag{5.58}$$

where

$$M = 2 \int_{\mathbf{R}^m} \mathrm{Nsc}\{[\underline{D}e^{\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}] e^{-\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}\} (\mathbf{i}\underline{x} - \langle \underline{x} \rangle) |f(\underline{x})|^2 d\underline{x}.$$

Proof. The proof of Theorem 5.5 is the same as that of Theorem 5.4, except that, instead of (5.54), we need to show

$$\mathrm{var}_{\underline{x}} \int_{\mathbf{R}^m} |\rho_2'(\underline{x})|^2 d\underline{x} \geq \frac{1}{4} |\mathbf{i}m + M|^2. \tag{5.59}$$

It is proceeded as

$$\begin{aligned}
&\mathrm{var}_{\underline{x}} \int_{\mathbf{R}^m} |\rho_2'(\underline{x})|^2 d\underline{x} \\
&= \int_{\mathbf{R}^m} (\mathbf{i}\underline{x} - \langle \underline{x} \rangle)^2 |f(\underline{x})|^2 d\underline{x} \cdot \int_{\mathbf{R}^m} |\mathrm{Nsc}\{[\underline{D}f(\underline{x})][f(\underline{x})]^{-1}\}|^2 |f(\underline{x})|^2 d\underline{x} \\
&\geq | \int_{\mathbf{R}^m} \mathrm{Nsc}\{[\underline{D}f(\underline{x})][f(\underline{x})]^{-1}\} (\mathbf{i}\underline{x} - \langle \underline{x} \rangle) |f(\underline{x})|^2 d\underline{x}|^2
\end{aligned}$$

$$= |\int_{\mathbf{R}^m} \mathrm{Nsc}\{[\underline{D}f(\underline{x})][f(\underline{x})]^{-1}|f(\underline{x})|^2\}\mathbf{i}\underline{x}d\underline{x}$$

$$- \int_{\mathbf{R}^m} \mathrm{Nsc}\{[\underline{D}f(\underline{x})][f(\underline{x})]^{-1}\}\langle\underline{x}\rangle|f(\underline{x})|^2 d\underline{x}|^2$$

$$= |\int_{\mathbf{R}^m} \mathrm{Nsc}\{[\underline{D}f(\underline{x})]\overline{f(\underline{x})}\}\mathbf{i}\underline{x}d\underline{x} - \int_{\mathbf{R}^m} \mathrm{Nsc}\{[\underline{D}f(\underline{x})][f(\underline{x})]^{-1}\}\langle\underline{x}\rangle|f(\underline{x})|^2 d\underline{x}|^2$$

$$= |\int_{\mathbf{R}^m} \left\{\frac{1}{2}\underline{D}[|f(\underline{x})|^2\mathbf{i}\underline{x}] - \frac{1}{2}|f(\underline{x})|^2[\underline{D}\mathbf{i}\underline{x}] + |f(\underline{x})|^2\mathrm{NSc}\{[\underline{D}e^{\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}]e^{-\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}\}\mathbf{i}\underline{x}\right\} d\underline{x}$$

$$- \int_{\mathbf{R}^m} \mathrm{Nsc}\{[\underline{D}f(\underline{x})][f(\underline{x})]^{-1}\}\langle\underline{x}\rangle|f(\underline{x})|^2 d\underline{x}|^2$$

$$= \frac{1}{4}|\mathbf{i}m + 2\int_{\mathbf{R}^m} \mathrm{Nsc}\{[\underline{D}e^{\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}]e^{-\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}\}(\mathbf{i}\underline{x} - \langle\underline{x}\rangle)|f(\underline{x})|^2 d\underline{x}|^2$$

where

$$\int_{\mathbf{R}^m} \mathrm{Nsc}\{[\underline{D}f(\underline{x})][f(\underline{x})]^{-1}\}\langle\underline{x}\rangle|f(\underline{x})|^2 d\underline{x}$$

$$= \int_{\mathbf{R}^m} \left\{\frac{\underline{D}\rho(\underline{x})}{\rho(\underline{x})} + \mathrm{Nsc}\{[\underline{D}e^{\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}]e^{-\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}\}\right\}|f(\underline{x})|^2 d\underline{x}\langle\underline{x}\rangle$$

$$= \int_{\mathbf{R}^m} \frac{\underline{D}\rho(\underline{x})}{\rho(\underline{x})}|f(\underline{x})|^2 d\underline{x}\langle\underline{x}\rangle + \int_{\mathbf{R}^m} \mathrm{Nsc}\{[\underline{D}e^{\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}]e^{-\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}\}|f(\underline{x})|^2 d\underline{x}\langle\underline{x}\rangle$$

$$= \int_{\mathbf{R}^m} [\underline{D}\rho(\underline{x})]\rho(\underline{x})d\underline{x}\langle\underline{x}\rangle + \int_{\mathbf{R}^m} \mathrm{Nsc}\{[\underline{D}e^{\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}]e^{-\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}\}|f(\underline{x})|^2 d\underline{x}\langle\underline{x}\rangle$$

$$= \frac{1}{2}\int_{\mathbf{R}^m} [\underline{D}\rho^2(\underline{x})]d\underline{x}\langle\underline{x}\rangle + \int_{\mathbf{R}^m} \mathrm{Nsc}\{[\underline{D}e^{\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}]e^{-\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}\}|f(\underline{x})|^2 d\underline{x}\langle\underline{x}\rangle$$

$$= \int_{\mathbf{R}^m} \mathrm{Nsc}\{[\underline{D}e^{\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}]e^{-\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}\}|f(\underline{x})|^2 d\underline{x}\langle\underline{x}\rangle,$$

where we used the relation $\int_{\mathbf{R}^m}[\underline{D}\rho^2(\underline{x})]d\underline{x} = 0$. □

Remark 5.6. By Remark 4.2 and Remark 4.4, we know that when $m = 1$,

$$\mathrm{COV} = \int_{\mathbf{R}} |\mathbf{i}x_1\mathbf{e}_1 - \langle\underline{x}\rangle||\theta_1'(\underline{x}) - \langle\underline{\xi}\rangle||f(\underline{x})|^2 dx_1$$

coincides with (1.9). Hence for $m = 1$ the lower-bounds of (5.53) and (5.58) both reduce to that of (1.8). In other words, the two uncertainty principles both correspond to the strongest form of the classical uncertainty principle in one dimension.

Remark 5.7. As we indicated in the introduction section, [22] gives the following type uncertainty principle for real-para-vector-valued signal $f(\underline{x}) = \rho(\underline{x})e^{\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}$:

$$\left(\int_{\mathbf{R}^m} |\underline{x}|^2|f(\underline{x})|^2 d\underline{x}\right)\left(\int_{\mathbf{R}^m} |\underline{\xi}|^2|\hat{f}(\underline{\xi})|^2 d\underline{\xi}\right) \geq \frac{m^2}{4} + \mathrm{COV}^2_{\underline{x}\underline{\xi}}. \tag{5.60}$$

The left-hand side of (5.60) is the same as that of (5.58), except that the means of time and frequency both are assumed to be zero. If we also assume that the means of time and frequency are zero in (5.53) and (5.58), they can be rewritten as

$$\left(\int_{\mathbf{R}^m} |\underline{x}|^2|f(\underline{x})|^2 d\underline{x}\right)\left(\int_{\mathbf{R}^m} [\theta_1'(\underline{x})]^2|f(\underline{x})|^2 d\underline{x} + \int_{\mathbf{R}^m} |\rho_1'(\underline{x})|^2 d\underline{x}\right) \geq \frac{m^2}{4} + \mathrm{COV}^2 \tag{5.61}$$

and

$$\left(\int_{\mathbf{R}^m} |\underline{x}|^2|f(\underline{x})|^2 d\underline{x}\right)\left(\int_{\mathbf{R}^m} |\underline{\xi}|^2|\hat{f}(\underline{\xi})|^2 d\underline{\xi}\right) \geq \frac{1}{4}|\mathbf{i}m + M|^2 + \mathrm{COV}^2, \tag{5.62}$$

where

$$M = 2\int_{\mathbf{R}^m} \mathrm{Nsc}\{[\underline{D}e^{\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}]e^{-\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}\}(\mathbf{i}\underline{x})|f(\underline{x})|^2 d\underline{x}$$

and

$$\mathrm{COV} = \int_{\mathbf{R}^m} |\mathbf{i}\underline{x}||\theta_1'(\underline{x})||f(\underline{x})|^2 d\underline{x}.$$

To compare the lower-bounds in (5.60), (5.61) and (5.62), we write COV and $\mathrm{COV}_{\underline{x}\underline{\xi}}$ precisely, as

$$\begin{aligned}
\mathrm{COV} &= \int_{\mathbf{R}^m} |\mathbf{i}\underline{x}||\theta_1'(\underline{x})||f(\underline{x})|^2 d\underline{x} \\
&= \int_{\mathbf{R}^m} |\mathbf{i}\underline{x}|\left|\mathrm{Sc}\left\{[\underline{D}f(\underline{x})][f(\underline{x})]^{-1}\right\}\right||f(\underline{x})|^2 d\underline{x} \\
&\overset{(3.33)}{=\!=} \int_{\mathbf{R}^m} |\mathbf{i}\underline{x}|\left|\mathrm{Sc}\left\{[\underline{D}e^{\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}]e^{-\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}\right\}\right||f(\underline{x})|^2 d\underline{x} \\
&= \int_{\mathbf{R}^m} (\sum_{k=1}^m x_k^2)^{\frac{1}{2}}\left|\mathrm{Sc}\left\{\sum_{k=1}^m \mathbf{e}_k\left(\frac{\partial}{\partial x_k}e^{\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}\right)e^{-\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}\right\}\right||f(\underline{x})|^2 d\underline{x} \\
&= \int_{\mathbf{R}^m} (\sum_{k=1}^m x_k^2)^{\frac{1}{2}}\left|\mathrm{Sc}\left\{\sum_{k=1}^m \mathbf{e}_k\mathrm{NSc}\left\{\left(\frac{\partial}{\partial x_k}e^{\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}\right)e^{-\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}\right\}\right\}\right||f(\underline{x})|^2 d\underline{x},
\end{aligned} \tag{5.63}$$

and

$$\begin{aligned}\mathrm{COV}_{\underline{x}\underline{\xi}} &= \sum_{k=1}^{m} \int_{\mathbf{R}^m} \left| x_k \mathrm{NSc}\left\{ \left(\frac{\partial}{\partial x_k} e^{\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})} \right) e^{-\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})} \right\} \right| |f(\underline{x})|^2 d\underline{x} \\ &= \int_{\mathbf{R}^m} \sum_{k=1}^{m} |x_k| \left| \mathrm{NSc}\left\{ \left(\frac{\partial}{\partial x_k} e^{\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})} \right) e^{-\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})} \right\} \right| |f(\underline{x})|^2 d\underline{x}. \end{aligned} \tag{5.64}$$

Based on the last parts of (5.63) and (5.64) we are unable to show which of COV and $\mathrm{COV}_{\underline{x}\underline{\xi}}$ is larger. However, an example in [22] shows that COV is greater than $\mathrm{COV}_{\underline{x}\underline{\xi}}$ at least for some signals. The used signal is

$$f(\underline{x}) = (\frac{\alpha}{\pi})^{\frac{m}{4}} e^{\frac{-\alpha|\underline{x}|^2}{2}} e^{\beta_1 x_1 \mathbf{e}_1},$$

where α is a positive real number and $\beta_1 \in \mathbf{R}$. The means of time and frequency of $f(\underline{x})$ are both zero. It can be calculated directly that

$$\left(\frac{\partial}{\partial x_1} e^{\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})} \right) e^{-\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})} = \beta_1 \mathbf{e}_1$$

and

$$\left(\frac{\partial}{\partial x_k} e^{\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})} \right) e^{-\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})} = 0, \qquad k = 2, 3, \cdots, m.$$

Then by (5.64) and (5.63), we have

$$\mathrm{COV}_{\underline{x}\underline{\xi}} = \int_{\mathbf{R}^m} |x_1\beta_1| \, |f(\underline{x})|^2 d\underline{x},$$

$$\mathrm{COV} = \int_{\mathbf{R}^m} (\sum_{k=1}^{m} x_k^2)^{\frac{1}{2}} \, |\beta_1| \, |f(\underline{x})|^2 d\underline{x} > \mathrm{COV}_{\underline{x}\underline{\xi}},$$

and

$$M = 2 \int_{\mathbf{R}^m} \mathrm{Nsc}\{[\underline{D} e^{\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}] e^{-\frac{u(\underline{x})}{|u(\underline{x})|}\theta(\underline{x})}\}(\mathbf{i}\underline{x}) |f(\underline{x})|^2 d\underline{x} = 0.$$

We can conclude that for this signal the lower-bound of (5.61) and that of (5.62) coincide, and they both are larger than that of (5.60).

6. Uncertainty principle for real-scalar-valued signals on $\mathbf{R}^m$

For a real-scalar-valued function $f(\underline{x}) \in L^2(\mathbf{R}^m)$, it is easy to see that $f^{\pm}(\underline{x})$ are real-para-vector-valued, $\hat{f}(\underline{\xi})$ is complex-scalar-valued and $\hat{f^{\pm}}(\underline{\xi})$ are complex-para-vector-valued.

We call $f^+(\underline{x}) = \frac{1}{2}[f(\underline{x}) + Hf(\underline{x})]$ the *monogenic signal associated with* f, where Hf is the Hilbert transform in $\mathbf{R}^m$. In the following, we will study f^+ instead of f. As a matter of fact, the analytic or monogenic signal f^+ is more informative than f itself in studying the space and frequency means. Moreover, f^+ has the great advantage being the non-tangential boundary limit of a Hardy space function in the related domain [11].

To define the phase and amplitude derivatives of the monogenic signal $f^+(\underline{x})$, we represent $f^+(\underline{x})$ in the polar coordinate form (see [24]).

$$\begin{aligned} f^+(\underline{x}) &= \frac{1}{2}[f(\underline{x}) + Hf(\underline{x})] \\ &= \rho_+(\underline{x})\left[\frac{f(\underline{x})}{\rho_+(\underline{x})} + \frac{Hf(\underline{x})}{\rho_+(\underline{x})}\right] \\ &= \rho_+(\underline{x})\left[\frac{f(\underline{x})}{\rho_+(\underline{x})} + \frac{Hf(\underline{x})}{|Hf(\underline{x})|}\frac{|Hf(\underline{x})|}{\rho_+(\underline{x})}\right] \\ &= \rho_+(\underline{x})\left[\cos\theta_+(\underline{x}) + \frac{Hf(\underline{x})}{|Hf(\underline{x})|}\sin\theta_+(\underline{x})\right] \\ &= \rho_+(\underline{x})e^{\frac{Hf(\underline{x})}{|Hf(\underline{x})|}\theta_+(\underline{x})}, \end{aligned} \tag{6.65}$$

where $\rho_+(\underline{x}) = \frac{1}{2}\sqrt{f^2(\underline{x}) + |Hf(\underline{x})|^2}$ is called the *amplitude*, $\theta_+(\underline{x}) = \arctan\frac{|Hf(\underline{x})|}{f(\underline{x})}$ the *phase angle*, $\frac{Hf(\underline{x})}{|Hf(\underline{x})|}\theta_+(\underline{x})$ the *phase vector*, and $e^{\frac{Hf(\underline{x})}{|Hf(\underline{x})|}\theta_+(\underline{x})}$ the *phase direction* of f.

The following explanation would be necessary. In the following definition we apply the homogeneous Dirac operator to the amplitude and phase-related functions in relation to the non-tangential boundary limit of the associated monogenic function, as well as to the boundary function itself. The boundary limit function, however, may not be smooth. The amplitude and phase-related functions, as consequence, may not be smooth either. Therefore, the required classical partial derivatives may not exist. The right understanding of the application of the homogeneous Dirac operator to f^+ is as follows (see [11]): We apply Dirac operator $\underline{D}$ to $f^+(x_0 + \underline{x})$, $x_0 > 0$, that, as a monogenic function on $\mathbf{R}_1^{m+}$, is smooth. Once we have defined $\underline{D}f^+(x_0 + \underline{x})$, we take non-tangential boundary limit to obtain $\underline{D}f^+(\underline{x})$. The definitions of $\underline{D}\theta_+(\underline{x})$ and $\underline{D}\rho_+(\underline{x})$ are similar. The existence of boundary limit is guaranteed by the assumption that f belongs to the relevant Sobolev space.

Now we introduce the phase derivative and the amplitude derivative of $f^+(\underline{x})$.

Definition 6.1. Let $f(\underline{x}) \in L^2(\mathbf{R}^m)$ be real scalar-valued with $\underline{\xi}\hat{f}(\underline{\xi}) \in L^2(\mathbf{R}^m)$. Then the phase derivative of $f^+(\underline{x}) = \rho_+(\underline{x})e^{\frac{Hf(\underline{x})}{|Hf(\underline{x})|}\theta_+(\underline{x})}$ can be defined by the following two ways

$$\theta'_{+,1}(\underline{x}) = \mathrm{Sc}\left\{[\underline{D}f^+(\underline{x})][f^+(\underline{x})]^{-1}\right\} \tag{6.66}$$

and

$$\theta'_{+,2}(\underline{x}) = \mathrm{Sc}\{[\underline{D}\theta_+(\underline{x})]\frac{Hf(\underline{x})}{|Hf(\underline{x})|}\}. \tag{6.67}$$

The amplitude derivative is also defined by two ways

$$\rho'_{+,1}(\underline{x}) \triangleq \underline{D}\rho_+(\underline{x}) \tag{6.68}$$

and

$$\rho'_{+,2}(\underline{x}) \triangleq \rho_+(\underline{x})\mathrm{Nsc}\left\{[\underline{D}f^+(\underline{x})][f^+(\underline{x})]^{-1}\right\}. \tag{6.69}$$

Definition 6.2. Let $f(\underline{x}) \in L^2(\mathbf{R}^m)$ be real scalar-valued with $\underline{\xi}\hat{f}(\underline{\xi}) \in L^2(\mathbf{R}^m)$, and $\underline{x}f^+(\underline{x}) \in L^2(\mathbf{R}^m)$ with $\|f^+\|_2 = 1$, where f^+ is given by (6.65). Then the *mean of time* is given by

$$\langle\underline{x}\rangle_+ = \int_{\mathbf{R}^m} \mathbf{i}\underline{x}|f^+(\underline{x})|^2 d\underline{x}, \tag{6.70}$$

the *variance of* $\underline{x}$ is

$$\mathrm{var}_{\underline{x},+} = \int\limits_{\mathbf{R}^m} (\mathbf{i}\underline{x} - \langle \underline{x} \rangle_+)^2 |f^+(\underline{x})|^2 d\underline{x}, \tag{6.71}$$

the *mean of frequency* is

$$\langle \underline{\xi} \rangle_+ = \int\limits_{\mathbf{R}^m} |\mathbf{i}\underline{\xi}||\hat{f^+}(\underline{\xi})|^2 d\underline{\xi}, \tag{6.72}$$

the *variance of frequency* is defined by the following two formulas:

$$\mathrm{var}_{\underline{\xi},+} = \int\limits_{\mathbf{R}^m} (|\mathbf{i}\underline{\xi}| - \langle \underline{\xi} \rangle_+)^2 |\hat{f^+}(\underline{\xi})|^2 d\underline{\xi}, \tag{6.73}$$

and

$$\mathrm{var}^*_{\underline{\xi},+} \triangleq \int\limits_{\mathbf{R}^m} [\theta'_{+,1}(\underline{x}) - \langle \underline{\xi} \rangle_+]^2 |f^+(\underline{x})|^2 d\underline{x} + \int\limits_{\mathbf{R}^m} |\rho'_{+,1}(\underline{x})|^2 d\underline{x}. \tag{6.74}$$

The *covariance* is defined by

$$\mathrm{Cov}_+ = \int\limits_{\mathbf{R}^m} (\mathbf{i}\underline{x} - \langle \underline{x} \rangle_+)[\theta'_{+,1}(\underline{x}) - \langle \underline{\xi} \rangle_+] |f^+(\underline{x})|^2 d\underline{x}, \tag{6.75}$$

and the *absolute covariance* is defined by

$$\mathrm{COV}_+ = \int\limits_{\mathbf{R}^m} |\mathbf{i}\underline{x} - \langle \underline{x} \rangle_+||\theta'_{+,1}(\underline{x}) - \langle \underline{\xi} \rangle_+||f^+(\underline{x})|^2 d\underline{x}. \tag{6.76}$$

Lemma 6.3. *Let $f(\underline{x}) \in L^2(\mathbf{R}^m)$ be real-scalar-valued with $\underline{\xi}\hat{f}(\underline{\xi}) \in L^2(\mathbf{R}^m)$ and $\|f^+\|_2 = 1$, where f^+ is given by (6.65). Then there hold*

$$\langle \underline{\xi} \rangle_+ = \int\limits_{\mathbf{R}^m} \theta'_{+,1}(\underline{x}) |f^+(\underline{x})|^2 d\underline{x} \tag{6.77}$$

and

$$\mathrm{var}_{\underline{\xi},+} = \int\limits_{\mathbf{R}^m} [\theta'_{+,1}(\underline{x}) - \langle \underline{\xi} \rangle_+]^2 |f^+(\underline{x})|^2 d\underline{x} + \int\limits_{\mathbf{R}^m} |\rho'_{+,2}(\underline{x})|^2 d\underline{x}. \tag{6.78}$$

Proof. The method to prove Lemma 6.3 is the same as that of Theorem 4.6 and Theorem 4.7. □

Note that the proofs of (6.77) and (6.78) also can be found in the papers [24] and [21], respectively.

Theorem 6.4. *Let $f(\underline{x}) \in L^2(\mathbf{R}^m)$ be real scalar-valued with $\underline{\xi}\hat{f}(\underline{\xi}) \in L^2(\mathbf{R}^m)$, and $\underline{x}f^+(\underline{x}) \in L^2(\mathbf{R}^m)$ with $\|f^+\|_2 = 1$, where f^+ is given by (6.65). Then there hold*

$$\mathrm{var}_{\underline{x},+}\mathrm{var}^*_{\underline{\xi},+} \geq \frac{m^2}{4} + \mathrm{COV}^2_+, \tag{6.79}$$

and

$$\mathrm{var}_{\underline{x},+}\mathrm{var}_{\underline{\xi},+} \geq \frac{1}{4}|\mathbf{i}m + M|^2 + \mathrm{COV}_+^2, \tag{6.80}$$

where

$$M = 2\int_{\mathbf{R}^m} \mathrm{Nsc}\{[\underline{D}e^{\frac{Hf(\underline{x})}{|Hf(\underline{x})|}\theta_+(\underline{x})}]e^{-\frac{Hf(\underline{x})}{|Hf(\underline{x})|}\theta_+(\underline{x})}\}(\mathbf{i}\underline{x} - \langle \underline{x}\rangle_+)|f^+(\underline{x})|^2 d\underline{x}.$$

Proof. The proof of Theorem 6.4 is analogous with that of Theorem 5.4 and Theorem 5.5. □

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