Complex Variables and Elliptic Equations

An International Journal

ISSN: 1747-6933 (Print) 1747-6941 (Online) Journal homepage: http://www.tandfonline.com/loi/gcov20

The generalized Matsaev theorem on growth of subharmonic functions admitting a lower bound in $\mathbb{R}^n$

Yan Hui Zhang, Kit Ian Kou, Guan Tie Deng & Tao Qian

To cite this article: Yan Hui Zhang, Kit Ian Kou, Guan Tie Deng & Tao Qian (2017) The generalized Matsaev theorem on growth of subharmonic functions admitting a lower bound in $\mathbb{R}^n$, Complex Variables and Elliptic Equations, 62:5, 642-653, DOI: 10.1080/17476933.2016.1234464

To link to this article: http://dx.doi.org/10.1080/17476933.2016.1234464

COMPLEX VARIABLES AND ELLIPTIC EQUATIONS, 2017
VOL. 62, NO. 5, 642–653
http://dx.doi.org/10.1080/17476933.2016.1234464

Taylor & Francis
Taylor & Francis Group

The generalized Matsaev theorem on growth of subharmonic functions admitting a lower bound in $\mathbb{R}^n$

Yan Hui Zhang[a], Kit Ian Kou[b], Guan Tie Deng[c] and Tao Qian[b]

[a]Department of Mathematics, Beijing Technology and Business University, Beijing, China; [b]Faculty of Science and Technology, Department of Mathematics, University of Macau, Macau; [c]Key Laboratory of Mathematics and Complex Systems of Ministry of Education, School of Mathematical Sciences, Beijing Normal University, Beijing, China

ABSTRACT
We generalize Matsaev's theorem for subharmonic functions from two to higher dimension. The proofs are nontrivial and constructive.

ARTICLE HISTORY
Received 11 July 2016
Accepted 30 August 2016

COMMUNICATED BY
Y. Xu

KEYWORDS
Lower bound; upper bound; Cartwright's class; growth

AMS SUBJECT CLASSIFICATIONS
31C05; 31C99

1. Introduction

In 1960, Matsaev [1] proved the following Theorem.

Theorem 1.1: *Suppose an entire function $f(z)$ in complex plane $\mathbb{C}$ has a lower bound*

$$|f(z)| \geq \exp\left\{-Mr^{\rho}\frac{1}{|\sin\alpha|^k}\right\}, \quad z = re^{i\alpha} \in \mathbb{C}, \qquad (1)$$
$$r > 0, \ \rho > 1, \ k \geq 0.$$

Then the function $f(z)$ is of order ρ and finite type.

Note: Throughout, M stands for various values which may depend on ρ or k, but not on z or $f(z)$, not necessarily the same on any two occurrences.

The Matsaev Theorem has been found various applications in mathematics.[2–7] The inequalities like (1) are crucial in many problems, since they are intrinsically connected with the estimates of the Cauchy-type integrals.[8] A special attention of the related studies has been paid to dealing with the Matsaev theorem on subharmonic functions in the half space of $\mathbb{R}^n$.[8,9]

The proof of Matsaev theorem consists of two steps, each having an independent interest. In the first step, a certain upper bound is significantly improved from the lower bound by using Carleman's and R.Nevanlinna's formulas studied in [10]. Then in the

CONTACT Yan Hui Zhang zhangyanhui@th.btbu.edu.cn

second step, the theorem is derived from this upper bound. This two-step procedure has also applied to many related studies by others.

Matseav proved the following result of subharmonic functions on the upper bound in the plane which plays a basic role in proving Matsaev's theorem ([11, p.212, Theorem 3]), which can be viewed as a far-reaching generalization of the well-known Liouville theorem on bounded entire functions.[8]

Theorem 1.2: *Let $u(z)$ be a subharmonic function in the complex plane $\mathbb{C}$ which satisfies the estimate*

$$u(z) \leq M\frac{1+r^{\rho}}{|\sin\alpha|^{l}}, \quad z = re^{i\alpha} \in \mathbb{C}, \rho > 1, l \geq 0. \tag{2}$$

Then $u(z)$ is of order ρ and finite type.

Other results of this type can be seen in [11]. Govorrov and Zhuravleva [5] generalized Theorem 1.2 to analytic functions in the upper half-plane. One form of related estimates are seen in [6], where Rashkovskii proved a version of Matsaev's theorem for subharmonic functions $u(z)$ in the complex plane $\mathbb{C}$ and his assumptions were imposed on an integral norm of the negative part $u^{-} = u^{+} - u$ with $u^{+}(z) = \max\{u(z), 0\}$.

Most recently Kheyfits [8] extended Rashkovskii's result to subharmonic functions (Theorem 1.3) with respect to the stationary Schrodinger operator L_c, i.e. the weak solutions of the inequality

$$-L_c u \equiv \Delta u - c(x)u \geq 0,$$

where Δ is the Laplace operator. Subsolutions of this inequality are called c-subharmonic functions. Correspondingly, solutions of the equation

$$\Delta u - c(x)u = 0$$

are called c-harmonic functions. Kheyfits's result on the upper bound is stated as follows.

Theorem 1.3: *Let u be a c-harmonic function in $\mathbb{H} \cup -\mathbb{H}$, c-subharmonic or continuous in $\mathbb{R}^n$, such that*

$$u(x) \leq M\frac{1+r^{\rho}}{|\cos\theta_1|^{l}}, \quad \rho > \rho_k, \, l \geq 0, \tag{3}$$

in which ρ_k is a constant with respect to n and $k < \infty$. Then

$$u(x) \leq M(1+r^{\rho}), \quad x \in \mathbb{R}^n, \tag{4}$$

where the spherical coordinates in $\mathbb{R}^n$ are defined by

$$x = (x_1, x_2, \ldots, x_{n-1}, x_n) = (r, \theta), \;\; \theta = (\theta_1, \theta_2, \ldots, \theta_{n-1}),$$

such that

$$\cos\theta_1 = x_n/r, \qquad 0 < \theta_1 < \pi.$$

Here $\mathbb{H} = \{x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n, x_n > 0\}$ is the upper half space of $\mathbb{R}^n$.

By using Theorem 1.3, Kheyfits obtained an interesting version of Matsaev theorem on c-harmonic functions. Let $B(r)$ be the open ball of radius r centred at the point $0 \in \mathbb{R}^n$, $S(r) = \partial B(r)$ and $B_+(r) = B(r) \cap \mathbb{H}$. Set

$$K(r) = \partial\mathbb{H}\backslash S(r), \; 1 \leq r < \infty, \;\; \text{and} \quad K(1,r) = K(r)\backslash\overline{K(1)}.$$

Kheyfits's version of Matsaev theorem is the following result.

Theorem 1.4: *Let c belong to $\mathcal{C}(\mathbb{H})$, and $u(x)$ be a c-harmonic function in $\mathbb{H}$, which is continuous up to the boundary $\partial\mathbb{H}$. Suppose*

$$|u(x)| \leq M$$

in the unit half-ball $\overline{B_+}$. Also suppose that the negative part of u has an integral estimate

$$\int_{S_+} u^-(r,\theta)\cos\theta_1 d\sigma(\theta) \leq M(1+r^\rho), \tag{5}$$

with $\rho > \rho_k = \frac{2-n+\chi_k}{2}$, and its boundary values satisfy

$$V_1(r)\int_{K(1,r)} u^-(y')\frac{W_1(|y'|)}{|y'|}\mathrm{d}y' \leq M(1+r^\rho), \tag{6}$$

in which

$$V_1(r) = Mr^{(2-n+\chi_k)/2}(1+o(1)),\ r\to\infty,$$

and

$$W_1(r) = Mr^{(2-n-\chi_k)/2}(1+o(1)),\ r\to\infty,$$

are two solutions of

$$-y'' - (n-1)r^{-1}y' + \lambda r^{-2} + q(r)y(r) = 0,\quad 0<r<\infty, \lambda = n-1,$$

where $\chi_k = \sqrt{n^2+4k}$. Then for all x in $\mathbb{H}$

$$\max_{\theta\in S} u(x) \leq M(1+|x|^\rho),$$

that is, u has the growth of at most order ρ and normal type in $\mathbb{H}$.

In this paper, we present generalized Matsaev results on growth of subharmonic functions admitting a lower bound in $\mathbb{R}^n$. Different from all the others, my approach is based on techniques developed in papers.[10,12–14]

The generalization in this paper not only includes Theorem 1.1 as a special case, but also generalizes the half space result in Theorem 1.4 to the entire space. The work in this paper is a continuation of the study on the growth of harmonic functions and subharmonic functions in the upper half-space conducted in [8,10,12–16]. It is also a further development of the study of the Maximum Principle (e.g. [8,10,17]) and operator theory (e.g. [2,4]).

The paper is organized as follows. In Section 2, we introduce some basic concepts to be used throughout the paper. The refinement of the upper bound for a subharmonic function in $\mathbb{H}$ will be presented and proved in Section 3. This is a high-dimensional version of Theorem 1.2.

The statement and proof of the generalized Matsaev theorem of subharmonic functions admitting a lower bound for $\rho > 1$ and $\rho \leq 1$ are provided in Sections 4 and 5, respectively. Both Sections 4 and 5 generalize the results in Theorems 1.1 and 1.4.

2. Preliminaries

This section introduces some notations, please refer to [10,18] for more details.

For $n \geq 2$, the hyperplane $\mathbb{R}^{n-1} \times \{0\} = \mathbb{R}^{n-1}$ is the boundary of $\mathbb{H}$, which is also denoted by $\partial\mathbb{H}$. One can define the lower half-space by

$$-\mathbb{H} = \{x = (x_1, x_2, \ldots, x_n),\ x_n < 0\}.$$

Taking

$$x' = (x_1, x_2, \ldots, x_{n-1})$$

into account, set

$$x = (x_1, x_2, \ldots, x_n) = (x', x_n).$$

In the sense of Lebesgue measure

$$\mathrm{d}x' = \mathrm{d}x_1 \cdots \mathrm{d}x_{n-1}, \quad \mathrm{d}x = \mathrm{d}x'\mathrm{d}x_n.$$

Let $|x|$ denote the Euclidean norm. Then

$$|x'|^2 = x_1^2 + x_2^2 + \cdots + x_{n-1}^2, \quad |x|^2 = |x'|^2 + x_n^2.$$

The unit vector based on $x(\neq 0)$ will be denoted by $\frac{x}{|x|}$, $x \neq 0$. For simplicity, a point $x' \in \mathbb{R}^{n-1}$ is often identified with $(x', 0)$ in $\mathbb{R}^n$ and is identified with the projection of x onto the hyperplane $\partial\mathbb{H}$. The notation $B(x_j, \rho_j)$ represents the open ball on $\mathbb{R}^n$ with centre $x_j \in \mathbb{R}^n$ and radius $\rho_j > 0$.

According to [19], let φ be the angle between $x \in \partial\mathbb{H}$ and the nth unit coordinate vector, i.e.

$$x_n = |x| \sin\varphi, |x'| = |x| \cos\varphi, \quad 0 \leq \varphi < \frac{\pi}{2}.$$

A function u defined in $\mathbb{H}$ with values in $[-\infty, \infty)$ is called *subharmonic* [20] if

(1) u is upper semicontinuous;
(2) for every compact subset K of $\mathbb{H}$ and every continuous function v on K which is harmonic in the interior of K, the inequality $u \leq v$ is valid in K if it holds in ∂K.

Let f be a complex-valued function defined in an open set D contained in the complex plane $\mathbb{C}$, i.e. $D \subset \mathbb{C}$. Write

$$f = u(x, y) + iv(x, y),$$

where u and v are real valued. We may induce a function $\overrightarrow{f}$ from f, defined on the induced set $\overrightarrow{D} \subset \mathbb{R}^n$, as follows:

$$\overrightarrow{f}(x) = u(|x'|, x_n) + \frac{x'}{|x'|} v(|x'|, x_n), x \in \overrightarrow{D}. \tag{7}$$

The function $\overrightarrow{f}$ will be called the *induced function* from f.

Let I be a domain on the unit sphere $S \subset \mathbb{R}^n$. We always assume that the boundary $\partial\mathbb{H}$ with respect to S is not a polar set in the light of the classical potential theory. Let

$$K^I = \{x = (r, \theta) \in \mathbb{R}^n, 0 < r < \infty, \theta \in I\}$$

be a cone generated by the domain I. Truncated cones are denoted by

$$K_r^I = K^I \cap B(0, r).$$

Recall that a subharmonic function $h(x)$ belongs to Cartwright's class C, if

$$|h(x)| < (\sigma + \varepsilon)|x|, \quad x > M, \tag{8}$$

in which $0 < \sigma < +\infty$ is a constant, and

$$\int_{\partial \mathbb{H}} \frac{h^+(x')}{1 + |x'|^n} \mathrm{d}x' < \infty. \tag{9}$$

The following two lemmas [14] will be used in this paper.

Lemma 2.1: *For a Cartwright's class C function $h(x)$,*

$$h(x) = \sigma_+ x_n + o(|x|), \; x_n \geq 0;$$
$$h(x) = \sigma_- x_n + o(|x|), \; x_n \leq 0,$$

hold in $\mathbb{H} \backslash G$ and $-\mathbb{H} \backslash G$, respectively, where $0 < \sigma_\pm < +\infty$, $G = \bigcup_{j=1}^{\infty} B(x_j, \rho_j)$, and $\rho_j > 0$ such that

$$\sum_{j=1}^{\infty} \frac{\rho_j}{|x_j|} \leq \rho_j, \quad j = 1, 2, \dots.$$

Lemma 2.2: *For a subharmonic function $h(x)$ in $\mathbb{R}^n$ to belong to the Cartwright's class C it is sufficient and necessary that the function $h(x)$ have positive harmonic majorants in the upper and lower half-space $\mathbb{H}$ and $-\mathbb{H}$, respectively, here $-\mathbb{H}$ is the lower half space defined as above.*

3. Refinement of the upper bound

We now investigate the refinement of the upper bound for subharmonic functions in $\mathbb{H}$.

To begin, we cite the following Phrágmen–Lindelöf theorem [21] for the subharmonic functions.

Theorem 3.1: *Let $u(x)$ be a subharmonic function in a cone K^I, where I is a regular domain on $\mathbb{R}^n$ and*

$$M(r, u) = \sup \left\{ u(x) : x \in K_r^I \right\}.$$

If

$$\liminf_{r \to \infty} r^{-\mu^+} M(r, u) \leq 0 \quad \textit{and} \quad \sup_{x \in \partial K_r^I} u(x) \leq M,$$

Then

$$u(x) \leq M, \quad x \in K^I.$$

The positive solution μ^+ of the quadratic equation

$$\mu(\mu + n - 2) = \lambda_0$$

is called the characteristics constants of the domain I, *here* λ_0 *is the least positive eigenvalue of the Beltrami operator in cone* $K^I \cap \{|x| = 1\}$ *for functions vanishing at the boundary of this region.*

By Theorem 3.1 and the methods of [8,11], we will be able to prove the following theorem, which is similar to Theorem 1.3.

Theorem 3.2: *Let* $u(x)$ *be a subharmonic function in* $\mathbb{R}^n$ *which satisfies the estimate*

$$u(x) \le M \frac{1+|x|^\rho}{|\sin\varphi|^l}, \quad \rho > 1, \quad l > 0, \quad x \in \mathbb{R}^n, \qquad (10)$$
$$|x'| = |x|\cos\varphi, x_n = |x|\sin\varphi, \quad 0 < \varphi < \pi.$$

Then

$$u(x) \le M\left(1 + r^{2\rho}\right), \ x \in \mathbb{R}^n. \qquad (11)$$

***Proof*:** Without loss of generality, we assume $l > \frac{1}{4}$. Then $\frac{\pi}{8l} < \frac{\pi}{2}$. We consider the triangle abc having the vertices

$$a = \left(2\csc\frac{\pi}{8l}, 0, \ldots, 0\right),$$
$$b = \left(0, 0, \ldots, 2\sec\frac{\pi}{8l}\right),$$
$$c = \left(-2\csc\frac{\pi}{8l}, 0, \ldots, 0\right),$$

and a point

$$d = \left(0, 0, \ldots, -2\sec\frac{\pi}{8l}\right),$$

denote by Γ the boundary of the rhombus with vertices at the points a, b, c, d.

Similar to [8], we set

$$h(x) = \frac{1}{|x+a|^{2l}|x-a|^{2l}}$$

for x on side ab.

If $l \le \frac{n}{2}$, $h(x)$ is subharmonic within Γ, and the following can be shown:

$$h(x) \ge \frac{M}{|\sin\theta|^{2l}} \qquad (12)$$

on Γ.

In fact, we first apply the Sine Rule in $\triangle oax$ to obtain

$$\frac{|x-a|}{\sin\theta} = \frac{|x|}{\sin\frac{\pi}{8l}},$$

θ is the angle between vectors $\overrightarrow{ox}$ and $\overrightarrow{oa}$. Applying the Sine Rule again in $\triangle axc$, we get

$$\frac{|x+a|}{\sin\frac{\pi}{8l}} = \frac{|x-a|}{\sin\psi},$$

where ψ is the angle between vectors $\overrightarrow{ox} + \overrightarrow{oa}$ and $\overrightarrow{oa}$. Then we have

$$h(x) = \frac{1}{|x+a|^{2l}|x-a|^{2l}} = \frac{(\sin\frac{\pi}{8l})^{2l}(\sin\psi)^{2l}}{|x|^{2l}|\sin^{2l}\theta|}.$$

Letting

$$M = \frac{(\sin\frac{\pi}{8l})^{2l}(\sin\psi)^{2l}}{|x|^{2l}}, \quad -\frac{\pi}{8l} \le \psi \le \frac{\pi}{8l}.$$

We see that (12) is proved.

Noting that Γ contains the projection of unit sphere to the plane of rhombus and applying the maximum principle to the function

$$u(rx) - h(x),$$

we see $u(x)$ is bounded above inside Γ and

$$u(x) \le \max_{|x|=1} h(x) + \max_{x\in\Gamma}[u(rx) - h(x)] \le M + \max_{0\le\theta\le\pi}\left(\frac{Mr^\rho}{|\sin\theta|^l} - \frac{M}{|\sin\theta|^{2l}}\right).$$

Since

$$\max_{0\le\theta\le\pi}\left(\frac{Mr^\rho}{|\sin\theta|^l} - \frac{M}{|\sin\theta|^{2l}}\right) = Mr^{2\rho},$$

applying Theorem 3.1 within cone K^I, we obtain the desired inequality in the theorem. □

4. Growth of subharmonic functions for $\rho > 1$

In this section, we present the generalized Matsaev theorem of subharmonic functions admitting a lower bound for $\rho > 1$. To prove the result, we need the following theorem [10] that provides a lower bound derived from an upper one for harmonic functions in $\mathbb{H}$.

Theorem 4.1: *Let $u(x)$ be a harmonic function in the half space $\mathbb{H}$ with continuous boundary values on the boundary hyperplane $\mathbb{R}^{n-1}$. Suppose*

$$u(x) \le Kr^\rho, \; x \in \mathbb{R}^n, r = |x| \ge 1, \rho > n-1, \tag{13}$$

and

$$|u(x)| \le K, \; |x| \le 1, x_n > 0. \tag{14}$$

Then

$$u(x) \ge -CK\frac{(1+r^\rho)}{|\sin\varphi|^{n-1}}, \tag{15}$$

where

$$x_n = |x|\sin\varphi, |x'| = |x|\cos\varphi, \; 0 < \varphi < \frac{\pi}{2}.$$

The estimate provided by the above theorem is important for studying harmonic functions and their growth properties since the assumption of the theorem is weaker than that of the Maximum Principle. Note that the theorem reduces to the well-known theorem in [11] when $n = 2$.

The following is the generalized Matsaev theorem on growth of subharmonic functions admitting a lower for $\rho > 1$.

Theorem 4.2: *Suppose a subharmonic function $u(x)$ has the lower bound*

$$u(x) \geq -Mr^{\rho}\frac{1}{|\sin\varphi|^{k}}, \;\; \rho > 1, k \geq 0, \;\; x \in \mathbb{R}^{n}, \qquad x_n = |x|\sin\varphi. \tag{16}$$

Then u has the growth of order ρ and normal type in $\mathbb{R}^{n}$.

***Proof*:** Let $u(x)$ be a subharmonic function satisfying (16). Without loss of generality, we assume that $u(x) \neq 0$ for $|x| \leq 1$, and hence

$$|u(x)| \geq M, \quad |x| \leq 1. \tag{17}$$

We choose γ and φ such that

$$1 < \gamma < \min(2, \rho), 0 < \varphi < \pi\left(1 - \frac{1}{\gamma}\right),$$

and consider the function

$$u_{\gamma,\varphi}(y) = u\left(y^{\frac{1}{\gamma}}\omega\right),$$

in which $y = (y', y_n) \in \mathbb{H}$ with

$$y' = |y|\cos\theta \;\; \text{and} \;\; y_n = |y|\sin\theta, \;\; y^{\frac{1}{\gamma}} = \left(y_1^{\frac{1}{\gamma}}, \ldots, y_n^{\frac{1}{\gamma}}\right),$$

and $\omega = (\omega', \omega_n) \in \mathbb{H}$ with

$$|\omega| = 1, |\omega'| = \cos\frac{\varphi}{2}, \;\; \text{and} \;\; \omega_n = \sin\frac{\varphi}{2},$$

here θ and $\frac{\varphi}{2}$ are the angles between y and $\hat{e}_n, \omega$ and $\hat{e}_n$, respectively. $y^{\frac{1}{\gamma}}\omega$ is the induced function from $\zeta^{\frac{1}{\gamma}}e^{i\frac{\varphi}{2}}, \zeta = re^{i\theta} \in \mathbb{C}$.

Note that $u_{\gamma,\varphi}(y)$ is a harmonic function in the closed upper half-space $\overline{\mathbb{H}}$, which satisfies the following

$$|u(x)| \leq \frac{M|x|^{\rho}}{|\sin\varphi|^{k}} \leq \frac{M}{|\sin\varphi|^{k}}, \qquad |x| = |y^{\frac{1}{\gamma}}\omega| \leq 1.$$

Recall that y and ω are in the induced space of $\mathbb{H}$,

$$|u(x)| \leq \frac{M}{|\sin\left(\frac{\theta}{\gamma} + \frac{\varphi}{2}\right)|^{k}} \leq \frac{M}{|\csc\frac{\varphi}{2}|^{k}} = K.$$

Therefore $u_{\gamma,\varphi}(y)$ satisfies the following estimate

$$u_{\gamma,\varphi}(y) \geq -|y|^{\frac{\rho}{\gamma}}|\csc\left(\frac{\varphi}{2} + \frac{\theta}{\gamma}\right)|^{k} \geq -M|y|^{\frac{\rho}{\gamma}}|\csc\left(\frac{\varphi}{2}\right)|^{k}. \tag{18}$$

By Theorem 4.1 and $-u_{\gamma,\varphi}(y)$, the estimates (17) and (18) yield the upper bound

$$u_{\gamma,\varphi}(y) \le M|y|^{\frac{\rho}{\gamma}} |\csc\left(\frac{\varphi}{2}\right)|^{k} \csc\theta.$$

In order to estimate the function $u(x)$, we know that $0 < \gamma\frac{\varphi}{2} < \pi$, and set $y = (r\omega)^{\gamma}$ in (18). By the definition of $u_{\gamma,\varphi}(y)$ we obtain the estimate

$$u(x) = u_{\gamma,\varphi}[(r\omega)^{\gamma}] \le Mr^{\rho}|\csc\left(\frac{\varphi}{2}\right)|^{k}\csc\frac{\gamma\varphi}{2} \le Mr^{\rho}(\csc\varphi)^{k+1}, \tag{19}$$

within the cone

$$\left\{x \in \mathbb{H}, 0 < \varphi < \pi\left(1 - \frac{1}{\gamma}\right)\right\}.$$

In the same way the estimate is proved for

$$0 > \varphi > -\pi\left(1 - \frac{1}{\gamma}\right).$$

If we replace $u(x)$ by $u(-x)$, we find that (19) holds within the cone

$$\left\{x \in \mathbb{H}, |\pi - \varphi| < \pi\left(1 - \frac{1}{\gamma}\right)\right\}.$$

It remains to obtain the estimate of $u(x)$ within the cone

$$\left\{x \in \mathbb{H}, |x'| = |x|\cos\varphi, x_n = |x|\sin\varphi, \pi\left(1 - \frac{1}{\gamma}\right) < \varphi < \frac{\pi}{\gamma}\right\}.$$

For this purpose we set $y = v^{\gamma}$ in (18), where

$$v = (v', v_n) \in \mathbb{H} \ \text{ with } \ v' = r\cos\left(\phi - \frac{\varphi}{2}\right) \ \text{ and } \ v_n = r\sin\left(\phi - \frac{\varphi}{2}\right),$$

and obtain

$$u(y) = [u_{\gamma,\varphi}(v)]^{\gamma} \le M(1 + r^{\rho}).$$

The same estimate is obviously true for

$$-\frac{\pi}{\gamma} < \pi < -\pi\left(1 - \frac{1}{\gamma}\right).$$

Combining all the estimates, we obtain

$$u(x) \le \frac{M(1 + |x|^{\rho})}{|\sin\varphi|^{l}},$$

with $l = k + 1$. Then, Theorem 4.2 follows immediately from Theorem 3.2. □

When $n = 2$, the above theorem reduces to Theorem 1.1 where $u(z) = \log|f(z)|$. In addition, the theorem generalizes the half space result in Theorem 1.4 to the entire space.

5. Growth of subharmonic functions for $\rho \leq 1$

As for the subharmonic functions admitting a lower bound for $\rho \leq 1$, we have the following generalized Matsaev result, whose proof utilizes an approach different from that in the previous section.

Theorem 5.1: *Suppose a subharmonic function $h(x)$ in $\mathbb{R}^n$ has the lower bound*

$$h(x) \geq -M\frac{1+|x|^{\rho}}{x_n^k}, \quad x_n = |x| \sin\varphi, \, 0 \leq \rho < 1, \, k > 0. \tag{20}$$

Then it belongs to Cartwright's class C.

***Proof*:** To begin with, we consider the subharmonic function $-h(x)$ in the half-space $\{x \in \mathbb{R}^n, x_n > 1\}$. By (20)

$$-h(x) \leq -M(1+|x|^{\rho}), \quad \rho < 1, \, x_n \geq 1. \tag{21}$$

Hence $-h(x)$ has a positive harmonic majorant in the upper half-space $\mathbb{H}$, and then has the following representation [20] (P_{128}, Theorem 3.1.8′ and P_{186}, Theorem 3.3.16)

$$-h(x) = -\frac{1}{(n-2)w_n}\int_{\mathbb{H}} h(y+e)\left[\frac{1}{|x--y+e|^{n-2}} - \frac{1}{|x-\widetilde{y}+e|^{n-2}}\right]\mathrm{d}y$$
$$-\frac{2(x_n-1)}{nw_n}\int_{\partial\mathbb{H}}\frac{h(y'+e)}{|x--y'+e|^n}\mathrm{d}y' + M(x_n-1), x_n > 1,$$

in which

$$e = (0, \ldots, 0, 1),$$

and

$$\widetilde{y} = (y_1, \ldots, y_{n-1}, -y_n)$$

is the reflection of y onto the hyperplane $\partial\mathbb{H}$, where

$$\int_{\mathbb{H}}\frac{h(y+e)}{1+|y|^n}\mathrm{d}y < \infty, \tag{22}$$

and

$$\int_{\partial\mathbb{H}}\frac{h(y'+e)}{1+|y'|^n}\mathrm{d}y' < \infty. \tag{23}$$

By Lemma 2.1

$$-h(x) < (\sigma+\varepsilon)|x|, \quad \sigma = \max(\sigma_+, \sigma_-),$$

holds for $|x| > M, x \in \mathbb{R}^n$. Therefore,

$$h(x) < (\sigma+\varepsilon)|x|, \quad |x| > M, \tag{24}$$

in the half-space

$$\mathbb{H}_1 = \{x = (x', x_n) \in \mathbb{R}^n, x_n \geq 1\}$$

and

$$\{x = (x', x_n) \in \mathbb{R}^n, x_n \leq -1\}.$$

Applying Theorem 4.1 in

$$\{|y_n| \leq 1\}$$

for

$$-h(x) - Mx',$$

we have

$$|-h(x)| < (\sigma + \varepsilon)|x|.$$

Note that condition (23) yields the convergence of the integral

$$\frac{2(x_n - 1)}{w_n} \int_{\partial \mathbb{H}} \frac{h(y' + e)}{|x - -y' + e|^n} \mathrm{d}y', \quad x_n > 1,$$

which defines a positive harmonic function in $\mathbb{H}_1$ with boundary values on the hyperplane

$$\{x = (x', x_n) \in \mathbb{R}^n, x_n = 1\}$$

coinciding with $h(y' + e)$. Thus $h(x)$ has a positive harmonic majorant in the half-space

$$\{x = (x', x_n) \in \mathbb{R}^n, x_n < 1\}.$$

Similarly, $h(x)$ has a positive harmonic majorant in the half-space

$$\{x = (x', x_n) \in \mathbb{R}^n, x_n > -1\}.$$

Therefore, $h(x)$ has a positive majorant in both $\mathbb{H}$ and $-\mathbb{H}$. Applying Lemma 2.2, we conclude that $h(x)$ belongs to the Cartwright's class C. □

Disclosure statement

No potential conflict of interest was reported by the authors.

Funding

The work was supported by NSFC [grant number 11501015]; NSFC [grant number 10901166]; Macau Government FDCT [grant number 099/2012/A3].

References

[1] Matsaev VI. On the growth of entire functions that admit a certain estimate from below. Dokl. Akad. Nauk. SSSR. 1960;132:283–286. [English translate in Soviet Math. 1, 1960].
[2] Gohberg IC, Krein MG. Introduce to the theory of linear nonselfadjoint operators in Hilbert Space. Translations of mathematical monographs. Vol. 18. Providence (RI): American Mathematical Society; 1969.
[3] Kheyfits AI. Indicators of functions of order less than one that are analytic in an open half-plane and have completely regular growth in interior angles. Math. USSR, Izv. 1975;9:850–860.
[4] Matsaev VI, Mogul'skii EZ. A division theorem for analytic functions with a given majorant, and some of its applications. Investigations on linear operators and theory of functions. VI. Zap. Naucn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI). 1976;56:73–89.

[5] Govorrov NV, Zhuravleva MI. On an upper bound of the module of a function analytic in a half-plane and in a plane with a cut. Severo-Kavkaz. Nauchn. Tsentra Vyssh. Shkoly Estestv. Nauk. 1973;4:102–103.
[6] Rashkovskii A. Classical and new loglog-theorems. Expo. Math. 2009;27:271–287.
[7] Sergienko EN. Growth of functions representable as the difference of subharmonic functions and admitting a special lower bound. Teor. Funktsii Funktsional. Anal. i Prilozhen. 1982;37:116–122.
[8] Kheyfits AI. Growth of Schrödingerian subharmonic functions admitting certain bounds. Oper. Theory: Adv. Appl. 2013;229:215–231.
[9] Yoshida H. A boundedness criterion for subharmonic functions. J. London Math. Soc. 1981;S2 24:148–160.
[10] Zhang YH, Deng GT, Kou KI. On the lower bound for a class of harmonic functions in the half space. Acta Mathematica Scientia. 2012;32B:1487–1494.
[11] Levin BY. Lectures on entire functions. Providence (RI): American Mathematical Society; 1996.
[12] Zhang YH, Kou KI, Deng GT. Integral representation and asymptotic behavior of harmonic functions in half space. J. Differ. Equ. 2014;257:2753–2764.
[13] Zhang YH, Deng GT, Qian T. Integral representations of a kind of harmonic functions in the half space. J. Differ. Equ. 2016;2:923–936.
[14] Zhang YH. Phrágmen-Lindelöf theorems of subharmonic functions and their applications in the half space. Sci. Sin. Math. 2015;45:1931–1938. Chinese.
[15] Zhang YH, Deng GT. Integral representation and asymptotic property of analytic functions with order less than two in the half-plane. Complex Variables Elliptic Equ. 2005;50:283–297.
[16] Zhang YH, Deng GT. Growth properties for a class of subharmonic functions in half space. Acta Mathematica Sinica. 2008;51:319–326. Chinese.
[17] Carleman T. Extension d'un théorème de Liouville [Extension of Liouville's Theorem]. Acta Math. 1926;48:25–61.
[18] Qian T. Fourier analysis on starlike Lipschitz surfaces. J. Funct. Anal. 2001;183:370–412.
[19] Siegel D, Talvila E. Sharp growth estimates for modified Poisson integrals in a half space. Potential Anal. 2001;15:333–360.
[20] Hörmander L. Notions of convexity. Boston: Birkhäuser; 1994.
[21] Deny J, Lelong P. Sur une gènèralisation de l'indicatrice de Phragmén-Lindelöf [A generation of indication of Phragmen-Lindelof]. C.R. Acad. Sci. Paris. 1947;224:1046–1048.