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Rational approximation in Hardy spaces on strips

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ABSTRACT

In this paper we consider rational approximation of functions in the Hardy spaces on the strip $S_a = \{z = x+iy \in \mathbb{C}; x \in \mathbb{R}, |y| < a\}$ with $a > 0$. Observing that $H^2(S_a)$ can be regarded as the direct sum of $H^2(\mathbb{C}^+_{-a})$ and $H^2(\mathbb{C}^-_a)$, we give three kinds of adaptive rational approximation in $H^2(S_a)$ enhancing the corresponding approximations in $H^2(\mathbb{C}^+_{-a})$ and $H^2(\mathbb{C}^-_a)$, where $\mathbb{C}_{+,-a} = \{z = x + iy \in \mathbb{C}; x \in \mathbb{R}, y > -a\}$ and $\mathbb{C}_{-,a} = \{z = x + iy \in \mathbb{C}; x \in \mathbb{R}, y < a\}$. We also obtain a type of approximation for subspaces in $H^p(S_a), 1 < p < \infty$ by making use of Riesz bases in these subspaces.

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1. Introduction

Rational approximation, a historical research topic, has been receiving great attention both in theory and practice. In this paper we will focus on the study of the topic in classical Hardy spaces. The so-called Takenaka-Malmquist (TM) system is naturally invoked when we consider rational approximation in Hardy spaces. In recent years Qian and his collaborators have published a series of papers (see [1–3] and the references therein) concerning adaptive rational approximations in Hardy spaces of various contexts. Among them, the so-called adaptive Fourier decomposition (AFD) is the core, which is based on the generalized backward shift operator leading to an adaptive TM system. It has been shown that AFD is not only theoretical but also practical (see e.g. [4] for its applications in signal analysis). The generalizations of AFD have been developed to the settings of Quaternionic and Clifford analysis, and several complex variables (see e.g. [3]). In [2] the so-called Pre-Orthogonal Greedy Algorithm (Pre-OGA) is proposed that generalizes the AFD theory to abstract Hilbert spaces. Pre-OGA in various contexts gives better approximations than the ordinary greedy algorithms. We note that the AFD-type expansions, equivalent with Pre-OGA in general contexts, can be used in a wide class of function spaces. They do not require delicate things like basis or Blaschke products, etc., to exist in the space, but give rise to approximations with fast convergence to the projected function in terms of linear combinations of the reproducing kernels. See for example the theory on matrix and n-torus ([2,5]). For an alternative treatment of Pre-OGA in reproducing kernel Hilbert spaces see e.g. [3].

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Motivated by the studies of AFD, in this paper we consider adaptive approximations in Hardy spaces on strips $H^2(S_a)$. On one hand, we observe that there exists a unique decomposition of functions in $H^2(S_a)$ and then conclude that rational approximations in $H^2(\mathbb{C}^+_{-a})$ and $H^2(\mathbb{C}^-_a)$ give rise to those in $H^2(S_a)$. On the other hand, $H^2(S_a)$ itself can be indeed regarded as a reproducing kernel Hilbert space in view of the Paley–Wiener theorem (see Section 2). Thus the theory of Pre-OGA can be directly applied to $H^2(S_a)$, which also yields a type of rational approximation of functions in $H^2(S_a)$. In the last section we study rational approximation in $H^p(S_a), 1 < p < \infty$. Similar to the H^2 case, we show that a function in $H^p(S_a)$ can be uniquely written as the sum of functions in $H^p(\mathbb{C}^+_{-a})$ and $H^p(\mathbb{C}^-_a)$. Using the uniform boundedness of partial sum operator in $H^p(\mathbb{C}^+_0)$ (see [6]), we provide the counterpart in $H^p(S_a)$. As an application of the above result, we give a sufficient condition on $\{z_k\} \subset S_a$ for the Szegö kernels associated with $\{z_k\}$ being a Riesz basis of the subspace spanned by those kernels. Consequently, we give the estimate of the difference between F and its m-th partial sum in the H^p-norm.

The present paper is organized as follows. In Section 2 we recall some fundamental results in $H^2(S_a)$. In Section 3 three kinds of rational approximation in $H^2(S_a)$ are given, which include AFD, unwinding AFD and Pre-OGA. In Section 4 rational approximation in $H^p(S_a), 1 < p < \infty$, is studied.

2. Preliminaries

Without loss of generality, we consider the Hardy spaces on S_a, where $S_a = \{z = x + iy \in \mathbb{C}; x \in \mathbb{R}, |y| < a\}$ with $a > 0$.

Denote by $H^p(S_a), 1 \leq p < \infty$, the Hardy spaces on S_a, which is defined by

$$H^p(S_a) = \left\{ F \text{ is analytic in } S_a;\ ||F||^p_{H^p} = \sup_{|y|<a} \int_{-\infty}^{\infty} |F(x+iy)|^p \mathrm{d}x < \infty \right\}.$$

In particular, $H^2(S_a)$ is a reproducing kernel Hilbert space which can be seen through the Paley-Wiener Theorem. The Paley-Wiener Theorem gives a very nice characterization of functions in $H^2(S_a)$ that is stated as follows.

Theorem 2.1 (see [7]): *$F \in H^2(S_a)$ if and only if there exists $f \in L^2(\mathbb{R})$ such that*

$$\sup_{|y|<a} \int_{-\infty}^{\infty} |f(t)|^2 e^{-4\pi yt} \mathrm{d}t < \infty \tag{2.1}$$

and

$$F(z) = \int_{-\infty}^{\infty} f(t) e^{2\pi izt} \mathrm{d}t, \tag{2.2}$$

which means f is the Fourier transform of the restriction of F to $\mathbb{R}$.

The generalization of Theorem 2.1 in the Hardy spaces on tubes can be found in [8]. Note that the condition (2.1) implies $e^{2\pi a|t|}f(t) \in L^2(\mathbb{R})$. Conversely, $e^{2\pi a|t|}f(t) \in L^2(\mathbb{R})$ also implies the condition (2.1). Thus Theorem 2.1 can be read as

Theorem 2.2: *$F \in H^2(S_a)$ if and only if $f \in L^2(\mathbb{R})$ such that*

$$e^{2\pi a|t|}f(t) \in L^2(\mathbb{R}) \tag{2.3}$$

and

$$F(z) = \int_{-\infty}^{\infty} f(t)e^{2\pi izt}\mathrm{d}t.$$

By Theorem 2.2, we can consider $H^2(S_a)$ as a reproducing kernel Hilbert space in the following sense. Denote by $L_a^2(\mathbb{R})$ the closed subspace in $L^2(\mathbb{R})$ whose elements are of the form

$$g_a(t) = e^{2\pi a|t|}g(t) \in L^2(\mathbb{R}),$$

where $g \in L^2(\mathbb{R})$. Let $h_a(z;t) = e^{-2\pi a|t|}e^{-2\pi i\bar{z}t}$. Obviously, $h_a(z;t) \in L_a^2(\mathbb{R})$. By Theorem 2.2, $H^2(S_a)$ can be regarded as the image space of functions defined by

$$F(z) = \langle f_a, h_a(z;\cdot)\rangle_{L^2}, \quad z \in S_a.$$

This induces an inner product on $H^2(S_a)$ by

$$\langle F, G\rangle = \langle f_a, g_a\rangle_{L^2}, \quad F, G \in H^2(S_a),$$

where f_a and g_a are the correspondences of F and G in $L_a^2(\mathbb{R})$. Then we can define the Szegö kernel for $H^2(S_a)$ by

$$K_a(w,\bar{z}) = \langle h_a(z;t), h_a(w;t)\rangle_{L^2}.$$

Directly calculating $K_a(w,\bar{z})$ gives

$$\begin{aligned} K_a(w,\bar{z}) &= \int_{-\infty}^{\infty} e^{-4\pi a|t|}e^{-2\pi i\bar{z}t}e^{2\pi iwt}\mathrm{d}t \\ &= \int_0^{\infty} e^{-4\pi at}e^{-2\pi i\bar{z}t}e^{2\pi iwt}\mathrm{d}t + \int_{-\infty}^{0} e^{4\pi at}e^{-2\pi i\bar{z}t}e^{2\pi iwt}\mathrm{d}t \\ &= \frac{1}{2\pi i}\left(\frac{1}{w-\bar{z}-2ia} - \frac{1}{w-\bar{z}+2ia}\right). \end{aligned}$$

In particular, one has

$$\begin{aligned} F(z) = \langle F, K_a(\cdot,\bar{z})\rangle &= \int_{-\infty}^{\infty} e^{2\pi a|t|}f(t)e^{-2\pi a|t|}e^{2\pi izt}\mathrm{d}t \\ &= \int_0^{\infty} e^{2\pi at}f(t)e^{-2\pi at}e^{2\pi izt}\mathrm{d}t + \int_{-\infty}^{0} e^{-2\pi at}f(t)e^{2\pi at}e^{2\pi izt}\mathrm{d}t \\ &= \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{F(x-ia)}{x-ia-z}\mathrm{d}x - \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{F(x+ia)}{x+ia-z}\mathrm{d}x, \end{aligned}$$

where the last equality is by Parseval's formula, and $F(x-ia)$ and $F(x+ia)$ are the non-tangential boundary limit functions of $F(z)$. From the last two formulas, we have that

$K_a(w,\bar{z})$ (as a function of w) is a rational function with poles outside S_a, and $\langle F, K_a(\cdot,\bar{z})\rangle$ coincides with the Cauchy integral.

3. The $p = 2$ cases

As shown in Section 2, we have

$$K_a(w,\bar{z}) = \frac{1}{2\pi i}\left(\frac{1}{w-\bar{z}-2ia} - \frac{1}{w-\bar{z}+2ia}\right).$$

We briefly write the Hardy spaces $H^2(\mathbb{R}\times(-\infty,a))$ as $H^2(\mathbb{C}_a^-)$, and $H^2(\mathbb{R}\times(-a,\infty))$ as $H^2(\mathbb{C}_{-a}^+)$. One should note that $-\frac{1}{2\pi i}\frac{1}{w-\bar{z}+2ia}$ and $\frac{1}{2\pi i}\frac{1}{w-\bar{z}-2ia}$ are respectively the Szegö kernels of $H^2(\mathbb{C}_{-a}^+)$ and $H^2(\mathbb{C}_a^-)$. Thus, for $F \in H^2(S_a)$, one has

$$F(z) = F^+(z) + F^-(z), \quad F^+ \in H^2(\mathbb{C}_{-a}^+), F^- \in H^2(\mathbb{C}_a^-). \tag{3.1}$$

The above decomposition is unique. In fact, if there exists $H^+ \in H^2(\mathbb{C}_{-a}^+)$ and $H^- \in H^2(\mathbb{C}_a^-)$ such that $F = H^+ + H^-$, we have $G = F^+ - H^+ = H^- - F^-$. Thus $G \in H^2(\mathbb{C}_a^-)\cap H^2(\mathbb{C}_{-a}^+)$. Hence, $G(x+iy) \in H^2(\mathbb{C}_0^+)$ for $y \in (0,\infty)$, and $G(x+iy) \in H^2(\mathbb{C}_0^-)$ for $y \in (-\infty, 0)$. The former means that $\hat{G}(x)$ is with support in $[0,\infty)$, and the latter means that $\hat{G}(x)$ is with support in $(-\infty,0]$, which implies $\hat{G}(x) \equiv 0$, and hence $G(z) \equiv 0$. See Lemma 4.1 for a general discussion. Thus the induced norm of $H^2(S_a)$ is actually given by

$$||F^+||_+^2 + ||F^-||_-^2 = ||F||^2,$$

where $||\cdot||_+$ is the norm of $H^2(\mathbb{C}_{-a}^+)$ and $||\cdot||_-$ is the norm of $H^2(\mathbb{C}_a^-)$. Note that $||F||_{H^2} \leq ||F^+||_+ + ||F^-||_-$, which is given by Minkowski's inequality. Therefore, approximations of functions in $H^2(\mathbb{C}_{-a}^+)$ and $H^2(\mathbb{C}_a^-)$ will give rise to those of functions in $H^2(S_a)$.

3.1. *Upper and lower Hardy spaces decomposition and AFD on the strip*

AFD was originally proposed in the classical Hardy spaces of the unit disc and the upper-half plane (see [1]). Without any difficulty, one can easily obtain AFD in $H^2(\mathbb{C}_{-a}^+)$ and $H^2(\mathbb{C}_a^-)$. In the following we provide the related results in $H^2(\mathbb{C}_{-a}^+)$ and $H^2(\mathbb{C}_a^-)$, and accordingly give rational approximations of functions in $H^2(S_a)$.

Let $\{z_k\}$ be a sequence of points in $\mathbb{C}_0^+$. It is known that the TM system associated with $\{z_k\}$ is defined as

$$\begin{aligned} B_1(z) &= B_{\{z_1\}}(z) \\ &= \sqrt{\frac{\Im z_1}{\pi}}\frac{i}{z-\bar{z}_1}, \ldots, B_k(z) = B_{\{z_1,\ldots,z_k\}}(z) = \sqrt{\frac{\Im z_k}{\pi}}\frac{i}{z-\bar{z}_k}\prod_{j=1}^{k-1}\frac{z-z_j}{z-\bar{z}_j}, \ldots \end{aligned} \tag{3.2}$$

where $z \in \mathbb{C}_0^+$. Accordingly, TM systems in $H^2(\mathbb{C}_{-a}^+)$ and $H^2(\mathbb{C}_a^-)$ are respectively given by

$$\{B_k^+(z)\} = \left\{ B^+_{\{z_1,\ldots,z_k\}}(z) = \sqrt{\frac{\Im z_k + a}{\pi}}\frac{i}{z-\bar{z}_k+2ia}\prod_{j=1}^{k-1}\frac{z-z_j}{z-\bar{z}_j+2ia} \right\},$$

where $\{z_k\} \subset \mathbb{C}^+_{-a}, z \in \mathbb{C}^+_{-a}$, and

$$\{B_k^-(z)\} = \left\{ B^-_{\{z_1,\ldots,z_k\}}(z) = \sqrt{\frac{a - \Im z_k}{\pi}} \frac{-i}{z - \overline{z}_k - 2ia} \prod_{j=1}^{k-1} \frac{z - z_j}{z - \overline{z}_j - 2ia} \right\},$$

where $\{z_k\} \subset \mathbb{C}^-_a, z \in \mathbb{C}^-_a$.

The main idea of AFD is the adaptive procedure of selecting z_k for each k according to a function or an approximation remainder. Precisely, given m points $\{z_k\}_{k=1}^m$ in $\mathbb{C}^+_{-a}$ (resp. $\mathbb{C}^-_a$), we are to select the $(m+1)$-th point in $\mathbb{C}^+_{-a}$ (resp. $\mathbb{C}^-_a$) such that

$$z_{m+1} = \arg \max_{z \in \mathbb{C}^+_{-a}} |\langle F, B^+_{\{z_1,\ldots,z_m,z\}}\rangle_+| \quad (\text{resp. } z_{m+1} = \arg \max_{z \in \mathbb{C}^-_a} |\langle F, B^-_{\{z_1,\ldots,z_m,z\}}\rangle_-|), \tag{3.3}$$

where F is a given function in $H^2(\mathbb{C}^+_{-a})$ (resp. $H^2(\mathbb{C}^-_a)$). We call (3.3) the maximal selection principle. The existence of such z_{m+1} follows from the so-called 'Boundary Vanishing Condition (BVC)' (see [9], and see also Lemmas 3.1 and 3.2). Then, by the general theory of AFD, we have

Theorem 3.1: *Suppose that $F \in H^2(\mathbb{C}^+_{-a})$ (resp. $H^2(\mathbb{C}^-_a)$), and $\{z_k\}$ is a sequence of points in $\mathbb{C}^+_{-a}$ (resp. $\mathbb{C}^-_a$), where each z_k is selected according to (3.3). Then we have*

$$\lim_{m\to\infty} ||F - \sum_{k=1}^m \langle F, B_k^+\rangle_+ B_k^+||_+ = 0 \quad (\textit{resp. } \lim_{m\to\infty} ||F - \sum_{k=1}^m \langle F, B_k^-\rangle_- B_k^-||_- = 0).$$

Consequently, we have

Corollary 3.2: *For $F \in H^2(S_a)$, let $F^\pm$ be, respectively, components of F in $H^2(\mathbb{C}^+_{-a})$ and $H^2(\mathbb{C}^-_a)$. Assume that $\{z_k^+\}$ and $\{z_k^-\}$ are, respectively, in $\mathbb{C}^+_{-a}$ and $\mathbb{C}^-_a$, where each $z_k^\pm$ is selected by the maximal selection principle according to $F^\pm$. Then we have*

$$\lim_{m\to\infty} ||F - (\sum_{k=1}^m \langle F^+, B_k^+\rangle_+ B_k^+ + \sum_{k=1}^m \langle F^-, B_k^-\rangle_- B_k^-)|| = 0.$$

3.2. Upper and lower Hardy spaces decomposition and Unwinding AFD

Unwinding AFD is a variation of AFD that is more efficient (in practice) than AFD. The main difference between unwinding AFD and the original AFD is the factorization of inner function at each step. It is known that for $F \in H^2(\mathbb{C}^+_0)$, one has

$$F(z) = B_F(z) S_F(z) O_F(z),$$

where B_F, S_F and O_F are respectively the Blaschke product part, the singular inner function part and the outer function part of F (see e.g. [10,11]). Usually, $I_F(z) = B_F(z)S_F(z)$ denotes the inner function part of F. For $F \in H^2(\mathbb{C}^+_0)$, one proceeds the following procedure:

$$G_1(z) = F(z), \quad G_k(z) = I_k(z) O_k(z), \quad G_{k+1} = \frac{O_k(z) - \langle O_k, e_{z_k}\rangle_{H^2(\mathbb{C}^+_0)} e_{z_k}(z)}{\frac{z - z_k}{z - \overline{z}_k}}, \tag{3.4}$$

then

$$F(z)=\sum_{k=1}^{m}\langle O_k,e_{z_k}\rangle_{H^2(\mathbb{C}_0^+)}I^{(k)}(z)B_k(z)+I_m(z)G_{m+1}(z)\prod_{k=1}^{m}\frac{z-z_k}{z-\overline{z}_k}, \tag{3.5}$$

where $I_k(z)$ and $O_k(z)$ are the inner and outer parts of $G_k(z)$, $e_{z_k}(z)$ is the normalized Szegö kernel for $H^2(\mathbb{C}_0^+)$, $I^{(k)}(z)=I_1(z)\cdots I_k(z)$, and each z_k is selected according to

$$z_k=\arg\max_{z\in\mathbb{C}_0^+}|\langle O_k,e_z\rangle_{H^2(\mathbb{C}_0^+)}|. \tag{3.6}$$

Generally, an approximating function given by the unwinding AFD is not rational, but we can give a rational one if replacing the factorization of I_k in (3.5) by a finite Blaschke product. The existence of z_k in (3.6) also follows from the BVC in $H^2(\mathbb{C}_0^+)$. The convergence of such a decomposition can be found in [4].

Now applying the above idea to $H^2(\mathbb{C}_{-a}^+)$ and $H^2(\mathbb{C}_a^-)$, one has

Theorem 3.3: *For $F\in H^2(S_a)$, let $F^{\pm}$ be respectively the components of F in $H^2(\mathbb{C}_{-a}^+)$ and $H^2(\mathbb{C}_a^-)$. Assume that $\{z_k^+\}$ and $\{z_k^-\}$ are respectively in $\mathbb{C}_{-a}^+$ and $\mathbb{C}_a^-$, where each $z_k^{\pm}$ is selected by the maximal selection principle according to $F^{\pm}$. Then we have*

$$\lim_{m\to\infty}||F-(\sum_{k=1}^{m}\langle O_k^+,e_{z_k^+}\rangle_+I^{+,(k)}B_k^++\sum_{k=1}^{m}\langle O_k^-,e_{z_k^-}\rangle_-I^{-,(k)}B_k^-)||=0,$$

where $O_k^{\pm}$, $e_{z_k^{j}\pm}$ and $I^{\pm,(k)}$, are correspondingly defined as (3.4) and (3.5).

3.3. $H^2(S_a)$ treated as a RKHS and Pre-OGA

In the previous parts we obtain approximations in $H^2(S_a)$ from those in $H^2(\mathbb{C}_{-a}^+)$ and $H^2(\mathbb{C}_a^-)$, but we can directly apply Pre-OGA to $H^2(S_a)$. The Pre-OGA essentially generalizes the sprit of AFD to reproducing kernel Hilbert spaces based on the fact that a TM system on the unit disc is generated by Szegö kernels. In the following we give a brief introduction to Pre-OGA in $H^2(S_a)$. Suppose that $\{z_k\}$ is a sequence of points in S_a. Let l_k be the cardinality of the set $\{j:z_j=z_k,j<k\}$. Define $\widetilde{K}_a(w,\overline{z}_k)$ as

$$\widetilde{K}_a(w,\overline{z}_k)=\frac{d^{l_k}}{d\overline{z}^{l_k}}K_a(w,\overline{z})|_{z=z_k}.$$

Denote by $\{\mathcal{B}_k=\mathcal{B}_{\{z_1,z_2,\ldots,z_k\}}\}$ the orthogonal system generated by applying the Gram–Schmidt orthonormal process to $\{\widetilde{K}_a(z,\overline{z}_k)\}$ (see [2] for details). By the Gram–Schmidt orthonormal process, we have

$$\mathcal{B}_1 = \mathcal{B}_{\{z_1\}} = \frac{\gamma_1}{||\gamma_1||} = \frac{\widetilde{K}_a(z,\overline{z}_1)}{||\widetilde{K}_a(z,\overline{z}_1)||};$$

$$\gamma_k = \widetilde{K}_a(z,\overline{z}_k) - \sum_{j=1}^{k-1}\langle \widetilde{K}_a(\cdot,\overline{z}_k),\mathcal{B}_j\rangle\mathcal{B}_j,$$

$$\mathcal{B}_k = \mathcal{B}_{\{z_1,\ldots,z_k\}} = \frac{\gamma_k}{||\gamma_k||}, \quad k \geq 2, k \in \mathbb{Z}.$$

Suppose that $\{z_k\}_{k=1}^m$ are m given points in S_a. We are to choose the $(m+1)$-th point in S_a according to the following criterion

$$z_{m+1} = \arg\max_{z\in S_a} |\langle F, \mathcal{B}_{\{z_1,\ldots,z_m,z\}}\rangle|. \tag{3.7}$$

The next lemma gives a set of sufficient conditions so that z_{m+1} in (3.7) exists.

Lemma 3.4: *Suppose that $\{z_k\}_{k=1}^{m+1}$ are $m+1$ points in S_a. Let $\{z_k\}_{k=1}^m$ be fixed. If*

$$\lim_{|y_{m+1}|\to a} \frac{|\langle F, \widetilde{K}_a(\cdot, z_{m+1})\rangle|}{||\widetilde{K}_a(\cdot,\overline{z}_{m+1})||} = 0, \quad z_{m+1} = x_{m+1} + iy_{m+1}, \tag{3.8}$$

holds uniformly for $x_{m+1} \in \mathbb{R}$, then

$$\lim_{|y_{m+1}|\to a} ||F - \sum_{k=1}^{m+1}\langle F,\mathcal{B}_k\rangle\mathcal{B}_k|| = ||F - \sum_{k=1}^{m}\langle F,\mathcal{B}_k\rangle\mathcal{B}_k||.$$

If

$$\lim_{|x_{m+1}|\to \infty} \frac{|\langle F, \widetilde{K}_a(\cdot, \overline{z}_{m+1})\rangle|}{||\widetilde{K}_a(\cdot,\overline{z}_{m+1})||} = 0, \quad z_{m+1} = x_{m+1} + iy_{m+1}, \tag{3.9}$$

holds uniformly for $|y_{m+1}| < a$, then

$$\lim_{|x_{m+1}|\to \infty} ||F - \sum_{k=1}^{m+1}\langle F,\mathcal{B}_k\rangle\mathcal{B}_k|| = ||F - \sum_{k=1}^{m}\langle F,\mathcal{B}_k\rangle\mathcal{B}_k||.$$

***Proof*:** See e.g. [2,9] for details. □

Under the assumption that (3.8) and (3.9) hold, the above lemma implies the existence of z_{m+1} in (3.7). As in [9], we call (3.8) and (3.9) the 'Boundary Vanishing Condition' (BVC) in $H^2(S_a)$. Note that since $\{z_k\}_{k=1}^m$ are previously fixed, (3.8) and (3.9) are reduced to

$$\lim_{|y_{m+1}|\to a} \frac{|F(z_{m+1})|}{||K_a(\cdot,\overline{z}_{m+1})||} = 0, \quad z_{m+1} = x_{m+1} + iy_{m+1}, \tag{3.10}$$

and

$$\lim_{|x_{m+1}|\to\infty} \frac{|F(z_{m+1})|}{||K_a(\cdot, \overline{z}_{m+1})||} = 0, \quad z_{m+1} = x_{m+1} + iy_{m+1}, \tag{3.11}$$

which is called the weak BVC in $H^2(S_a)$.

Under the assumption that z_{m+1} in (3.7) exists for each m, the convergent result follows from the theory of Pre-OGA.

Theorem 3.5: *Suppose that $F \in H^2(S_a)$, and $\{z_k\}$ is a sequence of points in S_a, where each element is selected according to (3.7). Then we have*

$$\lim_{m\to\infty} ||F - \sum_{k=1}^{m} \langle F, \mathcal{B}_k\rangle \mathcal{B}_k|| = 0.$$

The weak BVC in $H^2(S_a)$ is shown in the next lemmas.

Lemma 3.6: *For $F \in H^2(S_a)$, we have*

$$\lim_{|y|\to a} \frac{|F(z)|}{\sqrt{K_a(z, \overline{z})}} = 0, \quad z = x + iy \in S_a, \tag{3.12}$$

holds uniformly for $x \in \mathbb{R}$.

Proof: First note that $K_a(z, \overline{z}) = ||K_a(\cdot, \overline{z})||^2 = \frac{1}{4\pi}\left(\frac{1}{y+a} - \frac{1}{y-a}\right) = \frac{a}{2\pi}(\frac{1}{(a+y)(a-y)}) > 0$ for $|y| < a$. Using Theorem 2.2, we have

$$F(z) = \langle f_a, h_a(z; \cdot)\rangle.$$

Since $f_a \in L^2_a(\mathbb{R})$, we can find a function $g \in L^2(\mathbb{R}) \cap L^p(\mathbb{R}), 1 < p < 2$, such that for any given $\epsilon > 0$,

$$||f_a - g||_{L^2} < \epsilon.$$

Then

$$\begin{aligned} \frac{|F(z)|}{\sqrt{K_a(z, \overline{z})}} &\le \frac{|\langle f_a - g, h_a(z; \cdot)\rangle_{L^2}|}{\sqrt{K_a(z, \overline{z})}} + \frac{|\langle g, h_a(z; \cdot)\rangle_{L^2}|}{\sqrt{K_a(z, \overline{z})}} \\ &\le \epsilon + ||g||_{L^p} \frac{||h_a(z, \cdot)||_{L^q}}{\sqrt{K_a(z, \overline{z})}}, \end{aligned} \tag{3.13}$$

where $q = \frac{p}{p-1}$. Calculating $||h_a(z, \cdot)||_{L^q}$, we have

$$||h_a(z, \cdot)||^q_{L^q} = \int_{-\infty}^{\infty} |e^{-2\pi a|t|} e^{2\pi izt}|^q \mathrm{d}t = \frac{1}{q} K_a(z, \overline{z}).$$

Therefore, when $|y|$ tends to a,

$$(3.13) \le \epsilon + ||g||_{L^p} \frac{1}{q^{\frac{1}{q}}} K_a(z, \overline{z})^{\frac{1}{q} - \frac{1}{2}} < C\epsilon,$$

where C is a constant. □

Next we show

Lemma 3.7: *For $F \in H^2(S_a)$, we have that*

$$\lim_{|x|\to\infty} \frac{|F(z)|}{\sqrt{K_a(z,\overline{z})}} = 0, \quad z = x + iy \in S_a,$$

holds uniformly for $|y| < a$.

***Proof*:** By Lemma 3.6, we only need to show

$$\lim_{|x|\to\infty} \frac{|F(z)|}{\sqrt{K_a(z,\overline{z})}} = 0$$

holds uniformly for $|y| \le b < a$. Since $K_a(z,\overline{z})$ is independent of x, it suffices to show that

$$\lim_{|x|\to\infty} |F(z)| = 0$$

holds uniformly for $|y| \le b$. Let $d = a - b$. From the definition of $H^2(S_a)$ we have that

$$\int_{|\eta-y|\le\frac{d}{2}} \int_{-\infty}^{\infty} |F(\xi + i\eta)|^2 \mathrm{d}\xi \,\mathrm{d}\eta < \infty.$$

Thus, for any given ϵ, there exists a large $N > 0$ such that

$$\int_{|\eta-y|\le\frac{d}{2}} \int_{|\xi|>N} |F(\xi + i\eta)|^2 \mathrm{d}\xi \,\mathrm{d}\eta < \epsilon.$$

Then using the mean value theorem, for $|x| > N + \frac{d}{2}$, we have

$$\begin{aligned} |F(x+iy)|^2 &\le \frac{4}{\pi d^2} \int_{|w-z|\le\frac{d}{2}} |F(\xi + i\eta)|^2 \mathrm{d}\xi \,\mathrm{d}\eta \\ &\le \frac{4}{\pi d^2} \int_{|\eta-y|\le\frac{d}{2}} \int_{|x-\xi|\le\frac{d}{2}} |F(\xi + i\eta)|^2 \mathrm{d}\xi \,\mathrm{d}\eta \\ &< \epsilon, \end{aligned}$$

where the last inequality is due to the fact $|x| - \frac{d}{2} \le |\xi| \le |x| + \frac{d}{2}$. □

From next section we will deal with the H^p theory.

4. General $p \in (1,\infty)$ cases via upper and lower Hardy spaces decomposition

Motivated by the $H^2(S_a)$ case, in this section we show that functions in $H^p(S_a)$, $1 < p < \infty$, can also be written as sums of functions in $H^p(\mathbb{C}^+_{-a})$ and $H^p(\mathbb{C}^-_a)$. Subsequently, we obtain that functions in $H^p(S_a)$ can be approximated by sums of rational function in, respectively, $H^p(\mathbb{C}^+_{-a})$ and $H^p(\mathbb{C}^-_a)$.

First, we show that

Lemma 4.1: *For any $a > 0, b > 0$, and $0 < p < \infty$, $H^p(\mathbb{C}^+_{-a}) \cap H^p(\mathbb{C}^-_b) = \{0\}$.*

Proof: Suppose that $G \in H^p(\mathbb{C}^+_{-a}) \cap H^p(\mathbb{C}^-_b)$. In our proof we mainly use the following property, which follows from subharmonicity of $|G|^p$, i.e.

$$|G(x+iy)| \leq \frac{C_p}{\pi |y+a|^{\frac{1}{p}}} \left(\int_{-\infty}^{\infty} |G(\xi - ia)|^p \mathrm{d}\xi \right)^{\frac{1}{p}} \tag{4.1}$$

for $y \in (-a, \infty)$, and

$$|G(x+iy)| \leq \frac{C_p}{\pi |y-b|^{\frac{1}{p}}} \left(\int_{-\infty}^{\infty} |G(\xi + ib)|^p \mathrm{d}\xi \right)^{\frac{1}{p}} \tag{4.2}$$

for $y \in (-\infty, b)$, where C_p is a constant depending on p.

The proof is based on the Fourier spectrum characterization of H^∞ functions proved in [12]. The inequality (4.1) implies that $G(x), x \in \mathbb{R}$, can be considered as the nontangential boundary limit function of a function in $H^\infty(\mathbb{C}^+_0)$. Using the result in [12], we have that $\hat{G}$ is a distribution with support in $[0, \infty)$. Similarly, $\hat{G}$ is also a distribution with support in $(-\infty, 0]$. Thus $\hat{G}$ is either 0 or a finite linear combination of the Dirac delta function and its derivatives (see e.g. [13]). However, the latter implies that $G(x)$ has to be a finite degree polynomial of x in the distribution sense, which contradicts with the fact that $G(x)$ is L^p-integrable. Thus $\hat{G} \equiv 0$, and hence $G(z) \equiv 0$.

□

Remark:

(1) In fact, we can give another proof by using Liouville's theorem. Using (4.1) and (4.2) again, we know that $G(z)$ is entire and $|G(z)|$ is bounded in the whole complex plane. Hence, $G(z)$ is a constant by Liouville's theorem. By (4.1) and (4.2) again, we have that $|G(x+iy)| \to 0$ as $|y| \to \infty$, which implies $G(z) \equiv 0$.

(2) For $p = \infty$, we consider $\widetilde{H}^\infty(\mathbb{C}^+_{-a}) = H^\infty(\mathbb{C}^+_{-a})/\mathbb{C}$ and $\widetilde{H}^\infty(\mathbb{C}^-_b) = H^\infty(\mathbb{C}^-_b)/\mathbb{C}$ instead of $H^\infty(\mathbb{C}^+_{-a})$ and $H^\infty(\mathbb{C}^-_b)$. Then we still have $\widetilde{H}^\infty(\mathbb{C}^+_{-a}) \cap \widetilde{H}^\infty(\mathbb{C}^-_b) = \{0\}$.

(3) We also note that when $1 < p < \infty$, $H^p(\mathbb{C}^+_{-a}) \cap H^p(\mathbb{C}^-_b) \subset H^p(\mathbb{C}^+_0) \cap H^p(\mathbb{C}^-_0) = \{0\}$, which is essentially implied by the properties of the Hilbert transform (see e.g. [10]). In fact, by the Plemelj formula (see e.g. [10]), we have that for $F \in H^p(\mathbb{C}^+_0)$ and $G \in H^p(\mathbb{C}^-_0)$, there exist real-valued functions $f, g \in L^p(\mathbb{R})$ such that

$$F(x) = f(x) + i\mathcal{H}(f)(x)$$

and

$$G(x) = g(x) - i\mathcal{H}(g)(x),$$

where $F(x)$ and $G(x)$ are, respectively, nontangential boundary limit functions of F and G, and $\mathcal{H}$ is the Hilbert transform. Therefore, for $F \in H^p(\mathbb{C}^+_0) \cap H^p(\mathbb{C}^-_0)$, we have $\mathcal{H}(f) = 0$. Then, combining with the fact that $\mathcal{H}^2(f) = -f$, we have $f = 0$, and hence $F = 0$.

(4) When $0 < p < 1$, the conclusion, $H^p(\mathbb{C}_{-a}^+) \cap H^p(\mathbb{C}_b^-) = \{0\}$, cannot be obtained from the inclusion in (3) since $H^p(\mathbb{C}_0^+) \cap H^p(\mathbb{C}_0^-)$ contains nonzero elements (see e.g. [14]).

Consequently, one has

Theorem 4.2: *For $F \in H^p(S_a), 1 < p < \infty$, there exist $F_1 \in H^p(\mathbb{C}_{-a}^+)$ and $F_2 \in H^p(\mathbb{C}_a^-)$ such that*

$$F(z) = F_1(z) + F_2(z),$$

and the above decomposition is unique.

Proof: Using the Cauchy integral formula in contour (e.g. a rectangular $\square_{x_0,\epsilon} = \{z = x + iy \in S_a; |x| < x_0, -a + \epsilon < y < a - \epsilon\}$ with $0 < \epsilon < a$), we have, for $F \in H^p(S_a)$ and $z \in \square_{x_0,\epsilon}$,

$$\begin{aligned} F(z) &= \frac{1}{2\pi i}\int_{\partial\square_{x_0,\epsilon}} \frac{F(\xi)}{\xi - z}\mathrm{d}\xi \\ &= \frac{1}{2\pi i}\int_{-x_0}^{x_0} \frac{F(t + i(-a+\epsilon))}{t + i(-a+\epsilon) - z}\mathrm{d}t + \frac{1}{2\pi}\int_{-a+\epsilon}^{a-\epsilon} \frac{F(x_0 + is)}{x_0 + is - z}ds \\ &- \frac{1}{2\pi i}\int_{-x_0}^{x_0} \frac{F(t + i(a-\epsilon))}{t + i(a-\epsilon) - z}\mathrm{d}t - \frac{1}{2\pi}\int_{-a+\epsilon}^{a-\epsilon} \frac{F(-x_0 + is)}{-x_0 + is - z}\mathrm{d}s \\ &= I_1 + I_2 + I_3 + I_4. \end{aligned}$$

By suitably taking $x_0 \to \infty$ and $\epsilon \to 0$, one can show that $I_2 \to 0$ and $I_4 \to 0$, and

$$I_1 \to \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{F(t - ia)}{t - ia - z}\mathrm{d}t$$

and

$$I_3 \to -\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{F(t + ia)}{t + ia - z}\mathrm{d}t.$$

For the detailed proof of the result, we refer to [7] Paley and Wiener's original proof for $p = 2$, and [15] Li and Deng's proof for $1 < p < \infty$ in a similar way.

Therefore, we have the Cauchy integral formula for $F \in H^p(S_a)$, i.e.

$$F(z) = F^+(z) + F^-(z),$$

where

$$F^+(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{F(t - ia)}{t - ia - z}\mathrm{d}t, \quad z \in S_a, \tag{4.3}$$

and

$$F^-(z) = -\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{F(t + ia)}{t + ia - z}\mathrm{d}t, \quad z \in S_a, \tag{4.4}$$

where $F(t + ia)$ and $F(t - ia)$ are the nontangential boundary limit functions of F.

Since $1 < p < \infty$, one can easily conclude that $F^+ \in H^p(\mathbb{C}^+_{-a})$ and $F^- \in H^p(\mathbb{C}^-_a)$ by the L^p-boundedness of Hardy projection (see e.g. [10]).

As in Section 3, if F has another decomposition, then one must have a function $G \in H^p(\mathbb{C}^+_{-a}) \cap H^p(\mathbb{C}^-_a)$. By Lemma 4.1, $G \equiv 0$. Thus the decomposition is unique. □

Note that for $p = 1$, the decomposition $F(z) = F^+(z) + F^-(z)$ is still valid although we cannot conclude that $F^+ \in H^1(\mathbb{C}^+_{-a})$ and $F^- \in H^1(\mathbb{C}^-_a)$.

Motivated by the H^2 case, we give the induced norm on $H^p(S_a), 1 < p < \infty$, which is defined as

$$||F||^p_{*,H^p} = ||F^+||^p_{H^p(\mathbb{C}^+_{-a})} + ||F^-||^p_{H^p(\mathbb{C}^-_a)}, \tag{4.5}$$

where F^+ and F^- are, respectively, the components of F in $H^p(\mathbb{C}^+_{-a})$ and $H^p(\mathbb{C}^-_a)$. When there is no confusion, we still denote the norms of $H^p(\mathbb{C}^+_{-a})$ and $H^p(\mathbb{C}^-_a)$ by $||\cdot||_+$ and $||\cdot||_-$ for simplicity, and accordingly, the dual pair of $H^p(\mathbb{C}^+_{-a})$ and $H^q(\mathbb{C}^+_{-a})$ by $\langle\cdot,\cdot\rangle_+$, and the dual pair of $H^p(\mathbb{C}^-_a)$ and $H^q(\mathbb{C}^-_a)$ by $\langle\cdot,\cdot\rangle_-$, where $q = \frac{p}{p-1}$.
The convergence in this induced norm gives rise to the convergence in $||\cdot||_{H^p}$ since we have, by Minkowski's inequality,

$$\begin{aligned}\sup_{|y|<a}\left(\int_{-\infty}^{\infty}|F(x+iy)|^p\right)^{\frac{1}{p}} &\leq \sup_{|y|<a}\left(\int_{-\infty}^{\infty}|F^+(x+iy)|^p\right)^{\frac{1}{p}} + \sup_{|y|<a}\left(\int_{-\infty}^{\infty}|F^-(x+iy)|^p\right)^{\frac{1}{p}}\\ &\leq \sup_{y>-a}\left(\int_{-\infty}^{\infty}|F^+(x+iy)|^p\right)^{\frac{1}{p}} + \sup_{y<a}\left(\int_{-\infty}^{\infty}|F^-(x+iy)|^p\right)^{\frac{1}{p}}.\end{aligned}$$

By Theorem 4.2, $H^p(S_a)$ can be regarded as the direct sum of $H^p(\mathbb{C}^+_{-a})$ and $H^p(\mathbb{C}^-_a)$, i.e.

$$H^p(S_a) = H^p(\mathbb{C}^+_{-a}) \oplus_p H^p(\mathbb{C}^-_a).$$

Due to the Schauder-property of the TM system we have the following [6]

Theorem 4.3 ([6]): *Suppose that $\{z_k\}$ is a sequence of points in $\mathbb{C}^+_0$. Let $\overline{span}^p$ denote the H^p closure. Then, for $F \in \overline{span}^p\{B_k\}$, we have*

$$||S_m(F)||_{H^p(\mathbb{C}^+_0)} \leq K||F||_{H^p(\mathbb{C}^+_0)},$$

and

$$\lim_{m\to\infty} ||S_m(F) - F||_{H^p(\mathbb{C}^+_0)} = 0,$$

where $S_m(F) = \sum_{k=1}^m \langle F, B_k\rangle B_k$, $\langle\cdot,\cdot\rangle$ is the dual pair of H^p and $H^q, q = \frac{p}{p-1}$, $\{B_k\}$ is the TM system associated with $\{z_k\}$ defined as (3.2), and K is a constant.

If $\{z_k\}$ satisfies the non-separable condition

$$\sum_{k=1}^{\infty} \frac{\sqrt{\Im z_k}}{1 + |z_k|^2} = \infty,$$

Theorem 4.3 implies that $\{B_k\}$ is a Schauder basis for $H^p(\mathbb{C}^+_0)$.

Consequently, we have

Theorem 4.4: *Suppose that* $\{z_k^+\}$ *and* $\{z_k^-\}$ *are, respectively, sequences of points in* $\mathbb{C}_{-a}^+$ *and* $\mathbb{C}_a^-$. *Then, for* $F \in \overline{span}^p\{B_j^+\} \oplus_p \overline{span}^p\{B_k^-\}$, *we have*

$$||S_m(F)||_{*,H^p} \leq K||F||_{*,H^p},$$

and

$$\lim_{m\to\infty} ||S_m(F) - F||_{*,H^p} = 0,$$

where $S_m(F) = S_m^+(F^+) + S_m^-(F^-) = \sum_{k=1}^m \langle F^+, B_k^+\rangle_+ B_k^+ + \sum_{k=1}^m \langle F^-, B_k^-\rangle_- B_k^-$, F^+ *and* F^- *are, respectively, the components of* F *in* $H^p(\mathbb{C}_{-a}^+)$ *and* $H^p(\mathbb{C}_a^-)$, *and* K *is a constant.*

As an application of Theorem 4.4, we give a sufficient condition on $\{z_k\} \subset S_a$ for the Szegö kernels associated with $\{z_k\}$ being a Riesz basis of the subspace spanned by those kernels. Specifically, in the following we combine Theorem 4.4 with the result of the Calerson interpolation problem. The main result in this part is stated as follows.

Assume that $\{z_k\}$ is a sequence of points in S_a satisfying the δ-uniformly separated condition (see e.g. [10]), i.e.

$$\inf_k \prod_{j\neq k, j=1}^{\infty} \left|\frac{z_k - z_j}{z_k - \overline{z}_j + 2ia}\right| > \delta > 0, \quad \text{and } \inf_k \prod_{j\neq k, j=1}^{\infty} \left|\frac{z_k - z_j}{z_k - \overline{z}_j - 2ia}\right| > \delta > 0. \tag{4.6}$$

The next theorem is analogous to the result in [16].

Theorem 4.5: *Suppose that* $\{z_k\}$ *is a sequence of points in* S_a *satisfying* (4.6). *Then, for* $F \in \overline{span}^p\{B_j^+\} \oplus_p \overline{span}^p\{B_k^-\}$, *there exist two positive constants* A *and* B *such that*

$$A||F||_{*,H^p}^p \leq \sum_{k=1}^{\infty} (a - \Im z_k)(a + \Im z_k)|F(z_k)|^p \leq B||F||_{*,H^p}^p, \tag{4.7}$$

where A *and* B *only depend on* p.

Before proving Theorem 4.5, we first give an application of it.

Corollary 4.6: *Suppose that all conditions in Theorem 4.5 are satisfied. Then, we have*

$$\begin{aligned} A||F - S_m(F)||_{*,H^p}^p &\leq \sum_{k=m+1}^{\infty} (a - \Im z_k)(a + \Im z_k)|F(z_k) - S_m(F)(z_k)|^p \\ &\leq B||F - S_m(F)||_{*,H^p}^p, \end{aligned}$$

where $S_m(F) = S_m^+(F^+) + S_m^-(F^-) = \sum_{k=1}^m \langle F^+, B_k^+\rangle_+ B_k^+ + \sum_{k=1}^m \langle F^-, B_k^-\rangle_- B_k^-$, F^+ *and* F^- *are respectively the components of* F *in* $H^p(\mathbb{C}_{-a}^+)$ *and* $H^p(\mathbb{C}_a^-)$.

Proof: We first replace F by $(F - S_m(F))$ in (4.7). We also note that

$$S_m(F)(z_k) = F(z_k), \quad k = 1, 2, \ldots, m.$$

In fact, using the process given in [1], we have

$$F^{+}(z)=\sum_{k=1}^{m}\langle F^{+},B_k^{+}\rangle_{+}B_k^{+}+F_m^{+}(z)\prod_{j=1}^{m}\frac{z-z_j}{z-\overline{z}_j+2ia}$$

and

$$F^{-}(z)=\sum_{k=1}^{m}\langle F^{-},B_k^{-}\rangle_{-}B_k^{-}+F_m^{-}(z)\prod_{j=1}^{m}\frac{z-z_j}{z-\overline{z}_j-2ia},$$

where $F_m^{+}\in H^p(\mathbb{C}_{-a}^{+})$ and $F_m^{-}\in H^p(\mathbb{C}_{a}^{-})$. Hence we have the interpolation properties

$$S_m^{+}(F^{+})(z_k)=F^{+}(z_k),\quad k=1,2,\ldots,m,$$

and

$$S_m^{-}(F^{-})(z_k)=F^{-}(z_k),\quad k=1,2,\ldots,m.$$

Using these properties, we can eliminate the first m term of $\sum_{k=1}^{\infty}(a-\Im z_k)(a+\Im z_k)|F(z_k)-S_m(F)(z_k)|^p$, and then obtain the result. □

In the following we will prove Theorem 4.5. We first need two results for preparation. The first one is the solution to Calerson's interpolation problem.

Proposition 4.7 ([17]): *Suppose that $\{z_k\}$ is a sequence of points in $\mathbb{C}_0^{+}$. In $H^p(\mathbb{C}_0^{+}), 1\leq p<\infty$, the following statements are equivalent:*

1. *$\{z_k\}$ is uniformly separated;*
2. *$\{z_k\}$ is an interpolating sequence (i.e. for any $\{(\Im z_k)^{\frac{1}{p}}w_k\}\in l^p$, there exists a function $F\in H^p(\mathbb{C}_0^{+})$ such that $F(z_k)=w_k, k=1,2,\ldots$);*
3. *$\mu(z)=\sum_{k=1}^{\infty}(\Im z_k)\delta_{z_k}(z)$ is a Carleson measure, i.e.*

$$\sum_{k=1}^{\infty}(\Im z_k)|F(z_k)|^p\leq C||F||^p_{H^p(\mathbb{C}_0^{+})},\quad \textit{all } F\in H^p(\mathbb{C}_0^{+}),$$

where C a constant that is independent of F.

One can accordingly have Proposition 4.7 in $H^p(\mathbb{C}_{-a}^{+})$ and $H^p(\mathbb{C}_{a}^{-})$.
The second result we will need is Lemma 4.8. We will prove it by using the techniques in [18, p.12] (see also [16]). Define

$$\widetilde{B}_m^{+}(z)=\alpha\prod_{j=1}^{m}\frac{z-z_j}{z-\overline{z}_j+2ia}$$

and

$$\widetilde{B}_m^{-}(z)=\beta\prod_{j=1}^{m}\frac{z-z_j}{z-\overline{z}_j-2ia},$$

where $|\alpha|=|\beta|=1$. Moreover, we let

$$\widetilde{B}_m^{+,\prime}(z_k)=\alpha\prod_{j\neq k,j=1}^{m}\frac{z_k-z_j}{z_k-\overline{z}_j+2ia}$$

and

$$\widetilde{B}_m^{-,\prime}(z_k) = \beta \prod_{j\neq k, j=1}^{m} \frac{z_k - z_j}{z_k - \overline{z}_j - 2ia}.$$

Lemma 4.8: *Suppose that $\{z_k\}$ satisfies (4.6), and $\{(a-\Im z_k)^{\frac{1}{p}}(a+\Im z_k)^{\frac{1}{p}} w_k\} \in l^p, 1 < p < \infty$. Then there exists a function $G_m \in \overline{span}^p\{B_j^+\} \oplus_p \overline{span}^p\{B_k^-\}$ defined as*

$$G_m(z) = \sum_{k=1}^{m} \frac{i(a-\Im z_k)(a+\Im z_k)}{a} \left(\frac{\widetilde{B}_m^-(z)}{(z-z_k)\widetilde{B}_m^{-,\prime}(z_k)} - \frac{\widetilde{B}_m^+(z)}{(z-z_k)\widetilde{B}_m^{+,\prime}(z_k)} \right) w_k,$$

such that

$$G_m(z_k) = w_k, k = 1, 2, \ldots, m,$$

and

$$D||G_m||_{*,H^p}^p \leq \sum_{k=1}^{\infty} (a-\Im z_k)(a+\Im z_k)|w_k|^p, \tag{4.8}$$

where D is a positive constant that is independent of m.

Proof: Obviously, $G_m \in \overline{span}^p\{B_j^+\} \oplus_p \overline{span}^p\{B_k^-\}$, and satisfies $G_m(z_k) = w_k, k = 1, 2, \ldots, m$. To complete the proof, we need to prove (4.8). Let $H(\cdot - ia) \in L^q(\mathbb{R})$. We consider G_m^+ and G_m^- as linear functionals on $L^q(\mathbb{R})$, where G_m^+ and G_m^- are, respectively, the components of G_m in $H^p(\mathbb{C}_{-a}^+)$ and $H^p(\mathbb{C}_a^-)$. Then we have

$$\begin{aligned}
&|\langle G_m^+, H\rangle_+| \\
&\quad \leq \sum_{k=1}^{m} \frac{|w_k|(a-\Im z_k)(a+\Im z_k)}{a} \left| \langle \frac{\widetilde{B}_m^+(\cdot)}{(\cdot - z_k)\widetilde{B}_m^{+,\prime}(z_k)}, H(\cdot)\rangle_+ \right| \\
&\quad \leq \sum_{k=1}^{m} \frac{|w_k|(a-\Im z_k)(a+\Im z_k)}{a\delta} \left| \int_{-\infty}^{\infty} \frac{\widetilde{B}_m^+(x-ia)\overline{H(x-ia)}}{x-ia-z_k} \mathrm{d}x \right| \\
&\quad \leq 2\pi \sum_{k=1}^{m} \frac{|w_k|(a-\Im z_k)(a+\Im z_k)}{a\delta} \left(\left| T(\widetilde{B}_m^+\overline{H})(z_k) \right| \right) \\
&\quad \leq \frac{2\pi}{a\delta} \left(\sum_{k=1}^{m} |w_k|^p (a-\Im z_k)(a+\Im z_k) \right)^{\frac{1}{p}} \left(\sum_{k=1}^{m} (a-\Im z_k)(a+\Im z_k) \left| T(\widetilde{B}_m^+\overline{H})(z_k) \right|^q \right)^{\frac{1}{q}} \\
&\quad \leq \frac{2\pi}{a\delta} \left(\sum_{k=1}^{m} |w_k|^p (a-\Im z_k)(a+\Im z_k) \right)^{\frac{1}{p}} \left((2a)^{\frac{1}{q}} C^{\frac{1}{q}} ||T(\widetilde{B}_m^+\overline{H})||_{H^q(\mathbb{C}_{-a}^+)} \right) \\
&\quad \leq \frac{2\pi}{a\delta} \left(\sum_{k=1}^{m} |w_k|^p (a-\Im z_k)(a+\Im z_k) \right)^{\frac{1}{p}} \left((2a)^{\frac{1}{q}} C^{\frac{1}{q}} E||H(\cdot - ia)||_{L^q} \right),
\end{aligned}$$

where T projects functions in $L^q(\mathbb{R})$ to functions in $H^q(\mathbb{C}_{-a}^+)$, and E is a constant from

$$||T(H)||_{H^q(\mathbb{C}_{-a}^+)} \leq E||H(\cdot - ia)||_{L^q}, H(\cdot - ia) \in L^q(\mathbb{R}), 1 < q < \infty.$$

Then

$$||G_m^+||_+^p \leq \frac{(2\pi)^p(2a)^{\frac{p}{q}}C^{\frac{p}{q}}E^p}{a^p\delta^p}\sum_{k=1}^{\infty}|w_k|^p(a-\Im z_k)(a+\Im z_k).$$

Similarly,

$$||G_m^-||_-^p \leq \frac{(2\pi)^p(2a)^{\frac{p}{q}}C^{\frac{p}{q}}E^p}{a^p\delta^p}\sum_{k=1}^{\infty}|w_k|^p(a-\Im z_k)(a+\Im z_k).$$

Therefore,

$$D||G_m||_{*,H^p}^p \leq \sum_{k=1}^{\infty}|w_k|^p(a-\Im z_k)(a+\Im z_k),$$

where $D = \frac{a^p\delta^p}{2(2\pi)^p(2a)^{\frac{p}{q}}C^{\frac{p}{q}}E^p}$. □

Proof of Theorem 4.5: The right-hand side of (4.7) follows from Proposition 4.7. In fact, if $\{z_k\}$ satisfies (4.6), then by Proposition 4.7 we have

$$\sum_{k=1}^{\infty}(a+\Im z_k)|F^+(z_k)|^p \leq C||F^+||_+^p$$

and

$$\sum_{k=1}^{\infty}(a-\Im z_k)|F^-(z_k)|^p \leq C||F^-||_-^p,$$

where F^+ and F^- are, respectively, the components of F in $H^p(\mathbb{C}_{-a}^+)$ and $H^p(\mathbb{C}_a^-)$. Hence,

$$\begin{aligned}\sum_{k=1}^{\infty}(a-\Im z_k)(a+\Im z_k)|F(z_k)|^p &\leq \sum_{k=1}^{\infty}(a-\Im z_k)(a+\Im z_k)2^{p-1}(|F_1(z_k)|^p+|F_2(z_k)|^p)\\ &\leq 2^pa\sum_{k=1}^{\infty}(a+\Im z_k)|F^+(z_k)|^p+2^pa\sum_{k=1}^{\infty}(a-\Im z_k)|F^-(z_k)|^p\\ &\leq 2^paC(||F^+||_+^p+||F^-||_-^p)\\ &= B||F||_{*,H^p}^p,\end{aligned}$$

where $B = 2^paC$.

To prove the left-hand side of (4.7), it suffices to find a function $G_m \in \overline{span}^p\{B_j^+\} \oplus_p \overline{span}^p\{B_k^-\}$ to satisfy $G_m(z_k) = F(z_k), k = 1, 2, \ldots, m$, and

$$D||G_m||_{*,H^p}^p \leq \sum_{k=1}^{\infty}(a-\Im z_k)(a+\Im z_k)|F(z_k)|^p,$$

where D is a positive constant. Using Lemma 4.8, we can exactly construct such G_m according to $\{z_k\}$. Finally, we can complete the proof by Theorem 4.3. By Theorem 4.3, we have

$$||S_m(G_m)||_{*,H^p} \leq K||G_m||_{*,H^p}.$$

We also have $S_m(G_m)(z_k) = G_m(z_k) = F(z_k), k = 1, 2, \ldots, m$. Thus

$$||S_m(F)||^p_{*,H^p} = ||S_m(G_m)||^p_{*,H^p} \leq K^p||G_m||^p_{*,H^p} \leq \frac{K^p}{D}\sum_{k=1}^{\infty}(a - \Im z_k)(a + \Im z_k)|F(z_k)|^p.$$

Using Theorem 4.3 again, we get the desired conclusion

$$A||F||^p_{*,H^p} \leq \sum_{k=1}^{\infty}(a - \Im z_k)(a + \Im z_k)|F(z_k)|^p \leq B||F||^p_{*,H^p},$$

where $A = \frac{D}{K^p}$ and $B = 2^p aC$. □

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