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Complex Analysis and Operator Theory

Wavelets and Holomorphic Functions

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Abstract In this paper, we apply Meyer's holomorphic wavelets to study holomorphic function spaces systematically. The holomorphic function spaces under our setting include the classical function spaces as well as extensions of the latter. Holomorphic wavelets pay a role to connect real analysis and complex analysis.

Keywords Holomorphic wavelets · Holomorphic functions · Triebel-Lizorkin-Morrey spaces · Besov-Morrey spaces

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1 Introduction and Motivation

The object of this paper is to use Meyer wavelets to analyze holomorphic functions. We explore the Botchkariev-Meyer-Wojtasczcyk Theorem in a delicate way to the general cases and expand banks of specific function spaces in complex analysis. In classic complex analysis, we meet with only a few function spaces, including Hardy spaces, Lipschitz spaces, BMO spaces, Bergmann spaces, Bloch spaces and Q spaces. There has not been a systematic treatment. See [1,3,4,13,21,24,27,34]. Among others, K. Zhu, for instance, proposed to use wavelets to treat complex analysis problems in a series of talks. The holomorphic wavelet setting of complex function spaces to include all the above mentioned classical spaces and vastly expand to new spaces in complex analysis.

In real analysis, most function spaces are classified into Triebel-Lizorkin-Morrey spaces and Besov-Morrey spaces. Those spaces include Hardy spaces, BMO spaces, Bergmann spaces, Bloch spaces and Q spaces. They have been studied heavily in the recent years. See Cui-Yang [2], Essen-Janson-Peng-Xiao [4], Li-Yang [8,9], Li-Xiao-Yang [7], Lin-yang [11] and Li-Yang-Zheng [10], Liang-Sawano-Ullrich-Yang-Yuan [12], Yang-Yuan [28–30], Yuan-Sickel-Yang [33]. In this paper, we establish function spaces for holomorphic functions systematically with the helps of real methods and holomorphic Meyer wavelets.

Now, we present the definition of holomorphic function. The following Definitions 1.1 and 1.2 can be found in Meyer's book [13]. By the way, we explain how to expand holomorphic functions on the boundary to analytic functions in the half plane or in the disc. See also Yang-Qian-Li [32]. Let $S_0(\mathbb{R}) = \{f(x) \in S(\mathbb{R}) : \int x^\alpha f(x)dx = 0, \forall\alpha \in \mathbb{N}\}$ and let $S_0'(\mathbb{R})$ be the relative dual space.

Definition 1.1 A distribution $f(x) \in S_0'(\mathbb{R})$ on real axis is said to be holomorphic, if

$$\operatorname{Supp} \hat{f}(\xi) \subset [0, \infty).$$

For classic function f, the Hilbert operator H is defined as follows:

$$Hf(x) = \frac{1}{\pi} p.v. \int_{-\infty}^{\infty} \frac{f(y)}{x-y} dy.$$

When $f \in S_0'(\mathbb{R})$, the Hilbert operator H is accordingly defined by the inverse Fourier transform of the following Fourier transform:

$$\widehat{Hf}(\xi) = -i\,\mathrm{sgn}(\xi)\hat{f}(\xi).$$

For holomorphic distribution f, by comparing the relation between the Fourier transforms of Hf and f, we get $Hf = -if$. See also [22].

In the half plane $\mathbb{R}^2_+ = \{z = x + it : x \in \mathbb{R}, t \in \mathbb{R}_+\}$, in the classical case, the following Cauchy formulation establishes a one-to-one relation between a data holomorphic function on the real axis with the corresponding analytic function in the upper half plane

$$F(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{f(y)}{y - z}dy, \qquad \mathrm{Im}z > 0. \tag{1.1}$$

Denote

$$P_t(x) = \frac{t}{\pi(t^2 + x^2)} \text{ and } Q_t(x) = \frac{x}{\pi(x^2 + t^2)}, \tag{1.2}$$

where $P_t(x - y)$ and $Q_t(x - y)$ are respectively the real and the imaginary parts of 2 times of the Cauchy kernel. As $t \to 0$, the Eq. (1.1) becomes

$$F(x) = \frac{1}{2}f(x) + i\frac{1}{2}Hf(x).$$

With the above idea, we are to establish the relation of holomorphic distributions on the real axis with analytic functions in the upper half plane (see also [32]). By applying the ideas in this paper and those in [32], we can also consider the characterization relation between analytic functions in the half plane and their Carleson measures. The later mentioned results will be appeared in a forthcoming publication soon.

Definition 1.2 A distribution $f(x)$ on the unit interval is said to be holomorphic, if there exists C_k such that $f(x) = \sum_0^\infty C_k e^{2\pi kix}$ in the distribution sense.

We can similarly consider certain periodic holomorphic functions which correspond to analytic functions on the unit disc $D = B(0, 1) = \{z \in \mathbb{C}, |z| < 1\}$. Let

$$P_r^{[0,1]}(x) = \frac{1 - r^2}{1 - 2r\cos 2\pi x + r^2} \text{ and } Q_r^{[0,1]}(x) = \frac{2r\sin 2\pi x}{1 - 2r\cos 2\pi x + r^2}.$$

$P_r^{[0,1]}(x)$ and $Q_r^{[0,1]}(x)$ are, respectively, periodizations of the Poisson and conjugate Poisson kernels $P_r(x)$ and $Q_r(x)$ given in (1.2). The Cauchy Eq. (1.1) corresponds to the following Eq. (1.3), that establishes the relation between analytic functions in the disc with holomorphic boundary data functions on the circle.

$$F(re^{i2\pi x}) = \frac{1}{2}\int_0^1 \left(P_r^{[0,1]}(x - y) + iQ_r^{[0,1]}(x - y)\right) f(e^{i2\pi y})dy, \ \forall 0 \le r < 1. \tag{1.3}$$

Also, the related Carleson measures issues can be considered. Which will be appeared in another paper recently.

In Sect. 3, we construct certain special analytic function $F(z)$. With different senses of convergence, $F(z)$ converges to different boundary limits. On the other hand, holomorphic Meyer wavelets expand the function without losing any information lead to distributional boundary limits that can be based on to reconstruct the original function. Thus we avoid the ambiguity. Below we cite some related studies which analyze analytic functions by analytic bases. Botchkariev-Wojtasczcyk used Franklin system to consider analytic functions in $\mathbb{H}^p$, see [1,27]. Meyer analyzed holomorphic functions in $\mathbb{H}^p$ and $A(D)$, see [13]. Qian and his collaborators used rational systems to study analytic signals, see [20,21]. In the present paper, we give characterizations of a wide class of holomorphic function spaces by holomorphic Meyer wavelets.

Meyer indicates in [13] that holomorphic wavelets have different nature from the usual wavelets. New techniques are required when using holomorphic wavelets. We will use the continuity of Calderón-Zygmund operators and other skills to establish wavelet characterization. This systematically expands the bank of function spaces in complex analysis.

In Sect. 2, we present preliminaries on Meyer wavelets and holomorphic Meyer wavelets. In Sect. 3, we provide some remarks on convergence of analytic Hardy spaces H^p ($\frac{1}{2} < p < 1$). In Sect. 4, we present preliminaries, including technical lemmas on function spaces and operator continuities, needed in proving our main theorems. In the last two sections, we study systematically holomorphic function spaces with Meyer's holomorphic wavelets.

2 Preliminaries on Meyer Wavelets

Meyer wrote in his famous book [13] certain special discretization of Cauchy formula can be seen as wavelets. Qian and his collaborators have been using rational approximation to study many problems, cf [3,19,21,23] and the references therein. But up to now, to get holomorphic wavelets in $S(\mathbb{R})$, we have to use Meyer wavelets. We can find the following preliminaries on Meyer wavelets in different sections of the mentioned Meyer's book [13].

In the first two Sects. 2.1 and 2.2, we present Meyer wavelets in real analysis. These wavelets will be used to construct holomorphic wavelets in Sects. 2.3 and 2.4. In the last two subsections, we present the related holomorphic Meyer wavelets in complex analysis.

2.1 Wavelets $\{\psi_{j,k}(x)\}_{j,k\in\mathbb{Z}}$ on the Real Line

Let Ψ be an even function belonging to $C_0^\infty\big([-\frac{4\pi}{3}, \frac{4\pi}{3}]\big)$ satisfying

$$\begin{cases} 0 \le \Psi(\xi) \le 1; \\ \Psi(\xi) = 1, \qquad \forall |\xi| \le \frac{2\pi}{3}. \end{cases}$$

Then $\omega(\xi) = \sqrt{\Psi(\frac{\xi}{2})^2 - \Psi(\xi)^2}$ is an even function belonging to $C_0^\infty([-\frac{8\pi}{3}, \frac{8\pi}{3}])$ satisfying

$$\begin{cases} 0 \le \omega(\xi) \le 1, & \forall |\xi| \le \frac{2\pi}{3}; \\ \omega^2(\xi) + \omega^2(2\xi) = 1 = \omega^2(\xi) + \omega^2(2\pi - \xi), & \forall \xi \in [\frac{2\pi}{3}, \frac{4\pi}{3}]. \end{cases}$$

Let

$$\hat{\psi}^0(\xi) = \Psi(\xi),\ \hat{\psi}(\xi) = e^{-\frac{i\xi}{2}}\omega(\xi) \text{ and}$$
$$\psi_{j,k}(x) = 2^{\frac{j}{2}}\psi(2^j x - k), \forall j, k \in \mathbb{Z}.$$

Then we have (see [13] or [31] or other wavelet. books)

Lemma 2.1 (i) $\sum_{k\in\mathbb{Z}} \psi^0(x-k) = 1$.
(ii) $\psi(x) \in \mathcal{S}(\mathbb{R})$ *and* $\psi(x)$ *is real valued.*
(iii) $\text{supp}\hat{\psi} \subset \{\xi : \frac{2\pi}{3} \le |\xi| \le \frac{8\pi}{3}\}$.
(iv) $\psi(1-x) = \psi(x)$.
(v) $\{\psi_{j,k}(x)\}_{j,k\in\mathbb{Z}}$ *is an orthonormal basis of* $L^2(\mathbb{R})$.
(vi) $\{\psi^0(x-k)\}_{k\in\mathbb{Z}} \bigcup \{\psi_{j,k}(x)\}_{j\in\mathbb{N},k\in\mathbb{Z}}$ *is orthonormal basis of* $L^2(\mathbb{R})$.

2.2 Wavelets $\{g_\lambda(x)\}_{\lambda\in\Lambda}$ on Interval

To study analytic function in the disc, we need to use a variation of wavelets on the interval, and we present wavelets on the interval in this subsection. Define $g_0(x) = 1$. For $j \in \mathbb{N}, 0 \le k < 2^j$, define

$$g_{j,k}(x) = 2^{\frac{j}{2}} \sum_{l=-\infty}^{+\infty} \psi(2^j x + 2^j l - k).$$

$g_{j,k}(x)$ is a real valued function and it is symmetrical about $x = k2^{-j} + 2^{-j-1}$. Let $\Lambda = \{0\} \bigcup \{(j,k), j \in \mathbb{N}, 0 \le k < 2^j\}$. It is easy to see that

(i) $g_\lambda(x)(\lambda \in \Lambda)$ are functions of period 1.
(ii) $\langle g_\lambda(x), g_{\lambda'}(x)\rangle = \delta_{\lambda,\lambda'}, \forall \lambda, \lambda' \in \Lambda$.
(iii) $L^2[0,1]$ is the closure of the span of $\{g_\lambda(x) : \lambda \in \Lambda\}$.

See [13] and [31] or any other wavelet books. The following result is well-known.

Lemma 2.2 $g_\lambda(x)(\lambda \in \Lambda)$ *is an orthonormal basis in* $L^2[0,1]$.

2.3 Holomorphic Wavelets $\{S_{j,k}\}_{j\in\mathbb{Z},k\in\mathbb{N}}$ on the Real Line

In this subsection, we present Meyer's holomorphic wavelets $\{S_{j,k}\}_{j\in\mathbb{Z},k\in\mathbb{N}}$. Denote

$$\text{sgn}\xi = \begin{cases} -1, & \text{if } \xi \le 0; \\ 1, & \text{if } \xi > 0. \end{cases} \quad \text{and} \quad \chi_+(\xi) = \begin{cases} 0, & \text{if } \xi \le 0; \\ 1, & \text{if } \xi > 0. \end{cases}$$

Let

$$\mathbb{H}^2(\mathbb{R}) = \{f \in L^2(\mathbb{R}) : \hat{f}(\xi) = 0, \forall \xi < 0\}.$$

Generally speaking, a holomorphic function $f(x)$ can be seen as the trace function of an analytic function $F(z)$ and the extension of a holomorphic function through Cauchy formula (1.1) is an analytic function. Let H be the Hilbert transformation. It can be shown

$$\widehat{Hf}(\xi) = -i\,\mathrm{sgn}(\xi)\hat{f}(\xi).$$

Let I be the unit operator. Then $P = \frac{1}{2}(I + iH)$ is the orthogonal projection operator from $L^2(\mathbb{R})$ to $\mathbb{H}^2(\mathbb{R})$.

Let

$$\hat{\tau}(\xi) = e^{-\frac{i\xi}{2}}\omega(\xi)\chi_+(\xi) \text{ and } \tau(x) = (2\pi)^{-1}\int_0^{\infty} e^{i(x-\frac{1}{2})\xi}\omega(\xi)d\xi.$$

For $j, k \in \mathbb{Z}$, denote

$$\tau_{j,k}(x) = 2^{\frac{j}{2}}\tau(2^j x - k).$$

For $j \in \mathbb{Z}, k \in \mathbb{N}$, it is easy to see that

$$S_{j,k}(x) = \tau_{j,k}(x) + \tau_{j,-k-1}(x) \text{ are holomorphic functions .}$$

The following result is well-known. See [13].

Lemma 2.3 *The set of holomorphic functions* $\{S_{j,k}\}_{j\in\mathbb{Z},k\in\mathbb{N}}$ *is an orthonormal basis of* $\mathbb{H}^2(\mathbb{R})$.

Proof (i) Let E be the space composed by the even functions in $L^2(\mathbb{R})$. Hence all the functions in E can be written as limits of linear combinations of the functions $\{\psi_{j,k} + \psi_{j,-k-1}\}_{j\in\mathbb{Z},k\in\mathbb{N}}$.

(ii) The orthogonal projection operator P from $L^2(\mathbb{R})$ to $\mathbb{H}^2(\mathbb{R})$ enjoys the property

$$\widehat{Pf}(\xi) = \hat{f}\chi_+(\xi). \tag{2.1}$$

Then $2^{-\frac{1}{2}}P$ is a norm-preserving isomorphism from $E \to \mathbb{H}^2(\mathbb{R})$. Since $P(\psi) = \tau$, $f(x) \in \mathbb{H}^2(\mathbb{R})$ can be written as the limits of linear combinations of functions $\{\tau_{j,k} + \tau_{j,-k-1}\}_{j\in\mathbb{Z},k\in\mathbb{N}}$.

(iii) $\forall j, j' \in \mathbb{Z}, k, k' \in \mathbb{N}$, we have

$$\langle \tau_{j,k} + \tau_{j,-k-1}, \tau_{j',k'} + \tau_{j',-k'-1}\rangle = c\delta_{j,j'}\delta_{k,k'}.$$

Hence $\{S_{j,k}\}_{j\in\mathbb{Z},k\in\mathbb{N}}$ is an orthogonal basis in $\mathbb{H}^2(\mathbb{R})$. □

Remark 2.4 (i) As $k \to +\infty$, the support of the bimodal function $S_{j,k}$ tends to ∞ and $-\infty$ simultaneously.

(ii) For the usual wavelet $\{\psi_{j,k}\}_{j,k}$, the index for k runs over the whole $\mathbb{Z}$. For the holomorphic wavelet $\{S_{j,k}\}_{j,k}$, the index for k runs only over $\mathbb{N}$.

(iii) Hence when we use holomorphic wavelets to characterize analytic function spaces, as Meyer noted in [13], we need to introduce new methods other than those for the traditional wavelets in real analysis.

2.4 Holomorphic Wavelets $\{G_m(x)\}_{m\geq 0}$ on Interval

Definition 2.5 We call $f(x) \in \mathbb{H}^2[0, 1]$, if there exists $\{C_k\}_{k=0}^{\infty}$ such that $\sum_{k=0}^{\infty} |C_k|^2 < \infty$ and, in the L^2 convergence sense,

$$f(x) = \sum_{k=0}^{\infty} C_k e^{2\pi ikx}.$$

We note that in the case

$$F(z) = \sum_{0}^{\infty} C_k z^k$$

is an analytic function in the unit disc, and, with $z = te^{2\pi ix}$, we may write $f(x) = F(e^{2\pi ix})$. Let $G_0 = 1, G_1(x) = G_{0,0}(x) = e^{i2\pi x}$. $\forall j \geq 1, 0 \leq k < 2^{j-1}$ and $m = 2^{j-1} + k + 1$, define

$$G_m(x) = G_{j,k}(x) = \pi^{-1} 2^{-\frac{j}{2}} \sum_{k=0}^{\infty} \omega(2k\pi 2^{-j}) \cos\left(2k\pi(l + \frac{1}{2})2^{-j}\right) e^{2k\pi ix}.$$

Let Λ_a denote the set $\{0\} \bigcup \{(j, k), j \in \mathbb{N}, 0 \leq k < 2^{j-1}\}$. We note that Λ_a is different from the index set Λ in Sect. 2.2, and $\Lambda_a \subsetneqq \Lambda$. The holomorphic wavelets $\{G_m(x)\}_{m\geq 0}$ on the interval is different from the traditional wavelets on the interval in the real analysis. The following result is well-known. See [13].

Lemma 2.6 *$G_m(x) (m \in \mathbb{N})$ is an orthonormal basis of $\mathbb{H}^2[0, 1]$.*

Proof (i) $\forall j \in \mathbb{N}$, denote

$$\begin{aligned} g_j(x) &= (2\pi)^{-1} 2^{-\frac{j}{2}} \sum_{-\infty}^{\infty} \hat{\psi}(2l\pi 2^{-j}) e^{2l\pi ix} \\ &= (2\pi)^{-1} 2^{-\frac{j}{2}} \sum_{-\infty}^{\infty} \omega(2l\pi 2^{-j}) e^{2l\pi i(x - 2^{-j-1})} \\ h_j(x) &= (2\pi)^{-1} 2^{-\frac{j}{2}} \sum_{0}^{\infty} \omega(2l\pi 2^{-j}) e^{2l\pi i(x - 2^{-j-1})} \end{aligned}$$

Simple calculation shows that

$$g_{j,k}(x) = g_j(x - k2^{-j}),\ \ h_j(2^{-j} - x) = \overline{h}_j(x),$$

and

$$G_m(x) = G_{j,k}(x) = P(g_{j,k} + g_{j,2^j-k-1}).$$

By Lemma 2.2, $f \in \mathbb{H}^2[0,1]$ can be written as a linear combination of $G_m(x)(m \in \mathbb{N})$.

(ii) $\forall m, m' \in \mathbb{N}$, it is easy to see

$$\langle G_m(x), G_{m'}(x)\rangle = \delta_{m,m'}.$$

□

3 Some Remarks on Non-tangential Boundary Limits

For general complex Hardy space knowledge we refer to [5]. In this section, we construct some special analytic functions $F(z)$ that belong to the analytic Hardy spaces $H^p(\mathbb{R}^2_+)$ $(\frac{1}{2} < p < 1)$. Denote by $\mathrm{Re}F(z)$ and $\mathrm{Im}F(z)$ the real part and imaginary part of $F(z)$, respectively. For $\frac{1}{2} < p < 1$, when analyzing $F(z)$ under its non-tangential boundary limit in the L^p norm sense, we lose a part of the function through this classical method. The original analytic functions can not be recovered by the L^p-non-tangential boundary limit through the Cauchy's formula (1.1). Different senses of convergence give rise to different boundary limits, as indicated in the two propositions below. This is one of our motivations to use holomorphic wavelets. By adopting holomorphic wavelets we do not lose any information but establish the right and full correspondence.

We first present an example for the $p < 1$ case. Denote by δ_0 and δ_1 the Dirac functions at zero and at 1, respectively. Set

$$f(t) = \delta_0(t) - \delta_1(t) + iH(\delta_0 - \delta_1)(t) = \delta_0(t) - \delta_1(t) + \frac{i}{\pi}\left(\frac{1}{t} - \frac{1}{t-1}\right). \tag{3.1}$$

Then $\mathrm{Re} f(t) = \delta_0(t) - \delta_1(t)$ and $\mathrm{Im} f(t) = \frac{1}{\pi}(\frac{1}{t} - \frac{1}{t-1})$.

Observed from the Plemelj formula, two times of the formal Cauchy integral on the distributional boundary data $\delta_0 - \delta_1$ gives rise to an analytic function $F(z)$ in the upper-half complex plane that would correspond to the holomorphic distribution f (see the proof for (iv) of Proposition 3.1 below):

$$F(z) = \frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{\delta_0(t) - \delta_1(t)}{t - z}dt = \frac{i}{\pi}\left(\frac{1}{z} - \frac{1}{z-1}\right).$$

In fact, taking the boundary limit, while treating the Poisson kernels (at $t = 0$ and at $t = 1$) as the Dirac and shifted Dirac distribution, respectively, we obtain the holomorphic distribution f (see [6], or Proposition 3.2 below).

For the classical $p \geq 1$ cases one can reconstruct the analytic Hardy H^p functions by using either their real or imaginary parts, or both, of their non-tangential boundary limits via the Cauchy or the Poisson type integrals in the Lebesgue integration sense. For $\infty > p \geq 1$, let g be the pointwise or the L^p-wise non-tangential boundary limit of $G \in H^p(\mathbf{C}^+)$. Denoting by C the Cauchy integral operator from the boundary to the upper half space, we have, for $z \in \mathbf{C}^+$, $z = x + iy$ (see, for instance, [18]),

$$G(z) = Cg(z) = 2C(\mathrm{Re}g(z)) = i2C(\mathrm{Im}g)(z) = P_y * g(x) = iHQ_y * g(x),$$

where H is Hilbert transformation. Below we will sketch a short proof for the relation

$$G(z) = 2C(\mathrm{Re}g(z)), \tag{3.2}$$

for it will be recalled in the proof of the following proposition.

Since g is a holomorphic function in the sense specified in Definition 1.1, we have $Hg = -ig$. We hence have $H(\mathrm{Re}g) = \mathrm{Im}g$. By the Plemelj formula there hold the following relations

$$\lim_{y \to 0+} C(\mathrm{Re}g)(x + iy) = \frac{1}{2}\mathrm{Re}g(x) + i\frac{1}{2}H(\mathrm{Re}g)(x) = \frac{1}{2}g(x).$$

Therefore, the analytic extensions of $C(\mathrm{Re}g)(x)$ and $\frac{1}{2}g(x)$ to the upper-half complex plane must be the same, viz., $G(z) = 2C(\mathrm{Re}g)(z)$, showing that the corresponding analytic function in the Hardy space for $p \geq 1$ may be reconstructed by merely the real part of the holomorphic function on the boundary.

The following proposition shows that for $p < 1$ the above relation (3.2), in general, does not hold, that exhibits the fact that for $p < 1$ merely the pointwise or the L^p-wise boundary limit is not be sufficient, through using the Lebesgue type integrals, to reconstruct the original analytic function in the related domain.

Proposition 3.1 *For the analytic function $F(z)$ there hold*

(i) $F(z) \in H^p, \forall \frac{1}{2} < p < 1.$

(ii) *$F(z)$ has its non-tangential boundary limit almost everywhere. In fact, $\forall x \neq 0, 1$,*

$$\lim_{t \to 0} F(x + it) = i\,\mathrm{Im} f(x).$$

(iii) *Further, $i\,\mathrm{Im} f(x)$ is the L^p non-tangential boundary limit of $F(z)$. In fact,*

$$\lim_{t \to 0} \|F(x + it) - i\,\mathrm{Im} f\|_{L^p} = 0, \forall \frac{1}{2} < p < 1.$$

(iv) From none of the pointwise non-tangential boundary limit or the L^p-wise non-tangential boundary limit of F one can reconstruct F by using the Cauchy or Poisson integrals as in the classical case.

Proof (i) Denote

$$F(z) = \frac{i}{\pi}\left(\frac{1}{z} - \frac{1}{z-1}\right). \tag{3.3}$$

We have

$$|F(z)| \leq \frac{1}{\pi|z||z-1|} \leq \begin{cases} \frac{2}{\pi}\frac{1}{|x|}, & \text{if } |x| < \frac{1}{2}; \\ \frac{2}{\pi}\frac{1}{|x-1|}, & \text{if } |x-1| < \frac{1}{2}; \\ \frac{1}{\pi}\frac{1}{|x||x-1|}, & \text{if } x < -\frac{1}{2} \text{ or } x > \frac{3}{2}. \end{cases}$$

Hence, if $\frac{1}{2} < p < 1$, then

$$\sup_{t>0}\int_{-\infty}^{+\infty}|F(x+it)|^p dx < +\infty.$$

That is to say, $F(z) \in H^p, \forall \frac{1}{2} < p < 1$.

(ii) We note that

$$\text{Re}F(x+it) = \frac{1}{\pi}\left(\frac{t}{x^2+t^2} - \frac{t}{(x-1)^2+t^2}\right),$$
$$\text{Im}F(x+it) = \frac{1}{\pi}\left(\frac{x}{x^2+t^2} - \frac{x-1}{(x-1)^2+t^2}\right).$$

For $x \neq 0, 1$, we have

$$\lim_{t\to 0}\text{Re}F(x+it) = 0,$$
$$\lim_{t\to 0}\text{Im}F(x+it) = \text{Im}f.$$

We get then the conclusion (ii).

(iii) For $\frac{1}{2} < p < 1$, we have

$$\lim_{t\to 0}\|\text{Re}F(x+it)\|_{L^p} = 0,$$
$$\lim_{t\to 0}\|\text{Im}F(x+it) - \text{Im}f\|_{L^p} = 0.$$

We get then the conclusion (iii).

(iv) This is obvious as

$$\lim_{y\to 0+}\text{Re}F(x+iy) = 0, a.e.$$

and

$$\lim_{t\to 0}\|\text{Re}F(x+it)\|_{L^p} = 0, \forall\frac{1}{2} < p < 1.$$

□

We wish to indicate in the following proposition what happens with the above given holomorphic distribution and analytic function pair f and F.

Proposition 3.2 (i) *For $F(z)$ defined in the Eq. (3.3) and f defined in the Eq. (3.1), the following relation holds:*

$$\lim_{t\to 0}\langle F(\cdot + it), \psi\rangle = \langle f, \psi\rangle, \quad \forall\psi \in S(\mathbb{R}). \tag{3.4}$$

It amounts to saying that the non-tangential distributional limit is the right one to correspond to the analytic Hardy space function F.

(ii) *For $\phi(x) \in S(R^n)$ such that $\int \phi(x)dx = 1$, we have*

$$\lim_{t\to 0}\left\langle \frac{1}{t}\phi(\frac{\cdot}{t}), \psi\right\rangle = \psi(0), \quad \forall\psi \in S(R)). \tag{3.5}$$

$$\lim_{t\to 0}\left\| \frac{1}{t}\phi(\frac{\cdot}{t})\right\|_{L^p} = 0, \quad \forall 0 < p < 1. \tag{3.6}$$

That is to say, the Dirac function δ can be treated as the zero element in $L^p(0 < p < 1)$.

Proof (i) Since the Poisson kernel when t tending to zero is an approximation to identity [25], we have

$$\lim_{t\to 0}\langle \mathrm{Re}F(\cdot + it), \psi\rangle = \langle \mathrm{Re} f, \psi\rangle, \quad \forall\psi \in S(\mathbb{R}).$$

Since the conjugate Poisson kernel induces the Hilbert transformation [18], we have

$$\lim_{t\to 0}\langle \mathrm{Im}F(\cdot + it), \psi\rangle = \langle \mathrm{Im} f, \psi\rangle, \quad \forall\psi \in S(\mathbb{R}).$$

The above two equations imply the Eq. (3.4)

(ii) Like the Poisson kernel, the sequence $\frac{1}{t}\phi(\frac{\cdot}{t})$ is an approximation to identity, we immediately have

$$\lim_{t\to 0}\left\langle \frac{1}{t}\phi(\frac{\cdot}{t}), \psi\right\rangle = \langle \delta_0, \psi\rangle = \psi(0).$$

(iii) The following fact implies that the Eq. (3.6) is true.

$$\int \left|\frac{1}{t}\phi(\frac{x}{t})\right|^p dx = t^{1-p}\int |\phi(x)|^p dx.$$

□

The example of this section shows that for $p < 1$ merely pointwise and L^p boundary limits are insufficient to study H^p spaces, and insufficient to study other analytic function spaces. Distributional boundary limits are necessary. In the last two sections of this paper, we will show that holomorphic Meyer wavelets can be used to well analyze the general holomorphic function spaces. In particular, the extension of a holomorphic distribution by the Cauchy formula (1.1) or the Cauchy formula (1.3) in the respective contexts is an analytic function in the related region. The characterizations of analytic function spaces by using holomorphic Meyer's wavelets thus can be completely established.

4 Preliminaries on Generalized Morrey Spaces in Real Analysis

In this section we present the necessary knowledge on generalized Morrey spaces and the related operator skills in real analysis. We need to use those results to study holomorphic functions in Sects. 5 and 6.

4.1 Function Spaces and Wavelet Characterization

First we present the related definitions and wavelet characterizations of Triebel-Lizorkin-Morrey spaces and Besov-Morrey spaces in real analysis. See Li-Xiao-Yang [7], Li-Yang-Zheng [10], Lin-Yang [11] etc. When we analyze holomorphic functions in the real axis, we use homogeneous Triebel-Lizorkin-Morrey spaces and Besov-Morrey spaces. When we analyze holomorphic functions on the circle, we need to use the related non-homogeneous spaces.

Let $\phi \in C_0^\infty(B(0,2))$ and $\phi(x) = 1$ on the unit ball $B(0,1)$. For $Q = B(x_Q, r)$, denote $\phi_Q(x) = \phi(\frac{x-x_Q}{r})$. For $m_0 \in \mathbb{N}$ and for any functions $f(x)$, $S^{m_0,f}$ denote the set of polynomial functions $P_{Q,f}(x) = \sum_{0\leq\gamma\leq m_0} a_\gamma x^\gamma$ such that

$$\int x^\gamma \phi_Q(x) \left(f(x) - P_{Q,f}(x)\right) dx = 0, \forall 0 \leq \gamma \leq m_0.$$

The definitions of Besov spaces $\dot{B}_p^{\gamma,q}$ and Triebel-Lizorkin spaces $\dot{F}_p^{\gamma,q}$ can be found in [17,26]. The generalized homogeneous type Morrey spaces are defined as follows:

Definition 4.1 Given $m_0 \in \mathbb{N}$, $1 \leq q \leq \infty$, $\gamma_1 \in \mathbb{R}$, $|\gamma_1| < m_0 - 1$.

(i) $1 \leq p < \infty, 0 \leq \gamma_2 \leq \frac{1}{p}$. $f(x) \in \dot{F}_{p,q}^{\gamma_1,\gamma_2}$, if

$$\sup_Q \inf_{P_{Q,f}\in S^{m_0,f}} |Q|^{\gamma_2-\frac{1}{p}} \|\phi_Q\left(f - P_{Q,f}\right)\|_{\dot{F}_p^{\gamma_1,q}} < \infty.$$

(ii) $1 \leq p \leq \infty, 0 \leq \gamma_2 \leq \frac{1}{p}$. $f(x) \in \dot{B}_{p,q}^{\gamma_1,\gamma_2}$, if

$$\sup_Q \inf_{P_{Q,f}\in S^{m_0,f}} |Q|^{\gamma_2-\frac{1}{p}} \|\phi_Q\left(f - P_{Q,f}\right)\|_{\dot{B}_p^{\gamma_1,q}} < \infty.$$

The above spaces are already characterized by wavelets. We can find their wavelet characterizations in Li-Xiao-Yang [7], Li-Yang-Zheng [10] and Lin-Yang [11], etc.

Lemma 4.2 *Given* $m_0 \in \mathbb{N}$, $1 \le q \le \infty$, $\gamma_1 \in \mathbb{R}$, $|\gamma_1| < m_0 - 1$.

(i) *Let* $1 \le p < \infty$, $0 \le \gamma_2 \le \frac{1}{p}$. *Then* $f(x) \in \dot{F}^{\gamma_1,\gamma_2}_{p,q}$ *if and only if*

$$\sup_Q |Q|^{\gamma_2 - \frac{1}{p}} \left\| \left(\sum_{Q_{j,k} \subset Q} 2^{jq(\gamma_1 + \frac{1}{2})} |a_{j,k}|^q \chi(2^j x - k) \right)^{\frac{1}{q}} \right\|_{L^p} < \infty. \tag{4.1}$$

(ii) $1 \le p \le \infty$, $0 \le \gamma_2 \le \frac{1}{p}$. $f(x) \in \dot{B}^{\gamma_1,\gamma_2}_{p,q}$, *if and only if*

$$\sup_Q |Q|^{\gamma_2 - \frac{1}{p}} \left[\sum_{j \ge -\log_2 |Q|} 2^{jq(\gamma_1 + \frac{1}{2} - \frac{1}{p})} \left(\sum_{Q_{j,k} \subset Q} |a_{j,k}|^p \right)^{q/p} \right]^{\frac{1}{q}} < \infty. \tag{4.2}$$

The related non-homogeneous spaces are defined as follows.

Definition 4.3 Given $m_0 \in \mathbb{N}$, $1 \le q \le \infty$, $\gamma_1 \in \mathbb{R}$, $|\gamma_1| < m_0 - 1$.

(i) Let $1 \le p < \infty$, $0 \le \gamma_2 \le \frac{1}{p}$. Then $f(x) \in F^{\gamma_1,\gamma_2}_{p,q}$ if

$$\sup_{|Q| \le 1} |Q|^{\gamma_2 - \frac{1}{p}} \inf_{P_{Q,f} \in S^{m_0,f}} \|\phi_Q \left(f - P_{Q,f}\right)\|_{F^{\gamma_1,q}_p} < \infty.$$

(ii) Let $1 \le p \le \infty$, $0 \le \gamma_2 \le \frac{1}{p}$. Then $f(x) \in B^{\gamma_1,\gamma_2}_{p,q}$ if

$$\sup_{|Q| \le 1} |Q|^{\gamma_2 - \frac{1}{p}} \inf_{P_{Q,f} \in S^{m_0,f}} \|\phi_Q \left(f - P_{Q,f}\right)\|_{B^{\gamma_1,q}_p} < \infty.$$

For non-homogeneous spaces we can get the following wavelet characterization that is analogous with what we have for homogeneous spaces.

Lemma 4.4 *Given* $m_0 \in \mathbb{N}$, $1 \le q \le \infty$, $\gamma_1 \in \mathbb{R}$, $|\gamma_1| < m_0 - 1$.

(i) *For* $1 \le p < \infty$, $0 \le \gamma_2 \le \frac{1}{p}$, $f(x) \in F^{\gamma_1,\gamma_2}_{p,q}$ *if and only if*

$$\sup_{|Q| \le 1} |Q|^{\gamma_2 - \frac{1}{p}} \| \left(\sum_{Q_{j,k} \subset Q} 2^{jq(\gamma_1 + \frac{1}{2})} |a_{j,k}|^q \chi(2^j x - k) \right)^{\frac{1}{q}} \|_{L^p} < \infty. \tag{4.3}$$

(ii) *For* $1 \le p \le \infty$, $0 \le \gamma_2 \le \frac{1}{p}$, $f(x) \in B^{\gamma_1,\gamma_2}_{p,q}$ *if and only if*

$$\sup_{|Q| \le 1} |Q|^{\gamma_2 - \frac{1}{p}} \left[\sum_{j \ge -\log_2 |Q|} 2^{jq(\gamma_1 + \frac{1}{2} - \frac{1}{p})} \left(\sum_{Q_{j,k} \subset Q} |a_{j,k}|^p \right)^{q/p} \right]^{\frac{1}{q}} < \infty. \tag{4.4}$$

4.2 Continuity of Caldrón-Zygmund Operators

When we analyze holomorphic functions with holomorphic wavelets, we involve the Calderón-Zygmund operators. We recall some basic knowledge on the Calderón-Zygmund operators (cf. [13,16]). Assume that $K(x, y)$ is a smooth function for $x \neq y$, and there exists a positive integer N_0 such that

$$|\partial_x^\alpha \partial_y^\beta K(x, y)| \lesssim |x-y|^{-(1+|\alpha|+|\beta|)}, \quad \forall \ |\alpha|+|\beta| \leq N_0. \tag{4.5}$$

Definition 4.5 A linear operator

$$Tf(x) = \int K(x, y) f(y) dy$$

is said to be a Calderón-Zygmund operator if it is continuous from $C^1(\mathbb{R})$ to $(C^1(\mathbb{R}))'$, where the kernel $K(\cdot, \cdot)$ satisfies the condition (4.5) and

$$Tx^\alpha = T^* x^\alpha = 0, \quad \forall \ \alpha \in \mathbb{N} \ \text{ with } \ |\alpha| \leq N_0.$$

For such an operator, we denote $T \in CZO(N_0)$.

The kernel $K(\cdot, \cdot)$ may have high singularity on the diagonal $x = y$. So, according to the Schwartz kernel theorem, it is only a distribution in $S'(\mathbb{R}^2)$. For $(j, k), (j', k') \in \mathbb{Z}^2$, let

$$a_{j,k,j',k'} = \langle K(x, y), \psi_{j,k}(x) \psi_{j',k'}(y) \rangle.$$

If T is a Calderón-Zygmund operator, then its kernel $K(\cdot, \cdot)$ and the related coefficients satisfy the following relations (cf. [13,16,31]).

Lemma 4.6 (i) *If $T \in CZO(N_0)$, then the coefficients $a_{j,k,j',k'}$ satisfy*

$$|a_{j,k,j',k'}| \lesssim \frac{\left(\frac{2^{-j}+2^{-j'}}{2^{-j}+2^{-j'}+|k2^{-j}-k'2^{-j'}|} \right)^{1+N_0}}{2^{|j-j'|(\frac{1}{2}+N_0)}}, \quad \forall \, (j, k), (j', k') \in \mathbb{Z}^2. \tag{4.6}$$

(ii) *If $a_{j,k,j',k'}$ satisfy (4.6), then $K(\cdot, \cdot)$, the kernel of the operator T, can be written as*

$$K(x, y) = \sum_{(j,k),(j',k') \in \mathbb{Z}^2} a_{j,k,j',k'} \psi_{j,k}(x) \psi_{j',k'}(y)$$

in the distribution sense. Moreover, T belongs to $CZO(N_0 - \delta)$ for small positive numbers δ.

The following Lemma tells that Calderón-Zygmund operators are bounded on the Triebel-Lizorkin-Morrey spaces and Besov Morrey spaces. See Meyer [14], Meyer-Coifman [15], Li-Xiao-Yang [7], Li-Yang-Zheng [10], Lin-Yang [11], Stein [24] etc.

Lemma 4.7 *Given* $m_0 \in \mathbb{N}$, $1 \le q \le \infty$, $\gamma_1 \in \mathbb{R}$, $|\gamma_1| < m_0 - 1$. *Given* $N_0 > m_0 + 1$ *and* $T \in CZO(N_0)$.

(i) *For* $1 \le p < \infty$, $0 \le \gamma_2 \le \frac{1}{p}$, *we have* $\|Tf(x)\|_{\dot{F}^{\gamma_1,\gamma_2}_{p,q}} \le C\|f(x)\|_{\dot{F}^{\gamma_1,\gamma_2}_{p,q}}$.

(ii) *For* $1 \le p \le \infty$, $0 \le \gamma_2 \le \frac{1}{p}$, *we have* $\|Tf(x)\|_{\dot{B}^{\gamma_1,\gamma_2}_{p,q}} \le C\|f(x)\|_{\dot{B}^{\gamma_1,\gamma_2}_{p,q}}$.

5 Bimodal Wavelets and General Holomorphic Function Spaces

Meyer [13] proves that bimodal wavelet bases are unconditional bases in $\mathbb{H}^p$. In this section, we consider general Morrey spaces. We characterize their norms in terms of the absolute values of their wavelet coefficients. Hence Bimodal wavelets are unconditional bases for general Morrey spaces. Our general spaces include the following known spaces in complex analysis:

$$\begin{cases} \text{Hardy spaces} & \mathbb{H}^p = \dot{F}^{0,\frac{1}{p}}_{p,2}\ (0 < p < \infty); \\ \text{Bergman space} & B_p = \dot{B}^{0,p}_1 = \dot{B}^{0,\frac{1}{p}}_{1,p}\ (0 < p < \infty); \\ \text{Bloch space} & \dot{B}^{0,\infty}_\infty = \dot{B}^{0,0}_{\infty,\infty}; \\ \text{Lipschitz space} & C^1 = \dot{B}^{1,\infty}_\infty = \dot{B}^{1,0}_{\infty,\infty}; \\ \text{BMO space} & BMO = \dot{F}^{0,0}_{2,2} = \dot{B}^{0,0}_{2,2}; \\ \text{Q space} & Q_r = \dot{F}^{r,r}_{2,2} = \dot{B}^{r,r}_{2,2}\ (0 < r < \frac{1}{2}). \end{cases}$$

See Botchkariev [1], Dang-Qian-You [3], Essen-Janson-Peng-Xiao [4], Meyer [13], Qian-Wang [21], Stein [24], Wojtaszczyk [27] and Zhu [34]. Hence the following defined generalized holomorphic function spaces and the augments largely and systematically extend the function spaces bank in complex analysis.

Definition 5.1 Given $1 \le p \le \infty$, $1 \le q \le \infty$, $0 \le \gamma_2 \le \frac{1}{p}$ and $|\gamma_1| \le m_0 - 1$.

(i) For $p < \infty$, a holomorphic distribution $f(x)$ belongs to $\dot{F}^{\gamma_1,\gamma_2}_{p,q,hol}$ if

$$\sup_Q \inf_{P_{Q,f} \in S^{m_0,f}} |Q|^{\gamma_2 - \frac{1}{p}} \|\phi_Q\left(f - P_{Q,f}\right)\|_{\dot{F}^{\gamma_1,q}_p} < \infty.$$

(ii) A holomorphic distribution $f(x)$ belongs to $\dot{B}^{\gamma_1,\gamma_2}_{p,q,hol}$ if

$$\sup_Q \inf_{P_{Q,f} \in S^{m_0,f}} |Q|^{\gamma_2 - \frac{1}{p}} \|\phi_Q\left(f - P_{Q,f}\right)\|_{\dot{B}^{\gamma_1,q}_p} < \infty.$$

In this section, we consider the wavelet characterization of the above spaces by using Meyer's bimodal holomorphic wavelets. $\forall j \in \mathbb{Z}$, $k \in \mathbb{N}$, denote

$$a_{j,k} = \langle f(x), S_{j,k}(x)\rangle.$$

According to Remark 2.4, to consider holomorphic function with holomorphic wavelets, we must use methodology different from real analysis. Those include, for instance, boundedness of the operators and symmetry property of wavelets, etc. But we still have the following similar results as we have in real analysis.

Theorem 5.2 *Given* $m_0 \in \mathbb{N}$, $1 \leq q \leq \infty$, $\gamma_1 \in \mathbb{R}$, $|\gamma_1| < m_0 - 1$.

(i) $1 \leq p < \infty, 0 \leq \gamma_2 \leq \frac{1}{p}$. $f(x) \in \dot{F}^{\gamma_1,\gamma_2}_{p,q,hol}$ *if and only if*

$$\sup_Q |Q|^{\gamma_2 - \frac{1}{p}} \| \left(\sum_{Q_{j,k} \subset Q} 2^{jq(\gamma_1 + \frac{1}{2})} |a_{j,k}|^q \chi(2^j x - k) \right)^{\frac{1}{q}} \|_{L^p} < \infty. \tag{5.1}$$

(ii) $1 \leq p \leq \infty, 0 \leq \gamma_2 \leq \frac{1}{p}$. $f(x) \in \dot{B}^{\gamma_1,\gamma_2}_{p,q,hol}$ *if and only if*

$$\sup_Q |Q|^{\gamma_2 - \frac{1}{p}} \left[\sum_{j \geq -\log_2 |Q|} 2^{jq(\gamma_1 + \frac{1}{2} - \frac{1}{p})} \left(\sum_{Q_{j,k} \subset Q} |a_{j,k}|^p \right)^{q/p} \right]^{\frac{1}{q}} < \infty. \tag{5.2}$$

Proof (i) Denote

$$\overline{\tau_{j,k}(x)} = (2\pi)^{-1} 2^{\frac{j}{2}} \int_0^\infty e^{i(2^j x - k - \frac{1}{2})\xi} \omega(\xi) d\xi. \tag{5.3}$$

$$\overline{\tau_{j,-k-1}(x)} = (2\pi)^{-1} 2^{\frac{j}{2}} \int_0^\infty e^{i(2^j x + k + \frac{1}{2})\xi} \omega(\xi) d\xi. \tag{5.4}$$

Since $\hat{f}(\xi) = 0, \forall \xi < 0$, we have

$$\begin{aligned} a_{j,k} &= \langle f(x), \tau_{j,k}(x) + \tau_{j,-k-1}(x) \rangle \\ &= \langle f(x), \psi_{j,k}(x) + \psi_{j,-k-1}(x) \rangle \\ &= \langle f(x), \psi_{j,k}(x) + \psi_{j,k}(-x) \rangle \\ &= \langle f(x) + f(-x), \psi_{j,k}(x) \rangle. \end{aligned} \tag{5.5}$$

$f(x) \in \dot{F}^{\gamma_1,\gamma_2}_{p,q,hol}$ implies that its wavelet coefficients satisfy the Eq. (5.1).
Conversely, we observe, from the Eqs. (5.3) and (5.4),

$$a_{j,-k-1} = \langle f(x), \tau_{j,k}(x) + \tau_{j,-k-1}(x) \rangle = a_{j,k}. \tag{5.6}$$

Hence $a_{j,k}$ satisfies the Eq. (4.1) implies

$$g(x) = \sum_{j,k \in \mathbb{Z}} a_{j,k} \psi_{j,k}(x) \in \dot{F}^{\gamma_1,\gamma_2}_{p,q}.$$

Let

$$\begin{aligned} T_+ f(x) &= \sum_{j\in\mathbb{Z},k\in\mathbb{N}} \tau_{j,k}(x) \int \psi_{j,k}(y) f(y) dy, \\ T_- f(x) &= \sum_{j\in\mathbb{Z},k\leq -1} \tau_{j,k}(x) \int \psi_{j,k}(y) f(y) dy, \\ Sf(x) &= f(-x). \end{aligned} \tag{5.7}$$

The above operators T_+, T_- and S all Calderón-Zygmund operators and they are continuous operators on $\dot{F}^{\gamma_1,\gamma_2}_{p,q}$. Hence $a_{j,k}$ satisfies the Eq. (4.1) implies

$$\sum_{j\in\mathbb{Z},k\in\mathbb{N}} a_{j,k}(\tau_{j,k}(x) + \tau_{j,-k-1}(x)) \in \dot{F}^{\gamma_1,\gamma_2}_{p,q}.$$

That is to say, $f(x) \in \dot{F}^{\gamma_1,\gamma_2}_{p,q,hol}$.

(ii) Since $\hat{f}(\xi) = 0, \forall \xi < 0$, applying Eqs. (5.3) and (5.4), $a_{j,k}$ satisfies the Eqs. (5.5). Hence $f(x) \in \dot{B}^{\gamma_1,\gamma_2}_{p,q,hol}$ implies its wavelet coefficients of $f(x)$ satisfies the Eq. (5.2).

Conversely, by the symmetry property of τ, we observe that

$$a_{j,-k-1} = \langle f(x), \tau_{j,k}(x) + \tau_{j,-k-1}(x)\rangle = a_{j,k}.$$

Hence, $a_{j,k}$ satisfying the Eq. (5.1) implies

$$g(x) = \sum_{j,k\in\mathbb{Z}} a_{j,k}\psi_{j,k}(x) \in \dot{B}^{\gamma_1,\gamma_2}_{p,q}.$$

Since operators T_+, T_- and S defined in Eq. (5.7) all are continuous operators on $\dot{B}^{\gamma_1,\gamma_2}_{p,q}$, $a_{j,k}$ satisfying the Eq. (5.2) implies

$$\sum_{j\in\mathbb{Z},k\in\mathbb{N}} a_{j,k}(\tau_{j,k}(x) + \tau_{j,-k-1}(x)) \in \dot{B}^{\gamma_1,\gamma_2}_{p,q}.$$

Hence $f(x) \in \dot{B}^{\gamma_1,\gamma_2}_{p,q,hol}$. □

6 Generalized Botchkariev-Meyer-Wojtasczcyk Theorem

Botchkariev-Wojtasczcyk using Franklin system studied analytic functions in $\mathbb{H}^p$, see [1,27]. Meyer used holomorphic Meyer wavelets to study $A(D)$ in [13]. In this section, we prove a generalized Botchkariev-Meyer-Wojtasczcyk Theorem, which extends the function spaces on the interval in real analysis to the corresponding holomorphic function spaces on the circle systematically.

Definition 6.1 Given $1 \leq p \leq \infty$, $1 \leq q \leq \infty$, $\gamma_1 \in \mathbb{R}$, $0 \leq \gamma_2 \leq \frac{1}{p}$. Let m_0 be a sufficiently big integer, being bigger than some number depending on γ_1, γ_2, p, q.

(i) For $p < \infty$, a 1-periodic holomorphic distribution $f(x)$ belongs to $F^{\gamma_1,\gamma_2,hol}_{p,q,period}$ if

$$\sup_{|Q|\leq 1} \inf_{P_{Q,f}\in S^{m_0,f}} |Q|^{\gamma_2-\frac{1}{p}} \left\|\phi_Q\left(f - P_{Q,f}\right)\right\|_{F^{\gamma_1,q}_p} < \infty.$$

(ii) A 1-periodic holomorphic distribution $f(x)$ belongs to $B^{\gamma_1,\gamma_2,hol}_{p,q,period}$ if

$$\sup_{|Q|\leq 1} \inf_{P_{Q,f}\in S^{m_0,f}} |Q|^{\gamma_2-\frac{1}{p}} \left\|\phi_Q\left(f - P_{Q,f}\right)\right\|_{B^{\gamma_1,q}_p} < \infty.$$

$\forall m \in \mathbb{N}$, we denote

$$a_m = \langle f(x), G_m(x)\rangle.$$

Although holomorphic wavelets have different nature from the usual ones, we still have the following counterpart results as we have in real analysis.

Theorem 6.2 *Given* $m_0 \in \mathbb{N}$, $1 \leq q \leq \infty$, $\gamma_1 \in \mathbb{R}$, $|\gamma_1| < m_0 - 1$.

(i) *For* $1 \leq p < \infty$, $0 \leq \gamma_2 \leq \frac{1}{p}$, $f(x) \in F^{\gamma_1,\gamma_2,hol}_{p,q,period}$ *if and only if*

$$\sup_{|Q|\leq 1} |Q|^{\gamma_2-\frac{1}{p}} \left\| \left(\sum_{Q_{j,k}\subset Q} 2^{jq(\gamma_1+\frac{1}{2})} |a_{j,k}|^q \chi(2^j x - k) \right)^{\frac{1}{q}} \right\|_{L^p} < \infty. \tag{6.1}$$

(ii) *For* $1 \leq p \leq \infty$, $0 \leq \gamma_2 \leq \frac{1}{p}$, $f(x) \in B^{\gamma_1,\gamma_2,hol}_{p,q,period}$ *if and only if*

$$\sup_{|Q|\leq 1} |Q|^{\gamma_2-\frac{1}{p}} \left[\sum_{j\geq -\log_2|Q|} 2^{jq(\gamma_1+\frac{1}{2}-\frac{1}{p})} \left(\sum_{Q_{j,k}\subset Q} |a_{j,k}|^p \right)^{q/p} \right]^{\frac{1}{q}} < \infty. \tag{6.2}$$

Proof (i) We first prove that $f(x) \in F^{\gamma_1,\gamma_2,hol}_{p,q,period}$ implies that its wavelet coefficients satisfy the Eq. (6.1). For $m = 0$ and $m = 1$, the proof is easy. We only give the proof for the cases $m \geq 2$. In fact, for such cases we have

$$a_{j,k} = \int_0^1 f(x) G_{j,k}(x) dx$$

Since $f(x)$ is a holomorphic function, we have

$$a_{j,k} = \int_0^1 f(x)(g_{j,k} + g_{j,2^j-k-1})(x) dx$$

By the symmetry property of $g_{j,k}$, we perform change of variable and get

$$a_{j,k} = \int_0^1 f(x) g_{j,k}(x) dx + \int_0^1 f(-x) g_{j,k}(x) dx$$

That is to say, $f(x) \in F^{\gamma_1,\gamma_2,hol}_{p,q,period}$ implies that its wavelet coefficients satisfy the Eq. (6.1).

To prove the converse result, we need only to show

$$\sum_{j\geq 1, 0\leq k<2^{j-1}} a_{j,k} G_{j,k}(x) \in F^{\gamma_1,\gamma_2,hol}_{p,q,period}.$$

Since

$$G_{j,k}(x) = P\{g_{j,k}(x) + g_{j,2^j-k-1}(x)\},$$

we have

$$\sum_{j\geq 1, 0\leq k<2^{j-1}} a_{j,k} G_{j,k}(x) = P\Big\{ \sum_{j\geq 1, 0\leq k<2^{j-1}} a_{j,k} g_{j,k}(x) \Big\} + P\Big\{ \sum_{j\geq 1, 0\leq k<2^{j-1}} a_{j,k} g_{j,2^j-k-1}(x) \Big\}.$$

First, we note

$$\sum_{j\geq 1, 0\leq k<2^{j-1}} a_{j,k} g_{j,k}(x) \in F^{\gamma_1,\gamma_2}_{p,q}.$$

Further, by the periodicity property of the wavelets, we have

$$\sum_{j\geq 1, 0\leq k<2^{j-1}} a_{j,k} g_{j,2^j-k-1}(x) \in F^{\gamma_1,\gamma_2}_{p,q}.$$

Since the operator P is continuous on $F^{\gamma_1,\gamma_2}_{p,q}$, $a_{j,k}$ satisfying the Eq. (6.1) implies

$$\sum_{j\geq 1, 0\leq k<2^{j-1}} a_{j,k} G_{j,k}(x) \in F^{\gamma_1,\gamma_2,hol}_{p,q,period}.$$

Hence $f(x) \in F^{\gamma_1,\gamma_2,hol}_{p,q,period}$.

(ii) We next prove that $f(x) \in B^{\gamma_1,\gamma_2,hol}_{p,q,period}$ implies that its wavelet coefficients satisfy the Eq. (6.2). For $m = 0$ and $m = 1$, the proof is clear. We only consider the cases where $m \geq 2$. In such case, we have

$$a_{j,k} = \int_0^1 f(x) G_{j,k}(x) dx.$$

Since $f(x)$ is a holomorphic function, we have

$$a_{j,k} = \int_0^1 f(x)(g_{j,k} + g_{j,2^j-k-1})(x) dx.$$

By the symmetry property of $g_{j,k}$, performing change of variable we get

$$a_{j,k} = \int_0^1 f(x) g_{j,k}(x) dx + \int_0^1 f(-x) g_{j,k}(x) dx$$

That is to say, $f(x) \in B^{\gamma_1,\gamma_2,hol}_{p,q,period}$ implies that its wavelet coefficients satisfy the Eq. (6.2).

Conversely, we need only prove

$$\sum_{j\geq 1, 0\leq k<2^{j-1}} a_{j,k} G_{j,k}(x) \in B^{\gamma_1,\gamma_2,hol}_{p,q,period}.$$

Since we have

$$G_{j,k}(x) = P\big\{g_{j,k}(x) + g_{j,2^j-k-1}(x)\big\},$$

then

$$\sum_{j\geq 1, 0\leq k<2^{j-1}} a_{j,k} G_{j,k}(x) = P\Big\{\sum_{j\geq 1, 0\leq k<2^{j-1}} a_{j,k} g_{j,k}(x)\Big\} + P\Big\{\sum_{j\geq 1, 0\leq k<2^{j-1}} a_{j,k} g_{j,2^j-k-1}(x)\Big\}.$$

Since the operator P is continuous on $B^{\gamma_1,\gamma_2}_{p,q}$, the fact that $a_{j,k}$ satisfy the Eq. (6.2) implies

$$\sum_{j\geq 1, 0\leq k<2^{j-1}} a_{j,k} G_{j,k}(x) \in B^{\gamma_1,\gamma_2,hol}_{p,q,period}.$$

Hence $f(x) \in B^{\gamma_1,\gamma_2,hol}_{p,q,period}$. □

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