

# Spectra of rational orthonormal systems

Qihui Chen<sup>1</sup>, Pei Dang<sup>2,\*</sup> & Tao Qian<sup>3</sup>

<sup>1</sup>*Department of Mathematics, College of Mathematics and Informatics,  
South China Agricultural University, Guangzhou 510642, China;*

<sup>2</sup>*Faculty of Information Technology, Macau University of Science and Technology, Macao, China;*

<sup>3</sup>*Macao Institute of Systems Engineering, Macau University of Science and Technology, Macao, China*

*Email: chenqihui@hotmail.com, pdang@must.edu.mo, tqian@must.edu.mo*

Received October 20, 2016; accepted May 3, 2018; published online September 6, 2019

**Abstract** We use the Weyl correspondence approach to investigate the spectral operators of the Laguerre and the Kautz systems. We show that the basic functions in the Laguerre and the Kautz systems are the eigenvectors of modified Sturm-Liouville operators. The results are further generalized to multiple parameters of one complex variable in both the unit disc and the upper half-plane contexts.

**Keywords** spectral operator, Laguerre and Kautz systems, TM system, Caley transform, Heisenberg group, Weyl correspondence

**MSC(2010)** 42A38, 44A15, 62P30, 94A20

**Citation:** Chen Q H, Dang P, Qian T. Spectra of rational orthonormal systems. *Sci China Math*, 2019, 62: 1961–1976, <https://doi.org/10.1007/s11425-016-9433-4>

## 1 Introduction

In rational approximation of one complex variable, one cannot avoid the so-called rational orthogonal systems, or alternatively, Takenaka-Malmquist (TM) systems [20]. In the context of the unit disc, a TM system is an infinite sequence of parameterized rational functions

$$E_{\vec{a}_n}(z) = \frac{\sqrt{1-|a_n|^2}}{1-\bar{a}_n z} \prod_{k=0}^{n-1} \frac{z-a_k}{1-\bar{a}_k z}, \quad z \in \mathbb{D}, \quad n = 0, 1, \dots, \quad (1.1)$$

where  $a_k$  are any complex number in  $\mathbb{D}$  and  $\vec{a}_n = (a_0, \dots, a_{n-1}, a_n)^T$ . The multiple product part of each  $E_{\vec{a}_n}$  is a Blaschke product with  $n$  zeros, being the product of the explicitly given  $n$  Möbius transforms, while the remainder is an  $L^2$ -normalized Szegő kernel. Different choices for the zeros result in different model structures. When all the  $a_k$ 's are identical to a fixed real number  $a \in (-1, 1)$ , we get the Laguerre systems [10, 11, 21, 22]; and when all  $a_k$ 's are equal to a complex number  $a \in \mathbb{D}$ , it corresponds to the Kautz systems [21, 22]:

$$E_n(z) = \frac{\sqrt{1-|a|^2}}{1-\bar{a}z} \left( \frac{z-a}{1-\bar{a}z} \right)^n, \quad z \in \mathbb{D}, \quad n = 0, 1, \dots \quad (1.2)$$

\* Corresponding author

More specifically, when all  $a_k$ 's are zero the system reduces to the classical finite impulse response (FIR) models, a half of the Fourier system [5], namely,  $\{z^k\}_{k=0}^{\infty}$ , a basis of the Hardy spaces  $H^p(\mathbb{D})$ ,  $1 < p < \infty$ . For general parameters  $a_k$  is a system  $\{E_{\vec{a}_n}\}_{n=0}^{\infty}$  which is the basis of a Hardy  $H^p$  space for  $p \in (1, \infty)$  only if the hyperbolic non-separability condition is satisfied. For  $p = 2$ , a TM system is always an orthonormal system of  $H^2(\mathbb{D})$ , regardless of whether it is a basis, i.e., regardless of whether the hyperbolic non-separability condition is met.

Such systems have been well studied in pure mathematics and applied science [2, 3, 6, 8, 16, 19, 20, 23, 24]. In recent decades, engineers have paid a significant amount of attention to rational orthogonal systems in the unit disc case [1, 12–14, 18]. For system identification of stable linear time-invariant systems, it is crucial, in practice, to form dynamical models from measured data. Rational model structures, such as the ARX (autoregressive with exogenous terms) and ARMAX (autoregressive moving-average with exogenous terms) models are natural choices, because almost all systems can be described by rational transfer functions (Hardy space functions) [9, 10]. Recently, researchers in time-frequency analysis have shown an increasing interest in rational orthogonal systems to investigate analytic signals from a nonlinear phase in  $L^2(\partial\mathbb{D})$  or  $H_{\pm}^2(\partial\mathbb{D})$  with a positive instantaneous frequency. Here,  $H_{+}^2(\partial\mathbb{D})$  and  $H_{-}^2(\partial\mathbb{D})$  consist of the non-tangential boundary limits of the complex Hardy  $H^2$  functions inside and outside the unit disc, respectively. Due to  $E_{\vec{a}_n} \in H^2(\mathbb{D})$  and taking the non-tangential boundary limits of  $E_{\vec{a}_n}$ , we get the boundary TM systems  $\{e_{\vec{a}_n}\}_{n=0}^{\infty} \subset H_{+}^2(\partial\mathbb{D})$  defined by

$$e_{\vec{a}_n}(t) := \lim_{r \rightarrow 1^-} E_{\vec{a}_n}(re^{it}), \quad t \in \mathbb{T} = [-\pi, \pi] \quad (1.3)$$

for some vector  $\vec{a}_n = (a_0, \dots, a_{n-1}, a_n)^T \in \mathbb{D}^{n+1}$ , and in particular, the Laguerre and the Kautz systems

$$e_n(t) := \lim_{r \rightarrow 1^-} E_n(re^{it}), \quad t \in \mathbb{T}. \quad (1.4)$$

The system  $\{e_{\vec{a}_n}\}_{n=0}^{\infty}$  consisting of basic functions of nonlinear phases is an orthonormal basis of  $H_{+}^2(\partial\mathbb{D})$  if the hyperbolic non-separability condition is met. Here, a nonlinear phase is such that each  $e_{\vec{a}_n}$  has the polarized factorization  $e_{\vec{a}_n}(t) = \rho_n(t)e^{i\theta_n(t)}$  with the nonlinear phase function  $\theta_n$ . Since the space  $L^2(\partial\mathbb{D})$  can be expressed as the direct sum of the two relevant boundary Hardy spaces, namely,  $L^2(\partial\mathbb{D}) = H_{+}^2(\partial\mathbb{D}) \oplus H_{-}^2(\partial\mathbb{D})$ , the system  $\{e_{\vec{a}_n}\}_{n=0}^{\infty} \cup \{\overline{e_{\vec{a}_n}}\}_{n=1}^{\infty}$  is an orthonormal basis of  $L^2(\partial\mathbb{D})$  if the two systems are bases in their respective spaces.

The fundamentality of the Fourier system demonstrated by its explicit representations, generality, and effectiveness depends, to a large extent on the general entries of the system. Research results on TM systems and their general terms, namely, nonlinear Fourier atoms can be regarded as advances of Fourier theory. They include a Bedrosian identity, nonlinear phase basis, adaptive algorithm (see [4, 15, 17, 18]).

Notice that the operator  $\frac{d^2}{dt^2} : C^2[-\pi, \pi] \rightarrow C[-\pi, \pi]$  has a discrete spectrum, a discrete set of the eigenfunctions  $e^{in}$ , that is the Fourier system. (Notice that if the domain is changed to  $\mathbb{R}$ , the spectrum will be continuous and  $e^{ix}$  will be the generalized eigenvectors.) It is natural to ask whether any general rational system would have a spectral operator  $\mathcal{L}$ . This note gives a part of the answer to this question. We will use the Weyl correspondence theory to investigate the spectral operators of the Laguerre systems and the Kautz systems  $\{e_n\}_{n=0}^{\infty}$ . Concretely, we will look for  $F \in \mathcal{S}'(\mathbb{R}^2)$  to generate a differential operator  $\mathcal{L}$  through the Weyl transform (2.2) such that  $e_n$  is the eigenvector of  $\mathcal{L}$ . We also generalize the results to the cases of multiple parameters with a complex variable in both the unit disc and the upper half-plane contexts.

We found that the spectral operators of the Laguerre systems and the Kautz systems are closely related to the general Sturm-Liouville operators. We will deal with the general Sturm-Liouville operators under the framework of Heisenberg group and Weyl correspondence.

Section 2 reviews the Heisenberg group and Weyl correspondence, and then discusses how to generate the Sturm-Liouville operators. Section 3 focuses on the spectral operator of the Laguerre systems and the Kautz systems. Section 4 generalizes the results of Section 3 to the multiple-parameter cases through the one complex variable setting. Section 5 considers the upper half-plane. Section 6 discusses the Cayley transformation method that converts the results of the upper half-plane to the unit disc and vice versa.

## 2 Weyl correspondence and Sturm-Liouville operators

The group representation of Heisenberg group suggests an exponential type operator  $e^{2\pi i(p\mathcal{D}+q\mathcal{X})}$  (see (2.1)) and then leads to the Weyl correspondence theory to generate pseudo-differential operators. The full Heisenberg group is the set  $(\mathbb{R}^n)^2 \times \mathbb{R}$  with the multiplication

$$(p, q, d)(\tilde{p}, \tilde{q}, \tilde{d}) = \left( p + \tilde{p}, q + \tilde{q}, d + \tilde{d} + \frac{1}{2}[(p, q), (\tilde{p}, \tilde{q})] \right),$$

where the symplectic product on the phase space  $(\mathbb{R}^n)^2$  is defined by  $[(p, q), (\tilde{p}, \tilde{q})] = p\tilde{q} - q\tilde{p}$ . For  $p, q \in \mathbb{R}^n, d \in \mathbb{R}$ , denoted by  $\mathcal{R}_d, \mathcal{M}_q$  and  $\mathcal{T}_p$ , the usual rotation, modulation, and translation operators are defined, respectively, by

$$\mathcal{R}_d f(x) = e^{\pi i d} f(x), \quad \mathcal{M}_q f(x) = e^{2\pi i q x} f(x), \quad \mathcal{T}_p f(x) = f(x - p), \quad x \in \mathbb{R}^n.$$

The Schrödinger representation of the full Heisenberg group is the unitary operator  $\mathcal{R}_d \mathcal{M}_{\frac{q}{2}} \mathcal{T}_{-p} \mathcal{M}_{\frac{q}{2}}$ . Up to the rotation factor  $\mathcal{R}_d$ , the symmetric form  $\mathcal{M}_{\frac{q}{2}} \mathcal{T}_{-p} \mathcal{M}_{\frac{q}{2}}$  is crucial in harmonic analysis in the phase space. We adopt Folland's notation [7]:

$$\rho(p, q) = e^{2\pi i(p\mathcal{D}+q\mathcal{X})} = \mathcal{M}_{\frac{q}{2}} \mathcal{T}_{-p} \mathcal{M}_{\frac{q}{2}}. \tag{2.1}$$

Essentially,  $e^{2\pi i(p\mathcal{D}+q\mathcal{X})}$  is the operator mapping  $f(\cdot)$  to  $e^{\pi i p q} e^{2\pi i q(\cdot)} f(\cdot + p)$ . Here,  $\mathcal{D} = \frac{1}{2\pi i} (\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n})^T$  and  $\mathcal{X} f = (x_1 f, \dots, x_n f)^T$ .

Another symmetric form  $\mathcal{M}_{\frac{p}{2}} \mathcal{T}_q \mathcal{M}_{\frac{p}{2}}$  gives rise to the operator

$$\tilde{\rho}(p, q) = e^{2\pi i(p\mathcal{X}-q\mathcal{D})} = \mathcal{M}_{\frac{p}{2}} \mathcal{T}_q \mathcal{M}_{\frac{p}{2}},$$

which maps  $f(\cdot)$  to  $e^{-\pi i p q} e^{2\pi i p(\cdot)} f(\cdot - q)$ . The identity  $\mathcal{M}_{\frac{p}{2}} \mathcal{T}_q \mathcal{M}_{\frac{p}{2}} = \mathcal{F} \mathcal{M}_{\frac{q}{2}} \mathcal{T}_{-p} \mathcal{M}_{\frac{q}{2}} \mathcal{F}^{-1}$  implies the relationship between  $e^{2\pi i(p\mathcal{D}+q\mathcal{X})}$  and  $e^{2\pi i(p\mathcal{X}-q\mathcal{D})}$ ,

$$\tilde{\rho}(p, q) = e^{2\pi i(p\mathcal{X}-q\mathcal{D})} = \mathcal{F} e^{2\pi i(p\mathcal{D}+q\mathcal{X})} \mathcal{F}^{-1} = \mathcal{F} \rho(p, q) \mathcal{F}^{-1}.$$

The Fourier transform on  $\mathbb{R}^n$  is defined by  $\mathcal{F}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \xi} dx, \xi \in \mathbb{R}^n$ . The Weyl transform

$$\rho(F) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} F(p, q) \rho(p, q) dp dq = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} F(p, q) e^{2\pi i(p\mathcal{D}+q\mathcal{X})} dp dq \tag{2.2}$$

with  $F \in L^1(\mathbb{R}^{2n})$ , can be used to obtain pseudo-differential operators for function spaces defined on  $\mathbb{R}^n$ . Here, the integral is an ordinary Bochner integral if  $F \in L^1(\mathbb{R}^{2n})$ .

The explicit formula for the operator  $\rho(F)$  is

$$\begin{aligned} \rho(F)f(x) &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} F(p, q) e^{\pi i p q + 2\pi i q x} f(x + p) dp dq \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} F(y - x, q) e^{\pi i q(x+y)} f(y) dy dq. \end{aligned}$$

Naturally,  $\rho(F)$  can be regarded as an integral operator with the kernel

$$K_F(x, y) = \int_{\mathbb{R}^n} F(y - x, q) e^{\pi i q(x+y)} dq = (\mathcal{F}_2^{-1} F) \left( y - x, \frac{y + x}{2} \right), \tag{2.3}$$

where  $\mathcal{F}_j$  denotes the partial Fourier transform for the  $j$ -th variable.

**Proposition 2.1.** *The map  $\rho$  from  $L^1(\mathbb{R}^{2n})$  to the space of bounded operators on  $L^2(\mathbb{R}^n)$ , defined by (2.2), extends uniquely to a bijection from  $\mathcal{S}'(\mathbb{R}^{2n})$  to the spaces of continuous linear maps from  $\mathcal{S}(\mathbb{R}^n)$  to  $\mathcal{S}'(\mathbb{R}^n)$ . Moreover,  $\rho$  maps  $L^2(\mathbb{R}^{2n})$  unitarily onto the space of Hilbert-Schmidt operators on  $L^2(\mathbb{R}^n)$ , and  $\rho(F)$  is a compact operator on  $L^2(\mathbb{R}^n)$  for all  $F \in L^1(\mathbb{R}^{2n})$ .*

Weyl's prescription for assigning an operator  $\sigma(\mathcal{D}, \mathcal{X})$  to a function  $\sigma(\xi, x)$  amounts to postulating that the exponential function  $e^{2\pi i(p\xi + qx)}$  should correspond to the operator  $\rho(p, q) = e^{2\pi i(p\mathcal{D} + q\mathcal{X})}$  defined in (2.1). Once this is granted, one can expand an arbitrary  $\sigma(\xi, x)$  in terms of an exponential via the inverse Fourier transform  $\sigma(\xi, x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{\sigma}(p, q) e^{2\pi i(p\xi + qx)} dpdq$  to create a new operator

$$\sigma(\mathcal{D}, \mathcal{X}) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{\sigma}(p, q) e^{2\pi i(p\mathcal{D} + q\mathcal{X})} dpdq = \rho(\hat{\sigma}). \tag{2.4}$$

This integral is a Bochner integral if  $\hat{\sigma} \in L^1(\mathbb{R}^{2n})$ . Proposition 2.1 indicates that the notation  $\sigma(\mathcal{D}, \mathcal{X}) = \rho(\hat{\sigma})$  makes sense as an operator from  $\mathcal{S}(\mathbb{R}^n)$  to  $\mathcal{S}'(\mathbb{R}^n)$  whenever  $\hat{\sigma}$  and  $\sigma$  are both tempered distributions.

From (2.3),  $\sigma(\mathcal{D}, \mathcal{X}) = \rho(\hat{\sigma})$  is an integral operator whose distribution kernel is

$$\begin{aligned} K_\sigma(x, y) &= (\mathcal{F}_2^{-1} \hat{\sigma})\left(y - x, \frac{y + x}{2}\right) = (\mathcal{F}_1 \sigma)\left(y - x, \frac{y + x}{2}\right) \\ &= \int_{\mathbb{R}^n} \sigma\left(\xi, \frac{x + y}{2}\right) e^{2\pi i(x - y)\xi} d\xi. \end{aligned}$$

As a consequence, the operator  $\sigma(\mathcal{D}, \mathcal{X}) = \rho(\hat{\sigma})$  is given by

$$\sigma(\mathcal{D}, \mathcal{X})f(t) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \sigma\left(\xi, \frac{t + y}{2}\right) e^{2\pi i(t - y)\xi} f(y) dyd\xi. \tag{2.5}$$

In the rest of this section, we investigate differential operators of order 2 generated by (2.5), which maps  $\mathcal{S}(\mathbb{R})$  to  $\mathcal{S}'(\mathbb{R})$ .

**Theorem 2.2.** *Suppose that  $\sigma(\xi, t) = \sum_{k=0}^m r_k(t)\xi^k$ . Then the operator  $\sigma(\mathcal{D}, \mathcal{X})$  is given by*

$$\sigma(\mathcal{D}, \mathcal{X}) = \sum_{k=0}^m \left(\frac{1}{2\pi i}\right)^k \sum_{j=0}^k \binom{k}{j} \left(\frac{1}{2}\right)^{k-j} \frac{d^{k-j}}{dt^{k-j}} r_k(t) \frac{d^j}{dt^j}. \tag{2.6}$$

*Proof.* We write  $\sigma(\mathcal{D}, \mathcal{X})f(t)$  as

$$\begin{aligned} \sigma(\mathcal{D}, \mathcal{X})f(t) &= \int_{\mathbb{R}} \int_{\mathbb{R}} \sigma\left(\xi, \frac{t + y}{2}\right) e^{2\pi i(t - y)\xi} f(y) dyd\xi \\ &= \sum_{k=0}^m \int_{\mathbb{R}} \int_{\mathbb{R}} r_k\left(\frac{t + y}{2}\right) \xi^k e^{2\pi i(t - y)\xi} f(y) dyd\xi \\ &= \sum_{k=0}^m \int_{\mathbb{R}} \xi^k \left[ \int_{\mathbb{R}} f(y) r_k\left(\frac{t + y}{2}\right) e^{-2\pi iy\xi} dy \right] e^{2\pi it\xi} d\xi \\ &= \sum_{k=0}^m \int_{\mathbb{R}} \xi^k \mathcal{F}\left[f(\cdot) r_k\left(\frac{t + \cdot}{2}\right)\right](\xi) e^{2\pi it\xi} d\xi \\ &= \sum_{k=0}^m \mathcal{F}^{-1}\left\{ \xi^k \mathcal{F}\left[f(\cdot) r_k\left(\frac{t + \cdot}{2}\right)\right](\xi) \right\}(t). \end{aligned}$$

By using the relation  $\mathcal{F}^{-1}(\cdot^k \mathcal{F}f(\cdot))(x) = \left(\frac{1}{2\pi i}\right)^k \frac{d^k}{dx^k} f(x)$ , it follows that

$$\begin{aligned} \sigma(\mathcal{D}, \mathcal{X})f(t) &= \sum_{k=0}^m \left(\frac{1}{2\pi i}\right)^k \frac{d^k}{dx^k} \left[ f(x) r_k\left(\frac{t + x}{2}\right) \right]_{x=t} \\ &= \sum_{k=0}^m \left(\frac{1}{2\pi i}\right)^k \sum_{j=0}^k \binom{k}{j} \left(\frac{1}{2}\right)^{k-j} \frac{d^{k-j}}{dt^{k-j}} r_k(t) \frac{d^j}{dt^j} f(t). \end{aligned}$$

This completes the proof. □

In particular, when  $m = 2$ , we have

$$\begin{aligned} &\sigma(\mathcal{D}, \mathcal{X})f(t) \\ &= \sum_{k=0}^2 \left(\frac{1}{2\pi i}\right)^k \sum_{j=0}^k \binom{k}{j} \left(\frac{1}{2}\right)^{k-j} \frac{d^{k-j}}{dt^{k-j}} r_k(t) \frac{d^j}{dt^j} f(t) \\ &= r_0(t)f(t) + \frac{1}{2\pi i} \left(\frac{1}{2}r_1'(t)f(t) + r_1(t)f'(t)\right) - \frac{1}{4\pi^2} \left(\frac{1}{4}r_2''(t)f(t) + r_2'(t)f'(t) + r_2(t)f''(t)\right) \\ &= \left(r_0(t) + \frac{1}{4\pi i}r_1'(t) - \frac{1}{16\pi^2}r_2''(t)\right)f(t) + \left(\frac{1}{2\pi i}r_1(t) - \frac{1}{4\pi^2}r_2'(t)\right)f'(t) - \frac{1}{4\pi^2}r_2(t)f''(t). \end{aligned}$$

Therefore, we obtain the following corollary.

**Corollary 2.3.** Set  $\sigma(\xi, t) = r_0(t) + r_1(t)\xi + r_2(t)\xi^2$ . The operator  $\sigma(\mathcal{D}, \mathcal{X})$  defined in (2.5) is

$$\sigma(\mathcal{D}, \mathcal{X}) = -\frac{r_2(t)}{4\pi^2} \frac{d^2}{dt^2} + \left(\frac{1}{2\pi i}r_1(t) - \frac{1}{4\pi^2}r_2'(t)\right) \frac{d}{dt} + \left(r_0(t) + \frac{1}{4\pi i}r_1'(t) - \frac{1}{16\pi^2}r_2''(t)\right) \mathcal{I}. \tag{2.7}$$

**Theorem 2.4.** Suppose that  $\sigma(\mathcal{D}, \mathcal{X})$  is defined by (2.5). Then  $\sigma(\mathcal{D}, \mathcal{X}) = a(t)\frac{d^2}{dt^2} + b(t)\frac{d}{dt} + c(t)\mathcal{I}$  only if

$$\sigma(\xi, t) = \left(c(t) - \frac{1}{2}b'(t) + \frac{1}{4}a''(t)\right) + 2\pi i(b(t) - a'(t))\xi - 4\pi^2 a(t)\xi^2. \tag{2.8}$$

*Proof.* Set  $\sigma(\xi, t) = r_0(t) + r_1(t)\xi + r_2(t)\xi^2$ . Equation (2.7) leads to a relationship between  $(a, b, c)$  and  $(r_0, r_1, r_2)$  as follows:

$$\begin{aligned} a(t) &= -\frac{1}{4\pi^2}r_2(t), \\ b(t) &= \frac{1}{2\pi i}r_1(t) - \frac{1}{4\pi^2}r_2'(t), \\ c(t) &= r_0(t) + \frac{1}{4\pi i}r_1'(t) - \frac{1}{16\pi^2}r_2''(t). \end{aligned}$$

Solving these equations, we obtain that  $r_2(t) = -4\pi^2 a(t)$ ,  $r_1(t) = 2\pi i[b(t) + \frac{1}{4\pi^2}r_2'(t)] = 2\pi i[b(t) - a'(t)]$ , and

$$\begin{aligned} r_0(t) &= c(t) - \frac{1}{4\pi i}r_1'(t) + \frac{1}{16\pi^2}r_2''(t) \\ &= c(t) - \frac{1}{4\pi i}2\pi i[b(t) - a'(t)]' + \frac{1}{16\pi^2}[-4\pi^2 a(t)]'' \\ &= c(t) - \frac{1}{2}b'(t) + \frac{1}{4}a''(t). \end{aligned}$$

This completes the proof of this theorem. □

We now turn to the Sturm-Liouville operator. A general Sturm-Liouville operator  $\mathcal{L}_{u,v}$  has the form  $\mathcal{L}_{u,v} = u(t)\frac{d}{dt}v(t)\frac{d}{dt}$ , where  $u$  and  $v$  are defined on an interval  $I = (a, b)$ ,  $(a, +\infty)$ , or  $\mathbb{R}$ .

**Theorem 2.5.** The Sturm-Liouville operator  $\mathcal{L}_{u,v}$  can be generated through (2.5) with the kernel

$$\sigma(\xi, t) = \frac{1}{4} [u''(t)v(t) - u(t)v''(t)] - 2\pi i u'(t)v(t)\xi - 4\pi^2 u(t)v(t)\xi^2. \tag{2.9}$$

*Proof.* The identity (2.9) is a direct consequence of (2.8) for  $a(t) = u(t)v(t)$ ,  $b(t) = u(t)v'(t)$  and  $c(t) = 0$ . □

**Corollary 2.6.** The kernel function  $\sigma(\xi, t) = -2\pi i u(t)u'(t)\xi - 4\pi^2 u^2(t)\xi^2$  determines the operator

$$\mathcal{L}_u = u(t)\frac{d}{dt}u(t)\frac{d}{dt} = u^2(t)\frac{d^2}{dt^2} + u(t)u'(t)\frac{d}{dt}. \tag{2.10}$$

Throughout the paper, we will investigate the generalized Sturm-Liouville operators

$$\mathcal{L} = \mathcal{L}_u + \wp_{\alpha,\beta} \tag{2.11}$$

with  $\wp_{\alpha,\beta} = \alpha(t)\frac{d}{dt} + \beta(t)\mathcal{I}$  for some differential functions  $u, \alpha$  and  $\beta$ . For the kernel of the operator  $\mathcal{L}$ , we have the following theorem.

**Theorem 2.7.** *The kernel function*

$$\sigma(\xi, t) = \left( \beta(t) - \frac{1}{2}\alpha'(t) \right) + (2\pi i\alpha(t) - 2\pi i u(t)u'(t))\xi - 4\pi^2 u^2(t)\xi^2$$

determines the operator  $\mathcal{L}$  defined in (2.11).

*Proof.* By Theorem 2.4, we know that  $\sigma(\mathcal{D}, \mathcal{X}) = \alpha(t)\frac{d}{dt} + \beta(t)\mathcal{I}$  if and only if  $\sigma(\xi, t) = (\beta(t) - \frac{1}{2}\alpha'(t)) + 2\pi i\alpha(t)\xi$ . Combining this with Corollary 2.6 and (2.11), the proof is completed.  $\square$

### 3 Spectral operators of the Laguerre systems and the Kautz systems

For any parameter  $a$  in the unit disc,  $\mathbb{D} := \{z = x + iy : |z| < 1\}$ . Denote the Poisson kernel for the unit disc by

$$p_a(t) = \frac{1 - |a|^2}{|1 - \bar{a}e^{it}|^2}, \quad t \in \mathbb{T}. \tag{3.1}$$

Define the real-valued function  $\theta_a : \mathbb{T} = (-\pi, \pi) \rightarrow \mathbb{R}$  by  $e^{i\theta_a(t)} = \frac{z-a}{1-\bar{a}z}|_{z=e^{it}}$ ,  $t \in \mathbb{T}$  with the extension principle  $\theta_a(t + 2\pi) = 2\pi + \theta_a(t)$ . We note that the density function of the harmonic measure is  $\theta'_a(t) = p_a(t)$ .

Suppose that  $\mathcal{L}_u$  is defined in (2.10) with  $u = \frac{1}{p_a(\cdot)}$ . Define the differential operator  $\mathcal{L}$  by

$$\mathcal{L} = \mathcal{L}_u + \wp_u \tag{3.2}$$

with  $\wp_u = \wp_{\alpha,\beta}$  and  $\alpha(t) = \frac{2i}{(1-ae^{-it})p_a^2(t)}$  and  $\beta(t) = \frac{i}{(1-ae^{-it})p_a^2(t)}\left(i - \frac{p'_a(t)}{p_a(t)}\right)$ .

The main result of this section is the following.

**Theorem 3.1.** *Suppose that  $\mathcal{L}$  is defined in (3.2). Then*

$$\mathcal{L}(e_n(t)) = -(n + 1)^2 e_n(t). \tag{3.3}$$

*Proof.* To proceed with the proof, we need two identities. The first is

$$\frac{d}{dt}e_n(t) = i\left((n + 1)p_a(t) - \frac{1}{1 - ae^{-it}}\right)e_n(t), \quad t \in \mathbb{T}.$$

The second is

$$\begin{aligned} \frac{d^2}{dt^2}e_n(t) &= i(n + 1)\frac{p'_a(t)}{p_a^2(t)}e_n(t) - \frac{ae^{-it}}{(1 - ae^{-it})^2}e_n(t) - (n + 1)^2 p_a^2(t)e_n(t) \\ &\quad - \frac{1}{(1 - ae^{-it})^2}e_n(t) + \frac{2(n + 1)p_a(t)}{1 - ae^{-it}}e_n(t). \end{aligned}$$

Using the representations of  $\frac{d}{dt}e_n(t)$  and  $\frac{d^2}{dt^2}e_n(t)$ , we have

$$\begin{aligned} \frac{e''_n(t)}{p_a^2(t)} &= i(n + 1)\frac{p'_a(t)}{p_a^2(t)}e_n(t) - \frac{ae^{-it}}{(1 - ae^{-it})^2 p_a^2(t)}e_n(t) - (n + 1)^2 e_n(t) \\ &\quad - \frac{1}{(1 - ae^{-it})^2 p_a^2(t)}e_n(t) + \frac{2(n + 1)}{(1 - ae^{-it})p_a(t)}e_n(t) \end{aligned}$$

and

$$\begin{aligned} -\frac{p'_a(t)}{p_a^3(t)}e'_n(t) &= -i\frac{p'_a(t)}{p_a^3(t)}\left((n+1)p_a(t) - \frac{1}{1-ae^{-it}}\right)e_n(t) \\ &= -i\frac{p'_a(t)}{p_a^2(t)}(n+1)e_n(t) + i\frac{p'_a(t)}{p_a^3(t)}\frac{1}{1-ae^{-it}}e_n(t). \end{aligned}$$

On one hand, this gives

$$\begin{aligned} \mathcal{L}_u(e_n(t)) &= u^2(t)\frac{d^2}{dt^2}e_n(t) + u(t)u'(t)\frac{d}{dt}e_n(t) \\ &= i(n+1)\frac{p'_a(t)}{p_a^2(t)}e_n(t) - \frac{ae^{-it}}{(1-ae^{-it})^2p_a^2(t)}e_n(t) - (n+1)^2e_n(t) \\ &\quad - \frac{1}{(1-ae^{-it})^2p_a^2(t)}e_n(t) + \frac{2(n+1)}{(1-ae^{-it})p_a(t)}e_n(t) \\ &\quad - i\frac{p'_a(t)}{p_a^2(t)}(n+1)e_n(t) + i\frac{p'_a(t)}{p_a^3(t)}\frac{1}{1-ae^{-it}}e_n(t) \\ &= -\frac{ae^{-it}}{(1-ae^{-it})^2p_a^2(t)}e_n(t) - (n+1)^2e_n(t) \\ &\quad - \frac{1}{(1-ae^{-it})^2p_a^2(t)}e_n(t) + \frac{2(n+1)}{(1-ae^{-it})p_a(t)}e_n(t) + i\frac{p'_a(t)}{p_a^3(t)}\frac{1}{1-ae^{-it}}e_n(t) \\ &= -(n+1)^2e_n(t) + \frac{2(n+1)}{(1-ae^{-it})p_a(t)}e_n(t) - \frac{1+ae^{-it}}{(1-ae^{-it})^2p_a^2(t)}e_n(t) \\ &\quad + i\frac{p'_a(t)}{p_a^3(t)}\frac{1}{1-ae^{-it}}e_n(t). \end{aligned}$$

Conversely, we have

$$\begin{aligned} \frac{2i}{(1-ae^{-it})p_a^2(t)}\frac{d}{dt}e_n(t) &= \frac{2i}{(1-ae^{-it})p_a^2(t)}i\left((n+1)p_a(t) - \frac{1}{1-ae^{-it}}\right)e_n(t) \\ &= -\frac{2(n+1)}{(1-ae^{-it})p_a(t)}e_n(t) + \frac{2}{(1-ae^{-it})^2p_a^2(t)}e_n(t). \end{aligned}$$

Adding  $\mathcal{L}_u(e_n(t))$  and  $\frac{2i}{(1-ae^{-it})p_a^2(t)}\frac{d}{dt}e_n(t)$ , we have

$$\begin{aligned} \mathcal{L}_u(e_n(t)) + \frac{2i}{(1-ae^{-it})p_a^2(t)}\frac{d}{dt}e_n(t) &= -(n+1)^2e_n(t) + \frac{1}{(1-ae^{-it})p_a^2(t)}e_n(t) + i\frac{p'_a(t)}{p_a^3(t)}\frac{1}{1-ae^{-it}}e_n(t). \end{aligned}$$

Hence,

$$\begin{aligned} \mathcal{L}_u(e_n(t)) + \frac{2i}{(1-ae^{-it})p_a^2(t)}\frac{d}{dt}e_n(t) - \frac{1}{(1-ae^{-it})p_a^2(t)}e_n(t) - i\frac{p'_a(t)}{p_a^3(t)}\frac{1}{1-ae^{-it}}e_n(t) &= \\ = -(n+1)^2e_n(t), \end{aligned}$$

which is identical to

$$\mathcal{L}_u(e_n(t)) + \frac{2i}{(1-ae^{-it})p_a^2(t)}\left(\frac{d}{dt} + \frac{i}{2} - \frac{1}{2}\frac{p'_a(t)}{p_a(t)}\right)e_n(t) = -(n+1)^2e_n(t).$$

Therefore,  $\mathcal{L}_u(e_n(t)) + \wp_u(e_n(t)) = -(n+1)^2e_n(t)$ . This completes the proof. □

### 4 Multiple parameters

This section deals with three aspects. First, we establish a more general theorem as Theorem 3.1. Second, we offer a complex variable treatment. Third, we build a bridge between a boundary TM system and a TM system.

### 4.1 Boundary TM systems

Set

$$p_{\bar{a}_n}(t) = \sum_{j=0}^n p_{a_j}(t), \quad t \in \mathbb{T}, \tag{4.1}$$

where  $p_a$  is defined in (3.1).

Now recall the Sturm-Liouville operator  $\mathcal{L}_u$  in (2.10). For the sequence  $\{\bar{a}_n : n = 0, 1, \dots\}$  of parameter vectors in the TM system, define the operator sequence  $\mathcal{L}_{u_n, \mathbb{T}}$  by

$$\mathcal{L}_{u_n, \mathbb{T}} = u_n(t) \frac{d}{dt} u_n(t) \frac{d}{dt} = u_n^2(t) \frac{d^2}{dt^2} + u_n(t) u_n'(t) \frac{d}{dt} \tag{4.2}$$

with  $u_n = \frac{1}{p_{\bar{a}_n}}$ . Essentially,

$$\mathcal{L}_{u_n, \mathbb{T}} = \frac{1}{(p_{\bar{a}_n}(t))^2} \frac{d^2}{dt^2} - \frac{p'_{\bar{a}_n}(t)}{(p_{\bar{a}_n}(t))^3} \frac{d}{dt}.$$

Define the sequence  $\{\wp_{u_n} : n = 0, 1, \dots\}$  of differential operators by

$$\wp_{u_n, \mathbb{T}} = \frac{2i}{(1 - a_n e^{-it}) p_{\bar{a}_n}^2(t)} \left( \frac{d}{dt} + \frac{i}{2} - \frac{1}{2} \frac{p'_{\bar{a}_n}(t)}{p_{\bar{a}_n}(t)} \right). \tag{4.3}$$

Define

$$\mathcal{L}_{n, \mathbb{T}} = \mathcal{L}_{u_n, \mathbb{T}} + \wp_{u_n, \mathbb{T}}. \tag{4.4}$$

**Theorem 4.1.** Suppose that the operator  $\mathcal{L}_{n, \mathbb{T}}$  is defined in (4.4). Then

$$\mathcal{L}_{n, \mathbb{T}}(e_{\bar{a}_n}) = -e_{\bar{a}_n}. \tag{4.5}$$

*Proof.* The proof is like that of Theorem 3.1. The following two identities are crucial:

$$\frac{d}{dt} e_{\bar{a}_n}(t) = i \left( p_{\bar{a}_n}(t) - \frac{1}{1 - a_n e^{-it}} \right) e_{\bar{a}_n}(t), \quad t \in \mathbb{T}$$

and

$$\begin{aligned} \frac{d^2}{dt^2} e_{\bar{a}_n}(t) &= i p'_{\bar{a}_n}(t) e_{\bar{a}_n}(t) - \frac{a_n e^{-it}}{(1 - a_n e^{-it})^2} e_{\bar{a}_n}(t) - p_{\bar{a}_n}^2(t) e_{\bar{a}_n}(t) \\ &\quad - \frac{1}{(1 - a_n e^{-it})^2} e_{\bar{a}_n}(t) + \frac{2 p_{\bar{a}_n}(t)}{1 - a_n e^{-it}} e_{\bar{a}_n}(t). \end{aligned}$$

On one hand, by using the representations of  $\frac{d}{dt} e_{\bar{a}_n}(t)$  and  $\frac{d^2}{dt^2} e_{\bar{a}_n}(t)$ , we get

$$\begin{aligned} \frac{e''_{\bar{a}_n}(t)}{p_{\bar{a}_n}^2(t)} &= i \frac{p'_{\bar{a}_n}(t)}{p_{\bar{a}_n}^2(t)} e_{\bar{a}_n}(t) - \frac{a_n e^{-it}}{(1 - a_n e^{-it})^2 p_{\bar{a}_n}^2(t)} e_{\bar{a}_n}(t) - e_{\bar{a}_n}(t) \\ &\quad - \frac{1}{(1 - a_n e^{-it})^2 p_{\bar{a}_n}^2(t)} e_{\bar{a}_n}(t) + \frac{2}{(1 - a_n e^{-it}) p_{\bar{a}_n}(t)} e_{\bar{a}_n}(t) \end{aligned}$$

and

$$\begin{aligned} -\frac{p'_{\bar{a}_n}(t)}{p_{\bar{a}_n}^3(t)} e'_{\bar{a}_n}(t) &= -i \frac{p'_{\bar{a}_n}(t)}{p_{\bar{a}_n}^3(t)} \left[ p_{\bar{a}_n}(t) - \frac{1}{1 - a_n e^{-it}} \right] e_{\bar{a}_n}(t) \\ &= -i \frac{p'_{\bar{a}_n}(t)}{p_{\bar{a}_n}^2(t)} e_{\bar{a}_n}(t) + i \frac{p'_{\bar{a}_n}(t)}{p_{\bar{a}_n}^3(t)} \frac{1}{1 - a_n e^{-it}} e_{\bar{a}_n}(t). \end{aligned}$$

Therefore,

$$\mathcal{L}_{u_n, \mathbb{T}}(e_{\bar{a}_n}(t)) = \frac{e''_{\bar{a}_n}(t)}{p_{\bar{a}_n}^2(t)} - \frac{p'_{\bar{a}_n}(t)}{p_{\bar{a}_n}^3(t)} e'_{\bar{a}_n}(t)$$

$$\begin{aligned}
 &= i \frac{p'_{\bar{a}_n}(t)}{p_{\bar{a}_n}^2(t)} e_{\bar{a}_n}(t) - \frac{a_n e^{-it}}{(1 - a_n e^{-it})^2 p_{\bar{a}_n}^2(t)} e_{\bar{a}_n}(t) - e_{\bar{a}_n}(t) \\
 &\quad - \frac{1}{(1 - a_n e^{-it})^2 p_{\bar{a}_n}^2(t)} e_{\bar{a}_n}(t) + \frac{2}{(1 - a_n e^{-it}) p_{\bar{a}_n}(t)} e_{\bar{a}_n}(t) \\
 &\quad - i \frac{p'_{\bar{a}_n}(t)}{p_{\bar{a}_n}^2(t)} e_{\bar{a}_n}(t) + i \frac{p'_{\bar{a}_n}(t)}{p_{\bar{a}_n}^3(t)} \frac{1}{1 - a_n e^{-it}} e_{\bar{a}_n}(t) \\
 &= -\frac{a_n e^{-it}}{(1 - a_n e^{-it})^2 p_{\bar{a}_n}^2(t)} e_{\bar{a}_n}(t) - e_{\bar{a}_n}(t) - \frac{1}{(1 - a_n e^{-it})^2 p_{\bar{a}_n}^2(t)} e_{\bar{a}_n}(t) \\
 &\quad + \frac{2}{(1 - a_n e^{-it}) p_{\bar{a}_n}(t)} e_{\bar{a}_n}(t) + i \frac{p'_{\bar{a}_n}(t)}{p_{\bar{a}_n}^3(t)} \frac{1}{1 - a_n e^{-it}} e_{\bar{a}_n}(t) \\
 &= -e_{\bar{a}_n}(t) + \frac{2}{(1 - a_n e^{-it}) p_{\bar{a}_n}(t)} e_{\bar{a}_n}(t) - \frac{1 + a_n e^{-it}}{(1 - a_n e^{-it})^2 p_{\bar{a}_n}^2(t)} e_{\bar{a}_n}(t) \\
 &\quad + i \frac{p'_{\bar{a}_n}(t)}{p_{\bar{a}_n}^3(t)} \frac{1}{1 - a_n e^{-it}} e_{\bar{a}_n}(t).
 \end{aligned}$$

Conversely, we have

$$\begin{aligned}
 \frac{2i}{(1 - a_n e^{-it}) p_{\bar{a}_n}^2(t)} \frac{d}{dt} e_{\bar{a}_n}(t) &= \frac{2i}{(1 - a_n e^{-it}) p_{\bar{a}_n}^2(t)} i \left( p_{\bar{a}_n}(t) - \frac{1}{1 - a_n e^{-it}} \right) e_{\bar{a}_n}(t) \\
 &= -\frac{2}{(1 - a_n e^{-it}) p_{\bar{a}_n}(t)} e_{\bar{a}_n}(t) + \frac{2}{(1 - a_n e^{-it})^2 p_{\bar{a}_n}^2(t)} e_{\bar{a}_n}(t).
 \end{aligned}$$

The summation of  $\mathcal{L}_{u_n, \mathbb{T}}(e_{\bar{a}_n}(t))$  and  $\frac{2i}{(1 - a_n e^{-it}) p_{\bar{a}_n}^2(t)} \frac{d}{dt} e_{\bar{a}_n}(t)$  is

$$\begin{aligned}
 &\mathcal{L}_{u_n, \mathbb{T}}(e_{\bar{a}_n}(t)) + \frac{2i}{(1 - a_n e^{-it}) p_{\bar{a}_n}^2(t)} \frac{d}{dt} e_{\bar{a}_n}(t) \\
 &= -e_{\bar{a}_n}(t) + \frac{2}{(1 - a_n e^{-it}) p_{\bar{a}_n}(t)} e_{\bar{a}_n}(t) - \frac{1 + a_n e^{-it}}{(1 - a_n e^{-it})^2 p_{\bar{a}_n}^2(t)} e_{\bar{a}_n}(t) \\
 &\quad + i \frac{p'_{\bar{a}_n}(t)}{p_{\bar{a}_n}^3(t)} \frac{1}{1 - a_n e^{-it}} e_{\bar{a}_n}(t) - \frac{2}{(1 - a_n e^{-it}) p_{\bar{a}_n}(t)} e_{\bar{a}_n}(t) + \frac{2}{(1 - a_n e^{-it})^2 p_{\bar{a}_n}^2(t)} e_{\bar{a}_n}(t) \\
 &= -e_{\bar{a}_n}(t) + \frac{1}{(1 - a_n e^{-it}) p_{\bar{a}_n}^2(t)} e_{\bar{a}_n}(t) + i \frac{p'_{\bar{a}_n}(t)}{p_{\bar{a}_n}^3(t)} \frac{1}{1 - a_n e^{-it}} e_{\bar{a}_n}(t).
 \end{aligned}$$

Hence,

$$\begin{aligned}
 &\mathcal{L}_{u_n, \mathbb{T}}(e_{\bar{a}_n}(t)) + \frac{2i}{(1 - a_n e^{-it}) p_{\bar{a}_n}^2(t)} \frac{d}{dt} e_{\bar{a}_n}(t) - \frac{1}{(1 - a_n e^{-it}) p_{\bar{a}_n}^2(t)} e_{\bar{a}_n}(t) \\
 &\quad - i \frac{p'_{\bar{a}_n}(t)}{p_{\bar{a}_n}^3(t)} \frac{1}{1 - a_n e^{-it}} e_{\bar{a}_n}(t) = -e_{\bar{a}_n}(t),
 \end{aligned}$$

from which, we obtain

$$\mathcal{L}_{u_n, \mathbb{T}}(e_{\bar{a}_n}(t)) + \frac{2i}{(1 - a_n e^{-it}) p_{\bar{a}_n}^2(t)} \left( \frac{d}{dt} + \frac{i}{2} - \frac{1}{2} \frac{p'_{\bar{a}_n}(t)}{p_{\bar{a}_n}(t)} \right) e_{\bar{a}_n} = -e_{\bar{a}_n}.$$

By noting the definition of  $\wp_{u_n, \mathbb{T}}$  in (4.3), we see that the above equation is equivalent to

$$\mathcal{L}_{u_n, \mathbb{T}}(e_{\bar{a}_n}(t)) + \wp_{u_n, \mathbb{T}}(e_{\bar{a}_n}(t)) = -e_{\bar{a}_n}(t).$$

This confirms (4.5) and finishes the proof of the theorem. □

**Remark 4.2.** Equation (4.5) does not reduce to (3.3) when all the parameters  $a_j$ 's are identical. In fact, when all  $a_j$ 's are equal to  $a$ , the operators  $\mathcal{L}_{u_n, \mathbb{T}}$  and  $\wp_{u_n, \mathbb{T}}$ , respectively, reduce to  $\mathcal{L}_u$  and  $\wp$  up to the factor  $\frac{1}{n^2}$ , i.e.,  $\mathcal{L}_{u_n, \mathbb{T}} = \frac{1}{n^2} \mathcal{L}_u$  and  $\wp_{u_n, \mathbb{T}} = \frac{1}{n^2} \wp_u$ . It is clear that (4.5) gives rise to (3.3).

### 4.2 Complex variable setting

For  $a \in \mathbb{D}$ , denoted by  $P_a(z)$  the Poisson kernel for the unit disc can be written as follows:

$$P_a(z) = \frac{1 - |a|^2}{(z - a)(1 - \bar{a}z)}, \quad z \in \mathbb{D}. \tag{4.6}$$

Set

$$P_{\vec{a}_n}(z) = \sum_{j=0}^n P_{a_j}(z), \quad z \in \mathbb{D} \tag{4.7}$$

for vector  $\vec{a}_n = (a_0, \dots, a_{n-1}, a_n)^T \in \mathbb{D}^{n+1}$ . Define  $\mathcal{L}_{u_n, \mathbb{D}}$  by

$$\mathcal{L}_{u_n, \mathbb{D}} = u_n(z) \frac{d}{dz} u_n(z) \frac{d}{dz} = \frac{1}{P_{\vec{a}_n}^2(z)} \frac{d^2}{dz^2} - \frac{P'_{\vec{a}_n}(z)}{P_{\vec{a}_n}^3(z)} \frac{d}{dz} \tag{4.8}$$

with  $u_n = \frac{1}{P_{\vec{a}_n}}$ . Let

$$\wp_{u_n, \mathbb{D}} = \frac{2}{(z - a_n)P_{\vec{a}_n}^2(z)} \left( \frac{d}{dz} - \frac{1}{2} \frac{P'_{\vec{a}_n}(z)}{P_{\vec{a}_n}(z)} \right) \tag{4.9}$$

and

$$\mathcal{L}_{n, \mathbb{D}} = \mathcal{L}_{u_n, \mathbb{D}} + \wp_{u_n, \mathbb{D}}. \tag{4.10}$$

**Theorem 4.3.** *Suppose that the operator  $\mathcal{L}_{n, \mathbb{D}}$  and the system  $\{E_{\vec{a}_n}\}$  are defined in (4.10) and (1.1), respectively. Then*

$$\mathcal{L}_{n, \mathbb{D}}(E_{\vec{a}_n}(z)) = E_{\vec{a}_n}(z), \quad z \in \mathbb{D}. \tag{4.11}$$

*Proof.* With disintegrated computation, the proof is like that of Theorem 4.1 by invoking the identity

$$\frac{d}{dz} E_{\vec{a}_n}(z) = \left( P_{\vec{a}_{n-1}}(z) + \frac{\bar{a}_n}{1 - \bar{a}_n z} \right) E_{\vec{a}_n}(z) = \left( P_{\vec{a}_n}(z) - \frac{1}{z - a_n} \right) E_{\vec{a}_n}(z), \quad z \in \mathbb{D}.$$

This completes the proof. □

### 4.3 From the complex TM system to the boundary TM system

This section addresses the relationship between  $\mathcal{L}_{n, \mathbb{D}}$  and  $\mathcal{L}_{n, \mathbb{T}}$ . We will prove that by setting  $z = e^{it}$  in (4.11) so that we obtain (4.5). In fact, by setting

$$R(t) = \frac{1}{P_{\vec{a}_n}^2(e^{it})} E_{\vec{a}_n}''(e^{it}) - \frac{P'_{\vec{a}_n}(e^{it})}{P_{\vec{a}_n}^3(e^{it})} E_{\vec{a}_n}'(e^{it}) + \frac{2}{(e^{it} - a_n)P_{\vec{a}_n}^2(e^{it})} \left[ E_{\vec{a}_n}'(e^{it}) - \frac{1}{2} \frac{P'_{\vec{a}_n}(e^{it})}{P_{\vec{a}_n}(e^{it})} E_{\vec{a}_n}(e^{it}) \right],$$

and  $L(t) = E_{\vec{a}_n}(e^{it}) = e_{\vec{a}_n}(t)$ ,  $t \in \mathbb{T}$ , and by setting  $z = e^{it}$  in (4.11), the relation can be written as  $R(t) = L(t)$ .

The following theorem indicates that  $R(t) = -\mathcal{L}_{n, \mathbb{T}}(e_{\vec{a}_n}(t))$ , from which we conclude that (4.11) with  $z = e^{it}$  leads to (4.5).

**Theorem 4.4.** *Suppose that  $\mathcal{L}_{n, \mathbb{T}}$  is the operator defined in (4.4) and  $R(t)$  is the right-hand side of (4.11) with  $z = e^{it}$ . Then  $R(t) = -\mathcal{L}_{n, \mathbb{T}}(e_{\vec{a}_n}(t))$ .*

*Proof.* Using the relations between  $E_{\vec{a}_n}$  and  $e_{\vec{a}_n}$ ,

$$\begin{aligned} E_{\vec{a}_n}'(e^{it}) &= -ie^{-it} e_{\vec{a}_n}'(t), \quad t \in \mathbb{T}, \\ E_{\vec{a}_n}''(e^{it}) &= ie^{-2it} (e_{\vec{a}_n}'(t) + ie_{\vec{a}_n}''(t)), \quad t \in \mathbb{T}, \end{aligned}$$

and similar relations for  $P_{\vec{a}_n}$  and  $p_{\vec{a}_n}$ ,

$$P_{\vec{a}_n}(e^{it}) = e^{-it} p_{\vec{a}_n}(t), \quad t \in \mathbb{T},$$

$$P'_{\bar{a}_n}(e^{it}) = -e^{-2it}(p_{\bar{a}_n}(t) + ip'_{\bar{a}_n}(t)), \quad t \in \mathbb{T},$$

we have

$$\begin{aligned} R(t) &= \frac{1}{e^{-2it}p_{\bar{a}_n}^2(t)} ie^{-2it}(e'_{\bar{a}_n}(t) + ie''_{\bar{a}_n}(t)) - \frac{-e^{-2it}(p_{\bar{a}_n}(t) + ip'_{\bar{a}_n}(t))}{e^{-3it}p_{\bar{a}_n}^3(t)} (-i)e^{-it}e'_{\bar{a}_n}(t) \\ &\quad + \frac{2}{(e^{it} - a_n)e^{-2it}p_{\bar{a}_n}^2(t)} \left[ -ie^{-it}e'_{\bar{a}_n}(t) - \frac{1}{2} \frac{-e^{-2it}(p_{\bar{a}_n}(t) + ip'_{\bar{a}_n}(t))}{e^{-it}p_{\bar{a}_n}(t)} e_{\bar{a}_n}(t) \right] \\ &= \frac{1}{p_{\bar{a}_n}^2(t)} (ie'_{\bar{a}_n}(t) - e''_{\bar{a}_n}(t)) - \frac{i(p_{\bar{a}_n}(t) + ip'_{\bar{a}_n}(t))}{p_{\bar{a}_n}^3(t)} e'_{\bar{a}_n}(t) \\ &\quad + \frac{2}{(e^{it} - a_n)e^{-it}p_{\bar{a}_n}^2(t)} \left[ -ie'_{\bar{a}_n}(t) + \frac{1}{2} \frac{(p_{\bar{a}_n}(t) + ip'_{\bar{a}_n}(t))}{p_{\bar{a}_n}(t)} e_{\bar{a}_n}(t) \right] \\ &= -\frac{e''_{\bar{a}_n}(t)}{p_{\bar{a}_n}^2(t)} + \frac{p'_{\bar{a}_n}(t)}{p_{\bar{a}_n}^3(t)} e'_{\bar{a}_n}(t) \\ &\quad + \frac{2}{(1 - a_n e^{-it})p_{\bar{a}_n}^2(t)} \left[ -ie'_{\bar{a}_n}(t) + \frac{1}{2} e_{\bar{a}_n}(t) + \frac{i}{2} \frac{p'_{\bar{a}_n}(t)}{p_{\bar{a}_n}(t)} e_{\bar{a}_n}(t) \right] \\ &= -\frac{e''_{\bar{a}_n}(t)}{p_{\bar{a}_n}^2(t)} + \frac{p'_{\bar{a}_n}(t)}{p_{\bar{a}_n}^3(t)} e'_{\bar{a}_n}(t) \\ &\quad + \frac{2i}{(1 - a_n e^{-it})p_{\bar{a}_n}^2(t)} \left[ -e'_{\bar{a}_n}(t) - \frac{i}{2} e_{\bar{a}_n}(t) + \frac{1}{2} \frac{p'_{\bar{a}_n}(t)}{p_{\bar{a}_n}(t)} e_{\bar{a}_n}(t) \right] \\ &= -\mathcal{L}_{u_n, \mathbb{T}}(e_{\bar{a}_n}(t)) - \wp_{u_n, \mathbb{T}}(e_{\bar{a}_n}(t)) = -\mathcal{L}_{n, \mathbb{T}}(e_{\bar{a}_n}(t)). \end{aligned}$$

This completes the proof of the theorem. □

### 5 Upper half-plane

There is a parallel theory for TM systems on the upper half-plane. In the upper half-plane context, we say  $f \in H^p(\mathbb{C}^+)$ ,  $0 < p < \infty$ , if  $f$  is analytic on  $\mathbb{C}^+$  and  $\sup_{y>0} \int_{-\infty}^{\infty} |f(x + iy)|^p dx = \|f\|_{H^p(\mathbb{C}^+)}^p < \infty$ . When  $p = \infty$ , we write  $f \in H^\infty(\mathbb{C}^+)$  for the totality of all the bounded analytic functions on  $\mathbb{C}^+$ , and we give  $H^\infty(\mathbb{C}^+)$  the norm  $\|f\|_{H^\infty(\mathbb{C}^+)} = \sup_{w \in \mathbb{C}^+} |f(w)|$ . The relation between  $f \in H^p(\mathbb{C}^+)$  and their non-tangential boundary limits on  $\mathbb{R}$  is the same as for the unit disc.  $H^2(\mathbb{C}^+)$  has inner product

$$\langle f, g \rangle_{\mathbb{C}^+} = \int_{-\infty}^{\infty} f(t) \overline{g(t)} dt, \quad f, g \in H^2(\mathbb{C}^+).$$

For a given parameter sequence  $\{\lambda_n\}_{n=0}^\infty \subset \mathbb{C}^+$ , the corresponding TM system  $\{\beta_{\bar{\lambda}_n}\}_{n=0}^\infty$  on the upper half-plane  $\mathbb{C}^+$  is

$$\beta_{\bar{\lambda}_n}(z) = \frac{\sqrt{\frac{1}{\pi} \text{Im}\{\lambda_n\}}}{z - \bar{\lambda}_n} \prod_{j=0}^{n-1} \frac{z - \lambda_j}{z - \bar{\lambda}_j}, \quad z \in \mathbb{C}^+, \quad n \in \mathbb{Z}_+, \tag{5.1}$$

with vector  $\vec{\lambda}_n = (\lambda_0, \dots, \lambda_{n-1}, \lambda_n)^T \in (\mathbb{C}^+)^{n+1}$ . Under the condition  $\sum_{k=0}^\infty \frac{\sqrt{\text{Im}(\lambda_k)}}{1 + |\lambda_k|^2} = \infty$ ,  $\{\beta_{\bar{\lambda}_n}\}$  is an orthonormal basis of the Hardy space  $H^p(\mathbb{C}^+)$ ,  $1 < p < \infty$ .

The corresponding boundary TM system  $\{\beta_{\bar{\lambda}_n}\}_{n=0}^\infty$  on the upper-half plane case is

$$\beta_{\bar{\lambda}_n}(t) := \lim_{\text{Im}(z) \rightarrow 0^+} \beta_{\bar{\lambda}_n}(z) = \frac{\sqrt{\frac{1}{\pi} \text{Im}\{\lambda_n\}}}{t - \bar{\lambda}_n} \prod_{j=0}^{n-1} \frac{t - \lambda_j}{t - \bar{\lambda}_j}, \quad t \in \mathbb{R}, \quad n = 0, 1, \dots \tag{5.2}$$

### 5.1 For a boundary TM system on the upper half-plane

For  $\lambda \in \mathbb{C}^+$ , the Poisson kernel  $q_\lambda$  on the real line is defined by

$$q_\lambda(t) = \frac{2\text{Im}\{\lambda\}}{|t - \lambda|^2}, \quad t \in \mathbb{R}. \tag{5.3}$$

Generally, for vector  $\vec{\lambda}_n = (\lambda_0, \dots, \lambda_{n-1}, \lambda_n)^T$ , set

$$q_{\vec{\lambda}_n}(t) = \sum_{j=0}^n q_{\lambda_j}(t), \quad t \in \mathbb{R}. \tag{5.4}$$

For a sequence  $\{\vec{\lambda}_n : n = 0, 1, \dots\}$  of parameter vectors on the upper half-plane, define the operator sequence  $\mathcal{L}_{u_n}$  by

$$\mathcal{L}_{u_n, \mathbb{R}} = u_n(t) \frac{d}{dt} u_n(t) \frac{d}{dt} = u_n^2(t) \frac{d^2}{dt^2} + u_n(t) u_n'(t) \frac{d}{dt} \tag{5.5}$$

with  $u_n = \frac{1}{q_{\vec{\lambda}_n}}$ . Essentially,

$$\mathcal{L}_{u_n, \mathbb{R}} = \frac{1}{q_{\vec{\lambda}_n}^2(t)} \frac{d^2}{dt^2} - \frac{q'_{\vec{\lambda}_n}(t)}{q_{\vec{\lambda}_n}^3(t)} \frac{d}{dt}.$$

Set

$$\wp_{u_n, \mathbb{R}} = \frac{2}{(t - \lambda_n) q_{\vec{\lambda}_n}^2(t)} \left( \frac{d}{dt} - \frac{1}{2} \frac{q'_{\vec{\lambda}_n}(t)}{q_{\vec{\lambda}_n}(t)} \right) \tag{5.6}$$

and

$$\mathcal{L}_{n, \mathbb{R}} = \mathcal{L}_{u_n, \mathbb{R}} + \wp_{u_n, \mathbb{R}}. \tag{5.7}$$

**Theorem 5.1.** Suppose that the operator  $\mathcal{L}_{n, \mathbb{R}}$  and the system  $\{\beta_{\vec{\lambda}_n}\}$  are defined in (5.7) and (5.2), respectively. Then

$$\mathcal{L}_{n, \mathbb{R}}(\beta_{\vec{\lambda}_n}(t)) = -\beta_{\vec{\lambda}_n}(t), \quad t \in \mathbb{R}. \tag{5.8}$$

*Proof.* The identity

$$(t - \lambda_n) \beta_{\vec{\lambda}_n}(t) = \sqrt{\frac{1}{\pi} \text{Im}(\lambda_n)} \prod_{j=0}^n \frac{t - \lambda_j}{t - \bar{\lambda}_j}$$

implies that

$$\beta'_{\vec{\lambda}_n}(t) = \left( iq_{\vec{\lambda}_n}(t) - \frac{1}{t - \lambda_n} \right) \beta_{\vec{\lambda}_n}(t).$$

Furthermore, we have

$$\beta''_{\vec{\lambda}_n}(t) = \left[ iq'_{\vec{\lambda}_n}(t) + \frac{2}{(t - \lambda_n)^2} - q_{\vec{\lambda}_n}^2(t) - \frac{2i}{t - \lambda_n} q_{\vec{\lambda}_n}(t) \right] \beta_{\vec{\lambda}_n}(t).$$

Then

$$\begin{aligned} \mathcal{L}_{u_n, \mathbb{R}}(\beta_{\vec{\lambda}_n}(t)) &= \frac{1}{q_{\vec{\lambda}_n}^2(t)} \beta''_{\vec{\lambda}_n}(t) - \frac{q'_{\vec{\lambda}_n}(t)}{q_{\vec{\lambda}_n}^3(t)} \beta'_{\vec{\lambda}_n}(t) \\ &= \frac{1}{q_{\vec{\lambda}_n}^2(t)} \left( iq'_{\vec{\lambda}_n}(t) + \frac{2}{(t - \lambda_n)^2} - q_{\vec{\lambda}_n}^2(t) - \frac{2i}{t - \lambda_n} q_{\vec{\lambda}_n}(t) \right) \beta_{\vec{\lambda}_n}(t) \\ &\quad - \frac{q'_{\vec{\lambda}_n}(t)}{q_{\vec{\lambda}_n}^3(t)} \left( iq_{\vec{\lambda}_n}(t) - \frac{1}{t - \lambda_n} \right) \beta_{\vec{\lambda}_n}(t) \\ &= \left( \frac{2}{(t - \lambda_n)^2 q_{\vec{\lambda}_n}^2(t)} - 1 - \frac{2i}{(t - \lambda_n) q_{\vec{\lambda}_n}(t)} + \frac{q'_{\vec{\lambda}_n}(t)}{(t - \lambda_n) q_{\vec{\lambda}_n}^3(t)} \right) \beta_{\vec{\lambda}_n}(t) \end{aligned}$$

and

$$\begin{aligned} \wp_{u_n, \mathbb{R}}(\beta_{\vec{\lambda}_n}(t)) &= \frac{2}{(t - \lambda_n)q_{\vec{\lambda}_n}^2(t)} \left( \beta'_{\vec{\lambda}_n}(t) - \frac{1}{2} \frac{q'_{\vec{\lambda}_n}(t)}{q_{\vec{\lambda}_n}(t)} \beta_{\vec{\lambda}_n}(t) \right) \\ &= \frac{2}{(t - \lambda_n)q_{\vec{\lambda}_n}^2(t)} \left( \left( i q_{\vec{\lambda}_n}(t) - \frac{1}{t - \lambda_n} \right) \beta_{\vec{\lambda}_n}(t) - \frac{1}{2} \frac{q'_{\vec{\lambda}_n}(t)}{q_{\vec{\lambda}_n}(t)} \beta_{\vec{\lambda}_n}(t) \right) \\ &= \left( \frac{2i}{(t - \lambda_n)q_{\vec{\lambda}_n}(t)} - \frac{2}{(t - \lambda_n)^2 q_{\vec{\lambda}_n}^2(t)} - \frac{q'_{\vec{\lambda}_n}(t)}{(t - \lambda_n)q_{\vec{\lambda}_n}^3(t)} \right) \beta_{\vec{\lambda}_n}(t). \end{aligned}$$

Therefore,  $\mathcal{L}_{u_n, \mathbb{R}}(\beta_{\vec{\lambda}_n}(t)) + \wp_{u_n, \mathbb{R}}(\beta_{\vec{\lambda}_n}(t)) = -\beta_{\vec{\lambda}_n}(t)$ . This completes the proof of this theorem. □

### 5.2 For TM systems on the upper half-plane

For  $\lambda \in \mathbb{C}^+$ , we extend the Poisson kernel  $q_\lambda$  to the complex plane

$$Q_\lambda(z) = \frac{2\text{Im}\{\lambda\}}{(z - \lambda)(z - \bar{\lambda})}, \quad z \in \mathbb{C} \tag{5.9}$$

and

$$Q_{\vec{\lambda}_n}(z) = \sum_{j=0}^n Q_{\lambda_j}(z), \quad z \in \mathbb{C} \tag{5.10}$$

for vector  $\vec{\lambda}_n = (\lambda_0, \dots, \lambda_{n-1}, \lambda_n)^T \in (\mathbb{C}^+)^{n+1}$ .

For the sequence  $\{\vec{\lambda}_n : n = 0, 1, \dots\}$  of parameter vectors, define the operator sequence  $\mathcal{L}_{u_n, \mathbb{C}^+}$  by

$$\mathcal{L}_{u_n, \mathbb{C}^+} = u_n(z) \frac{d}{dz} u_n(z) \frac{d}{dz} = \frac{1}{Q_{\vec{\lambda}_n}^2(z)} \frac{d^2}{dz^2} - \frac{Q'_{\vec{\lambda}_n}(z)}{Q_{\vec{\lambda}_n}^3(z)} \frac{d}{dz} \tag{5.11}$$

with  $u_n = \frac{1}{Q_{\vec{\lambda}_n}}$ . Set

$$\wp_{u_n, \mathbb{C}^+} = \frac{2}{(z - \lambda_n)Q_{\vec{\lambda}_n}^2(z)} \left( \frac{d}{dz} - \frac{1}{2} \frac{Q'_{\vec{\lambda}_n}(z)}{Q_{\vec{\lambda}_n}(z)} \right) \tag{5.12}$$

and

$$\mathcal{L}_{n, \mathbb{C}^+} = \mathcal{L}_{u_n, \mathbb{C}^+} + \wp_{u_n, \mathbb{C}^+}. \tag{5.13}$$

Using the identity

$$\frac{d}{dz} \beta_{\vec{\lambda}_n}(z) = \left( i Q_{\vec{\lambda}_{n-1}}(z) - \frac{1}{z - \bar{\lambda}_n} \right) \beta_{\vec{\lambda}_n}(z) = \left( i Q_{\vec{\lambda}_n}(z) - \frac{1}{z - \lambda_n} \right) \beta_{\vec{\lambda}_n}(z), \quad z \in \mathbb{C},$$

and applying a similar discussion to that in the proof of Theorem 5.1, we obtain the following theorem.

**Theorem 5.2.** For the operator  $\mathcal{L}_{n, \mathbb{C}^+}$  and the system  $\{\beta_n\}$  as defined in (5.13) and (5.1), respectively, it holds that

$$\mathcal{L}_{n, \mathbb{C}^+}(\beta_{\vec{\lambda}_n}(z)) = -\beta_{\vec{\lambda}_n}(z), \quad z \in \mathbb{C}^+. \tag{5.14}$$

Notice that the operators  $\mathcal{L}_{n, \mathbb{R}}$  and  $\mathcal{L}_{n, \mathbb{C}^+}$ , defined respectively in (5.7) and (5.13) are almost the same because they are connected by the natural mapping  $z \rightarrow \text{Re}(z)$ . Essentially, this is due to the analyticity of  $\beta_{\vec{\lambda}_n}(z), z \in \mathbb{C}^+$  and the real analyticity of  $\beta_{\lambda_n}(t), t \in \mathbb{R}$ .

### 6 From the upper half-plane to the unit disc

This section will establish a relationship between the operators  $\mathcal{L}_{n,\mathbb{C}^+}$  and  $\mathcal{L}_{n,\mathbb{D}}$ . To build a bridge between  $\mathcal{L}_{n,\mathbb{C}^+}$  and  $\mathcal{L}_{n,\mathbb{D}}$ , we use the Cayley transform  $\kappa : \mathbb{D} \rightarrow \mathbb{C}^+$ :

$$\kappa(z) = i \frac{1-z}{1+z}, \quad z \in \mathbb{D},$$

which is a bijection between  $\mathbb{D}$  and  $\mathbb{C}^+$  with inverse  $\kappa^{-1} : \mathbb{C}^+ \rightarrow \mathbb{D}$ ,

$$\kappa^{-1}(z) = \frac{i-z}{i+z}, \quad z \in \mathbb{C}^+.$$

The following identity is crucial. It establishes a relationship between  $\beta_{\bar{\lambda}_n}$  and  $E_{\bar{a}_n}$  through the composition of  $\kappa$ :

$$\beta_{\bar{\lambda}_n}(\kappa(z)) = C_1(1+z)E_{\bar{a}_n}(z), \quad z \in \mathbb{D},$$

where  $(\lambda_j, a_j) \in \mathbb{C}^+ \times \mathbb{D}$  is the  $\kappa$ -pair ruled by  $\lambda_j = \kappa(a_j) = i \frac{1-a_j}{1+a_j}$  or  $a_j = \kappa^{-1}(\lambda_j) = \frac{i-\lambda_j}{i+\lambda_j}$  for  $j \in \{0, 1, \dots, n\}$ ,  $C_1 = \frac{1}{2\sqrt{\pi}}e^{i\alpha}$ , and

$$e^{i\alpha} = (-1)^{n+1}i \frac{1+\bar{a}_n}{|1+a_n|} \prod_{j=0}^{n-1} \frac{1+\bar{a}_j}{1+a_j}.$$

Replacing  $z \in \mathbb{C}^+$  by  $\kappa(z) \in \mathbb{C}^+, z \in \mathbb{D}$ , in (5.14), we get  $R(z) = L(z), z \in \mathbb{D}$ , with

$$\begin{aligned} R(z) &= \frac{1}{Q_{\bar{\lambda}_n}^2(\kappa(z))} \beta_{\bar{\lambda}_n}''(\kappa(z)) - \frac{Q'_{\bar{\lambda}_n}(\kappa(z))}{Q_{\bar{\lambda}_n}^3(\kappa(z))} \beta'_{\bar{\lambda}_n}(\kappa(z)) \\ &\quad + \frac{2}{(\kappa(z) - \lambda_n)Q_{\bar{\lambda}_n}^2(\kappa(z))} \left[ \beta'_{\bar{\lambda}_n}(\kappa(z)) - \frac{1}{2} \frac{Q'_{\bar{\lambda}_n}(\kappa(z))}{Q_{\bar{\lambda}_n}(\kappa(z))} \beta_{\bar{\lambda}_n}(\kappa(z)) \right] \end{aligned}$$

and

$$L(z) = -\beta_{\bar{\lambda}_n}(\kappa(z)) = -C_1(1+z)E_{\bar{a}_n}(z), \quad z \in \mathbb{D}.$$

The following theorem shows that  $R(z) = -C_1(1+z)\mathcal{L}_{n,\mathbb{D}}(E_{\bar{a}_n}(z))$  so that (5.14) reduces to (4.11) when we replace  $z \in \mathbb{C}^+$  by  $\kappa(z)$  for  $z \in \mathbb{D}$  in (5.14).

**Theorem 6.1.** *Suppose that  $\mathcal{L}_{n,\mathbb{D}}$  is the operator defined in (4.10) and  $R(z)$  is the right-hand side of (5.14) when we replace  $z \in \mathbb{C}^+$  by  $\kappa(z), z \in \mathbb{D}$ . Then  $R(z) = -C_1(1+z)\mathcal{L}_{n,\mathbb{D}}(E_{\bar{a}_n}(z)), z \in \mathbb{D}$ .*

*Proof.* We first prove two identities associated with the derivatives of  $\beta_{\bar{\lambda}_n}$ :

$$\beta'_{\bar{\lambda}_n}(\kappa(z)) = \frac{i}{2}C_1(1+z)^2[E_{\bar{a}_n}'(z) + (1+z)E'_{\bar{a}_n}(z)], \quad z \in \mathbb{D}$$

and

$$\beta''_{\bar{\lambda}_n}(\kappa(z)) = -C_1 \left[ \frac{1}{4}(1+z)^5 E''_{\bar{a}_n}(z) + (1+z)^4 E'_{\bar{a}_n}'(z) + \frac{1}{2}(1+z)^3 E_{\bar{a}_n}(z) \right], \quad z \in \mathbb{D}.$$

Similarly, by noting that  $\frac{2\text{Im}(\lambda)}{|1-i\lambda|^2} = \frac{1}{2}(1-|a|^2)$  for  $\kappa$ -pair  $(\lambda, a)$ ,  $Q_{\bar{\lambda}_n}$  satisfies both

$$Q_{\bar{\lambda}_n}(\kappa(z)) = \frac{1}{2}(1+z)^2 P_{\bar{a}_n}(z), \quad z \in \mathbb{D},$$

and

$$Q'_{\bar{\lambda}_n}(\kappa(z)) = \frac{i}{2}(1+z)^3 P_{\bar{a}_n}'(z) + \frac{i}{4}(1+z)^4 P'_{\bar{a}_n}(z), \quad z \in \mathbb{D}.$$

By substituting the expressions of  $Q_{\lambda_n}(\kappa(z))$ ,  $Q'_{\lambda_n}(\kappa(z))$  and  $\beta_{\lambda_n}^{(j)}(\kappa(z)), j = 0, 1, 2$ , into the above equation, it follows that

$$\begin{aligned}
 R(z) = & 4(1+z)^{-4} \frac{1}{P_{\bar{a}_n}^2(z)} (-C_1) \left[ \frac{1}{4}(1+z)^5 E''_{\bar{a}_n}(z) + (1+z)^4 E'_{\bar{a}_n}(z) + \frac{1}{2}(1+z)^3 E_{\bar{a}_n}(z) \right] \\
 & - \frac{\frac{i}{2}(1+z)^3 P_{\bar{a}_n}(z) + \frac{i}{4}(1+z)^4 P'_{\bar{a}_n}(z)}{\frac{1}{8}(1+z)^6 P_{\bar{a}_n}^3(z)} \frac{i}{2} C_1 (1+z)^2 [E_{\bar{a}_n}(z) + (1+z) E'_{\bar{a}_n}(z)] \\
 & + \frac{2}{(\kappa(z) - \lambda_n) \frac{1}{4}(1+z)^4 P_{\bar{a}_n}^2(z)} \left\{ \frac{i}{2} C_1 (1+z)^2 [E_{\bar{a}_n}(z) + (1+z) E'_{\bar{a}_n}(z)] \right. \\
 & \left. - \frac{1}{2} \frac{\frac{i}{2}(1+z)^3 P_{\bar{a}_n}(z) + \frac{i}{4}(1+z)^4 P'_{\bar{a}_n}(z)}{\frac{1}{2}(1+z)^2 P_{\bar{a}_n}(z)} C_1 (1+z) E_{\bar{a}_n}(z) \right\}.
 \end{aligned}$$

After some calculations, we get

$$\begin{aligned}
 -\frac{1}{C_1(1+z)} R(z) = & \frac{1}{P_{\bar{a}_n}^2(z)} [E''_{\bar{a}_n}(z) + 4(1+z)^{-1} E'_{\bar{a}_n}(z) + 2(1+z)^{-2} E_{\bar{a}_n}(z)] \\
 & - 2(1+z)^{-2} \frac{E_{\bar{a}_n}(z)}{P_{\bar{a}_n}^2(z)} - (1+z)^{-1} \frac{P'_{\bar{a}_n}(z)}{P_{\bar{a}_n}^3(z)} E_{\bar{a}_n}(z) - 2(1+z)^{-1} \frac{E'_{\bar{a}_n}(z)}{P_{\bar{a}_n}^2(z)} - \frac{P'_{\bar{a}_n}(z)}{P_{\bar{a}_n}^3(z)} E'_{\bar{a}_n}(z) \\
 & - \frac{2}{(\kappa(z) - \lambda_n) \frac{1}{4}(1+z)^2 P_{\bar{a}_n}^2(z)} \left[ \frac{i}{2} E'_{\bar{a}_n}(z) - \frac{i}{4} \frac{P'_{\bar{a}_n}(z)}{P_{\bar{a}_n}(z)} E_{\bar{a}_n}(z) \right] \\
 = & \frac{1}{P_{\bar{a}_n}^2(z)} [E''_{\bar{a}_n}(z) + 2(1+z)^{-1} E'_{\bar{a}_n}(z)] - (1+z)^{-1} \frac{P'_{\bar{a}_n}(z)}{P_{\bar{a}_n}^3(z)} E_{\bar{a}_n}(z) - \frac{P'_{\bar{a}_n}(z)}{P_{\bar{a}_n}^3(z)} E'_{\bar{a}_n}(z) \\
 & - \frac{2}{(\kappa(z) - \lambda_n) \frac{1}{4}(1+z)^2 P_{\bar{a}_n}^2(z)} \left[ \frac{i}{2} E'_{\bar{a}_n}(z) - \frac{i}{4} \frac{P'_{\bar{a}_n}(z)}{P_{\bar{a}_n}(z)} E_{\bar{a}_n}(z) \right] \\
 = & E_{\bar{a}_n}(z).
 \end{aligned}$$

Recalling the operator  $\mathcal{L}_{u_n, \mathbb{D}}$  with  $u_n = P_{\bar{a}_n}(z)$ , noting that  $\kappa(z) - \lambda_n = -\frac{2i}{1+a_n} \frac{z-a_n}{1+z}$  for the  $\kappa$ -pair  $(\lambda_n, a_n)$  and

$$\begin{aligned}
 & -\frac{2}{(\kappa(z) - \lambda_n) \frac{1}{4}(1+z)^2 P_{\bar{a}_n}^2(z)} \left[ \frac{i}{2} E'_{\bar{a}_n}(z) - \frac{i}{4} \frac{P'_{\bar{a}_n}(z)}{P_{\bar{a}_n}(z)} E_{\bar{a}_n}(z) \right] \\
 = & \frac{2(1+a_n)}{(z-a_n) P_{\bar{a}_n}^2(z)} (1+z)^{-1} \left[ E'_{\bar{a}_n}(z) - \frac{1}{2} \frac{P'_{\bar{a}_n}(z)}{P_{\bar{a}_n}(z)} E_{\bar{a}_n}(z) \right] \\
 = & \frac{2}{(z-a_n) P_{\bar{a}_n}^2(z)} (1+a_n) \left( 1 - \frac{z}{1+z} \right) \left[ E'_{\bar{a}_n}(z) - \frac{1}{2} \frac{P'_{\bar{a}_n}(z)}{P_{\bar{a}_n}(z)} E_{\bar{a}_n}(z) \right] \\
 = & \frac{2}{(z-a_n) P_{\bar{a}_n}^2(z)} \left( 1 - \frac{z-a_n}{1+z} \right) \left[ E'_{\bar{a}_n}(z) - \frac{1}{2} \frac{P'_{\bar{a}_n}(z)}{P_{\bar{a}_n}(z)} E_{\bar{a}_n}(z) \right] \\
 = & \wp_{u_n, \mathbb{D}}(E_{\bar{a}_n}(z)) - \frac{2}{(1+z) P_{\bar{a}_n}^2(z)} \left[ E'_{\bar{a}_n}(z) - \frac{1}{2} \frac{P'_{\bar{a}_n}(z)}{P_{\bar{a}_n}(z)} E_{\bar{a}_n}(z) \right],
 \end{aligned}$$

we obtain

$$\begin{aligned}
 -\frac{1}{C_1(1+z)} R(z) = & \mathcal{L}_{u_n, \mathbb{D}}(E_{\bar{a}_n}(z)) + 2(1+z)^{-1} \frac{E'_{\bar{a}_n}(z)}{P_{\bar{a}_n}^2(z)} - (1+z)^{-1} \frac{P'_{\bar{a}_n}(z)}{P_{\bar{a}_n}^3(z)} E_{\bar{a}_n}(z) \\
 & + \wp_{u_n, \mathbb{D}}(E_{\bar{a}_n}(z)) - \frac{2}{(1+z) P_{\bar{a}_n}^2(z)} \left[ E'_{\bar{a}_n}(z) - \frac{1}{2} \frac{P'_{\bar{a}_n}(z)}{P_{\bar{a}_n}(z)} E_{\bar{a}_n}(z) \right] \\
 = & \mathcal{L}_{n, \mathbb{D}}(E_{\bar{a}_n}(z)) + \wp_{u_n, \mathbb{D}}(E_{\bar{a}_n}(z)) \\
 = & \mathcal{L}_{n, \mathbb{D}}(E_{\bar{a}_n}(z)).
 \end{aligned}$$

This completes the proof of the theorem. □

**Acknowledgements** This work was supported by National Natural Science Foundation of China (Grant Nos. 11571083, 11971178 and 11701597), the starting grant of South China Agricultural University, the Science and Technology Development Fund, Macao SAR (Grant Nos. 154/2017/A3, 079/2016/A2 and FDCT 0123/2018/A3).

## References

- 1 Akcay H, Ninness B. Orthonormal basis functions for modelling continuous-time systems. *Signal Process*, 1999, 77: 261–274
- 2 Baratchart L, Olivi M, Wielonsky F. On a rational approximation problem in the real Hardy space  $H^2$ . *Theoret Comput Sci*, 1992, 94: 175–197
- 3 Bultheel A, Carrette P. Algebraic and spectral properties of general Toeplitz matrices. *SIAM J Control Optim*, 2003, 41: 1413–1439
- 4 Chen Q, Mai W, Zhang L, et al. System identification by discrete rational atoms. *Automatica*, 2015, 56: 53–59
- 5 Edwards R E. *Fourier Series, A Modern Introduction, Volume 2*. Graduate Texts in Mathematics. New York: Springer-Verlag, 1979
- 6 Epstein B. *Orthogonal Families of Analytic Functions*. New York: Macmillan, 1965
- 7 Folland G B. *Harmonic Analysis on Phase Space*. Princeton: Princeton University Press, 1989
- 8 Hoffman K. *Banach Spaces of Analytic Functions*. Englewood Cliffs: Prentice-Hall, 1962
- 9 Ljung L. *System Identification: Theory for the User*, 2nd ed. Upper Saddle River: Prentice-Hall, 1999
- 10 Mäkilä P. Laguerre series approximation of infinite dimensional systems. *Automatica*, 1990, 26: 333–345
- 11 Mäkilä P. Laguerre methods and  $H^\infty$  identification of continuous-time systems. *Internat J Control*, 1991, 53: 689–707
- 12 Malti R, Ekongolo S B, Ragot J. Dynamic SISO and MISO system approximation based on optimal Laguerre models. *IEEE Trans Automat Control*, 1998, 43: 1317–1322
- 13 Mendel J. A unified approach to the synthesis of orthonormal exponential functions useful in systems analysis. *IEEE Trans Systems Sci Cybernet*, 1966, 2: 54–62
- 14 Mi W, Qian T. Frequency domain identification: An algorithm based on adaptive rational orthogonal system. *Automatica*, 2012, 48: 1154–1162
- 15 Qian T. Intrinsic mono-component decomposition of functions: An advance of Fourier theory. *Math Methods Appl Sci*, 2010, 33: 880–891
- 16 Qian T, Chen Q, Tan L. The rational orthogonal systems are Schauder bases. *Complex Var Elliptic Equ*, 2014, 59: 841–846
- 17 Qian T, Wang R, Xu Y, et al. Orthonormal bases with nonlinear phase. *Adv Comput Math*, 2010, 33: 75–95
- 18 Qian T, Wang Y B. Adaptive decomposition into basic signals of non-negative instantaneous frequencies: A variation and realization of greedy algorithm. *Adv Comput Math*, 2011, 34: 279–293
- 19 Shen X C. *Approximation of Complex Functions*. Beijing: Science Press, 1992
- 20 Takenaka S. On the orthogonal functions and a new formula of interpolation. *Jpn J Math*, 1925, 2: 129–145
- 21 Wahlberg B. System identification using Kautz models. *IEEE Trans Automat Control*, 1994, 39: 1276–1282
- 22 Wahlberg B, Mäkilä P. On approximation of stable linear dynamical systems using Laguerre and Kautz functions. *Automatica*, 1996, 32: 693–708
- 23 Walsh J L. *Interpolation and Approximation by Rational Functions in the Complex Domain*, 4th ed. Providence: Amer Math Soc, 1965
- 24 Young R. *An Introduction to Nonharmonic Fourier Series*. New York: Academic Press, 1980