Complex Analysis and Operator Theory
https://doi.org/10.1007/s11785-019-00928-9

Complex Analysis and Operator Theory

Fourier Spectrum Characterizations of Hardy Spaces H^p on Tubes for $0 < p < 1$

Guantie Deng[1] · Haichou Li[2] · Tao Qian[3]

Received: 21 December 2018 / Accepted: 6 May 2019

Abstract

As continuation of the study of Fourier spectrum characterization of higher-dimensional Hardy spaces $H^p(T_\Gamma)$ on tubes for $1 \le p \le \infty$, this paper aims to obtain analogous Fourier spectrum characterizations and integral representation formulas of higher-dimensional Hardy spaces $H^p(T_\Gamma)$ on tubes for the index range $0 < p < 1$. For $1 \le p \le \infty$, the $H^p(T_\Gamma)$ are well understood via the Poisson and conjugate Poisson integrals. However, for $0 < p < 1$, those integrals are no longer defined that requires more delicate analysis.

Keywords Hardy spaces · Fourier transform · Tube domain · Integral representation

---

Communicated by Irene Sabadini, Michael Shapiro, Daniele Struppa.

Guantie Deng: This work was partially supported by NSFC (Grant No. 11271045) and by SRFDP (Grant No. 20100003110004). Haichou Li: This work was supported by KQNCX (Grant No. 2017KQNCX018). Tao Qian: The work was partially supported by Macao Government FDCT 079/2016/A2 and MYRG2016-00053-FST.

✉ Haichou Li
hcl2016@scau.edu.cn

Guantie Deng
denggt@bnu.edu.cn

Tao Qian
fsttq@umac.mo

1 School of Mathematical Sciences, Beijing Normal University, Beijing 100875, China

2 College of Mathematics and Informatics, South China Agricultural University, Guangzhou 510640, China

3 Macau University of Science and Technology, Macao, China

Published online: 16 May 2019 Birkhäuser

1 Introduction

The Fourier spectrum properties of the functions as non-tangential boundary limits of those in the classical complex Hardy spaces $H^p(\mathbb{C}^+)$, $1 \le p \le \infty$, are completely characterized (see, for instance, [10,11]). The characterization is in terms of the location of the supports of the classical or distributional Fourier transforms of these boundary limit functions. It is shown that, for $f \in L^p(\mathbb{R})$, $1 \le p \le \infty$, f is the non-tangential boundary limit of a function in a $H^p(\mathbb{C}^+)$ if and only if $\operatorname{supp}\hat{f}$, for $1 \le p \le 2$; or, the distributional $\operatorname{supp}\hat{f}$, if $2 < p \le \infty$, is contained in $[0, \infty)$. For $p = 2$ this property of H^p is known as one of the Paley–Wiener Theorems. We now recall the formulation in the 1-D case.

Theorem A (Paley–Wiener) [6] *The function $F \in H^2(\mathbb{C}^+)$ if and only if there exists a function $f \in L^2(0, \infty)$, such that the integral representation holds*

$$F(z) = \int_0^\infty f(t)e^{2\pi itz}dt$$

for $z \in \mathbb{C}^+$, and, furthermore, $\|f\|_{L^2} = \|F\|_{H^2}$.

Generalizations to higher dimensions of the above famous Paley–Wiener theorem are obtained in [10,11] *and* [13]. *There holds the following related result.*

Theorem B [10] *If $f \in H^p(\mathbb{C}^+)$, $1 \le p \le \infty$, then, as a tempered distribution, $\hat{f}$ is supported in* $[0, \infty)$. *That means that $(\hat{f}, \varphi) = 0$ for all $\varphi \in S(\mathbb{R})$ with* $\operatorname{supp}\hat{\varphi} \subset (-\infty, 0]$. *Moreover, in the range $1 \le p \le 2$ there holds $\hat{f}(x) = 0$ for almost all* $x \in (-\infty, 0]$.

The converse result of Theorem B *holds as follows.*

Theorem C [11] *If $f \in L^p(\mathbb{R})$, $1 \le p \le \infty$, and d-*$\operatorname{supp}\hat{f} \subset [0, \infty)$*, then f is the boundary limit of a function in $H^p(\mathbb{C}^+)$.*

To the authors' knowledge, very little literature has addressed the Fourier spectrum aspect for higher-dimensional Hardy spaces $H^p(T_\Gamma)$ on tubes T_Γ, except in [1,13] *where it is shown that such characterization result holds for $p = 2$. They show in* [13] *that a $L^p(\mathbb{R}^n)$ function is the non-tangential boundary limit of a function in $H^2(T_\Gamma)$ if and only if its Fourier transform vanishes outside the dual cone of* Γ.

We next recall some basic results in higher dimensions, for any open and connected subset B of $\mathbb{R}^n$, Stein and Weiss [13] *obtained the following fundamental representation theorem restricted to only $H^2(T_B)$.*

Theorem D [13] *$F(z)$ is a function of higher-dimensional Hardy spaces $H^2(T_B)$, where T_B is a tube on B which is an open connected subset of $\mathbb{R}^n$, if and only if it has the integral representation*

$$F(z) = \int_{\mathbb{R}^n} e^{2\pi iz\cdot t} f(t)dt$$

for $z \in T_\Gamma$, where f is a function satisfying

$$\sup_{y \in B} \int_{\mathbb{R}^n} e^{-2\pi y \cdot t} |f(t)|^2 dt \le A^2 < \infty.$$

Stein and Weiss [13] commented that the theory of $H^2(T_B)$ becomes richer if the bases B are with more restrictions. When the base B is restricted to be an open cone Γ, one obtains the following sharper representation theorem for $H^2(T_\Gamma)$ functions. We note that, the following Theorem E is about the regular open cones. The result also gives rise to the Fourier spectrum characterization of the Hardy H^2-functions.

Theorem E [13] *$F(z)$ is a function of $H^2(T_\Gamma)$, where Γ is a regular open cone in $\mathbb{R}^n$, if and only if it has the integral representation*

$$F(z) = \int_{\Gamma^*} e^{2\pi i z \cdot t} f(t) dt, \quad z \in T_\Gamma,$$

where Γ^* is the dual cone of Γ and $f(t)$ is a measurable function on $\mathbb{R}^n$ satisfying

$$\int_{\Gamma^*} |f(t)|^2 dt < \infty.$$

Moreover, $\|F\|_{H^2} = \|F\|_2 = \left(\int_{\Gamma^*} |f(t)|^2 dt\right)^{1/2}$.

Open cones *are those satisfying the following two conditions:*

(1) 0 does not belong to Γ;
(2) For any $x, \ y \in \Gamma$, and any $\alpha, \ \beta > 0$, there holds $\alpha x + \beta y \in \Gamma$.

We note that a cone Γ is a convex set. The dual cone of Γ, *denoted by Γ^*, is defined as*

$$\Gamma^* = \{y \in \mathbb{R}^n : y \cdot x \ge 0, \text{ for any } x \in \Gamma\}.$$

A cone Γ is said to be regular *if the interior of its dual cone Γ^* is nonempty.*

For instance, when $n = 1$, there are only two open cones, the open half-lines $\{x \in \mathbb{R} : \ x > 0\}$ and $\{x \in \mathbb{R} : \ x < 0\}$. Their dual cones are the closed half-lines $\{x \in \mathbb{R} : \ x \ge 0\}$ and $\{x \in \mathbb{R} : \ x \ge 0\}$, both having non-empty interiors.

When $n = 2$, the open cones are the angular regions between two rays meeting at the origin. Such a cone is regular if and only if the corresponding angle is strictly less than π.

When $n \ge 2$, the first octant of $\mathbb{R}^n$ is a particular open convex regular cone whose dual cone is the closure of itself.

We will be using the following technical results.

Theorem F [13] *Suppose that $F \in H^p(T_\Gamma)$, where Γ is a regular open cone in $\mathbb{R}^n$, then the following properties hold for $1 \le p \le \infty$.*

(a) *If* Γ *is a cone whose closure is contained in* $\Gamma \cup \{0\}$, *then*

$$\lim_{y \in \Gamma,\ y \to 0} F(x+iy) = F(x)$$

for $x \in \mathbb{R}^n$;

(b)

$$\lim_{y \in \Gamma,\ y \to 0} \|F(x+iy) - F(x)\|_p = 0;$$

(c)

$$F(z) = \int_{\mathbb{R}^n} P(x-t, y) F(t) dt,$$

where $F(x)$ *is called the* restricted (non-tangential) boundary limit *of* $F(z)$.

We note that the Fourier spectrum characterization result (Theorem E) obtained by Stein and Weiss is restricted to $p = 2$. It is natural to ask in what extent they can be generalized to all the cases $0 < p \leq \infty$. Our recent work [8] proved that the Fourier spectrum characterization is also valid for the cases $1 \leq p \leq \infty$. That is

Theorem G [8] *Let* $F(x)$ *be a function of* $L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$. $F(x)$ *is the restricted boundary limit function of some function* $F(x+iy)$ *in* $H^p(T_\Gamma)$ *if and only if* d-supp $\widehat{F} \subset \Gamma^*$, *where* T_Γ *is a tube on a regular open cone* Γ *and* Γ^* *is the dual cone of* Γ. *Moreover,*

(1) *For* $1 \leq p \leq 2$, $\hat{F}$ *is locally integrable, and*

$$F(z) = \int_{\mathbb{R}^n} \mathcal{X}_{\Gamma^*}(t) e^{2\pi i z \cdot t} \hat{F}(t)\, dt = \int_{\mathbb{R}^n} F(t) K(z-t) dt,$$

where $K(z)$ *is the Cauchy kernels associated with the tube* T_Γ.

(2) *For* $1 \leq p \leq \infty$,

$$F(z) = \int_{\mathbb{R}^n} P(x-t, y) F(t) dt.$$

where $P(x, y)$ *is the Poisson kernels associated with the tube* T_Γ.

In case Γ *is a polygonal cone, there holds*

(3) *For* $2 < p < \infty$,

$$F(z) = \int_{\mathbb{R}^n} F(t) K(z-t) dt.$$

All the above mentioned results for higher dimensions are for the index range $1 \leq p \leq \infty$. One would concern what happen for the index range $0 < p < 1$?

This paper is devoted to answer this question. The obstacle for the cases $0 < p < 1$ is that there is no Fourier transform of such L^p functions, and the Hardy space functions cannot be defined via Poisson and conjugate Poisson integral when $0 < p < 1$. Deng and Qian [3] have obtained certain Fourier spectrum characterization and integral representation results about the classical one dimensional Hardy spaces $H^p(\mathbb{C}^+)$ for index range $0 < p < 1$ as follows.

Theorem H [3] *If* $F \in H^p(\mathbb{C}^+),\ 0 < p < 1$, *then there exist a positive constant* A_p, *depending only on* p, *and a slowly increasing continuous function* f *whose support is contained in* $[0, \infty)$, *satisfying*

$$(f, \varphi) = \lim_{y>0, y\to 0} \int_{\mathbb{R}} F(x+iy)\hat{\varphi}(x)dx$$

for φ *in the Schwarz class* $\mathbb{S}$, *and*

$$|f(t)| \leqslant A_p \|F\|_{H^p} |t|^{\frac{1}{p}-1}, \quad t \in \mathbb{R}, \tag{1}$$

$$F(z) = \frac{1}{\sqrt{2\pi}} \int_0^\infty f(t)e^{itz}dt, \qquad z \in \mathbb{C}^+. \tag{2}$$

With detailed analysis it turns out that the analogous characterization and integral representation results hold for higher dimensional Hardy spaces $H^p(T_\Gamma)$ on tubes for $0 < p < 1$. The purpose of this paper is to achieve two generalizations in respectively two directions: one is to generalize Theorem G to the index range $0 < p < 1$, and the other is to generalize Theorem H to multi-dimensions. We note that both the generalizations are nontrivial, for there are technical difficulties to overcome for each of them.

For the first generalization, the methods in proving the analogous Fourier spectrum characterization of $H^p(T_B)$ for the cases of $0 < p < 1$ is more difficult than the cases of $1 \le p < \infty$. For $1 \le p \le \infty$, the spaces $H^p(T_\Gamma)$ are well understood via the Poisson and conjugate Poisson integrals. However, for $0 < p < 1$, those integrals are no longer defined that makes the discussion more complicated.

For the second, the subject of several complex variables in our treatment is not merely an iterative one-complex variable theory. In fact, some basic, or natural, or classical results of one-complex variable, are not easy to be obtained in higher dimensional cases. For example, Lemma 4.4 in Sect. 4 is a well known result for one dimension. In our paper the higher dimensional counterpart is proved by using the pseudoconvexity and plurisubharmonicity of several complex variables [7].

The writing plan of this paper is arrange as follows: In Sect. 2, some basic notation and terminology are recalled. In Sect. 3, the main results are stated. In Sect. 4, we prove some useful lemmas. The last Sect. 5 is devoted to proving the main theorems.

2 Preliminary Knowledge

We begin with some basic definitions and well-known results from the theory of classical one dimensional Hardy spaces of upper-half or lower-half complex plane (see [4,6]).

For $0 < p < +\infty$, let $H^p(\mathbb{C}^{\pm})$ denote the space of the functions f analytic on the upper-half or lower-half complex plane $\mathbb{C}^{\pm} := \{z = x + iy : \pm y > 0\}$ for which the quantity

$$\|f\|_{H^p_{\pm}} := \sup_{\pm y > 0} \left(\int_{-\infty}^{\infty} |f(x+iy)|^p dx \right)^{\frac{1}{p}}$$

is finite.

The tube T_B with base B, where B is an open subset of $\mathbb{R}^n$, is the set

$$T_B = \{z = x + iy \in \mathbb{C}^n : \ x \in \mathbb{R}^n, \ y \in B\}.$$

For example, when $n = 1$, the classical upper-half complex plane $\mathbb{C}^+$ and lower-half complex plane $\mathbb{C}^-$ are the tubes in $\mathbb{C}$ with the base $B_+ = \{y \in \mathbb{R} : \ y > 0\}$ and the base $B_- = \{y \in \mathbb{R} : \ y < 0\}$, respectively. That is, $\mathbb{C}^+ = T_{B_+} = \{z = x + iy \in \mathbb{C} : x \in \mathbb{R}, \ y \in B_+\}$ and $\mathbb{C}^- = T_{B_-} = \{z = x + iy \in \mathbb{C} : x \in \mathbb{R}, \ y \in B_-\}$. Obviously, the tube T_B are generalizations of $\mathbb{C}^+$ and $\mathbb{C}^-$.

It is known that n-dimensional real Euclidean space $\mathbb{R}^n$ has 2^n octants. To denote the octants, we adopt the following notations.

First, we define and fix $\sigma_1(j) = 1$ for all $j = 1, 2, \ldots, n$ and denote by Γ_{σ_1} as the first octant of $\mathbb{R}^n$. That is

$$\Gamma_{\sigma_1} = \{y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n : \ y_j > 0, \ j = 1, 2, \ldots, n\}.$$

for all $k = 1, 2, \ldots, 2^n$.

Similarly, we define $\sigma_k(j) = +1$ or -1 for all $j = 1, 2, \ldots, n$, and $k = 1, 2, \ldots, 2^n$, and let $\sigma_k = (\sigma_k(1), \sigma_k(2), \ldots, \sigma_k(n))$. The 2^n octants of $\mathbb{R}^n$ are denoted by Γ_{σ_k} are defined as

$$\Gamma_{\sigma_k} = \{y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n : \ \sigma_k(j) y_j > 0, \ j = 1, 2, \ldots, n\},$$

for all $k = 1, 2, \ldots, 2^n$.

Correspondingly, $\mathbb{C}^n$ is decomposed into 2^n tubes, denoted by $T_{\Gamma_{\sigma_k}}, \ k = 1, 2, \ldots, 2^n$. That is

$$T_{\Gamma_{\sigma_k}} = \{z = x + iy \in \mathbb{C}^n : \ x \in \mathbb{R}^n, \ y \in \Gamma_{\sigma_k}\}.$$

A holomorphic function $F(z)$ is said to belong to the higher dimensional Hardy spaces $H^p(T_B),\ 0 < p < \infty$, if it satisfies

$$\|F\|_{H^p} = \sup\left\{\left(\int_{\mathbb{R}^n} |F(x+iy)|^p dx\right)^{\frac{1}{p}} \ :\ y \in B\right\} < \infty.$$

That is

$$H^p(T_B) = \{F : F \text{ holomorphic on } \ T_B \text{ and } \ \|F\|_{H^p} < \infty\}.$$

The spaces $H^p(T_{\Gamma_{\sigma_k}})$ are defined through replacing B by $\Gamma_{\sigma_k},\ k = 1, \ldots, 2^n$. Let Γ be one of the Γ_{σ_j}, a function, f, defined in tube T_Γ, is said to have non-tangential boundary limit (NTBL) l in each component of the variable at $x_0 \in \mathbb{R}^n$ if $f(z) = f(x+iy) = f(x_1+iy_1, \ldots, x_n+iy_n)$ tends to l as the point $z = (x_1, y_1; x_2, y_2; \ldots; x_n, y_n)$ tends to $x_0 = (x_0^{(1)}, 0; x_0^{(2)}, 0; \ldots; x_0^{(n)}, 0)$ within the Cartesian product

$$\gamma_\alpha(x_0) = \Gamma_{\alpha_1}(x_0^{(1)}) \times \Gamma_{\alpha_2}(x_0^{(2)}) \times \cdots \times \Gamma_{\alpha_n}(x_0^{(n)}) \subset T_\Gamma,$$

for all n-tuples $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ of positive real numbers, where

$$\Gamma_{\alpha_j}(x_0^{(j)}) = \left\{(x_j, y_j) \in \mathbb{C}^+ :\ |x_j - x_0^{(j)}| < \alpha_j y_j\right\},\ j = 1, 2, \ldots, n.$$

As an important property of the Hardy spaces, it is shown that if f is a function in a Hardy space $H^p, 0 < p < \infty$, then for almost all $x_0,\ f$ has NTBL [13]

Since the mapping that maps the functions in the Hardy spaces to their NTBLs is an isometric isomorphism, we denote by $H^p_{\sigma_k}(\mathbb{R}^n)$ for $0 < p < 1$ the NTBLs of the functions in $H^p(T_{\Gamma_{\sigma_k}})$, that is

$$H^p_{\sigma_k}(\mathbb{R}^n) = \left\{f :\ f \text{ is the NTBL of a function in } H^p(T_{\Gamma_{\sigma_k}})\right\}$$

for all $k = 1, 2, \ldots, 2^n$. The non-tangential boundary limit of $F(z) \in H^p(T_{\Gamma_{\sigma_k}})$ as $y \to 0$ in the tube are denoted by

$$F_{\sigma_k}(x) = \lim_{y\in\Gamma_{\sigma_k}, y\to 0} F(x+iy) = \lim_{\sigma_k(1)y_1\to 0^+,\ldots,\sigma_k(n)y_n\to 0^+} F(x_1+iy_1, \ldots, x_n+iy_n). \tag{3}$$

The Schwarz class $S(\mathbb{R}^n)$ is the space consisting of all those C^∞ function φ on $\mathbb{R}^n$ (i.e., all the partial derivatives of φ exist and are continuous) such that

$$\sup_{x\in\mathbb{R}^n} |x^\alpha(D^\beta\varphi(x))| < \infty$$

for all n-tuples $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$ of nonnegative integers, where

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}; \quad D^\beta = \frac{\partial^{|\beta|}}{\partial x_1^{\beta_1} \partial x_2^{\beta_2} \cdots \partial x_n^{\beta_n}}, \quad |\beta| = \beta_1 + \beta_2 + \cdots + \beta_n.$$

A measurable function f such that

$$\frac{f(x)}{(1+|x|^2)^k} \in L^p(\mathbb{R}^n)$$

for some positive integer k, where $1 < p \leq \infty$, is called a tempered L^p function (when $p = \infty$ such a function is often also called a slowly increasing function).

For function f in $L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, a tempered distribution T_f, can be defined through the relation

$$T_f(\varphi) = (T_f, \varphi) = \int_{\mathbb{R}^n} f(t)\varphi(t)\, dt$$

for φ in the Schwarz class $\mathbb{S}(\mathbb{R}^n)$. It is clear that T_f is a continuous linear functional on Schwarz class $\mathbb{S}(\mathbb{R}^n)$(see, [13]).

Let $F \in (S(\mathbb{R}^n))'$. If there exists a holomorphic function $f(x+iy)$ in T_B such that for any φ in the Schwarz class $S(\mathbb{R}^n)$, there holds

$$F_f(\varphi) = (F, \varphi) = \lim_{y>0,\ y\to 0} \int_{\mathbb{R}^n} f(x+iy)\varphi(x)\, dx$$

then we say that F is a holomorphic distribution and $f(x+iy)$ is an holomorphic representation of F.

The Fourier transform $\hat{f}$ of a function $f \in L^1(\mathbb{R}^n)$ is defined by

$$\hat{f}(x) = \int_{\mathbb{R}^n} f(t) e^{-2\pi i x \cdot t}\, dt,$$

where $x = (x_1, x_2, \ldots, x_n)$, $t = (t_1, t_2, \ldots, t_n)$ are elements of $\mathbb{R}^n$. The inner product of $x,\ t \in \mathbb{R}^n$ is the number $x \cdot t = \sum_{j=1}^{n} x_j t_j$, and $|x| = \sqrt{x \cdot x}$.

The Fourier transform of a function $f \in L^2(\mathbb{R}^n)$ is defined as the L^2 limit of the sequence $\hat{g}_k$, where g_k is any sequence in $L^1 \cap L^2$ converging to f in the L^2 norm. The definition of the Fourier transform can be extended to functions in $L^p(\mathbb{R}^n)$ for $1 < p < 2$ by decomposing such functions into a fat tail part in L^2 plus a fat body part in L^1 on $(\mathbb{R}^n)$. Since the function class $L^1(\mathbb{R}^n) + L^2(\mathbb{R}^n)$ is easily seen to contain all the spaces $L^p(\mathbb{R}^n)$, $1 \leq p \leq 2$, the Fourier transform $\hat{f}$ is defined for all $f \in L^p(\mathbb{R}^n)$ [13].

The Fourier transformation $\widehat{T}$ of tempered distribution T is defined as the continuous linear functional through the relation

$$(\widehat{T}, \varphi) = (T, \hat{\varphi})$$

for φ in the Schwarz class $\mathbb{S}(\mathbb{R}^n)$. We observe that if $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq 2$, the Fourier transform of f as a distribution coincides with the function $\hat{f}$ defined as the above. However, for any $p > 2$, there exists a function $f \in L^p(\mathbb{R}^n)$, whose Fourier transform, as a tempered distribution, is not a function.

3 Main Results

First, we obtain the following integral representation results about any n dimensions for index range $0 < p < 1$.

Theorem 3.1 *If $f \in H^p(T_{\Gamma_{\sigma_1}})$, $0 < p \leq 1$, where Γ_{σ_1} is the first octant of $\mathbf{R}^n$, then there exist a constant C_p which is independent of f, and a slowly increasing continuous function F whose support is contained in $\overline{\Gamma}_{\sigma_1}$, such that*

$$(F, \varphi) = \lim_{y>0,\ y\to 0} \int_{\mathbb{R}^n} f(x+iy)\hat{\varphi}(x)dx \tag{4}$$

for φ in the Schwarz class $S(\mathbb{R}^n)$, and t

$$f(z) = \int_{\Gamma_{\sigma_1}} F(t)e^{2\pi i t\cdot z}dt \tag{5}$$

for $z \in T_{\Gamma_1}$. Moreover, for $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, there holds

$$|F(x)| \leq C_p e^{nB_p} B_p^{-nB_p} \prod_{k=1}^{n} x_k^{B_p} \|f\|_{H^p}, \tag{6}$$

where $C_p = (\frac{\pi}{2})^{\frac{n}{p}}$, $B_p = \frac{1}{p} - 1 \geq 0$.

It is similarly to prove the following corollary.

Corollary 3.1 *If $f \in H^p(T_{\Gamma_{\sigma_j}})$, $0 < p \leq 1$, for any $j = 2, 3, \ldots, 2^n$, where $T_{\Gamma_{\sigma_j}}$ is any other octant of $\mathbf{R}^n$ except the first octant, then there also exist a constant C_p which is independent of f, and a slowly increasing continuous function F whose support is contained in $\overline{\Gamma}_{\sigma_1}$, such that*

$$(F, \varphi) = \lim_{\sigma_j y>0,\ \sigma_j y\to 0} \int_{\mathbb{R}^n} f(x+iy)\hat{\varphi}(x)\,dx \tag{7}$$

for φ in the Schwarz class $S(\mathbb{R}^n)$, and

$$f(z) = \int_{\Gamma_{\sigma_j}} F(t)e^{2\pi i t\cdot z}dt \tag{8}$$

for $z \in T_{\Gamma_j}$. Moreover, for $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, there holds

$$|F(x)| \le C_p e^{nB_p} B_p^{-nB_p} \prod_{k=1}^{n} x_k^{B_p} \|f\|_{H^p} \tag{9}$$

where $C_p = (\frac{\pi}{2})^{\frac{n}{p}},\ B_p = \frac{1}{p} - 1 \ge 0.$

In order to obtain the analogous Fourier spectrum characterization result of $H^p,\ 0 < p < 1$, we need to consider an interest and important property of Hardy spaces. It is well known that, for any $0 < p < q < \infty$, the relation $H^q \subset H^p$ holds. However, when the indexs p and q satisfy that $0 < p,\ q < \infty$, how about the relation of H^q and H^p? For this question, our recently research work [8] has answered some part such as $1 \le p,\ q < \infty$ in the following Theorem I.

Theorem I [8] *Suppose that $F(z) \in H^p(T_\Gamma),\ 1 \le p \le \infty$, where Γ is a regular open cone in $\mathbb{R}^n$, and $F(x)$ is the boundary limit function of $F(z)$. If $F(x) \in L^q(\mathbb{R}^n),\ 1 \le q \le \infty$, then $F(z) \in H^q(T_\Gamma)$.*

With different method from Theorem I, we obtain the following Theorem 3.2 for more general index range $0 < p,\ q \le \infty$.

Theorem 3.2 *Let $T_{\Gamma_{\sigma_1}}$ be the first octant of $\mathbf{C}^n$. Suppose that $F(z) \in H^p(T_{\Gamma_{\sigma_1}}),\ 0 < p \le \infty$, and $F(x)$ is the boundary limit function of $F(z)$. If $F(x) \in L^q(\mathbb{R}^n),\ 0 < q \le \infty$, then $F(z) \in H^q(T_{\Gamma_{\sigma_1}})$.*

As an analogous Fourier spectrum characterization result of Hardy spaces $H^p,\ 0 < p < 1$, Theorem 3.3 is obtained as follows, which is also an application of Theorems 3.1 and 3.2

Theorem 3.3 *Let $0 < p < 1,\ f \in L^p(\mathbb{R}^n)$. Then $f \in H^p_{\sigma_1}(\mathbb{R}^n)$ if and only if there exist a sequence of functions $\{f_n\}$ satisfying $f_n \in L^2(\mathbb{R}^n) \bigcap L^p(\mathbb{R}^n)$, supp $\hat{f}_n \subset \overline{\Gamma}_{\sigma_1}$, and*

$$\lim_{n\to\infty} ||f - f_n||_{L^p} = 0 \tag{10}$$

where $H^p_{\sigma_1}(\mathbb{R}^n)$ is the set of the boundary limits of the functions in $H^p(T_{\Gamma_{\sigma_1}})$.

4 Some Useful Lemmas

We need the following important Lemmas.

Lemma 4.1 [13] *Let B be an open cone in $\mathbb{R}^n$. Suppose $F \in H^p(T_B),\ p > 0$, and $B_0 \subset B$ satisfies $d(B_0, B^c) = \inf\{|y_1 - y_2|;\ y_1 \in B_0,\ y_2 \in B^c\} \geq \varepsilon > 0$, then there exists a constant $C_p(\varepsilon)$, depending on ε and p but not on F, such that*

$$\sup_{z \in T_{B_0}} |F(z)| \leq C_p(\varepsilon)\|F\|_{H^p}.$$

Given below offers a more precise estimation than that obtained in the above lemma.

Lemma 4.2 *Suppose that $f \in H^p(T_\Gamma),\ p > 0$, where Γ is the first octant of $\mathbb{R}^n$. If let $f_\delta(z) = f(z + i\delta)$, for any $z = x + iy \in T_\Gamma$ and $\delta = (\delta_1, \delta_2, \ldots, \delta_n) \in \Gamma$, then there holds*

$$\sup_{z \in T_\Gamma} |f_\delta(z)| \leq C_p(\delta_1 \cdot \delta_2 \cdot \cdots \cdot \delta_n)^{-\frac{1}{p}} \|f\|_{H^p},$$

where $C_p = (\frac{2}{\pi})^{\frac{n}{p}}$.

Proof Let $z_0 = x_0 + iy_0 = (z_{01}, z_{02}, \ldots, z_{0n}) \in T_\Gamma$ and

$$\mathbb{D}^n = \{(z_1, z_n, \ldots, z_n) \in \mathbb{C}^n :\ |z_1 - z_{01}| < y_{01},\ |z_2 - z_{02}| < y_{02}, \ldots,\ |z_n - z_{0n}| < y_{0n}\}$$

be open polydisc. Since $f \in H^p(T_\Gamma)$, we know that $|f|^p$ is subharmonic as a function of $2n$ variables. Hence,

$$\begin{aligned}
|f(z_0)|^p &\leq \frac{1}{\pi^n y_{01}^2 y_{02}^2 \cdots y_{0n}^2} \int_{|z_1 - z_{01}| < y_{01}} \cdots \int_{|z_n - z_{0n}| < y_{0n}} |f(z_1, \ldots, z_n)|^p dz_1 \cdots dz_n \\
&\leq \frac{1}{\pi^n y_{01}^2 y_{02}^2 \cdots y_{0n}^2} \int_0^{2y_{01}} \cdots \int_0^{2y_{0n}} \int_{\mathbb{R}^n} |f(x + iy)|^p dy_1 \cdots dy_n dx \\
&\leq \frac{2^n}{\pi^n y_{01} y_{02} \cdots y_{0n}} \sup_{y \in \Gamma} \int_{\mathbb{R}^n} |f(x + iy)|^p dx \\
&\leq \frac{2^n}{\pi^n y_{01} y_{02} \cdots y_{0n}} \|f\|_{H^p}^p.
\end{aligned}$$

Therefore, for any $\delta = (\delta_1, \delta_2, \ldots, \delta_n) \in \Gamma$, we obtain that

$$|f_\delta(x + iy)| \leq C_p(\delta_1 \cdot \delta_2 \cdot \cdots \cdot \delta_n)^{-\frac{1}{p}} \|f\|_{H^p},$$

where $C_p = (\frac{2}{\pi})^{\frac{n}{p}}$. □

The following Theorem J is needed by the proof of Theorem 2.2.

Theorem J [6] *If $F(z) \in H^p(\mathbb{C}^+),\ 0 < p \leq \infty$, and $F(z_0) \neq 0$, then*

$$\log|F(z_0)| \leq \int_{-\infty}^{\infty} \log|F(t)| P_{z_0}(t) dt,$$

where $P_{z_0}(t)$ is the Poisson kernel associated with the upper-half plane $\mathbb{C}^+$.

As generalization of a result in [13], we have.

Lemma 4.3 *If $F(z) \in H^p(T_\Gamma)$, where Γ is the first octant of $\mathbb{R}^n$. Then, for any $j = 1, 2, \ldots, n$ and fixed $(z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n) \in T_\Gamma^{n-1}$, where T_Γ^{n-1} denotes the tube with its base as the first octant of $\mathbb{R}^{n-1}$, we can prove that the function $F(z_1, \ldots, z_n)$ as a function of one complex variable $z_j = x_j + iy_j \in \mathbb{C}^+$, belongs to $H^p(\mathbb{C}^+)$, that is*

$$F(z_1, \ldots, z_j, \ldots, z_n) \in H^p(\mathbb{C}^+).$$

Proof Fixing $(z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n) \in T_\Gamma^{n-1}$, we consider the function f of one complex variable defined by letting

$$f(\zeta_j) = F(z_1, \ldots, z_{j-1}, \zeta_j, z_{j+1}, \ldots, z_n),$$

whenever $\zeta_j = \xi_j + i\eta_j \in \mathbb{C}^+$.

Since $F(z_1, \ldots, z_{j-1}, \zeta_j, z_{j+1}, \ldots, z_n) \in H^p(T_\Gamma)$, $|F(z_1, \ldots, z_{j-1}, \zeta_j, z_{j+1}, \ldots, z_n)|^p$ is subharmonic as a function of $z_1 = x_1 + iy_1 \in \mathbb{C}^+$. Then we have, by writing $w_1 = u_1 + iv_1$,

$$\begin{aligned}
&|F(z_1, \ldots, z_{j-1}, \zeta_j, z_{j+1}, \ldots, z_n)|^p \\
&\quad \leq \frac{1}{\pi y_1^2} \int_{|w_1 - z_1| < y_1} |F(w_1, z_2 \ldots, z_{j-1}, \zeta_j, z_{j+1}, \ldots, z_n)|^p du_1 dv_1 \\
&\quad \leq \frac{1}{\pi y_1^2} \int_0^{2y_1} \int_{-\infty}^{\infty} |F(w_1, z_2 \ldots, z_{j-1}, \zeta_j, z_{j+1}, \ldots, z_n)|^p du_1 dv_1.
\end{aligned}$$

Repeating this argument for $z_2, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n$, we can obtain

$$\begin{aligned}
&|F(z_1, \ldots, z_{j-1}, \zeta_j, z_{j+1}, \ldots, z_n)|^p \\
&\leq \left(\pi^{n-1} \prod_{l=1, l \neq j}^{n} y_l^2 \right)^{-1} \int_0^{2y_1} \cdots \int_0^{2y_{j-1}} \int_0^{2y_{j+1}} \cdots \int_0^{2y_n} \int_{\mathbb{R}^{n-1}} \\
&|F(w_1, \ldots, w_{j-1}, \zeta_j, w_{j+1}, \ldots, w_n)|^p \, du_1 \cdots du_{j-1} du_{j+1} \cdots du_n dv_1 \cdots dv_{j-1} dv_{j+1} \cdots dv_n.
\end{aligned}$$

Now integrating both sides of the last inequality with respect to ξ_j, we have

$$\begin{aligned}
&\int_{\mathbb{R}} |f(\xi_j + i\eta_j)|^p d\xi_j \\
&\quad = \int_{\mathbb{R}} |F(z_1, \ldots, z_{j-1}, \xi_j + i\eta_j, z_{j+1}, \ldots, z_n)|^p d\xi_j \\
&\quad \leq \int_{\mathbb{R}} \left(\pi^{n-1} \prod_{l=1, l \neq j}^{n} y_l^2 \right)^{-1} \int_0^{2y_1} \cdots \int_0^{2y_{j-1}} \int_0^{2y_{j+1}} \cdots \int_0^{2y_n} \int_{\mathbb{R}^{n-1}}
\end{aligned}$$

$$
\begin{aligned}
&|F(w_1,\ldots,w_{j-1},\zeta_j,w_{j+1},\ldots,w_n)|^p\, du_1\cdots du_{j-1}du_{j+1}\cdots du_n dv_1\\
&\quad\cdots dv_{j-1}dv_{j+1}\cdots dv_n d\xi_j\\
&\quad\le\left(\pi^{n-1}\prod_{l=1,l\neq j}^{n} y_l^2\right)^{-1}\int_0^{2y_1}\cdots\int_0^{2y_{j-1}}\int_0^{2y_{j+1}}\cdots\int_0^{2y_n}\int_{\mathbb{R}^n}\\
&|F(u+iv)|^p du dv_1\cdots dv_{j-1}dv_{j+1}\cdots dv_n\\
&\quad\le\frac{1}{\pi^{n-1}\prod\limits_{l=1,l\neq j}^{n} y_l^2}\prod_{l=1,l\neq j}^{n}2y_l\sup_{v\in T_\Gamma}\int_{\mathbb{R}^n}|F(u+iv)|^p du\\
&\quad\le 2^{n-1}\pi^{1-n}\prod_{l=1,l\neq j}^{n} y_l^{-1}\|F\|_{H^p}=C<\infty,
\end{aligned}
$$

where $u+iv=(u_1+v_1,\ldots,u_{j-1}+iv_{j-1},\xi_j+i\eta,u_{j+1}+iv_{j+1},\ldots,u_n+iv_n)$. This shows that $f(\zeta_j)\in H^p(\mathbb{C}^+)$, for any $j=1,2,\ldots,n$, which implies that the proof of this Lemma is complete. □

In order to prove Theorem 3.3, we need the following key Lemma.

Lemma 4.4 *If $f(z)\in H^p(T_{\Gamma_{\sigma_j}})$, $0<p<\infty$, $j=1,2,\ldots,2^n$, where σ_j is the octants of $\mathbb{R}^n$, and $f(x)$ is the boundary limit of $f(z)$. Then $\varphi(y)$ is continuous convex and bounded in $\overline{\Gamma}_{\sigma_j}$, where $\varphi(y)=\int_{\mathbb{R}^n}|f(x+iy)|^p dx$, $y\in\overline{\Gamma}_{\sigma_j}$, moreover,*

$$
\|f\|^p_{H^p_{\sigma_j}}=\sup_{y\in\Gamma_{\sigma_j}}\varphi(y)=\varphi(0,\ldots,0)=\|f\|^p_{L_p}.
$$

Proof Let

$$
\widetilde{\varphi}(z)=\int_{\mathbb{R}^n}|f(t+z)|^p dt,\quad z\in T_{\overline{\Gamma}_{\sigma_j}},
$$

then

$$
\widetilde{\varphi}(z)=\varphi(y),\quad z=x+iy\in T_{\overline{\Gamma}_{\sigma_j}}.
$$

Similarly to the proof (see [13]) of Part (b) in Theorem F above, we can obtain that

$$
\lim_{z\to z_0}\int_{\mathbb{R}^n}|f(t+z)-f(t+z_0)|^p dt=0
$$

for any $z_0\in T_{\Gamma_{\sigma_j}}$. Thus,

$$
|\widetilde{\varphi}(z)-\widetilde{\varphi}(z_0)|\le\int_{\mathbb{R}^n}|f(t+z)-f(t+z_0)|^p dt\to 0,\quad z\to z_0,
$$

which implies that $\varphi(y)$ is continuous in $\overline{\Gamma}_{\sigma_j}$.

For any $a \in T_{\Gamma_{\sigma_j}},\ w \in \mathbb{C}^n,\ \lambda \in \mathbb{C}$, let

$$u_{a,w}(\lambda) = \widetilde{\varphi}(a + \lambda w).$$

$f(z) \in H^p(T_{\Gamma_{\sigma_j}})$ implies that $|f(z)|^p$ is plurisubharmonic. For $\lambda_0 \in \mathbb{C}$, there exists $\delta > 0$ such that $\{a + \lambda w : |\lambda - \lambda_0| < \delta\} \in T_{\Gamma_{\sigma_j}}$. Hence, for $0 < r < \delta$,

$$\begin{aligned}
\widetilde{\varphi}(a + \lambda_0 w) &= \int_{\mathbb{R}^n} |f(t + a + \lambda_0 w)|^p dt \\
&\le \int_{\mathbb{R}^n} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(t + a + (\lambda_0 + re^{i\theta})w)|^p \, d\theta \right) dt \\
&= \frac{1}{2\pi} \int_0^{2\pi} \left(\int_{\mathbb{R}^n} |f(t + a + (\lambda_0 + re^{i\theta})w)|^p \, dt \right) d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} \widetilde{\varphi}(a + (\lambda_0 + re^{i\theta})w) d\theta.
\end{aligned} \tag{11}$$

Therefore, function $u_{a,w}(\lambda) = \widetilde{\varphi}(a + \lambda w)$ is subharmonic in $D(\lambda, \delta) = \{\lambda : |\lambda - \lambda_0| < \delta\}$. By the definition of plurisubharmonic function [7], $\widetilde{\varphi}(z)$ is a plurisubharmonic function in $T_{\Gamma_{\sigma_j}}$.

Next, we are to show that $\varphi(y) = \widetilde{\varphi}(z)$ is convex in $\overline{\Gamma}_{\sigma_j}$.

For any $a, b \in \Gamma_{\sigma_j}$, $\{ia + \lambda i(b - a) :\ \lambda = \zeta + i\eta \in \mathbb{C},\ 0 \le \zeta \le 1\} \subset \Gamma_{\sigma_j}$,

$$\begin{aligned}
u_{a,b}(\lambda) &= \widetilde{\varphi}(ia + \lambda i(b - a)) = \widetilde{\varphi}(-\eta(b - a) + i(a + \zeta(b - a)) \\
&= \varphi(a + \zeta(b - a)) = \widetilde{\varphi}(ia + i\zeta(b - a)) = u_{a,b}(\zeta).
\end{aligned} \tag{12}$$

Since $u_{a,b}(\lambda)$ is subharmonic and continuous in Ω, where

$$\Omega = \{\lambda = \zeta + i\eta \in \mathbb{C} :\ ia + \lambda i(b - a) \in T_{\Gamma_{\sigma_j}}\} \supset \{\lambda = \zeta + i\eta \in \mathbb{C} : 1 \ge \zeta \ge 0\}.$$

Therefore, there exists a function $\alpha(t) \in C_c^\infty(\mathbb{R})$ satisfying supp$\alpha \subset [0, 1]$ and

$$\frac{1}{2\pi} \int_0^1 t\alpha(t^2) dt = 1$$

for $0 < \delta < 1/2$, such that the function

$$\widetilde{u}_\delta(\lambda) = \int_{-\infty}^{\infty} u_{a,b}(\xi)\alpha_\delta(\lambda - \xi) dm(\xi) \tag{13}$$

is subharmonic and infinitely differentiable in Ω_δ, and $\widetilde{u}_\delta(\lambda)$ is monotone decreasing to $u_{a,b}(\lambda)$ as δ is monotone decreasing to 0, where

$$\Omega_\delta = \{\lambda \in \mathbb{C} :\ d(\lambda, \Omega^c) > \delta > 0\} \supset \{\lambda = \zeta + i\eta \in \mathbb{C} : 1 - \delta > \zeta > \delta\},$$

and $\alpha_\delta(\lambda) = \frac{1}{\delta^2}\alpha(\frac{|\lambda|^2}{\delta^2})$, $dm(\xi)$ is the Lebesgue measure.

The subharmonicity of $\widetilde{u}_\delta(\lambda)$ implies that ,

$$\Delta\widetilde{u}_\delta = \frac{d^2}{d\zeta^2}\widetilde{u}_\delta + \frac{d^2}{d\eta^2}\widetilde{u}_\delta \geq 0.$$

(12) implies that, for $1 - \delta > \zeta > \delta$,

$$\Delta\widetilde{u}_\delta = \frac{d^2}{d\zeta^2}\widetilde{u}_\delta(\zeta) \geq 0.$$

Therefore, $\widetilde{u}_\delta(\zeta)$ is convex in $(\delta, 1-\delta)$. Then $u_{a,b}(\zeta)$ is convex in $(0, 1)$. Since $u_{a,b}(\zeta)$ is continuous in $[0, 1]$, $u_{a,b}(\zeta)$ is convex in $[0, 1]$. Thus, for $0 \leq \zeta \leq 1$,

$$u_{a,b}(\zeta) \leq (1 - \zeta)u_{a,b}(0) + \zeta u_{a,b}(1).$$

From the last formula and (12), we obtain

$$\varphi(a + \zeta(b - a)) = \varphi((1 - \zeta)a + \zeta b) \leq (1 - \zeta)\varphi(a) + \zeta\varphi(b).$$

This shows that $\varphi(y)$ is convex in Γ_{σ_j}. Together with that $\varphi(y)$ is continuous in $\overline{\Gamma}_{\sigma_j}$, we know that $\varphi(y)$ is convex in $\overline{\Gamma}_{\sigma_j}$. Then $\varphi(y)$ is a convex bounded function about $y \in \overline{\Gamma}_{\sigma_j}$. Therefore, $\varphi(y)$ can attain the maximum value at the origin point (0,0,...,0), that is

$$\max_{y\in\Gamma_{\sigma_j}} \varphi(y) = \varphi(0, \ldots, 0) = \int_{\mathbb{R}^n} |f(x)|^p\, dx = \|f\|^p_{L_p}.$$

Thus, $\|f\|^p_{H^p} = \sup_{y\in\Gamma_{\sigma_j}} \varphi(y) = \|f\|^p_{L_p}$. So the proof of Lemma is complete. □

5 Proofs of Main Theorems

Proof of Theorem 3.1 Let $0 < p \leq 1$, $f \in H^p(T_{\Gamma_{\sigma_1}})$, where Γ_{σ_1} is the first octant of $\mathbb{R}^n$. For $y_0 \in \Gamma_{\sigma_1}$, let $f_{y_0}(z) = f(z + iy_0)$, $\Gamma_0 = \{y + y_0 :\ y \in \Gamma_{\sigma_1}\} \subset \Gamma_{\sigma_1}$. It is clearly that $f_{y_0}(z) \in H^p(T_{\Gamma_{\sigma_1}})$, and there exists $0 < \varepsilon \leq \min\{y_{01}, \ldots, y_{0n}\}$, such that

$$d(\Gamma_0, \Gamma^c_{\sigma_1}) = \inf\{|y_1 - y_2|;\ y_1 \in \Gamma_0,\ y_2 \in \Gamma^c_{\sigma_1}\} \geq \varepsilon > 0.$$

By Lemma 4.1, there exists a positive constant $C_p(\varepsilon)$, such that

$$\sup_{z\in T_{\Gamma_{\sigma_1}}} |f_{y_0}(z)| \leq C_p(\varepsilon)\|f\|_{H^p} := M.$$

From the last inequality, we have

$$\int_{\mathbb{R}^n} |f_{y_0}(x+iy)|^2 dx = \int_{\mathbb{R}^n} |f_{y_0}(x+iy)|^p |f_{y_0}(x+iy)|^{2-p} dx$$
$$\leq \int_{\mathbb{R}^n} |f_{y_0}(x+iy)|^p (C_p(\varepsilon)\|f\|_{H^p})^{2-p} dx \leq M^{2-p}\|f_{y_0}\|_{H^p}^p < \infty.$$

Therefore,

$$f_{y_0}(z) \in H^2(T_{\Gamma_{\sigma_1}}).$$

By higher-dimensional Paley–Wiener Theorem, we obtain

$$\operatorname{supp} \hat{f}_{y_0} \subset \overline{\Gamma}_{\sigma_1}. \tag{14}$$

Moreover, By Lemma 4.2, there exists a constant C_p such that

$$\begin{aligned} \int_{\mathbb{R}^n} |f_{y_0}(x+iy)| dx &= \int_{\mathbb{R}^n} |f_{y_0}(x+iy)|^p |f_{y_0}(x+iy)|^{1-p} dx \\ &\leq \int_{\mathbb{R}^n} |f_{y_0}(x+iy)|^p (C_p\|f\|_{H^p}(y_{01}\cdots y_{0n})^{-\frac{1}{p}})^{1-p} dx \\ &\leq C_p^{1-p}\|f\|_{H^p}(y_{01}\cdots y_{0n})^{(1-\frac{1}{p})}. \end{aligned} \tag{15}$$

where $C_p = (\frac{2}{\pi})^{\frac{n}{p}}$.

Since that $\hat{f}_{y_0}(t)$ is continuous in $\mathbb{R}^n$, as obtained from (15), and

$$f_{y_0}(z) = \int_{\Gamma_{\sigma_1}} e^{2\pi i z\cdot t} \hat{f}_{y_0}(t) dt = \int_{\mathbb{R}^n} e^{2\pi i x\cdot t}(e^{-2\pi y\cdot t}\mathcal{X}_{\overline{\Gamma}_{\sigma_1}}(t)\hat{f}_{y_0}(t)) dt.$$

We have

$$\hat{f}_{y_0+y}(t) = e^{-2\pi y\cdot t}\mathcal{X}_{\overline{\Gamma}_{\sigma_1}}(t)\hat{f}_{y_0}(t) = e^{2\pi y_0\cdot t}\mathcal{X}_{\overline{\Gamma}_{\sigma_1}}(t)\hat{f}_{y_0}(t)e^{-2\pi(y_0+y)\cdot t}.$$

Hence,

$$\hat{f}_{y_0+y}(t)e^{2\pi(y_0+y)\cdot t} = \hat{f}_{y_0}(t)e^{2\pi y_0\cdot t}, \quad \text{a.e.} \quad t \in \overline{\Gamma}_{\sigma_1}, \; y, \; y_0 \in \Gamma_{\sigma_1}.$$

This implies that $\hat{f}_y e^{2\pi y\cdot t}$ is independent of y.

Moreover, for $y_0 \in \Gamma_{\sigma_1}$, if let

$$F(t) = \hat{f}_{y_0}(t)e^{2\pi y_0\cdot t}, \qquad t \in \overline{\Gamma}_{\sigma_1},$$

then

$$F(t) = \hat{f}_y(t)e^{2\pi y\cdot t}, \quad t \in \overline{\Gamma}_{\sigma_1},$$

for any $y \in \Gamma_{\sigma_1}$

So that

$$\hat{f}_y(t) = \mathcal{X}_{\overline{\Gamma}_{\sigma_1}}(t)F(t)e^{-2\pi y\cdot t} \to F(t), \tag{16}$$

as $y \to 0$, and

$$\begin{aligned} f_{y_0}(z) &= \int_{\Gamma_{\sigma_1}} e^{2\pi i z\cdot t} \hat{f}_{y_0}(t)dt \\ &= \int_{\mathbb{R}^n} \mathcal{X}_{\overline{\Gamma}_{\sigma_1}}(t)F(t)e^{-2\pi y_0\cdot t}e^{2\pi i z\cdot t}dt \\ &= \int_{\mathbb{R}^n} \mathcal{X}_{\overline{\Gamma}_{\sigma_1}}(t)F(t)e^{2\pi i(z+iy_0)\cdot t}dt. \end{aligned}$$

Replace $(z + iy_0)$ by z in the last formula, we can obtain

$$f(z) = \int_{\Gamma_{\sigma_1}} e^{2\pi i z\cdot t} F(t)dt, \quad z \in T_{\Gamma_{\sigma_1}},$$

which shows that the (5) holds. Moreover, together with (14) and (16), there is

$$\mathrm{supp} F \subset \overline{\Gamma}_{\sigma_1}. \tag{17}$$

From (15), for any $x = (x_1, \ldots, x_n) \in \Gamma_{\sigma_1}$, we obtain

$$\begin{aligned} |F(x)| &= |\hat{f}_{y_0}(x)e^{2\pi y_0\cdot x}| \le \|f_{y_0}\|_1 e^{2\pi y_0\cdot x} \\ &\le C_p\|f\|_{H^p}\left(\prod_{k=1}^{n} y_{0k}\right)^{-B_p} e^{2\pi \sum\limits_{k=1}^{n} y_{0k}x_k} \\ &\le C_p\|f\|_{H^p} e^{2\pi \sum\limits_{k=1}^{n} y_{0k}x_k - B_p \log \prod\limits_{k=1}^{n} y_{0k}} \\ &\le C_p\|f\|_{H^p} e^{\sum\limits_{k=1}^{n}(2\pi y_{0k}x_k - B_p \log y_{0k})} \end{aligned} \tag{18}$$

where $B_p = \frac{1}{p} - 1 \ge 0$. Since $F(x)$ is independent of y_0 and

$$\inf\{2\pi y_{0k}x_k - B_p \log y_{0k})\} = B_p - B_p(\log B_p - \log 2\pi x_k),$$

for each $k = 1, 2, \ldots, n$. Together with (18), we obtain

$$\begin{aligned} |F(x)| &\le \inf \left\{ C_p \|f\|_{H^p} e^{\sum_{k=1}^{n} (2\pi y_{0k} x_k - B_p \log y_{0k})} \ : \ y_0 \in \Gamma_{\sigma_1} \right\} \\ &= C_p \|f\|_{H^p} e^{\sum_{k=1}^{n} (B_p - B_p (\log B_p - \log 2\pi x_k))} \\ &= C_p \|f\|_{H^p} e^{n B_p} B_p^{-n B_p} \prod_{k=1}^{n} x_k^{B_p} \end{aligned} \tag{19}$$

for $x_k \ge 0, \quad k = 1, 2, \ldots, n$.

From (17) and (19), we know that F is a slowly increasing continuous function whose support supp$F$ is contained in $\overline{\Gamma}_{\sigma_1}$. F can be regarded as a tempered distribution, which satisfies

$$(F, \varphi) = \int_{\mathbb{R}^n} F(x)\varphi(x)dx$$

for φ in the Schwarz class $S(\mathbb{R}^n)$.

By Lebesgue Dominated Convergence Theorem, we obtain

$$\begin{aligned} \lim_{y_0 > 0,\ y_0 \to 0} \int_{\mathbb{R}^n} f_{y_0}(x)\hat{\varphi}(x)dx &= \lim_{y_0 \to 0} \int_{\mathbb{R}^n} \hat{f}_{y_0}(x)\varphi(x)dx \\ &= \lim_{y_0 \to 0} \int_{\Gamma_{\sigma_1}} e^{-2\pi y_0 \cdot x} F(x)\varphi(x)dx \\ &= \int_{\mathbb{R}^n} F(x)\varphi(x)dx \\ &= (F, \varphi). \end{aligned}$$

Thus, the proof of Theorem 3.1 is complete. □

Proof of Theorem 3.2 Let us fix $(z_2, \ldots, z_n) \in T_{\Gamma_{n-1}}$, where Γ_{n-1} denotes the first octant of $\mathbb{R}^{n-1}$, and consider the function G of one complex variable defined by letting

$$G(z_1) = F(z_1, z_2, \ldots, z_n)$$

whenever z_1 belongs to the upper-half plane $\mathbb{C}^+$.

The fact $F(z) \in H^p(T_{\Gamma_{\sigma_1}}),\ 0 < p \le \infty$, and Lemma 4.3 together show that $G(z_1) \in H^p(\mathbb{C}^+),\ 0 < p \le \infty$. Then, by Theorem J,

$$\begin{aligned}\log|F(z)| = \log|G(z_1)| &\leq \int_{-\infty}^{\infty} \log|G(t_1)| P_{z_1}(t_1) dt_1 \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} \log|F(t_1, z_2, \ldots, z_n)| \frac{y_1}{(x_1 - t_1)^2 + y_1^2} dt_1.\end{aligned} \tag{20}$$

Similarly, let

$$G(z_2) = F(t_1, z_2, \ldots, z_n)$$

whenever z_2 belongs to the upper-half plane $\mathbb{C}^+$.

By Lemma 4.3 again, $F(z) \in H^p(T_{\Gamma_{\sigma_1}}),\ 0 < p \leq \infty$, shows that $G(z_2) \in H^p(\mathbb{C}^+),\ 0 < p \leq \infty$. Then, by Theorem J again,

$$\begin{aligned}\log|F(t_1, z_2, \ldots, z_n)| = \log|G(z_2)| &\leq \int_{-\infty}^{\infty} \log|G(t_2)| P_{z_2}(t_2) dt_2 \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} \log|F(t_1, t_2, z_3, \ldots, z_n)| \frac{y_2}{(x_2 - t_2)^2 + y_2^2} dt_2.\end{aligned} \tag{21}$$

Together with (20) and (21), we obtain

$$\begin{aligned}\log|F(z)| &\leq \frac{1}{\pi^2}\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \log|F(t_1, t_2, z_3, \ldots, z_n)| \frac{y_2}{(x_2 - t_2)^2 + y_2^2}\, dt_2\right) \\ &\quad \frac{y_1}{(x_1 - t_1)^2 + y_1^2} dt_1 \\ &= \frac{1}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \log|F(t_1, t_2, z_3, \ldots, z_n)| \frac{y_2}{(x_2 - t_2)^2 + y_2^2} \\ &\quad \frac{y_1}{(x_1 - t_1)^2 + y_1^2} dt_2 dt_1.\end{aligned}$$

Repeating this argument for $z_3, \ldots, z_n$, we obtain

$$\begin{aligned}\log|F(z)| &\leq \frac{1}{\pi^n}\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \log|F(t_1, t_2, \ldots, t_n)| \prod_{j=1}^{n} \frac{y_j}{(x_j - t_j)^2 + y_j^2} dt_n \cdots dt_1 \\ &\leq \int_{\mathbb{R}^n} \log|F(t)| P(x - t, y) dt.\end{aligned} \tag{22}$$

Applying Jensen's inequality, with the convex increasing function $f(x) = e^{qx},\ 0 < q \leq \infty$, and the probability measure $P(x - t, y)dt$, to (22) of above, we obtain

$$
\begin{aligned}
|F(z)|^q &= e^{\log|F(z)|^q} = e^{q\log|F(z)|} = f(\log|F(z)|) \\
&\le f\left(\int_{\mathbb{R}^n} \log|F(t)|P(x-t,y)dt\right) \\
&\le \int_{\mathbb{R}^n} f(\log|F(t)|)P(x-t,y)dt \\
&\le \int_{\mathbb{R}^n} |F(t)|^q P(x-t,y)dt.
\end{aligned}
$$

So that,

$$\int_{\mathbb{R}^n} |F(x+iy)|^q \, dt \le \|f\|_q \|P_y\|_1 = \|f\|_q < \infty.$$

Therefore,

$$\sup_{y\in\Gamma_{\sigma_1}} \int_{\mathbb{R}^n} |F(x+iy)|^q \, dt \le \|f\|_q < \infty.$$

This, together with $F(z) \in H^p(T_{\Gamma_{\sigma_1}})$, imply $F(z) \in H^q(T_{\Gamma_{\sigma_1}})$. Thus, the proof of Theorem 3.2 is complete. □

Proof of Theorem 3.3 "Only if" part: If $0 < p < 1$, $f \in L^p(\mathbb{R}^n)$ and $f \in H^p_{\sigma_1}(\mathbb{R}^n)$, then there exists $f(z) \in H^p(T_{\Gamma_{\sigma_1}})$ such that $f(x)$ is the non-tangential boundary limit of function $f(z)$. By Lemma 4.1, there exists a constant $C(\varepsilon)$ which is independent on f, such that

$$|f(x+iy)| \le C(\varepsilon)\|f\|_{H^p}, \quad z = x+iy \in T_{\Gamma_{\sigma_1}}.$$

Let $f_n(z) = f(z+z_n)$, $z_n = (\frac{i}{n},\ldots,\frac{i}{n})$, then $f_n \in H^p(T_{\Gamma_{\sigma_1}}) \cap H^2(T_{\Gamma_{\sigma_1}})$. So, by Theorem 3.1, we obtain that $f_n(x) \in L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$, and $\operatorname{supp}\hat{f}_n \subset \overline{\Gamma}_{\sigma_1}$. By Lemma 4.4, there holds

$$\lim_{n\to\infty} ||f - f_n||_{L_p} = \lim_{n\to\infty} ||f - f_n||_{H^p} = 0.$$

The necessity of Theorem 3.3 is proved.

"If" part: If there exists a sequence of functions $\{f_n\}$ satisfying $f_n \in L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$, and $\operatorname{supp}\hat{f}_n \subset \overline{\Gamma}_{\sigma_1}$. Then by Paley–Wiener Theorem, we have

$$f_n(z) = \int_{\Gamma_{\sigma_1}} \hat{f}_n(t) e^{2\pi i t\cdot z} dt \quad (z \in T_{\Gamma_{\sigma_1}}),$$

and $f_n(z) \in H^2(T_{\Gamma_{\sigma_1}})$. So, by Theorem 3.2, we get that $f_n(z) \in H^p(T_{\Gamma_{\sigma_1}})$. Moreover, by Lemma 4.4, $||f_m - f_n||_{L^p} = ||f_m - f_n||_{H^p}$. Thus, there exists a $f(z) \in H^p(T_{\Gamma_{\sigma_1}})$ such that

$$\lim_{n\to\infty} ||f - f_n||_{L^p} = \lim_{n\to\infty} ||f - f_n||_{H^p} = 0.$$

This implies that $f(x) \in H^p_{\sigma_1}(T_{\Gamma_{\sigma_1}})$. Thus, the proof of Theorem 3.3 is complete. □

References

1. Bochner, S.: Bounded analytic functions in several variables and multiple laplace integrals. Am. J. Math. **59**(4), 732–738 (1937)
2. Deng, G.T.: Complex analysis. Beijing Normal University Press, Beijing **(in Chinese)** (2010)
3. Deng, G.T., Qian, T.: Rational approximation of functions in hardy spaces. Complex Anal. Oper. Theory **10**(5), 903–920 (2016)
4. Duren, P.L.: Theory of H^p Spaces. Pure and Applied Mathematics, vol. 38. Academic Press, New York (1970)
5. Deng, G.T., Li, H.C., Qian, T.: Rational Approximation of Functions in Hardy Spaces, New Trends in Analysis and Interdisciplinary Applications, pp. 189–197. Birkhäuser, Cham (2017)
6. Garnett, J.B.: Bounded Analytic Functions. Pure and Applied Mathematics, vol. 96. Academic Press, Inc., New York (1981)
7. Hörmander, L.: An Introduction to Complex Analysis in Several Variables. North-Holland Mathematical Library, vol. 7, 3rd edn. North-Holland Publishing Co., Amsterdam (1990)
8. Li, H.C., Deng, G.T., Qian, T.: Fourier spectrum characterization of H^p spaces on tubes. Complex Anal. Oper. Theory **5**, 1190–1218 (2018)
9. Pandey, J.N., Singh, O.P.: Characterization of functions with Fourier transform supported on orthants. J. Math. Anal. Appl. **185**(2), 438–463 (1994)
10. Qian, T.: Characterization of boundary values of functions in Hardy spaces with applications in signal analysis. J. Integral Equ. Appl. **17**(2), 159–198 (2005)
11. Qian, T., Xu, Y.S., Yan, D.Y., Yan, L.X., Yu, B.: Fourier spectrum characterization of Hardy spaces and applications. Proc. Am. Math. Soc. **137**(3), 971–980 (2009)
12. Rudin, W.: Real and Complex Analysis. McGraw-Hill Book Co., New York (1966)
13. Stein, E.M., Weiss, G.: Introduction to Fourier Analysis on Euclidean Spaces. Princeton Mathematical Series, No. 32. Princeton University Press, Princeton (1971)
14. Walsh, J.L.: Interpolation and Approximation by Rational Functions in the Complex Domain. American Mathematical Society Colloquium Publications. American Mathematical Society, Providence (1965)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.