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Paley–Wiener-type theorem for analytic functions in tubular domains

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A B S T R A C T

Herein, a weighted version of the Paley–Wiener-type theorem for analytic functions in a tubular domain over a regular cone is obtained by using H^p space methods. Then, the classical n-dimensional Paley–Wiener theorem is generalized to a case wherein $0 < p < 2$. Finally, a version of the edge-of-the-wedge theorem is obtained as an application of the weighted theorems.

1. Introduction

1.1. Background

The Paley–Wiener theorem describes the properties of the Fourier spectrum of a function, which is the non-tangential limit of one in the classic Hardy space H^p associated with the upper half-plane $\mathbb{C}^+ = \{z = x + iy : y > 0\}$, in terms of the location of the support of its Fourier transform. When $p = 2$, it is the classical one-dimensional Paley–Wiener Theorem.

Theorem A *(Paley–Wiener). ([4,11]). $F \in H^2(\mathbb{C}^+)$ if and only if there exists a function $f \in L^2[0, \infty)$ such that $F(z) = \int_0^\infty f(t)e^{2\pi it\cdot z}dt$ for $z \in \mathbb{C}^+$.*

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In order to introduce the following results, we recall the definition of the Fourier transform. Assume that $f \in L^1(\mathbb{R}^n)$. The Fourier transform of f, denoted by $\hat{f}$, is defined as $\hat{f}(x) = F = \int_{\mathbb{R}^n} f(t)e^{-2\pi i x\cdot t}dt$ for all $x \in \mathbb{R}^n$.

In a previous study [9], Theorem A was generalized to a one-dimensional distribution case for $1 \le p \le \infty$. In the distributional case, the support of function f is denoted by d-supp$f$.

Theorem B. *$f \in H^p(\mathbb{C}^+)$, where $1 \le p \le \infty$. Then, as a tempered distribution, $\hat{f}$ is supported in $[0, \infty)$.*

In another research [10], Qian et al. proved the converse of the above theorem.

Theorem C. *For $1 \le p \le \infty$, $f \in L^p(\mathbb{R})$ and d-supp$\hat{f} \subset [0, \infty)$. Then, f is the boundary limit of a function in $H^p(\mathbb{C}^+)$.*

Higher-dimensional cases can be naturally considered. We first introduce some definitions in the n-dimensional complex Euclidean space $\mathbb{C}^n$.

We denote the elements of $\mathbb{C}^n$ by $z = (z_1, z_2, ..., z_n)$. The product of $z, w \in \mathbb{C}^n$ is $z \cdot w = z_1w_1 + z_2w_2 + \ldots + z_nw_n$. The Euclidean norm of $z \in \mathbb{C}^n$ is $|z| = \sqrt{z \cdot \bar{z}}$, where $\bar{z} = (\overline{z_1}, \overline{z_2}, ..., \overline{z_n})$.

A nonempty subset $\Gamma \subset \mathbb{R}^n$ is called an open cone, if it satisfies (i) $0 \notin \Gamma$, and (ii) whenever $x, y \in \Gamma$ and $\alpha, \beta > 0$, the expression $\alpha x + \beta y \in \Gamma$ holds.

The dual cone of Γ is expressed as $\Gamma^* = \{y \in \mathbb{R}^n : y \cdot x \ge 0, \text{ for any } x \in \Gamma\}$, which is clearly a closed convex cone with vertex at 0. Next, $(\Gamma^*)^* = \overline{\mathrm{ch}\Gamma}$, where ch$\Gamma$ is the convex hull of Γ. We say that the cone Γ is regular if the interior of its dual cone Γ^* is non-empty.

The tube T_Γ with base Γ is the set of all points $z = (z_1, z_2, ..., z_n) = (x_1 + iy_1, ..., x_n + iy_n) = x + iy \in \mathbb{C}^n$ with $y \in \Gamma$.

A function F belongs to a Hardy space $H^p(T_\Gamma)$, if it is holomorphic in T_Γ, and satisfies

$$\|F\|_{H^p} = \sup\left\{\left(\int\limits_{\mathbb{R}^n} |F(x+iy)|^p dx\right)^{\frac{1}{p}} : y \in \Gamma\right\} < \infty.$$

In Ref. [12], Stein and Weiss obtained a representation theorem that claims the above characterization for an n-dimensional case. Note that the set supp$f$ is the support of a measurable function f on $\mathbb{R}^n$, which is the closure of the set $\{x : f(x) \neq 0\}$.

Theorem D. *Suppose Γ is an open cone. Then $F \in H^2(T_\Gamma)$ if and only if $F(z) = \int_{\Gamma^*} e^{2\pi i z\cdot t} f(t)dt$, where f is a measurable function on $\mathbb{R}^n$ satisfying supp$f \subset \Gamma^*$ and $\|F\|_{H^2} = \|f\|_{L^2(\Gamma^*)} = \left(\int_{\Gamma^*} |f(t)|^2 dt\right)^{\frac{1}{2}}$.*

Related generalizations of this result were obtained. Especially, Li et al. got some characterization conclusions in Ref. [7] for $H^p(T_\Gamma)$ with the index range $1 \le p \le 2$.

Theorem E. *Assume that Γ is a regular open cone in $\mathbb{R}^n$ and $F(x) \in L^p(\mathbb{R}^n)$, where $1 \le p \le 2$. Then, F is the boundary limit function of $F(x + iy) \in H^p(T_\Gamma)$ if and only if d-supp$\hat{F} \subset \Gamma^*$.*

All the results mentioned herein are for one or higher dimensional Hardy spaces H^p, where $1 \le p \le \infty$. Since some formulas and methods are not available when $0 < p < 1$, by using some other techniques, Deng and Qian proved an analogous one-dimensional result for the case when $0 < p < 1$ in Ref. [2]. Recall that a measurable function f on $\mathbb{R}^n$ is called a slowly increasing function, if there exists a positive constant a such that $f(x)(1 + |x|)^{-a} \in L^1(\mathbb{R}^n)$.

Theorem F. *If $0 < p < 1$, $F \in H^p(\mathbb{C}^+)$, then there exists a positive constant A_p depending only on p, and a slowly increasing continuous function f, which is supported in $[0, \infty)$, such that, for φ in the Schwartz class S,*

$$(f, \varphi) = \lim_{y>0, y\to 0} \int_{\mathbb{R}} F(x + iy)\hat{\varphi}(x)dx,$$

and $|f(t)| \le A_p \|F\|_{H^p_+} |t|^{\frac{1}{p}-1}$ holds for $t \in \mathbb{R}$, and $F(z) = \int_0^\infty f(t)e^{2\pi itz}dt$ for $z \in \mathbb{C}^+$.

It is also natural to generalize this result to the higher dimensional case. Restricting the cone to be the first octant $\Gamma_{\sigma_1} = \{y = (y_1, \dots, y_n) : y_i > 0 \text{ for all } i = 1, \dots, n\}$, Li proved the following representation result.

Theorem G. *([6]). If $0 < p < 1$, $F \in H^p(T_{\Gamma_{\sigma_1}})$, then there exists a constant C_p, which is independent of F, and a slowly increasing continuous function f, whose support is in $\overline{\Gamma}_{\sigma_1}$, such that, for φ in the Schwartz class S,*

$$(f, \varphi) = \lim_{y\in\Gamma_{\sigma_1}, y\to\infty} \int_{\mathbb{R}^n} F(x + iy)\hat{\varphi}(x)dx$$

and $|f(x)| \le C_p \|F\|_{H^p} e^{nB_p} B_p^{-nB_p} \prod_{k=1}^n |x_k|^{B_p}$, and $F(z) = \int_{\overline{\Gamma}_{\sigma_1}} f(t)e^{2\pi it\cdot z}dt$, where $C_p = (\frac{\pi}{2})^{\frac{n}{p}}$, $B_p = \frac{1}{p} - 1 \ge 0$.

Some weighted versions of the Paley–Wiener theorem were considered previously, including one by Genchev in Ref. [5]. Suppose that $D = \{z \in \mathbb{C}^n, \operatorname{Im} z_j < 0, 1 \le j \le n\}$ is the last octant in $\mathbb{C}^n$ and $\sigma = (\sigma_1, \dots, \sigma_n) \in \mathbb{R}^n$ is a vector with non-negative components. Let $E_\sigma(D)$ be the set of holomorphic functions on D that satisfy $|F(z)| \le A_\varepsilon \exp\{\sum_{j=1}^n (\sigma_j + \varepsilon)|z_j|\}$ for $\varepsilon > 0$ and $z \in D$. Integral representations of functions in $E_\sigma(D)$ with boundary values $F(x) \in L^p(\mathbb{R}^n)$ are studied in [5], being separated into two cases, namely, $p \ge 2$ and $1 \le p \le 2$, corresponding to the Theorems H and I given in the sequel.

Theorem H. *([5]). Let $F(z) \in E_\sigma(D)$ have boundary values $F(x)$ and suppose that the condition*

$$\int_{\mathbb{R}^n} (1 + |x|^{n(p-2)})|F(x)|^p dx < \infty \tag{1}$$

holds, where $p \ge 2$ and $|x|^2 = \sum_{j=1}^n x_j^2$. Then F has the form

$$F(z) = \int_{-G(\sigma)} e^{2\pi iz\cdot t} f(t)dt, \tag{2}$$

where $G(\sigma) = \{t \in \mathbb{R}^n, 2\pi t_j \ge -\sigma_j, 1 \le j \le n\}$ and f is a measurable function satisfying $\operatorname{supp} f \subset -G(\sigma)$ and $f \in L^p(-G(\sigma))$.

When $1 \le p \le 2$, the following theorem was established.

Theorem I. *([5]). Suppose that $F(z) \in E_\sigma(D)$ have boundary values $F(x) \in L^p(\mathbb{R}^n)$, where $1 \le p \le 2$. Then (2) is again satisfied, with f continuous if $p = 1$ and with f measurable and satisfying the conditions $|t|^{n(1-\frac{2}{p})} f(t) \in L^p(\mathbb{R}^n)$ and $f(t) \in L^q(\mathbb{R}^n)$, where $\frac{1}{p} + \frac{1}{q} = 1$, if $p > 1$.*

These two theorems were generalized to a larger class of convex domains in $\mathbb{C}^n$. In Ref. [8], $a(z)$ is denoted as a non-negative convex function continuous in $T_{\overline{\Gamma}}$ and homogeneous of degree 1. Let $P_a(T_\Gamma)$ be the class of functions that are holomorphic in T_Γ and satisfy $|F(z)| \le c_\varepsilon e^{a(z)+\varepsilon|z|}$ for $\varepsilon > 0$, $c_\varepsilon > 0$ and $z \in T_\Gamma$. Musin obtained two results in [8] for functions in $P_a(T_\Gamma)$ with boundary values $F(x) \in L^p(\mathbb{R}^n)$. When $p \ge 2$, a representation result was stated as follows.

Theorem J. *([8]). Suppose that $F \in P_a(T_\Gamma)$ have boundary values $F(x)$ which satisfy $|x|^{n(1-\frac{2}{p})} F(x) \in L^p(\mathbb{R}^n)$ for $p \ge 2$. Then there exists $f \in L^p(\mathbb{R}^n)$ such that $F(z) = \int_{U(\tilde{a},\Gamma)} e^{2\pi i z\cdot t} f(t)dt$ holds for $z \in T_\Gamma$, where $U(\tilde{a}, \Gamma) = \{\xi \in \mathbb{R}^n : -2\pi\xi \cdot y \le \tilde{a}(y)$ for all $y \in \overline{\Gamma}\}$ and $\tilde{a}(y) = a(iy)$ for $y \in \Gamma$.*

Musin established the following result for $1 \le p < 2$.

Theorem K. *([8]). Suppose that $F \in P_a(T_\Gamma)$ have boundary values $F(x) \in L^p(\mathbb{R}^n)$ for $1 \le p < 2$. Then $F(z) = \int_{U(\tilde{a},\Gamma)} e^{2\pi i z\cdot t} f(t)dt$ holds for $z \in T_\Gamma$. For $p = 1$ we have $f \in C(\mathbb{R}^n)$, while for $p > 1$ we have supp$f \subset U(\tilde{a}, \Gamma)$, $f \in L^q(\mathbb{R}^n)$ and $|t|^{n(1-\frac{2}{p})} f(t) \in L^p(\mathbb{R}^n)$.*

1.2. Statement of main results

Herein, we consider Paley–Wiener-type theorems for functions in the weighted class defined as follows.

For the first time, we consider the following type of generalization. Let ψ be a measurable function in $\mathbb{R}^n$. A function $F(z)$ holomorphic in tube T_Γ is said to belong to space $H^p(\Gamma, \psi)$ if

$$\|F\|_{H^p(\Gamma,\psi)} = \sup\left\{ e^{-2\pi\psi(y)} \left(\int_{\mathbb{R}^n} |F(x+iy)|^p dx \right)^{\frac{1}{p}} : \ y \in \Gamma \right\} < \infty$$

for $0 < p < \infty$ and

$$\|F\|_{H^\infty(\Gamma,\psi)} = \sup\left\{ e^{-2\pi\psi(y)} |F(x+iy)| : x \in \mathbb{R}^n, y \in \Gamma \right\} < \infty$$

for $p = \infty$.

In the main results, we assume that $\psi \in C(\overline{\Gamma})$ and satisfies

$$R_\psi = \overline{\lim_{y\in\Gamma, y\to\infty}} \frac{\psi(y)}{|y|} < \infty, \tag{3}$$

and let

$$U(\psi, \Gamma) = \{\xi \in \mathbb{R}^n : \underline{\lim_{y\in\Gamma, y\to\infty}} (\psi(y) - \xi \cdot y) > -\infty\}. \tag{4}$$

Then we establish the following representation theorems.

Theorem 1. *Assume that $1 \le p \le 2$, $\frac{1}{p} + \frac{1}{q} = 1$, Γ is a regular open cone in $\mathbb{R}^n$. If $F(z) \in H^p(\Gamma, \psi)$, then there exists $f(t) \in L^q(\mathbb{R}^n)$ with supp$f \subseteq -U(\psi, \Gamma)$ such that*

$$\int_{\mathbb{R}^n} |f(t)|^q dt \le \|F\|^q_{H^p(\Gamma,\psi)} \tag{5}$$

and

$$F(z) = \int_{\mathbb{R}^n} f(t)e^{2\pi i t\cdot z} dt \tag{6}$$

hold for $z \in T_\Gamma$.

Theorem 2. *Assume that* $p > 2$, Γ *is a regular open cone in* $\mathbb{R}^n$. *If* $F(z) \in H^p(\Gamma, \psi)$ *satisfies*

$$\varliminf_{y\in\Gamma, y\to 0} \int_{\mathbb{R}^n} |F(x+iy)|^p |x|^{n(p-2)} dx < \infty, \tag{7}$$

then there exists $f(t) \in L^p(\mathbb{R}^n)$ *with* $\mathrm{supp} f \subseteq -U(\psi, \Gamma)$ *such that (6) holds for* $z \in T_\Gamma$.

Theorem 3. *Assume that* $F(z) \in H^p(\Gamma, \psi)$, *where* $0 < p < 1$ *and* Γ *is a regular open cone in* $\mathbb{R}^n$. *Then, there exists a real constant* R_ψ *defined as (3) and a slowly increasing continuous function* $f(t)$ *with* $\mathrm{supp} f \subseteq (\Gamma^* + \overline{D(0, R_\psi)})$ *such that (6) holds for* $z \in T_\Gamma$.

In the above theorems, take $\psi(y) = \frac{a(iy)}{2\pi}$, where $a(z)$ is defined as in Theorem J and Theorem K, and $a(iky) = ka(iy)$ for $y \in \Gamma$ and $k > 0$. By applying Theorem 1 and Theorem 2, we can obtain the same results as those derived from Theorem K and Theorem J. And in the case, $\mathrm{supp} f \subset -U(\frac{a(iy)}{2\pi}, \Gamma) = \{t : -2\pi t\cdot y - a(iy) \le 0\}$.

By restricting Γ to be the last octant D, we can define $\psi(y) = -\frac{\sigma\cdot y}{2\pi}$ as in Theorem H and Theorem I for $y \in D$ and $\sigma \in \mathbb{R}^n$. Theorem 1 and Theorem 2 imply the same conclusions as those by Theorem I and Theorem H. In the case, $\mathrm{supp} f \subset -U(-\frac{\sigma\cdot y}{2\pi}, \Gamma) = \{-2\pi t_j + \sigma_j \ge 0, 1 \le j \le n\}$.

In addition, for any $\psi(y)$ defined in the form of $c|y| + \phi(y)$ satisfying (3), where $c \ge 0$ and $\phi(y) = o(|y|)$ when $|y| \to \infty$, analogous integral representations hold.

On the other hand, suppose that $F \in P_a(T_\Gamma)$ with boundary value $F(x) \in L^p(\mathbb{R}^n)$ when $1 \le p < 2$. For any $\varepsilon > 0$ and $z \in T_{\Gamma\bigcup\{0\}}$, let ω be a non-negative $C^\infty(\mathbb{R}^n)$ function supported in the unit ball with $\|\omega\|_{L^1(\mathbb{R}^n)} = 1$ and $\omega_\varepsilon(t) = \varepsilon^{-n}\omega(\varepsilon^{-1}t)$, and set $F_\varepsilon(z) = \int_{\mathbb{R}^n} F(z+u)\omega_\varepsilon(u)du$. Then $F_\varepsilon(z) \in H^p(\Gamma, \frac{a(iy)}{2\pi})$, where $1 \le p \le 2$. According to Theorem 1, there exist $f_\varepsilon, f \in L^q(\mathbb{R}^n)$ such that f_ε weakly* converges to f along with $\varepsilon \to 0$ and (6) holds for F_ε with $\mathrm{supp} f_\varepsilon \subset -U(\frac{a(iy)}{2\pi}, \Gamma)$. Sending $\varepsilon \to 0$, (6) holds for F and $\mathrm{supp} f \subset -U(\frac{a(iy)}{2\pi}, \Gamma)$. Theorem J can also be obtained by applying Theorem 2 when $p > 2$. For the same reason, when $F(z) \in E_\sigma(D)$ with boundary value $F(x)$ satisfying certain conditions, Theorem H and Theorem I can be concluded as corollaries of Theorem 1 and Theorem 2. Therefore, by applying Theorem 1 and 2, Theorem H, I, J and K can be generalized to cases in which $F(z)$ satisfies $|F(z)| \le C_\varepsilon e^{\psi(z)+\varepsilon|z|}$ with boundary value $F(x) \in L^p(\mathbb{R}^n)$ for $1 \le p < 2$ and $|x|^{n(1-\frac{2}{p})}F(x) \in L^p(\mathbb{R}^n)$ for $p \ge 2$.

If we set $\psi(y) = 0$, then $\mathrm{supp} f \subset \Gamma^*$. When $1 \le p \le 2$, Theorem 1 implies Theorem B and E in the one and higher dimensional cases respectively. In the particular case $p = 2$, it reduces to the classical Paley–Wiener Theorems, which are Theorem A and D herein. When $0 < p < 1$, Theorem F and G are special cases of Theorem 3.

2. Lemmas

In order to prove Paley–Wiener-type results for holomorphic functions in tubular domains, we need the following lemmas.

Lemma 1 *([1]). Assume that a is a real number and u is subharmonic in the upper half-plane $\mathbb{C}^+$, which satisfies $\sigma = \overline{\lim}_{z\in\mathbb{C}^+,|z|\to\infty} |z|^{-1}u(z)$ and $\overline{\lim}_{z=x+iy\in\mathbb{C}^+,y\to 0} u(z) \le a$, then $u(x+iy) \le a+\sigma y$ for all $z = x+iy \in \mathbb{C}^+$.*

Proof. The proof of Lemma 1 refers to [1]. $\square$

Lemma 2. *Assume that Γ is a regular open convex cone of $\mathbb{R}^n$. Let $\psi \in C(\overline{\Gamma})$ satisfy (3). By defining $U(\psi,\Gamma)$ as in (4), we have $U(\psi,\Gamma) \subset (-\Gamma^* + \overline{D(0,R_\psi)})$.*

Proof. Assume that $\xi \in U(\psi,\Gamma)$. If $\xi \in -\Gamma^*$, it is clear that $\xi \in -\Gamma^* + \overline{D(0,R_\psi)}$. Otherwise, for $\xi \notin -\Gamma^*$, there exists $\xi_1 \in -\Gamma^*$ such that

$$|\xi-\xi_1| = \inf\{|\xi - x| : x \in \partial(-\Gamma^*)\}$$

and $\xi_1 \cdot (\xi-\xi_1) = 0$. Then, for any $\tilde{y} \in -\Gamma^*$,

$$(\tilde{y}-\xi)\cdot(\frac{\xi_1-\xi}{|\xi_1-\xi|}) \ge |\xi_1-\xi|.$$

It follows that $\tilde{y}\cdot(\xi_1-\xi) \ge 0$, which implies $\xi_1 - \xi \in (-\Gamma^*)^* = -\overline{\Gamma}$. Thus, $\xi-\xi_1 \in \overline{\Gamma}$. For any $\varepsilon > 0$, based on (3), there exists $r_\varepsilon > 0$ such that $\psi(y) \le (R_\psi+\varepsilon)|y|$ holds for $y \in \overline{\Gamma}$ with $|y| \ge r_\varepsilon$. Since $\xi \in U(\psi,\Gamma)$, according to (4), there exists A_ξ and $r_0 > r_\varepsilon$ such that $\psi(y) - \xi\cdot y \ge A_\xi$ holds for any $y \in \overline{\Gamma}$, where $|y| \ge r_0$. Letting $e_0 = \frac{\xi-\xi_1}{|\xi-\xi_1|}$, then $\xi\cdot e_0 = (\xi-\xi_1)\cdot\frac{\xi-\xi_1}{|\xi-\xi_1|} = |\xi-\xi_1|$. Set $y = \rho e_0$ with $\rho \ge r_0$, then $y \in \overline{\Gamma}$. We can observe that

$$(R_\psi+\varepsilon)\rho \ge \psi(\rho e_0) \ge A_\xi + \rho\xi\cdot e_0 = A_\xi + \rho|\xi-\xi_1|,$$

which implies that $|\xi-\xi_1| \le R_\psi + \varepsilon$ for considerably small $\varepsilon > 0$. It follows that $\xi - \xi_1 \in \overline{D(0,R_\psi)}$. Thus, $\xi = \xi-\xi_1+\xi_1 \in \overline{D(0,R_\psi)} - \Gamma^*$. Then we obtain $U(\psi,\Gamma) \subset (-\Gamma^* + \overline{D(0,R_\psi)})$. $\square$

Lemma 3 *([3]). Let $K \subset int\,\Gamma^*$ be a compact set. Then there exists a positive constant δ_K such that, for all $y \in \Gamma$ and all $u \in K$, $y\cdot u \ge \delta_K|y|$.*

Lemma 4. *Assume that $F(z) \in H^p(\Gamma,\psi)$, $0 < p < \infty$, Γ is a regular open cone in $\mathbb{R}^n$, and $\psi \in C(\Gamma\bigcup\{0\})$ satisfies (3). Then,*

$$\int\limits_{\mathbb{R}^n} |F(x+iy)|^p dx \le e^{2p\pi(|y|R_\psi+\psi(0))}\|F\|^p_{H^p(\Gamma,\psi)}. \tag{8}$$

Moreover, when $1 < p < \infty$, there exist $F_0(x) \in L^p(\mathbb{R}^n)$ and a sequence $\{y_k\}$ in Γ tending to zero as $k\to\infty$, such that

$$\lim_{k\to\infty}\int\limits_{\mathbb{R}^n} F(x+iy_k)h(x)dx = \int\limits_{\mathbb{R}^n} F_0(x)h(x)dx \tag{9}$$

holds for any $h \in L^q(\mathbb{R}^n)$.

Proof. Assume that $1 < p < \infty$, the unit ball of $L^p(\mathbb{R}^n)$ is weakly compact, which implies that, for $F \in H^p(T_\Gamma)$, there exists $F_0(x) \in L^p(\mathbb{R}^n)$ and a sequence $\{y_k\}$ in Γ tending to zero as $y_k \to 0$ such that (9) holds for any $h \in L^q(\mathbb{R}^n)$.

Next, we prove (8). Given an integer $N > 0$ and $y_0 \in \Gamma$, let $E(N) = [-N, N] \times \cdots \times [-N, N]$ be a cube in $\mathbb{R}^n$ and $y \in \Gamma$, $\{e_1, e_2, \ldots, e_n\}$ be the standard basis vectors in the Euclidean space $\mathbb{R}^n$. The function $g_{j,N}(w)$ $(j = 1, 2, \ldots)$ defined by

$$\sum_{k_1=-j}^{j} \sum_{k_2=-j}^{j} \cdots \sum_{k_n=-j}^{j} \left| F\left(\frac{N}{j}(k_1 e_1 + k_2 e_2 + \cdots + k_n e_n) + wy + iy_0 \right) \right|^p \left(\frac{N}{j} \right)^n$$

is continuous in $\overline{\mathbb{C}^+}$ and converges uniformly for w in every compact subset of $\mathbb{C}^+$ to the function

$$h_N(w) = \int_{E(N)} |F(yw + t + iy_0)|^p dt,$$

where $\overline{\mathbb{C}^+}$ is the closure of $\mathbb{C}^+$. For $k \in \mathbb{N}^n$, $y_0, y \in \Gamma$, $F\left(\frac{N}{j}k + wy + iy_0\right)$ is a holomorphic function of $w \in \mathbb{C}^+$ for fixed $y, y_0 \in \Gamma$. Thus, the function

$$\log \left(\left| F\left(\frac{N}{j}k + wy + iy_0 \right) \right|^p \frac{N^n}{j^n} \right)$$

is subharmonic in $\mathbb{C}^+$, which indicates that $\log g_{j,N}(w)$ is subharmonic in $\mathbb{C}^+$. Then, the function $\log |h_N(w)|$ is subharmonic in $\mathbb{C}^+$. For fixed $y \in \Gamma$, where $|y| > R_0$, the set $\{vy + y_0 : 0 \le v \le |y|^{-1}(R_0 + |y_0|)\}$ is compact in Γ. By the continuity of ψ in Γ, we have

$$\sup\{\frac{\psi(vy + y_0)}{|vy + y_0|} : 0 \le v \le |y|^{-1}(R_0 + |y_0|)\} < \infty.$$

Therefore,

$$\overline{\lim_{w \in \mathbb{C}^+, |w| \to \infty}} \frac{\log |h_N(w)|}{|w|} \le 2\pi p |y| R_\psi$$

and

$$|h_N(u)| \le \int_{\mathbb{R}^n} |F(x + iy_0)|^p dx.$$

Applying Lemma 1 to the subharmonic function $\log |h_N(w)|$ in $\mathbb{C}^+$, it follows that

$$\int_{E(N)} |F(yw + x + iy_0)|^p dx \le e^{2\pi p |y| R_\psi v} \int_{\mathbb{R}^n} |F(x + iy_0)|^p dx.$$

For $y \in \Gamma$, letting $w = i$ and $N \to \infty$, we observe that

$$\begin{aligned} \int_{\mathbb{R}^n} |F(x + iy + iy_0)|^p dx &\le e^{2p\pi |y| R_\psi} \int_{\mathbb{R}^n} |F(x + iy_0)|^p dx \\ &\le e^{2\pi p |y| R_\psi} e^{2p\pi \psi(y_0)} \|F\|_{H^p(\Gamma,\psi)}. \end{aligned}$$

Thus, by sending $y_0 \to 0$ and based on (3), Fatou's lemma and the continuity of ψ at 0, we obtain the desired estimate

$$\int_{\mathbb{R}^n} |F(x+iy)|^p dx \le e^{2p\pi(|y|R_\psi+\psi(0))}\|F\|^p_{H^p(\Gamma,\psi)}.$$

Consequently, (8) holds for any $y \in \Gamma$ and the proof is complete. □

3. Proof of the theorems

3.1. Proof of Theorem 1

Proof. We divide the proof of Theorem 1 into the following steps.

Step 1. Let ω be a non-negative $C^\infty(\mathbb{R}^n)$ function with compact support in the unit ball and $\|\omega\|_{L^1(\mathbb{R}^n)} = 1$. Let $\omega_\varepsilon(t) = \varepsilon^{-n}\omega(\varepsilon^{-1}t)$. Since Γ is regular, we choose $u_0 \in \Gamma^*$ and $\varepsilon_0 > 0$ such that the ball $D(u_0, 2\varepsilon_0) \subseteq \Gamma^*$. Furthermore, let $\tilde{\omega}(u) = \omega_{\varepsilon_0}(u-u_0)$, then the function $\Omega(z) = \int_{\mathbb{R}^n} e^{2\pi iz\cdot u}\tilde{\omega}(u)du$ is an entire function. For any $y \in \Gamma, x \in \mathbb{R}^n$, we have $|\Omega(x+iy)| \le 1$. Notice that the Hausdorff–Young inequality implies

$$\begin{aligned}
&\left(\int_{\mathbb{R}^n} |\Omega(\varepsilon x + i\varepsilon y) - \Omega(\varepsilon x)|^q dx\right)^{\frac{1}{q}}\\
&\le \left(\int_{\mathbb{R}^n} \left|(e^{2\pi y\cdot u} - 1)\tilde{\omega}(\frac{-u}{\varepsilon})\varepsilon^{-n}\right|^p du\right)^{\frac{1}{p}}\\
&\le (\exp\{2\pi|y|\varepsilon(|u_0|+1)\} - 1)\varepsilon^{-n+\frac{n}{p}} \left(\int_{\mathbb{R}^n} (\tilde{\omega}(-u))^p\, du\right)^{\frac{1}{p}}
\end{aligned}$$

for $y \in \Gamma$.

For $\Omega(\varepsilon z) = \int_{\mathbb{R}^n} e^{2\pi i\varepsilon z\cdot u}\tilde{\omega}(u)du$, integrating by parts and taking the derivative with respect to z under the integral, the following formula holds,

$$(-2\pi\varepsilon i)^{|\alpha|} z^\alpha D_z^\beta(\Omega(\varepsilon z)) = \int_{\mathbb{R}^n} (2\pi\varepsilon iu)^\beta e^{2\pi\varepsilon iz\cdot u} D_u^\alpha\left(\tilde{\omega}(u)\right) du,$$

wherein $\alpha = (\alpha_1, \dots, \alpha_n)$, $\beta = (\beta_1, \dots, \beta_n)$, $D_u^\alpha = D_{u_1}^{\alpha_1} \cdots D_{u_n}^{\alpha_n}$, $D_z^\beta = D_{z_1}^{\beta_1} \cdots D_{z_n}^{\beta_n}$, and $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$. This implies that, for all $\varepsilon > 0$, α, and β, there exists a constant $M_{\alpha,\beta,\varepsilon} > 0$ such that

$$|z^\alpha D_z^\beta(\Omega(\varepsilon z))| \le e^{2\pi\gamma(y)} M_{\alpha,\beta,\varepsilon} < \infty$$

for $z = x + iy \in \mathbb{C}^n$, where $\gamma(y) = \max\{-y\cdot u : |u - u_0| \le \varepsilon_0\}$. Let $K = D(u_0, 2\varepsilon)$. Then by Lemma 3, there exists a positive constant δ_K such that $\gamma(y) \le -\delta_K|y|$ for y that satisfies $\frac{y}{|y|} \in K$. Therefore, for each $N \ge |\alpha|/2$, there exists a constant $M_{N,\beta,\varepsilon} \ge 0$ such that

$$|D_z^\beta(\Omega(\varepsilon z))| \le \frac{M_{N,\beta,\varepsilon} e^{-2\pi\varepsilon\delta_K|y|}}{(1+|x|^2)^N}. \tag{10}$$

Step 2. Let $F_\varepsilon(z) = \int_{\mathbb{R}^n} F(z+u)\omega_\varepsilon(u)du$ for $z \in T_\Gamma$ and let $F_\varepsilon(x) = \int_{\mathbb{R}^n} F_0(x+u)\omega_\varepsilon(u)du$, where $F_0(x)$ is defined as in Lemma 4. It is clear that $F_\varepsilon(z)$ is holomorphic in T_Γ. Since $|F_\varepsilon(z) - F(z)| \le \max\{|F(z+\varepsilon t) - F(z)| : |t| \le 1\}$, we have

$$\lim_{\varepsilon\to 0} F_\varepsilon(z) = F(z) \tag{11}$$

uniformly on any compact subset of T_Γ. Hölder's inequality and (8) imply that

$$|F_\varepsilon(z)| \le \|F\|_{H^p(\Gamma,\psi)} e^{2\pi(R_\psi|y|+\psi(0))}\|\omega_\varepsilon\|_{L^q}. \tag{12}$$

Based on the Minkowski inequality,

$$\begin{aligned}&\left(\int\limits_{\mathbb{R}^n}\left(\int\limits_{\mathbb{R}^n}|F(x+u+iy)\omega_\varepsilon(u)|du\right)^p dx\right)^{\frac{1}{p}}\\ &\le \int\limits_{\mathbb{R}^n}\left(\int\limits_{\mathbb{R}^n}|F(x+u+iy)\omega_\varepsilon(u)|^p dx\right)^{\frac{1}{p}} du \le e^{2\pi\psi(y)}\|F\|_{H^p(\Gamma,\psi)},\end{aligned} \tag{13}$$

which implies that $F_\varepsilon(z) \in H^p(\Gamma, \psi)$. Then based on (8),

$$\begin{aligned}\left(\int\limits_{\mathbb{R}^n}|F_\varepsilon(x+iy)|^p dx\right)^{\frac{1}{p}} &\le \left(\int\limits_{\mathbb{R}^n}\left(\int\limits_{\mathbb{R}^n}|F(x+u+iy)\omega_\varepsilon(u)|du\right)^p dx\right)^{\frac{1}{p}}\\ &\le \|F\|_{H^p(\Gamma,\psi)} e^{2\pi(R_\psi|y|+\psi(0))}.\end{aligned} \tag{14}$$

Since $\omega_\varepsilon \in L^q(\mathbb{R}^n)$, it follows from (9) that

$$\lim_{k\to\infty} F_\varepsilon(x+iy_k) = \int\limits_{\mathbb{R}^n} F_0(x+u)\omega_\varepsilon(u)du = F_\varepsilon(x). \tag{15}$$

Step 3. Let

$$g_{\varepsilon,t}(y) = g_\varepsilon(t,y) = \int\limits_{\mathbb{R}^n} G_\varepsilon(u+iy)e^{2\pi i(u+iy)\cdot t}du, \tag{16}$$

where $y \in \Gamma \bigcup \{0\}$, $t \in \mathbb{R}^n$, $G_\varepsilon(z) = F_\varepsilon(z)\Omega(\varepsilon z)$. Clearly, (10) and (12) indicate that $g_\varepsilon(t,y)$ is a continuous function of $t \in \mathbb{R}^n$. We now prove that $g_{\varepsilon,t}(y)$ is a constant in Γ. Let $H_\varepsilon(u+iy) = G_\varepsilon(u+iy)e^{2\pi i(u+iy)\cdot t}$. According to the Cauchy integral formula, (10) and (12), for all $y \in \overline{D(y_0,\delta_0)} \subset \overline{\Gamma}$ and $x \in \mathbb{R}^n$, there exists a constant $M_{y_0,\delta_0,t,\varepsilon} > 0$ such that $\left|\frac{\partial}{\partial y_k}H_\varepsilon(z)\right| \le M_{y_0,\delta_0,t,\varepsilon}(1+|x|)^{-n-1}$. The Cauchy–Riemann equations imply that $\frac{\partial}{\partial y_k}H_\varepsilon(u+iy) = i\frac{\partial}{\partial u_k}H_\varepsilon(u+iy)$ for $z = u+iy$. Thus, taking the derivative with respect to y under the integral, for $k = 1, 2, \ldots, n$,

$$\frac{\partial}{\partial y_k} g_\varepsilon(t,y) = \int\limits_{\mathbb{R}^n}\frac{\partial}{\partial y_k}H_\varepsilon(u+iy)du = \int\limits_{\mathbb{R}^n} i\frac{\partial}{\partial u_k}H_\varepsilon(u+iy)du = 0. \tag{17}$$

Therefore, $g_{\varepsilon,t}(y)$ is a constant in Γ.

Step 4. Hölder's inequality and (13) imply that

$$\begin{aligned}&\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n}|F(u+x+iy)\omega_\varepsilon(u)\Omega(\varepsilon(x+iy))|dudx\\ &\le \|F\|_{H^p(\Gamma,\psi)}e^{2\pi\psi(y)}\left(\int\limits_{\mathbb{R}^n}|\Omega(\varepsilon(x+iy))|^q dx\right)^{\frac{1}{q}}.\end{aligned}$$

Using Fubini's theorem, for $y \in \Gamma$, we obtain

$$\begin{aligned} g_{\varepsilon,t}(y) &= \int\limits_{\mathbb{R}^n} F_\varepsilon(x+iy)\Omega(\varepsilon(x+iy))e^{2\pi i(x+iy)\cdot t}dx \\ &= \int\limits_{\mathbb{R}^n}\left(\int\limits_{\mathbb{R}^n} F(u+s+iy)\omega_\varepsilon(s)ds\right)\Omega(\varepsilon(u+iy))e^{2\pi i(u+iy)\cdot t}du \\ &= \int\limits_{\mathbb{R}^n} F(x+iy)h_\varepsilon(x,y,t)dx, \end{aligned}$$

where

$$h_\varepsilon(x,y,t) = \int\limits_{\mathbb{R}^n} \omega_\varepsilon(x-u)\Omega(\varepsilon(u+iy))e^{2\pi i(u+iy)\cdot t}du$$

is a continuous function of $x, y, t \in \mathbb{R}^n$. Since

$$\begin{aligned} & h_\varepsilon(x,y,t) \\ &= \int\limits_{\mathbb{R}^n} \omega_\varepsilon(x-u)\left(\int\limits_{\mathbb{R}^n} e^{2\pi i(\varepsilon u+i\varepsilon y)\cdot s}\tilde{\omega}(s)ds\right)e^{2\pi i(u+iy)\cdot t}du \\ &= e^{2\pi i(x+iy)\cdot t}\int\limits_{\mathbb{R}^n}\left(\varepsilon^{-n}\tilde{\omega}(\frac{-u}{\varepsilon})\int\limits_{\mathbb{R}^n} e^{-2\pi iv\cdot t}e^{2\pi i(v-iy)\cdot u}\omega_\varepsilon(v)dv\right)e^{-2\pi iu\cdot x}du, \end{aligned}$$

the function $h_\varepsilon(x,y,t)e^{-2\pi i(x+iy)\cdot t}$ of $x \in \mathbb{R}^n$ is the Fourier transform of the function

$$h_{\varepsilon,y,t}(u) = \varepsilon^{-n}\tilde{\omega}(\frac{-u}{\varepsilon})e^{2\pi y\cdot u}\int\limits_{\mathbb{R}^n} e^{2\pi iv\cdot(u-t)}\omega_\varepsilon(v)dv$$

and the function $h_\varepsilon(x,0,t)e^{-2\pi ix\cdot t}$ of $x \in \mathbb{R}^n$ is the Fourier transform of the function

$$h_{\varepsilon,0,t}(u) = \varepsilon^{-n}\tilde{\omega}(\frac{-u}{\varepsilon})\int\limits_{\mathbb{R}^n} e^{2\pi iv\cdot(u-t)}\omega_\varepsilon(v)dv.$$

The Hausdorff–Young inequality implies that

$$\begin{aligned} & \left(\int\limits_{\mathbb{R}^n} |h_\varepsilon(x,y,t)|^q e^{2\pi qy\cdot t}dx\right)^{\frac{1}{q}} \\ &\le \left(\int\limits_{\mathbb{R}^n}\left|e^{2\pi y\cdot u}\varepsilon^{-n}\tilde{\omega}(\frac{-u}{\varepsilon})\int\limits_{\mathbb{R}^n} e^{2\pi iv\cdot(u-t)}\omega_\varepsilon(v)dv\right|^p du\right)^{\frac{1}{p}} \\ &\le \left(\int\limits_{\mathbb{R}^n}\left|e^{-2\pi\varepsilon y\cdot u}\tilde{\omega}(u)\right|^p du\right)^{\frac{1}{p}}\varepsilon^{n(\frac{1}{p}-1)} \end{aligned} \tag{18}$$

and

$$\begin{aligned}
&\left(\int\limits_{\mathbb{R}^n} |h_\varepsilon(x,y,t)e^{2\pi y\cdot t} - h_\varepsilon(x,0,t)|^q dx \right)^{\frac{1}{q}} \\
&\leq \left(\int\limits_{\mathbb{R}^n} \left| (e^{2\pi y\cdot u} - 1)\varepsilon^{-n}\tilde{\omega}(\frac{-u}{\varepsilon}) \int\limits_{\mathbb{R}^n} e^{2\pi i v\cdot(u-t)}\omega_\varepsilon(v)dv \right|^p du \right)^{\frac{1}{p}} \\
&\leq \left(\int\limits_{\mathbb{R}^n} \left|(e^{-2\pi\varepsilon y\cdot u} - 1)\tilde{\omega}(u)\right|^p du \right)^{\frac{1}{p}} \varepsilon^{n(\frac{1}{p}-1)}.
\end{aligned} \tag{19}$$

On the other hand, letting $G_\varepsilon(u) = F_\varepsilon(u)\Omega(\varepsilon u)$ and $g_\varepsilon(t) = \int_{\mathbb{R}^n} F_\varepsilon(u)\Omega(\varepsilon u)e^{2\pi i u\cdot t}du$, we have

$$\begin{aligned}
|g_{\varepsilon,t}(y) - g_\varepsilon(t)| &= \left| \int\limits_{\mathbb{R}^n} (F(x+iy)h_\varepsilon(x,y,t) - F_0(x)h_\varepsilon(x,0,t))dx \right| \\
&\leq |I_1(\varepsilon,t,y)| + |I_2(\varepsilon,t,y)| + |I_3(\varepsilon,t,y)|,
\end{aligned}$$

where $y \in \Gamma$,

$$\begin{aligned}
I_1(\varepsilon,t,y) &= \int\limits_{\mathbb{R}^n} F(x+iy)h_\varepsilon(x,y,t)(1 - e^{2\pi y\cdot t})dx, \\
I_2(\varepsilon,t,y) &= \int\limits_{\mathbb{R}^n} F(x+iy)(h_\varepsilon(x,y,t)e^{2\pi y\cdot t} - h_\varepsilon(x,0,t))dx, \\
I_3(\varepsilon,t,y) &= \int\limits_{\mathbb{R}^n} F(x+iy)h_\varepsilon(x,0,t)dx - \int\limits_{\mathbb{R}^n} F_0(x)h_\varepsilon(x,0,t)dx.
\end{aligned}$$

Based on (9) and (18), we have $I_3(\varepsilon,t,y_k) \to 0$ as $k \to \infty$. Hölder's inequality, (18) and (19) imply that $|I_1(\varepsilon,t,y)| + |I_2(\varepsilon,t,y)| \to 0$ as $y \to 0$, where $y \in \Gamma$. We deduce from (17) that for $y \in \Gamma$, there holds:

$$g_{\varepsilon,t}(y) = g_\varepsilon(t) = \int\limits_{\mathbb{R}^n} G_\varepsilon(u+iy)e^{2\pi i(u+iy)\cdot t}du. \tag{20}$$

As a result, for all $t \in \mathbb{R}^n$, the following estimate holds:

$$|g_\varepsilon(t)| \leq \|F\|_{H^p(\Gamma,\psi)} e^{2\pi(\psi(y)-y\cdot t)} \left(\int\limits_{\mathbb{R}^n} |\Omega(\varepsilon(x+iy))|^q dx \right)^{\frac{1}{q}}. \tag{21}$$

Notice that

$$\left(\int\limits_{\mathbb{R}^n} |\Omega(\varepsilon(x+iy))|^q dx \right)^{\frac{1}{q}} \leq \left(\int\limits_{\mathbb{R}^n} |e^{-2\pi\varepsilon y\cdot u}\tilde{\omega}(u)|^p du \right)^{\frac{1}{p}} \leq \|\tilde{\omega}\|_{L^p(\mathbb{R}^n)}$$

for $y \in \Gamma$. Next, we show that $g_\varepsilon(t) = 0$ for $t \notin U(\psi, \Gamma)$. To this end, assume that $t_0 \notin (U(\Gamma, \psi))$. Then, based on (4), there is a sequence $\{y_k\}$ in Γ tending to zero as $k \to \infty$, such that $\psi(y_k) - t_0 \cdot y_k \to -\infty$. It follows from (21) that $g_\varepsilon(t_0) = 0$ for $t_0 \notin U(\Gamma, \psi)$.

Step 5. The Hausdorff–Young inequality implies that

$$\left(\int\limits_{U(\Gamma,\psi)} |g_\varepsilon(t)|^q e^{2\pi q y \cdot t} dt \right)^{\frac{1}{q}} \leq \left(\int\limits_{\mathbb{R}^n} |F_\varepsilon(u+iy)\Omega(\varepsilon(u+iy))|^p du \right)^{\frac{1}{p}}$$

for $\varepsilon > 0$, $y \in \Gamma$. Based on (10), (14), (16), Fatou's lemma and the fact that $|\Omega(\varepsilon(u+iy))| \leq 1$, for $\varepsilon > 0$, $y \in \Gamma$,

$$\|g_\varepsilon\|^q_{L^q(\mathbb{R}^n)} \leq \varliminf_{y \in \Gamma, y \to 0} \|e^{2\pi y \cdot t} g_\varepsilon\|^q_{L^q(\mathbb{R}^n)} \leq e^{2\pi\psi(0)} \|F\|^q_{H^p(\Gamma,\psi)}. \tag{22}$$

Therefore, there exists $g(t) \in L^q(\mathbb{R}^n)$ and a sequence $\{\varepsilon_k\}$ such that

$$\lim_{k \to \infty} \int\limits_{\mathbb{R}^n} g_{\varepsilon_k}(t) h(t) dt = \int\limits_{\mathbb{R}^n} g(t) h(t) dt \tag{23}$$

holds for any $h \in L^p(\mathbb{R}^n)$ as $\varepsilon_k \to 0$ along with $k \to \infty$. We rewrite (20) as

$$g_\varepsilon(t) e^{2\pi y \cdot t} = \int\limits_{\mathbb{R}^n} F_\varepsilon(u+iy)\Omega(\varepsilon(u+iy)) e^{2\pi i t \cdot u} du.$$

Then for $y \in \Gamma$, $g_{\varepsilon,t}(y) e^{2\pi y \cdot t}$ is the inverse Fourier transform of $G_\varepsilon(u+iy)$ considered as a function of u.

Recall that $g_\varepsilon(t) = 0$ for $t \notin U(\psi, \Gamma)$. For fixed $y_0 \in \Gamma$, there exists a $\delta_1 > 0$ such that $\overline{D(y_0, \delta_1)} \subseteq \Gamma$. Thus,

$$\delta_2 = \inf\{x \cdot y : x \in \Gamma^*, |x| = 1, |y - y_0| \leq \delta_1\} > 0.$$

Consequently, by Lemma 2, $U(\psi, \Gamma) \subseteq -\Gamma^* + \overline{D(0, R_\psi)}$. For $t \in U(\psi, \Gamma)$, there exist $t_1 \in -\Gamma^*$, and $t_2 \in \mathbb{R}^n$ satisfying $|t_2| \leq R_\psi$ such that $t = t_1 + t_2$. Therefore, for $y \in D(y_0, \delta_1)$,

$$\begin{aligned} t \cdot y &= t_1 \cdot y + t_2 \cdot y \leq -|t_1|\delta_2 + |t_2||y| \\ &\leq -(|t| - |t_2|)\delta_2 + R_\psi |y| \leq -|t|\delta_2 + R_\psi(\delta_2 + |y_0| + \delta_1). \end{aligned}$$

As a result,

$$|g_\varepsilon(t) e^{2\pi y \cdot t}| \leq |g_\varepsilon(t)| e^{2\pi(-|t|\delta_2 + R_\psi(\delta_2 + |y_0| + \delta_1))}.$$

Combining with (22), this implies that $g_\varepsilon(t) e^{2\pi y \cdot t} \in L^1(\mathbb{R}^n)$ for $y \in \Gamma$. Note that $F_\varepsilon(u+iy)\Omega(\varepsilon(u+iy)) \in L^1(\mathbb{R}^n)$ for $y \in \Gamma$, we then have the following inverse Fourier transform:

$$F_\varepsilon(x+iy)\Omega(\varepsilon(x+iy)) = \int\limits_{\mathbb{R}^n} g_\varepsilon(t) e^{-2\pi i t \cdot (x+iy)} dt, \tag{24}$$

which is holomorphic in T_Γ since $g_\varepsilon(t) e^{2\pi y \cdot t} \in L^1(\mathbb{R}^n)$. Let $\chi(t)$ be the characteristic function of set $U(\psi, \Gamma)$. Then, the function $\chi(t) e^{-2\pi i (x+iy) \cdot t}$ belongs to $L^p(\mathbb{R}^n)$. According to (23),

$$\lim_{k\to\infty} F_{\varepsilon_k}(z)\Omega(\varepsilon_k z) = \lim_{k\to\infty} \int\limits_{\mathbb{R}^n} g_{\varepsilon_k}(t)\chi(t)e^{-2\pi it\cdot z}dt = \int\limits_{\mathbb{R}^n} g(t)\chi(t)e^{-2\pi it\cdot z}dt. \tag{25}$$

Sending ε to zero, we have $\Omega(\varepsilon z) \to 1$ for $z \in T_\Gamma$. Consequently, based on (11) and (25),

$$F(z) = \int\limits_{\mathbb{R}^n} g(t)\chi(t)e^{-2\pi it\cdot z}dt. \tag{26}$$

We see that (6) holds by letting $f(t) = g(-t)$ and $\mathrm{supp} f \subseteq -U(\psi,\Gamma)$. The proof of Theorem 1 is complete. $\square$

3.2. Proof of Theorem 2

Proof. Following the proof of Theorem 1, we have

$$g_\varepsilon(t) = \int\limits_{\mathbb{R}^n} G_\varepsilon(u+iy)e^{2\pi i(u+iy)\cdot t}du \tag{27}$$

for $y \in \Gamma$, $t \in \mathbb{R}^n$. And $g_\varepsilon(t) = 0$ for $t \notin U(\psi,\Gamma)$.

The Hardy–Littlewood inequality ([13]), (7), and (27) indicate that there exists a constant c_p such that

$$\int\limits_{\mathbb{R}^n} |e^{2\pi y\cdot t}g_\varepsilon(x)|^p dx \le c_p \int\limits_{\mathbb{R}^n} |G_\varepsilon(x+iy)|^p |x|^{n(p-2)}dx,$$

where $G_\varepsilon(x) = F_\varepsilon(x)\Omega(\varepsilon x)$. Based on Hölder's inequality,

$$|F_\varepsilon(x+iy)|^p \le \Bigg(\int\limits_{D(0,1)} |F(x+\varepsilon t+iy)|^p dt\Bigg)\Bigg(\int\limits_{D(0,1)} |\omega(t)|^q dt\Bigg)^{\frac{p}{q}}.$$

It follows from Fatou's lemma and (10) that

$$\begin{aligned}
&\int\limits_{\mathbb{R}^n} |g_\varepsilon(x)|^p dx \\
&\le \varliminf_{y\in\Gamma, y\to 0} c_p\|\omega\|^p_{L^p(\mathbb{R}^n)} \int\limits_{\mathbb{R}^n} \Bigg(\int\limits_{D(0,1)} |F(x+\varepsilon t+iy)|^p dt\Bigg) |\Omega(\varepsilon(x+iy))|^p |x|^{n(p-2)}dx \\
&\le c_p\|\omega\|^p_{L^p(\mathbb{R}^n)} M_{0,0,\varepsilon} \varliminf_{y\in\Gamma, y\to 0} \int\limits_{\mathbb{R}^n} |F(x+iy)|^p \Bigg(\int\limits_{D(0,1)} |x-\varepsilon t|^{n(p-2)}dt\Bigg)dx \\
&\le C \varliminf_{y\in\Gamma, y\to 0} \int\limits_{\mathbb{R}^n} (1+|x|^{n(p-2)})|F(x+iy)|^p dx < \infty,
\end{aligned}$$

where $C = c_p\|\omega\|^p_{L^p(\mathbb{R}^n)} M_{0,0,\varepsilon} 2^{n(p-2)-1} V_n$ with $0 < \varepsilon < 1$, and V_n is the volume of an n-dimensional ball in $\Gamma^* \subset \mathbb{R}^n$. Therefore, there exists $g(t) \in L^p(\mathbb{R}^n)$ and a sequence of $\{\varepsilon_k\}$ such that

$$\lim_{k\to\infty} \int\limits_{\mathbb{R}^n} g_{\varepsilon_k}(t)h(t)dx = \int\limits_{\mathbb{R}^n} g(t)h(t)dx \tag{28}$$

holds for any $h \in L^q(\mathbb{R}^n)$ as $\varepsilon_k \to 0$ along with $k \to \infty$. We rewrite (27) as

$$g_\varepsilon(t)e^{2\pi y\cdot t} = \int\limits_{\mathbb{R}^n} F_\varepsilon(u+iy)\Omega(\varepsilon(u+iy))e^{2\pi it\cdot u}du.$$

Note that $g_\varepsilon(t)e^{2\pi y\cdot t} \in L^1(\mathbb{R}^n)$ for $y \in \Gamma$, which can be certified by the same way as that of Theorem 1. The inverse Fourier transform formula is given as

$$F_\varepsilon(x+iy)\Omega(\varepsilon x + i\varepsilon y) = \int\limits_{\mathbb{R}^n} g_\varepsilon(t)e^{-2\pi it\cdot(x+iy)}dt.$$

Then, for a sequence of $\{\varepsilon_k\}$ tending to zero as $k \to \infty$,

$$\lim_{k\to\infty} F_{\varepsilon_k}(z)\Omega(\varepsilon z) = \lim_{k\to\infty} \int\limits_{\mathbb{R}^n} g_{\varepsilon_k}(t)\chi(t)e^{-2\pi it\cdot z}dt = \int\limits_{\mathbb{R}^n} g(t)\chi(t)e^{-2\pi it\cdot z}dt,$$

where $\chi(t)$ is the characteristic function of set $U(\psi,\Gamma)$. As a result, $F(z) = \int_{U(\psi,\Gamma)} g(t)e^{-2\pi it\cdot z}dt$. We can see that (6) holds by letting $f(t) = g(-t)$, and $\text{supp} f \subseteq -U(\Gamma,\psi)$. □

3.3. Proof of Theorem 3

Proof. For momentarily fixed $y_0 \in \Gamma$, let $F_{y_0}(z) = F(z+iy_0)$. Then F_{y_0} is holomorphic in T_Γ. Let $r = d(y_0,\partial\Gamma) = \inf\{|y_0 - y| : y_0 \in \Gamma, y \in \partial\Gamma\}$ and $\delta = \delta_{y_0} = r/2$. It follows from the subharmonic property of function $|F_{y_0}(z)|^p$ and Lemma 4 that

$$\begin{aligned}|F_{y_0}(z)|^p &\le \frac{1}{\Omega_{2n}\delta^{2n}} \int\limits_{|\eta|\le\delta} \left(\left(\int\limits_{\mathbb{R}^n} |F(\tau + i(y+y_0+\eta))|^p d\tau\right)^{\frac{1}{p}}\right)^p d\eta \\ &\le \frac{1}{\Omega_{2n}\delta^{2n}} \int\limits_{|\eta|\le\delta} \left(e^{2\pi\psi(y+y_0+\eta)}\|F\|_{H^p(\Gamma,\psi)}\right)^p d\eta \le C_{n,p,\delta}e^{2p\pi\psi_{y_0}(y)},\end{aligned}$$

where $\psi_{y_0}(y) = \sup\{\psi(y+y_0+\eta) : |\eta| \le \delta\}$, $C_{n,p,\delta} = \frac{\Omega_n}{\Omega_{2n}\delta^n}\|F\|^p_{H^p(\Gamma,\psi)}$ and Ω_m is the volume of the unit ball in $\mathbb{R}^m$. Therefore,

$$\begin{aligned}\int\limits_{\mathbb{R}^n} |F_{y_0}(x+iy)|^2 dx &\le C^{\frac{2-p}{p}}_{n,p,\delta} e^{2(2-p)\pi\psi_{y_0}(y)} \int\limits_{\mathbb{R}^n} |F_{y_0}(x+iy)|^p dx \\ &\le C^{\frac{2-p}{p}}_{n,p,\delta}\|F_{y_0}\|^p_{H^p(\Gamma,\psi)}e^{4\pi\psi_{y_0}(y)},\end{aligned}$$

which implies that $F_{y_0} \in H^2(\Gamma,\psi_{y_0})$. Similarly, we have

$$\int\limits_{\mathbb{R}^n} |F_{y_0}(x+iy)|dx \le C^{\frac{1-p}{p}}_{n,p,\delta}\|F_{y_0}\|^p_{H^p(\Gamma,\psi)}e^{2\pi\psi_{y_0}(y)}. \tag{29}$$

Thus, $F_{y_0} \in H^1(\Gamma,\psi_{y_0}) \cap H^2(\Gamma,\psi_{y_0})$. Let $g_y(t)$ be the inverse Fourier transform of $F_y(x)$. Applying Theorem 1 to $F_{y_0}(z)$, we obtain

$$g_{y_0}(t)e^{-2\pi y_0\cdot t} = g_{y+y_0}(t)e^{-2\pi(y+y_0)\cdot t}$$

for $y, y_0 \in \Gamma$, which shows that $g_y(t)e^{-2\pi y\cdot t}$ is independent of y. We denote it by $g(t)$. Then

$$g(t) = g_y(t)e^{-2\pi y\cdot t} \tag{30}$$

is continuous on $\mathbb{R}^n$. It follows that $F(z) = \int_{\mathbb{R}^n} g(t)e^{-2\pi iz\cdot t}dt$ for $z \in T_\Gamma$. Combining with (29), we obtain

$$|g(t)| = |g_{y_0+y}(t)e^{-2\pi(y_0+y)\cdot t}| \leq \widetilde{C}_{n,p}\exp\{J(y_0, y, t)\},$$

where $\widetilde{C}_{n,p} = (\frac{\Omega_n}{\Omega_{2n}})^{\frac{1-p}{p}}\|F_{y_0}\|_{H^p(\Gamma,\psi)}$ and

$$J(y_0, y, t) = -n(\frac{1}{p} - 1)\log\delta_{y_0} - 2\pi(y_0 + y)\cdot t + 2\pi\psi_{y_0}(y)$$

for $z \in T_\Gamma$. We can now prove supp$g(t) \subset U(\psi_{y_0}, \Gamma)$. To this end, we show that $g(t) = 0$ for $t \notin U(\psi_{y_0}, \Gamma)$. In fact, when $t \notin U(\psi_{y_0}, \Gamma)$, based on (4), there is a sequence $\{y_k\}$ in Γ tending to zero as $k \to \infty$, such that $\psi(y_k) - t\cdot y_k \to -\infty$. Then $g(t) = 0$ for $t \notin U(\psi_{y_0}, \Gamma)$. Letting $f(t) = g(-t)$, the representation

$$F(z) = \int\limits_{\mathbb{R}^n} f(t)e^{2\pi iz\cdot t} \tag{31}$$

holds and supp$f \subseteq -U(\psi_{y_0}, \Gamma)$. According to Lemma 2, $-U(\psi_{y_0}, \Gamma) \subset (\Gamma^* + \overline{D(0, R_{\psi_{y_0}})})$. Since $R_{\psi_{y_0}} = R_\psi$ for any fixed $y_0 \in \Gamma$, we see that $-U(\psi_{y_0}, \Gamma)$ is also a subset of $\Gamma^* + \overline{D(0, R_\psi)}$. Hence, supp$f \subseteq (\Gamma^* + \overline{D(0, R_\psi)})$.

Next, we prove that $f(t)$ is a slowly increasing function on $\Gamma^* + \overline{D(0, R_\psi)}$. Let

$$J(t) = \inf\{J(y_0, y, t) : y_0 \in \Gamma, y \in \overline{\Gamma}\},$$

then $|f(t)| = |g(-t)| \leq \widetilde{C}_{n,p}\exp\{J(-t)\}$. The fact that $\psi \in C(\overline{\Gamma})$ and (3) indicate that there exists a positive constant $A > R_\psi$, which is independent of y_0, y, such that $\psi_{y_0}(y) \leq A(1+|y_0|+|y|)$ for any $y_0, y \in \Gamma$. Taking $y_0 = \rho v$ with $\rho > 0$ and a fixed $v \in \Gamma$ with $|v| = 1$, we have $\delta_{y_0} = d(\rho v, \partial\Gamma)/2 = \rho\varepsilon$, where $\varepsilon = d(v, \partial\Gamma)/2$. Thus,

$$J(-t) = \inf_{\rho>0}\{-n(\frac{1}{p} - 1)\log(\varepsilon\rho) + 2\pi\rho|t| + 2\pi A(1 + \rho)\}.$$

The above infimum can be attained when $\rho = n(\frac{1}{p} - 1)(2\pi(|t| + A))^{-1}$. Then

$$J(-t) \leq 2\pi A + n(\frac{1}{p} - 1)\left(-\log\varepsilon - \log(n(\frac{1}{p} - 1)) + 1 + \log(2\pi(A + |t|))\right).$$

Hence, for $t \in \Gamma^* + \overline{D(0, R_\psi)}$, there exists a positive $A_{n,p,v}$ such that

$$|f(t)| \leq \widetilde{C}_{n,p}e^{J(-t)} \leq A_{n,p,v}(1 + |t|)^{n(\frac{1}{p}-1)},$$

which shows that f is a slowly increasing function. Thus, the proof is complete. □

4. Application

Let K be a compact subset of $\mathbb{R}^n$, we denote the support function of K by $\varphi_K(y)$, which is defined as $\varphi_K(y) = \sup\{x \cdot y : x \in K\}$. It is convex and continuous on $\mathbb{R}^n$ and satisfies condition (3). For any $s \geq 0, y \in \mathbb{R}^n$, $\varphi_K(sy) = s\varphi_K(y)$. We define the polar set of K as $K^* = \{y \in \mathbb{R}^n : \varphi_K(y) \leq 1\}$.

If K is convex, closed and $0 \in K$, then $K^{**} = (K^*)^* = K$ ([12], Chapter 3, Lemma 4.7). Moreover, $\varphi_{K^*}(x) = \sup\{x \cdot y : y \in K^*\}$ and $\varphi_{K^{**}}(x) = \varphi_K(x)$.

For all $z \in \mathbb{C}^n$, define $\tilde{\varphi}_K(z) = \sup\{|z \cdot t| : t \in K\}$. An entire function F on $\mathbb{C}^n$ is of exponential type K^*, where K is compact, if for each $\varepsilon > 0$ there exists a constant A_ε such that

$$|F(z)| \leq A_\varepsilon e^{2\pi(1+\varepsilon)\tilde{\varphi}_K(z)} \tag{32}$$

for all $z \in \mathbb{C}^n$.

If K is convex, compact and symmetric (that is, $x \in K$ implies $-x \in K$), and it has a non-empty interior, it is called a symmetric body. The class of entire functions satisfying (32) is denoted by $\mathscr{E}(K^*)$ ([12]).

Theorem L *(Paley–Wiener in $\mathbb{C}^n$). ([12], Chapter 3, Theorem 4.9). Suppose K is a symmetric body and $F \in L^2(\mathbb{R}^n)$. Then F is the Fourier transform of a function, $f \in L^2(K)$, vanishing outside K if and only if F is the restriction to $\mathbb{R}^n$ of a function in $\mathscr{E}(K^*)$.*

We will generalize the Paley–Wiener theorem for band-limited functions defined in $\mathbb{C}^n$ to the case when $0 < p \leq 2$. We first introduce the following lemmas.

Lemma 5. *Assume that $0 < p < \infty$, K is compact and symmetric, Γ is a regular open cone in $\mathbb{R}^n$, $F(z)$ is holomorphic in the tube T_Γ and continuous in the closed tube $T_{\overline{\Gamma}}$. For each $\varepsilon > 0$, if there exists a constant A_ε such that (32) holds for all $z \in T_\Gamma$ and $F \in L^p(\mathbb{R}^n)$, then*

$$\int_{\mathbb{R}^n} |F(x+iy)|^p dx \leq e^{2\pi p\varphi_K(y)} \int_{\mathbb{R}^n} |F(x)|^p dx \tag{33}$$

for all $y \in \Gamma$.

Proof. The proof is similar to that of Lemma 4. Given an integer $N > 0$, let $E(N) = [-N, N] \times \cdots \times [-N, N]$ be a cube in $\mathbb{R}^n$ and $b \in \Gamma$. Then, function $g_{j,N}(w)(j = 1, 2, \cdots)$ defined by

$$\sum_{k_1=-j}^{j} \sum_{k_2=-j}^{j} \cdots \sum_{k_n=-j}^{j} \Big|F\Big(\frac{N}{j}(k_1e_1 + k_2e_2 + \ldots + k_ne_n) + wb\Big)\Big|^p \Big(\frac{N}{j}\Big)^n$$

is continuous in $\overline{\mathbb{C}^+}$ and converges uniformly for w in every compact subset of $\mathbb{C}^+$ to the function

$$h_N(w) = \int_{E(N)} |F(bw + t)|^p dt.$$

For fixed $N > 1, j > 1, b \in \Gamma$, and $k_1, k_2, \ldots, k_n \in \mathbb{N}$, the function

$$\log\left(\Big|F\Big(\frac{N}{j}(k_1e_1 + k_2e_2 + \cdots + k_ne_n) + a + wb\Big)\Big|^p \frac{N^n}{j^n}\right)$$

is subharmonic in $\mathbb{C}^+$, which implies that $\log g_{j,N}(w)$ is subharmonic in $\mathbb{C}^+$. Hence function $\log|h_N(w)|$ is subharmonic in $\mathbb{C}^+$ and satisfies

$$\overline{\lim_{w\in\mathbb{C}^+,|w|\to\infty}}\frac{\log|h_N(w)|}{|w|}\leqslant 2\pi p\varphi_K(b)$$

and

$$|h_N(u)|\leqslant\int\limits_{\mathbb{R}^n}|F(x)|^p dx.$$

Applying Lemma 1 to the subharmonic function $\log|h_N(w)|$ in $\mathbb{C}^+$, there holds

$$\int\limits_{E(N)}|F(bw+x)|^p dx\leqslant e^{2\pi p\varphi_K(b)v}\int\limits_{\mathbb{R}^n}|F(x)|^p dx.$$

Since φ_K is homogeneous of degree one, letting $w=vi$, $y=vb\in\Gamma$, and sending $N\to\infty$, we obtain the desired estimate (33) for $y\in\Gamma$ and the proof is complete. $\square$

The following lemma shows that inequality (33) holds for any $y\in\mathbb{R}^n$.

Lemma 6. *Assume that $0<p<\infty$, K is compact and symmetric, F is an entire function in $\mathbb{C}^n$ such that $F\in L^p(\mathbb{R}^n)$. If $F\in\mathscr{E}(K^*)$, then (33) holds for any $y\in\mathbb{R}^n$.*

Proof. Note that $\mathbb{R}^n$ can be decomposed into a finite union of non-overlapping convex regular cones, $\Gamma_1,\Gamma_2,\dots,\Gamma_k$, with vertexes at the origin 0. Based on Lemma 5, (33) holds for any $y\in\mathbb{R}^n=\bigcup_{j=1}^{k}\Gamma_j$. Then the desired formula can be proved. $\square$

We can now state an n-dimensional version of the Paley–Wiener theorem for $0<p\le 2$:

Theorem 4. *Assume that $0<p\le 2$, K is a symmetric body, $F\in L^p(\mathbb{R}^n)$. Then F is the Fourier transform of a function $f\in L^1(\mathbb{R}^n)$ when $0<p<2$ and $f\in L^2(\mathbb{R}^n)$ when $p=2$, vanishing outside K if and only if F is the restriction to $\mathbb{R}^n$ of a function in $\mathscr{E}(K^*)$.*

Proof. If F is the Fourier transform of function $f\in L^1(\mathbb{R}^n)$ vanishing outside K, then it is easy to check that

$$F(z)=\int\limits_{\mathbb{R}^n}e^{-2\pi iz\cdot t}f(t)dt=\int\limits_{K}e^{-2\pi ix\cdot t}e^{2\pi y\cdot t}f(t)dt$$

extends F to a function in $\mathscr{E}(K^*)$.

The converse can be deduced from Lemma 6 and Theorem 1. Assume that F is the restriction to $\mathbb{R}^n$ of a function in $\mathscr{E}(K^*)$. For simplicity, we also denote the latter by F. Based on Lemma 6, (33) holds for all $y\in\mathbb{R}^n$. The subharmonic property of function $|F(z)|^p$ and Lemma 6 imply that

$$\begin{aligned}|F(z)|^p&\le\frac{1}{\Omega_{2n}}\int\int\limits_{|\tau+i\eta|\le 1}|F(z+\tau+i\eta)|^p d\tau d\eta\\&\le\frac{1}{\Omega_{2n}}\int\limits_{D_n(0,1)}d\eta\int\limits_{\mathbb{R}^n}|F(z+\tau+i\eta)|^p d\tau\le C_n^p e^{2p\pi|y|R_1}\int\limits_{\mathbb{R}^n}|F(x)|^p dx,\end{aligned}$$

where Ω_n is the volume of the unit ball $D_n(0,1)$ in $\mathbb{R}^n$, Ω_{2n} is the volume of the unit ball $D_{2n}(0,1)$ in $\mathbb{C}^n$, $R_1=\max\{\varphi_K(y):|y|=1\}$ and $C_n^p=\Omega_n e^{2p\pi R_1}\Omega_{2n}^{-1}$. Therefore,

$$
\begin{aligned}
\int\limits_{\mathbb{R}^n} |F(x+iy)|^2 dx &= \int\limits_{\mathbb{R}^n} |F(x+iy)|^{p+2-p} dx \\
&\leq e^{2(2-p)\pi|y|R_1} C_n^{2-p} \Big(\int\limits_{\mathbb{R}^n} |F(x)|^p dx \Big)^{\frac{2-p}{p}} \int\limits_{\mathbb{R}^n} |F(x+iy)|^p dx \\
&\leq e^{4\pi|y|R_1} C_n^{2-p} \Big(\int\limits_{\mathbb{R}^n} |F(x)|^p dx \Big)^{\frac{2}{p}}.
\end{aligned}
$$

By Lemma 6, we then have $F \in H^2(T_B)$ for all bounded bases B. Thus, there exists g such that

$$
F(z) = \int\limits_{\mathbb{R}^n} e^{2\pi i z\cdot t} g(t) dt, \quad \int\limits_{\mathbb{R}^n} |F(x+iy)|^2 dx = \int\limits_{\mathbb{R}^n} |g(t)|^2 e^{-4\pi y\cdot t} dt
$$

for all $z = x + iy \in T_B$ ([12], Chapter 3, Theorem 2.3). We can assume $0 \in B$, then Plancherel's theorem asserts that $g \in L^2(\mathbb{R}^n)$ and $\|g\|_2^2 = \int_{\mathbb{R}^n} |F(x)|^2 dx$. Thus, we see that $F(x) = \hat{f}(t)$ is the Fourier transform of $f(t) = g(-t)$. Based on Lemma 6,

$$
\int\limits_{\mathbb{R}^n} |f(t)|^2 e^{4\pi y\cdot t} dt \leq e^{4\pi\varphi_K(y)} \int\limits_{\mathbb{R}^n} |F(x)|^2 dx. \tag{34}
$$

By using the same method as in the end of the proof of Theorem 4.9 of Chapter 3 in [12], we can prove that the inequality (34) holds only when f vanishes almost everywhere outside K. Then Theorem 4 can be stated. □

The following three theorems are versions of the edge-of-the-wedge theorem (see in [14–16]). First, we introduce some definitions:

Let Γ be a regular open cone in $\mathbb{R}^n$. We denote by $\mathfrak{A}(\Gamma)$ the space of functions $\psi \in C(\Gamma)$, which satisfy

$$
\underline{\lim}_{y\in\Gamma, y\to 0} \psi(y) < \infty \quad \text{and} \quad R(\psi, \Gamma) = \overline{\lim}_{y\in\Gamma, |y|\to\infty} \frac{\psi(y)}{|y|} < \infty.
$$

Theorem 5. *Assume that Γ is a regular open cone in $\mathbb{R}^n$, $\psi_1 \in \mathfrak{A}(\Gamma)$, and $\psi_2 \in \mathfrak{A}(-\Gamma)$. If $F_1 \in H^{p_1}(\Gamma, \psi_1)$ and $F_2 \in H^{p_2}(-\Gamma, \psi_2)$ $(1 \leq p_1, p_2 \leq 2)$ satisfy*

$$
\underline{\lim}_{y\in\Gamma, y\to 0} \int\limits_{\mathbb{R}^n} |F_1(x+iy) - F_2(x-iy)|^2 dx = 0, \tag{35}
$$

then F_1 and F_2 can be analytically extended to each other and further form an entire function $F \in \mathscr{E}(K^*)$. Furthermore, there exists a measurable function $f(t) \in L^2(\mathbb{R}^n)$ with $\mathrm{supp} f \subseteq K = (-U(\psi_1, \Gamma)) \bigcap (-U(\psi_2, -\Gamma))$, such that

$$
F(z) = \int\limits_K f(t) e^{2\pi i t\cdot z} dt
$$

holds for $z \in \mathbb{C}^n$.

Proof. Theorem 1 implies that there exists a measurable function $f_j \in L^{q_j}(\mathbb{R}^n)$ $(j = 1, 2)$ with $\mathrm{supp} f_j \subseteq -U(\psi_j, (-1)^{j-1}\Gamma)$ such that

$$F_j(z) = \int\limits_{\mathbb{R}^n} f_j(t) e^{2\pi i t\cdot z} dt$$

holds for $z \in T_{(-1)^{j-1}\Gamma}$. Plancherel's theorem and (35) imply that

$$\int\limits_{\mathbb{R}^n} |f_1(t)e^{2\pi t\cdot y} - f_2(t)e^{-2\pi t\cdot y}|^2 dt = \int\limits_{\mathbb{R}^n} |F_1(x+iy) - F_2(x-iy)|^2 dx.$$

The finiteness of the right hand side can be deduced from (26). Fatou's lemma implies that $\|f_1 - f_2\|_{L^2(\mathbb{R}^n)} = 0$, and hence $f_1(t) = f_2(t)$ almost everywhere on $\mathbb{R}^n$. If we let $f(t) = f_1(t) = f_2(t)$, then $\mathrm{supp} f \subseteq K$. Let $R = \max\{R(\psi_1, \Gamma), R(\psi_2, -\Gamma)\}$. Then, Lemma 2 implies that

$$K \subseteq (\Gamma^* + \overline{D(0,R)}) \cap (-\Gamma^* + \overline{D(0,R)}).$$

Thus, set K is a bounded convex set. Consequently,

$$F(z) = \int\limits_{K} e^{2\pi i z\cdot t} f(t) dt$$

is an entire function, where $F(z) = F_1(z)$ for $z \in T_\Gamma$ and $F(z) = F_2(z)$ for $z \in T_{-\Gamma}$. Moreover,

$$|F(z)| \le C_0 \exp\{2\pi \varphi_K(\mathrm{Im} z)\} \le C_0 e^{2\pi R_0 |y|}, \quad z \in \mathbb{C}^n,$$

where $\varphi_K(y)$ is the support function of the convex set K and

$$C_0 = \int\limits_{K} |f(t)| dt, \quad R_0 = \sup\{\frac{\varphi_K(b)}{|b|} : b \in \mathbb{R}^n, b \neq 0\}.$$

The proof is complete. $\square$

Applying the same method as for cases $p > 2$ and $0 < p < 1$, we can obtain the following two theorems. Therefore. The proofs are omitted.

Theorem 6. *Assume that Γ is a regular open cone in $\mathbb{R}^n$, $p_1, p_2 > 2$, $\psi_1 \in \mathfrak{A}(\Gamma)$, and $\psi_2 \in \mathfrak{A}(-\Gamma)$. If $F_1 \in H^{p_1}(\Gamma, \psi_1)$ and $F_2 \in H^{p_2}(-\Gamma, \psi_2)$ satisfy the conditions of Theorem 2 and (35) holds on $\mathbb{R}^n$, then F_1 and F_2 can be analytically extended to each other and further form an entire function $F \in \mathscr{E}(K^*)$. Furthermore, there exists a measurable function $f(t) \in L^2(\mathbb{R}^n)$, which is the Fourier transform of $F(x)$ with $\mathrm{supp} f \subseteq K = (-U(\psi_1, \Gamma)) \bigcap (-U(\psi_2, -\Gamma))$, such that the representation*

$$F(z) = \int\limits_{K} f(t) e^{2\pi i t\cdot z} dt$$

holds for $z \in \mathbb{C}^n$.

Theorem 7. *Assume that Γ is a regular open cone in $\mathbb{R}^n$, $0 < p_1, p_2 < 1$, $\psi_1 \in \mathfrak{A}(\Gamma)$, and $\psi_2 \in \mathfrak{A}(-\Gamma)$. If $F_1 \in H^{p_1}(\Gamma, \psi_1)$ and $F_2 \in H^{p_2}(-\Gamma, \psi_2)$ satisfy conditions of Theorem 3 and (35) holds almost everywhere on $\mathbb{R}^n$, then F_1 and F_2 can be analytically extended to each other and further form an entire function $F \in \mathscr{E}(K^*)$. Moreover, there exists a slowly increasing continuous function $f(t) \in L^2(\mathbb{R}^n)$, which is the Fourier transform of $F(x)$ with $\mathrm{supp} f \subseteq K = (\Gamma^* + \overline{D(0,R)}) \bigcap (-\Gamma^* + \overline{D(0,R)})$, such that the representation*

$$F(z) = \int_K f(t) e^{2\pi i t \cdot z} dt$$

holds for $z \in \mathbb{C}^n$.

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