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Uncertainty Principle and Phase–Amplitude Analysis of Signals on the Unit Sphere

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Abstract. This paper is devoted to studying uncertainty principle of Heisenberg type for signals on the unit sphere in the Clifford algebra setting. In the Clifford algebra setting we propose two forms of uncertainty principle for spherical signals, of which both correspond to the strongest form of uncertainty principle for periodic signals. The lower-bounds of the proven uncertainty principles are in terms of a scalar-valued phase derivative.

Keywords. Uncertainty principle, Phase derivative, Spherical signals, Spherical Hilbert transform, Spherical Dirac operator.

1. Introduction

The Heisenberg uncertainty principle originated in quantum mechanics. In time–frequency analysis, the classical uncertainty principle states that a function and its Fourier transform cannot be simultaneously well localized, where the function is always assumed to live on the real axis, being phrased as *non-periodic signals*. There exists an ample amount of literature that focus on uncertainty principles for non-periodic signals (for example, [4–6, 8, 13, 14, 16, 17, 19, 20]). There will be different forms of uncertainty principles when we study the subject for signals in different function spaces. Recently, a number of authors deal with uncertainty principles for periodic and spherical signals.

Breitenberger studies uncertainty principles for periodic functions in [3] from the physics point of view. For detailed information on periodic uncertainty principles we refer the interested readers to [23, 25, 26, 29, 30]. The

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cited references give the earliest uncertainty principles for periodic signals $s(e^{it}) = \rho(t)e^{i\varphi(t)} \in L^2([0, 2\pi))$, and those are essentially of the form

$$\sigma_t^2 \sigma_k^2 \geq \frac{1}{4}|t_0|^2, \tag{1.1}$$

where σ_t^2, σ_k^2 and t_0 are, respectively, the circular variance, the variance of Fourier frequency and the mean angle of t, with the definitions

$$\sigma_t^2 \triangleq \int_0^{2\pi} |(e^{it} - t_0)s(e^{it})|^2 dt, \tag{1.2}$$

$$\sigma_k^2 \triangleq \sum_{k=-\infty}^{\infty} (k - k_0)^2 |c_k|^2, \tag{1.3}$$

and

$$t_0 \triangleq \int_0^{2\pi} e^{it}|s(e^{it})|^2 dt, \tag{1.4}$$

where k_0 is the mean of Fourier frequency k, given by

$$k_0 \triangleq \sum_{k=-\infty}^{\infty} k|c_k|^2, \tag{1.5}$$

and c_k's are the Fourier coefficients,

$$c_k = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} s(e^{it})e^{-ikt} dt, \quad k = 0, \pm 1, \pm 2, \ldots. \tag{1.6}$$

Recently, [11] proposes two uncertainty principles for periodic signals, both being stronger than (1.1), read as

$$\sigma_t^2 \sigma_k^2 \geq \frac{1}{4}|t_0|^2 + |\mathrm{Cov_p}|^2, \tag{1.7}$$

and

$$\sigma_t^2 \sigma_k^2 \geq \frac{1}{4}|t_0|^2 + \mathrm{COV_p}^2, \tag{1.8}$$

where $\mathrm{Cov_p}$ is the *covariance* defined by

$$\begin{aligned} \mathrm{Cov_p} &= \int_0^{2\pi} e^{it}\varphi'(t)|s(e^{it})|^2 dt - t_0 k_0 \\ &= \int_0^{2\pi} [\varphi'(t) - k_0] \left(e^{it} - t_0\right) |s(e^{it})|^2 dt, \end{aligned} \tag{1.9}$$

and $\mathrm{COV_p}$ is the *absolute covariance* defined by

$$\mathrm{COV_p} = \int_0^{2\pi} |\varphi'(t) - k_0||e^{it} - t_0||s(e^{it})|^2 dt, \tag{1.10}$$

where $\varphi'(t)$ is the classical phase derivative, or otherwise suitably defined (see [11]). It is easy to see that the lower-bound of (1.8) is larger than that of (1.7), that is, (1.8) is stronger than (1.7).

Recently, there arises some interest in uncertainty principles of spherical signals [7,21,24,28,29]. Those uncertainty principles are in the form of a lower bound on the product of variances that are defined as follows:

$$\mathrm{V}_{x,\psi}\mathrm{V}_{\Omega,\psi} \geq \|\tau_\psi\|^2, \tag{1.11}$$

where $\mathrm{V}_{x,\psi}, \mathrm{V}_{\Omega,\psi}, \tau_\psi$ are given in Definition 1.1. $\|\cdot\|$ denotes the Euclidean norm in $\mathbf{R}^3$. The uncertainty principle (1.11) is proved in the vector space setting.

Definition 1.1. If $\psi(x)$ is a twice-continuously differentiable complex-valued function, and $\int_{\mathbb{S}^2} |\psi(x)|^2 d\sigma(x) = 1$, then the spherical mean, or mean of the space vector variable x, is defined to be

$$\tau_\psi \triangleq \int_{\mathbb{S}^2} x|\psi(x)|^2 d\sigma(x). \tag{1.12}$$

We note that τ_ψ is a vector. The variance of x is defined to be

$$\mathrm{V}_{x,\psi} \triangleq \int_{\mathbb{S}^2} \|x-\tau_\psi\|^2 |\psi(x)|^2 d\sigma(x) = 1 - \|\tau_\psi\|^2. \tag{1.13}$$

The mean of frequency is defined as

$$a(\psi) \triangleq \int_{\mathbb{S}^2} \Omega\psi(x)\overline{\psi(x)} d\sigma(x), \tag{1.14}$$

and the variance of frequency is defined as

$$\mathrm{V}_{\Omega,\psi} \triangleq \int_{\mathbb{S}^2} \|[\Omega - a(\psi)]\psi\|^2 d\sigma(x) = \int_{\mathbb{S}^2} (-\Delta^*_{\mathbb{S}^2})\psi \cdot \overline{\psi} d\sigma(x) - \|a(\psi)\|^2, \tag{1.15}$$

where $\mathbb{S}^2$ is the unit sphere embedded in $\mathbf{R}^3$, the surface variable $x \in \mathbb{S}^2$ is regarded as position operator and the *angular momentum operator* $\Omega = -\mathbf{i}L^* = -\mathbf{i}x \times \nabla^*$ as momentum operator, where ∇^* denotes the surface gradient and L^* the surface curl gradient. Note that both of them are roots of the Laplace–Beltrami operator of the unit sphere in the sense

$$\Delta^*_{\mathbb{S}^2} = L^* \cdot L^* = \nabla^* \cdot \nabla^*.$$

More details are referred to [18].

Uncertainty principle for signals on sphere $\mathbb{S}^2$ can be regarded as a generalization of that for periodic signals, or signals on circle $\mathbb{S}^1$. Based on the lower bound of uncertainty principle, up to now the development of uncertainty principle for periodic signals can be essentially represented by the three inequalities (1.1), (1.7) and (1.8). From the lower bound of (1.11), we can conclude that the uncertainty principle (1.11) for the sphere case, in fact, corresponds to (1.1) for periodic signals. It is natural to think whether uncertainty principles for spherical signals have forms that correspond to (1.7) and (1.8). In the study, we will pursue those forms of uncertainty principle for signals on the sphere. We are to study uncertainty principles on the sphere in the Clifford algebra setting not in the vector space setting. Indeed, Clifford algebra offers a complex structure with Cauchy's theory that brings a precise analogy of the unit circle context to the unit sphere: a Fourier–Laplace series on the sphere can be further decomposed into a Laurent-wise Fourier series consisting of two parts of which one corresponds to the Hardy space inside the sphere and the other corresponds to the Hardy space outside the sphere.

It is noticeable that besides phase derivatives the proof of (1.8) also involves amplitude derivatives of signals [11]. To obtain counterpart results, right definitions of phase and amplitude derivatives of spherical signals are crucial. In [33] the authors propose a scalar-valued phase derivative in the Clifford algebra setting that is shown in the paper as a right replacement of the 1-D phase derivative in higher dimensions. In the Clifford algebra setting there are formally more than one formulation of phase or amplitude derivative as discussed in Sect. 2.3. In the one complex variable setting, corresponding to the lowest degree of Clifford algebras, the different ways defining phase and amplitude derivative reduce to the classical one. The higher dimensions are different. The two ways of defining the amplitude derivatives in higher dimensions, in particular, lead to two alternative ways to define variance of frequency for spherical signals, viz., var_k and var_k^*, as given in Definition 3.3. Based on the two alternative ways we obtain two forms of uncertainty principles, both correspond to the strongest form of uncertainty principles for periodic signals, (1.8).

For applications of uncertainty principles of the classical and the generalized types we refer the readers to [10] and [31].

The writing of the paper is organized as follows. In Sect. 2, we recall some basic knowledge in Clifford algebra, define and analyze phase and amplitude derivatives of spherical signals in the Clifford algebra setting. Section 3 is devoted to studying the spherical means and variance of "time" and "frequency" in both the Clifford algebra and the vector space settings. Section 4 discusses uncertainty principles on the sphere in the Clifford algebra setting and deduces two different types of uncertainty principles.

2. Preliminaries

2.1. Some Basic Knowledge of Clifford Algebra

We review some basic knowledge of Clifford algebra (see [1,12]). Let $\mathbf{e}_1, \ldots, \mathbf{e}_m$ be *basic elements* satisfying $\mathbf{e}_i\mathbf{e}_j + \mathbf{e}_j\mathbf{e}_i = -2\delta_{ij}$, where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise, $i, j = 1, 2, \ldots, m$. Let

$$\mathbf{R}^m = \{\underline{x} = x_1\mathbf{e}_1 + \cdots + x_m\mathbf{e}_m : x_j \in \mathbf{R}, j = 1, 2, \ldots, m\}$$

be identical with the usual m-dimensional Euclidean space. We similarly define

$$\mathbf{C}^m = \{\underline{x} = x_1\mathbf{e}_1 + \cdots + x_m\mathbf{e}_m : x_j \in \mathbf{C}, j = 1, 2, \ldots, m\}.$$

An element in $\mathbf{R}^m$ (or in $\mathbf{C}^m$) is called a *vector*. The real (complex) Clifford algebra generated by $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m$, denoted as $\mathbf{R}_m$ ($\mathbf{C}_m$), is the non-commutative algebra generated by $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m$, over the real (complex) field $\mathbf{R}$ ($\mathbf{C}$). A general element in $\mathbf{R}_m$, therefore, is of the form $x = \sum_T x_T\mathbf{e}_T$, where $x_T \in \mathbf{R}$, and $\mathbf{e}_T = \mathbf{e}_{i_1}\mathbf{e}_{i_2}\ldots\mathbf{e}_{i_l}$, being called *induced products*, where T runs over all the ordered subsets of $\{1,\ldots,m\}$, namely,

$$T = \{(i_1,...,i_l)\ :\ 1 \le i_1 < \cdots < i_l \le m,\ 1 \le l \le m\}.$$

When $T = \emptyset$, we set $\mathbf{e}_\emptyset = \mathbf{e}_0 = 1$. We denote $|T| = l$ where l is the number of the indices involved. A general Clifford number x may be decomposed into

$$x = \sum_{l=0}^{m} x^{(l)},\quad x^{(l)} = \sum_{|T|=l} x_T\mathbf{e}_T.$$

A Clifford number of the form $x^{(l)}$ is called an l-form Clifford number. A 2-form Clifford number is also called a *bi-vector.*

Let

$$\mathbf{R}_1^m \text{ (or } \mathbf{C}_1^m) = \{x = x_0 + \underline{x} : x_0 \in \mathbf{R} \text{ (or } \mathbf{C}), \underline{x} \in \mathbf{R}^m \text{ (or } \mathbf{C}^m)\}.$$

Elements in $\mathbf{R}_1^m$ or $\mathbf{C}_1^m$ are called *para-vectors.* A sum of a 0-form and a 2-form is called a *para-bivector.*

The natural inner product between $x = \Sigma_T x_T\mathbf{e}_T$ and $y = \sum_T y_T\mathbf{e}_T$ in $\mathbf{C}_m$, denoted by $\langle x, y\rangle$, is the complex number $\Sigma_T x_T\overline{y_T}$. The norm associated with this inner product is

$$|x| = \langle x, x\rangle^{\frac{1}{2}} = \left(\Sigma_T |x_T|^2\right)^{\frac{1}{2}}.$$

The multiplication of two vectors $\underline{x} = \sum_{j=1}^m x_j\mathbf{e}_j \in \mathbf{R}^m$ and $\underline{y} = \sum_{j=1}^m y_j\mathbf{e}_j \in \mathbf{R}^m$ is given by

$$\underline{xy} = -\langle \underline{x}, \underline{y}\rangle + \underline{x} \wedge \underline{y}$$

where $-\langle \underline{x}, \underline{y}\rangle$, being the negative value of the usual inner product, is a scalar, denoted by $\mathrm{Sc}(\underline{xy})$ and given by

$$-\langle \underline{x}, \underline{y}\rangle = -\sum_{j=1}^{m} x_j y_j = \frac{1}{2}(\underline{xy} + \underline{yx}),$$

and $\underline{x} \wedge \underline{y}$ is the non-scalar part of $\underline{xy}$, denoted by $\mathrm{NSc}(\underline{xy})$ and given by

$$\underline{x} \wedge \underline{y} = \sum_{i<j} e_{ij}(x_i y_j - x_j y_i) = \frac{1}{2}(\underline{xy} - \underline{yx}),$$

that is a bi-vector, also denoted as $\mathrm{Bi}(\underline{xy})$.

The Clifford conjugation and reversion of $\mathbf{e}_T = \mathbf{e}_{i_1}\ldots\mathbf{e}_{i_l}$ are $\bar{\mathbf{e}}_T = \bar{\mathbf{e}}_{i_l}\ldots\bar{\mathbf{e}}_{i_1}, \bar{\mathbf{e}}_j = -\mathbf{e}_j$ and $\tilde{\mathbf{e}}_T = \mathbf{e}_{i_l}\ldots\mathbf{e}_{i_1}$. The Clifford conjugation of a vector $\underline{x} \in \mathbf{R}^m$ is $\overline{\underline{x}} = -\underline{x}$.

It is easy to verify that $0 \ne \underline{x} \in \mathbf{R}^m$ implies

$$\underline{x}^{-1} = \frac{\overline{\underline{x}}}{|\underline{x}|^2}.$$

The open ball with center 0 and radius 1 in $\mathbf{R}^m$ is denoted by B^m. The unit sphere in $\mathbf{R}^m$ is denoted by $\mathbb{S}^{m-1}$, whose surface area, denoted by σ_{m-1}, is of value $2\pi^{\frac{m}{2}}/\Gamma(\frac{m}{2})$.

Let $f(\underline{x})$ be defined on $\mathbf{R}^m$ taking values in $\mathbf{R}_m$ and thus of the form $\Sigma_T f_T(\underline{x})\mathbf{e}_T$, where f_T are real-valued functions. We will use the *homogeneous Dirac operator*, $\underline{D}$, where

$$\underline{D} = \frac{\partial}{\partial x_1}\mathbf{e}_1 + \cdots + \frac{\partial}{\partial x_m}\mathbf{e}_m.$$

We define the "left" and "right" role of the operators $\underline{D}$, respectively, as

$$\underline{D}f = \sum_{i=1}^{m}\sum_{T}\frac{\partial f_T}{\partial x_i}\mathbf{e}_i\mathbf{e}_T \text{ and } f\underline{D} = \sum_{i=1}^{m}\sum_{T}\frac{\partial f_T}{\partial x_i}\mathbf{e}_T\mathbf{e}_i.$$

If f has all continuous first order partial derivatives and $\underline{D}f = 0$ in a (connected and open) domain Ω, then we say that f is *left-monogenic* in Ω; and, if $f\underline{D} = 0$ in Ω, we say that f is *right-monogenic* in Ω. If f is both left- and right-monogenic, then we say that f is *monogenic*.

We call

$$E(\underline{x}) = \frac{\overline{\underline{x}}}{|\underline{x}|^m}$$

the *Cauchy kernel* in $\mathbf{R}^m$. It is easy to verify that $E(\underline{x})$ is a monogenic function in $\mathbf{R}^m\backslash\{0\}$.

For $\underline{x} = |\underline{x}|\underline{\xi} = r\underline{\xi}$, the Dirac operator can be represented by the spherical form

$$\underline{D} = \underline{\xi}\partial_r + \frac{1}{r}\partial_{\underline{\xi}} = \underline{\xi}\partial_r + \frac{1}{r}\underline{\xi}\,\overline{\underline{\xi}}\partial_{\underline{\xi}} = \frac{1}{r}\underline{\xi}(r\partial_r + \overline{\underline{\xi}}\partial_{\underline{\xi}}) = \frac{1}{r}\underline{\xi}(r\partial_r + \Gamma_{\underline{\xi}}),$$

where Γ_ξ is the bi-vector-valued spherical Dirac operator

$$\Gamma_{\underline{\xi}} = \overline{\underline{\xi}}\partial_{\underline{\xi}} = -\sum_{i<j}\mathbf{e}_{ij}(x_i\partial_{x_j} - x_j\partial_{x_i}).$$

We will need the following properties.

Lemma 2.1. *Let* $x_{\text{Bi}} = x_1\mathbf{e}_1\mathbf{e}_2 + x_2\mathbf{e}_2\mathbf{e}_3 + x_3\mathbf{e}_3\mathbf{e}_1, y_{\text{Bi}} = y_1\mathbf{e}_1\mathbf{e}_2 + y_2\mathbf{e}_2\mathbf{e}_3 + y_3\mathbf{e}_3\mathbf{e}_1$, $x = x_0 + x_{\text{Bi}}, y = y_0 + y_{\text{Bi}}$ *and* $\underline{z} = z_1\mathbf{e}_1 + z_2\mathbf{e}_2 + z_3\mathbf{e}_3$ *with* x_0, y_0, x_i, y_i *and* $z_i, i = 1, 2, 3 \in \mathbf{R}$. *Then*

$$|x|^2|y|^2 = |xy|^2, \tag{2.1}$$

$$|x_{\text{Bi}}|^2|\underline{z}|^2 = |x_{\text{Bi}}\underline{z}|^2, \tag{2.2}$$

$$x_{\text{Bi}}\overline{x_{\text{Bi}}} = |x_{\text{Bi}}|^2 = \sum_{i=1}^{3}x_i^2, \tag{2.3}$$

$$x\overline{x} = |x|^2 = \sum_{i=0}^{3}x_i^2, \tag{2.4}$$

$$x_{\text{Bi}}y_{\text{Bi}} + y_{\text{Bi}}x_{\text{Bi}} = -2\langle x_{\text{Bi}}, y_{\text{Bi}}\rangle = -2\sum_{i=1}^{3}x_iy_i. \tag{2.5}$$

Proof of Lemma 2.1. (2.1) is by direct computation or by invoking existing knowledge on the Clifford group. (2.2) and (2.4) are by direct computation. To prove (2.3) and (2.5) we let $f_1 = \mathbf{e}_1\mathbf{e}_2, f_2 = \mathbf{e}_2\mathbf{e}_3, f_3 = \mathbf{e}_3\mathbf{e}_1$. It is easy to verify that $f_i f_j + f_j f_i = -2\delta_{ij}$. Then (2.3) and (2.5) follow from the same properties for vectors. $\square$

2.2. Fourier Expansion and Spherical Hilbert Transforms

We use the notation $L^2(\mathbb{S}^{m-1})$ to denote the square-integrable function space on the unit sphere $\mathbb{S}^{m-1}$ embedded in $\mathbf{R}^m$. For $f \in L^2(\mathbb{S}^{m-1})$, we have the *Fourier expansion*

$$f(\underline{\xi}) = \sum_{k=0}^{\infty} P_k(f)(\underline{\xi}) + Q_{k-1}(f)(\underline{\xi}), \tag{2.6}$$

where $P_0(f)$ is a constant, $Q_{-1} = 0$, and

$$P_k(f)(\underline{\xi}) = \frac{1}{\sigma_{m-1}} \int_{\mathbb{S}^{m-1}} C^+_{m,k}(\underline{\xi}, \underline{y}) f(\underline{y}) d\sigma(\underline{y}),$$

and

$$Q_{k-1}(f)(\underline{\xi}) = \frac{1}{\sigma_{m-1}} \int_{\mathbb{S}^{m-1}} C^-_{m,k-1}(\underline{\xi}, \underline{y}) f(\underline{y}) d\sigma(\underline{y})$$

where

$$\begin{aligned} C^+_{m,k}(\underline{\xi}, \underline{y}) = \frac{1}{2-m} \Big[& -(m+k-2) C_k^{\frac{m-2}{2}}(\langle \underline{\xi}, \underline{y} \rangle) \\ & + (2-m) C_{k-1}^{\frac{m}{2}}(\langle \underline{\xi}, \underline{y} \rangle)(\underline{\xi} \wedge \underline{y}) \Big], \end{aligned}$$

and

$$\begin{aligned} C^-_{m,k-1}(\underline{\xi}, \underline{y}) = \frac{1}{m-2} \Big[& k C_k^{\frac{m-2}{2}}(\langle \underline{\xi}, \underline{y} \rangle) \\ & + (2-m) C_{k-1}^{\frac{m}{2}}(\langle \underline{\xi}, \underline{y} \rangle)(\underline{\xi} \wedge \underline{y}) \Big], \quad k \geq 1, \end{aligned}$$

C_k^ν is the Gegenbauer polynomial of degree k associated with ν (see [12,33]).

The component $P_k(f)(\underline{\xi})$ is an *inner spherical monogenic of degree* k, which is the restriction to the unit sphere of the k-homogeneous left-monogenic function $P_k(f)(r\underline{\xi})$ in $\mathbf{R}^m$. The component $Q_{k-1}(f)(\underline{\xi})$ is an *outer spherical monogenic* of degree $k-1$, which is the restriction to the unit sphere of the $-(m+k-2)$-homogeneous left-monogenic function $Q_{k-1}(f)(r\underline{\xi})$ in $\mathbf{R}^m \backslash \{0\}$. Therefore, $P_k(f)(\underline{\xi}) \in H_2^+(\mathbb{S}^{m-1})$ and $Q_{k-1}(f)(\underline{\xi}) \in H_2^-(\mathbb{S}^{m-1})$, where $H_2^+(\mathbb{S}^{m-1})$ and $H_2^-(\mathbb{S}^{m-1})$ are the non-tangential boundary values of the Hardy spaces functions on B^m and $\overline{B^m}^c$. Moreover, we have

$$-\Gamma_{\underline{\xi}} P_k(f)(\underline{\xi}) = k P_k(f)(\underline{\xi}) \quad \text{and} \quad -\Gamma_{\underline{\xi}} Q_{k-1}(f)(\underline{\xi}) = -(k+m-2) Q_{k-1}(f)(\underline{\xi}).$$

In [27], Hilbert transform on the sphere is studied. Below we review some knowledge about it. We take $\Omega^+ = B^m$ and $\Omega^- = \overline{B^m}^c$. For a scalar-valued function f in $L^2(\mathbb{S}^{m-1})$, the two Cauchy integrals are given by

$$\begin{aligned}
M^{\pm}f(\underline{x}) &= \frac{1}{\sigma_{m-1}}\int_{\mathbb{S}^{m-1}} E(\underline{y}-\underline{x})(\pm\underline{y})f(\underline{y})d\sigma(\underline{y}) \\
&= \frac{1}{\sigma_{m-1}}\int_{\mathbb{S}^{m-1}} \langle E(\underline{y}-\underline{x}), \pm\underline{y}\rangle f(\underline{y})d\sigma(\underline{y}) \\
&\quad + \frac{1}{\sigma_{m-1}}\int_{\mathbb{S}^{m-1}} [E(\underline{y}-\underline{x})\wedge \pm\underline{y}\rangle] f(\underline{y})d\sigma(\underline{y}) \\
&= U^{\pm}+V^{\pm}, \quad \underline{x}\in\Omega^{\pm}.
\end{aligned} \tag{2.7}$$

Due to the Plemelj formula in the context there exist the non-tangential boundary limits of $M^{\pm}f(\underline{x})$, denoted by $f^{\pm}$, given by

$$f^{\pm}(\underline{\xi}) = \frac{1}{2}[f(\underline{\xi}) \pm \mathcal{C}f(\underline{\xi})], \quad \text{a.e. } \underline{\xi}\in\mathbb{S}^{m-1}, \tag{2.8}$$

where

$$f^{+}(\underline{\xi}) = \sum_{k=0}^{\infty} P_k(f)(\underline{\xi}), \quad f^{-}(\underline{\xi}) = \sum_{k=0}^{\infty} Q_{k-1}(f)(\underline{\xi}) \tag{2.9}$$

and $\mathcal{C}$ is the *principle value Cauchy singular integral operator* on the sphere given by

$$\begin{aligned}
\mathcal{C}f(\underline{\xi}) &= \frac{2}{\sigma_{m-1}}\lim_{\varepsilon\to 0}\int_{|\underline{y}-\underline{\xi}|>\varepsilon, \underline{y}\in\mathbb{S}^{m-1}} E(\underline{y}-\underline{\xi})\underline{y}f(\underline{y})d\sigma(\underline{y}) \\
&= \frac{2}{\sigma_{m-1}} p.v. \int_{\mathbb{S}^{m-1}} \langle E(\underline{y}-\underline{\xi}), \underline{y}\rangle f(\underline{y})d\sigma(\underline{y}) \\
&\quad + \frac{2}{\sigma_{m-1}} p.v. \int_{\mathbb{S}^{m-1}} [E(\underline{y}-\underline{\xi})\wedge\underline{y}\rangle] f(\underline{y})d\sigma(\underline{y}), \quad \text{a.e. } \underline{\xi}\in\mathbb{S}^{m-1},
\end{aligned} \tag{2.10}$$

showing that $\mathcal{C}f(\underline{\xi})$ is divided into its scalar and bi-vector parts, and $f^{\pm}$ is a para-bivector-valued function.

For any Clifford valued function g we will use the mappings Sc: $g\to \mathrm{Sc}[g]$ and NSc: $g\to\mathrm{NSc}[g]$, where $\mathrm{Sc}[g]$ and $\mathrm{NSc}[g]$ denote the scalar and non-scalar parts of g, that is,

$$\mathrm{Sc}[g] = \frac{1}{2}[g+\overline{g}], \qquad \mathrm{NSc}[g] = \frac{1}{2}[g-\overline{g}].$$

Using this notation, for scalar-valued function f, we can rewrite the relation (2.8) as

$$f^{\pm}(\underline{\xi}) = \frac{1}{2}\{f(\underline{\xi}) \pm \mathrm{Sc}[\mathcal{C}f] \pm \mathrm{NSc}[\mathcal{C}f]\} = u^{\pm}+v^{\pm}, \tag{2.11}$$

where $u^{\pm}$ and $v^{\pm}$ are, respectively, non-tangential boundary values of $U^{\pm}$ and its Cauchy-type harmonic conjugation $V^{\pm}$, $\pm$ refers to "inner" or "outer" part of the sphere, respectively. With $\mathrm{Sc}[\mathcal{C}]f = \mathrm{Sc}[\mathcal{C}f]$ and $\mathrm{NSc}[\mathcal{C}]f = \mathrm{NSc}[\mathcal{C}f]$, we have the operator equations

$$u^{\pm} = \frac{1}{2}(I\pm\mathrm{Sc}[\mathcal{C}])f, \quad v^{\pm} = \pm\frac{1}{2}\mathrm{NSc}[\mathcal{C}]f, \tag{2.12}$$

and, therefore, at least formally,

$$v^{\pm} = H^{\pm}u^{\pm} = \pm \mathrm{NSc}[\mathcal{C}](I \pm \mathrm{Sc}[\mathcal{C}])^{-1}u^{\pm},$$

being the Hilbert transforms of $u^{\pm}$, respectively.

The inner spherical Hilbert transform of f is given by

$$Hf(\underline{\xi}) = \lim_{r\to 1^-} \int_{\mathbb{S}^{m-1}} Q(r\underline{\xi}, \underline{\omega}) f(\underline{\omega}) d\sigma(\underline{\omega}),$$

where

$$Q(r\underline{\xi}, \underline{\omega}) = \frac{1}{\sigma_{m-1}} \left(\frac{2}{|r\underline{\xi} - \underline{\omega}|^m} - \frac{m-2}{r^{m-1}} \int_0^r \frac{\rho^{m-2}}{|\rho\underline{\xi} - \underline{\omega}|^m} d\rho \right) r\underline{\xi} \wedge \underline{\omega},$$

where $\underline{\xi}, \underline{\omega} \in \mathbb{S}^{m-1}$, $0 \le r < 1$.

The Fourier series form of the inner spherical Hilbert transform is given by (see [2,27])

$$Hf(\underline{\xi}) = \sum_{k=1}^{\infty} \frac{k}{k+m-2} P_k(f)(\underline{\xi}) - Q_{k-1}(f)(\underline{\xi}).$$

2.3. Phase and Amplitude Derivatives of Spherical Signals

To define amplitude and phase derivatives of spherical signals we first review some related knowledge in the periodic signal case. Let s be a complex-valued signal on the circle. Assume that the classical sense derivatives $s'(e^{it}), \rho_s'(t)$ and $\varphi'(t)$ of the given signal $s(e^{it}) = \rho_s(t)e^{i\varphi(t)}$ all exist at all points. Take derivative with respect to t and divide the both sides of $[s(e^{it})]' = [\rho_s(t)e^{i\varphi(t)}]'$ by $s(e^{it})$. By separating the real and the imaginary parts, we have

$$\rho_s'(t) = -\rho_s(t)\mathrm{Im}\left[\frac{e^{it}s'(e^{it})}{s(e^{it})}\right], \tag{2.13}$$

and

$$\varphi'(t) = \mathrm{Re}\left[\frac{e^{it}s'(e^{it})}{s(e^{it})}\right]. \tag{2.14}$$

For real-valued function $f \in L^2(\mathbb{S}^{m-1})$, as stated in Sect. 2.2, there exist functions $u^{\pm} \in L^2(\mathbb{S}^{m-1})$ such that

$$\begin{aligned} f^+(\underline{\xi}) &= u^+(\underline{\xi}) + H^+u^+(\underline{\xi}) \in H_2^+(\mathbb{S}^{m-1}) \text{ and } f^-(\underline{\xi}) \\ &= u^-(\underline{\xi}) + H^-u^-(\underline{\xi}) \in H_2^-(\mathbb{S}^{m-1}), \end{aligned}$$

being the non-tangential boundary limits of some left-monogenic functions respectively inside and outside the unit ball. We call $f^+(\underline{\xi})$ the *monogenic signal associated with* f (see [33]). Instead of defining phase and amplitude derivatives of f directly, we, instead, define those of $f^{\pm}$ as what is defined in [33]. In the process of defining the phase and amplitude derivatives the property $f^+(\underline{\xi})\overline{f^+(\underline{\xi})} = |f^+(\underline{\xi})|^2$ for the para-bivecter function $f^+(\underline{\xi})$ is crucial. The required relation, however, is only valid for $m \le 3$. In the rest of the paper we restrict ourselves to the case $m = 3$, that is $f \in L^2(\mathbb{S}^2)$.

To define the phase and amplitude derivatives of $f^{\pm}(\underline{\xi})$, we represent $f^{\pm}(\underline{\xi})$ in the amplitude-phase form [33]. We take f^+ as example and the case for f^- is similar. For simplicity, we use the notations u and H for, respectively, u^+ and H^+, and have

$$\begin{aligned} f^+(\underline{\xi}) &= u(\underline{\xi}) + Hu(\underline{\xi}) \\ &= \rho(\underline{\xi})\left[\frac{u(\underline{\xi})}{\rho(\underline{\xi})} + \frac{Hu(\underline{\xi})}{\rho(\underline{\xi})}\right] \\ &= \rho(\underline{\xi})\left[\frac{u(\underline{\xi})}{\rho(\underline{\xi})} + \frac{Hu(\underline{\xi})}{|Hu(\underline{\xi})|}\frac{|Hu(\underline{\xi})|}{\rho(\underline{\xi})}\right] \\ &= \rho(\underline{\xi})\left[\cos\theta(\underline{\xi}) + \frac{Hu(\underline{\xi})}{|Hu(\underline{\xi})|}\sin\theta(\underline{\xi})\right] \\ &= \rho(\underline{\xi})e^{\frac{Hu(\underline{\xi})}{|Hu(\underline{\xi})|}\theta(\underline{\xi})}, \end{aligned} \tag{2.15}$$

where $\rho(\underline{\xi}) = \sqrt{(u)^2(\underline{\xi}) + |Hu(\underline{\xi})|^2}$ is called the *amplitude* of f^+, $\theta(\underline{\xi}) = \arctan\frac{|Hu(\underline{\xi})|}{u(\underline{\xi})}$ the *phase* of f^+, $\frac{Hu(\underline{\xi})}{|Hu(\underline{\xi})|}\theta(\underline{\xi})$ the *phase vector*, and $e^{\frac{Hu(\underline{\xi})}{|Hu(\underline{\xi})|}\theta(\underline{\xi})}$ the *phase direction*. We note that when $m = 3$, Hu is a bivector, thus by (2.3) we have $\left\{\frac{Hu(\underline{\xi})}{|Hu(\underline{\xi})|}\right\}^2 = -1$, that is, $\frac{Hu(\underline{\xi})}{|Hu(\underline{\xi})|}$ plays the same role as the imaginary unit i in the case $m = 2$. In the Euler formula the imaginary unit can be substituted by $\frac{Hu(\underline{\xi})}{|Hu(\underline{\xi})|}$, that is why we have the last line in the formula (2.15) (see [15,33]).

The definitions of phase and amplitude derivatives of $f^+(\underline{\xi})$ are as follows. Those of $f^-(\underline{\xi})$ are similar.

Definition 2.2. Let $f(\underline{\xi}) \in L^2(\mathbb{S}^2)$ be scalar-valued and f^+ the Hardy space projection of f into $H^2(\mathbb{S}^2)$ with the expression

$$f^+(\underline{\xi}) = \rho(\underline{\xi})e^{\frac{Hu(\underline{\xi})}{|Hu(\underline{\xi})|}\theta(\underline{\xi})},$$

where ρ and θ are defined through (2.15), u is given by (2.12). Then a phase derivative can be defined in one of the following two ways:

$$\theta_1'(\underline{\xi}) \triangleq \mathrm{Sc}\left\{\left[-\Gamma_{\underline{\xi}} f^+(\underline{\xi})\right]\left[f^+(\underline{\xi})\right]^{-1}\right\} \tag{2.16}$$

and

$$\theta_2'(\underline{\xi}) \triangleq \mathrm{Sc}\left\{[-\Gamma_{\underline{\xi}}\theta(\underline{\xi})]\frac{Hu(\underline{\xi})}{|Hu(\underline{\xi})|}\right\}. \tag{2.17}$$

The amplitude derivative $\rho'(\underline{\xi})$ is also defined through two ways

$$\rho_1'(\underline{\xi}) \triangleq \frac{Hu(\underline{\xi})}{|Hu(\underline{\xi})|}[-\Gamma_{\underline{\xi}}\rho(\underline{\xi})], \tag{2.18}$$

and

$$\rho_2'(\underline{\xi}) \triangleq \frac{Hu(\underline{\xi})}{|Hu(\underline{\xi})|}\rho\mathrm{Nsc}\left\{\left[-\Gamma_{\underline{\xi}} f^+(\underline{\xi})\right][f^+(\underline{\xi})]^{-1}\right\}. \tag{2.19}$$

We note that the phase derivatives $\theta_1'(\underline{\xi})$ and $\theta_2'(\underline{\xi})$ are defined first in [33]. In [33], the authors give the detailed reason why the phase derivatives are defined as $\theta_1'(\underline{\xi})$ and $\theta_2'(\underline{\xi})$. Simply speaking, the definitions of $\theta_1'(\underline{\xi})$ and $\theta_2'(\underline{\xi})$ are in analogy with the left-hand and right-hand sides of (2.14). By the same method, we define *amplitude derivative* $\rho_1'(\underline{\xi})$ and $\rho_2'(\underline{\xi})$ in this paper based on the left-hand and right-hand sides of (2.13).

The following explanation is necessary. In the above definition we apply the spherical Dirac differential operator to various functions related to the non-tangential boundary limit function on the sphere of the Hardy space function f^+. The boundary limit function, however, is not necessarily smooth, and, as consequence, may not have the required partial derivatives. The right understanding of the application of the spherical Dirac differential operator to f^+ is as follows (see [9]): we apply $\Gamma_{\underline{\xi}}$ to $f^+(r\underline{\xi}), 0 < r < 1$, that, as a monogenic function inside the unit ball, is smooth. Once we have defined $\Gamma_{\underline{\xi}} f^+(r\underline{\xi})$, we take non-tangential boundary limit to obtain $\Gamma_{\underline{\xi}} f^+(\underline{\xi})$. The definitions of $\Gamma_{\underline{\xi}}\theta(\underline{\xi})$ and $\Gamma_{\underline{\xi}}\rho(\underline{\xi})$ are similar. The existence of each involved boundary limit is guaranteed by the assumption that f belongs to the relevant Sobolev space.

Below through directed calculation we obtain the other representation of $\rho_1'(\underline{\xi})$ and the relationship between $\rho_1'(\underline{\xi})$ and $\rho_2'(\underline{\xi})$.

Since $f^+(\underline{\xi})$ is a para-bivector, by using (2.4), we have

$$f^+(\underline{\xi})\overline{f^+(\underline{\xi})} = |f^+(\underline{\xi})|^2 \quad \text{and} \quad [f^+(\underline{\xi})]^{-1} = \frac{\overline{f^+(\underline{\xi})}}{|f^+(\underline{\xi})|^2} = \frac{1}{\rho(\underline{\xi})} e^{-\frac{Hu(\underline{\xi})}{|Hu(\underline{\xi})|}\theta(\underline{\xi})}.$$

Through direct computation we have

$$\begin{aligned} -\Gamma_{\underline{\xi}} f^+(\underline{\xi}) &= -\Gamma_{\underline{\xi}} \left[\rho(\underline{\xi}) e^{\frac{Hu(\underline{\xi})}{|Hu(\underline{\xi})|}\theta(\underline{\xi})}\right] \\ &= \left[-\Gamma_{\underline{\xi}}\rho(\underline{\xi})\right] e^{\frac{Hu(\underline{\xi})}{|Hu(\underline{\xi})|}\theta(\underline{\xi})} + \rho(\underline{\xi}) \left[-\Gamma_{\underline{\xi}} e^{\frac{Hu(\underline{\xi})}{|Hu(\underline{\xi})|}\theta(\underline{\xi})}\right]. \end{aligned}$$

Then,

$$\left[-\Gamma_{\underline{\xi}} f^+(\underline{\xi})\right] \left[f^+(\underline{\xi})\right]^{-1} = \frac{-\Gamma_{\underline{\xi}}\rho(\underline{\xi})}{\rho(\underline{\xi})} + \left[-\Gamma_{\underline{\xi}} e^{\frac{Hu(\underline{\xi})}{|Hu(\underline{\xi})|}\theta(\underline{\xi})}\right] e^{-\frac{Hu(\underline{\xi})}{|Hu(\underline{\xi})|}\theta(\underline{\xi})}. \tag{2.20}$$

Based on (2.20), we have

$$\begin{aligned} -\Gamma_{\underline{\xi}}\rho(\underline{\xi}) = \rho(\underline{\xi}) \Big\{ &\left[-\Gamma_{\underline{\xi}} f^+(\underline{\xi})\right] [f^+(\underline{\xi})]^{-1} \\ &- \left[-\Gamma_{\underline{\xi}} e^{\frac{Hu(\underline{\xi})}{|Hu(\underline{\xi})|}\theta(\underline{\xi})}\right] e^{-\frac{Hu(\underline{\xi})}{|Hu(\underline{\xi})|}\theta(\underline{\xi})} \Big\}. \end{aligned} \tag{2.21}$$

Since $-\Gamma_{\underline{\xi}}\rho(\underline{\xi})$ is bi-vector-valued, (2.18) can be further represented as

$$\begin{aligned}
&\rho_1'(\underline{\xi})\\
&= \frac{Hu(\underline{\xi})}{|Hu(\underline{\xi})|}[-\Gamma_{\underline{\xi}}\rho(\underline{\xi})]\\
&= \frac{Hu(\underline{\xi})}{|Hu(\underline{\xi})|}\rho(\underline{\xi})\left\{\mathrm{Nsc}\left\{[-\Gamma_{\underline{\xi}}f^+(\underline{\xi})][f^+(\underline{\xi})]^{-1}\right\}\right.\\
&\quad\left.-\mathrm{Nsc}\left\{\left[-\Gamma_{\underline{\xi}}e^{\frac{Hu(\underline{\xi})}{|Hu(\underline{\xi})|}\theta(\underline{\xi})}\right]e^{-\frac{Hu(\underline{\xi})}{|Hu(\underline{\xi})|}\theta(\underline{\xi})}\right\}\right\}\\
&= \rho_2'(\underline{\xi}) - \frac{Hu(\underline{\xi})}{|Hu(\underline{\xi})|}\rho(\underline{\xi})\mathrm{Nsc}\left\{\left[-\Gamma_{\underline{\xi}}e^{\frac{Hu(\underline{\xi})}{|Hu(\underline{\xi})|}\theta(\underline{\xi})}\right]e^{-\frac{Hu(\underline{\xi})}{|Hu(\underline{\xi})|}\theta(\underline{\xi})}\right\}.
\end{aligned} \tag{2.22}$$

3. Means and Variances of Time and Frequency for Spherical Signals

In this section, we aim to give appropriate definitions of means of time and frequency for signals on $\mathbb{S}^2$. We first review the related knowledge for periodic signals in Sect. 3.1. Then in Sect. 3.2 we give our definitions of the means and variances of time and frequency for signals on $\mathbb{S}^2$ in the Clifford algebra setting. In the vector space settings [24] propose certain definitions of the means and variances of time and frequency for signals on the sphere. In Sect. 3.3 we make comparisons between the definitions given in [24] and those in the Clifford algebra setting.

3.1. Mean and Variance of Time and Frequency for Periodic Signals

Expanding $s(e^{it}) \in L^2([0, 2\pi))$ into its

Fourier series, we have, in the L^2-convergence sense,

$$s(e^{it}) = \frac{1}{\sqrt{2\pi}}\sum_{k=-\infty}^{\infty} c_k e^{ikt},$$

where c_k's are the Fourier coefficients,

$$c_k = \frac{1}{\sqrt{2\pi}}\int_0^{2\pi} s(e^{it})e^{-ikt}dt, \quad k = 0, \pm1, \pm2, \ldots. \tag{3.1}$$

There exist different definitions for means and variances of time and frequency for periodic square-integrable functions (see [3,11,25,26,29,30]). We adopt the method used in [11] to define means and variances of time and frequency, that are t_0, σ_t^2, k_0 and σ_k^2, given in the introduction section.

In [11], the author represents k_0 and σ_k^2 in the time domain. The results in [11] give reasons for the means, as well as the phase and amplitude derivatives, as defined in formulas (2.14) and (2.13). For the self containing purpose we include the results here.

Lemma 3.1. *Assume* $s(e^{it}) = \rho_s(t)e^{i\varphi(t)} \in L^2([0, 2\pi))$ *and* $\|s\|_2 = 1$. *Assume that the classical derivatives* $\rho_s'(t), \varphi'(t), s'(e^{it})$ *exist at all points, and* $s'(e^{it})$ *is in* $L^2([0, 2\pi))$. *Then there hold*

$$k_0 = \int_0^{2\pi} \varphi'(t)\rho_s^2(t)dt, \tag{3.2}$$

and

$$\sigma_k^2 = \int_0^{2\pi} [\varphi'(t) - k_0]|s(e^{it})|^2 dt + \int_0^{2\pi} {\rho_s'}^2(t)dt. \tag{3.3}$$

3.2. Mean and Variance for Spherical Signals in the Clifford Algebra Setting

In the classical one dimensional cases we study time–frequency analysis. In higher dimensions the counterpart concepts for time and frequency are space and suitably defined frequency concepts. We will adopt similar notation and terminology. As an example, when we say "mean of time", we mean "mean of space".

We first proceed to define the mean of frequency of f. As for the classical real-valued signal case we will show that for any real-valued signal on the sphere the mean of its frequency is identical to zero. As discussed in [33], the Fourier frequencies k for a periodic signal $s(e^{it})$ are the phase derivatives of e^{ikt}, and the mean of Fourier frequency of a periodic signal $s(e^{it})$ is defined by (1.5). The formula (3.2) exhibits the relation between the Fourier frequency and the phase derivative of signal. To define the mean of frequency of f on the sphere, we need the Fourier frequencies of f and the energy distributions on the respective frequencies. The phase derivatives of $P_k(f)(\underline{\xi})$ and $Q_{k-1}(f)(\underline{\xi})$, obtained through the formula (2.16), are respectively k and $-(k + m - 2)|_{m=3} = -(k + 1)$, and k and $-(k + 1), k = 0, 1, \ldots$ are regarded as the Fourier frequencies of f^+ and f^-, respectively. The mean of frequency of a real-valued signal $f(\underline{\xi}) \in L^2(\mathbb{S}^2)$ is defined as

$$\langle k\rangle_f \triangleq \sum_{k=0}^{\infty} k\|P_k(f)(\underline{\xi})\|^2 + \sum_{k=0}^{\infty}[-(k + 1)]\|Q_{k-1}(f)(\underline{\xi})\|^2, \tag{3.4}$$

where $P_k(f)(\underline{\xi})$ and $Q_{k-1}(f)(\underline{\xi})$ are defined in (2.6). We have the following significant fact.

Proposition 3.2. *Let* $f(\underline{\xi}) \in L^2(\mathbb{S}^2)$ *be any real-valued signal with* $\|f\|_2 = 1$. *Then there holds*

$$\langle k\rangle_f = 0.$$

Proof of Proposition. Since $P_k(f)(\underline{\xi})$ and $Q_{k-1}(f)(\underline{\xi})$ are para-bivectors, by using (2.4), we have

$$|P_k(f)(\underline{\xi})|^2 = P_k(f)(\underline{\xi})\overline{P_k(f)(\underline{\xi})} \text{ and } |Q_{k-1}(f)(\underline{\xi})|^2 = Q_{k-1}(f)(\underline{\xi})\overline{Q_{k-1}(f)(\underline{\xi})}.$$

Then

$$\begin{aligned}
\langle k\rangle_f &= \sum_{k=0}^{\infty} k\|P_k(f)(\underline{\xi})\|^2 + \sum_{k=0}^{\infty}[-(k+1)]\|Q_{k-1}(f)(\underline{\xi})\|^2 \\
&= \sum_{k=1}^{\infty} k\int_{\mathbb{S}^2} |P_k(f)(\underline{\xi})|^2 d\sigma(\underline{\xi}) + \sum_{k=1}^{\infty}[-(k+1)]\int_{\mathbb{S}^2} |Q_{k-1}(f)(\underline{\xi})|^2 d\sigma(\underline{\xi}) \\
&= \int_{\mathbb{S}^2} \sum_{k=1}^{\infty} kP_k(f)(\underline{\xi})\overline{P_k(f)(\underline{\xi})}d\sigma(\underline{\xi}) \\
&\quad + \int_{\mathbb{S}^2} \sum_{k=1}^{\infty}[-(k+1)]Q_{k-1}(f)(\underline{\xi})\overline{Q_{k-1}(f)(\underline{\xi})}d\sigma(\underline{\xi}) \\
&= \int_{\mathbb{S}^2} \sum_{k=1}^{\infty}[-\Gamma_{\underline{\xi}} P_k(f)(\underline{\xi})]\overline{P_k(f)(\underline{\xi})}d\sigma(\underline{\xi}) \\
&\quad + \int_{\mathbb{S}^2} \sum_{k=1}^{\infty}[-\Gamma_{\underline{\xi}} Q_{k-1}(f)(\underline{\xi})]\overline{Q_{k-1}(f)(\underline{\xi})}d\sigma(\underline{\xi}) \\
&= \int_{\mathbb{S}^2} \sum_{k=1}^{\infty}\{-\Gamma_{\underline{\xi}}[P_k(f)(\underline{\xi}) + Q_{k-1}(f)(\underline{\xi})]\}\overline{[P_k(f)(\underline{\xi}) + Q_{k-1}(f)(\underline{\xi})]}d\sigma(\underline{\xi}) \\
&= \int_{\mathbb{S}^2} [-\Gamma_{\underline{\xi}} f(\underline{\xi})]\overline{f(\underline{\xi})}d\sigma(\underline{\xi}) \\
&= 0,
\end{aligned}$$

where since $[-\Gamma_{\underline{\xi}} f(\underline{\xi})]\overline{f(\underline{\xi})}$ is bi-vector-valued, and $\langle k\rangle_f$ should be a real number, so the integral $\int_{\mathbb{S}^2}[-\Gamma_{\underline{\xi}} f(\underline{\xi})]\overline{f(\underline{\xi})}d\sigma(\underline{\xi})$ must be zero. The proof is completed. □

In signal analysis, no matter in the classical one dimensional cases [4,9] or the higher dimensional cases [32], one studies the analytic signal f^+ instead of studying the original real-valued signal f. There are at least two good reasons for this. The first is that for any real-valued signal its mean of frequency is zero. The basic fact is that for a real-valued signal its Fourier coefficients, or the Fourier transform values, at a positive and corresponding the negative spectrum are conjugate to each other. As a result, in the expression of the mean of frequency the positive part and the negative part are cancelled out. The celebrating Proposition 3.2 shows that on the sphere the same result holds. Due to the zero mean property of frequency the quantity of deviation of the frequencies of a real valued function does not reflect the true derivation of the frequencies. But the derivation of f^+ does [4]. The second reason of working with f^+ is that f^+ is defined through the analytic function theory that deals with functions whose Fourier coefficients or Fourier transform values are non-zero only at the non-negative Fourier spectra (the positive Fourier spectrum property). Operations of analytic functions preserve the analyticity property and thus preserve the positive Fourier spectrum property.

Definition 3.3. Let $f(\underline{\xi}) \in L^2(\mathbb{S}^2)$ be real-valued and $\|f^+\|_2 = 1$. Then the *spherical mean*, or *mean of time* $\underline{\xi}$, is defined to be

$$\langle \underline{\xi} \rangle \triangleq \int_{\mathbb{S}^2} \underline{\xi} |f^+(\underline{\xi})|^2 d\sigma(\underline{\xi}). \tag{3.5}$$

The *spherical variance*, or *variance of time* $\underline{\xi}$, is

$$\mathrm{var}_{\underline{\xi}} \triangleq \int_{\mathbb{S}^2} |\underline{\xi} - \langle \underline{\xi} \rangle|^2 |f^+(\underline{\xi})|^2 d\sigma(\underline{\xi}). \tag{3.6}$$

The *mean of frequency* is defined by

$$\langle k \rangle \triangleq \sum_{k=0}^{\infty} k \|P_k(f)(\underline{\xi})\|^2, \tag{3.7}$$

where $f^+ = \sum_{k=0}^{\infty} P_k(f)$ as given in (2.9).
The *variance of frequency* has two formulations, defined respectively by the following two formulas

$$\mathrm{var}_k^* \triangleq \sum_{k=0}^{\infty} (k - \langle k \rangle)^2 \|P_k(f)(\underline{\xi})\|^2, \tag{3.8}$$

and

$$\mathrm{var}_k \triangleq \int_{\mathbb{S}^2} [\theta_1'(\underline{\xi}) - \langle k \rangle]^2 |f^+(\underline{\xi})|^2 d\sigma(\underline{\xi}) + \int_{\mathbb{S}^2} |\rho_1'(\underline{\xi})|^2 d\sigma(\underline{\xi}). \tag{3.9}$$

The *covariance* is defined by

$$\mathrm{Cov} = \int_{\mathbb{S}^2} (\underline{\xi} - \langle \underline{\xi} \rangle)[\theta_1'(\underline{\xi}) - \langle k \rangle] |f^+(\underline{\xi})|^2 d\sigma(\underline{\xi}). \tag{3.10}$$

Finally, the *absolute covariance* is defined by (see [8])

$$\mathrm{COV} = \int_{\mathbb{S}^2} |\underline{\xi} - \langle \underline{\xi} \rangle| |\theta_1'(\underline{\xi}) - \langle k \rangle| |f^+(\underline{\xi})|^2 d\sigma(\underline{\xi}). \tag{3.11}$$

Remark 3.4. Although we have two ways to define phase derivatives, we choose to use $\theta_1'(\underline{\xi})$ but not $\theta_2'(\underline{\xi})$ (see Definition 3.3). For periodic signals, from (3.2), we can see that the mean of phase derivative $\varphi'(t)$ against $|s(e^{it})|^2$ is equal to the mean of Fourier frequency. For spherical signals, we obtain the same result, that is (3.14), only when we adopt $\theta_1'(\underline{\xi})$. For this reason we use $\theta_1'(\xi)$ but not $\theta_2'(\xi)$, too, when we further study variance and covariance.

Remark 3.5. In Definition 3.3, we use two methods to define the variance of frequency. Those two definitions are both inspired by the periodic signal case.

The variance of frequency var_k^* can be regarded as a counterpart of (1.3) in the frequency domain. By Theorem 3.8, var_k^* has a representation in the time domain given by

$$\mathrm{var}_k^* = \int_{\mathbb{S}^2} [\theta_1'(\underline{\xi}) - \langle k \rangle]^2 |f^+(\underline{\xi})|^2 d\sigma(\underline{\xi}) + \int_{\mathbb{S}^2} |\rho_2'(\underline{\xi})|^2 d\sigma(\underline{\xi}), \tag{3.12}$$

that is a counterpart of (3.3). This gives support to use $\rho_2'(\underline{\xi})$. Replacing $\rho_2'(\underline{\xi})$ with $\rho_1'(\underline{\xi})$ in (3.12) we have an alternative counterpart of (3.3), namely,

$$\int_{\mathbb{S}^2}[\theta_1'(\underline{\xi})-\langle k\rangle]^2|f^+(\underline{\xi})|^2d\sigma(\underline{\xi})+\int_{\mathbb{S}^2}|\rho_1'(\underline{\xi})|^2d\sigma(\underline{\xi}).$$

That is just the definition of var_k. When $m=2$, $\sigma_k^2, \mathrm{var}_k^*$ and var_k coincide. When $m>2$ they are, unfortunately, not. In the following we will consider both formulations var_k and var_k^* in relation to uncertainty principle.

The following lemma is an application of Minkovski's inequality.

Lemma 3.6. *Let $g(\underline{x})=g_0(\underline{x})+\sum_{|T|=1}^m g_T(\underline{x})\mathbf{e}_T\in L^1(\mathbf{R}^m;\mathbf{R}_m)$. Then for any positive measure $d\mu(\underline{x})$ there holds*

$$\int_{\mathbf{R}^m}|g(\underline{x})|d\mu(\underline{x})\geq\left|\int_{\mathbf{R}^m}g(\underline{x})d\mu(\underline{x})\right|. \tag{3.13}$$

Remark 3.7. By using Lemma 3.6, we immediately obtian

$$\begin{aligned}\mathrm{COV}^2&=\left\{\int_{\mathbb{S}^2}|\underline{\xi}-\langle\underline{\xi}\rangle||\mathrm{Sc}\left\{\left[-\Gamma_{\underline{\xi}}f^+(\underline{\xi})\right][f^+(\underline{\xi})]^{-1}\right\}-\langle k\rangle||f^+(\underline{\xi})|^2d\sigma(\underline{\xi})\right\}^2\\&\geq\left|\int_{\mathbb{S}^2}(\underline{\xi}-\langle\underline{\xi}\rangle)\left\{\mathrm{Sc}\left\{\left[-\Gamma_{\underline{\xi}}f^+(\underline{\xi})\right][f^+(\underline{\xi})]^{-1}\right\}-\langle k\rangle\right\}|f^+(\underline{\xi})|^2d\sigma(\underline{\xi})\right|^2\\&=|\mathrm{Cov}|^2,\end{aligned}$$

that is

$$\mathrm{COV}^2\geq|\mathrm{Cov}|^2.$$

This result will be used to obtain Corollary 4.5.

Theorem 3.8. *Let $f(\underline{\xi})\in L^2(\mathbb{S}^2)$ be real-valued, $\Gamma_{\underline{\xi}}f^+(\underline{\xi})\in L^2(\mathbb{S}^2)$ and $\|f^+\|_2=1$. Then there hold*

$$\langle k\rangle=\int_{\mathbb{S}^2}\theta_1'(\underline{\xi})|f^+(\underline{\xi})|^2d\sigma(\underline{\xi}) \tag{3.14}$$

and

$$\mathrm{var}_k^*=\int_{\mathbb{S}^2}\left[\theta_1'(\underline{\xi})-\langle k\rangle\right]^2|f^+(\underline{\xi})|^2d\sigma(\underline{\xi})+\int_{\mathbb{S}^2}|\rho_2'(\underline{\xi})|^2d\sigma(\underline{\xi}). \tag{3.15}$$

Proof of Theorem. We are to represent the mean and variance of frequency, $\langle k\rangle$ and var_k^*, in the time domain. In the following computations we will use the property $-\Gamma_{\underline{\xi}}P_k(f)(\underline{\xi})=kP_k(f)(\underline{\xi})$. Since $P_k(f)(\underline{\xi})$ and $f^+(\underline{\xi})$ are para-bivector-valued, by using (2.4), we have

$$|P_k(f)(\underline{\xi})|^2=P_k(f)(\underline{\xi})\overline{P_k(f)(\underline{\xi})}\text{ and }|f^+(\underline{\xi})|^2=f^+(\underline{\xi})\overline{f^+(\underline{\xi})}.$$

Then

$$\begin{aligned}\langle k\rangle &= \sum_{k=0}^{\infty} k \int_{\mathbb{S}^2} |P_k(f)(\underline{\xi})|^2 d\sigma(\underline{\xi}) \\ &= \int_{\mathbb{S}^2} \sum_{k=0}^{\infty} kP_k(f)(\underline{\xi})\overline{P_k(f)(\underline{\xi})} d\sigma(\underline{\xi}) \\ &= \int_{\mathbb{S}^2} \sum_{k=0}^{\infty}[-\Gamma_{\underline{\xi}} P_k(f)(\underline{\xi})]\overline{P_k(f)(\underline{\xi})} d\sigma(\underline{\xi}) \\ &= \int_{\mathbb{S}^2} [-\Gamma_{\underline{\xi}} f^+(\underline{\xi})]\overline{f^+(\underline{\xi})} d\sigma(\underline{\xi}) \\ &= \int_{\mathbb{S}^2} \left[-\Gamma_{\underline{\xi}} f^+(\underline{\xi})\right] \left[f^+(\underline{\xi})\right]^{-1} f^+(\underline{\xi})\overline{f^+(\underline{\xi})} d\sigma(\underline{\xi}) \\ &= \mathrm{Sc}\left\{\int_{\mathbb{S}^2} \left[-\Gamma_{\underline{\xi}} f^+(\underline{\xi})\right] [f^+(\underline{\xi})]^{-1} |f^+(\underline{\xi})|^2 d\sigma(\underline{\xi})\right\} \\ &= \int_{\mathbb{S}^2} \mathrm{Sc}\left\{\left[-\Gamma_{\underline{\xi}} f^+(\underline{\xi})\right] [f^+(\underline{\xi})]^{-1}\right\} |f^+(\underline{\xi})|^2 d\sigma(\underline{\xi}).\end{aligned} \tag{3.16}$$

Now we explain the last two lines of the above aligned formula. From the definition of $\langle k\rangle$, we know $\langle k\rangle$ is scalar-valued, thus the non-scalar part of $\int_{\mathbb{S}^2}[-\Gamma_{\underline{\xi}} f^+(\underline{\xi})][f^+(\underline{\xi})]^{-1} f^+(\underline{\xi})\overline{f^+(\underline{\xi})} d\sigma(\underline{\xi})$ must be zero, then the sixth equality in the above aligned formula holds. Since $|f^+(\underline{\xi})|^2$ is scalar-valued, we get the last equality.

Next we prove (3.15). In fact,

$$\begin{aligned}\mathrm{var}_k^* &= \sum_{k=0}^{\infty}(k-\langle k\rangle)^2 \int_{\mathbb{S}^2} |P_k(f)(\underline{\xi})|^2 d\sigma(\underline{\xi}) \\ &= \int_{\mathbb{S}^2} \sum_{k=0}^{\infty}(k-\langle k\rangle)^2 P_k(f)(\underline{\xi})\overline{P_k(f)(\underline{\xi})} d\sigma(\underline{\xi}) \\ &= \int_{\mathbb{S}^2} \sum_{k=0}^{\infty}(k-\langle k\rangle)P_k(f)(\underline{\xi})\overline{(k-\langle k\rangle)P_k(f)(\underline{\xi})} d\sigma(\underline{\xi}) \\ &= \int_{\mathbb{S}^2} \sum_{k=0}^{\infty}[-\Gamma_{\underline{\xi}} P_k(f)(\underline{\xi}) - \langle k\rangle P_k(f)]\overline{[-\Gamma_{\underline{\xi}} P_k(f)(\underline{\xi})] - \langle k\rangle P_k(f)} d\sigma(\underline{\xi}) \\ &= \int_{\mathbb{S}^2} [-\Gamma_{\underline{\xi}} f^+(\underline{\xi}) - \langle k\rangle f^+(\underline{\xi})]\overline{-\Gamma_{\underline{\xi}} f^+(\underline{\xi}) - \langle k\rangle f^+(\underline{\xi})} d\sigma(\underline{\xi}) \\ &\overset{(2.4)}{=\!=} \int_{\mathbb{S}^2} |-\Gamma_{\underline{\xi}} f^+(\underline{\xi}) - \langle k\rangle f^+(\underline{\xi})|^2 d\sigma(\underline{\xi}).\end{aligned}$$

Since $f^+(\underline{\xi})$ is para-vector, we have $|[f^+(\underline{\xi})]^{-1}|^2 = \frac{1}{|f^+(\underline{\xi})|^2}$, thus

$$|-\Gamma_{\underline{\xi}} f^+(\underline{\xi}) - \langle k\rangle f^+(\underline{\xi})|^2 = |-\Gamma_{\underline{\xi}} f^+(\underline{\xi}) - \langle k\rangle f^+(\underline{\xi})|^2 |[f^+(\underline{\xi})]^{-1}|^2 |f^+(\underline{\xi})|^2.$$

By the Property (2.1), we have

$$|-\Gamma_{\underline{\xi}} f^+(\underline{\xi}) - \langle k\rangle f^+(\underline{\xi})|^2 = |[-\Gamma_{\underline{\xi}} f^+(\underline{\xi}) - \langle k\rangle f^+(\underline{\xi})][f^+(\underline{\xi})]^{-1}|^2 |f^+(\underline{\xi})|^2.$$

Now we proceed to calculate var_k^*.

$$\begin{aligned}
\mathrm{var}_k^* &= \int_{\mathbb{S}^2} |-\Gamma_{\underline{\xi}} f^+(\underline{\xi}) - \langle k\rangle f^+(\underline{\xi})|^2 d\sigma(\underline{\xi}) \\
&= \int_{\mathbb{S}^2} |[-\Gamma_{\underline{\xi}} f^+(\underline{\xi}) - \langle k\rangle f^+(\underline{\xi})][f^+(\underline{\xi})]^{-1}|^2 |f^+(\underline{\xi})|^2 d\sigma(\underline{\xi}) \\
&= \int_{\mathbb{S}^2} |[-\Gamma_{\underline{\xi}} f^+(\underline{\xi})][f^+(\underline{\xi})]^{-1} - \langle k\rangle|^2 |f^+(\underline{\xi})|^2 d\sigma(\underline{\xi}) \\
&\overset{(2.3)}{\underset{(2.4)}{=\!=}} \int_{\mathbb{S}^2} \left\{\mathrm{Sc}\{[-\Gamma_{\underline{\xi}} f^+(\underline{\xi})][f^+(\underline{\xi})]^{-1}\} - \langle k\rangle\right\}^2 |f^+(\underline{\xi})|^2 d\sigma(\underline{\xi}) \\
&\quad + \int_{\mathbb{S}^2} |\mathrm{Nsc}\{[-\Gamma_{\underline{\xi}} f^+(\underline{\xi})][f^+(\underline{\xi})]^{-1}\}|^2 |f^+(\underline{\xi})|^2 d\sigma(\underline{\xi}) \\
&= \int_{\mathbb{S}^2} [\theta_1'(\underline{\xi}) - \langle k\rangle]^2 |f^+(\underline{\xi})|^2 d\sigma(\underline{\xi}) + \int_{\mathbb{S}^2} |\rho_2'(\underline{\xi})|^2 d\sigma(\underline{\xi}).
\end{aligned}$$

The proof is completed. □

We note that in [33], the authors already consider the mean of frequency $\langle k\rangle$ and obtain formula (3.14). The reason we include the proof of (3.14) is that we will recall the relation (3.16) in the proof of Theorem 4.4, and, there is a mistake with the definition of $\langle k\rangle$ in the paper [33].

3.3. Mean and Variance for Spherical Signals in the Vector Space Setting

In the introduction part we have reviewed the definition of means and variances of time and frequency for signals on $\mathbb{S}^2$ proposed in [24,28,29]. In what follows we will compare the definition of means and variances of time and frequency for signals on $\mathbb{S}^2$ in the vector space setting with that in the Clifford algebra setting.

An $x \in \mathbb{S}^2$ can be represented by the polar coordinate form, that is

$$x = \varepsilon^r = \begin{bmatrix} \sqrt{1-t^2}\cos\theta_2 \\ \sqrt{1-t^2}\sin\theta_2 \\ t \end{bmatrix} \overset{t=\cos\theta_1}{=\!=\!=} \begin{bmatrix} \sin\theta_1\cos\theta_2 \\ \sin\theta_1\sin\theta_2 \\ \cos\theta_1 \end{bmatrix}, \tag{3.17}$$

where $\theta_2 \in [0, 2\pi]$ is the *longitude*, $t \in [-1, 1]$ is the *polar distance*, $\theta_1 \in [0, \pi]$ is the *latitude* (see P86, [22], for details).

In [22], the local coordinate expression of the surface curl gradient L^* is given by

$$L^* = -\varepsilon^{\theta_2}\sqrt{1-t^2}\frac{\partial}{\partial t} + \varepsilon^t \frac{1}{\sqrt{1-t^2}}\frac{\partial}{\partial\theta_2},$$

where

$$\varepsilon^{\theta_2} = \begin{bmatrix} -\sin\theta_2 \\ \cos\theta_2 \\ 0 \end{bmatrix}, \qquad \varepsilon^t = \begin{bmatrix} -t\cos\theta_2 \\ -t\sin\theta_2 \\ \sqrt{1-t^2} \end{bmatrix} \overset{t=\cos\theta_1}{=\!=\!=} \begin{bmatrix} -\cos\theta_1\cos\theta_2 \\ -\cos\theta_1\sin\theta_2 \\ \sin\theta_1 \end{bmatrix}.$$

It can be calculated that

$$L^* = \begin{bmatrix} -\frac{\cos\theta_1\cos\theta_2}{\sin\theta_1}\frac{\partial}{\partial\theta_2} - \sin\theta_2\frac{\partial}{\partial\theta_1} \\ -\frac{\cos\theta_1\sin\theta_2}{\sin\theta_1}\frac{\partial}{\partial\theta_2} + \cos\theta_2\frac{\partial}{\partial\theta_1} \\ \frac{\partial}{\partial\theta_2} \end{bmatrix},$$

and

$$\Delta^*_{\mathbb{S}^2} = L^*\cdot L^* = \frac{\partial^2}{\partial\theta_1^2} + \frac{\cos\theta_1}{\sin\theta_1}\frac{\partial}{\partial\theta_1} + \frac{1}{\sin^2\theta_1}\frac{\partial^2}{\partial\theta_2^2}.$$

The vector x corresponds to the Clifford number $\cos\theta_1\mathbf{e}_1+\sin\theta_1\cos\theta_2\mathbf{e}_2$ $+\sin\theta_1\sin\theta_2\mathbf{e}_3\in\mathbb{S}^2$, denoted by $\underline{x}=\cos\theta_1\mathbf{e}_1+\sin\theta_1\cos\theta_2\mathbf{e}_2+\sin\theta_1\sin\theta_2\mathbf{e}_3$ $\in\mathbb{S}^2$. Now we write $\Gamma_{\underline{x}}$ in the Clifford algebra setting into the polar coordinate form (see [1]). Through direct computation we obtain

$$\begin{aligned}\Gamma_{\underline{x}} = \overline{\underline{x}}\partial_{\underline{x}} = &\left(\frac{\cos\theta_1\sin\theta_2}{\sin\theta_1}\frac{\partial}{\partial\theta_2} - \cos\theta_2\frac{\partial}{\partial\theta_1}\right)\mathbf{e}_1\mathbf{e}_2 - \frac{\partial}{\partial\theta_2}\mathbf{e}_2\mathbf{e}_3 \\ &+\left(\frac{\cos\theta_1\cos\theta_2}{\sin\theta_1}\frac{\partial}{\partial\theta_2} + \sin\theta_2\frac{\partial}{\partial\theta_1}\right)\mathbf{e}_3\mathbf{e}_1,\end{aligned}$$

and

$$\begin{aligned}\Gamma_{\underline{x}}\Gamma_{\underline{x}} &= -\left(\frac{\cos\theta_1}{\sin\theta_1}\frac{\partial}{\partial\theta_1} + \frac{\partial^2}{\partial\theta_1^2} + \frac{1}{\sin^2\theta_1}\frac{\partial^2}{\partial\theta_2^2}\right) \\ &\quad+\left(\frac{\cos\theta_1\sin\theta_2}{\sin\theta_1}\frac{\partial}{\partial\theta_2} - \cos\theta_2\frac{\partial}{\partial\theta_1}\right)\mathbf{e}_1\mathbf{e}_2 \\ &\quad-\frac{\partial}{\partial\theta_2}\mathbf{e}_2\mathbf{e}_3 + \left(\frac{\cos\theta_1\cos\theta_2}{\sin\theta_1}\frac{\partial}{\partial\theta_2} + \sin\theta_2\frac{\partial}{\partial\theta_1}\right)\mathbf{e}_3\mathbf{e}_1 \\ &= -\left(\frac{\cos\theta_1}{\sin\theta_1}\frac{\partial}{\partial\theta_1} + \frac{\partial^2}{\partial\theta_1^2} + \frac{1}{\sin^2\theta_1}\frac{\partial^2}{\partial\theta_2^2}\right) + \Gamma_{\underline{x}}.\end{aligned}$$

Then we have

$$\Gamma_{\underline{x}} - \Gamma_{\underline{x}}^2 = (I-\Gamma_{\underline{x}})\Gamma_{\underline{x}} = \frac{\cos\theta_1}{\sin\theta_1}\frac{\partial}{\partial\theta_1} + \frac{\partial^2}{\partial\theta_1^2} + \frac{1}{\sin^2\theta_1}\frac{\partial^2}{\partial\theta_2^2} = \Delta^*_{\mathbb{S}^2}.$$

If x is written as a Clifford number $\underline{x}$, $\psi(x)$ can also be written as a Clifford-valued function. To compare the means and variances of time and frequency given in Definitions 1.1 and 3.3, we write the mean and variance of time and frequency in Definition 1.1 in the Clifford algebra setting, and let $\psi(x)=f^+(\underline{x})$.

It is easy to see, from Definitions 1.1 and 3.3,

$$\tau_\psi \text{ and } \mathrm{V}_{x,\psi} \text{ coincide with } \langle\underline{x}\rangle \text{ and } \mathrm{var}_{\underline{x}}, \text{ respectively.} \tag{3.18}$$

In Definition 1.1, the mean of frequency is

$$a(\psi) \triangleq \int_{\mathbb{S}^2}\Omega\psi(x)\overline{\psi(x)}d\sigma(x).$$

We write Ω in the Clifford setting, that is,

$$\begin{aligned}\Omega &= -\mathbf{i}L^* \\ &= -\mathbf{i}\left[\left(-\frac{\cos\theta_1\sin\theta_2}{\sin\theta_1}\frac{\partial}{\partial\theta_2}+\cos\theta_2\frac{\partial}{\partial\theta_1}\right)\mathbf{e}_1\mathbf{e}_2+\frac{\partial}{\partial\theta_2}\mathbf{e}_2\mathbf{e}_3\right. \\ &\quad \left.+\left(-\frac{\cos\theta_1\cos\theta_2}{\sin\theta_1}\frac{\partial}{\partial\theta_2}-\sin\theta_2\frac{\partial}{\partial\theta_1}\right)\mathbf{e}_3\mathbf{e}_1\right] \\ &= -\mathbf{i}(-\Gamma_{\underline{x}}).\end{aligned}$$

As consequence,

$$\begin{aligned}a(\psi) &= \int_{\mathbb{S}^2}\Omega\psi(x)\overline{\psi(x)}d\sigma(x) \\ &= \int_{\mathbb{S}^2}[-\mathbf{i}(-\Gamma_{\underline{x}})]f^+(\underline{x})\overline{f^+(\underline{x})}d\sigma(\underline{x}) \\ &= -\mathbf{i}\int_{\mathbb{S}^2}[-\Gamma_{\underline{x}}f^+(\underline{x})]\overline{f^+(\underline{x})}d\sigma(\underline{x}) = -\mathbf{i}\langle k\rangle.\end{aligned}$$

The variance of frequency in the vector space setting

$$\begin{aligned}\mathrm{V}_{\Omega,\psi} &\triangleq \int_{\mathbb{S}^2}\|[\Omega-a(\psi)]\psi\|^2 d\sigma(x) \\ &= \int_{\mathbb{S}^2}[\Omega\psi-a(\psi)\psi]\cdot\overline{[\Omega\psi-a(\psi)\psi]}d\sigma(x) \\ &= \int_{\mathbb{S}^2}[-\mathbf{i}(-\Gamma_{\underline{x}})f^+(\underline{x})-(-\mathbf{i}\langle k\rangle)f^+(\underline{x})] \\ &\quad\cdot\overline{[-\mathbf{i}(-\Gamma_{\underline{x}})f^+(\underline{x})-(-\mathbf{i}\langle k\rangle)f^+(\underline{x})]}d\sigma(\underline{x}) \\ &= \int_{\mathbb{S}^2}\left[-\Gamma_{\underline{x}}f^+(\underline{x})-\langle k\rangle f^+(\underline{x})\right]\cdot\overline{[-\Gamma_{\underline{x}}f^+(\underline{x})-\langle k\rangle f^+(\underline{x})]}d\sigma(\underline{x}) \\ &= \mathrm{var}_k^*.\end{aligned}$$

Remark 3.9. From the above computation, we can see that if we write (1.12), (1.13), (1.14) and (1.15) in the Clifford setting, then the definitions of mean and variance of time and frequency in the vector space setting all coincide with those defined in Definition 3.3 with the Clifford setting. Note that there are two formulations for variance in Definition 3.3: var_k^* and var_k. The variance in Definition 1.1 coincides with var_k^*, but not with var_k.

4. Uncertainty Principle for Spherical Signals in the Clifford Algebra Setting

By Lemma 3.6 and the Hölder inequality, we immediately obtain the following lemma.

Lemma 4.1. *Let $f(\underline{\xi}) \in L^2(\mathbb{S}^2)$ be real-valued, $\underline{\xi} f^+(\underline{\xi}), \Gamma_{\underline{\xi}} f^+(\underline{\xi}) \in L^2(\mathbb{S}^2)$ and $\|f^+\|_2 = 1$. Then there holds*

$$\begin{aligned}
&\mathrm{var}_{\underline{\xi}} \cdot \int_{\mathbb{S}^2} |\rho_1'(\underline{\xi})|^2 d\sigma(\underline{\xi}) \\
&\geq \left| \int_{\mathbb{S}^2} \mathrm{Nsc}\left\{ \left[-\Gamma_{\underline{\xi}} f^+(\underline{\xi})\right] [f^+(\underline{\xi})]^{-1} - \left[-\Gamma_{\underline{\xi}} e^{\frac{Hu}{|Hu|}\theta(\underline{\xi})}\right] e^{-\frac{Hu}{|Hu|}\theta(\underline{\xi})} \right\} \right. \\
&\left. \times (\underline{\xi} - \langle \underline{\xi} \rangle) |f^+(\underline{\xi})|^2 d\sigma(\underline{\xi}) \right|^2 .
\end{aligned} \tag{4.1}$$

Proof of Lemma. By the Hölder inequality, we obtain the first inequality in the following aligned formula.

$$\begin{aligned}
&\mathrm{var}_{\underline{\xi}} \cdot \int_{\mathbb{S}^2} |\rho_1'(\underline{\xi})|^2 d\sigma(\underline{\xi}) \\
&\overset{(2.22)}{=\!=} \int_{\mathbb{S}^2} |\underline{\xi} - \langle \underline{\xi} \rangle|^2 |f^+(\underline{\xi})|^2 d\sigma(\underline{\xi}) \\
&\quad \cdot \int_{\mathbb{S}^2} |\mathrm{Nsc}\left\{ \left[-\Gamma_{\underline{\xi}} f^+(\underline{\xi})\right] [f^+(\underline{\xi})]^{-1} - \left[-\Gamma_{\underline{\xi}} e^{\frac{Hu}{|Hu|}\theta(\underline{\xi})}\right] e^{-\frac{Hu}{|Hu|}\theta(\underline{\xi})} \right\} |^2 \\
&\quad \times |f^+(\underline{\xi})|^2 d\sigma(\underline{\xi}) \\
&\geq \left[\int_{\mathbb{S}^2} |\mathrm{Nsc}\left\{ \left[-\Gamma_{\underline{\xi}} f^+(\underline{\xi})\right] [f^+(\underline{\xi})]^{-1} - \left[-\Gamma_{\underline{\xi}} e^{\frac{Hu}{|Hu|}\theta(\underline{\xi})}\right] e^{-\frac{Hu}{|Hu|}\theta(\underline{\xi})} \right\} \right. \\
&\quad \left. \times ||(\underline{\xi} - \langle \underline{\xi} \rangle|| f^+(\underline{\xi})|^2 d\sigma(\underline{\xi}) \right]^2 \\
&\overset{(2.2)}{=\!=} \left[\int_{\mathbb{S}^2} |\mathrm{Nsc}\left\{ [-\Gamma_{\underline{\xi}} f^+(\underline{\xi})][f^+(\underline{\xi})]^{-1} - \left[-\Gamma_{\underline{\xi}} e^{\frac{Hu}{|Hu|}\theta(\underline{\xi})}\right] e^{-\frac{Hu}{|Hu|}\theta(\underline{\xi})} \right\} \right. \\
&\quad \left. \times \, (\underline{\xi} - \langle \underline{\xi} \rangle)|| f^+(\underline{\xi})|^2 d\sigma(\underline{\xi}) \right]^2 \\
&\geq \left| \int_{\mathbb{S}^2} \mathrm{Nsc}\left\{ \left[-\Gamma_{\underline{\xi}} f^+(\underline{\xi})\right] [f^+(\underline{\xi})]^{-1} - \left[-\Gamma_{\underline{\xi}} e^{\frac{Hu}{|Hu|}\theta(\underline{\xi})}\right] e^{-\frac{Hu}{|Hu|}\theta(\underline{\xi})} \right\} \right. \\
&\quad \left. \times (\underline{\xi} - \langle \underline{\xi} \rangle) |f^+(\underline{\xi})|^2 d\sigma(\underline{\xi}) \right|^2 ,
\end{aligned}$$

where we use Lemma 3.6 in the last inequality. □

Lemma 4.2. *Let $f(\underline{\xi}) \in L^2(\mathbb{S}^2)$ be real-valued. Then*

$$\begin{aligned}
&\left\{ \left[-\Gamma_{\underline{\xi}} f^+(\underline{\xi})\right] \overline{f^+(\underline{\xi})} - f^+(\underline{\xi}) \overline{\left[-\Gamma_{\underline{\xi}} f^+(\underline{\xi})\right]} \right\} \underline{\xi} \\
&= -\Gamma_{\underline{\xi}} [|f^+(\underline{\xi})|^2 \underline{\xi}] - |f^+(\underline{\xi})|^2 [-\Gamma_{\underline{\xi}} \underline{\xi}] + 2|f^+(\underline{\xi})|^2 \\
&\quad \times \mathrm{NSc}\left\{ \left[-\Gamma_{\underline{\xi}} e^{\frac{Hu}{|Hu|}\theta(\underline{\xi})}\right] e^{-\frac{Hu}{|Hu|}\theta(\underline{\xi})} \right\} \underline{\xi}.
\end{aligned}$$

Proof of Lemma. Since

$$-\Gamma_{\underline{\xi}} f^+(\underline{\xi}) = \left[-\Gamma_{\underline{\xi}}\rho(\underline{\xi})\right] e^{\frac{Hu}{|Hu|}\theta(\underline{\xi})} + \rho(\underline{\xi}) \left[-\Gamma_{\underline{\xi}} e^{\frac{Hu}{|Hu|}\theta(\underline{\xi})}\right] \quad \text{and}$$
$$\overline{f^+(\underline{\xi})} = \rho(\underline{\xi}) e^{-\frac{Hu}{|Hu|}\theta(\underline{\xi})},$$

then

$$[-\Gamma_{\underline{\xi}} f^+(\underline{\xi})]\overline{f^+(\underline{\xi})} = \rho(\underline{\xi})[-\Gamma_{\underline{\xi}}\rho(\underline{\xi})] + |f^+(\underline{\xi})|^2 \left[-\Gamma_{\underline{\xi}} e^{\frac{Hu}{|Hu|}\theta(\underline{\xi})}\right] e^{-\frac{Hu}{|Hu|}\theta(\underline{\xi})}.$$

It is easy to see

$$\begin{aligned} f^+(\underline{\xi})\overline{\left[-\Gamma_{\underline{\xi}} f^+(\underline{\xi})\right]} &= \overline{\left[-\Gamma_{\underline{\xi}} f^+(\underline{\xi})\right] \overline{f^+(\underline{\xi})}} \\ &= -\rho(\underline{\xi}) \left[-\Gamma_{\underline{\xi}}\rho(\underline{\xi})\right] + |f^+(\underline{\xi})|^2 \overline{\left[-\Gamma_{\underline{\xi}} e^{\frac{Hu}{|Hu|}\theta(\underline{\xi})}\right] e^{-\frac{Hu}{|Hu|}\theta(\underline{\xi})}}. \end{aligned}$$

Then we have

$$\begin{aligned} &\left[-\Gamma_{\underline{\xi}} f^+(\underline{\xi})\right] \overline{f^+(\underline{\xi})} - f^+(\underline{\xi})\overline{\left[-\Gamma_{\underline{\xi}} f^+(\underline{\xi})\right]} \\ &= 2\rho(\underline{\xi})[-\Gamma_{\underline{\xi}}\rho(\underline{\xi})] + |f^+(\underline{\xi})|^2 \\ &\quad \times \left\{ \left[-\Gamma_{\underline{\xi}} e^{\frac{Hu}{|Hu|}\theta(\underline{\xi})}\right] e^{-\frac{Hu}{|Hu|}\theta(\underline{\xi})} - \overline{\left[-\Gamma_{\underline{\xi}} e^{\frac{Hu}{|Hu|}\theta(\underline{\xi})}\right] e^{-\frac{Hu}{|Hu|}\theta(\underline{\xi})}} \right\} \\ &= 2\rho(\underline{\xi})[-\Gamma_{\underline{\xi}}\rho(\underline{\xi})] + 2|f^+(\underline{\xi})|^2 \mathrm{NSc}\left\{ \left[-\Gamma_{\underline{\xi}} e^{\frac{Hu}{|Hu|}\theta(\underline{\xi})}\right] e^{-\frac{Hu}{|Hu|}\theta(\underline{\xi})} \right\}. \end{aligned}$$

Hence,

$$\begin{aligned} &\left\{ \left[-\Gamma_{\underline{\xi}} f^+(\underline{\xi})\right] \overline{f^+(\underline{\xi})} - f^+(\underline{\xi})\overline{\left[-\Gamma_{\underline{\xi}} f^+(\underline{\xi})\right]} \right\} \underline{\xi} \\ &= 2\rho(\underline{\xi})[-\Gamma_{\underline{\xi}}\rho(\underline{\xi})]\underline{\xi} + 2|f^+(\underline{\xi})|^2 \mathrm{NSc}\left\{ \left[-\Gamma_{\underline{\xi}} e^{\frac{Hu}{|Hu|}\theta(\underline{\xi})}\right] e^{-\frac{Hu}{|Hu|}\theta(\underline{\xi})} \right\} \underline{\xi} \\ &= -\Gamma_{\underline{\xi}}[|f^+(\underline{\xi})|^2\underline{\xi}] - |f^+(\underline{\xi})|^2[-\Gamma_{\underline{\xi}}\underline{\xi}] + 2|f^+(\underline{\xi})|^2 \\ &\quad \times \mathrm{NSc}\left\{ \left[-\Gamma_{\underline{\xi}} e^{\frac{Hu}{|Hu|}\theta(\underline{\xi})}\right] e^{-\frac{Hu}{|Hu|}\theta(\underline{\xi})} \right\} \underline{\xi}. \end{aligned}$$

□

The three formulas in Lemma 4.3 are needed in the proof of Theorem 4.4.

Lemma 4.3. *Let* $f(\underline{\xi}) \in L^2(\mathbb{S}^2)$ *be real-valued,* $\Gamma_{\underline{\xi}} f^+(\underline{\xi}) \in L^2(\mathbb{S}^2)$ *and* $\|f^+\|_2 = 1$. *Then*

$$\int_{\mathbb{S}^2} \mathrm{Nsc}\left\{ \left[-\Gamma_{\underline{\xi}} f^+(\underline{\xi})\right] [f^+(\underline{\xi})]^{-1} |f^+(\underline{\xi})|^2 \right\} \langle \underline{\xi} \rangle d\sigma(\underline{\xi}) = 0, \tag{4.2}$$

$$\int_{\mathbb{S}^2} \mathrm{Nsc}\left\{ \left[-\Gamma_{\underline{\xi}} e^{\frac{Hu}{|Hu|}\theta(\underline{\xi})}\right] e^{-\frac{Hu}{|Hu|}\theta(\underline{\xi})} \right\} \langle \underline{\xi} \rangle |f^+(\underline{\xi})|^2 d\sigma(\underline{\xi}) = 0, \tag{4.3}$$

and

$$\begin{aligned} &\int_{\mathbb{S}^2} \left\{ \left[-\Gamma_{\underline{\xi}} f^+(\underline{\xi})\right] \overline{f^+(\underline{\xi})}\underline{\xi} - f^+(\underline{\xi})\overline{\left[-\Gamma_{\underline{\xi}} f^+(\underline{\xi})\right]\underline{\xi}} \right\} d\sigma(\underline{\xi}) \\ &= 2\langle \underline{\xi} \rangle + 2\int_{\mathbb{S}^2} \mathrm{NSc}\left\{ \left[-\Gamma_{\underline{\xi}} e^{\frac{Hu}{|Hu|}\theta(\underline{\xi})}\right] e^{-\frac{Hu}{|Hu|}\theta(\underline{\xi})} \right\} \underline{\xi} |f^+(\underline{\xi})|^2 d\sigma(\underline{\xi}). \end{aligned} \tag{4.4}$$

Proof of Lemma. We first prove (4.2). We recall, by invoking (3.16), that $\int_{\mathbb{S}^2}[\Gamma_{\underline{\xi}} f^+(\underline{\xi})]\overline{f^+(\underline{\xi})}d\sigma(\underline{\xi}) = -\langle k\rangle$ is real valued. Then

$$\begin{aligned}&\int_{\mathbb{S}^2} \mathrm{Nsc}\left\{\left[-\Gamma_{\underline{\xi}} f^+(\underline{\xi})\right][f^+(\underline{\xi})]^{-1}|f^+(\underline{\xi})|^2\right\}\langle\underline{\xi}\rangle d\sigma(\underline{\xi})\\ &= \langle\underline{\xi}\rangle\int_{\mathbb{S}^2} \mathrm{Nsc}\left\{\left[\Gamma_{\underline{\xi}} f^+(\underline{\xi})\right]\overline{f^+(\underline{\xi})}\right\}d\sigma(\underline{\xi}) = 0.\end{aligned}$$

Now we prove (4.3).

$$\begin{aligned}&\int_{\mathbb{S}^2} \mathrm{Nsc}\left\{\left[-\Gamma_{\underline{\xi}} e^{\frac{Hu}{|Hu|}\theta(\underline{\xi})}\right] e^{-\frac{Hu}{|Hu|}\theta(\underline{\xi})}\right\}|f^+(\underline{\xi})|^2 d\sigma(\underline{\xi})\\
&= \int_{\mathbb{S}^2} \mathrm{Nsc}\left\{\left[-\Gamma_{\underline{\xi}} e^{\frac{Hu}{|Hu|}\theta(\underline{\xi})}\right] e^{-\frac{Hu}{|Hu|}\theta(\underline{\xi})}|f^+(\underline{\xi})|^2\right\} d\sigma(\underline{\xi})\\
&= \int_{\mathbb{S}^2} \mathrm{Nsc}\left\{\left[-\Gamma_{\underline{\xi}} e^{\frac{Hu}{|Hu|}\theta(\underline{\xi})}\right] \rho(\underline{\xi})\overline{f^+(\underline{\xi})}\right\} d\sigma(\underline{\xi})\\
&= \int_{\mathbb{S}^2} \mathrm{Nsc}\left\{\left\{-\Gamma_{\underline{\xi}}[\rho(\underline{\xi})e^{\frac{Hu}{|Hu|}\theta(\underline{\xi})}] - [-\Gamma_{\underline{\xi}}\rho(\underline{\xi})]e^{\frac{Hu}{|Hu|}\theta(\underline{\xi})}\right\}\overline{f^+(\underline{\xi})}\right\} d\sigma(\underline{\xi})\\
&= \int_{\mathbb{S}^2} \mathrm{Nsc}\left\{\left[-\Gamma_{\underline{\xi}} f^+(\underline{\xi})\right]\overline{f^+(\underline{\xi})} - [-\Gamma_{\underline{\xi}}\rho(\underline{\xi})]e^{\frac{Hu}{|Hu|}\theta(\underline{\xi})}\overline{f^+(\underline{\xi})}\right\} d\sigma(\underline{\xi})\\
&= \int_{\mathbb{S}^2} \mathrm{Nsc}\left\{\left[-\Gamma_{\underline{\xi}} f^+(\underline{\xi})\right]\overline{f^+(\underline{\xi})} - [-\Gamma_{\underline{\xi}}\rho(\underline{\xi})]\rho(\underline{\xi})\right\} d\sigma(\underline{\xi})\\
&= \frac{1}{2}\mathrm{Nsc}\int_{\mathbb{S}^2}\Gamma_{\underline{\xi}}[\rho^2(\underline{\xi})]d\sigma(\underline{\xi}) = 0.\end{aligned}$$

The proof of (4.4) is as follows. By Lemma 4.2, we have

$$\begin{aligned}&\int_{\mathbb{S}^2}\left\{\left[-\Gamma_{\underline{\xi}} f^+(\underline{\xi})\right]\overline{f^+(\underline{\xi})}\underline{\xi} - f^+(\underline{\xi})\overline{\left[-\Gamma_{\underline{\xi}} f^+(\underline{\xi})\right]}\underline{\xi}\right\}d\sigma(\underline{\xi})\\
&= \int_{\mathbb{S}^2}\left\{-\Gamma_{\underline{\xi}}[|f^+(\underline{\xi})|^2\underline{\xi}] - |f^+(\underline{\xi})|^2[-\Gamma_{\underline{\xi}}\underline{\xi}] + 2|f^+(\underline{\xi})|^2\right.\\
&\quad\left.\times\,\mathrm{NSc}\left\{\left[-\Gamma_{\underline{\xi}} e^{\frac{Hu}{|Hu|}\theta(\underline{\xi})}\right]e^{-\frac{Hu}{|Hu|}\theta(\underline{\xi})}\right\}\underline{\xi}\right\}d\sigma(\underline{\xi})\\
&= \int_{\mathbb{S}^2}\left\{-\Gamma_{\underline{\xi}}\left[|f^+(\underline{\xi})|^2\underline{\xi}\right]\right\}d\sigma(\underline{\xi}) - \int_{\mathbb{S}^2}[-\Gamma_{\underline{\xi}}\underline{\xi}]|f^+(\underline{\xi})|^2 d\sigma(\underline{\xi})\\
&\quad+2\int_{\mathbb{S}^2}\mathrm{NSc}\left\{\left[-\Gamma_{\underline{\xi}} e^{\frac{Hu}{|Hu|}\theta(\underline{\xi})}\right]e^{-\frac{Hu}{|Hu|}\theta(\underline{\xi})}\right\}\underline{\xi}|f^+(\underline{\xi})|^2 d\sigma(\underline{\xi})\\
&= 2\int_{\mathbb{S}^2}\underline{\xi}|f^+(\underline{\xi})|^2 d\sigma(\underline{\xi}) + 2\int_{\mathbb{S}^2}\mathrm{NSc}\left\{\left[-\Gamma_{\underline{\xi}} e^{\frac{Hu}{|Hu|}\theta(\underline{\xi})}\right]e^{-\frac{Hu}{|Hu|}\theta(\underline{\xi})}\right\}\\
&\quad\times\underline{\xi}|f^+(\underline{\xi})|^2 d\sigma(\underline{\xi})\\
&= 2\langle\underline{\xi}\rangle + 2\int_{\mathbb{S}^2}\mathrm{NSc}\left\{\left[-\Gamma_{\underline{\xi}} e^{\frac{Hu}{|Hu|}\theta(\underline{\xi})}\right]e^{-\frac{Hu}{|Hu|}\theta(\underline{\xi})}\right\}\underline{\xi}|f^+(\underline{\xi})|^2 d\sigma(\underline{\xi}),\end{aligned}$$

where we use

$$\int_{\mathbb{S}^2}\left\{-\Gamma_{\underline{\xi}}\left[|f^+(\underline{\xi})|^2\underline{\xi}\right]\right\}d\sigma(\underline{\xi}) = 0 \quad \text{and} \quad -\Gamma_{\underline{\xi}}\underline{\xi} = -2\underline{\xi}.$$

□

The following is one of our main results.

Theorem 4.4. *Let $f(\underline{\xi}) \in L^2(\mathbb{S}^2)$ be real-valued, $\underline{\xi} f^+(\underline{\xi}), \Gamma_{\underline{\xi}} f^+(\underline{\xi}) \in L^2(\mathbb{S}^2)$ and $\|f^+\|_2 = 1$. Then there holds*

$$\mathrm{var}_{\underline{\xi}} \mathrm{var}_k \geq |\langle \underline{\xi} \rangle|^2 + \mathrm{COV}^2. \tag{4.5}$$

Proof of Theorem. To prove the inequality (4.5), due to (3.9), we just need to prove the following two inequalities:

$$\mathrm{var}_{\underline{\xi}} \int_{\mathbb{S}^2} |\rho_1'(\underline{\xi})|^2 d\sigma(\underline{\xi}) \geq |\langle \underline{\xi} \rangle|^2, \tag{4.6}$$

and

$$\mathrm{var}_{\underline{\xi}} \int_{\mathbb{S}^2} [\theta_1'(\underline{\xi}) - \langle k \rangle]^2 |f^+(\underline{\xi})|^2 d\sigma(\underline{\xi}) \geq \mathrm{COV}^2. \tag{4.7}$$

Now we prove the inequality (4.6). Using Lemma 4.1, we have

$$\begin{aligned}
&\mathrm{var}_{\underline{\xi}} \cdot \int_{\mathbb{S}^2} |\rho_1'(\underline{\xi})|^2 d\sigma(\underline{\xi}) \\
&\geq \Bigg| \int_{\mathbb{S}^2} \Big\{ \mathrm{Nsc} \Big\{ \Big[-\Gamma_{\underline{\xi}} f^+(\underline{\xi}) \Big] [f^+(\underline{\xi})]^{-1} \Big\} - \mathrm{Nsc} \\
&\quad \times \Big\{ \Big[-\Gamma_{\underline{\xi}} e^{\frac{Hu}{|Hu|}\theta(\underline{\xi})} \Big] e^{-\frac{Hu}{|Hu|}\theta(\underline{\xi})} \Big\} \Big\} \\
&\quad \times (\underline{\xi} - \langle \underline{\xi} \rangle) |f^+(\underline{\xi})|^2 d\sigma(\underline{\xi}) \Bigg|^2 \\
&= \Bigg| \int_{\mathbb{S}^2} \mathrm{Nsc} \Big\{ \Big[-\Gamma_{\underline{\xi}} f^+(\underline{\xi}) \Big] [f^+(\underline{\xi})]^{-1} |f^+(\underline{\xi})|^2 \Big\} (\underline{\xi} - \langle \underline{\xi} \rangle) d\sigma(\underline{\xi}) \\
&\quad - \int_{\mathbb{S}^2} \mathrm{Nsc} \Big\{ \Big[-\Gamma_{\underline{\xi}} e^{\frac{Hu}{|Hu|}\theta(\underline{\xi})} \Big] e^{-\frac{Hu}{|Hu|}\theta(\underline{\xi})} \Big\} (\underline{\xi} - \langle \underline{\xi} \rangle) |f^+(\underline{\xi})|^2 d\sigma(\underline{\xi}) \Bigg|^2 \\
&\overset{(4.2)}{\underset{(4.3)}{=}} \Bigg| \int_{\mathbb{S}^2} \mathrm{Nsc} \Big\{ \Big[-\Gamma_{\underline{\xi}} f^+(\underline{\xi}) \Big] \overline{f^+(\underline{\xi})} \Big\} \underline{\xi} d\sigma(\underline{\xi}) \\
&\quad - \int_{\mathbb{S}^2} \mathrm{Nsc} \Big\{ \Big[-\Gamma_{\underline{\xi}} e^{\frac{Hu}{|Hu|}\theta(\underline{\xi})} \Big] e^{-\frac{Hu}{|Hu|}\theta(\underline{\xi})} \Big\} \underline{\xi} |f^+(\underline{\xi})|^2 d\sigma(\underline{\xi}) \Bigg|^2 \\
&= \frac{1}{4} \Bigg| \int_{\mathbb{S}^2} \Big\{ \Big[-\Gamma_{\underline{\xi}} f^+(\underline{\xi}) \Big] \overline{f^+(\underline{\xi})} - f^+(\underline{\xi}) \overline{\Big[-\Gamma_{\underline{\xi}} f^+(\underline{\xi}) \Big]} \Big\} \underline{\xi} d\sigma(\underline{\xi}) \\
&\quad -2 \int_{\mathbb{S}^2} \mathrm{Nsc} \Big\{ \Big[-\Gamma_{\underline{\xi}} e^{\frac{Hu}{|Hu|}\theta(\underline{\xi})} \Big] e^{-\frac{Hu}{|Hu|}\theta(\underline{\xi})} \Big\} \underline{\xi} |f^+(\underline{\xi})|^2 d\sigma(\underline{\xi}) \Bigg|^2 \\
&\overset{(4.4)}{=} |\langle \underline{\xi} \rangle|^2.
\end{aligned}$$

Finally, we prove (4.7) through Hölder inequality

$$\begin{aligned}
&\operatorname{var}_{\underline{\xi}} \int_{\mathbb{S}^2} \{\mathrm{Sc}\{[-\Gamma_{\underline{\xi}} f^+(\underline{\xi})] \\
&\quad \times [f^+(\underline{\xi})]^{-1}\} - \langle k\rangle\}^2 |f^+(\underline{\xi})|^2 d\sigma(\underline{\xi}) \\
&= \int_{\mathbb{S}^2} |\underline{\xi} - \langle \underline{\xi}\rangle|^2 |f^+(\underline{\xi})|^2 d\sigma(\underline{\xi}) \int_{\mathbb{S}^2} \\
&\quad \times \left\{\mathrm{Sc}\left\{\left[-\Gamma_{\underline{\xi}} f^+(\underline{\xi})\right] [f^+(\underline{\xi})]^{-1}\right\} - \langle k\rangle\right\}^2 |f^+(\underline{\xi})|^2 d\sigma(\underline{\xi}) \\
&\geq \left\{\int_{\mathbb{S}^2} |\underline{\xi} - \langle \underline{\xi}\rangle| |\mathrm{Sc}\left\{\left[-\Gamma_{\underline{\xi}} f^+(\underline{\xi})\right] [f^+(\underline{\xi})]^{-1}\right\} - \langle k\rangle| |f^+(\underline{\xi})|^2 d\sigma(\underline{\xi})\right\}^2 \\
&= \mathrm{COV}^2. \qquad (4.8)
\end{aligned}$$

□

By using Remark 3.7, we immediately have the following corollary.

Corollary 4.5. *Let $f(\underline{\xi}) \in L^2(\mathbb{S}^2)$ be real-valued, $\underline{\xi} f^+(\underline{\xi}), \Gamma_{\underline{\xi}} f^+(\underline{\xi}) \in L^2(\mathbb{S}^2)$ and $\|f^+\|_2 = 1$. Then there holds*

$$\operatorname{var}_{\underline{\xi}} \operatorname{var}_k \geq |\langle \underline{\xi}\rangle|^2 + |\mathrm{Cov}|^2. \qquad (4.9)$$

Remark 4.6. In Theorem 4.4 and Corollary 4.5 we use var_k as the variance of frequency and obtain two forms of uncertainty principle on the sphere, (4.5) and (4.9). The lower bound of (4.5) is larger than that of (4.9). Although both (4.5) and (4.9) have one more positive term than (1.11), we can not say (4.5) and (4.9) are stronger uncertainty principles than (1.11). That is because the variance of frequency used in (4.5) and (4.9) is different from what is used in (1.11). However, we can say that, the two forms of uncertainty principle of spherical signals, (4.5) and (4.9), essentially, correspond to (1.8) and (1.7).

If we use var_k^* as the variance of frequency, then we have

Theorem 4.7. *Let $f(\underline{\xi}) \in L^2(\mathbb{S}^2)$ be real-valued, $\underline{\xi} f^+(\underline{\xi}), \Gamma_{\underline{\xi}} f^+(\underline{\xi}) \in L^2(\mathbb{S}^2)$ and $\|f^+\|_2 = 1$. Then there holds*

$$\operatorname{var}_{\underline{\xi}} \operatorname{var}_k^* \geq |\langle \underline{\xi}\rangle + M|^2 + \mathrm{COV}^2, \qquad (4.10)$$

where

$$M = \int_{\mathbb{S}^2} \mathrm{Nsc}\left\{\left[-\Gamma_{\underline{\xi}} e^{\frac{Hu}{|Hu|}\theta(\underline{\xi})}\right] e^{-\frac{Hu}{|Hu|}\theta(\underline{\xi})}\right\} \underline{\xi} |f^+(\underline{\xi})|^2 d\sigma(\underline{\xi})$$

is a Clifford number containing terms of 1-form and 3-form.

Proof of Theorem. The proof of Theorem 4.7 is same with that of Theorem 4.4 except one point, that is, here we need to prove

$$\operatorname{var}_{\underline{\xi}} \int_{\mathbb{S}^2} |\rho_2'(\underline{\xi})|^2 d\sigma(\underline{\xi}) \geq |\langle \underline{\xi}\rangle + M|^2 \qquad (4.11)$$

instead of (4.6).

Now we prove (4.11).

$$
\begin{aligned}
&\mathrm{var}_{\underline{\xi}} \int_{\mathbb{S}^2} |\rho_2'(\underline{\xi})|^2 d\sigma(\underline{\xi}) \\
&\quad = \int_{\mathbb{S}^2} |\underline{\xi} - \langle \underline{\xi} \rangle|^2 |f^+(\underline{\xi})|^2 d\sigma(\underline{\xi}) \cdot \\
&\qquad \times \int_{\mathbb{S}^2} |\mathrm{Nsc}\left\{\left[-\Gamma_{\underline{\xi}} f^+(\underline{\xi})\right][f^+(\underline{\xi})]^{-1}\right\}|^2 |f^+(\underline{\xi})|^2 d\sigma(\underline{\xi}) \\
&\quad \geq |\int_{\mathbb{S}^2} \mathrm{Nsc}\left\{\left[-\Gamma_{\underline{\xi}} f^+(\underline{\xi})\right][f^+(\underline{\xi})]^{-1}\right\}(\underline{\xi} - \langle \underline{\xi} \rangle)|f^+(\underline{\xi})|^2 d\sigma(\underline{\xi})|^2 \\
&\quad = |\int_{\mathbb{S}^2} \mathrm{Nsc}\left\{\left[-\Gamma_{\underline{\xi}} f^+(\underline{\xi})\right][f^+(\underline{\xi})]^{-1}|f^+(\underline{\xi})|^2\right\}(\underline{\xi} - \langle \underline{\xi} \rangle) d\sigma(\underline{\xi}) \\
&\quad = |\int_{\mathbb{S}^2} \mathrm{Nsc}\left\{\left[-\Gamma_{\underline{\xi}} f^+(\underline{\xi})\right]\overline{f^+(\underline{\xi})}\right\}\underline{\xi} d\sigma(\underline{\xi})|^2 \\
&\quad = \frac{1}{4}|\int_{\mathbb{S}^2} \left\{\left[-\Gamma_{\underline{\xi}} f^+(\underline{\xi})\right]\overline{f^+(\underline{\xi})} - f^+(\underline{\xi})\overline{\left[-\Gamma_{\underline{\xi}} f^+(\underline{\xi})\right]}\right\}\underline{\xi} d\sigma(\underline{\xi})|^2 \\
&\quad = \frac{1}{4}|2\langle \underline{\xi} \rangle + 2\int_{\mathbb{S}^2} \mathrm{Nsc}\left\{\left[-\Gamma_{\underline{\xi}} e^{\frac{Hu}{|Hu|}\theta(\underline{\xi})}\right] e^{-\frac{Hu}{|Hu|}\theta(\underline{\xi})}\right\}\underline{\xi}|f^+(\underline{\xi})|^2 d\sigma(\underline{\xi})|^2 \\
&\quad = |\langle \underline{\xi} \rangle + \int_{\mathbb{S}^2} \mathrm{Nsc}\left\{\left[-\Gamma_{\underline{\xi}} e^{\frac{Hu}{|Hu|}\theta(\underline{\xi})}\right] e^{-\frac{Hu}{|Hu|}\theta(\underline{\xi})}\right\}\underline{\xi}|f^+(\underline{\xi})|^2 d\sigma(\underline{\xi})|^2. \qquad (4.12)
\end{aligned}
$$

□

Corollary 4.8. *Let $f(\underline{\xi}) \in L^2(\mathbb{S}^2)$ be real-valued, $\underline{\xi} f^+(\underline{\xi}), \Gamma_{\underline{\xi}} f^+(\underline{\xi}) \in L^2(\mathbb{S}^2)$ and $\|f^+\|_2 = 1$. Then there holds*

$$
\mathrm{var}_{\underline{\xi}} \mathrm{var}_k^* \geq |\langle \underline{\xi} \rangle + M|^2 + |\mathrm{Cov}|^2, \qquad (4.13)
$$

where

$$
M = \int_{\mathbb{S}^2} \mathrm{Nsc}\left\{\left[-\Gamma_{\underline{\xi}} e^{\frac{Hu}{|Hu|}\theta(\underline{\xi})}\right] e^{-\frac{Hu}{|Hu|}\theta(\underline{\xi})}\right\}\underline{\xi}|f^+(\underline{\xi})|^2 d\sigma(\underline{\xi})
$$

is a Clifford number containing terms of 1-form and 3-form.

Remark 4.9. In Theorem 4.7 and Corollary 4.8 we obtain other two forms of uncertainty principle of spherical signals by using var_k^* as the variance of frequency. In Remark 3.9, we note that (1.12), (1.13), (1.14) and (1.15) in the vector space setting coincide with (3.5), (3.6), (3.7) and (3.8). Hence the left-hand sides of (1.11) and (4.10) are just the same. Since the term $|\langle \underline{\xi} \rangle + M|^2$ in (4.10) and (4.13) cannot be clearly compared with $|\langle \underline{\xi} \rangle|^2$ in their values, the right hand side of (4.10) and (4.13) cannot be clearly compared with $|\langle \underline{\xi} \rangle|^2$, either. According to (3.18), the quantity $|\langle \underline{\xi} \rangle|^2$, however, coincides with the right hand side of (1.11). Hence, the related uncertainty principles are all incomparable.

Although the new proposed uncertainty principles (4.5) and (4.10) are incomparable with the existing one (1.11), both of them correspond to the strongest form of uncertainty principle (1.8) for periodic signals.

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Uncertainty Principle and Phase–Amplitude

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