Annali di Matematica pura ed applicata
(IV), Vol. CLXV (1993), pp. 369-394

# A Martingale Proof of $L_2$ Boundedness of Clifford-Valued Singular Integrals (*).

G. I. GAUDRY (**) - R. LONG (**) (***) - T. QIAN (*$_*$*)

**Summary.** – *This paper presents a theory of Clifford algebra-valued martingales on a $\sigma$-finte measure space, with respect to a pseudoaccretive weight. A novel dual pair system of Haar functions associated with the Clifford martingale is constructed, and Littlewood-Paley estimates are established. The dual pair system of Clifford Haar functions is used to give a new proof of the boundedness of the Cauchy principal value integral on Lipschitz surfaces, and of the Clifford-valued T(b) theorem.*

## 0. – Introduction.

In this paper we present a martingale proof of the $L_2$-boundedness of the Cauchy integral operator on Lipschitz surfaces. We then indicate how one proves the Clifford $T(b)$ theorem by using the same method. The method is in the spirit of [3]. Nevertheless, there are new features. One defines a suitable sequence of atomic $\sigma$-fields on $\mathbb{R}^d$; since the Clifford algebra is noncommutative, it is necessary to associate with each atom a *pair* of Clifford-valued «Haar» functions. Thus, the appropriate «Haar» system is in fact a system of pairs of Clifford-valued functions. We use only martingale techniques to prove the $L_2$-norm equivalence between the function $f$ and its Littlewood-Paley function $S(f)$. The first part of the paper is devoted to developing the relevant martingale theory and Littlewood-Paley estimates for Clifford-valued functions.

Versions of the $T(b)$ theorem have been formulated by a number of other authors ([3], [6], [8], [16] for instance). It will be clear to the reader familiar with [6] that our proof borrows a number of important ideas from that paper. Nevertheless, our approach stands on its own merits and offers a novel and unified treatment of the themes in question.

---

(*) Entrata in Redazione il 15 luglio 1991.
Indirizzo degli AA.: Mathematics Discipline, School of Information Science and Technology, The Flinders University of South Australia, P.O.Box 2100, Adelaide S.A. 5001, Australia.
(**) Research supported by the Australian Research Council.
(***) Research supported by the National Science Foundation of China.
(*$_*$*) Research carried out as a National Research Fellow.

The idea of using Clifford algebras in connection with singular integral operators is due to R. COIFMAN. References [11]-[14] and [17] provide ample evidence of the fruitfulness of this idea.

Thanks are due to ALAN MCINTOSH and MICHAEL COWLING for encouraging us to pursue a Clifford-martingale approach to these problems and for helpful conversations on the subject of this paper. The second author would like to express his gratitude to NEIL TRUDINGER for his kind invitation to visit the Centre for Mathematical Analysis, where part of this work was carried out.

## 1. - Preparation.

### 1.1. *Clifford algebras.*

For the convenience of the reader, we include a brief overview of the basic ideas of Clifford algebras. For more detailed accounts see [1], [2] and [17].

Let $d$ be a nonnegative integer, and $e_0, e_1, \ldots, e_d$ the standard basis of $\mathbb{R}^{1+d}$.

DEFINITION. – The Clifford algebra $\mathbb{A}_d$ is the noncommutative algebra over $\mathbb{R}$ generated by $e_0, e_1, \ldots, e_d$ subject to the relations

(i) $e_0 = 1$;

(ii) $e_j^2 = -1$ for $1 \leq j \leq d$;

(iii) $e_j e_k = -e_k e_j$ for $1 \leq j < k \leq d$.

If $S$ is a nonempty subset of $\{1, \ldots, d\}$, define the element $e_S$ of $\mathbb{A}_d$ as follows. Write $S$ in increasing order, say $S = \{j_1 < j_2 < \ldots < j_s\}$, and let $e_S = e_{j_1} e_{j_2} \ldots e_{j_s}$. For completeness, we write $e_\emptyset = e_0 = 1$. The elements $e_S$, $S \subseteq \{1, \ldots, d\}$ form a basis of $\mathbb{A}_d$. The algebra $\mathbb{A}_d$ is given an inner-product structure by declaring the basis $\{e_S\}_{S \subseteq \{1, \ldots, d\}}$ to be orthonormal. Thus, if $x = \sum_S x_S e_S$, then $|x| = \left( \sum_S |x_S|^2 \right)^{1/2}$. Note that $\mathbb{A}_d$ is a normed algebra, since there exists a constant $C$, depending on the dimension, such that

$$|xy| \leq C|x|\,|y| \tag{1}$$

for all $x, y \in \mathbb{A}_d$.

When $d = 0, 1, 2$, $\mathbb{A}_d$ is isomorphic to $\mathbb{R}$, $\mathbb{C}$, and the space of quaternions, respectively. Although $\mathbb{A}_d$ is not a skew field when $d > 2$, it is nevertheless the case that *every nonzero element of* $\mathbb{R}^{1+d}$ $(\subseteq \mathbb{A}_d)$ has a multiplicative inverse. In fact, if $x = x_0 e_0 + x_1 e_1 + \ldots + x_d e_d \in \mathbb{R}^{1+d}$, then $x^{-1} = \bar{x}|x|^{-2}$, where the *conjugate vector* is $\bar{x} = x_0 e_0 - x_1 e_1 - \ldots - x_d e_d$. Note also the elementary identity

$$a^{-1} - b^{-1} = a^{-1}(b-a)b^{-1} = b^{-1}(b-a)a^{-1} \tag{2}$$

for all nonzero elements $a, b \in \mathbb{R}^{1+d}$. The group generated by the set of nonzero elements of $\mathbb{R}^{1+d}$ is called the *Clifford group*. If $x$ and $y$ are two elements of the Clifford group, then $|xy| = |x|\,|y|$: see [1].

Let $\Omega$ denote an open subset of $\mathbb{R}^{1+d}$ and $f$ a $C^1$-function on $\Omega$ with values in $\mathbb{A}_d$. The operator

$$D_l = \sum_{j=0}^{d} \frac{\partial}{\partial x_j} e_j$$

acts on $f = \sum_S f_S e_S$ as follows:

$$D_l f = \sum_{j=0}^{d} \sum_S \frac{\partial f_S}{\partial x_j} e_j e_S .$$

DEFINITION. – The function $f$ is said to be *left-monogenic* if $D_l f = 0$.

If we let

$$D_r f = \sum_{j=0}^{d} \sum_S \frac{\partial f_S}{\partial x_j} e_S e_j ,$$

we can similarly define a *right-monogenic* function to be one for which $D_r f = 0$. When $f$ is both left- and right-monogenic, we say it is *monogenic*.

The most basic monogenic functions are obtained by fixing a point $y \in \Omega$ and letting

$$g_y(x) = \frac{\overline{y - x}}{|y - x|^{d+1}}, \qquad x \neq y .$$

Further examples can be constructed from the basic ones as follows. Let $\Sigma$ be a smooth $d$-dimensional oriented submanifold of $\mathbb{R}^{1+d}$, the unit normal to the manifold at the point $y$ consistent with the orientation being denoted $n(y)$. If $f$ is an $\mathbb{A}_d$-valued function that is absolutely integrable with respect to surface measure $d\sigma$, and $x \notin \Sigma$, we define

$$T_\Sigma f(x) = \frac{1}{\alpha_d} \int_\Sigma \frac{\overline{y - x}}{|y - x|^{d+1}}\, n(y)\, f(y)\, d\sigma(y), \tag{3}$$

where $\alpha_d$ is the volume of the unit $d$-sphere. Monogenicity of $T_\Sigma f$ follows by differentiating under the integral sign.

It turns out that $T_\Sigma f = f$ on $\Omega$ if $f$ is monogenic on a neighbourhood of $\overline{\Omega}$, where $\Omega$ is an open bounded set with smooth boundary, $\Omega \subsetneq \mathbb{R}^{1+d}$, $\Sigma$ is the boundary of $\Omega$, $\Omega$ lies on one side of $\Sigma$, and $n(y)$ is the unit exterior normal to $\Sigma = \partial\Omega$ at $y$. See [2]. For this reason, the operator (3) is called the *Cauchy operator associated to* $\Sigma$ (for say integrable functions $f$).

This paper deals not with (3) but with the corresponding singular integral operators on $\Sigma$. Define the *Cauchy singular integral operator* to be $T_\Sigma$ where

$$(T_\Sigma f)(x) = \frac{2}{\alpha_d}\,\text{p.v.}\int_\Sigma \frac{\overline{y-x}}{|y-x|^{d+1}}\, n(y)\, f(y)\, d\sigma(y) \tag{4}$$

for $x \in \Sigma$, whenever the principal value integral exists. The principal value is taken to be $\lim_{\varepsilon\to 0} \int_{y\in\Sigma:\ |x-y|>\varepsilon} \dots$ . The question is whether the Cauchy singular integral operator $f \mapsto T_\Sigma f$ given by (4) is bounded on some function space carried by $\Sigma$, for a given class of surfaces.

DEFINITION. – The surface $\Sigma \subseteq \mathbb{R}^{1+d}$ is a *Lipschitz graph* if

$$\Sigma = \{A(u)\, e_0 + u \colon u \in \mathbb{R}^d\}$$

where $A$ is a real-valued function which is differentiable a.e. and for which $\sup_j \|\partial A/\partial x_j\|_\infty < +\infty$.

DEFINITION. – The space $L_2(\Sigma;\ \mathbb{A}_d)$ is the space of equivalence classes of measurable functions $f$ on $\Sigma$ with values in $\mathbb{A}_d$:

$$f(x) = \sum_S f_S(x)\, e_S$$

such that

$$\|f\|_2 = \left(\int_\Sigma \sum_S |f_S(x)|^2\, d\sigma(x)\right)^{1/2} < +\infty .$$

One of the principal aims of this paper is to prove the following result.

THEOREM 1. – *If $\Sigma$ is a Lipschitz graph, then the Cauchy singular integral operator is bounded from* $L_2(\Sigma;\ \mathbb{A}_d)$ *to* $L_2(\Sigma;\ \mathbb{A}_d)$.

Notice that, if $f$ is a real-valued $L_2$-function on $\Sigma$, then $T_\Sigma$ is $\mathbb{A}_d$-valued, and that its *scalar part* $(T_\Sigma f)_0$, viz. the $e_0$-component of $T_\Sigma f$, is the singular double-layer potential operator

$$(T_\Sigma f)_0(x) = \frac{2}{\alpha_d}\,\text{p.v.}\int_\Sigma \frac{\langle y-x,\, n(y)\rangle}{|y-x|^{d+1}}\, f(y)\, d\sigma(y) . \tag{5}$$

Theorem 1 therefore has the following consequence.

COROLLARY. – *If $\Sigma$ is a Lipschitz graph, the singular double-layer potential operator* (5) *is bounded from* $L_2(\Sigma)$ *to* $L_2(\Sigma)$.

Both Theorem 1 and its Corollary are already known [4], [14], but with different proofs.

1.2. *Clifford-valued martingales.*

Let $X$ be a set, $\mathcal{B}$ a $\sigma$-field in $X$, $\nu$ be a non negative measure on $\mathcal{B}$, and $\{\mathcal{F}_n\}_{-\infty}^{\infty}$ a non-decreasing family of $\sigma$-fields in $X$ satisfying the following conditions:

(i) $\bigcup_{-\infty}^{\infty} \mathcal{F}_n$ generates $\mathcal{B}$;

(ii) $\bigcap_{-\infty}^{\infty} \mathcal{F}_n = \{\emptyset, X\}$;

(iii) the measure $\nu$ is $\sigma$-finite on $\mathcal{B}$ and on each $\mathcal{F}_n$.

Let $\mathcal{F}$ be a sub-$\sigma$-field of $\mathcal{B}$ such that $\nu$ is $\sigma$-finite on $\mathcal{F}$. Since $(X, \mathcal{F})$ is $\sigma$-finite, $X$ can be written $X = \bigcup_j U_j$ where $U_j \in \mathcal{F}$ and $\nu(U_j) < +\infty$. If $f$ is a locally integrable, scalar-valued function on $(X, \mathcal{B}, \nu)$, i.e. a $\mathcal{B}$-measurable function whose integral is finite on every set of finite $\nu$-measure, its conditional expectation $\widetilde{E}(f|\mathcal{F})$ is well-defined by specifying that, on each $U_j$, $\widetilde{E}(f|\mathcal{F})$ is equal to the conditional expectation of $f_{|U_j}$ with respect to $(\mathcal{F}_{|U_j}, \nu_{|U_j})$. It then follows that, if $A$ is any set in $\mathcal{F}$ of finite $\nu$-measure, then

$$\int_A \widetilde{E}(f|\mathcal{F})\, d\nu = \int_A f d\nu. \tag{6}$$

If $f$ is integrable, then (6) also holds for every $A \in \mathcal{F}$, whether of finite $\nu$-measure or not.

The definition of conditional expectation can be extended to locally integrable $\mathbb{A}_d$-valued functions, by specifying that, if $f = \sum_S f_S e_S$, then

$$\widetilde{E}(f|\mathcal{F}) = \sum_S \widetilde{E}(f_S|\mathcal{F})\, e_S .$$

The characteristic martingale property (6) holds also for $\mathbb{A}_d$-valued functions $f$.

We write $L^p(\mathcal{F}, d\nu; \mathbb{A}_d)$, or simply $L^p(d\nu; \mathbb{A}_d)$, $1 \leq p \leq \infty$, for the Lebesgue spaces of $\mathbb{A}_d$-valued $\mathcal{F}$-measurable functions on $X$. The space $L^1_{\mathrm{loc}}(d\nu; \mathbb{A}_d)$ has the obvious interpretation.

Suppose that $\psi$ is an $L^\infty$ function on $X$ with values in $\mathbb{R}^{1+d}$.

DEFINITION. – Suppose that $\widetilde{E}(\psi|\mathcal{F}) \neq 0$ a.e., and let $f \in L^1_{\mathrm{loc}}(d\nu; \mathbb{A}_d)$. The *left- and right-conditional expectations* $E^l$ and $E^r$ of $f$ with respect to $\mathcal{F}$ are given by the formulas

$$E^l(f) = E^l(f|\mathcal{F}) = \widetilde{E}(\psi|\mathcal{F})^{-1}\widetilde{E}(\psi f|\mathcal{F}) \tag{7}$$

$$E^r(f) = E^r(f|\mathcal{F}) = \widetilde{E}(f\psi|\mathcal{F})\widetilde{E}(\psi|\mathcal{F})^{-1} . \tag{8}$$

The conditional expectations with respect to $\mathcal{F}_n$ are denoted $E^l(f|\mathcal{F}_n)$ or $E_n^l(f)$, and $E^r(f|\mathcal{F}_n)$ or $E_n^r(f)$.

The mapping properties of $E^l$ and $E^r$ are good only under further assumptions on the function $\psi$.

PROPOSITION 1. – *Suppose* $1 \leq p \leq \infty$. *The operator* $E^l$ *(resp.* $E^r$*) is bounded on* $L^p$ *if and only if there exists a constant* $c_0 > 0$ *such that*

$$c_0^{-1} \leq |\widetilde{E}(\psi|\mathcal{F})(x)| \leq c_0 \quad \text{for a.e. } x. \tag{9}$$

PROOF. – This follows by modifying the corresponding argument in [5]. ■

A function $\psi \in L^\infty(X; \mathbb{R}^{1+d})$ that satisfies (9) is said to be *pseudoaccretive with respect to* $\mathcal{F}$. We take as a standing assumption from now on that the condition (9) is satisfied for the generic $\mathcal{F}$ and for all $\mathcal{F}_n$, the constant in (9) being independent of $n$. That being so, it follows that, if $f \in L^1_{\text{loc}}(d\nu; \mathbb{A}_d)$, then $E^l(f)$ and $E^r(f)$ are locally integrable also.

The main elementary properties of $E^l$ and $E^r$ are as follows.

PROPOSITION 2. – *Let the notation be as above. Then*

(*a*) *If* $g \in L^\infty(\mathcal{F}, d\nu; \mathbb{A}_d)$, *then*

$$E^l(fg) = E^l(f)g.$$

*Similarly, the right-conditional expectation* $E^r$ *commutes with multiplication on the left by* $g$.

(*b*)
$$E^l(1) = E^r(1) = 1.$$

(*c*) *If* $f \in L^1_{\text{loc}}(d\nu; \mathbb{A}_d)$, *and* $A$ *is of finite measure (resp.* $f \in L^1(d\nu; \mathbb{A}_d)$, *and* $A$ *is measurable), then*

$$\int_A \psi E^l(f)\, d\nu = \int_A \psi f\, d\nu, \tag{10}$$

$$\int_A E^r(f)\psi\, d\nu = \int_A f\psi\, d\nu. \tag{11}$$

(*d*) *For* $n \leq m$, *we have*

$$E_n(E_m(f)) = E_n(f) \tag{12}$$

*where* $E_n$ *denotes the left- (or right-) conditional expectation with respect to* $\mathcal{F}_n$.

(*e*) *Writing* $\Delta_n^l = E_n^l - E_{n-1}^l$, *and* $\Delta_n^r = E_n^r - E_{n-1}^r$, *and*

$$\langle f, g\rangle_\psi = \int f\psi g\, d\nu,$$

*we have*

$$\langle \Delta_n^r f, \Delta_m^l g\rangle_\psi = 0\,, \qquad \textit{for all } n \neq m, \textit{ and } f, g \in L^2(d\nu;\ \mathbb{A}_d).$$

PROOF. – Statements (*a*) and (*b*) are obvious. To prove (*c*), suppose that $A \in \mathcal{F}$. Then

$$\int_A \psi E^l f\, d\nu = \int_X \chi_A \psi E^l f\, d\nu = \int_X \widetilde{E}(\chi_A \psi E^l f)\, d\nu = \int_X \chi_A \widetilde{E}(\psi)\, E^l f\, d\nu = \int_A \psi f\, d\nu$$

since $E^l f$ and $A$ are $\mathcal{F}$-measurable. Similarly for $E^r$.

Part (*d*) is proved as follows: for the left-conditional expectation, for example,

$$E_n^l(E_m^l(f)) = \widetilde{E}_n(\psi)^{-1}\widetilde{E}_n(\psi \widetilde{E}_m(\psi)^{-1}\widetilde{E}_m(\psi f)) =$$

$$= \widetilde{E}_n(\psi)^{-1}\widetilde{E}_n(\widetilde{E}_m[\psi \widetilde{E}_m(\psi)^{-1}\widetilde{E}_m(\psi f)]) = \widetilde{E}_n(\psi)^{-1}\widetilde{E}_n(\psi f) = E_n^l(f)\,.$$

The argument for the right-conditional expectation is similar.

Finally we prove (*e*). For $n > m$,

$$\langle \Delta_n^r f, \Delta_m^l g\rangle_\psi = \int \Delta_n^r f \psi \Delta_m^l g\, d\nu = \int \widetilde{E}_{n-1}(\Delta_n^r f \psi \Delta_m^l g)\, d\nu = \int \widetilde{E}_{n-1}(\Delta_n^r f \psi)\, \Delta_m^l g\, d\nu =$$

$$= \int \widetilde{E}_{n-1}(\Delta_n^r f\psi)\, \widetilde{E}_{n-1}(\psi)^{-1}\widetilde{E}_{n-1}(\psi)\, \Delta_m^l g\, d\nu = \int \widetilde{E}_{n-1}^r(\Delta_n^r f)\, \widetilde{E}_{n-1}(\psi)\, \Delta_m^l g\, d\nu = 0\,,$$

where the last step used (12). For $m > n$, the proof is similar. ■

DEFINITION. – Let $f \in L^1_{\mathrm{loc}}(d\nu;\ \mathbb{A}_d)$. The *left-martingale with respect to* $\{\mathcal{F}_n\}_{-\infty}^{\infty}$ *generated by* $f$ is the sequence $\{f_n^l\}_{-\infty}^{\infty} = \{E_n^l(f)\}_{-\infty}^{\infty}$. The left-Littlewood-Paley square function is

$$S^l(f) = \left(|f_{-\infty}^l|^2 + \sum_{-\infty}^{\infty} |\Delta_n^l f|^2\right)^{1/2}$$

if the limit $f_{-\infty}^l = \lim_{n \to -\infty} E_n^l(f)$ exists pointwise a.e..

The right-martingale and right-Littlewood-Paley square functions are defined similarly. Note that if $f \in \bigcup_{\infty > p \geq 1} L^p(d\nu;\ \mathbb{A}_d)$, and $\nu(X) = +\infty$, then $f_{-\infty}^l = 0$. See [13].

If $f \in L^1_{\mathrm{loc}}(d\nu;\ \mathbb{A}_d)$, the BMO-norm of $f$ is defined to be

$$\|f\|_{\mathrm{BMO}} = \sup_n \|\widetilde{E}_n(|f - \widetilde{E}_{n-1} f|^2)\|_\infty^{1/2}\,. \tag{13}$$

We shall need to use the fact that, if $\psi \in L^\infty(d\nu; \mathbb{R}^{1+d})$, then $\psi \in \mathrm{BMO}$, and

$$\widetilde{E}_n\left(\sum_{k=n}^{\infty} |\widetilde{\Delta}_k(\psi)|^2\right) \leqslant C\|\psi\|^2_{\mathrm{BMO}} \leqslant C\|\psi\|^2_\infty \tag{14}$$

for every $n$. (As usual, «tilde» refers to the ordinary conditional expectation.) By the John-Nirenberg inequality, the right-hand side of (13) is equivalent to $\sup_n \|\widetilde{E}_n(|f - \widetilde{E}_n f|)\|_\infty$. See [9] and [10].

The following Littlewood-Paley result is one of the essential ingredients of this paper. In the proof of Lemma 1, and elsewhere in this paper, $C$ will denote a constant which may vary from line to line.

LEMMA 1. – *There exists a constant $c > 0$, depending only on $c_0$ and $d$, such that*

$$c^{-1}\|S(f)\|_{L_2} \leqslant \|f\|_{L_2} \leqslant c\|S(f)\|_{L_2}, \tag{15}$$

*for all $f \in L^2_{\mathrm{loc}}(d\nu; \mathbb{A}_d)$, where $S$ denotes either $S^l$ or $S^r$.*

PROOF. – We consider the left-martingale case only. The other case is treated similarly. Let $n_0$ be fixed, and consider the sequence $\{\mathcal{F}_n\}_{n \geqslant n_0}$ and the corresponding part of the square function: $\left(\sum_{n \geqslant n_0+1} |\Delta_n^l f|^2\right)^{1/2}$. If $n \geqslant n_0 + 1$, we have

$$\Delta_n^l f = \widetilde{E}(\psi|\mathcal{F}_n)^{-1}\widetilde{E}(\psi f|\mathcal{F}_n) - \widetilde{E}(\psi|\mathcal{F}_{n-1})^{-1}\widetilde{E}(\psi f|\mathcal{F}_{n-1}) = \tag{16}$$

$$= [\widetilde{E}(\psi|\mathcal{F}_n)^{-1} - \widetilde{E}(\psi|\mathcal{F}_{n-1})^{-1}]\widetilde{E}(\psi f|\mathcal{F}_n) + \widetilde{E}(\psi|\mathcal{F}_{n-1})^{-1}[\widetilde{E}(\psi f|\mathcal{F}_n) - \widetilde{E}(\psi f|\mathcal{F}_{n-1})].$$

Therefore,

$$|\Delta_n^l(f)|^2 \leqslant C(|\widetilde{\Delta}_n(\psi)|^2\,|\widetilde{E}(\psi f|\mathcal{F}_n)|^2 + |\widetilde{\Delta}_n(\psi f)|^2), \tag{17}$$

by (1), (2) and (9). Since $\nu$ is $\sigma$-finite on $\mathcal{F}_{n_0}$, we may write $X = \bigcup_{j=1}^{\infty} U_j$, where $U_1 \subseteq U_2 \subseteq \ldots$ and the sets $U_j$ are in $\mathcal{F}_{n_0}$ and of finite measure. Fix $M \geqslant 1$. Then by (17),

$$\int_{U_M} \sum_{n\geqslant n_0+1} |\Delta_n^l f|^2 \leqslant C\left(\int_{U_M} \sum_{n\geqslant n_0+1} |\widetilde{E}(\psi f|\mathcal{F}_n)|^2\,|\widetilde{\Delta}_n\psi|^2\,d\nu + \int_{U_M} \sum_{n\geqslant n_0+1} |\widetilde{\Delta}_n(\psi f)|^2\,d\nu\right) \leqslant \tag{18}$$

$$\leqslant C\left(\int_{U_M} \sum_{n\geqslant n_0+1} \widetilde{E}_n^{*2}(\psi f)\,|\widetilde{\Delta}_n(\psi)|^2\,d\nu + \int_X |\psi f|^2\,d\nu\right) \leqslant$$

$$\leqslant C\left(\int_{U_M} \sum_{n\geqslant n_0+1} \widetilde{E}_n^{*2}(\psi f)\,|\widetilde{\Delta}_n(\psi)|^2\,d\nu + \int_X |f|^2\,d\nu\right)$$

by the standard Littlewood-Paley estimates [9], where

$$\tilde{E}_n^*(f) = \sup_{n_0+1 \leq j \leq n} |\tilde{E}(f|\mathcal{F}_j)|.$$

Let $T_n = \sum_{k=n}^{\infty} |\tilde{\Delta}_k \psi|^2$ for $n \geq n_0 + 1$, and set $T_{n_0} = 0$. If $N > n_0$, we have

$$\sum_{n=n_0+1}^{N} \tilde{E}_n^{*2}(\psi f)|\tilde{\Delta}_n(\psi)|^2 = \sum_{n=n_0+1}^{N} \tilde{E}_n^{*2}(\psi f)(T_n - T_{n+1}) = \tag{19}$$

$$= \sum_{n=n_0}^{N-1} T_{n+1}(\tilde{E}_{n+1}^{*2}(\psi f) - \tilde{E}_n^{*2}(\psi f)) - \tilde{E}_N^{*2}(\psi f)\, T_{N+1}\,.$$

It follows from (14) and (19) that

$$\int_{U_M} \sum_{n \geq n_0+1} \tilde{E}_n^{*2}(\psi f)\,|\tilde{\Delta}_n(\psi)|^2 d\nu \leq \int_{U_M} \sum_{n=n_0}^{\infty}\left(\sum_{k=n+1}^{\infty} |\tilde{\Delta}_k \psi|^2\right)(\tilde{E}_{n+1}^{*2}(\psi f) - \tilde{E}_n^{*2}(\psi f))\, d\nu = \tag{20}$$

$$= \int_{U_M} \sum_{n=n_0}^{\infty} \tilde{E}_{n+1}\left(\sum_{k=n+1}^{\infty} |\tilde{\Delta}_k \psi|^2\right)(\tilde{E}_{n+1}^{*2}(\psi f) - \tilde{E}_n^{*2}(\psi f))\, d\nu \leq$$

$$\leq \|\psi\|_{\mathrm{BMO}}^2 \int_{U_M} |\psi f|^{*2} d\nu \leq C\|\psi\|_\infty^2 \int_{U_M} |f|^2 d\nu.$$

In the last step, boundedness of the maximal function on $L^2(U_m)$ is used. The constant does not depend on $M$ nor on $n_0$.

Using (18) and (20), we conclude that

$$\int_{U_M} \sum_{n \geq n_0+1} |\Delta_n^l f|^2 d\nu \leq C \int_{U_M} |f|^2 d\nu. \tag{21}$$

The left-hand inequality of (15) follows by letting $M \to +\infty$ and then $n_0 \to -\infty$ in (21).

In proving the right-hand inequality in (15), we need to use the following facts. If $g \in L^2(d\nu; \mathbb{A}_d)$, then

(*a*) $\lim_{n \to +\infty} E_n^l g = g = \lim_{n \to +\infty} E_n^r g$ in the $L^2$-sense.

(*b*) $\lim_{n \to -\infty} E_n^l g = 0 = \lim_{n \to -\infty} E_n^r g$ in the $L^2$-sense.

(*c*) $g = \sum_{-\infty}^{\infty} \Delta_n^l g = \sum_{-\infty}^{\infty} \Delta_n^r g$.

These results can be established in much the same way as the corresponding scalar-valued results are proved in [9], Chapter 5. Of course the condition (9) is crucial in the proofs.

Suppose that $f, g \in L^2(d\nu; \mathbb{A}_d)$. Then

$$\left|\int_X f\psi g\, d\nu\right| = \left|\int_X \left(\sum_{-\infty}^{\infty} \Delta_n^r g\right)\psi\left(\sum_{-\infty}^{\infty} \Delta_m^l f\right) d\nu\right| = \left|\int_X \left(\sum_{-\infty}^{\infty} \Delta_n^r g \psi \Delta_n^l f\right) d\nu\right| \leq C\|S^r g\|_2 \|S^l f\|_2 \tag{22}$$

by Proposition 2(*e*), (9) and the right-hand inequality in (15).

The proof is completed by taking the supremum in (22) over $g$ such that $\|g\|_2 \leq 1$ and using condition (9) again. ■

REMARK. – See also [13] for further results about Clifford-valued martingales.

1.3. *A family of σ-fields and an associated pair-basis-system.*

The development in § 1.2 deals with general σ-fields and martingales. We now construct the particular example, and associated «Haar» functions appropriate to the analysis of the Cauchy integral.

Let $X = \mathbb{R}^d$, $\mathcal{B}$ the Borel σ-field, and let $d\nu$ be Lebesgue measure, also denoted $dx$. The Lebesgue measure of a measurable set $U$ will be denoted $|U|$. Let $\mathcal{F}_0$ be the σ-field generated by the family $\mathfrak{J}_0$ of cubes of side length 1, whose corners lie at the points of the integer lattice. Divide each cube $I \in \mathfrak{J}_0$ equally by the hyperplane that bisects the edges parallel to the $x_1$-axis, and let $\mathfrak{J}_1$ be the family of «dyadic-quasi-cubes» so produced. Let $\mathcal{F}_1$ be the σ-field generated by $\mathfrak{J}_1$. Now subdivide each dyadic-quasi-cube of $\mathfrak{J}_1$ into two dyadic-quasi-cubes by the hyperplane that bisects the edges parallel to the $x_2$-axis, and let $\mathcal{F}_2$ be the σ-field generated by the new family of dyadic-quasi-cubes. Continue in this manner, at each stage bisecting each dyadic-quasi-cube of the previous family by the hyperplane perpendicular to the next coordinate axis. This produces the sequence $\{\mathcal{F}_n\}_{n=0}^{\infty}$. The σ-fields $\mathcal{F}_n$, $n < 0$, are produced by the reverse procedure to the one just described–successive doubling in the coordinate directions. Note that each dyadic-quasi-cube (i.e. atom) in $\mathcal{F}_{kd}$, $k \in \mathbb{Z}$ is actually a standard dyadic cube, of side length $2^{-k}$. Finally, let $\mathfrak{J} = \bigcup_{n=-\infty}^{\infty} \mathfrak{J}_n$. Note that every $I \in \mathfrak{J}$ is a dyadic-quasi-cube, say $I \in \mathfrak{J}_{n-1}$, and so can be written $I = I_1 \cup I_2$, where $I_1$ and $I_2$ are dyadic-quasi-cubes in $\mathfrak{J}_n$.

From now on, we work only with left-martingales, so we simplify notation by writting $E_n$, $\Delta_n$, $f_n$, etc. in place of $E_n^l$, $\Delta_n^l$, $f_n^l$, etc. The function $\psi \in L^\infty(X; \mathbb{R}^{1+d}) = L^\infty(\mathbb{R}^d; \mathbb{R}^{1+d})$ is still assumed to satisfy (9), but with respect to the particular sequence $\{\mathcal{F}_n\}_{-\infty}^{\infty}$ of σ-fields just constructed. The folowing lemma is another essential ingredient of this paper.

LEMMA 2. – *For each $I \in \mathfrak{J}_{n-1}$, $I = I_1 \cup I_2$, where $I_1, I_2 \in \mathfrak{J}_n$, there is a pair of $\mathbb{A}_d$-valued functions $\alpha_I$, $\beta_I$ on $\mathbb{R}^d$ and a positive constant $C$ such that*

(i) $$\alpha_I = a_1\chi_{I_1} + a_2\chi_{I_2} \quad (a_j \in \mathbb{A}_d);$$

$$\beta_I = b_1\chi_{I_1} + b_2\chi_{I_2} \quad (b_j \in \mathbb{A}_d);$$

(ii) *for all* $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d; \mathbb{A}_d)$,

$$\Delta_n f(x) = \alpha_I(x)\langle \beta_I, f\rangle_\psi \qquad (x \in I),$$

(iii) $C^{-1}|I|^{-1/2} \leqslant |\alpha_I(x)| \leqslant C|I|^{-1/2}$ *and* $C^{-1}|I|^{-1/2} \leqslant |\beta_I(x)| \leqslant C|I|^{-1/2}$ *for all* $x \in I$ *and all* $I$;

(iv)
$$\int \psi \alpha_I \, dx = \int \beta_I \psi \, dx = 0 .$$

PROOF. – Define $\alpha_I$ and $\beta_I$ as in (i). It is then a matter of choosing $a_1, a_2, b_1$, and $b_2$ so that (ii)-(iv) hold.

Consider (ii). Since $\mathcal{F}_n$ and $\mathcal{F}_{n-1}$ are atomic, we have

$$\widetilde{E}_{n-1} f = \left( \frac{1}{|I|} \int_I f(y)\, dy \right) \chi_I$$

on $I$, with similar formulas for $\widetilde{E}_n(f)$, etc. Let $u = \int_I \psi(t)\, dt$, $u_j = \int_{I_j} \psi(t)\, dt$ $(j = 1, 2)$. Then on $I$,

$$\Delta_n f = \widetilde{E}(\psi | \mathcal{F}_n)^{-1} \widetilde{E}(\psi f | \mathcal{F}_n) - \widetilde{E}(\psi | \mathcal{F}_{n-1})^{-1} \widetilde{E}(\psi f | \mathcal{F}_{n-1}) =$$
$$= u_1^{-1}\left( \int_{I_1} \psi f dx \right)\chi_{I_1} + u_2^{-1}\left( \int_{I_2} \psi f dx \right)\chi_{I_2} - u^{-1}\left( \int_{I_1} \psi f dx + \int_{I_2} \psi f dx \right)(\chi_{I_1} + \chi_{I_2}) =$$
$$= \left( (u_1^{-1} - u^{-1}) \int_{I_1} \psi f dx - u^{-1} \int_{I_2} \psi f dx \right)\chi_{I_1} + \left( (u_2^{-1} - u^{-1}) \int_{I_2} \psi f dx - u^{-1} \int_{I_1} \psi f dx \right)\chi_{I_2} .$$

On the other hand,

$$\alpha_I \langle \beta_I, f\rangle_\psi = \left( a_1 b_1 \int_{I_1} \psi f dx + a_1 b_2 \int_{I_2} \psi f dx \right)\chi_{I_1} + \left( a_2 b_2 \int_{I_2} \psi f dx + a_2 b_1 \int_{I_1} \psi f dx \right)\chi_{I_2} .$$

Comparing these last two expressions, we see that we should choose $a_i, b_i$ $(i = 1, 2)$ so that

$$a_1 b_1 = u_1^{-1} - u^{-1}, \qquad a_2 b_2 = u_2^{-1} - u^{-1}, \qquad a_1 b_2 = -u^{-1} = a_2 b_1 .$$

Keeping in mind that $u = u_1 + u_2$, and applying (2), we see that this system of equations can be expressed more simply as

$$(23) \qquad a_1 b_1 = u^{-1} u_2 u_1^{-1}, \qquad a_2 b_2 = u^{-1} u_1 u_2^{-1}, \qquad a_1 b_2 = -u^{-1}, \qquad a_2 b_1 = -u^{-1} .$$

The solutions of (23) are of the form

$$(24) \qquad a_1 = u^{-1} u_2 c, \qquad a_2 = -u^{-1} u_1 c, \qquad b_1 = c^{-1} u_1^{-1}, \qquad b_2 = -c^{-1} u_2^{-1}$$

where $c$ is an arbitrary invertible element of $\mathbb{A}_d$. We now wish to choose $c$ so that (iii) holds. In fact, it is apparent from (i) and (24) that (iii) holds if $c$ is taken to be $|I|^{-1/2}$.

It remains to check (iv). From (i) and (24) we have

$$\int \psi\alpha_I\,dx = \int \psi(a_1\chi_{I_1} + a_2\chi_{I_2})\,dx = u_1 a_1 + u_2 a_2 =$$

$$= (u_1 u^{-1} u_2 - u_2 u^{-1} u_1)\,c = u_1 u^{-1}(u - u_1)\,c - (u - u_1)\,u^{-1} u_1 c = 0\,.$$

From (24), we have $\int \beta_I \psi\,dx = 0$. ■

## 2. - Proof of Theorem 1.

In what follows, we suppress the fact that the Cauchy (singular) integral is a principal value by writing our operators in terms of ordinary integrals. It is of course necessary to attend to the question of existence of these (principal value) integrals at appropriate places. Most of the analysis is carried out on $\mathbb{R}^d$ rather than on $\Sigma$. The principal values are to be interpreted as the ones obtained by projecting the Euclidean balls in $\Sigma$ onto $\mathbb{R}^d$ and integrating over their complements.

Let $\phi(v) = A(v)\,e_0 + v$ $(v \in \mathbb{R}^d)$ be the coordinate system on $\Sigma$ defined by $A$. Then the unit normal vector to $\Sigma$ is $n(\phi(v)) = (e_0 - \nabla A(v))/\sqrt{1 + |\nabla A(v)|^2}$. In terms of these coordinates, we have

$$T_\Sigma h(\phi(u)) = \int_{\mathbb{R}^d} \frac{\overline{\phi(v) - \phi(u)}}{|\phi(v) - \phi(u)|^{1+d}}\, n(\phi(v))\, h(\phi(v)) \sqrt{1 + |\nabla A(v)|^2}\, dv =$$

$$= \int_{\mathbb{R}^d} \frac{\overline{\phi(v) - \phi(u)}}{|\phi(v) - \phi(u)|^{1+d}}\, \psi(v)\, h(\phi(v))\, dv$$

say, where $\psi(v) = e_0 - \nabla A(v)$. Since $|\nabla A(v)| \le C$, we see that $T_\Sigma$ is bounded on $L_2(\Sigma; \mathbb{A}_d)$ if and only if the operator

$$T\colon f \mapsto \int_{\mathbb{R}^d} \frac{\overline{\phi(v) - \phi(u)}}{|\phi(v) - \phi(u)|^{1+d}}\, f(v)\, dv \tag{25}$$

is bounded from $L_2(\mathbb{R}^d; \mathbb{A}_d)$ into $L_2(\mathbb{R}^d; \mathbb{A}_d)$.

We note that, if $I$ is a dyadic-quasi-cube, then the principal value integral

$$T(\psi\chi_I)(u) = \text{p.v.} \int_{\mathbb{R}^d} \frac{\overline{\phi(v) - \phi(u)}}{|\phi(v) - \phi(u)|^{1+d}}\, \psi(v)\, \chi_I(v)\, dv$$

exists and defines a locally integrable function. The existence and local integrability of $T(\psi\chi_I)(u)$ on the set $\mathbb{R}^d \setminus I$ are straightforward. Moreover, the singularity of

$T(\psi\chi_I)(u)$ as $u \to \partial I$ in $\mathbb{R}^d \setminus I$ is $O(\log(\operatorname{dist}(u, \partial I)))$. To treat the case where $u \in I$, it suffices to consider

$$T_\Sigma F(x) = \text{p.v.}\int_\Sigma \frac{\overline{y-x}}{|y-x|^{1+d}}\, n(y) F(y)\, d\sigma(y),$$

where $F$ vanishes off $\phi(I)$ and satisfies a uniform Lipschitz condition. Write

$$T_\Sigma F(x) = \text{p.v.}\int_\Sigma \frac{\overline{y-x}}{|y-x|^{1+d}}\, n(y)\{F(y)-F(x)\}\, d\sigma(y) + \text{p.v.}\int_\Sigma \frac{\overline{y-x}}{|y-x|^{1+d}}\, n(y) F(x)\, d\sigma(y).$$

The Lipschitz condition on $F$ gives an appropriate control on the first integral, whilst the monogenicity and cancellation properties of the kernel $\overline{y-x}/|y-x|^{1+d}$, combined with Cauchy's theorem, give a suitable control on the second integral.

Write the operator in (25) in the form

$$Tf(u) = \int_{\mathbb{R}^d} K(u, v) f(v)\, dv.$$

The essential properties of the kernel $K$ are contained in the following lemma.

LEMMA 3. – *The kernel $K$ satisfies the inequalities*

$$|K(x, y)| \leqslant C|x-y|^{-d} \qquad (x \neq y) \tag{26}$$

$$|K(x, y) - K(x', y)| \leqslant C\frac{|x-x'|}{|x-y|^{1+d}} \tag{27}$$

*and*

$$|K(y, x) - K(y, x')| \leqslant C\frac{|x-x'|}{|x-y|^{1+d}} \tag{28}$$

*for all* $x \neq y$, $|x-x'| < 1/2|x-y|$.

Let $\mathcal{S}$ denote the span over $\mathbb{A}_d$ of the set of characteristic functions of dyadic-quasi-cubes. The space $\psi\mathcal{S}$ of pointwise products with the function $\psi$ is a right-linear space over $\mathbb{A}_d$. Similarly, $\mathcal{S}\psi$ is a left-linear space over $\mathbb{A}_d$. Using the ideas of [7], it is possible to define $T\psi$ as a Clifford-left linear functional on the subspace $(\mathcal{S}\psi)_0$ of $\mathcal{S}\psi$ consisting of functions having integral 0: fix $g\psi \in (\mathcal{S}\psi)_0$, and choose $N$ so large that the ball $B_N$ of radius $N$ about 0 contains the support of $g$. Then define

$$T\psi(g\psi) = T(\psi\chi_{B_N})(g\psi) + \int\!\!\int g(x)\psi(x)\{K(x,y) - K(0,y)\}\{1-\chi_{B_N}(y)\}\psi(y)\, dx\, dy = I_N^{(1)} + I_N^{(2)}.$$

This definition is meaningful because of the properties (27) and (28) of the kernel $K$. It is an important fact that

$$\langle \beta_J, T\psi \rangle_\psi = T\psi(\beta_J \psi) = 0 . \tag{29}$$

This can be proved by taking account of the following observations. (*a*) $I_N^{(2)} \to 0$ as $N \to \infty$. (*b*) By using the monogenicity of the Cauchy kernel, along with Cauchy's theorem [2], it can be shown that $\lim_{N\to\infty} T(\psi\chi_{B_N})(x)$ exists and is independent of $x \in \operatorname{supp} \beta_J$. Since $\beta_J \psi$ has integral 0, it follows that $\lim_{N\to\infty} T(\psi\chi_{B_N})(\beta_J\psi) = 0$. In establishing (*b*), one works on the surface $\Sigma$.

We note that, if $T^t$ is the operator $f \mapsto \int f(y) K(y, x)\, dy$, then

$$\langle T^t(\chi_I \psi), \chi_J \rangle_\psi = \langle \chi_I, T(\psi\chi_J) \rangle_\psi$$

for all dyadic-quasi-cubes $I$, $J$. Just as for the operator $T$, we have

$$\langle T^t \psi, \beta_J \rangle_\psi = T^t \psi(\psi\beta_J) = 0 . \tag{30}$$

By Lemma 2, if $f \in L_2(\mathbb{R}^d; \mathbb{A}_d)$, we have

$$f = \sum_{-\infty}^{\infty} \Delta_n f = \sum_I \alpha_I \langle \beta_I, f \rangle_\psi$$

and formally,

$$T(\psi f) = \sum_{J \in \mathfrak{J}} T(\psi\alpha_J) \langle \beta_J, f \rangle_\psi = \sum_{J, I} \alpha_I \langle \beta_I, T(\psi\alpha_J) \rangle_\psi \langle \beta_J, f \rangle_\psi = \sum_I \alpha_I \sum_J \langle \beta_I, T(\psi\alpha_J) \rangle_\psi \langle \beta_J, f \rangle_\psi .$$

Let $u_{IJ} = \langle \beta_I, T(\psi\alpha_J) \rangle_\psi$. By Lemmas 1 and 2, it suffices to show that the linear transformation on $l^2(\mathfrak{J}; \mathbb{A}_d)$ defined by the matrix $(u_{IJ})$ is bounded. We shall do so by using the following variant of Schur's lemma.

LEMMA 4 (Schur). – *Suppose there exists a family* $(\omega_I)$ *of positive numbers and a constant* $C$ *such that*

$$\text{(i)} \qquad \sum_J |\omega_J u_{IJ}| \leqslant C\omega_I \qquad (I \in \mathfrak{J}) \tag{31}$$

and

$$\text{(ii)} \qquad \sum_I |\omega_I u_{IJ}| \leqslant C\omega_J \qquad (I \in \mathfrak{J}) . \tag{32}$$

*Then the matrix* $(u_{IJ})$ *defines a bounded operator on* $l^2(\mathfrak{J}; \mathbb{A}_d)$.

PROOF. – This is a natural modification of the proof of the scalar version. ■

We shall apply Lemma 4 by taking $\omega_I = |I|^t$ for an appropriate positive number $t$. (Recall that $|I|$ denotes the Lebesgue measure of $I$.) We begin with some basic facts relevant to the estimation of the terms $|\langle \beta_I, T(\psi\alpha_J) \rangle_\psi|$.

NOTATION. – Suppose that $I$ and $J$ are atoms in $\mathcal{F}_n$ and $\mathcal{F}_m$ respectively, and that $n \geq m$. An atom $A \in \mathcal{F}_n$ is said to be *contiguous to* $J$ (resp. *contiguous to* $J^c$) if $A$ is not contained in $J$ (resp. $J^c$), but has part of its boundary in common with the boundary of $J$. Using a certain licence in notation, we denote by $I + J$ the union of $J$ and the atoms $A$, in the same $\sigma$-field as $I$, that are contiguous to $J$. In particular, $2J$ denotes the union of $J$ with all of the atoms in $\mathcal{F}_n$ that are contiguous to $J$. The *bottom-left corner* $x_J$ *of* $J$ is the vertex of $J$ having minimal coordinates.

LEMMA 5. – *Let $I$ and $J$ be atoms, in $\mathcal{F}_n$ and $\mathcal{F}_m$ respectively, with $n \geq m$. There is a constant $C$, independent of $m$ and $n$, such that, if $I \subseteq 2J \setminus J$, then*

$$\int_{I \times J} |x-y|^{-d}\,dx\,dy \leq C|I|\left(\log\frac{|J|}{|I|}+1\right).$$

PROOF. – This is an elementary calculation. ∎

LEMMA 6. – *Let $I$ and $J$ be atoms in $\bigcup_{j=-\infty}^{\infty} \mathcal{F}_j$. Then*

(i) *for all $x \notin 2J$,*

$$|T(\psi\alpha_J)| \leq C|J|^{1/2+1/d}|x-x_J|^{-1-d}; \tag{33}$$

(ii) *if $I \subseteq (2J)^c$, then*

$$|\langle\beta_I, T(\psi\alpha_J)\rangle_\psi| \leq C|I|^{-1/2}|J|^{1/2+1/d}\int_I |x-x_J|^{-1-d}\,dx; \tag{34}$$

(iii) *for all $x \notin J$,*

$$|T(\psi\alpha_J)(x)| \leq C|J|^{-1/2}\int_J |x-y|^{-d}\,dy;$$

(iv) *if $I \subseteq 2J \setminus J$, then*

$$|\langle\beta_I, T(\psi\alpha_J)\rangle_\psi| \leq C\frac{|I|^{1/2}}{|J|^{1/2}}\left(\log\frac{|J|}{|I|}+1\right).$$

*(In all of the statements, the constant $C$ is independent of $I$ and $J$).*

PROOF. – (i) This derives from the cancellation properties of the Haar function $\alpha_J$. Thus,

$$T(\psi\alpha_J) = \int K(x,y)\psi(y)\alpha_J(y)\,dy = \int_J [K(x,y)-K(x,x_J)]\psi(y)\alpha_J(y)\,dy.$$

So it follows from (28) that if $x \notin 2J$, then

$$|T(\psi\alpha_J)(x)| \leqslant C|J|^{-1/2} \int_J \frac{|y - x_J|}{|x - x_J|^{1+d}}\, dy \leqslant$$

$$\leqslant C|J|^{1/2}\, |x - x_J|^{-1-d} \sup_{y \in J} |y - x_J| \leqslant C|J|^{1/2+1/d}\, |x - x_J|^{-1-d}.$$

(ii) This is clear from (i) and Lemma 2 (iii).

(iii) This follows from (26).

(iv) This is clear from (iii) and Lemma 5. ∎

The estimation of $\sum_I |I|^t\, |\langle \beta_I,\, T(\psi\alpha_J)\rangle_\psi|$ will be divided into three parts, each with a number of separate cases, depending on the relative sizes and disposition of the atoms $I$ and $J$.

*Case* 1. *The sum with respect to atoms $I$ larger than $J$.*

Fix $J \in \mathcal{F}_m$ and consider the set $2J$. Let $x_J$ be the bottom-left corner of $J$. Consider atoms $I \in \mathcal{F}_n$, $n < m$.

($a$) If $I$ lies outside $2J$, then, by Lemma 6(ii) and Lemma 2(iii),

$$|\langle \beta_I,\, T(\psi\alpha_J)\rangle_\psi| \leqslant C|I|^{-1/2}\, |J|^{1/2+1/d} \int_I |x - x_J|^{-1-d}\, dx.$$

Consequently, the estimate of the part of the Schur sum corresponding to this case is

$$\sum_{I \in \bigcup_{n<m} \mathcal{F}_n,\, I \subseteq (2J)^c} |I|^t |\langle \beta_I, T(\psi\alpha_J)\rangle_\psi| \leqslant C \sum_{k=1}^{\infty} (2^k |J|)^{t-1/2} \sum_{I \in \mathcal{F}_{m-k},\, I \subseteq (2J)^c} |J|^{1/2+1/d} \int_I |x - x_J|^{-1-d}\, dx \leqslant$$

$$\leqslant C \sum_{k=1}^{\infty} 2^{k(t-1/2)} |J|^{t+1/d} \int_{(2J)^c} |x - x_J|^{-1-d}\, dx \leqslant C \sum_{k=1}^{\infty} 2^{k(t-1/2)} |J|^t \leqslant C|J|^t$$

if $t < 1/2$.

($b$) For a fixed $n < m$, the dyadic-quasi-cubes that meet $2J$ are of two kinds: those that lie in $2J \setminus J$, and one that contains $J$. If $I$ lies in $2J \setminus J$, then it follows from Lemma 6(iv) that

$$|I|^t |\langle \beta_I,\, T(\psi\alpha_J)\rangle_\psi| \leqslant C\, \frac{|I|^{t+1/2}}{|J|^{1/2}} \left(\log \frac{|J|}{|I|} + 1\right) \leqslant C|J|^t$$

since the ratio of the measures of $I$ and $J$ is bounded above and away from 0, independently of $I$ and $J$. Since the number of such terms is bounded, independently of $I$ and $J$, the corresponding part of the Schur sum is $O(|J|^t)$

If $I$ contains $J$ and is larger than $J$, then $I$ can be written $I = I_1 \cup I_2$, where $I_1$ and $I_2$ are atoms in $\mathcal{F}_{n+1}$. Suppose that $J \subseteq I_1$, and write $\beta_I = \beta_1 \chi_{I_1} + \beta_2 \chi_{I_2}$. Then (see (29) and (30)),

$$\langle \beta_1 \chi_{I_1}, T(\psi\alpha_J)\rangle_\psi = -\langle \beta_1 \chi_{I_1^c}, T(\psi\alpha_J)\rangle_\psi .$$

Now $I_1^c$ contains part of the region $2J \setminus J$, on which we can use Lemma 6(iii), and part of $(2J)^c$, on which we can use Lemma 6(i). Specifically,

$$(35)\qquad \langle \beta_1 \chi_{I_1}, T(\psi\alpha_J)\rangle_\psi \,| = \left| \beta_1 \int_{I_1^c} \psi(x)\, T(\psi\alpha_J)(x)\, dx \right| \leqslant$$

$$\leqslant C|\beta_1| \left( \int_{2J\setminus J} |T(\psi\alpha_J)(x)|\, dx + \int_{(2J)^c} |T(\psi\alpha_J)(x)|\, dx \right) \leqslant$$

$$\leqslant C|I|^{-1/2}|J|^{-1/2} \int_{2J\setminus J} dx \int_J |x-y|^{-d}\, dy + C|I|^{-1/2}|J|^{1/2+1/d} \int_{(2J)^c} |x - x_J|^{-1-d}\, dx \leqslant$$

$$\leqslant C\{|I|^{-1/2}|J|^{-1/2} + |I|^{-1/2}|J|^{1/2}\} \leqslant C\frac{|J|^{1/2}}{|I|^{1/2}},$$

the second-last step by Lemma 5. As for $\langle \beta_2 \chi_{I_2}, T(\psi\alpha_J)\rangle_\psi$, we have that $I_2$ is disjoint from $J$; so the same kind of estimate as in (35) can be established.

The estimate of the part of the Schur sum corresponding to dyadic-quasi-cubes $I \supsetneq J$ ($n$ varying) is

$$\sum_{I \in \bigcup_{n<m} \mathcal{F}_n,\, I \supsetneq J} |I|^t |\langle \beta_I, T(\psi\alpha_J)\rangle_\psi| \leqslant C \sum_{k=1}^{\infty} (2^k|J|)^{t-1/2}|J|^{1/2} \leqslant C|J|^t$$

provided $t < 1/2$.

*Case 2. The sum with respect to atoms I smaller than J.*

In this case, we are dealing with atoms $J \in \mathcal{F}_m$ and $I \in \mathcal{F}_n$ with $n > m$.

(*a*) If $I$ lies outside $2J$, then $J$ lies outside of $2I$. So by Lemma 6(i) applied to $T^t$,

$$|T^t(\beta_I\psi)(x)| \leqslant C|I|^{1/2+1/d}|x - x_I|^{-1-d}$$

and so

$$|\langle T^t(\beta_I\psi), \alpha_J\rangle_\psi| \leq C|I|^{1/2+1/d}|J|^{-1/2}\int_J |x-x_I|^{-1-d}\,dx \leq$$

$$\leq C|I|^{1/2+1/d}|J|^{1/2}|x_I-x_J|^{-1-d} \leq C|I|^{1/d-1/2}|J|^{1/2}\int_I |x-x_J|^{-1-d}\,dx,$$

the middle step because $I \subsetneq (2J)^c$. The corresponding Schur sum estimate is

$$\sum_{I\in \bigcup_{n>m}\mathcal{F}_n,\, I\cap 2J=\emptyset} |I|^{t+1/d-1/2}|J|^{1/2}\int_I |x-x_J|^{-1-d}\,dx \leq$$

$$\leq C\sum_{k=1}^{\infty}(2^{-k}|J|)^{t+1/d-1/2}|J|^{1/2}\int_{(2J)^c}|x-x_J|^{-1-d}\,dx \leq C\sum_{k=1}^{\infty}(2^{-k})^{t+1/d-1/2}|J|^t \leq C|J|^t$$

provided $t > 1/2 - 1/d$.

(*b*) If $I\cap J=\emptyset$ and $I\subsetneq 2J\setminus(I+J)$, then $J\subsetneq(2I)^c$. So by the analogue of Lemma 6(ii) for $T^t$

$$|\langle\beta_I, T(\psi\alpha_J)\rangle_\psi| = |\langle T^t(\beta_I\psi), \alpha_J\rangle_\psi| \leq C|J|^{-1/2}|I|^{1/2+1/d}\int_J |x-x_I|^{-1-d}\,dx. \tag{36}$$

Let $d(x, J)$ denote the distance of the point $x$ from $J$. The atom $I$ may have unequal side-lengths. Let $l(I)$ be its smallest side-length. Then it follows from (36) that

$$|\langle\beta_I, T(\psi\alpha_J)\rangle_\psi| \leq C|J|^{-1/2}|I|^{1/2+1/d}\frac{1}{d(x_I, J)} \leq \tag{37}$$

$$\leq C|J|^{-1/2}|I|^{1/2+1/d}|I|^{-1}\int_I \frac{dx}{d(x,J)+l(I)}.$$

Let $L$ and $l$ be the maximal and minimal side-lengths of $J$. Then $L\leq 2l$ and $l^d\leq|J|\leq 2^d l^d$. The dyadic-quasi-cube $I\in\mathcal{F}_{m+k}$ has minimum side-length $l(I)\geq l/2^{k/d+1}$. It follows from (37) that the estimate of the relevant part of the Schur sum is therefore

$$\sum_{I\in\bigcup_{n>m}\mathcal{F}_n,\ I\subseteq 2J\setminus(I+J)} |I|^t|\langle\beta_I, T(\psi\alpha_J)\rangle_\psi| \leq$$

$$\leq C\sum_{k=1}^{\infty}(2^{-k}|J|)^{t+1/d-1/2}|J|^{-1/2}\int_{2J\setminus(I+J)}\frac{dx}{d(x,J)+l2^{-k/d-1}} \leq$$

$$\leq C\sum_{k=1}^{\infty}(2^{-k}|J|)^{t+1/d-1/2}|J|^{-1/2}\int_0^{3L}dx_1\ldots\int_0^{3L}dx_{d-1}\int_{l2^{-k/d-1}}^{2l}\frac{du}{u+l2^{-k/d-1}} \leq$$

$$\leqslant C\sum_{k=1}^{\infty}(2^{-k}|J|)^{t+1/d-1/2}|J|^{-1/2}|J|^{(d-1)/d}\log\left(\frac{2l+l2^{-k/d-1}}{l2^{-k/d}}\right)\leqslant$$

$$\leqslant C\sum_{k=1}^{\infty}(2^{-k})^{t+1/d-1/2}\frac{k}{d}|J|^{t}\leqslant C|J|^{t}$$

provided $t>1/2-1/d$.

(*c*) If $I\subsetneq(I+J)\setminus J$, we have $I\subsetneq 2J\setminus J$ and so, by Lemma 6(iv),

$$|\langle\beta_I,\,T(\psi\alpha_J)\rangle_\psi|\leqslant C\frac{|I|^{1/2}}{|J|^{1/2}}\left(\log\frac{|J|}{|I|}+1\right). \tag{38}$$

In the region $(I+J)\setminus J$, there are $O(L^{d-1}/(l2^{-k/d-1})^{d-1})$ atoms that belong to $\mathcal{F}_n$. (Divide the maximal face area of $J$ by the minimal face area of $I$). In other words, there are $O(2^{k(1-1/d)})$ such atoms. The corresponding Schur sum estimate is, from (38),

$$C\sum_{k=1}^{\infty}(2^{-k}|J|)^{t+1/2}|J|^{-1/2}k\,2^{k(1-1/d)}=C|J|^{t}\sum_{k=1}^{\infty}k(2^{-k})^{t-1/2+1/d}\leqslant C|J|^{t}$$

provided $t>1/2-1/d$.

(*d*) If $I\subsetneq J$, and $L$ is contiguous to $J^c$, we write $J=J_1\cup J_2$, where $J_1$ and $J_2$ are atoms in $\mathcal{F}_{n+1}$. Let $\alpha_J=\alpha_1\chi_{J_2}+\alpha_2\chi_{J_2}$, and suppose that $I\subsetneq J_1$.

Consider first those atoms $I\subsetneq J_1$ that are contiguous to $J_1^c$. We have

$$|\langle\beta_I,\,T(\psi\alpha_1\chi_{J_1})\rangle_\psi|=|\langle\beta_I,\,T(\psi\alpha_1\chi_{J_1^c})\rangle_\psi|=|\langle T(\beta_1\psi),\,\alpha_I\chi_{J_1^c}\rangle_\psi|\leqslant$$

$$\leqslant\left|\int_{J_1^c\cap 2I}T^t(\beta_I\psi)(x)\psi(x)\alpha_1\,dx\right|+\left|\int_{J_1^c\setminus 2I}\dots\right|.$$

So

$$|\langle\beta_I,\,T(\psi\alpha_1\chi_{J_1})\rangle_\psi|\leqslant \tag{39}$$

$$\leqslant C|I|^{-1/2}|J|^{-1/2}\int_{2I\setminus I}dx\int_I|x-y|^{-d}dy+C|I|^{1/2+1/d}\int_{(2I)^c}|x-x_I|^{-1-d}dx\leqslant$$

$$\leqslant C|I|^{-1/2}|J|^{-1/2}|I|\log\left(\frac{|I|}{|I|}+1\right)+C|I|^{1/2}|J|^{-1/2}\leqslant C\frac{|I|^{1/2}}{|J|^{1/2}}$$

by Lemma 5 and the analogue of Lemma 6(i) for $T^t(\beta_I\psi)$. Since $J_2\subsetneq J_1^c$, the same argument as in (39) shows that

$$|\langle\beta_I,\,T(\psi\alpha_2\chi_{J_2})\rangle_\psi|\leqslant C\frac{|I|^{1/2}}{|J|^{1/2}}. \tag{40}$$

There are $O(2^{k(1-1/d)})$ atoms in $\mathcal{F}_{m+k}$ that are contiguous to $J_1^c$. See the argument in

1(*c*) above. It follows from (39) and (40) that the Schur sum estimate, appropriate to the atoms that are contiguous to $J_1^c$, reduces to

$$C\sum_{k=1}^{\infty}(2^{-k}|J|)^{t+1/2}|J|^{-1/2}|J|^{-1/2}2^{k(1-1/d)} = C|J|^t\sum_{k=1}^{\infty}(2^{-k})^{t-1/2+1/d} \leqslant C|J|^t,$$

which holds provided $t > 1/2 - 1/d$.

(*e*) If $I \subseteq J_1$ and $I$ is not contiguous to $J_1^c$, then by the analogue of Lemma 6(i),

$$|\langle\beta_I, T(\psi\alpha_1\chi_{J_1})\rangle_\psi| = \left|\int T^t(\beta_I\psi)(x)\,\psi(x)\,\alpha_1\chi_{J_1^c}(x)\,dx\right| \leqslant$$

$$\leqslant C|J|^{-1/2}\int_{J_1^c}|T^t(\beta_I\psi)(x)|\,dx \leqslant C|I|^{1/2+1/d}|J|^{-1/2}\int_{J_1^c}|x-x_I|^{-1-d}\,dx \leqslant$$

$$\leqslant C|I|^{1/2+1/d}|J|^{-1/2}\frac{1}{d(x_I, J_1^c)}.$$

A similar estimate holds for $|\langle\beta_I, T(\psi\alpha_2\chi_{J_2})\rangle_\psi|$. So the relevant Schur estimate is (cf. 1(*b*) above)

$$\sum_{k=1}^{\infty}(2^{-k}|J|)^{t+1/2+1/d}|J|^{-1/2}\sum_{j=1}^{2^{k(1-1/d)}}\frac{1}{jl2^{-k/d}} \leqslant$$

$$\leqslant C\sum_{k=1}^{\infty}(2^{-k}|J|)^{t+1/2+1/d}|J|^{-1/2-1/d}2^k\log(2^{k(1-1/d)}) \leqslant C\sum_{k=1}^{\infty}k(2^{-k})^{t-1/2+1/d}|J|^t| \leqslant C|J|^t$$

provided $t > 1/2 - 1/d$.

*Case* 3. *Atoms of the same size.*

Here we need estimate only the term $\langle\beta_I, T(\psi\alpha_I)\rangle_\psi$ since the arguments for Case 1 can be used to estimate the other parts of the Schur sum.

According to Lemma 2, it suffices to prove that

$$|\langle\chi_I, T(\psi\chi_I)\rangle_\psi| \leqslant C|I|$$

for all dyadic-quasi-cubes $I$. To do so, it is necessary to use the monogenicity of the Cauchy kernel. So we pass from $T$ back to $T_\Sigma$. Recall that the coordinate mapping is $\phi(v) = A(v)e_0 + v$. Consider, for small $\varepsilon > 0$ and $x = \phi(u)$ $(u \in I)$,

$$\int_{|x-y|>\varepsilon}\frac{\overline{y-x}}{|y-x|^{1+d}}\,n(y)\,\chi_{\phi(I)}(y)\,d\sigma(y). \tag{41}$$

Let $P_x$ be the tangent hyperplane to $\Sigma$ at $x$; set $a(u) = \mathrm{dist}(u, \partial\phi I)$, and $b = b(x) =$

$= \operatorname{dist}(x, \partial\phi(I))$. Write (41) as

$$\int_{b > |x-y| > \varepsilon} \dots + \int_{|x-y| > b} \dots = I_1 + I_2$$

say. Then

$$|I| \leqslant C \log\left(\frac{C|I|^{1/d}}{a(u)}\right).$$

Using Cauchy's theorem [2], we can write

$$I_1 = \int_{S_b} + \int_{S_\varepsilon} + \int_{x,\, y \in P_x,\ b > |x-y| > \varepsilon} \tag{42}$$

where $S_b$ and $S_\varepsilon$ are the portions of the spheres of radii $b$ and $\varepsilon$ respectively that lie between $\Sigma$ and $P_x$. The third integral in (42) is 0, since the kernel is anti-symmetric, and the integrals over $S_b$ and $S_\varepsilon$ are bounded by a constant, independent of $x$, $\varepsilon$ and $b$. Therefore

$$|\langle \chi_I, T(\psi\chi_I)\rangle_\psi| \leqslant C|I| + C\int_I \log\left(\frac{C|I|^{1/d}}{a(u)}\right) du \leqslant C|I|. \qquad \blacksquare$$

NOTE ADDED IN PROOF. – The Cases 1-3 do not exhaust all possibilities. The omitted cases can be treated by modifications of the arguments given above.

### 3. – The $T(b)$ theorem.

The version of the $T(b)$ theorem stated below is formulated for an operator $T$ from (a subspace of) $L_2(\mathbb{R}^d; \mathbb{A}_d)$ to $L_2(\mathbb{R}^d; \mathbb{A}_d)$. The proof we give can be modified in the obvious way to prove a more general theorem for operators from $L_2(\mathbb{R}^d; \mathbb{A}_{d_1})$ into $L_2(\mathbb{R}^d; \mathbb{A}_{d_1})$, where $d_1$ is not necessarily the same as $d$. The case $d_1 = 1$ was treated by DAVID [6].

Suppose that $b_1$ and $b_2$ are two pseudoaccretive functions. The space $b_1 L_2(\mathbb{R}^d; \mathbb{A}_d)$ is defined as the space of products of the form $b_1 f$, $f \in L_2(\mathbb{R}^d; \mathbb{A}_d)$. Similarly for $L_2(\mathbb{R}^d; \mathbb{A}_d)\, b_2$. These spaces are isomorphic to $L_2(\mathbb{R}^d; \mathbb{A}_d)$. Let $\mathcal{S}$ denote the space of finite linear combinations over $\mathbb{A}_d$ of characteristic functions of dyadic-quasi-cubes. Then $b_1 \mathcal{S}$ is dense in $b_1 L_2(\mathbb{A}_d)$. Denote by $(\mathcal{S}b_2)^*$ the space of Clifford-left-linear functionals on $\mathcal{S}b_2$, with values in $\mathbb{A}_d$. Similarly, $(b_1\mathcal{S})^*$ is the space of Clifford-right-linear functionals on $b_1\mathcal{S}$.

Let $T$ be a mapping from $b_1\mathcal{S}$ into $(\mathcal{S}b_2)^*$ that is Clifford-right-linear, and let $\Delta = \{(x, y)\colon x = y\}$. As in [7], we say that $T$ *is associated with a standard Calderón-Zygmund kernel* if there is a $C^\infty$ function $K$ on $\mathbb{R}^d \times \mathbb{R}^d \setminus \Delta$, with values in $\mathbb{A}_d$, and a number $\delta$ such that $0 < \delta \leq 1$,

$$|K(x, y)| \leq C \frac{1}{|x-y|^d} \qquad (x \neq y) \tag{43}$$

$$|K(x, y) - K(x, y_0)| + |K(y, x) - K(y_0, x)| \leq \tag{44}$$

$$\leq C \frac{|y-y_0|^\delta}{|x-y|^{d+\delta}} \qquad \text{if } |y-y_0| < \frac{1}{2}|y-x|$$

and

$$T(b_1 f)(gb_2) = \int\int g(x)\, b_2(x)\, K(x, y)\, b_1(y) f(y)\, dx\, dy \tag{45}$$

for all $f, g \in \mathcal{S}$ having disjoint supports. In conformity with (45) we write, in general,

$$T(b_1 f)(gb_2) = \langle g, T(b_1 f)\rangle_{b_2}.$$

If $T^t$ is a left-linear mapping from $\mathcal{S}b_2$ to $(b_1 \mathcal{S})^*$ such that

$$\langle g, T(b_1 f)\rangle_{b_2} = \langle T^t(gb_2), f\rangle_{b_1}$$

for all $f, g \in \mathcal{S}$, and $T$ is associated with the kernel $K$, then $T^t$ is associated with the kernel $K(y, x)$ in the sense that

$$T^t(gb_2)(b_1 f) = \int\left(\int g(x)\, b_2(x)\, K(x, y)\, dx\right) b_1(y) f(y)\, dy.$$

We say that $T$ is *weakly bounded with respect to* $b_1$ *and* $b_2$ if there is a constant $C$ such that

$$|T(b_1 \chi_Q)(\chi_Q b_2)| \leq C|Q|$$

for all dyadic-quasi-cubes $Q$. This definition is formally different from the usual one ([7], [8]), in which the test functions are taken to be smooth. However, the two definitions are equivalent [6].

If $h \in L^\infty(\mathbb{R}^d; \mathbb{R}^{1+d})$, then $Th$ can be defined as in [7] as a linear functional on the subspace $(\mathcal{S}b_2)_0$ of $\mathcal{S}b_2$ consisting of functions having integral 0. In the statement of the Theorem below, the statement that $T(b_1) \in \mathrm{BMO}$ is taken to mean that there is a function $\phi$, say, that is locally integrable, belongs to BMO, and is such that $\langle g, T(b_1)\rangle_{b_2} = \langle g, \phi\rangle_{b_2}$ for all $g \in (\mathcal{S}b_2)_0$. A similar interpretation applies to $T^t(b_2)$. The space BMO is the one defined in (13), for the sequence of $\sigma$-fields of § 2.

THEOREM ($T(b)$ Theorem). – *Let $T$ and $T^t$ be as above, $T$ being associated with the standard Calderón-Zygmund kernel $K$. Then $T$ is extendible to a bounded linear* *operator from* $b_1 L_2(\mathbb{R}^d; \mathbb{A}_d)$ *to* $L_2(\mathbb{R}^d; \mathbb{A}_d) b_2$ *if and only if*

(i) $T(b_1)$, $T^t(b_2) \in \mathrm{BMO}$;

(ii) $T$ *is weakly bounded with respect to* $b_1$ *and* $b_2$.

PROOF. – The necessity of conditions (i)-(ii) was proved in the classical case by PETRE, SPANNE and STEIN ([18], [20] and [21]). Their proof can be adapted to the more general Clifford-algebra setting.

To prove the sufficiency, we treat first the case where $T(b_1) = T^t(b_2) = 0$. To each of the pseudoaccretive functions $b_1$ and $b_2$ we associate a Haar basis, as in § 1. We denote the respective pair-bases by $\{(\alpha_I^{(1)}, \beta_I^{(1)})\}_{I \in \mathfrak{J}}$ and $\{(\alpha_I^{(2)}, \beta_I^{(2)})\}_{I \in \mathfrak{J}}$. Then we have the formal expansion

$$T(b_1 f) = \sum_{I, J} \alpha_I^{(2)} \langle \beta_I^{(2)}, T b_1 \alpha_J^{(1)} \rangle_{b_2} \langle \beta_J^{(1)}, f \rangle_{b_1}.$$

Let

$$u_{IJ} = \langle \beta_I^{(2)}, T b_1 \alpha_J^{(1)} \rangle_{b_2}.$$

It suffices to show that the conditions of Lemma 4 are satisfied when $\omega_I$ is taken to be $|I|^t$ for a suitable positive number $t$.

Since $T(b_1) = T^t(b_2) = 0$, and the kernel associated to $T$ satisfies (43) and (44), the statement and proof of Lemma 6 hold unchanged for the present more general operator $T$. An examination of the estimates in Cases 1 and 2 shows that they go through unchanged, thanks to the assumption that $T(b_1) = T^t(b_2) = 0$. Cf. especially Case 1($b$) and Case 2($d$). The estimates of the part of the Schur sum corresponding to Case 3 hold by virtue of the weak boundedness assumption. (In § 2, the conditions $T(b_1) = T^t(b_2) = 0$ and weak boundedness are consequences of the Cauchy integral theorem.)

*The general case*: $T(b_1)$, $T^t(b_2) \in \mathrm{BMO}$. This can be treated by adapting the ideas of DAVID [6] to the martingale setting. Let $T(b_1) = \phi_1$, and $T^t(b_2) = \phi_2$. We define

$$U_i f = \sum_{k=-\infty}^{\infty} \Delta_k^{(j)}(\phi_i) E_{k-1}^{(i)}(b_i^{-1} f) \tag{46}$$

$i, j = 1, 2$, $i \neq j$, where $E_k^{(i)}$ and $\Delta_k^{(i)}$ are the left-conditional expectation operators and left-martingale difference with respect to the pseudoaccretive function $b_i$. It is obvious that $U_i b_i = \phi_i$ ($i = 1, 2$). The kernel $K_i$ of the operator $U_i$ is given formally by the expression

$$K_i(x, y) = \sum_{k=-\infty}^{\infty} \sum_{I \in \mathfrak{J}_{k-1}} \chi_I(x) \alpha_I^{(j)}(x) \langle \beta_I^{(j)}, \phi_i \rangle_{b_j} \left( \int_I b_i \right)^{-1} \chi_I(y). \tag{47}$$

It is easy to check, using the expression (46), that

$$\Delta_m^{(i)} U_i f = \Delta_m^{(j)}(\phi_i) E_{m-1}^{(i)}(b_i^{-1} f).$$

We claim that

$$\|S^{(i)}(U_i f)\|_2 \leqslant C\|f\|_2, \tag{48}$$

$S^{(i)}$ denoting the Littlewood-Paley square function with respect to $b_i$ and that therefore $U_i$ is bounded on $L_2$. To prove (48) note first that

$$\|S^{(i)}(U_i f)\|_2^2 = \int \sum_k |\Delta_k^{(j)}(\phi_i) E_{k-1}^{(i)}(b_i^{-1} f)|^2 dx \leqslant C \int \sum_k |\Delta_k^{(j)}(\phi_i)|^2 (E_{k-1}^{(i)*}(b_i^{-1} f))^2 dx \leqslant \tag{49}$$

$$\leqslant C \int \sum_{k=-\infty}^{\infty} \widetilde{E}_{k-1}\left(\sum_{m=k}^{\infty} |\Delta_m^{(j)}(\phi_i)|^2\right)[(E_{k-1}^{(i)*}(b_i^{-1} f))^2 - (E_{k-2}^{(i)*}(b_i^{-1} f))^2]\, dx$$

where $E_k^{(i)*} g = \sup_{m \leqslant k} |E_m^{(i)} g|$. Now

$$\widetilde{E}_{k-1}\left(\sum_{m=k}^{\infty} |\Delta_m^{(j)}(\phi_i)|^2\right) \leqslant C\|\phi_i\|_{\mathrm{BMO}}^2 \tag{50}$$

for every $k$. This is because, if $I \in \mathfrak{J}_{k-1}$, then we can restrict the $\sigma$-fields $\{\mathcal{F}_m\}_{m=k-1}^{\infty}$ to $I$ and conclude that, on $I$,

$$\widetilde{E}_{k-1}\left(\sum_{m=k}^{\infty} |\Delta_m^{(j)}(\phi_i)|^2\right) = \frac{1}{|I|}\int_I \sum_{m=k}^{\infty} |\Delta_m^{(j)}(\phi_i)|^2 dx =$$

$$= \frac{1}{|I|}\int_I \sum_{m=k}^{\infty} |\Delta_m^{(j)}(\phi_i - E_{k-1}^{(j)}(\phi_i))|^2 dx \leqslant \frac{C}{|I|}\int_I \left|\phi_i - \frac{1}{|I|^{(j)}}\int_I b_j \phi_i\right|^2 dx =$$

$$= \frac{C}{|I|}\int_I \left|\phi_i - \frac{1}{|I|}\int_I \phi_i\, dy + \frac{1}{|I|^{(j)}}\int_I b_j\left(\phi_i - \frac{1}{|I|}\int_I \phi_i\, dz\right) dx\right|^2 \leqslant C\|\phi_i\|_{\mathrm{BMO}}^2$$

where we have used the notation $|I|^{(j)} = \int_I b_j\, dx$. This establishes (50). Returning to (48), we have therefore that

$$\|S^{(i)}(U_i f)\|_2^2 \leqslant C\|\phi_i\|_{\mathrm{BMO}}^2 \int (Mf)^2 dx \leqslant C\|f\|_2^2,$$

$Mf$ denoting the usual Hardy-Littlewood maximal function. This proves (48).

By Lemma 1, $U_i$ is bounded on $L_2$. The transpose operator $U_i^t$ is also bounded on $L_2$. If $i \neq j$,

$$\langle U_i^t(b_j), f\rangle_{b_i} = \langle b_j, U_i(b_i f)\rangle = \sum_{k=-\infty}^{\infty} \sum_{I \in \mathfrak{J}_{k-1}} \left(\int b_j \alpha_I^{(j)}\right)\langle \beta_I^{(j)}, \phi_i\rangle_{b_j}\left(\int_I b_i\right)^{-1}\left(\int_I b_i f\right) = 0$$

since $\int b_j \alpha_I^{(j)}\, dx = 0$. So $U_i^t(b_j) = 0$ if $i \neq j$. Now let $R = T - U_1 - U_2^t$. We have

$$R(b_1) = R^t(b_2) = 0. \tag{51}$$

The operator $R$ is also weakly bounded. We wish to show that $R$, and hence $T$, is bounded on $L_2$ by applying the methods of proof of Theorem 1. This effectively reduces to checking that the operators $R$ and $R^t$ satisfy the same kind of conditions as those given in Lemma 6. Now the proof of Lemma 6(iii) and (iv) uses only the property (26) of the kernel $K$. Consider the kernels associated to $U_1$ and $U_2^t$. They are given by (47) for $i = 1, 2$. Now for fixed $x \neq y$, and fixed $k$, there is at most one $I \in \mathfrak{J}_{k-1}$, say $I_{k-1}$, for which the summand in (47) is nonzero. For such a term,

$$|x - y| \leq C 2^{-k} \tag{52}$$

where $C$ is independent of $x$, $y$, and $k$. Let $k_0$ be the largest integer for which (52) holds. The sum in (47) is then, in norm, at most

$$C \sum_{k=-\infty}^{k_0} |I_{k-1}|^{-1/2} \frac{1}{|I_{k-1}|} \int_{I_{k-1}} |\beta_{I_{k-1}}^{(j)}(y)\, b_j(y)|\, |\phi_i - (\phi_i)_{I_{k-1}}|\, dy \leq$$

$$\leq C \|\phi_i\|_{\mathrm{BMO}} \sum_{k=-\infty}^{k_0} |I_{k-1}|^{-1} \leq C \|\phi_i\|_{\mathrm{BMO}} \sum_{-\infty}^{k_0} 2^{dk} \leq C \|\phi_i\|_{\mathrm{BMO}} 2^{dk_0} \leq C \|\phi_i\|_{\mathrm{BMO}} |x - y|^{-d}$$

by (52).

As to the analogues of (i) and (ii) of Lemma 6, we note that, if $J$ is a dyadic-quasi-cube, and $x \notin 2J$, then

$$U_1(b_1 \alpha_J^{(1)})(x) = \sum_{-\infty}^{\infty} \sum_{I \in \mathfrak{J}_{k-1}} \alpha_I^{(2)}(x)\, \chi_I(x)\, \langle \beta_I^{(2)}, \phi_1 \rangle_{b_2} \left( \int_I b_1 \right)^{-1} \left( \int_I b_1 \alpha_J^{(1)} \right)$$

is zero. In fact, the last factor in a term of the double sumation is nonzero only when $I \subseteq J$. But then $\chi_I(x) = 0$ since $x \notin 2J$. So the term is zero. A similar argument applies to $U_2^t$. Therefore conclusions (i) and (ii), as well as (iii) and (iv), of Lemma 6 hold for the operator $R$. The operator $R^t$ is treated similarly. Given that $R(b_1) = R^t(b_2) = 0$, it follows that the proof of Theorem 1 applies *mutatis mutandis* to the operator $R$. ∎

## REFERENCES

[1] L. V. AHLFORS, *Möbius transforms and Clifford numbers*, Differential Geometry and Complex Analysis: H. E. Rauch Memorial Volume, Springer-Verlag, Berlin-Heidelberg-New York (1985), pp. 65-73.

[2] F. BRACKX - R. DELANGHE - F. SOMMEN, *Clifford Analysis*, Research Notes in Mathematics, Vol. 76, Pitman Advanced Publishing Company, Boston, London, Melbourne (1982).

[3] R. R. COIFMAN - P. W. JONES - S. SEMMES, *Two elementary proofs of the* $L_2$ *boundedness of Cauchy integrals on Lipschitz curves*, J. Amer. Math. Soc., 2 (1989), pp. 553-564.

[4] R. R. COIFMAN - A. MCINTOSH - Y. MEYER, *L'intégrale de Cauchy définit un opérateur borné sur* $L^2$ *pour les courbes lipschitziennes*, Ann. of Math., **116** (1982), pp. 361-387.

[5] M. G. COWLING - G. I. GAUDRY - T. QIAN, *A note on martingales with respect to complex measures*, Miniconference on Operators in Analysis, Macquarie University, September 1989, Proceedings of the Centre for Mathematical Analysis, Australian National University, 24 (1989), pp. 10-27.

[6] G. DAVID, *Wavelets, Calderón-Zygmund operators, and singular integrals on curves and surfaces*, Proceedings of the Special Year on Harmonic Analysis at Nankai Institute of Mathematics, Tianjin, China, Lecture Notes in Mathematics, Springer-Verlag, Berlin, to appear.

[7] G. DAVID - J.-L. JOURNÉ, *A boundedness criterion for generalized Calderón-Zygmund operators*, Ann. of Math., **120** (1984), pp. 371-397.

[8] G. DAVID - J.-L. JOURNÉ - S. SEMMES, *Opérateurs de Calderón-Zygmund sur les espaces de nature homogène*, preprint.

[9] R. E. EDWARDS - G. I. GAUDRY, *Littlewood-Paley and Multiplier Theory*, Springer-Verlag, Berlin-Heidelberg-New York (1977).

[10] A. M. GARSIA, *Martingale Inequalities*, W. A. Benjamin, Inc., New York (1973).

[11] C. LI - A. MCINTOSH - T. QIAN, *Singular integral operators on Lipschitz surfaces*, in preparation.

[12] C. LI - A. MCINTOSH - S. SEMMES, *Convolution singular integrals on Lipschitz surfaces*, preprint.

[13] R. LONG - T. QIAN, *Clifford martingale $\Phi$-equivalence betwee $S(f)$ and $f$*, preprint.

[14] A. MCINTOSH, *Clifford algebras and the high-dimensional Cauchy integral*, Approximation and Function Spaces, Banach Centre Publications, Vol. 22, Pwn-Polish Scientific Publishers, Warsaw (1989).

[15] A. MCINTOSH - T. QIAN, *Convolution singular integral operators on Lipschitz curves*, Proceedings of the Special Year on Harmonic Analysis at Nankai Institute of Mathematics, Tianjin, China, to appear in Lecture Notes in Mathematics, Springer-Verlag, Berlin.

[16] Y. MEYER, *Odelettes et opérateurs*. II: *Opérateurs de Calderón-Zygmund*, Hermann et Cie, Paris (1990).

[17] M. A. M. MURRAY, *The Cauchy integral, Calderón commutators and conjugations of singular integrals in* $\mathbb{R}^n$, Trans. Amer. Math. Soc., **289** (1985), pp. 497-518.

[18] J. PEETRE, *On convolution operators leaving* $L^{p,\lambda}$ *invariant*, Ann. Mat. Pura Appl., **72** (1966), pp. 295-304.

[19] S. SEMMES, *A criterion for the boundedness of singular integrals on hypersurfaces*, preprint.

[20] S. SPANNE, *Sur l'interpolation entre les espaces* $\mathcal{L}_k^{p,\Phi}$, Ann. Scuola Norm. Sup. Pisa, **20** (1966), pp. 625-648.

[21] E. M. STEIN, *Singular integrals, harmonic functions, and differentiability properties of functions of several variables*, Proc. Symp. in Pure Math., **10** (1967), pp. 316-335.