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Tighter Uncertainty Principles Based on Quaternion Fourier Transform

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Abstract. The quaternion Fourier transform (QFT) and its properties are reviewed in this paper. Under the polar coordinate form for quaternion-valued signals, we strengthen the stronger uncertainty principles in terms of covariance for quaternion-valued signals based on the right-sided quaternion Fourier transform in both the directional and the spatial cases. We also obtain the conditions that give rise to the equal relations of two uncertainty principles. Examples are given to verify the results.

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1. Introduction

The quaternion Fourier transform (QFT) plays a valued role in representation of signals. It transforms a real (or quaternionic) 2D signal into a quaternion-valued frequency domain signal. The four components of the QFT separate four cases of symmetry into real signals instead of only two as in the complex FT. In [6,7,35] the authors used the QFT to proceed color image analysis. The paper ([2]) implements the QFT to design a color image digital watermarking scheme. The authors in [3] applied the QFT to image pre-processing and neural computing techniques for speech recognition. Recently, the certain asymptotic properties of the QFT were analyzed and straightforward generalizations of classical Bochner–Minlos theorems to the framework of quaternionic analysis were derived in [16,17].

The uncertainty principle in the time-frequency plane plays an important role in signal processing [9,12,18,19,26,27,32,38,41]. This principle states that for a given unit energy signal $f(t)$ with Fourier transform

$$\hat{f}(\omega) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-\mathbf{i}\omega t} dt,$$

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the product of spreads of the signal in the time domain and the frequency domain is bounded by a lower bound

$$\sigma_t^2\sigma_\omega^2 \geq \frac{1}{4}, \tag{1.1}$$

where σ_t and σ_ω are, precisely, the duration and bandwidth of a signal $f(t)$ defined by

$$\sigma_t^2 := \int_{-\infty}^{\infty}(t- < t >)^2|f(t)|^2dt$$

and

$$\sigma_\omega^2 := \int_{-\infty}^{\infty}(\omega- < \omega >)^2|\hat{f}(\omega)|^2d\omega,$$

respectively. Here

$$< t >:= \int_{-\infty}^{\infty} t|f(t)|^2dt$$

is the mean time and

$$< \omega >:= \int_{-\infty}^{\infty} \omega|\hat{f}(\omega)|^2d\omega$$

is the mean frequency. Without loss of generality, let $< t >= 0$ and $< \omega >= 0$.

If $f(t)$ is expressed in the polar form $f(t) = |f(t)|e^{\mathbf{i}\theta(t)} = \rho(t)e^{\mathbf{i}\theta(t)}$, then the stronger version of the uncertainty principle ([8]) is

$$\sigma_t\sigma_\omega \geq \left|-\frac{1}{2} + \mathbf{i}\mathrm{Cov}_{\mathrm{t}\omega}\right| = \frac{1}{2}\sqrt{1 + 4\mathrm{Cov}_{\mathrm{t}\omega}^2}, \tag{1.2}$$

where $\mathrm{Cov}_{\mathrm{t}\omega}$ is the covariance of a signal defined by

$$\mathrm{Cov}_{\mathrm{t}\omega} := \int_{-\infty}^{\infty} \mathrm{t}\theta'(\mathrm{t})\rho^2(\mathrm{t})\mathrm{dt}.$$

The covariance is to be an indication of how instantaneous frequency, $\theta'(t)$, and time are related. When the instantaneous frequency does not change the covariance is zero ([8]).

Recently, in [10], Dang, Deng and Qian strengthen the result of (1.2) , they obtained:

$$\sigma_t\sigma_\omega \geq \left|-\frac{1}{2} + \mathbf{i}\mathrm{COV}_{\mathrm{t}\omega}\right| = \frac{1}{2}\sqrt{1 + 4\mathrm{COV}_{\mathrm{t}\omega}^2}, \tag{1.3}$$

where $\mathrm{COV}_{\mathrm{t}\omega}$ is the absolute covariance of a signal defined by

$$\mathrm{COV}_{\mathrm{t}\omega} := \int_{-\infty}^{\infty} |\mathrm{t}\theta'(\mathrm{t})|\rho^2(\mathrm{t})\mathrm{dt}.$$

Since $\int_{-\infty}^{\infty} t\theta'(t)\rho^2(t)dt \leq \int_{-\infty}^{\infty} |t\theta'(t)|\rho^2(t)dt$, (1.3) is stronger than (1.2). In [11], they extend the result to linear canonical transform.

Because of the importance of the classical uncertainty principle in physics [1,8,22–24,28,29,34,40], there have been many efforts to extend it to various types of functions and integral transforms, such as [30,36,39]. Since 1994, some studies [4,21,31] develop the uncertainty relations with the Quaternionic Fourier transform (QFT) in Hamiltonian quaternion analysis. The uncertainty principle for the Quaternion linear canonical transform (QLCT), the generalization of the QFT in the Hamiltonian quaternion algebra, are derived in [25]. All those papers obtained their uncertainty bounds without covariance in the spatial case. Recently, in [42], under the polar coordinate form of quaternion signals, we first give stronger uncertainty principles associated with covariance based on the right-sided quaternion Fourier transform both in the directional and the spatial cases.

In the present paper, we extend the results (1.3) to Quaternion-valued signals. The most advantage of this theory is that for quaternion-valued signals, if we write them into the polar coordinate form, we can obtain a tighter bound. Furthermore, we also deduce the sufficient and necessary conditions under which two uncertainty principles hold. These conditions are easily verified.

The article is organized as follows. In Sect. 2, Quaternion algebra is introduced and the polar representation of a quaternion-valued signal is presented. The quaternion Fourier transform and its properties are reviewed in Sect. 3. Two tighter uncertainty principles are generalized for the right-sided quaternion Fourier transform of quaternion-valued signal in Sect. 4. We give examples to illustrate the results in Sect. 5.

2. Preliminaries

The quaternion algebra $\mathcal{H}$ was first invented by W. R. Hamilton in 1843 for extending complex numbers to a 4D non-commutative field ([37]). A real quaternion $q \in \mathcal{H}$ can be written in form

$$q = q_0 + \underline{q} = q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3,\ q_k \in \mathbf{R},\ k = 0, 1, 2, 3,$$

where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ satisfy Hamilton's multiplication rules

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1, \mathbf{ij} = -\mathbf{ji} = \mathbf{k},$$

$$\mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \mathbf{ki} = -\mathbf{ik} = \mathbf{j}.$$

The *scalar part* of q is q_0 denoted by $\mathrm{Sc[q]} = \mathrm{q}_0$, The *non scalar part* (or *pure quaternion*) of q is $\underline{q}$ denoted by $\mathrm{NSc[q]} = \underline{\mathrm{q}}$.

Using the Hamilton's multiplication rules, the multiplication of two quaternion numbers $p = p_0 + \underline{p}$ and $q = q_0 + \underline{q}$ can be expressed as

$$pq = p_0q_0 + \underline{p} \cdot \underline{q} + p_0\underline{q} + q_0\underline{p} + \underline{p} \times \underline{q},$$

where

$$\underline{p} \cdot \underline{q} = -(p_1q_1 + p_2q_2 + p_3q_3)$$

and

$$\underline{p} \times \underline{q} = \mathbf{i}(p_3q_2 - p_2q_3) + \mathbf{j}(p_1q_3 - p_3q_1) + \mathbf{k}(p_2q_1 - p_1q_2).$$

We define the conjugation of $q \in \mathcal{H}$ by $\overline{q} = q_0 - \mathbf{i}q_1 - \mathbf{j}q_2 - \mathbf{k}q_3$. The quaternion conjugation is a linear anti-involution

$$\overline{\overline{q}} = q, \ \overline{p+q} = \overline{p} + \overline{q}, \ \overline{pq} = \overline{q}\overline{p}.$$

Clearly, $q\overline{q} = q_0^2 + q_1^2 + q_2^2 + q_3^2$. So the modulus of a quaternion q is defined by

$$|q| = \sqrt{q\overline{q}} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}.$$

It is easy to verify that $0 \neq q \in \mathcal{H}$ implies

$$q^{-1} = \frac{\overline{q}}{|q|^2}.$$

In this paper, we will study quaternion-valued signals $f : \mathbf{R}^2 \to \mathcal{H}$ that can be expressed as

$$f(\underline{x}) = f_0(\underline{x}) + \mathbf{i}f_1(\underline{x}) + \mathbf{j}f_2(\underline{x}) + \mathbf{k}f_3(\underline{x}),$$

where $\underline{x} = x_1\mathbf{i} + x_2\mathbf{j} \in \mathbf{R}^2$ and $f_k, k = 0, 1, 2, 3$ are real-valued functions. Here $\mathcal{H}$ is the quaternion algebra.

It is well-known that a complex signal $f(t) = u(t) + \mathbf{i}v(t)$ can be expressed in the polar coordinate form $|f(t)|e^{\mathbf{i}\theta(t)}$, where the *amplitude* $|f(t)| := \sqrt{u^2(t) + v^2(t)}$ and the *phase* $\theta(t) := \arctan \frac{v(t)}{u(t)}$.

We will be using the polar coordinate form of quaternion-valued signals ([5]), viz.,

$$\begin{aligned} f(\underline{x}) &= f_0(\underline{x}) + \mathbf{i}f_1(\underline{x}) + \mathbf{j}f_2(\underline{x}) + \mathbf{k}f_3(\underline{x}) \\ &= |f(\underline{x})|e^{\underline{u}(\underline{x})\theta(\underline{x})} \\ &= \rho(\underline{x})e^{\underline{u}(\underline{x})\theta(\underline{x})}, \end{aligned}$$

where $e^{\underline{u}(\underline{x})\theta(\underline{x})}$ is understood in accordance with Euler's formula $e^{\underline{u}(\underline{x})\theta(\underline{x})} = \cos\theta(\underline{x}) + \underline{u}(\underline{x})\sin\theta(\underline{x})$ and

$$\rho(\underline{x}) := \sqrt{f_0^2(\underline{x}) + f_1^2(\underline{x}) + f_2^2(\underline{x}) + f_3^2(\underline{x})}.$$

$$\underline{u}(x) := \frac{\mathbf{i}f_1(\underline{x}) + \mathbf{j}f_2(\underline{x}) + \mathbf{k}f_3(\underline{x})}{\sqrt{f_1^2(\underline{x}) + f_2^2(\underline{x}) + f_3^2(\underline{x})}}$$

belongs to the unit sphere $S^2 := \{\underline{x} \in \mathcal{H} \,|\, |\underline{x}|^2 = 1\}$ of 3D Euclidean space $\mathbf{R}^3$. Here $\underline{u}(\underline{x})$ can be written in the spherical coordinate form

$$\underline{u}(\underline{x}) = \mathbf{i}\cos\phi + \mathbf{j}\sin\phi\sin\tau + \mathbf{k}\sin\phi\cos\tau,$$

$\phi \in [0, \pi], \tau \in [0, 2\pi]$. The *quaternionic phase* is

$$\theta(\underline{x}) := \arctan \frac{\sqrt{f_1^2(\underline{x}) + f_2^2(\underline{x}) + f_3^2(\underline{x})}}{f_0(\underline{x})} \in [0, \pi].$$

Note that some researchers in [14,15] study monogenic signals f_M of the form

$$\begin{aligned} f_M(\underline{x}) :&= f_0(\underline{x}) + \mathbf{i}f_1(\underline{x}) + \mathbf{j}f_2(\underline{x}) \\ &= |f(\underline{x})|e^{\underline{u}_M(\underline{x})\theta_M(\underline{x})}, \end{aligned}$$

where $|f(\underline{x})| := \sqrt{f_0^2(\underline{x}) + f_1^2(\underline{x}) + f_2^2(\underline{x})}$ is the *amplitude*, $\theta_M(\underline{x}) := \arctan \frac{\sqrt{f_1^2(\underline{x})+f_2^2(\underline{x})}}{f_0(\underline{x})}$ is the *phase* and $\underline{u}_M(\underline{x}) := \mathbf{i}\cos\phi + \mathbf{j}\sin\phi$ is considered as the *orientation.*

Let the inner product of $f(\underline{x}), g(\underline{x}) \in L^2(\mathbf{R}^2, \mathcal{H})$ be defined by

$$< f(\underline{x}), g(\underline{x}) >:= \int_{\mathbf{R}^2} f(\underline{x})\overline{g(\underline{x})}d\underline{x}.$$

Clearly, $\|f\|_{L^2}^2 =< f, f >$.

3. Quaternion Fourier Transforms

The quaternion Fourier transform (QFT) is an extension of Fourier transform proposed by Ell [13]. Due to the non-commutative properties of quaternions, there are three different types of QFT, the left-sided QFT, the right-sided QFT and the two-sided QFT [33]. In this paper we only treat the right-sided QFT, the left-sided is similar. We now review the definition and some properties of the right-sided QFT ([4,20]).

Definition 3.1. *If $f \in L^1(\mathbf{R}^2, \mathcal{H})$, the quaternion Fourier transform (QFT) of f is defined by*

$$F\{f\}(\underline{\xi}) = \frac{1}{2\pi}\int_{\mathbf{R}^2} f(\underline{x})e^{-\mathbf{i}x_1\xi_1}e^{-\mathbf{j}x_2\xi_2}d\underline{x}$$

and if in addition, $F\{f\} \in L^1(\mathbf{R}^2, \mathcal{H})$, the inverse Fourier transform is defined by

$$f(\underline{x}) = \frac{1}{2\pi}\int_{\mathbf{R}^2} F\{f\}(\underline{\xi})e^{\mathbf{j}x_2\xi_2}e^{\mathbf{i}x_1\xi_1}d\underline{\xi}.$$

Lemma 3.1. ([4]) (Plancherel Theorem for QFT) *If $f, g \in L^2(\mathbf{R}^2, \mathcal{H})$, then*

$$< f, g >=< F\{f\}, F\{g\} > .$$

In particular, with $f = g$, we get the Parseval theorem, i.e.

$$\|f\|^2 = \|F\{f\}\|^2.$$

Lemma 3.2. ([4]) *If $f \in L^1 \bigcap L^2(\mathbf{R}^2, \mathcal{H})$ and for $k = 1, 2$, $\frac{\partial}{\partial x_k}f$ exists and is also in $L^2(\mathbf{R}^2, \mathcal{H})$, then*

$$\int_{\mathbf{R}^2} \xi_k^2|F\{f\}(\underline{\xi})|^2d\underline{\xi} = \int_{\mathbf{R}^2}\left|\frac{\partial}{\partial x_k}f(\underline{x})\right|^2 d\underline{x}. \tag{3.1}$$

4. Uncertainty Principles

In this section, we will give two uncertainty relations in terms of absolute covariance. We need the following technical Lemmas.

Lemma 4.1. ([42]) *For any quaternion signal* $f(\underline{x}) = \rho(\underline{x})e^{\underline{u}(\underline{x})\theta(\underline{x})}$, *if* $\frac{\partial \underline{u}}{\partial x_k}$ *and* $\frac{\partial \theta}{\partial x_k}$ *exists for* $k = 1, 2$, *then the scalar part of*

$$\left[\frac{\partial}{\partial x_k} e^{\underline{u}(\underline{x})\theta(\underline{x})}\right]\left[e^{-\underline{u}(\underline{x})\theta(\underline{x})}\right]$$

is zero.

Proof. The proof is given in [42]. To make the paper self-containing, we cite the proof here as well.

By the generalized Euler formula of quaternion $e^{\underline{u}(\underline{x})\theta(\underline{x})} = \cos\theta(\underline{x}) + \underline{u}(\underline{x})\sin\theta(\underline{x})$, we have

$$\begin{aligned}
&\frac{\partial}{\partial x_k}\left(e^{\underline{u}(\underline{x})\theta(\underline{x})}\right)\left(e^{-\underline{u}(\underline{x})\theta(\underline{x})}\right) \\
&= \frac{\partial}{\partial x_k}\left[\cos\theta(\underline{x}) + \underline{u}\sin\theta(\underline{x})\right]\left[\cos\theta(\underline{x}) - \underline{u}\sin\theta(\underline{x})\right] \\
&= \left[-\sin\theta(\underline{x})\frac{\partial\theta(\underline{x})}{\partial x_k} + \frac{\partial \underline{u}}{\partial x_k}\sin\theta(\underline{x}) + \underline{u}\cos\theta(\underline{x})\frac{\partial\theta(\underline{x})}{\partial x_k}\right]\left[\cos\theta(\underline{x}) - \underline{u}\sin\theta(\underline{x})\right] \\
&= \underline{u}(\underline{x})\frac{\partial\theta(\underline{x})}{\partial x_k} + \sin\theta(\underline{x})\cos\theta(\underline{x})\frac{\partial \underline{u}}{\partial x_k} - \sin^2\theta(\underline{x})\frac{\partial \underline{u}(\underline{x})}{\partial x_k}\underline{u}(\underline{x}) \qquad (4.1)
\end{aligned}$$

Clearly, the scalar part of

$$\frac{\partial}{\partial x_k}\left(e^{\underline{u}(\underline{x})\theta(\underline{x})}\right)\left(e^{-\underline{u}(\underline{x})\theta(\underline{x})}\right)$$

is decided by the third part of the formula (4.1). Now we prove it is zero.

For $\underline{u}(\underline{x}) \in S^2$, we have $[\underline{u}(\underline{x})]^2 = -1$. Therefore, we obtain

$$\begin{aligned}
\frac{\partial[\underline{u}(\underline{x})]^2}{\partial x_k} &= \frac{\partial \underline{u}(\underline{x})}{\partial x_k}\underline{u}(\underline{x}) + \underline{u}(\underline{x})\frac{\partial \underline{u}(\underline{x})}{\partial x_k} \\
&= 2\mathrm{Sc}\left[\frac{\partial \underline{u}(\underline{x})}{\partial x_k}\underline{u}(\underline{x})\right] \qquad (4.2) \\
&= 0.
\end{aligned}$$

This completes the proof. □

Remark 4.1. *In one dimensional cases, for signal* $f(x) = \rho(x)e^{\mathbf{i}\theta(x)}$, *it is easy to see that*

$$\left(\frac{\partial}{\partial x}e^{\mathbf{i}\theta(x)}\right)e^{-\mathbf{i}\theta(x)} = \mathbf{i}\theta'(x).$$

Lemma 4.2. *For any quaternion signal* $f(\underline{x}) = \rho(\underline{x})e^{\underline{u}(\underline{x})\theta(\underline{x})}$, *if* $\frac{\partial}{\partial x_k}f(\underline{x})$ *exists for* $k = 1, 2$, *then*

$$\left|\frac{\partial}{\partial x_k}f(\underline{x})\right|^2 = \left[\frac{\partial}{\partial x_k}\rho(\underline{x})\right]^2 + \rho^2(\underline{x})\left|\mathrm{NSc}\left[\left(\frac{\partial}{\partial x_k}e^{\underline{u}(\underline{x})\theta(\underline{x})}\right)\left(e^{-\underline{u}(\underline{x})\theta(\underline{x})}\right)\right]\right|^2. \qquad (4.3)$$

Proof. For $f(\underline{x}) = \rho(\underline{x})e^{\underline{u}(\underline{x})\theta(\underline{x})}$, we have

$$\frac{\partial}{\partial x_k} f(\underline{x}) = \frac{\partial}{\partial x_k}[\rho(\underline{x}) e^{\underline{u}(\underline{x})\theta(\underline{x})}]$$
$$= \left(\frac{\partial}{\partial x_k}\rho(\underline{x})\right) e^{\underline{u}(\underline{x})\theta(\underline{x})} + \rho(\underline{x})\frac{\partial}{\partial x_k}(e^{\underline{u}(\underline{x})\theta(\underline{x})}).$$

Therefore,

$$|\frac{\partial}{\partial x_k} f(\underline{x})|^2$$
$$= \frac{\partial}{\partial x_k} f(\underline{x}) \overline{\frac{\partial}{\partial x_k} f(\underline{x})}$$
$$= \left[\left(\frac{\partial}{\partial x_k}\rho\right) e^{\underline{u}\theta} + \rho\frac{\partial}{\partial x_k}(e^{\underline{u}\theta})\right]\left[\left(\frac{\partial}{\partial x_k}\rho\right) e^{-\underline{u}\theta} + \rho\overline{\frac{\partial}{\partial x_k}(e^{\underline{u}\theta})}\right]$$
$$= \left(\frac{\partial}{\partial x_k}\rho\right)^2 + \rho^2\frac{\partial}{\partial x_k}(e^{\underline{u}\theta})\overline{\frac{\partial}{\partial x_k}(e^{\underline{u}\theta})}$$
$$+\rho\left(\frac{\partial}{\partial x_k}\rho\right)\left[\frac{\partial}{\partial x_k}(e^{\underline{u}\theta})e^{-\underline{u}\theta} + \overline{\frac{\partial}{\partial x_k}(e^{\underline{u}\theta})e^{-\underline{u}\theta}}\right].$$

By Lemma 4.1, we have

$$\frac{\partial}{\partial x_k}(e^{\underline{u}\theta})e^{-\underline{u}\theta} + \overline{\frac{\partial}{\partial x_k}(e^{\underline{u}\theta})e^{-\underline{u}\theta}} = 0$$

and

$$\rho^2\frac{\partial}{\partial x_k}(e^{\underline{u}\theta})\overline{\frac{\partial}{\partial x_k}(e^{\underline{u}\theta})} = \rho^2\frac{\partial}{\partial x_k}(e^{\underline{u}\theta})e^{-\underline{u}\theta}\overline{\frac{\partial}{\partial x_k}(e^{\underline{u}\theta})e^{-\underline{u}\theta}}$$
$$= \rho^2\left|\frac{\partial}{\partial x_k}(e^{\underline{u}\theta})e^{-\underline{u}\theta}\right|^2$$
$$= \rho^2\left|\mathrm{NSc}\left[\left(\frac{\partial}{\partial x_k}e^{\underline{u}\theta}\right)e^{-\underline{u}\theta}\right]\right|^2.$$

This completes the proof. □

Clearly, using (3.1) and (4.3), we have

Theorem 4.1. *For any quaternion signal* $f(\underline{x}) = \rho(\underline{x})e^{\underline{u}(\underline{x})\theta(\underline{x})}$, *if* $f \in L^1 \bigcap L^2(\mathbf{R}^2, \mathcal{H})$, *and for* $k = 1, 2$, $\frac{\partial}{\partial x_k} f$ *exists and is also in* $L^2(\mathbf{R}^2, \mathcal{H})$, *then*

$$\int_{\mathbf{R}^2} \xi_k^2 |F\{f\}(\underline{\xi})|^2 d\underline{\xi} = \int_{\mathbf{R}^2}\left[\frac{\partial}{\partial x_k}\rho(\underline{x})\right]^2 d\underline{x}$$
$$+ \int_{\mathbf{R}^2}\rho^2(\underline{x})\left|\mathrm{NSc}\left[\left(\frac{\partial}{\partial x_k}e^{\underline{u}(\underline{x})\theta(\underline{x})}\right)\left(e^{-\underline{u}(\underline{x})\theta(\underline{x})}\right)\right]\right|^2 d\underline{x}. \tag{4.4}$$

Remark 4.2. *(4.4) is an effective formula to compute* $\int_{\mathbf{R}^2}\xi_k^2|F\{f\}(\underline{\xi})|^2 d\underline{\xi}$. *Using this formula, we can avoid computing the Fourier transform of* $f(\underline{x})$. *Due to the non-commutative property of quaternions, it is complicated to compute the Fourier transforms of quaternion-valued signals.*

Due to Remark 4.1, in the complex case we have ([8])*:*

$$\sigma_\omega^2 = \int_{-\infty}^{\infty} \rho'^2(x)dx + \int_{-\infty}^{\infty} \rho^2(x)\theta'^2(x)dx.$$

Theorem 4.2. (Uncertainty Principle in spatial case)
Let $f(\underline{x}) = |f(\underline{x})|e^{\underline{u}(\underline{x})\theta(\underline{x})}$*. If* $f(\underline{x}) \in L^2(\mathbf{R}^2, \mathcal{H})$, $x_k f(\underline{x}), \frac{\partial}{\partial x_k} f(\underline{x}) \in L^2(\mathbf{R}^2, \mathcal{H})$, $k = 1, 2$ *and* $\|f\|_{L^2} = 1$*, then*

$$\left(\int_{\mathbf{R}^2} x_k^2 |f(\underline{x})|^2 d\underline{x}\right)\left(\int_{\mathbf{R}^2} \xi_k^2 |F\{f\}(\underline{\xi})|^2 d\underline{\xi}\right) \geq \frac{1}{4} + \mathrm{COV}^2_{x_k\xi_k}, \tag{4.5}$$

where the absolute covariance

$$\mathrm{COV}_{x_k\xi_k} := \int_{\mathbf{R}^2} \left| x_k \mathrm{NSc}\left[\left(\frac{\partial}{\partial x_k} e^{\underline{u}(\underline{x})\theta(\underline{x})}\right) e^{-\underline{u}(\underline{x})\theta(\underline{x})}\right]\right| \rho^2(\underline{x})d\underline{x}.$$

The equality (4.5) *holds if and only if* $f(\underline{x}) = e^{-\frac{\alpha_1}{2}x_1^2 - \frac{\alpha_2}{2}x_2^2} e^{\underline{u}(\underline{x})\theta(\underline{x})}$ *and* $(\frac{\partial}{\partial x_k} e^{\underline{u}(\underline{x})\theta(\underline{x})})e^{-\underline{u}(\underline{x})\theta(\underline{x})} = \underline{\beta_k} x_k$*. Here* $\alpha_1, \alpha_2 > 0$ *and* $\underline{\beta_1}, \underline{\beta_2}$ *are pure quaternions.*

Proof. Applying formula (3.1) and (4.4), we have

$$\begin{aligned}
&\left(\int_{\mathbf{R}^2} x_k^2 |f(\underline{x})|^2 d\underline{x}\right)\left(\int_{\mathbf{R}^2} \xi_k^2 |F\{f\}(\underline{\xi})|^2 d\underline{\xi}\right) = \left(\int_{\mathbf{R}^2} x_k^2 \rho^2 d\underline{x}\right) \\
&\quad \times \left(\int_{\mathbf{R}^2} \left[\frac{\partial}{\partial x_k}\rho(\underline{x})\right]^2 d\underline{x} + \int_{\mathbf{R}^2} \rho^2 \left|\mathrm{NSc}\left[\left(\frac{\partial}{\partial \mathrm{x_k}} \mathrm{e}^{\underline{\mathrm{u}}(\underline{\mathrm{x}})\theta(\underline{\mathrm{x}})}\right)\left(\mathrm{e}^{-\underline{\mathrm{u}}(\underline{\mathrm{x}})\theta(\underline{\mathrm{x}})}\right)\right]\right|^2 d\underline{x}\right) \\
&= \left(\int_{\mathbf{R}^2} x_k^2 \rho^2 d\underline{x}\right)\left(\int_{\mathbf{R}^2} \left[\frac{\partial}{\partial x_k}\rho(\underline{x})\right]^2 d\underline{x}\right) \\
&\quad + \left(\int_{\mathbf{R}^2} x_k^2 \rho^2 d\underline{x}\right)\left(\int_{\mathbf{R}^2} \rho^2(\underline{x}) \left|\mathrm{NSc}\left[\left(\frac{\partial}{\partial x_k} e^{\underline{u}(\underline{x})\theta(\underline{x})}\right)\left(e^{-\underline{u}(\underline{x})\theta(\underline{x})}\right)\right]\right|^2 d\underline{x}\right)
\end{aligned} \tag{4.6}$$

Using Hölder inequality, we have

$$\begin{aligned}
&\left(\int_{\mathbf{R}^2} x_k^2 \rho^2 d\underline{x}\right)\left(\int_{\mathbf{R}^2} \left[\frac{\partial}{\partial x_k}\rho(\underline{x})\right]^2 d\underline{x}\right) \\
&\geq \left(\int_{\mathbf{R}^2} \left| x_k \rho[\frac{\partial}{\partial x_k}\rho(\underline{x})]\right| d\underline{x}\right)^2 \\
&\geq \left|\int_{\mathbf{R}^2} x_k \rho[\frac{\partial}{\partial x_k}\rho(\underline{x})] d\underline{x}\right|^2 \\
&= \left|\int_{\mathbf{R}^2} \frac{1}{2}\frac{\partial}{\partial x_k}\left(\rho^2 x_k\right) d\underline{x} - \int_{\mathbf{R}^2} \frac{1}{2}\rho^2 d\underline{x}\right|^2 \\
&= \frac{1}{4}.
\end{aligned} \tag{4.7}$$

The first term of (4.7) is a perfect differential and integrates to zero. The second term gives one half since we assume the signal is unit energy.

Similarly, we have

$$\left(\int_{\mathbf{R}^2} x_k^2\rho^2 d\underline{x}\right)\left(\int_{\mathbf{R}^2}\rho^2(\underline{x})\left|\mathrm{NSc}\left[\left(\frac{\partial}{\partial x_k}e^{\underline{u}(\underline{x})\theta(\underline{x})}\right)\left(e^{-\underline{u}(\underline{x})\theta(\underline{x})}\right)\right]\right|^2 d\underline{x}\right)$$
$$\geq\left(\int_{\mathbf{R}^2}\left|x_k\mathrm{NSc}\left[\left(\frac{\partial}{\partial x_k}e^{\underline{u}(\underline{x})\theta(\underline{x})}\right)\left(e^{-\underline{u}(\underline{x})\theta(\underline{x})}\right)\right]\right|\rho^2 d\underline{x}\right)^2$$
$$=\mathrm{COV}^2_{x_k\xi_k}. \tag{4.8}$$

connecting (4.7), (4.8) and (4.6), the inequality (4.5) holds.

Next we deduce the conditions under which the equation holds in (4.5). The equation in (4.7) holds if and only if $\frac{\partial}{\partial x_k}\rho(\underline{x})=\pm\alpha_k x_k\rho(\underline{x})$, where $\alpha_k>0$. That is $\rho(\underline{x})=e^{\pm\frac{\alpha_k}{2}x_k^2}$. For $f(\underline{x})\in L^2(\mathbf{R}^2)$, then we choose $\rho(\underline{x})=e^{-\frac{\alpha_k}{2}x_k^2}$.

Clearly, the equation holds in (4.8) if and only if

$$\mathrm{NSc}\left[\left(\frac{\partial}{\partial x_k}e^{\underline{u}(\underline{x})\theta(\underline{x})}\right)e^{-\underline{u}(\underline{x})\theta(\underline{x})}\right]$$
$$=\left(\frac{\partial}{\partial x_k}e^{\underline{u}(\underline{x})\theta(\underline{x})}\right)e^{-\underline{u}(\underline{x})\theta(\underline{x})} \tag{4.9}$$
$$=\underline{\beta_k}x_k.$$

Lemma 4.1 is used in the first equation of (4.9). This completes the proof. □

Corollary 4.1. ([42]) *Let $f(\underline{x})=|f(\underline{x})|e^{\underline{u}(\underline{x})\theta(\underline{x})}$. If $f(\underline{x})\in L^2(\mathbf{R}^2,\mathcal{H})$, $x_kf(\underline{x})$, $\frac{\partial}{\partial x_k}f(\underline{x})\in L^2(\mathbf{R}^2,\mathcal{H})$, for $k=1,2$ and $\|f\|_{L^2}=1$, then*

$$\left(\int_{\mathbf{R}^2}x_k^2|f(\underline{x})|^2d\underline{x}\right)\left(\int_{\mathbf{R}^2}\xi_k^2|F\{f\}(\underline{\xi})|^2d\underline{\xi}\right)$$
$$\geq\frac{1}{4}+\left|\mathrm{Cov}_{x_k\xi_k}\right|^2,$$

where the covariance

$$\mathrm{Cov}_{x_k\xi_k}:=\int_{\mathbf{R}^2}\mathrm{NSc}\left[\left(\frac{\partial}{\partial x_k}e^{\underline{u}(\underline{x})\theta(\underline{x})}\right)e^{-\underline{u}(\underline{x})\theta(\underline{x})}\right]\rho^2(\underline{x})x_kd\underline{x}.$$

Theorem 4.3. (Uncertainty Principle in directional case)

Let $f(\underline{x})=|f(\underline{x})|e^{\underline{u}(\underline{x})\theta(\underline{x})}$. If $f(\underline{x})\in L^2(\mathbf{R}^2,\mathcal{H}), x_kf(\underline{x}), \frac{\partial}{\partial x_k}f(\underline{x})\in L^2(\mathbf{R}^2,\mathcal{H}), k=1,2$ and $\|f\|_{L^2}=1$, then

$$\left(\int_{\mathbf{R}^2}|\underline{x}|^2|f(\underline{x})|^2d\underline{x}\right)\left(\int_{\mathbf{R}^2}|\underline{\xi}|^2|F\{f\}(\underline{\xi})|^2d\underline{\xi}\right)$$
$$\geq 1+\mathrm{COV}^2_{\underline{x}\underline{\xi}}, \tag{4.10}$$

where the absolute covariance is

$$\mathrm{COV}_{\underline{x}\underline{\xi}}:=\sum_{k=1}^{2}\int_{\mathbf{R}^2}\left|x_k\mathrm{NSc}\left[\left(\frac{\partial}{\partial x_k}e^{\underline{u}\theta}\right)e^{-\underline{u}\theta}\right]\right|\rho^2(\underline{x})d\underline{x}.$$

The equality (4.10) holds if and only if $f(\underline{x})=e^{-\frac{\alpha}{2}|\underline{x}|^2}e^{\underline{u}(\underline{x})\theta(\underline{x})}$ and $(\frac{\partial}{\partial x_k}e^{\underline{u}(\underline{x})\theta(\underline{x})})e^{-\underline{u}(\underline{x})\theta(\underline{x})}=\underline{\beta}x_k$. Here $\alpha>0$ and $\underline{\beta}$ are pure quaternion.

Proof. Applying (3.1) and (4.3), we have

$$\begin{aligned}
&\left(\int_{\mathbf{R}^2} |\underline{x}|^2 |f(\underline{x})|^2 d\underline{x}\right)\left(\int_{\mathbf{R}^2} |\underline{\xi}|^2 |F\{f\}(\underline{\xi})|^2 d\underline{\xi}\right)\\
&\quad = \left(\int_{\mathbf{R}^2} \sum_{k=1}^{2} x_k^2 \rho^2 d\underline{x}\right)\left(\int_{\mathbf{R}^2} \sum_{k=1}^{2} \xi_k^2 |F\{f\}(\underline{\xi})|^2 d\underline{\xi}\right)\\
&\quad = \left(\int_{\mathbf{R}^2} \sum_{k=1}^{2} x_k^2 \rho^2 d\underline{x}\right)\left(\int_{\mathbf{R}^2} \sum_{k=1}^{2} \left|\frac{\partial}{\partial x_k} f(\underline{x})\right|^2 d\underline{x}\right) = \left(\int_{\mathbf{R}^2} \sum_{k=1}^{2} x_k^2 \rho^2 d\underline{x}\right)\\
&\qquad \times \left(\int_{\mathbf{R}^2} \sum_{k=1}^{2} \left[\frac{\partial}{\partial x_k}\rho(\underline{x})\right]^2 + \rho^2(\underline{x}) \left|\mathrm{NSc}\left[\left(\frac{\partial}{\partial x_k} e^{\underline{u}(\underline{x})\theta(\underline{x})}\right)\left(e^{-\underline{u}(\underline{x})\theta(\underline{x})}\right)\right]\right|^2 d\underline{x}\right)\\
&\quad = \left(\int_{\mathbf{R}^2} \sum_{k=1}^{2} x_k^2 \rho^2 d\underline{x}\right)\left(\int_{\mathbf{R}^2} \sum_{k=1}^{2} \left[\frac{\partial}{\partial x_k}\rho(\underline{x})\right]^2 d\underline{x}\right)\\
&\qquad + \left(\int_{\mathbf{R}^2} \sum_{k=1}^{2} x_k^2 \rho^2 d\underline{x}\right)\left(\int_{\mathbf{R}^2} \sum_{k=1}^{2} \rho^2(\underline{x}) \left|\mathrm{NSc}\left[\left(\frac{\partial}{\partial x_k} e^{\underline{u}(\underline{x})\theta(\underline{x})}\right)\left(e^{-\underline{u}(\underline{x})\theta(\underline{x})}\right)\right]\right|^2 d\underline{x}\right).
\end{aligned}$$

Applying the Schwarz inequality of continuous and discrete cases, we have

$$\begin{aligned}
&\left(\int_{\mathbf{R}^2} \sum_{k=1}^{2} x_k^2 \rho^2 d\underline{x}\right)\left(\int_{\mathbf{R}^2} \sum_{k=1}^{2} \left[\frac{\partial}{\partial x_k}\rho(\underline{x})\right]^2 d\underline{x}\right)\\
&\quad \geq \left|\int_{\mathbf{R}^2} \left(\sum_{k=1}^{2} x_k^2 \rho^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^{2} (\frac{\partial}{\partial x_k}\rho)^2\right)^{\frac{1}{2}} d\underline{x}\right|^2\\
&\quad \geq \left|\int_{\mathbf{R}^2} \sum_{k=1}^{2} \left(\frac{\partial}{\partial x_k}\rho\right) x_k \rho d\underline{x}\right|^2\\
&\quad = 1.
\end{aligned}$$

(4.7) is used in the last step. Similarly, we have

$$\begin{aligned}
&\left(\int_{\mathbf{R}^2} \sum_{k=1}^{2} x_k^2 \rho^2 d\underline{x}\right)\left(\int_{\mathbf{R}^2} \sum_{k=1}^{2} \rho^2(\underline{x}) \left|\mathrm{NSc}\left[\left(\frac{\partial}{\partial x_k} e^{\underline{u}(\underline{x})\theta(\underline{x})}\right)\left(e^{-\underline{u}(\underline{x})\theta(\underline{x})}\right)\right]\right|^2 d\underline{x}\right)\\
&\quad \geq \left(\sum_{k=1}^{2} \int_{\mathbf{R}^2} \left|x_k \mathrm{NSc}\left[\left(\frac{\partial}{\partial x_k} e^{\underline{u}\theta}\right) e^{-\underline{u}\theta}\right]\right| \rho^2 d\underline{x}\right)^2.
\end{aligned}$$

Similarly, like Theorem 4.2, the equality (4.10) holds if and only if $f(\underline{x}) = e^{-\frac{\alpha}{2}|\underline{x}|^2} e^{\underline{u}(\underline{x})\theta(\underline{x})}$ and $(\frac{\partial}{\partial x_k} e^{\underline{u}(\underline{x})\theta(\underline{x})}) e^{-\underline{u}(\underline{x})\theta(\underline{x})} = \underline{\beta} x_k$. Here $\alpha > 0$ and $\underline{\beta}$ is a pure quaternion. This completes the proof. □

Corollary 4.2. ([42]) *Let* $f(\underline{x}) = |f(\underline{x})| e^{\underline{u}(\underline{x})\theta(\underline{x})}$. *If* $f(\underline{x}) \in L^2(\mathbf{R}^2, \mathcal{H})$, $x_k f(\underline{x})$, $\frac{\partial}{\partial x_k} f(\underline{x}) \in L^2(\mathbf{R}^2, \mathcal{H})$, *for* $k = 1, 2$, *then*

$$\begin{aligned}
&\left(\int_{\mathbf{R}^2} |\underline{x}|^2 |f(\underline{x})|^2 d\underline{x}\right)\left(\int_{\mathbf{R}^2} |\underline{\xi}|^2 |F\{f\}(\underline{\xi})|^2 d\underline{\xi}\right)\\
&\quad \geq 1 + \left|\mathrm{Cov}_{\underline{x}\underline{\xi}}\right|^2,
\end{aligned}$$

where the covariance is

$$\mathrm{Cov}_{\underline{x}\underline{\xi}} := \sum_{k=1}^{2} \int_{\mathbf{R}^2} \mathrm{NSc}\left[\left(\frac{\partial}{\partial x_k} e^{\underline{u}\theta}\right) e^{-\underline{u}\theta}\right] \rho^2(\underline{x}) x_k d\underline{x}.$$

5. Examples

Example 5.1. Consider a quaternionic valued signal of unit energy

$$f(\underline{x}) = \sqrt{\frac{\alpha}{\pi}} e^{\frac{-\alpha|\underline{x}|^2}{2}} e^{\underline{u}\frac{|\underline{x}|^2}{2}},$$

where α is a positive real number and $\underline{u} \in S^2$ is a pure quaternionic constant.

Computing directly, we have

$$\int_{\mathbf{R}^2} x_k^2 |f(\underline{x})|^2 d\underline{x} = \frac{\alpha}{\pi} \int_{\mathbf{R}^2} x_k^2 e^{-\alpha|\underline{x}|^2} d\underline{x} = \frac{1}{2\alpha},$$

and

$$\int_{\mathbf{R}^2} \xi_k^2 |F\{f\}(\underline{\xi})|^2 d\underline{\xi} = \int_{\mathbf{R}^2} \left(\frac{\partial}{\partial x_k}\rho\right)^2 d\underline{x} + \int_{\mathbf{R}^2} \rho^2 \left|\mathrm{NSc}\left[\left(\frac{\partial}{\partial x_k} e^{\underline{u}\theta}\right) e^{-\underline{u}\theta}\right]\right|^2 d\underline{x}$$
$$= \frac{\alpha^3}{\pi} \int_{\mathbf{R}^2} x_k^2 e^{-\alpha|\underline{x}|^2} d\underline{x} + \frac{\alpha}{\pi} \int_{\mathbf{R}^2} x_k^2 e^{-\alpha|\underline{x}|^2} d\underline{x}$$
$$= \frac{\alpha}{2} + \frac{1}{2\alpha}.$$

It is easy to see that $\mathrm{Cov}_{x_k\xi_k} = \frac{\underline{u}}{2\alpha}$, $\mathrm{COV}_{x_k\xi_k} = \frac{1}{2\alpha}$, $k = 1, 2$. Then

$$\left(\int_{\mathbf{R}^2} x_k^2 |f(\underline{x})|^2 d\underline{x}\right)\left(\int_{\mathbf{R}^2} \xi_k^2 |F\{f\}(\underline{\xi})|^2 d\underline{\xi}\right)$$
$$= \frac{1}{4} + \frac{1}{4\alpha^2}$$
$$> \frac{1}{4} + \left(\frac{1}{2\alpha}\right)^2 = \frac{1}{4} + \mathrm{COV}_{x_1\xi_1}^2$$
$$> \frac{1}{4} + \left|\frac{\underline{u}}{2\alpha}\right|^2 = \frac{1}{4} + |\mathrm{Cov}_{x_1\xi_1}|^2$$

and

$$\left(\int_{\mathbf{R}^2} |\underline{x}|^2 |f(\underline{x})|^2 d\underline{x}\right)\left(\int_{\mathbf{R}^2} |\underline{\xi}|^2 |F\{f\}(\underline{\xi})|^2 d\underline{\xi}\right)$$
$$= 1 + \frac{1}{\alpha^2}$$
$$= 1 + \left(\frac{1}{2\alpha} + \frac{1}{2\alpha}\right)^2 = 1 + \mathrm{COV}_{\underline{x}\underline{\xi}}^2$$
$$= 1 + \left|\frac{\underline{u}}{2\alpha} + \frac{\underline{u}}{2\alpha}\right|^2 = 1 + \left|\mathrm{Cov}_{\underline{x}\underline{\xi}}\right|^2.$$

Note that, in this case, the stronger forms of uncertainty principle of Theorems 4.2 and 4.3 become equalities. In fact, $(\frac{\partial}{\partial x_k}e^{\underline{u}(\underline{x})\theta(\underline{x})})e^{-\underline{u}(\underline{x})\theta(\underline{x})} = \underline{u}x_k$, which satisfies the conditions as given in (4.5) and (4.10).

Example 5.2. Consider a quaternionic valued signal of unit energy

$$f(\underline{x}) = \sqrt{\frac{\alpha}{\pi}}e^{\frac{-\alpha|\underline{x}|^2}{2}}e^{\mathbf{j}\frac{\beta_2 x_2^2}{2}}e^{\mathbf{i}\beta_1 x_1},$$

where α is a positive real number and $\beta_1, \beta_2 \in \mathbf{R}$.

By Example 5.1, we have

$$\begin{aligned}\int_{\mathbf{R}^2} x_k^2|f(\underline{x})|^2 d\underline{x} &= \frac{\alpha}{\pi}\int_{\mathbf{R}^2} x_k^2 e^{-\alpha|\underline{x}|^2}d\underline{x} \\ &= \frac{1}{2\alpha}, \ \ k=1,2. \end{aligned} \tag{5.1}$$

By direct calculation, we have

$$\begin{aligned}\int_{\mathbf{R}^2} \xi_1^2|F\{f\}(\underline{\xi})|^2 d\underline{\xi} &= \int_{\mathbf{R}^2}\left(\frac{\partial}{\partial x_1}\rho\right)^2 d\underline{x} + \int_{\mathbf{R}^2}\rho^2\left|\mathrm{NSc}\left[\left(\frac{\partial}{\partial x_1}e^{\underline{u}\theta}\right)e^{-\underline{u}\theta}\right]\right|^2 d\underline{x} \\ &= \frac{\alpha^3}{\pi}\int_{\mathbf{R}^2} x_1^2 e^{-\alpha|\underline{x}|^2}d\underline{x} + \frac{\alpha\beta_1^2}{\pi}\int_{\mathbf{R}^2} e^{-\alpha|\underline{x}|^2}d\underline{x} \\ &= \frac{\alpha}{2}+\beta_1^2, \end{aligned} \tag{5.2}$$

and

$$\begin{aligned}\int_{\mathbf{R}^2} \xi_2^2|F\{f\}(\underline{\xi})|^2 d\underline{\xi} &= \int_{\mathbf{R}^2}\left(\frac{\partial}{\partial x_2}\rho\right)^2 d\underline{x} + \int_{\mathbf{R}^2}\rho^2\left|\mathrm{NSc}\left[\left(\frac{\partial}{\partial x_2}e^{\underline{u}\theta}\right)e^{-\underline{u}\theta}\right]\right|^2 d\underline{x} \\ &= \frac{\alpha^3}{\pi}\int_{\mathbf{R}^2} x_2^2 e^{-\alpha|\underline{x}|^2}d\underline{x} + \frac{\alpha\beta_2^2}{\pi}\int_{\mathbf{R}^2} x_2^2 e^{-\alpha|\underline{x}|^2}d\underline{x} \\ &= \frac{\alpha}{2}+\frac{\beta_2^2}{2\alpha}. \end{aligned} \tag{5.3}$$

Clearly, we have $\mathrm{Cov}_{x_1\xi_1} = 0$, $\mathrm{Cov}_{x_2\xi_2} = \mathbf{j}\frac{\beta_2}{2\alpha}$ and $\mathrm{COV}_{x_1\xi_1} = \frac{|\beta_1|}{\sqrt{\pi\alpha}}$, $\mathrm{COV}_{x_2\xi_2} = \frac{|\beta_2|}{2\alpha}$. Therefore,

$$\begin{aligned}&\left(\int_{\mathbf{R}^2} x_1^2|f(\underline{x})|^2 d\underline{x}\right)\left(\int_{\mathbf{R}^2} \xi_1^2|F\{f\}(\underline{\xi})|^2 d\underline{\xi}\right) \\ &= \frac{1}{4}+\frac{\beta_1^2}{2\alpha} \\ &> \frac{1}{4}+\frac{\beta_1^2}{\pi\alpha} = \frac{1}{4}+\mathrm{COV}_{x_1\xi_1}^2 \\ &> \frac{1}{4} = \frac{1}{4}+|\mathrm{Cov}_{x_1\xi_1}|^2 \end{aligned} \tag{5.4}$$

and

$$\begin{aligned}&\left(\int_{\mathbf{R}^2} x_2^2|f(\underline{x})|^2 d\underline{x}\right)\left(\int_{\mathbf{R}^2} \xi_2^2|F\{f\}(\underline{\xi})|^2 d\underline{\xi}\right) \\ &= \frac{1}{4}+\frac{\beta_2^2}{4\alpha^2} = \frac{1}{4}+\mathrm{COV}_{x_2\xi_2}^2 \end{aligned}$$

$$= \frac{1}{4} + |\mathrm{Cov}_{x_2\xi_2}|^2$$
$$> \frac{1}{4}. \tag{5.5}$$

Expressions (5.4) and (5.5) verify Theorem 4.2. Note that, in formula (5.5), the stronger form of uncertainty principles of Theorem 4.2 becomes equality. In fact, $(\frac{\partial}{\partial x_2} e^{\underline{u}(\underline{x})\theta(\underline{x})})e^{-\underline{u}(\underline{x})\theta(\underline{x})} = \mathbf{j}\beta_2 x_2$, which satisfies the conditions of equation (4.10) holds.

Applying (5.1) and (5.2), (5.3), we have

$$\int_{\mathbf{R}^2} |\underline{x}|^2 |f(\underline{x})|^2 d\underline{x} = \frac{1}{\alpha},$$
$$\int_{\mathbf{R}^2} |\underline{\xi}|^2 |F\{f\}(\underline{\xi})|^2 d\underline{\xi} = \alpha + \beta_1^2 + \frac{\beta_2^2}{2\alpha}.$$

Then

$$\left(\int_{\mathbf{R}^2} |\underline{x}|^2 |f(\underline{x})|^2 d\underline{x}\right)\left(\int_{\mathbf{R}^2} |\underline{\xi}|^2 |F\{f\}(\underline{\xi})|^2 d\underline{\xi}\right)$$
$$= 1 + \frac{\beta_1^2}{\alpha} + \frac{\beta_2^2}{2\alpha^2}$$
$$> 1 + \left(\frac{|\beta_1|}{\sqrt{\pi\alpha}} + \frac{|\beta_2|}{2\alpha}\right)^2 = 1 + \mathrm{COV}_{\underline{x}\underline{\xi}}^2$$
$$> 1 + \left(\frac{\beta_2}{2\alpha}\right)^2 = 1 + \left|\mathrm{Cov}_{\underline{x}\underline{\xi}}\right|^2. \tag{5.6}$$

Here $\mathrm{Cov}_{\underline{x}\underline{\xi}} = \mathbf{j}\frac{\beta_2}{2\alpha}$ and $\mathrm{COV}_{\underline{x}\underline{\xi}} = \frac{|\beta_1|}{\sqrt{\pi\alpha}} + \frac{|\beta_2|}{2\alpha}$. (5.6) verifies Theorem 4.3.

Example 5.3. *Consider a quaternionic valued signal of unit energy*

$$f(\underline{x}) = \sqrt{\frac{\alpha}{\pi}} e^{\frac{-\alpha|\underline{x}|^2}{2}} e^{\mathbf{j}\frac{\beta_2 x_2^2}{2}} e^{\mathbf{i}\left(\frac{x_1^2}{2} + \beta_1 x_1\right)},$$

where α is a positive real number and $\beta_1, \beta_2 \in \mathbf{R}$.

Clearly, by Example 5.1 and 5.2, we have

$$\int_{\mathbf{R}^2} x_k^2 |f(\underline{x})|^2 d\underline{x} = \frac{1}{2\alpha}, \qquad k = 1, 2, \tag{5.7}$$
$$\int_{\mathbf{R}^2} \xi_2^2 |F\{f\}(\underline{\xi})|^2 d\underline{\xi} = \frac{\alpha}{2} + \frac{\beta_2^2}{2\alpha}, \tag{5.8}$$

and $\mathrm{Cov}_{x_2\xi_2} = \mathbf{j}\frac{\beta_2}{2\alpha}$, $\mathrm{COV}_{x_2\xi_2} = \frac{|\beta_2|}{2\alpha}$.

So we only calculate the quantities about variables x_1 and ξ_1. In fact,

$$\int_{\mathbf{R}^2} \xi_1^2 |F\{f\}(\underline{\xi})|^2 d\underline{\xi} = \int_{\mathbf{R}^2} \left(\frac{\partial}{\partial x_1}\rho\right)^2 d\underline{x} + \int_{\mathbf{R}^2} \rho^2 \left| NSc\left[\left(\frac{\partial}{\partial x_1} e^{\underline{u}\theta}\right) e^{-\underline{u}\theta}\right]\right|^2 d\underline{x}$$
$$= \frac{\alpha^3}{\pi} \int_{\mathbf{R}^2} x_1^2 e^{-\alpha|\underline{x}|^2} d\underline{x} + \frac{\alpha}{\pi} \int_{\mathbf{R}^2} |x_1 + \beta_1|^2 e^{-\alpha|\underline{x}|^2} d\underline{x}$$
$$= \frac{\alpha}{2} + \frac{1}{2\alpha} + \beta_1^2 + \frac{2|\beta_1|}{\sqrt{\alpha\pi}}. \tag{5.9}$$

and

$$\begin{aligned}
\mathrm{Cov}_{x_1\xi_1} &= \frac{\alpha}{\pi}\int_{\mathbf{R}^2} x_1 e^{-\alpha|\underline{x}|^2} e^{\mathbf{j}\frac{\beta_2 x_2^2}{2}}\mathbf{i}(x_1+\beta_1)e^{-\mathbf{j}\frac{\beta_2 x_2^2}{2}}d\underline{x}\\
&= \frac{\alpha}{\pi}\int_{-\infty}^{\infty}(x_1^2+x_1\beta_1)e^{-\alpha x_1^2}dx_1\int_{-\infty}^{\infty} e^{-\alpha x_2^2}e^{\mathbf{j}\frac{\beta_2 x_2^2}{2}}\mathbf{i}e^{-\mathbf{j}\frac{\beta_2 x_2^2}{2}}dx_2\\
&= \frac{\alpha}{\pi}\int_{-\infty}^{\infty} x_1^2 e^{-\alpha x_1^2}dx_1\int_{-\infty}^{\infty} e^{-\alpha x_2^2}e^{\mathbf{j}\frac{\beta_2 x_2^2}{2}}\mathbf{i}e^{-\mathbf{j}\frac{\beta_2 x_2^2}{2}}dx_2\\
&= \frac{\alpha}{\pi}\int_{-\infty}^{\infty} x_1^2 e^{-\alpha x_1^2}dx_1\int_{-\infty}^{\infty} e^{-\alpha x_2^2}e^{\mathbf{j}\beta_2 x_2^2}\mathbf{i}dx_2\\
&= \frac{\alpha}{\pi}\frac{\sqrt{\pi}}{2\alpha\sqrt{\alpha}}\sqrt{\frac{\pi}{\alpha-\mathbf{j}\beta_2}}\mathbf{i}\\
&= \frac{1}{2\alpha\sqrt{1-\mathbf{j}\frac{\beta_2}{\alpha}}}\mathbf{i}, \qquad (5.10)
\end{aligned}$$

$$\begin{aligned}
\mathrm{COV}_{x_1\xi_1} &= \frac{\alpha}{\pi}\int_{\mathbf{R}^2} e^{-\alpha|\underline{x}|^2}|x_1(x_1+\beta_1)|d\underline{x}\\
&\le \frac{\alpha}{\pi}\int_{\mathbf{R}^2} x_1^2 e^{-\alpha|\underline{x}|^2}d\underline{x}+\frac{\alpha|\beta_1|}{\pi}\int_{\mathbf{R}^2}|x_1|e^{-\alpha|\underline{x}|^2}d\underline{x}\\
&= \frac{1}{2\alpha}+\frac{|\beta_1|}{\sqrt{\pi\alpha}}. \qquad (5.11)
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\left(\int_{\mathbf{R}^2} x_1^2|f(\underline{x})|^2d\underline{x}\right)\left(\int_{\mathbf{R}^2}\xi_1^2|F\{f\}(\underline{\xi})|^2d\underline{\xi}\right)\\
&= \frac{1}{4}+\frac{1}{4\alpha^2}+\frac{\beta_1^2}{2\alpha}+\frac{|\beta_1|}{\alpha\sqrt{\pi\alpha}}\\
&> \frac{1}{4}+\left(\frac{1}{2\alpha}+\frac{|\beta_1|}{\sqrt{\pi\alpha}}\right)^2 \ge \frac{1}{4}+\mathrm{COV}^2_{\mathrm{x}_1\xi_1}\\
&> \frac{1}{4}+\Big|\frac{\mathbf{i}}{2\alpha\sqrt{1+\mathbf{j}\frac{\beta_2}{\alpha}}}\Big|^2 = \frac{1}{4}+|\mathrm{Cov}_{\mathrm{x}_1\xi_1}|^2\\
&> \frac{1}{4}. \qquad (5.12)
\end{aligned}$$

(5.12) *verifies Theorem 4.2. Using* (5.7) *and* (5.8), (5.9), *we have*

$$\int_{\mathbf{R}^2}|\underline{x}|^2|f(\underline{x})|^2d\underline{x} = \frac{1}{\alpha},$$

$$\int_{\mathbf{R}^2}|\underline{\xi}|^2|F\{f\}(\underline{\xi})|^2d\underline{\xi} = \alpha+\frac{\beta_2^2}{2\alpha}+\frac{1}{2\alpha}+\beta_1^2+\frac{2|\beta_1|}{\sqrt{\pi\alpha}}.$$

Then

$$\begin{aligned}&\left(\int_{\mathbf{R}^2}|\underline{x}|^2|f(\underline{x})|^2d\underline{x}\right)\left(\int_{\mathbf{R}^2}|\underline{\xi}|^2|F\{f\}(\underline{\xi})|^2d\underline{\xi}\right)\\&=1+\frac{\beta_2^2}{2\alpha^2}+\frac{1}{2\alpha^2}+\frac{\beta_1^2}{\alpha}+\frac{2|\beta_1|}{\alpha\sqrt{\pi\alpha}}\\&>1+\left(\frac{1}{2\alpha}+\frac{|\beta_1|}{\sqrt{\pi\alpha}}+\frac{|\beta_2|}{2\alpha}\right)^2=1+\mathrm{COV}^2_{\underline{x}\underline{\xi}}\\&>1+|\mathbf{i}\frac{\beta_2}{2\alpha}+\frac{\mathbf{i}}{2\alpha\sqrt{1+\mathbf{j}\frac{\beta_2}{\alpha}}}|^2=1+\left|\mathrm{Cov}_{\underline{x}\underline{\xi}}\right|^2.\end{aligned}\tag{5.13}$$

(5.13) *verifies Theorem 4.3.*

6. Appendix

Proof of (5.6): We are to show

$$1+\frac{\beta_1^2}{\alpha}+\frac{\beta_2^2}{2\alpha^2}>1+\left(\frac{|\beta_1|}{\sqrt{\pi\alpha}}+\frac{|\beta_2|}{2\alpha}\right)^2.$$

We have the following equivalent relations:

$$\begin{aligned}&\frac{\beta_1^2}{\alpha}+\frac{\beta_2^2}{2\alpha^2}>\left(\frac{|\beta_1|}{\sqrt{\pi\alpha}}+\frac{|\beta_2|}{2\alpha}\right)^2\\&\iff\frac{\beta_1^2}{\alpha}+\frac{\beta_2^2}{2\alpha^2}>\frac{\beta_1^2}{\pi\alpha}+\frac{\beta_2^2}{2\alpha^2}+\frac{|\beta_1||\beta_2|}{\alpha\sqrt{\pi\alpha}}\\&\iff\frac{\beta_1^2}{\alpha}+\frac{\beta_2^2}{4\alpha^2}>\frac{\beta_1^2}{\pi\alpha}+\frac{|\beta_1||\beta_2|}{\alpha\sqrt{\pi\alpha}}\\&\iff\frac{(\pi-1)\beta_1^2}{\pi\alpha}+\frac{\beta_2^2}{4\alpha^2}>\frac{|\beta_1||\beta_2|}{\alpha\sqrt{\pi\alpha}}.\end{aligned}$$

It is easy to see

$$\frac{(\pi-1)\beta_1^2}{\pi\alpha}+\frac{\beta_2^2}{4\alpha^2}>\frac{\beta_1^2}{\pi\alpha}+\frac{\beta_2^2}{4\alpha^2}\geq\frac{|\beta_1||\beta_2|}{\alpha\sqrt{\pi\alpha}}.$$

This completes the proof.

Proof of (5.13): We are to show

$$1+\frac{\beta_2^2}{2\alpha^2}+\frac{1}{2\alpha^2}+\frac{\beta_1^2}{\alpha}+\frac{2|\beta_1|}{\alpha\sqrt{\pi\alpha}}>1+\left(\frac{1}{2\alpha}+\frac{|\beta_1|}{\sqrt{\pi\alpha}}+\frac{|\beta_2|}{2\alpha}\right)^2.$$

We have the following equivalent relations:

$$\begin{aligned}&\frac{\beta_2^2}{2\alpha^2}+\frac{1}{2\alpha^2}+\frac{\beta_1^2}{\alpha}+\frac{2|\beta_1|}{\alpha\sqrt{\pi\alpha}}>(\frac{1}{2\alpha}+\frac{|\beta_1|}{\sqrt{\pi\alpha}}+\frac{|\beta_2|}{2\alpha})^2\\&\iff\frac{\beta_2^2}{2\alpha^2}+\frac{1}{2\alpha^2}+\frac{\beta_1^2}{\alpha}+\frac{2|\beta_1|}{\alpha\sqrt{\pi\alpha}}>\frac{1}{4\alpha^2}+\frac{|\beta_1|}{\alpha\sqrt{\pi\alpha}}+\frac{|\beta_2|}{2\alpha^2}+\frac{\beta_1^2}{\pi\alpha}+\frac{|\beta_1||\beta_2|}{\alpha\sqrt{\pi\alpha}}+\frac{\beta_2^2}{4\alpha^2}\end{aligned}$$

$$\Longleftrightarrow \frac{\beta_2^2}{4\alpha^2}+\frac{1}{4\alpha^2}+\frac{\beta_1^2}{\alpha}+\frac{|\beta_1|}{\alpha\sqrt{\pi\alpha}} > \frac{|\beta_2|}{2\alpha^2}+\frac{\beta_1^2}{\pi\alpha}+\frac{|\beta_1||\beta_2|}{\alpha\sqrt{\pi\alpha}}$$
$$\Longleftrightarrow \frac{(1-|\beta_2|)^2}{4\alpha^2}+\frac{(\pi-1)\beta_1^2}{\pi\alpha} > \frac{|\beta_1|(|\beta_2|-1)}{\alpha\sqrt{\pi\alpha}}.$$

Clearly

$$\frac{(1-|\beta_2|)^2}{4\alpha^2}+\frac{(\pi-1)\beta_1^2}{\pi\alpha} > \frac{(1-|\beta_2|)^2}{4\alpha^2}+\frac{\beta_1^2}{\pi\alpha} \geq \frac{|\beta_1|(|\beta_2|-1)}{\alpha\sqrt{\pi\alpha}}.$$

This completes the proof.

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