Coherent State Transforms and the Weyl Equation in Clifford Analysis

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Abstract

We study a transform, inspired by coherent state transforms, from the Hilbert space of Clifford algebra valued square integrable functions $L^2(\mathbb{R}^m, dx) \otimes \mathbb{C}_m$ to a Hilbert space of solutions of the Weyl equation on $\mathbb{R}^{m+1} = \mathbb{R} \times \mathbb{R}^m$, namely to the Hilbert space $\mathcal{M}L^2(\mathbb{R}^{m+1}, d\mu)$ of $\mathbb{C}_m$-valued monogenic functions on $\mathbb{R}^{m+1}$ which are L^2 with respect to an appropriate measure $d\mu$. We prove that this transform is a unitary isomorphism of Hilbert spaces and that it is therefore an analog of the Segal-Bargmann transform for Clifford analysis. As a corollary we obtain an orthonormal basis of monogenic functions on $\mathbb{R}^{m+1}$. We also study the case when $\mathbb{R}^m$ is replaced by the m-torus $\mathbb{T}^m$. Quantum mechanically, this extension establishes the unitary equivalence of the Schrödinger representation on M, for $M = \mathbb{R}^m$ and $M = \mathbb{T}^m$, with a representation on the Hilbert space $\mathcal{M}L^2(\mathbb{R} \times M, d\mu)$ of solutions of the Weyl equation on the space-time $\mathbb{R} \times M$.

Keywords: Mathematical physics; Coherent state transforms; Clifford analysis.

Contents

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1 Introduction

In this work we continue to explore the extensions of coherent state transforms to the context of Clifford analysis initiated in [KMNQ].

Clifford analysis (see [BDS, DSS]) extends the theory of complex analysis of holomorphic functions to functions of Clifford algebra variables, obeying generalized Cauchy-Riemann conditions and called monogenic functions. In the context of the present paper, monogenic functions correspond to solutions of the Weyl equation in the euclidean space-time $\mathbb{R} \times M$, for $M = \mathbb{R}^m$ or $M = \mathbb{T}^n$. In quantum physics, Clifford algebra or spinor representation valued functions describe some systems with internal degrees of freedom, such as particles with spin. Notice that spinor valued solutions of the Dirac equation can be described by Clifford algebra valued solutions of the same equation, by decomposing the algebra in a sum of minimal left ideals. (See, for example, Chapter 2 of [DSS].)

On the other hand, the Segal-Bargmann transform [Ba, Se1, Se2], for a particle on $\mathbb{R}^m$, establishes the unitary equivalence of the Schrödinger representation with Hilbert space $L^2(\mathbb{R}^m, dx)$, with (Fock space-like) representations with Hilbert spaces, $\mathcal{H}L^2(\mathbb{C}^m, d\nu)$, of holomorphic functions on the phase space, $\mathbb{R}^{2m} \cong \mathbb{C}^m$ which are L^2 with respect to an appropriate measure $d\nu$. In the Schrödinger representation, the position operators $\hat{x}_j, j = 1, \ldots, m$, act diagonally while the momentum operator $\hat{p}_j = -i\frac{\partial}{\partial x_j}$. In the Segal-Bargmann representation, on the other hand, it is the operators $\widehat{x_j + ip_j}$ that act on the Hilbert space $\mathcal{H}L^2(\mathbb{C}^m, d\nu)$ as multiplication by the holomorphic functions $x_j + ip_j, j = 1, \ldots, m$. In [Ha1], Hall has defined coherent state transforms (CSTs) for compact Lie groups G which are analogs of the Segal-Bargmann transform.

Let $\mathbb{R}_m$ (respectively $\mathbb{C}_m$) be the real (complex) Clifford algebra with m generators, see section 2.2). In [KMNQ], we presented a generalization of the Segal-Bargmann transform to a transform taking functions in $L^2(\mathbb{R}, dx)$ to Hilbert spaces of slice or axial monogenic Clifford algebra valued functions on $\mathbb{R}^{m+1}$. The unitarity of these transforms, with respect to appropriate measures, was established. See also [DG], where a similar transform (for $m = 2$) was studied, but with range a Hilbert space of slice monogenic functions on the full quaternionic algebra $\mathbb{R}_2 = \mathbb{H}$.

In the present work, we give a different generalization of the Segal-Bargmann transform. Instead of going from functions on $\mathbb{R}$ to functions on $\mathbb{R}^{m+1} \subset \mathbb{R}_m$, as in [KMNQ], this transform adds a single time variable to functions on $\mathbb{R}^m$ and maps to solutions of the Weyl (or Cauchy-Riemann) equation on $\mathbb{R}^{m+1}$. Namely, the transforms studied in sections 3 and 4 give a unitary equivalence between Schrödinger quantization on $L^2(M, d\underline{x}) \otimes \mathbb{C}_m$ and a Hilbert space of solutions of the Weyl equation in the euclidean space-time $\mathbb{R} \times M$.

2 Preliminaries

2.1 Coherent state transforms (CST)

In [Ha1], Hall introduced a class of unitary integral transforms from the Hilbert spaces of square integrable functions on compact Lie groups G, with respect to the Haar measure, to spaces of holomorphic functions on the complexification $G_{\mathbb{C}}$, which are L^2 with respect to an appropriate measure. These are known as coherent state transforms (CSTs) or generalized Segal–Bargmann transforms. These transforms were extended to Lie groups of compact type, which include the case of $G = \mathbb{R}^m$ considered in the present paper, by Driver in [Dr]. General Lie groups of compact type are products of compact Lie groups and $\mathbb{R}^m$, see Corollary 2.2 of [Dr].

We will briefly recall now the case $G = \mathbb{R}^m$ for which the Hall transform coincides with the classical Segal–Bargmann transform [Ba, Se1, Se2].

Let $\rho_t(x)$ denote the fundamental solution of the heat equation.

$$\frac{\partial}{\partial t}\rho_t = \frac{1}{2}\Delta\,\rho_t,$$

i.e.

$$\rho_t(x) = \frac{1}{(2\pi t)^{m/2}}\, e^{-\frac{|x|^2}{2t}},$$

where Δ is the Laplacian for the euclidean metric. The Segal–Bargman or coherent state transform

$$U\ : L^2(\mathbb{R}^m, dx) \longrightarrow \mathcal{H}(\mathbb{C}^m)$$

is defined by

$$\begin{aligned} U(f)(z) &= \int_{\mathbb{R}^m} \rho_{t=1}(z-x) f(x)\, dx = \\ &= \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} e^{-\frac{|z-x|^2}{2}}\, f(x)\, dx\,. \end{aligned} \tag{2.1}$$

where ρ_1 has been analytically continued to $\mathbb{C}^m$. We see that the transform U in (2.1) factorizes according to the following diagram

$$\begin{tikzcd} & \mathcal{H}(\mathbb{C}^m) \\ L^2(\mathbb{R}^m, dx) \arrow[ur, hook, "U"] \arrow[r, hook, "e^{\frac{\Delta}{2}}"'] & \mathcal{A}(\mathbb{R}^m) \arrow[u, "\mathcal{C}"'] \end{tikzcd} \tag{2.2}$$

where $\mathcal{C}$ denotes the analytic continuation from $\mathbb{R}^m$ to $\mathbb{C}^m$ and $e^{\frac{\Delta}{2}}(f)$ is the (real analytic) heat kernel evolution of the function $f \in L^2(\mathbb{R}^m, dx)$ at time $t = 1$, that is the solution of

$$\left\{ \begin{array}{lcl} \frac{\partial}{\partial t} h_t &=& \frac{1}{2}\Delta\, h_t \\ h_0 &=& f \end{array} \right. , \tag{2.3}$$

evaluated at time $t=1$,

$$e^{\frac{\Delta}{2}}(f) = h_1.$$

$\mathcal{A}(\mathbb{R}^m)$ in (2.2) is the space of (complex valued) real analytic functions on $\mathbb{R}^m$ with unique analytic continuation to entire functions on $\mathbb{C}^m$. Let $\widetilde{\mathcal{A}}(\mathbb{R}^m) \subset \mathcal{A}(\mathbb{R}^m)$ denote the image of $L^2(\mathbb{R}^m, dx)$ by the operator $e^{\frac{\Delta}{2}}$. The analytic continuation $\mathcal{C}$ on $\widetilde{\mathcal{A}}(\mathbb{R}^m)$ can be writen in the form

$$\mathcal{C}(f)(x,y) = f(x+iy) = e^{i\sum_{j=1}^m y_j\,\partial_{x_j}}\left(f(x)\right). \tag{2.4}$$

Then, the Segal–Bargmann theorem reads as follows.

Theorem 2.1 *The transform U in the diagram*

$$\begin{tikzcd} & \mathcal{H}L^2(\mathbb{C}^m, \nu\, dxdy) \\ L^2(\mathbb{R}^m, dx) \arrow[ur, "U"] \arrow[r, "e^{\frac{\Delta}{2}}"'] & \widetilde{\mathcal{A}}(\mathbb{R}^m) \arrow[u, "e^{i\sum_{j=1}^m y_j\,\partial_{x_j}}"'] \end{tikzcd} \tag{2.5}$$

is a unitary isomorphism, where $\nu(y) = e^{-|y|^2}$.

2.2 Clifford analysis

Let us briefly recall from [BDS, DSS, CSS1, CSS2, CSS3, DS, LMQ, Q1, Q2, Sou], some definitions and results from Clifford analysis. Let $\mathbb{R}_{m+1}$ denote the real Clifford algebra with $(m+1)$ generators, $\tilde{e}_j, j = 0, \ldots, m$, identified with the canonical basis of $\mathbb{R}^{m+1} \subset \mathbb{R}_{m+1}$ and satisfying the relations $\tilde{e}_i\tilde{e}_j + \tilde{e}_j\tilde{e}_i = -2\delta_{ij}$. Let $\mathbb{C}_{m+1} = \mathbb{R}_{m+1} \otimes \mathbb{C}$. We have that $\mathbb{R}_{m+1} = \oplus_{k=0}^{m+1}\mathbb{R}_{m+1}^k$, where $\mathbb{R}_{m+1}^k$ denotes the space of k-vectors, defined by $\mathbb{R}_{m+1}^0 = \mathbb{R}$ and $\mathbb{R}_{m+1}^k = \mathrm{span}_{\mathbb{R}}\{\tilde{e}_A \,:\, A \subset \{0, \ldots, m\}, |A| = k\}$.

Notice also that $\mathbb{R}_1 \cong \mathbb{C}$ and $\mathbb{R}_2 \cong \mathbb{H}$. The inner product in $\mathbb{R}_{m+1}$ is defined by

$$(u,v) = \left(\sum_A u_A\tilde{e}_A, \sum_B v_B\tilde{e}_B\right) = \sum_A u_A v_A.$$

The Dirac operator is defined as

$$\tilde{D} = \sum_{j=0}^m \partial_{x_j}\tilde{e}_j.$$

Let

$$e_j = -\tilde{e}_0\tilde{e}_j,\ j = 1, \ldots, m.$$

Note that $e_ie_j + e_je_i = -2\delta_{ij}$, so that $\{e_j\}_{j=1,\ldots,m}$ is a set of generators for a subalgebra $\mathbb{C}_{m+1}^+ \subset \mathbb{C}_{m+1}$ with $\mathbb{C}_{m+1}^+ \cong \mathbb{C}_m$. We will henceforth consider $\mathbb{C}_m$-valued functions.

One defines the Cauchy-Riemann operator by

$$D = \partial_{x_0} + \underline{D},$$

where

$$\underline{D} = \sum_{j=1}^m \partial_{x_j} e_j.$$

We have that $\underline{D}^2 = -\Delta_m$ and $D\overline{D} = \Delta_{m+1}$.

Consider the subspace of $\mathbb{R}_m$ of 1-vectors

$$\{\underline{x} = \sum_{j=1}^{m} x_j e_j :\ x = (x_1, \ldots, x_m) \in \mathbb{R}^m\} \cong \mathbb{R}^m,$$

which we identify with $\mathbb{R}^m$. Note that $\underline{x}^2 = -|\underline{x}|^2 = -(x, x)$.

Recall that a continuously differentiable function f on an open domain $\mathcal{O} \subset \mathbb{R}^{m+1}$, with values on $\mathbb{C}_m$, is called (left) monogenic on $\mathcal{O}$ if it satisfies the Weyl, or Cauchy-Riemann, equation (see, for example, [BDS, DSS, Sou])

$$Df(x_0, \underline{x}) = (\partial_{x_0} + \underline{D}) f(x_0, \underline{x}) = 0.$$

For $m = 1$, monogenic functions on $\mathbb{R}^2$ correspond to holomorphic functions of the complex variable $x_0 + e_1 x_1$.

In order to describe monogenic functions let us, following [LMQ, Q2], introduce the following projectors

$$\chi_{\pm}(\underline{p}) = \frac{1}{2}\left(1 \pm \frac{i\underline{p}}{|\underline{p}|}\right),$$

satisfying

$$Q(i\underline{p})\, \chi_{\pm}(\underline{p}) = \chi_{\pm}(\underline{p})\, Q(i\underline{p}) = Q(\pm|\underline{p}|)\, \chi_{\pm}(\underline{p})$$

for any polynomial in one variable with complex coefficients,

$$Q(\lambda) = \sum_{k=0}^{\ell} c_k \lambda^k.$$

For any function B, of one real variable, one naturally defines the following Clifford algebra valued function on $\mathbb{R}^m$,

$$b(p_1, \ldots, p_m) := B(|\underline{p}|)\chi_+(\underline{p}) + B(-|\underline{p}|)\chi_-(\underline{p}).$$

By abuse of notation, using the analogy for the case when B is a polynomial, we will still denote the right-hand side by $B(i\underline{p})$. Then, for $B_{x_0}(y) = e^{-x_0 y}$, the Clifford algebra valued function

$$e^{-ix_o\underline{p}} = B_{x_0}(i\underline{p}) = b_{x_0}(p_1, \ldots, p_m) = e^{-x_0|\underline{p}|}\chi_+(\underline{p}) + e^{x_0|\underline{p}|}\chi_-(\underline{p}), \tag{2.6}$$

satisfies the equation

$$\frac{\partial}{\partial x_0} e^{-ix_o\underline{p}} = -i\underline{p}\, e^{-ix_o\underline{p}},$$

which implies that the functions

$$e(x, \underline{p}) = e^{i((\underline{x}, \underline{p}) - x_0\underline{p})}$$

are monogenic in $(x_0, \underline{x})$ for all $\underline{p} \in \mathbb{R}^m$. In fact, this is the Cauchy-Kowalevski extension

$$e(x, \underline{p}) = e^{-x_0\underline{D}}\, e^{i(\underline{p}, \underline{x})} = e^{i(\underline{p}, \underline{x})}\, e^{ix_0\underline{p}},$$

which follows straightforwardly from

$$\underline{D}e^{i(\underline{p},\underline{x})} = ie^{i(\underline{p},\underline{x})}\underline{p}.$$

These functions play a very important role in the analysis of monogenic functions. We then obtain the following corollary of [LMQ, Q2]. Let $\mathcal{S}(\mathbb{R}^m)$ be the space of Schwarz functions on $\mathbb{R}^m$.

Proposition 2.2 *Let b_{x_0} be as in (2.6) and let $f \in \mathcal{S}(\mathbb{R}^m) \otimes \mathbb{C}_m$ be such that for all $x_0 \in \mathbb{R}$*

$$b_{x_0} \hat{f} \in \mathcal{S}(\mathbb{R}^m) \otimes \mathbb{C}_m, \tag{2.7}$$

where $\hat{f}$ denotes the Fourier transform of f. Then,

$$F(x_0, \underline{x}) = \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} e^{i((\underline{p},\underline{x})-x_0\underline{p})}\, \widehat{f}(\underline{p})\, d\underline{p}, \tag{2.8}$$

defines a monogenic function satisfying the Cauchy problem

$$\left\{ \begin{array}{rcl} \frac{\partial F}{\partial x_0} & = & -\underline{D}\, F \\ F(0, \underline{x}) & = & f(\underline{x})\,. \end{array} \right. \tag{2.9}$$

Proof. We have

$$f(\underline{x}) = \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} e^{i(\underline{p},\underline{x})}\, \widehat{f}(\underline{p})\, d\underline{p} = F(0, \underline{x}).$$

From above, $e^{i((\underline{p},\underline{x})-x_0\underline{p})}$ are monogenic functions, for all $p \in \mathbb{R}^m$, and under the conditions of the proposition, we can differentiate under the integral sign in (2.8) so that F is also a monogenic function. ■

Following [DS, DSS] (see [DSS], chapter III, Section 2 and Theorem 6 in [Som]), one also has

Proposition 2.3 *Let F be a monogenic function on $\mathbb{R}^{m+1}$, with $F(x_0 = 0, \underline{x}) = f(\underline{x})$. Then,*

$$F(x_0, \underline{x}) = \left(e^{-x_0\underline{D}} f\right)(x_0, \underline{x}) := \left(\sum_{k=0}^{\infty} (-1)^k \frac{x_0^k}{k!} \underline{D}^k f \right)(\underline{x}), \tag{2.10}$$

where the series converges uniformly on compact subsets.

3 A unitary transform from $L^2(\mathbb{R}^m, d^m x) \otimes \mathbb{C}_m$ to a Hilbert space of monogenic functions on $\mathbb{R} \times \mathbb{R}^m$

We can view the CST unitary map U in the diagram (2.5) as a unitarization of the analytic continuation $\mathcal{C}$ from $\mathbb{R}^m$ to $\mathbb{C}^m$. Indeed, by precomposing $\mathcal{C}$ with the smoothening contracting map $e^{\frac{\Delta}{2}}$ one obtains a unitary isomorphism from a L^2 space on $\mathbb{R}^m$ to a space of holomorphic square integrable functions on $\mathbb{C}^m$, the complexification of $\mathbb{R}^m$.

Aiming at obtaining an analogous unitarization of the Cauchy-Kowalewsky (CK) extension (2.10), we precompose it with the same smoothening contracting map $e^{\frac{\Delta}{2}}$ or, equivalently, we substitute the vertical arrow in the diagram (2.5) by the CK extension,

$$
\begin{tikzcd}
 & \mathcal{M}(\mathbb{R}^{m+1}) \\
L^2(\mathbb{R}^m, d\underline{x}) \otimes \mathbb{C}_m \arrow[ur, hook, "V"] \arrow[r, hook, "e^{\frac{\Delta}{2}}"'] & \widetilde{\mathcal{A}}(\mathbb{R}^m) \otimes \mathbb{C}_m, \arrow[u, "e^{-x_0\underline{D}}"']
\end{tikzcd}
\tag{3.1}
$$

where $\mathcal{M}(\mathbb{R}^{m+1})$ denotes the space of $\mathbb{C}_m$-valued monogenic functions on $\mathbb{R}^{m+1}$. Let ρ_1 be the heat kernel in $\mathbb{R}^m$, as in Section 2.1,

$$\rho_1(x) = (2\pi)^{-m/2} e^{-\frac{x^2}{2}} = (2\pi)^{-m} \int_{\mathbb{R}^m} e^{-\frac{p^2}{2}} e^{i(\underline{p},\underline{x})} d\underline{p}. \tag{3.2}$$

From Proposition 2.2, we then have the Cauchy-Kowalevski extension of ρ_1,

$$e^{-x_0\underline{D}}\rho_1(\underline{x}) = (2\pi)^{-m} \int_{\mathbb{R}^m} e^{-\frac{p^2}{2}} e^{i(\underline{p},\underline{x})} e^{-ix_0\underline{p}} d\underline{p}.$$

Our main result in this Section is then

Theorem 3.1 *For $\varphi \in L^2(\mathbb{R}^m, d\underline{x}) \otimes \mathbb{C}_m$ define*

$$V(\varphi)(x_0, \underline{x}) = (2\pi)^{-m} \int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^m} e^{-\frac{p^2}{2}} e^{i(\underline{p},\underline{x}-\underline{y})} e^{-ix_0\underline{p}} d\underline{p} \right) \varphi(y) d\underline{y}, \tag{3.3}$$

which can be abbreviated by

$$V(\varphi) = e^{-x_0\underline{D}} \circ e^{\frac{\Delta}{2}} \varphi.$$

Then, in the diagram

$$
\begin{tikzcd}
 & \mathcal{M}L^2(\mathbb{R}^{m+1}, d\mu) \\
L^2(\mathbb{R}^m, d\underline{x}) \otimes \mathbb{C}_m \arrow[ur, "V"] \arrow[r, hook, "e^{\frac{\Delta}{2}}"'] & \widetilde{\mathcal{A}}(\mathbb{R}^m) \otimes \mathbb{C}_m, \arrow[u, "e^{-x_0\underline{D}}"']
\end{tikzcd}
\tag{3.4}
$$

the map $V : L^2(\mathbb{R}^m, d\underline{x}) \otimes \mathbb{C}_m \to \mathcal{M}L^2(\mathbb{R}^{m+1}, d\mu)$ is a unitary isomorphism of Hilbert spaces, where $\mathcal{M}L^2(\mathbb{R}^{m+1}, d\mu)$ is the Hilbert space of monogenic functions on $\mathbb{R}^{m+1}$ which are L^2 with respect to the measure

$$d\mu = \frac{1}{\sqrt{\pi}} e^{-x_0^2} dx_0 d\underline{x},$$

and where the standard inner product on $\mathbb{C}_m$ is considered.

Let us now list some direct consequences of this theorem.

Corollary 3.2 *The subspace of monogenic functions which are in $L^2(\mathbb{R}^{m+1}, d\mu) \otimes \mathbb{C}_m$, $\mathcal{M}L^2(\mathbb{R}^{m+1}, d\mu)$, is a closed subspace of $L^2(\mathbb{R}^{m+1}, d\mu) \otimes \mathbb{C}_m$.*

We also obtain a characterization of the range of the heat operator in terms of monogenic functions, which is analogous to the one given in terms of holomorphic functions by the Segal-Bargmann theorem (see, for example, [Ha3]).

Corollary 3.3 *A real analytic function F on $\mathbb{R}^m$ is of the form*

$$F = e^{\frac{\Delta}{2}} f,$$

with $f \in L^2(\mathbb{R}^m, dx)$, iff its monogenic extension to $\mathbb{R}^{m+1}$ exists and is $d\mu$-square-integrable,

$$e^{-x_0 \underline{D}} f \in \mathcal{M}L^2(\mathbb{R}^{m+1}, d\mu).$$

Let now $\{H_k, k \in \mathbb{N}_0^m\}$ denote the orthogonal basis of $L^2(\mathbb{R}^m, e^{-x^2}dx)$ consisting of Hermite polynomials on $\mathbb{R}^m$, with

$$||H_k||^2 = \pi^{m/2} 2^k k!,$$

where we use multi-index notation: $k \in \mathbb{N}_0^m$,

$$H_k(x) := H_{k_1}(x_1) \cdots H_{k_m}(x_m), \tag{3.5}$$

$2^k := 2^{k_1 + \cdots + k_m}$, $k! := k_1! \cdots k_m!$ and $x^k = x_1^{k_1} \cdots x_m^{k_m}$.

Defining,

$$\varphi_k(x) = H_k(x) e^{-\frac{x^2}{2}}, \tag{3.6}$$

we have that $\{\varphi_k, k \in \mathbb{N}_0^m\}$ is an orthogonal basis for $L^2(\mathbb{R}^m, d\underline{x})$. From the isometricity of V and Lemma 3.7 below, we also obtain the following

Corollary 3.4 *Let*

$$\psi_k = 2^{m/2} V(\varphi_k) = e^{-x_0 \underline{D}} \left(x^k e^{-\frac{x^2}{4}} \right), \quad k \in \mathbb{N}_0^m.$$

Then, the set

$$\{\psi_k e_A, k \in \mathbb{N}_0^m, A \subset \{1, \ldots, m\}\}$$

is an orthogonal basis for the Hilbert space $\mathcal{M}L^2(\mathbb{R}^{m+1}, d\mu)$, where $e_\emptyset = 1$.

Remark 3.5 The coherent state transform of Hall is onto $\mathcal{H}L^2(\mathbb{C}^m, \nu dxdy)$. Let us consider its inverse

$$U^{-1} : \mathcal{H}L^2(\mathbb{C}^m, \nu dxdy) \to L^2(\mathbb{R}^m, dx).$$

By composing this operator with the operator V above we obtain the operator

$$V \circ U^{-1} : \mathcal{H}L^2(\mathbb{C}^m, \nu dxdy) \to \mathcal{M}L^2(\mathbb{R}^{m+1}, d\mu)$$

which is a unitary isomorphism. $\Diamond$

We will prove theorem 3.1 by a sequence of lemmas.

Lemma 3.6 *If $\varphi \in L^2(\mathbb{R}^m, d\underline{x}) \otimes \mathbb{C}_m$ then $V(\varphi)$ is a monogenic function on $\mathbb{R}^{m+1}$.*

Proof. By Leibniz rule, both

$$\frac{\partial}{\partial x_0} V(\varphi), \text{ and } \underline{D} V(\varphi)$$

can be computed by taking the differential operators inside both integral symbols in (3.3), due to the presence of the gaussian factor in the integrand which ensures the integrability of all of its derivatives. This then implies that

$$\left(\frac{\partial}{\partial x_0} + \underline{D}\right) V(\varphi) = 0,$$

so that $V(\varphi)$ is a monogenic function on $\mathbb{R}^{m+1}$. ■

Lemma 3.7 *We have*

$$e^{\frac{\Delta}{2}} \varphi_k = 2^{-m/2} x^k e^{-\frac{x^2}{4}}.$$

Proof. This follows from the well-known identity

$$e^{2xy-y^2} = \sum_{l \in \mathbb{N}_0^m} H_l(x) \frac{y^l}{l!}, \quad x, y \in \mathbb{R}^m$$

and from

$$\langle H_k, H_l \rangle_{L^2(\mathbb{R}^m, e^{-x^2} dx)} = \pi^{m/2} 2^k k! \delta_{kl}.$$

A simple evaluation of gaussian integrals then gives the result

$$e^{\frac{\Delta}{2}} \varphi_k = \int_{\mathbb{R}^m} e^{-\frac{(x-y)^2}{2}} H_k(y) e^{-\frac{y^2}{2}} dy = 2^{-m/2} x^k e^{-\frac{x^2}{4}}.$$

■

Lemma 3.8 *Let $f = \sum_A f_A e_A \in \mathcal{S}(\mathbb{R}^m, d\underline{x}) \otimes \mathbb{C}_m$, with Fourier transform $\widehat{f}$. Then,*

$$V(f)(x_0, \underline{x}) = \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} e^{-\frac{|\underline{p}|^2}{2}} e^{i(\langle \underline{p}, \underline{x}\rangle - x_0 \underline{p})} \widehat{f}(\underline{p}) \, d\underline{p}.$$

Proof. For $f \in \mathcal{S}(\mathbb{R}^m) \otimes \mathbb{C}_m$, we have

$$f(\underline{x}) = \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} e^{i\langle \underline{p}, \underline{x}\rangle} \widehat{f}(\underline{p}) \, d\underline{p} = \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} e^{i\langle \underline{p}, \underline{x}\rangle} \sum_A \widehat{f}_A(\underline{p}) \, e_A \, d\underline{p} \ .$$

and the result follows from (2.10),

$$e^{\frac{\Delta}{2}} e^{i\langle \underline{x}, \underline{p}\rangle} = e^{-\frac{|\underline{p}|^2}{2}} e^{i\langle \underline{x}, \underline{p}\rangle},$$

and the fact that under the conditions of the proposition the heat operator can be taken inside the integral. ■

Remark 3.9 We also note the following useful expression,

$$V(f)(x_0,\underline{x}) = \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} e^{-\frac{|\underline{p}|^2}{2}}\, e^{i(\underline{p},\underline{x})} \left(\cosh(x_0|\underline{p}|) - i\sinh(x_0|\underline{p}|)\, \frac{\underline{p}}{|\underline{p}|}\right) \widehat{f}(\underline{p})\, d\underline{p}.$$

$\diamondsuit$

Lemma 3.10 *For $f,h \in \mathcal{S}(\mathbb{R}^m)\otimes\mathbb{C}_m$, with $f=\sum_A f_A e_A, h=\sum_A h_A e_A$, we have*

$$\langle V(f), V(g)\rangle_{L^2(\mathbb{R}^{m+1},d\mu)\otimes\mathbb{C}_m} = \langle f,g\rangle_{L^2(\mathbb{R}^m,d\underline{x})\otimes\mathbb{C}_m}.$$

Proof. For any 1–vector $\underline{p}=\sum_{j=1}^m p_j e_j \in \mathbb{R}^1_m$ one has

$$(\underline{p}\, u, v) = -(u, \underline{p}\, v)\,, \quad \forall u,v\in\mathbb{C}_m \tag{3.7}$$

and therefore

$$(e^{i\underline{p}}\, u, v) = (u, e^{i\underline{p}}\, v)\,, \quad \forall u,v\in\mathbb{C}_m,$$

where the hermiticity of the standard inner product in $\mathbb{C}_m$, $(\cdot,\cdot)$, is used.

Then, for $f,h \in \mathcal{S}(\mathbb{R}^m)\otimes\mathbb{C}_m$, with $f=\sum_A f_A e_A, h=\sum_A h_A e_A$ we have, from Lemma 3.8,

$$\begin{aligned}
&\langle V(f), V(h)\rangle_{L^2(\mathbb{R}^{m+1},d\mu)\otimes\mathbb{C}_m} = \\
=\ & \frac{1}{\sqrt{\pi}(2\pi)^m}\int_{\mathbb{R}^{m+1}\times\mathbb{R}^{2m}} e^{i(\underline{p}-\underline{q},\underline{x})}\, e^{-\frac{|p|^2+|q|^2}{2}} \left(e^{-ix_0\underline{p}}\,\widehat{f}(\underline{p}), e^{-ix_0\underline{q}}\,\widehat{h}(\underline{q})\right) e^{-x_0^2}\, dx_0 d\underline{x} d\underline{p} d\underline{q} = \\
=\ & \frac{1}{\sqrt{\pi}}\int_{\mathbb{R}\times\mathbb{R}^m} e^{-|p|^2} \left(e^{-ix_0\underline{p}}\,\widehat{f}(\underline{p}), e^{-ix_0\underline{p}}\,\widehat{h}(\underline{p})\right) e^{-x_0^2}\, dx_0 d\underline{p} = \\
=\ & \frac{1}{\sqrt{\pi}}\int_{\mathbb{R}\times\mathbb{R}^m} e^{-|p|^2} \left(e^{-2ix_0\underline{p}}\,\widehat{f}(\underline{p}), \widehat{h}(\underline{p})\right) e^{-x_0^2}\, dx_0 d\underline{p} = \\
=\ & \frac{1}{\sqrt{\pi}}\int_{\mathbb{R}\times\mathbb{R}^m} e^{-|p|^2} \left[\cosh(2x_0|\underline{p}|)\left(\widehat{f}(\underline{p}), \widehat{h}(\underline{p})\right) - i\frac{\sinh(2x_0|\underline{p}|)}{|\underline{p}|}\left(\underline{p}\,\widehat{f}(\underline{p}), \widehat{h}(\underline{p})\right)\right] e^{-x_0^2}\, dx_0 d\underline{p} = \\
=\ & \frac{1}{\sqrt{\pi}}\int_{\mathbb{R}\times\mathbb{R}^m} e^{-|p|^2} \cosh(2x_0|\underline{p}|)\left(\widehat{f}(\underline{p}), \widehat{h}(\underline{p})\right) e^{-x_0^2}\, dx_0 d\underline{p} = \\
=\ & \int_{\mathbb{R}^m} \left(\widehat{f}(\underline{p}), \widehat{h}(\underline{p})\right) d\underline{p} = \\
=\ & \langle f,h\rangle_{L^2(\mathbb{R}^m,d\underline{x})\otimes\mathbb{C}_m}.
\end{aligned}$$

■

Proof. *(of Theorem 3.1)* From the denseness of $\mathcal{S}(\mathbb{R}^m)$ in $L^2(\mathbb{R}^m,d\underline{x})$ we conclude from Lemma 3.10 that V is an isometry onto its image which is, therefore, closed in $L^2(\mathbb{R}^{m+1},d\mu)\otimes\mathbb{C}_m$. Moreover, Lemma 3.6 ensures that the image of V contains only functions which are monogenic on $\mathbb{R}^{m+1}$. Therefore, $V(L^2(\mathbb{R}^m,d\underline{x})\otimes\mathbb{C}_m)$ is a Hilbert space of monogenic functions.

To prove that the image of V is all of $\mathcal{M}L^2(\mathbb{R}^{m+1},d\mu)$, note that the restriction of $f\in\mathcal{M}L^2(\mathbb{R}^{m+1},d\mu)$ to the hyperplane $x_0=0$, $f_0(\underline{x})=f(x_0=0,\underline{x})$, determines f uniquely.

Since entire monogenic functions have a Taylor series with infinite radius of convergence (see, for example, [BDS, Som]), it follows that f_0 can be expressed uniquely in the form

$$f_0 = \sum_A \sum_{k \in \mathbb{N}_0^m} \alpha_{k,A} x^k e^{-\frac{x^2}{4}} e_A, \quad \alpha_{k,A} \in \mathbb{C}.$$

Now, from Proposition 2.3,

$$f(x_0, \underline{x}) = \sum_A \sum_{j=0}^{\infty} \frac{(-x_0)^j}{j!} \underline{D}^j \left(\sum_{k \in \mathbb{N}_0^m} \alpha_{k,A} x^k e^{-\frac{x^2}{4}} \right) e_A = \sum_A \sum_{j=0}^{\infty} \sum_{k \in \mathbb{N}_0^m} \frac{(-x_0)^j}{j!} \alpha_{k,A} \underline{D}^j \left(x^k e^{-\frac{x^2}{4}} \right) e_A,$$

since convergent power series can be differentiated term by term. (The Gaussian factor $e^{-\frac{x^2}{4}}$ could, of course, also be expanded in power series.) This series converges absolutely in all of $\mathbb{R}^{m+1}$ so that the two summations can be interchanged giving

$$f(x_0, \underline{x}) = \sum_A \sum_{k \in \mathbb{N}_0^m} \alpha_{k,A} e^{-x_0 \underline{D}} \left(x^k e^{-\frac{x^2}{4}} \right) e_A.$$

From Lemma 3.7, all partial sums

$$\sum_A \sum_{k \in \mathbb{N}_0^m : ||k|| < N} \alpha_{k,A} e^{-x_0 \underline{D}} \left(x^k e^{-\frac{x^2}{4}} \right) e_A$$

are in the image of V which is closed. Moreover, see also Corollary 3.4 above, from Lemma 3.10 and from the orthogonality of the set $\{\varphi_k e_A, k \in \mathbb{N}_0^m, A \subset \{1, \ldots, m\}\}$ in $L^2(\mathbb{R}^m, d\underline{x}) \otimes \mathbb{C}_m$, we obtain that the L^2 condition for f in $\mathbb{R}^{m+1}$ with respect to the measure $d\mu$ is equivalent to

$$\sum_A \sum_{k \in \mathbb{N}_0^m} \alpha_{k,A} \varphi_k e_A \in L^2(\mathbb{R}^m, d\underline{x}) \otimes \mathbb{C}_m.$$

This completes the proof of Theorem 3.1. ■

We finish this section with a Proposition on explicit expressions for $V(P_k e^{-\frac{x^2}{2}})$ where P_k is an homogeneous monogenic polynomial of degree k in $\mathbb{R}^m$.

Proposition 3.11 *Let P_k be an homogeneous monogenic polynomial of degree k in $\mathbb{R}^m$. Then*

$$V\left(P_k e^{-\frac{x^2}{2}}\right) = e^{-\frac{x^2}{2}} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\sum_{j=0}^{[n/2]} \frac{(-1)^j}{2^j j!} u(n,j)!\, x_0^{u(n,j)} \right) H_{n,m,k}(\underline{x}) P_k(\underline{x}),$$

where $u(n,j) = (n-2j)$ for n even and $u(n,j) = (n-2j-1)$ for n odd and the $H_{n,m,k}$ are the so-called generalized Hermite polynomials.

Proof. From Chapter III of [DSS], we have

$$e^{-x_0 \underline{D}} \left(P_k e^{-\frac{x^2}{2}} \right)(x_0, \underline{x}) = e^{-\frac{x^2}{2}} \sum_{n=0}^{\infty} \frac{x_0^n}{n!} H_{n,m,k}(\underline{x}) P_k(\underline{x}).$$

Since the operator $\underline{D}$ commutes with the Laplace operator on $\mathbb{R}^m$, Δ, we have

$$V = e^{-x_0\underline{D}} \circ e^{\frac{\Delta}{2}} = e^{\frac{\Delta}{2}} \circ e^{-x_0\underline{D}}.$$

On the other hand, since monogenic functions are harmonic we have

$$\Delta f = -\frac{\partial^2 f}{\partial x_0^2},$$

for f monogenic on $\mathbb{R}^{m+1}$. Therefore,

$$V\left(P_k e^{-\frac{x^2}{2}}\right) = e^{-\frac{1}{2}\frac{\partial^2}{\partial x_0^2}}\left(e^{-\frac{x^2}{2}}\sum_{n=0}^{\infty}\frac{x_0^n}{n!}H_{n,m,k}(\underline{x})P_k(\underline{x})\right) = \left(e^{-\frac{x^2}{2}}\sum_{n=0}^{\infty}\left(e^{-\frac{1}{2}\frac{\partial^2}{\partial x_0^2}}\frac{x_0^n}{n!}\right)H_{n,m,k}(\underline{x})P_k(\underline{x})\right)$$

which proves the proposition. ■

4 A unitary transform from $L^2(\mathbb{T}^m, d^m x) \otimes \mathbb{C}_m$ to a Hilbert space of monogenic functions on $\mathbb{R} \times \mathbb{T}^m$

In this section, we generalize the coherent state transform of the last section to a transform on $\mathbb{C}_m$-valued L^2 functions on the compact Lie group defined by the m-dimensional torus

$$\mathbb{T}^m = \mathbb{R}^m/\mathbb{Z}^m,$$

where $x \sim x + 2\pi k, k \in \mathbb{Z}^m, x \in \mathbb{R}^m$. We will still denote by $(x_1, \ldots, x_m)$ the periodic coordinates on $\mathbb{T}^m$, with $x_j \in [0, 2\pi], j = 1, \ldots, m$. Note that the definitions of the Dirac ($\tilde{D}$), Cauchy-Riemann (D) and $\underline{D}$ operators generalize straightforwardly. Likewise, the Cauchy-Kowaleski extension of section 2.2 is obtained from the same expression. Let Δ denote the Laplacian on $\mathbb{T}^m$ with respect to an invariant metric and let dx denote the unit volume Haar measure.

We then define the operator V by the following diagram

$$\begin{tikzcd} & \mathcal{M}(\mathbb{T}^m \times \mathbb{R}) \\ L^2(\mathbb{T}^m, dx) \otimes \mathbb{C}_m \arrow[ur, hook, "V"] \arrow[r, hook, "e^{\frac{\Delta}{2}}"'] & \widetilde{\mathcal{A}}(\mathbb{T}^m) \otimes \mathbb{C}_m, \arrow[u, "e^{-x_0\underline{D}}"'] \end{tikzcd} \tag{4.1}$$

where $\mathcal{A}(\mathbb{T}^m)$ denotes the space of real analytic functions on $\mathbb{T}^m$ and $\mathcal{M}(\mathbb{T}^m \times \mathbb{R})$ is the space of $\mathbb{C}_m$-valued monogenic functions on $\mathbb{T}^m \times \mathbb{R}$.

As in the previous section, we obtain an explicit expression for this operator using the Fourier, or Peter-Weyl, decomposition for functions in $L^2(\mathbb{T}^m, dx)$. For $k \in \mathbb{Z}^m$, define $\underline{k} = \sum_{j=1}^m k_j e_j \in \mathbb{R}^1_m$.

Proposition 4.1 *Let $f = \sum_A f_A\, e_A \in L^2(\mathbb{T}^m, \underline{d}x) \otimes \mathbb{C}_m$, with Fourier decomposition*

$$f(\underline{x}) = \frac{1}{(2\pi)^{m/2}}\sum_{k\in\mathbb{Z}^m} f_k e^{i(\underline{k},\underline{x})} = \frac{1}{(2\pi)^{m/2}}\sum_{k\in\mathbb{Z}^m} e^{i(\underline{k},\underline{x})}\sum_A f_{k,A}\, e_A \quad .$$

Then,

$$V(f)(x_0,\underline{x}) = \frac{1}{(2\pi)^{m/2}} \sum_{k\in\mathbb{Z}^m} e^{-\frac{|k|^2}{2}} \, e^{i((\underline{k},\underline{x})-x_0\underline{k})} \, f_m.$$

Proof. The proof is analogous to the one of Lemma 3.10. ■

Remark 4.2 We note the following useful formula,

$$V(f)(x_0,\underline{x}) = \frac{1}{(2\pi)^{m/2}} \sum_{k\in\mathbb{Z}^m} e^{-\frac{|\underline{k}|^2}{2}} \, e^{i(\underline{k},\underline{x})} \left(\cosh(x_0|\underline{k}|) - i\sinh(x_0|\underline{k}|)\,\frac{\underline{k}}{|\underline{k}|}\right) f_k.$$

◊

Consider now the measure on $\mathbb{R}^m\times\mathbb{R}$ given by

$$d\mu = \frac{1}{\sqrt{\pi}}\, e^{-x_0^2}\, dx_0 dx,$$

and let $\mathcal{M}L^2(\mathbb{T}^m\times\mathbb{R},d\mu)$ be the corresponding Hilbert space of L^2 monogenic functions.

The analog of Theorem 3.1 is now

Theorem 4.3 *The map V in diagram (4.1) is unitary with respect to the measure $d\mu$, i.e. the map V in the diagram*

$$\begin{tikzcd} & \mathcal{M}L^2(\mathbb{T}^m\times\mathbb{R},d\mu) \\ L^2(\mathbb{T}^m,dx)\otimes\mathbb{C}_m \arrow[ur, "V"] \arrow[r, hook, "e^{\frac{\Delta}{2}}"'] & \widetilde{\mathcal{A}}(\mathbb{T}^m)\otimes\mathbb{C}_m \arrow[u, "e^{-x_0\underline{D}}"'] \end{tikzcd} \tag{4.2}$$

is a unitary isomorphism.

Proof. Then, for $f,h\in L^2(\mathbb{T}^m)\otimes\mathbb{C}_m$, with $f=\sum_A f_Ae_A$, $h=\sum_A h_Ae_A$ we have

$$\begin{aligned}
&\langle V(f),V(h)\rangle_{L^2(\mathbb{T}^m)\otimes\mathbb{C}_m} = \\
&= \frac{1}{\sqrt{\pi}(2\pi)^m}\int_{\mathbb{T}^m\times\mathbb{R}} \sum_{k,k'\in\mathbb{Z}^m} e^{i(\underline{k}-\underline{k}',\underline{x})}\, e^{-\frac{|k|^2+|k'|^2}{2}} \left(e^{-ix_0\underline{k}}\, f_k, e^{-ix_0\underline{k}'}\, h_{k'}\right) e^{-x_0^2}\, dx_0 d\underline{x} = \\
&= \frac{1}{\sqrt{\pi}}\int_{\mathbb{R}} \sum_{k\in\mathbb{Z}^m} e^{-|k|^2} \left(e^{-ix_0\underline{k}}\, f_k, e^{-ix_0\underline{k}}\, h_k\right) e^{-x_0^2}\, dx_0 = \\
&= \frac{1}{\sqrt{\pi}}\int_{\mathbb{R}} \sum_{k\in\mathbb{Z}^m} e^{-|k|^2} \left(e^{-2ix_0\underline{k}}\, f_k,\, h_k\right) e^{-x_0^2}\, dx_0 = \\
&= \frac{1}{\sqrt{\pi}}\int_{\mathbb{R}} \sum_{k\in\mathbb{Z}^m} e^{-|k|^2} \left[\cosh(2x_0|\underline{k}|)\,(f_k,\, h_k) - i\frac{\sinh(2x_0|\underline{k}|)}{|\underline{k}|}\,(\underline{k}\, f_k,\, h_k)\right] e^{-x_0^2}\, dx_0 = \\
&= \frac{1}{\sqrt{\pi}}\int_{\mathbb{R}} \sum_{k\in\mathbb{Z}^m} e^{-|k|^2} \cosh(2x_0|\underline{k}|)\,(f_k,\, h_k)\, e^{-x_0^2}\, dx_0 = \\
&= \sum_{k\in\mathbb{Z}^m} (f_k,\, h_k) = \\
&= \langle f,h\rangle_{L^2(\mathbb{T}^m,dx)\otimes\mathbb{C}_m}.
\end{aligned}$$

The proof of ontoness is analogous to the one in the proof of Theorem 3.1. ■

5 Quantum mechanical interpretation

As is well known, the Schrödinger representation in quantum mechanics is the one for which the position operator $\hat{x}$ acts by multiplication on $L^2(\mathbb{R}^m, dx)$. The momentum operator is then given by

$$\hat{p}_j = i\frac{\partial}{\partial x_j},\ j = 1, \ldots, m.$$

The CST from Section 2.1 intertwines the Schrödinger representation with the Segal-Bargmann representation, on which the operators $\hat{x}_j + i\hat{p}_j$ acts as the operator of multiplication by the holomorphic function $x_j + ip_j$ (see Theorem 6.3 of [Ha2]),

$$\left(U \circ (\hat{x}_j + i\hat{p}_j) \circ U^{-1}\right)(f)(x,p) = (x_j + ip_j)f(x,p),\ j = 1, \ldots, m. \tag{5.1}$$

We will prove now the analogous result for the coherent state transform of section 3.

Theorem 5.1 *The unitary map V induces a representation of the observable $\underline{x} + i\underline{p}$ on the Hilbert space of monogenic functions $\mathcal{M}L^2(\mathbb{R}^{m+1}, d\mu)$ given by*

$$\left(V \circ (\underline{\hat{x}} + i\underline{\hat{p}}) \circ V^{-1}\right)(f) = \left(e^{-x_0\underline{D}} \circ \underline{x}\right)(f_0), \tag{5.2}$$

where on the right hand side the operator $\underline{x}$ acts by Clifford multiplication on the left and $f_0(\underline{x}) = f(x_0 = 0, \underline{x})$.

Proof. Let $h = \sum_A h_A\, e_A \in \mathcal{S}(\mathbb{R}^m) \otimes \mathbb{C}_m$ and let $f = V(h)$. Then,

$$(\underline{\hat{x}} + i\underline{\hat{p}})(h)(\underline{x}) = \left(\frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} (\underline{x} + i\underline{p})\, e^{i\langle \underline{p}, \underline{x}\rangle}\, \widehat{h}(\underline{p})\, d\underline{p}\right).$$

From the above result of Hall (see also [Ha2]),

$$e^{\frac{\Delta}{2}}((\underline{x} + i\underline{p})e^{i\langle \underline{p}, \underline{x}\rangle}) = \underline{x}e^{\frac{\Delta}{2}}e^{i\langle \underline{p}, \underline{x}\rangle},$$

from which the result follows from the denseness of $\mathcal{S}(\mathbb{R}^m) \otimes \mathbb{C}_m$ in $L^2(\mathbb{R}^m, d\underline{x}) \otimes \mathbb{C}_m$. ■

Remark 5.2 Notice that the Segal-Bargmann transform can be expressed as, from (2.4),

$$e^{i\sum_{k=1}^m y_k \frac{\partial}{\partial x_k}} \circ e^{\frac{\Delta}{2}}$$

while the transform V of Section 3 is given by

$$e^{-\sum_{k=1}^m x_0\, e_k \frac{\partial}{\partial x_k}} \circ e^{\frac{\Delta}{2}}.$$

We therefore see that V is obtained from the Segal-Bargmann transform by replacing, in the operator of analytic continuation (2.4), the momentum variables y_k by the non-commutative variables $-ix_0e_k$, where x_0 is the euclidean time. ♢

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