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Tao Qian
Macau Institute of Systems Engineering
Macau University of Science and Technology
Macao, China

Pengtao Li
School of Mathematics and Statistics
Qingdao University
Qingdao, Shandong, China

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This book is a sincere dedication to
Professor Alan McIntosh

Preface

From the idea to the content, this book is basically Alan McIntosh's theory. In this book, we state systemically the theory of singular integrals and Fourier multipliers on Lipschitz graphs and surfaces which stems largely from the famous "Coifman-McIntosh-Meyer Theorem" since 1980s. The book elaborates the basic framework, essential thoughts, and main results of the theory. At the same time, this book also serves as a comprehensive reference on recent developments of this topic.

The subject of Fourier multipliers on Lipschitz surfaces has a profound background in harmonic analysis and partial differential equations. When we study boundary value problems of second-order elliptic operators, we need to deal with L^2-boundedness of the Cauchy-type integral operators on Lipschitz curves γ. Because the kernel of the Cauchy integral is nonlinear and non-smooth, there exists an essential difficulty on the study of the corresponding singular integral operators. In 1977, by using techniques of complex analysis, C. P. Calderón first proved that the singular Cauchy integral operator is bounded on $L^2(\gamma)$ under the assumption that the Lipschitz constant is sufficiently small. For the general cases, R. Coifman, A. McIntosh, and Y. Meyer applied the method of multilinear operators to get rid of the restriction on the Lipschitz constant and obtained the L^2-boundedness of the singular Cauchy integral operator on any Lipschitz curve γ. In considering the L^p-boundedness, $1 < p < \infty$, of a linear or non-linear operator, from the view of point of harmonic analysis, its L^2-boundedness would be the core. In fact, the L^p-boundedness of an operator may be deduced from its L^2-boundedness by using the interpolation theorem and the duality of L^p.

The corresponding problem on higher dimensional spaces is the L^p-boundedness of the singular Cauchy integral operators on Lipschitz surfaces Σ. The increase of the spatial dimension requires to apply a truly original approach. To introduce a Cauchy-type complex structure on the Euclidean spaces $\mathbb{R}^n$, the most efficient way is to embed $\mathbb{R}^n$ into the Hamilton quaternions or the Clifford algebra $\mathbb{R}_{(n)}$. The L^2-boundedness of the singular integral operators with holomorphic kernels on the Lipschitz surfaces was proved independently by Li et al. [1] and Gaudry et al. [2]. In this book, we adopt the method of Gaudry et al.

There exists a one to one correspondence between the convolution integrals T_ϕ and the Fourier multipliers M_b. In 1994, C. Li, A. McIntosh, and T. Qian established an explicit and one-to-one correspondence between the Clifford monogenic kernels ϕ and the complex holomorphic symbols b on Lipschitz graphs Σ (see [3]), and obtained the Cauchy-Dunford functional calculus of the Dirac operator on Σ. Such functional calculus has three equivalent forms: the Cauchy-Dunford integrals, the singular integrals with holomorphic kernels, and the bounded holomorphic Fourier multipliers. Since 1996, T. Qian began to consider the analogy on the high-dimensional spheres, tours, and their Lipschitz perturbations, i.e., the theories on starlike Lipschitz surfaces. For the cases of the spheres in the quaternionic and the Clifford algebras $\mathbb{R}_{(n)}$, by a generalized Fueter's theorem, Qian [4, 5] obtained a correspondence between a class of H^∞-Fourier multipliers and a class of holomorphic kernels, and proved that the corresponding class of H^∞-Fourier multipliers, the corresponding singular integral operators, and the induced Cauchy-Dunford functional calculus of the spherical Dirac operators are equivalent. Moreover, the mentioned operators are all bounded on $L^p(\Sigma)$. We note that, as necessary technical preparations of proving the correspondence and the boundedness of the operators, generalizations of the quaternionic Fueter theorem to the Clifford algebras $\mathrm{R}(n)$ were achieved: the cases n being odd were obtained by M. Sce [6], while the cases n being even were done by Qian [7], in the latter the fractional Laplace operators were defined via the corresponding Fourier multipliers. So far, the Fueter's theorem and its n-dimensional generalizations seem to be the unique approach to dealing with the singular integrals on Lipschitz perturbations of the spheres. The approach to analysis on the spheres offered by the Fueter theorem and its generalizations is an art of mathematics that may be viewed as the Clifford algebra version of "Starting from the unit disc" (see [8]). The further generalizations of Fueter's theorem and inverse Fueter's theorem have independent interest and applications; we refer the reader to [9].

This book establishes singular integral and Fourier multiplier theories in three different contexts: the Lipschitz curve context in the one complex variable setting; the graph type Lipschitz surface context; and the starlike Lipschitz surface context. The later two contexts are with the Clifford algebra setting. Chapters 1 and 2 are devoted to the theory of singular integrals and Fourier multipliers on Lispchitz curves. In Chap. 1, we state the boundedness, the singular integral expression, and the functional calculus of the Fourier multipliers. The analogous theory on the Lipschitz perturbations of the unit disc is given in Chap. 2.

In Chaps. 3–5, we will state systemically how to deal with the singular integrals and Fourier multipliers on the Lipschitz surfaces Σ by the technique of Clifford analysis. In Chap. 3, in order to make it self-containing, we state some basic facts and necessary backgrounds, including the Dirac operators, the Fourier transform, and monogenic functions on the sectors. At the same time, as a preliminary of the holomorphic Fourier multipliers on the Lipschitz surfaces, we introduce the generalizations of Fueter's theorem. In Chap. 4, we prove a Clifford martingale $T(b)$ theorem which implies the boundedness of the Cauchy-type singular integral operators. As is indicated above, for the main results of this chapter, there exists a parallel but different proof. We refer the interested reader to [1]. Chapter 5 includes the correspondence between H^∞-Fourier multipliers, the singular integral operators of monogenic kernels the Lipschitz surface Σ, and the H^∞-functional calculus of the spherical Dirac operator. The results of this chapter indicate that the Fourier multipliers and the monogenic kernels on the sectors play important roles in the theoretical framework of the Fourier multipliers and the singular integral operators.

The primary purpose of Chaps. 6–8 is to present the theory of the holomorphic Fourier multipliers on the starlike Lipschitz surfaces. In Chap. 6, we expatiate the results on the H^∞-Fourier multipliers on the starlike Lipschitz surfaces via the high-dimensional generalization of Fueter's theorem obtained in Chap. 3. In this chapter, we will give a detailed account of the estimate of the kernels of the operators with monogenic kernels. Chapter 7 is based on some new results on the fractional holomorphic Fourier multipliers on the starlike Lipschitz surfaces. The research on this topic is inextricably linked with the recent developments in the hyperbolic Clifford analysis. Theoretically speaking, the occurrence of the so-called "Photogenic Cauchy transform" implies that the study of the fractional Fourier multipliers is necessary. A well-known example of such class of Fourier multipliers is the fractional differential and integral operators with respect to the Dirac operator on the starlike Lipschitz surfaces. In addition, our study is significant for boundary values problems of differential operators on the starlike Lipschitz surfaces. In Chap. 8, using the complex analysis of several variables, we generalize the results of Chaps. 6 and 7 to the case of n-tours and the n-dimensional complex spheres. Particularly, the Cauchy-type singular integrals obtained by Gong [10] were extended to a family of singular integrals with holomorphic kernels. We also obtain the corresponding results of the fractional integrals and differentials.

In this book, we give a panorama-like and detailed demonstration of the theory of the holomorphic Fourier multipliers on the Lipschitz curves and surfaces. Through the writing of this book, we attempt to bring out the following core idea. Although the disposing technicalities vary with the different settings, the theories of different contexts all obey the same philosophy: the equivalence between the operator algebra of the singular integrals, Fourier multiplier operators, and the Cauchy-Dunford functional calculus of the Dirac operators.

The writing and the publication of this book received the great supports of two academicians of the Chinese Academy of Sciences, Lan Wen and Xiangyu Zhou. We should express our gratitude to them. Several teachers and graduate students of University of Macau and Qingdao University helped the authors diagram the illustrations of this book. Hereon, the authors wish to say a word of hearty thanks to them.

Macao, China Tao Qian
Qingdao, China Pengtao Li
September 2018

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Contents

Nomenclature

$\mathbb{C}$	The complex plane
$\mathbb{C}^n$	The n-fold product of complex numbers
$\mathbb{C}_{(M)}$	The complex 2^M-dimensional Clifford algebra
$\mathbb{R}^n$	Euclidean n-spaces
$\mathbb{R}_{(M)}$	The real 2^M-dimensional Clifford algebra
$\{\mathcal{F}_m\}_{m=-\infty}^{\infty}$	A nondecreasing family of σ-field
$\{f_m^l\}_{m=-\infty}^{\infty}$	The left martingale generated by f
$B(x, R)$	Ball of radius R centered at x in $\mathbb{R}^n$
$C_{\omega,\pm}$	The heart-shaped regions on the complex plane
dx	Lebesgue measure
$E^l(E^r)$	The left (right) conditional expectation
$H^\infty\left(S_\omega^c\right)$	The class of H^∞ Fourier multipliers on sector $\left(S_\omega^c\right)$
$H^{p0}(\Delta)(H^{p0}(\Delta^c))$	The Hardy space on the bounded (unbounded) connected components of $\mathbb{R}_1^n \backslash \Sigma$
$H^s\left(S_{\omega,\pm}^c\right)\left(H^s\left(S_\omega^c\right)\right)$	The class of fractional holomorphic Fourier multipliers on $S_{\omega,\pm}^C\left(S_\omega^C\right)$
$K\left(H_\omega^c\right)$	The class of integral kernels related to H^∞ Fourier multipliers
$K^s\left(H_{\omega,\pm}^c\right)$	The class of integral kernels related to fractional holomorphic Fourier multipliers
$L^p(\gamma)$	The Lebesgue spaces on the curve γ
$L^p(\Sigma)$	Lebesgue spaces on the surface Σ
$N_\alpha^c(f)$	The exterior non-tangential maximal function of f
$N_\alpha(f)$	The interior non-tangential maximal function of f
$S_{\omega,\pm}$	The right- and left-half sector with the angle 2ω
S_ω	The sector with the angle ω
$W_{\omega,+}\left(W_{\omega,-}\right)$	The "W" (M)-shaped region on the complex plane

Chapter 1
Singular Integrals and Fourier Multipliers on Infinite Lipschitz Curves

The main contents of this chapter are closely related with harmonic analysis and operator theory. Let γ denote a Lipschitz graph on the complex plane $\mathbb{C}$:

$$\gamma = \Big\{x + ig(x) \in \mathbb{C} : \ x \in \mathbb{R}\Big\},$$

where g is a Lipschitz function satisfying $\|g'\|_\infty \leqslant N < \infty$. We will prove the L^p-boundedness of certain singular convolution integral operators on γ. The main results of this chapter are based on the theory of Fourier multipliers and the H^∞-functional calculus of type ω operators on the curve γ which are established by A. McIntosh and T. Qian in [1]. Roughly speaking, the type ω operators can be represented as $b(D_\gamma)$, where D_γ is the differential operator on γ, and b is a bounded holomorphic function defined on some sector S^0_ν, $\nu > \tan^{-1} N$. With the additional assumption g being bounded, A. McIntosh and T. Qian studied a class of generalized Fourier multipliers on γ, see [2, 3] for the related results.

For the boundedness of singular convolution integrals, there exist several different methods. In this chapter, we apply the method introduced by A. McIntosh and T. Qian. The proof depends on the quadratic estimates of the type ω operators on sectors. Precisely, we first prove that the quadratic estimates of the type ω operators are equivalent to the inverse quadratic estimates of the dual operators (see Theorem 1.2.1). Then we prove, if an operator T satisfies the quadratic estimates and the related inverse quadratic estimates, then for a bounded holomorphic function b, the holomorphic functional calculus $b(T)$ is bounded, see Theorem 1.2.3.

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1.1 Convolutions and Differentiation on Lipschitz Graphs

In this section, denote by $\mathbb{C}$ and $\mathbb{R}$ the complex number field and the real number field, respectively. We use γ to denote the following Lipschitz graph :

$$\gamma = \left\{x + ig(x) \in \mathbb{C},\ \text{ where } g \text{ is a Lipschitz function satisfying } \|g'\|_\infty \leqslant N < \infty\right\}.$$

We will use the following complex-valued function spaces.

Definition 1.1.1 (i) Let $1 \leqslant p \leqslant \infty$. $L^p(\gamma)$ denotes the space consisting of all equivalent classes of functions: $u : \gamma \to \mathbb{C}$ which are measurable for the measure $|dz|$ and satisfy

$$\|u\|_p = \left(\int_\gamma |u(z)|^p |dz|\right)^{1/p} < \infty,\ 1 \leqslant p < \infty$$

and

$$\|u\|_\infty = \text{ ess-sup}|u(z)| < \infty,$$

where "ess-sup" denotes the essential supremum.

(ii) Denote by $C_0(\gamma)$ the space of all continuous functions on γ which converge to 0 at infinity. The norm of $C_0(\gamma)$ is defined by

$$\|u\|_\infty = \max_{z\in\gamma} |u(z)|.$$

For $1 \leqslant p \leqslant \infty$, let $p' = p/(p-1)$. Define the pairing between $L^p(\gamma)$ and $L^{p'}(\gamma)$ as follows:

$$\langle u, v\rangle = \int_\gamma u(z)v(z)dz.$$

It can be proved that for $1 < p < \infty$, $(L^p(\gamma), L^{p'}(\gamma))$ is a dual pair of Banach spaces. For $p = 1$, $(L^1(\gamma), C_0(\gamma))$ is a dual pair of Banach spaces. Here

$$\|u\|_p = \sup\left\{|\langle u, v\rangle|, v \in L^{p'}(\gamma), \|v\|_{p'} = 1\right\}$$

and

$$\|u\|_1 = \sup\left\{|\langle u, v\rangle|, v \in C_0(\gamma), \|v\|_\infty = 1\right\}.$$

Suppose that ϕ is a function defined on a subset of $\mathbb{C}$ which contains $\Gamma = \{z - \xi,\ z \in \gamma, \xi \in \gamma\}$ and u is a measurable function on γ. If $\phi(z - \cdot)u(\cdot) \in L^1(\gamma)$, then the convolution of u and ϕ is defined by

$$(\phi * u)(z) = \int_\gamma \phi(z - \xi)u(\xi)d\xi.$$

Theorem 1.1.1 *Let* $1 \leqslant p \leqslant \infty$. *Assume that* $u \in L^p(\gamma)$ *and* $\phi(\cdot - z) \in L^1(\gamma)$ *for almost all* $z \in \gamma$. *Then*

$$\|\phi * u\|_p \leqslant \sup_{z\in\gamma}\Big(\int_\gamma |\phi(z-\xi)||d\xi|\Big)^{1/p'} \sup_{\xi\in\gamma}\Big(\int_\gamma |\phi(z-\xi)||dz|\Big)^{1/p}\|u\|_p,$$

where $1/p' = 1 - 1/p$.

Proof At first, notice that for almost $z \in \gamma$, $\phi(z - \cdot)u(\cdot)$ is measurable. Then if $1 < p < \infty$, we have

$$\begin{aligned}
\|\phi * u\|_p &\leqslant \Big[\int_\gamma \Big(\int_\gamma |\phi(z-\xi)u(\xi)||d\xi|\Big)^p |dz|\Big]^{1/p}\\
&\leqslant \Big[\int_\gamma \Big(\int_\gamma |\phi(z-\xi)||d\xi|\Big)^{p/p'}\Big(\int_\gamma |\phi(z-\xi)||u(\xi)|^p|d\xi|\Big)|dz|\Big]^{1/p}\\
&\leqslant \sup_{z\in\gamma}\Big(\int_\gamma |\phi(z-\xi)||d\xi|\Big)^{1/p'}\Big(\int_\gamma\int_\gamma |\phi(z-\xi)||u(\xi)|^p|d\xi||dz|\Big)^{1/p}\\
&\leqslant \sup_{z\in\gamma}\Big(\int_\gamma |\phi(z-\xi)||d\xi|\Big)^{1/p'} \sup_{\xi\in\gamma}\Big(\int_\gamma\int_\gamma |\phi(z-\xi)||dz|\Big)^{1/p}\|u\|_p.
\end{aligned}$$

The cases $p = 1$ and $p = \infty$ can be dealt with similarly. In fact, for $p = 1$,

$$\begin{aligned}
\|\phi * u\|_1 &\leqslant \Big[\int_\gamma \Big(\int_\gamma |\phi(z-\xi)u(\xi)||d\xi|\Big)|dz|\Big]\\
&\leqslant \sup_{\xi\in\gamma}\Big(\int_\gamma\int_\gamma |\phi(z-\xi)||dz|\Big)\|u\|_1.
\end{aligned}$$

For $p = \infty$,

$$\begin{aligned}
\|\phi * u\|_\infty &\leqslant \sup_{z\in\gamma}\Big(\int_\gamma |\phi(z-\xi)u(\xi)||d\xi|\Big)\\
&\leqslant \sup_{z\in\gamma}\Big(\int_\gamma |\phi(z-\xi)||d\xi|\Big)\|u\|_\infty.
\end{aligned}$$

□

Let $\omega = \arctan N$. Denote by S_ω the following closed double sector (Fig. 1.1):

$$S_\omega = \Big\{z \in \mathbb{C},\ |\arg z| \leqslant \omega \text{ or } |\arg(-z)| \leqslant \omega\Big\} \cup \Big\{0\Big\}.$$

If $\mathrm{Im}\lambda > 0$, let

$$\phi_\lambda(z) = \begin{cases} ie^{i\lambda z}, & \mathrm{Re} z > 0,\\ 0, & \mathrm{Re} z \leqslant 0.\end{cases} \tag{1.1}$$

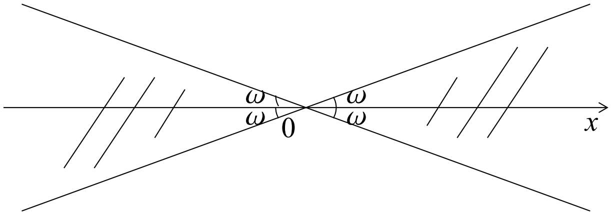


Fig. 1.1 S_ω

If Im$\lambda < 0$, let

$$\phi_\lambda(z) = \begin{cases} -ie^{i\lambda z}, & \text{Re}z < 0, \\ 0, & \text{Re}z \geqslant 0. \end{cases} \tag{1.2}$$

We have the following theorem:

Theorem 1.1.2 *Assume that $\lambda \in S_\omega$. Then the convolution operator defined on γ:*

$$R_\lambda u = \phi_\lambda * u$$

is bounded on $L^p(\gamma)$, $1 \leqslant p < \infty$, and $C_0(\gamma)$. For the two cases,

$$\|R_\lambda\| \leqslant \left\{\text{dist}(\lambda, S_\omega)\right\}^{-1}.$$

Moreover, for $u \in L^p(\gamma), v \in L^{p'}(\gamma)$, $1 \leqslant p \leqslant \infty$, (and, hence $u \in L^1(\gamma)$, $v \in C_0(\gamma)$*),*

$$\langle R_\lambda u, v\rangle = \langle u, R_{-\lambda} v\rangle.$$

Proof This theorem is a direct consequence of Theorem 1.1.1. Denote, respectively, by $\gamma^-(z)$ and $\gamma^+(z)$ the following two curves:

$$\gamma^-(z) = \{\xi \in \gamma,\ \text{Re}\xi \leqslant \text{Re}z\}$$

and

$$\gamma^+(z) = \{\xi \in \gamma,\ \text{Re}\xi > \text{Re}z\}.$$

For $u \in L^p(\gamma)$ or $C_0(\gamma)$, if $\lambda \notin S_\omega$, $R_\lambda u$ can be represented as

$$R_\lambda u(z) = \begin{cases} i\int_{\gamma^-(z)} e^{i\lambda(z-\xi)}u(\xi)d\xi, & \text{Im}\lambda > N|\lambda|, \\ -i\int_{\gamma^-(z)} e^{i\lambda(z-\xi)}u(\xi)d\xi, & \text{Im}\lambda < -N|\lambda|. \end{cases}$$

If $\lambda \notin \gamma$, then $|\tan\lambda| \leqslant \tan\omega$. Without loss of generality, let Im$\lambda > N|\text{Re}\lambda|$ and $z \in \gamma$, then by (1.1),

$$\int_\gamma |\phi_\lambda(z-\xi)||d\xi| \leqslant \{\text{dist}(\lambda, S_\omega)\}^{-1}.$$

Similarly, by invoking Theorem 1.1.1, we obtain the desired result. □

Let u be a Lipschitz function on γ. The derivative of u is defined as

$$u'(z) = \frac{d}{dz}\Big|_\gamma u(z) = \lim_{h\to 0, z+h\in\gamma} \frac{u(z+h)-u(z)}{h}.$$

It can be deduced from a simple computation that

$$\frac{d}{dz}\Big|_\gamma u(x+ig(x)) = (1+ig'(x))^{-1}\frac{d}{dx}u(x+ig(x)).$$

By duality, $D_{\gamma,p}$ can be defined as the closed linear operator on $L^p(\gamma)$, $1 \leqslant p \leqslant \infty$, and $C_0(\gamma)$ with the largest domain $\mathfrak{D}(D_{\gamma,p})$ in $L^p(\gamma)$, $1 \leqslant p \leqslant \infty$, and $C_0(\gamma)$. For all compactly supported Lipschitz functions v,

$$\langle D_{\gamma,p}u, v\rangle = \langle u, iv'\rangle.$$

The following properties of $D_{\gamma,p}$ can be proved on γ directly, and can alternatively be obtained via the related operators D_p on $L^p(\mathbb{R})$ or $C_0(\mathbb{R})$.

Theorem 1.1.3 *(i) $D_{\gamma,p}u(x+ig(x)) = (1+ig'(x))^{-1}D_p u(x+ig(x))$ and*

$$\begin{aligned}\mathfrak{D}(D_{\gamma,p}) &= \left\{u:\ u(\cdot+ig(\cdot)) \in \mathcal{D}(D_p)\right\}\\ &= \begin{cases} W_p^1(\gamma),\ 1\leqslant p\leqslant\infty \\ \Lambda_0(\gamma) = \{u\in C_0(\gamma):\ u'\in C_0(\gamma)\},\ p=0.\end{cases}\end{aligned}$$

Except when $p=\infty$, $\mathfrak{D}(D_p)$ is dense in $L^p(\gamma)$ (or $C_0(\gamma)$). Moreover, for all p, the space of compactly supported Lipschitz functions on γ is a dense subspace of $\mathfrak{D}(D_{\gamma,p})$ under the norm $\|u\|_p + \|D_{\gamma,p}\|_p$ (or $\|u\|_\infty + \|D_{\gamma,0}u\|_\infty$).

(ii) If $1\leqslant p\leqslant\infty$, $1\leqslant p'\leqslant\infty$ with $1/p+1/p'=1$, then

$$\langle D_{\gamma,p}u,\ v\rangle = -\langle u, D_{\gamma,p'}v\rangle,\ u\in W_p^1(\gamma), v\in W_{p'}^1(\gamma)$$

and

$$\langle D_{\gamma,1}u,\ v\rangle = -\langle u, D_{\gamma,0}v\rangle,\ u\in W_1^1(\gamma), v\in \Lambda_0(\gamma).$$

Also, each operator has the largest domain under which the equality holds.

(iii) If $\lambda\notin S_w$, then for all $u\in\mathfrak{D}(D_{\gamma,p})$ and v in the related dual space,

$$\langle -(D_{\gamma,p}+\lambda I)u, R_\lambda v\rangle = \langle u, v\rangle.$$

Hence, for λ *not belonging to the spectrum of* $D_{\gamma,p}$,

$$-(D_{\gamma,p}+\lambda I)^{-1}=R_\lambda,$$

and, in $L^p(\gamma)$, $1\leqslant p\leqslant\infty$, *or* $C_0(\gamma)$,

$$\|(D_{\gamma,p}+\lambda I)^{-1}\|\leqslant\{\mathrm{dist}(\lambda,S_w)\}.$$

1.2 Quadratic Estimates for Type ω Operators

We first state some backgrounds for bounded linear operators. Let T be a bounded linear operator on a Banach space X. The resolvent set of T is defined by

$$p(T)=\Big\{z\in\mathbb{C},\ (T-zI)\text{ is one-one, onto, and }(T-zI)^{-1}\text{ is bounded}\Big\}.$$

The spectrum of T is defined by $\sigma(T)=\mathbb{C}\backslash p(T)$. We can see that $\sigma(T)$ is a non-empty compact subset of $\mathbf{B}(0,\|T\|)$. The resolvent operators $R_\lambda=(T-\lambda I)^{-1}$ depend holomorphically on λ in $p(T)$ and satisfy

$$R_\lambda R_\mu=(\lambda-\mu)^{-1}(R_\lambda-R_\mu).$$

For a function f, there exist a number of methods to define operator algebras of f, $f(T)$, which satisfy

(i) $c_1f_1(T)+c_2f_2(T)=\{c_1f_1+c_2f_2\}(T)$,
(ii) $(f_1f_2)(T)=f_1(T)f_2(T)$.

Here we list several methods to define $f(T)$, where the norms satisfy different estimates:

(a) If $T=\int\lambda dE_\lambda$ is a self-adjoint operator on a Hilbert space H, then $f(T)=\int f(\lambda)dE_\lambda$, and for all bounded Borel functions f on γ, $\|f(T)\|\leqslant$ ess-sup(f), where "ess-sup" denotes the essential supremum with respect to the spectral measure.
(b) Let $X=L^p(\gamma)$, $1\leqslant p\leqslant\infty$. For some $L^\infty(\gamma)$-function w, let $Tu(z)=w(z)u(z)$. Then $\sigma(T)=$ ess-range(w), and if f is a bounded Borel function defined on $\sigma(T)$, $f(T)u(z)=f(w(z))u(z)$. Moreover, $\|f(T)\|=\|f\|_\infty$.
(c) Let $f(z)=\sum_{i=0}^{\infty}c_iz^i$, $|z|<r$, and $\|T\|<r$. Then $f(T)=\sum c_iT^i$ defines a bounded linear operator, and

$$\|f(T)\|\leqslant\sum c_i\|T\|^i<\infty.$$

(d) Suppose f is holomorphic on an open set $\Omega\supset\sigma(T)$ and δ is a path containing $\sigma(T)$. Let

$$f(T) = (2\pi i)^{-1} \int_{\delta} (T - \lambda I)^{-1} f(\lambda) d\lambda.$$

Because $(T - \lambda I)^{-1}$ depends on λ holomorphically, the integral is independent of the precise path.

Although the related formulas give a good estimate only in the examples (a) and (c), (i) and (ii) hold for all the cases. In particular, the above four methods give the same $f(T)$. Below we turn to define and discuss $f(D_\gamma)$ for the unbounded operator D_γ.

A closed linear operator T in a Banach space X is a linear mapping from a linear subspace $D(T)$ to X for which the graph $\{(u, Tu),\ u \in D(T)\}$ is a closed subspace of $X \times X$. Similarly, the spectrum $\sigma(T)$ and the resolvent set $p(T)$ are defined, respectively, as follows:

$$p(T) = \Big\{ z \in \mathbb{C},\ (T - zI) \text{ is one-one, onto, and } (T - zI)^{-1} \text{ is bounded} \Big\},$$
$$\sigma(T) = \mathbb{C} \setminus p(T),$$
$$R_\lambda = (T - \lambda I) \text{ depends holomorphically on } \lambda \in p(T).$$

Definition 1.2.1 For $0 \le \omega \leqslant \pi/2$, we define the sets

$$S_{\omega,+} = \Big\{ z \in \mathbb{C},\ |\arg(z)| < \omega \text{ or } z = 0 \Big\},\ S_{\omega,-} = -S_{\omega,+},$$

$$S_\omega = S_{\omega,+} \cup S_{\omega,-}.$$

The above sets are closed, whose interiors are denoted by $S^0_{\omega,+}$, $S^0_{\omega,-}$ and S^0_ω, see Fig. 1.2:

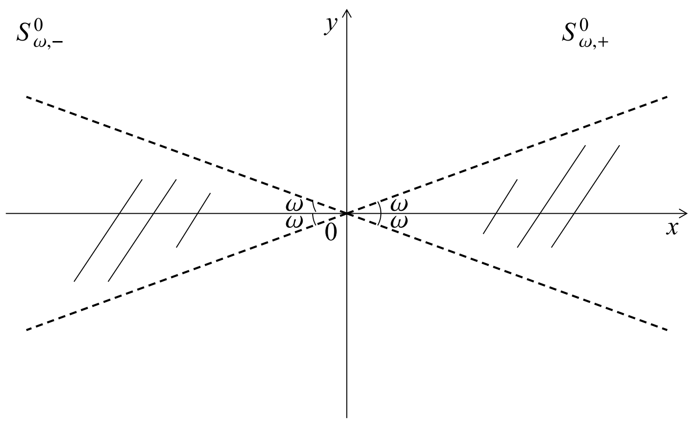


Fig. 1.2 S^0_ω

Definition 1.2.2 If T is an operator on the Banach space X satisfying:

(i) T is a closed operator on X;
(ii) $\sigma(T)$ is a closed subset of S_ω;
(iii) for all $\mu > \omega$, there exists a constant c_μ such that $\|R_\lambda\| \leqslant c_\mu |\lambda|^{-1}, \ \lambda \notin S_\mu^0$,

then T is called a type ω operator.

We give some examples of type ω operators.

(a) If T is a (unbounded) self-adjoint operator on a Hilbert space, the T is a type 0 operator.
(b) If $X = L^p(\gamma)$, $1 \leqslant p \leqslant \infty$ and for some Lebesgue measurable function w with its essential range in S_ω, $Tu(z) = w(z)u(z)$, then $\sigma(T) =$ ess-range(w) and T is a type ω operator.
(c) The operator D_γ defined in Sect. 1.1 is a type ω operator on $L^p(\gamma)$, $1 \leqslant p \leqslant \infty$ and $C_0(\gamma)$, where $\tan\omega = N$.

Let $H^\infty(S_\mu^0)$ denote the Banach space of all holomorphic functions with finite L^∞-norms, where the L^∞-norm is defined as $\|b\|_\infty = \sup|b(z)|$. Let $b \in H^\infty(S_\mu^0)$, $\mu > \omega$. For a type ω operator T, we define $b(T)$ and would like to obtain $\|b(T)\| \leqslant C\|b\|_\infty$. However, such estimate is not valid for all T. Hence in the next section, we will give some conditions on T such that the desired estimation holds, and then we apply these results to D_γ. For this purpose, let

$$\Psi(S_\mu^0) = \left\{\psi \in H^\infty(S_\mu^0), \ |\psi(z)| \leqslant \frac{c_s|z|^s}{(1+|z|)^{2s}} \text{ for some } c_s \geqslant 0, s > 0\right\}.$$

For $\psi \in \Psi(S_\mu^0)$, we consider $\psi(T)$. These operators are similar to the bounded operators in (d) and can be defined via the contour integral:

$$\psi(T) = (2\pi)^{-1}\int_\delta (T - \lambda I)^{-1}\psi(\lambda)d\lambda,$$

where δ is the sum of the paths δ_+ and δ_- (making an angle with θ with the real axis). Because T is a type ω operator and $\psi \in \Psi(S_\mu^0)$, it is easy to see that the integral converges in the operator norm and

$$\|\psi(T)\| \leqslant (2\pi)^{-1}\int_\delta c_\theta|\lambda|^{-1}\frac{|\lambda|^s}{(1+|\lambda|^{2s})}|d\lambda| < \infty.$$

The definition of $\psi(T)$ is independent of the choice of the paths δ. Moreover,

$$(\psi_1\psi_2)(T) = \psi_1(T)\psi_2(T),$$
$$\psi_1(T) + \psi_2(T) = (\psi_1 + \psi_2)(T).$$

Specially, if $\psi(z) = z(1+z^2)^{-1}$, then $\psi(T) = T(T+iI)^{-1}(T-iI)^{-1}$. Also, when $Tu = 0$, $\psi(T) = 0$. We refer the reader to [4] for the details. In the next section, we need the following estimate. For $\tau > 0$, define $\psi_\tau(z) = \psi(\tau z)$.

Lemma 1.2.1 *Let T is a type ω operator in X and let ψ, $\psi \in \Psi(S_\mu^0)$. Then there exists a constant c such that*

(i) *For all $b \in H^\infty(S_\mu^0)$ and $\tau \in (0,\infty)$, $\|(b\psi_\tau)(T)\| \leqslant c\|b\|_\infty$;*
(ii) *For all Borel functions $f : [\alpha, \beta] \to X$ and for all $0 < \alpha < \beta < \infty$,*

$$\Big(\int_0^\infty \Big\| \int_\alpha^\beta \psi_\tau(T)\psi_t(T)f(\tau)\tau^{-1}d\tau \Big\|^2 \frac{dt}{t}\Big)^{1/2} \leqslant c\Big(\int_\alpha^\beta \|f(\tau)\|^2 \frac{d\tau}{\tau}\Big)^{1/2}.$$

Proof (i) Because $(b\psi_\tau)(T) = (2\pi i)^{-1}\int_\delta R_\lambda \psi(\tau\lambda)b(\lambda)d\lambda$, we have

$$\begin{aligned}\|(b\psi_\tau)(T)\| &\leqslant (2\pi)^{-1}\int_\delta c_\theta|\lambda|^{-1}c_s|\tau\lambda|^s(1+|\tau\lambda|^{2s})^{-1}|d\lambda| \\ &\leqslant c\|b\|_\infty.\end{aligned}$$

(ii)

$$\begin{aligned}\|(\psi_\tau\psi_t)(T)\| &\leqslant (2\pi i)^{-1}\int_\delta c_\theta|\lambda|^{-1}\frac{c_s|\tau\lambda|^s}{(1+|\tau\lambda|^{2s})}\frac{c_s|t\lambda|^s}{(1+|t\lambda|^{2s})}|d\lambda| \\ &\leqslant \begin{cases} \pi = C\Big(\dfrac{t}{\tau}\Big)^s\Big(1+\log(\dfrac{\tau}{t})\Big), 0 < t < \tau < \infty; \\ C\Big(\dfrac{\tau}{t}\Big)^s\Big(1+\log(\dfrac{t}{\tau})\Big), 0 < \tau < t < \infty. \end{cases}\end{aligned}$$

Hence, we can obtain

$$\begin{aligned}&\int_0^\infty \Big\| \int_\alpha^\beta \psi_\tau(T)\psi_t(T)f(\tau)\tau^{-1}d\tau \Big\|^2 \frac{dt}{t} \\ &\leqslant \sup_t\Big(\int_\alpha^\beta \Big\|(\psi_\tau\psi_t)(T)\Big\|\frac{d\tau}{\tau}\Big)\sup_\tau\Big(\int_0^\infty \|(\psi_\tau\psi_t)(T)\|\frac{dt}{t}\Big)\Big(\int_\alpha^\beta \|f(t)\|^2\frac{dt}{t}\Big) \\ &\leqslant c\int_\alpha^\beta \|f(t)\|^2\frac{dt}{t}.\end{aligned}$$

□

A dual pair $\langle X, Y\rangle$ of Banach spaces consists of two Banach spaces X, Y and a bounded bilinear form $\langle u, v\rangle$ which satisfies:

$$\|u\|_X \leqslant C\sup\frac{|\langle u, v\rangle|}{\|v\|_Y}, u \in X$$

and

$$\|v\|_Y \leqslant C \sup \frac{|\langle u, v\rangle|}{\|u\|_X}, \quad v \in Y.$$

A dual pair $\langle T, T'\rangle$ of type ω operators concerns a type ω operator T in X and a type ω operator T' in Y which satisfy that for all $u \in D(T)$ and $v \in D(T')$,

$$\langle Tu, \ v\rangle = \langle u, T'v\rangle.$$

The following result can be verified easily.

Lemma 1.2.2 *If $\langle T, \ T'\rangle$ is a dual pair of type ω operators on $\langle X, \ Y\rangle$ and $\psi \in \Psi(S^0_\mu)$ for some $\mu > \omega$, then for all $u \in X$ and $v \in Y$, $\langle \psi(T)u, \ v\rangle = \langle u, \psi(T')v\rangle$. Moreover, there exists a constant c such that for $\psi \in \Psi(S^0_\mu)$, $\|\psi(T')\| \leqslant c\|\psi(T)\|$.*

Example 1.2.1 (a) Assume that $\langle X, \ Y\rangle = \langle L^p(\gamma), \ L^{p'}(\gamma)\rangle$, $1 \leqslant p \leqslant \infty$, and for some measurable function w with essential range in S_ω, $Tu = wu$ and $T'v = wv$. Then $\langle T, \ T'\rangle$ forms a dual pair of type ω operators. For all u, $\psi(T)u(z) = \psi(w(z))u(z)$. Obviously, $\langle \psi(T)u, \ v\rangle = \langle u, \ \psi(T')v\rangle$.

(b) For the operator D_γ defined in Sect. 1.1, $\langle D_\gamma, \ -D_\gamma\rangle$ is a dual pair of type ω operators on $\langle L^p(\gamma), \ L^{p'}(\gamma)\rangle$, $1 \leqslant p \leqslant \infty$, and $\langle C_0(\gamma), \ L^1(\gamma)\rangle$.

Lemma 1.2.3 *Let $\langle X, \ Y\rangle$ be a dual pair of Banach spaces. If Z is a dense linear subspace of Y, and f is a continuous function from a compact interval $[a, b]$ to X, then there exists a Borel function v from $[a, b]$ to Z such that for all t, $\|v(t)\| = 1$ and $\|f(t)\| \leqslant 2C\langle f(t), \ v(t)\rangle$.*

In fact, we can choose the function v as follows:

$$v(t) = \sum_k h_k(t)x_k(t)z_k,$$

where $z_k \in Z$, x_k is the characteristic function of an interval, h_k is a continuous function on this interval with $|h_k| = 1$, k takes over the natural numbers.

If for all $u \in X$ and a constant q,

$$Q(\psi): \ \left(\int_0^\infty \|\psi_\tau(T)u\|^2 \frac{d\tau}{\tau}\right)^{1/2} \leqslant q\|u\|,$$

then we say that the type ω operator T satisfies the quadratic estimate $Q(\psi)$ with respect to $\psi \in \Psi(S^0_\mu)$, $\mu > \omega$. For example,

(a) If T is a self-adjoint operator on a Hilbert space, then T satisfies $Q(\psi)$ for $\psi \in \Psi(S^0_\mu)$ and all $\mu > 0$. For this case,

$$q = \max\left\{\left(\int_0^\infty |\psi(s)|^2 \frac{ds}{s}\right)^{1/2}, \ \left(\int_0^\infty |\psi(-s)|^2 \frac{ds}{s}\right)^{1/2}\right\}.$$

(b) Let $X = L^p(\gamma)$ and $Tu(z) = w(z)u(z)$ for some Lebesgue measurable function w with the essential range in S_ω, then if $1 \leqslant p \leqslant 2$. For any $\psi \in \Psi(S_\mu^0)$, $\mu > \omega$, T satisfies $Q(\psi)$. However, for all $L^p(\gamma)$, T does not satisfy the related square function estimate:

$$S(\psi): \ \left\| \left(\int_0^\infty |\psi_\tau(T)u(\cdot)|^2 \frac{d\tau}{\tau} \right)^{1/2} \right\|_p \leqslant q' \|u\|_p.$$

For $p = 2$, $S(\psi)$ and $Q(\psi)$ are equivalent.

The following result gives the dual form of $Q(\psi)$.

Lemma 1.2.4 *Let $(T, \ T')$ be a dual pair of type ω operators on $\langle X, \ Y \rangle$ and Z be a dense linear subspace of Y. Then T satisfies the quadratic estimate $Q(\psi)$ with respect to $\psi \in \Psi(S_\mu^0)$, $\mu > \omega$ if and only if for a constant q_1, all Borel functions f from $[\alpha, \beta]$ to Z and all $0 < \alpha < \beta < \infty$,*

$$\left\| \int_\alpha^\beta \psi_\tau(T') f(\tau) \frac{d\tau}{\tau} \right\| \leqslant q_1 \left(\int_\alpha^\beta \|f(\tau)\|^2 \frac{d\tau}{\tau} \right)^{1/2}. \tag{1.4}$$

Proof We first assume that the operator T satisfies the quadratic estimate $Q(\psi)$. Let $g \in X$ and $f \in Y$. We have

$$\begin{aligned} \left\langle g, \ \int_\alpha^\beta \psi_\tau(T') f(\tau) \frac{d\tau}{\tau} \right\rangle &= \int_\alpha^\beta \left\langle \psi_\tau(T)g, \ f(\tau) \right\rangle \frac{d\tau}{\tau} \\ &\leqslant \int_\alpha^\beta \|f(\tau)\| \|\psi_\tau(T)g\| \frac{d\tau}{\tau} \\ &\leqslant \left(\int_\alpha^\beta \|\Psi_\tau(T)g\|^2 \frac{d\tau}{\tau} \right)^{1/2} \left(\int_\alpha^\beta \|f(\tau)\|^2 \frac{d\tau}{\tau} \right)^{1/2} \\ &\leqslant q_1 \left(\int_\alpha^\beta \|f(\tau)\|^2 \frac{d\tau}{\tau} \right)^{1/2} \|g\|. \end{aligned}$$

This implies that the operator T' satisfies the estimate (1.4).

Conversely, assume that (1.4) holds. Let $u \in X$ and $0 < \alpha < \beta < \infty$. By Lemma 1.2.3, there exists a Borel function v from $[\alpha, \beta]$ to Z such that for all τ, $\|v(\tau)\| = 1$ and

$$\|\psi_\tau(T)u\| \leqslant 2C \langle \psi_\tau(T)u, v(\tau) \rangle.$$

Write $g(\tau) = \langle \psi_\tau(T)u, \ v(\tau) \rangle$, by (1.4), we have

$$\begin{aligned} \int_\alpha^\beta \left\langle \psi_\tau(T)u, \ v(\tau) \right\rangle^2 \frac{d\tau}{\tau} &= \int_\alpha^\beta \left\langle u, \ \psi_\tau(T')g(\tau)v(\tau) \right\rangle \frac{d\tau}{\tau} \\ &\leqslant \|u\| \left\| \int_\alpha^\beta \psi_\tau(T')g(\tau)v(\tau) \frac{d\tau}{\tau} \right\| \end{aligned}$$

$$\leqslant \|u\|_{q_1}\Big(\int_\alpha^\beta \|g(\tau)v(\tau)\|^2\frac{d\tau}{\tau}\Big)^{1/2}$$
$$\leqslant \|u\|_{q_1}\Big(\int_\alpha^\beta \langle\psi_\tau u, v(\tau)\rangle^2\frac{d\tau}{\tau}\Big)^{1/2}.$$

Hence

$$\Big(\int_\alpha^\beta \|\psi_\tau(T)u\|^2\frac{d\tau}{\tau}\Big)^{1/2} \leqslant 2C\Big(\int_\alpha^\beta \langle\psi_\tau(T)u, v(\tau)\rangle^2\frac{d\tau}{\tau}\Big)^{1/2}$$
$$\leqslant 2Cq_1\|u\|,$$

where the constant in the above estimate is independent of α and β. So the quadratic estimate $Q(\psi)$ holds. □

We use $\Psi(S^0_{\mu,+})$ to denote the set:

$$\left\{\psi \in \Psi(S^0_\mu):\ \psi = 0 \text{ on } S^0_{\mu,-}\right\}.$$

Assume that $\langle T,\ T'\rangle$ is a dual pair of type ω operators in $\langle X,\ Y\rangle$. Let Z be a dense linear subspace of Y.

Definition 1.2.3 Let Y be a Banach space.
(i) Define Y_+ as the linear subspace of Y which consists of all Borel functions $v_+ \in Y$ from $[\alpha, \beta]$ to Z satisfying the following condition: there exists a function $\psi_+ \in \Psi(S^0_{\mu,+})$, $\mu > \omega$, such that

$$v_+ = \int_\alpha^\beta \psi_+(\tau T')f(\tau)\frac{d\tau}{\tau}.$$

(ii) Define Y_- as the linear subspace of Y which consists of all Borel functions $v_- \in Y$ from $[\alpha, \beta]$ to Z satisfying the following condition: there exists a function $\psi_- \in \Psi(S^0_{\mu,-})$, $\mu > \omega$, such that

$$v_- = \int_\alpha^\beta \psi_-(\tau T')f(\tau)\frac{d\tau}{\tau}.$$

Similarly, we can define the linear subspaces X_+ and X_-. Let $\psi \in \Psi(S^0_\mu)$, $\mu > \omega$. If for a q_+ and all $v_+ \in Y_+$,

$$R_+(\psi):\ \|v_+\| \leqslant q_+\Big(\int_0^\infty \|\psi_\tau(T')v_+\|^2\frac{d\tau}{\tau}\Big)^{1/2},$$

we say that T' satisfies the reverse quadratic estimate $R_+(\psi)$ with respect to the function ψ (this definition is independent of the choice of the dense linear subspaces Z). Similarly, we can define $R_-(\psi)$.

Theorem 1.2.1 *Let $\langle T,\ T'\rangle$ be a dual pair of type ω operators in $\langle X, Y\rangle$. If for some $\Psi \in \Psi(S^0_\mu)$, $\mu > \omega$, T' satisfies the reverse quadratic estimate $R_+(\Psi)$, then for any $\psi_+ \in \Psi(S^0_{\nu,+})$ and all $\nu > \omega$, T satisfies the quadratic estimate $Q(\psi_+)$.*

Proof By Lemma 1.2.4, we only need to verify that the dual operator T' satisfies $Q(\psi)$. Let f be a Borel function from $[\alpha, \beta]$ to the dense linear subspace Z of Y. Then

$$v_+ = \int_\alpha^\beta \psi_+(\tau T')f(\tau)\frac{d\tau}{\tau} \in Y_+.$$

Therefore, by $R_+(\Psi)$ and (ii) of Lemma 1.2.1, we have

$$\begin{aligned}\Big\| \int_\alpha^\beta \psi_+(\tau T')f(\tau)\frac{d\tau}{\tau}\Big\| &\leqslant cq_+\Big(\int_0^\infty \Big\|\int_\alpha^\beta \psi_t(T')\psi_+(\tau T')f(\tau)\frac{d\tau}{\tau}\Big\|^2\frac{dt}{t}\Big)^{1/2}\\ &\leqslant cq_+\Big(\int_\alpha^\beta \|f(\tau)\|^2\frac{d\tau}{\tau}\Big)^{1/2}.\end{aligned}$$

By Lemma 1.2.4, we can obtain that T satisfies $Q(\psi)$. □

Theorem 1.2.2 *Let $\langle T,\ T'\rangle$ be a dual pair of type ω operators in $\langle X, Y\rangle$. Assume that for ψ^+ and $\psi^- \in \Psi(S^0_\mu)$, $\mu > \omega$, T' satisfies the reverse quadratic estimate $R_+(\psi^+)$ and $R_-(\psi^-)$, respectively. Then for any $\psi \in \Psi(S^0_\nu)$ and all $\nu > \omega$, T satisfies the quadratic estimate $Q(\psi)$.*

This result is a direct consequence of the former theorem. It gives rather surprising conditions under which T satisfies quadratic estimate. The reason is that the hypotheses of the theorem only involve estimates on the subspaces Y_+ and Y_- (together with the assumption that $\langle T, T'\rangle$ is a dual pair of type ω operators). We will further investigate this topic in the sequel. The quadratic estimate implies that T has a H^∞ functional calculus. Below we give some result along the line.

Theorem 1.2.3 *Let $\langle T, T'\rangle$ be a dual pair of type ω operators in $\langle X, Y\rangle$, where $0 < \omega \leqslant \mu < \pi/2$. Assume that for some odd or even functions $\psi \in \Psi(S^0_\mu)$ satisfying $\psi(t) > 0, t > 0$, T and T' satisfies the quadratic estimate $Q(\psi)$. Then there exists a constant such that $\|b(T)\| \leqslant c\|b\|_\infty$ for all $b \in \Psi(S^0_\mu)$.*

Proof Let $\phi \in \Psi(S^0_\mu)$ be an even function satisfying

$$\int_0^\infty \phi(\tau)\psi^2(\tau)\frac{d\tau}{\tau} = 1.$$

Then for all $z \in S^0_\mu$,

$$b(z) = \int_0^\infty (b\phi_t)(z)\psi^2(tz)\frac{dt}{t}.$$

Hence for $u \in X$ and $v \in Y$,

$$\langle b(T)u,\ v\rangle = \Big\langle \int_0^\infty (b\phi_t)(T)\psi^2(tT)\frac{dt}{t}u,\ v\Big\rangle$$
$$= \int_0^\infty \Big\langle (b\phi_t)(T)\psi(tT)u,\ \psi(tT')v\Big\rangle \frac{dt}{t}.$$

Therefore,

$$|\langle b(T)u,\ v\rangle| \leqslant \sup \|(b\phi_t)(T)\| \Big(\int_0^\infty \|\psi(tT)u\|^2 \frac{dt}{t}\Big)^{1/2} \Big(\int_0^\infty \|\psi(tT')v\|^2 \frac{dt}{t}\Big)^{1/2}.$$

Applying (i) of Lemma 1.2.1 and the assumptions of T and T', we can see that $\|b(T)\| \leqslant c\|b\|_\infty$. □

This result was obtained by A. McIntosh, see [4], where it is shown that if T is a one-one operator with a dense domain and a dense range, and $b \in H^\infty(S^0_\mu)$, then the operator $b(T)$ is closed and has a dense domain, where

$$b(T) = T^{-1}(T^2+1)(b\psi)(T)$$

and $\psi(\xi) = \xi(\xi^2+1)^{-1}$. The following result is also obtained in [4].

Lemma 1.2.5 *Let T be a one-one type ω operator with a dense domain and a dense range in X. Assume that b_α is a uniformly bounded net of functions in $\Psi(S^0_\mu)$ which converges to a function $b \in H^\infty(S^0_\mu)$ uniformly on every set of the form*

$$\Big\{z \in S^0_\mu,\ 0 < \delta \leqslant |z| \leqslant \Delta < \infty\Big\}.$$

Suppose the operators $b_\alpha(T)$ are uniformly bounded. Then for all $u \in X$, $b_\alpha(T)u$ converges to $b(T)u$ uniformly and $\|b(T)\| \leqslant \sup_\alpha \|b_\alpha(T)\|$.

By Lemma 1.2.5, the former theorem and Theorem 1.2.2, we obtain the following result.

Theorem 1.2.4 *Let $\langle T,\ T'\rangle$ be a dual pair of type ω operators in $\langle X, Y\rangle$, T and T' have the dense domains and dense ranges. Assume that for some ψ^+ and some $\psi^- \in \Psi(S^0_\mu)$, $\mu > \omega$, T satisfies the reverse quadratic estimates $R_+(\psi^+)$ and $R_-(\psi^-)$, respectively, and that T' does too. Then for all $b \in H^\infty(S^0_\nu)$ and all $\nu > \omega$, $b(T)$ is a bounded operator in X. For some constant c_ν,*

$$\|b(T)\| \leqslant c_\nu \|b\|_\infty.$$

Moreover, for all $b_1, b_2 \in H^\infty(S^0_\nu)$, $(b_1b_2)(T) = b_1(T)b_2(T)$ and $(b_1+b_2)(T) = b_1(T) + b_2(T)$.

When $\langle X, Y \rangle$ is a pairing between a Hilbert space and itself, the converse of the above result also holds. But for L^p, $p \neq 2$, the converse result is not ture.

Let χ_+ define on S^0_ν and satisfy

$$\begin{cases} \chi_+(z) = 1, & \mathrm{Re} z > 0, \\ \chi_+(z) = 0, & \mathrm{Re} z < 0. \end{cases}$$

Notice that when T satisfies the assumption of Theorem 1.2.4, then $P_+ = \chi_+(T)$ and $P_- = I - P_+ = \chi_-(T)$ are bounded projections. We also note that χ_+ is contained in $R(P_+)$ and X_- is contained in $R(P_-)$. This indicates that Theorem 7.3.2 is a surprising result, since it implies the decomposition $X = R(P_+) \oplus R(P_-)$ as a consequence of the estimates on the spaces $R(P_+)$, $R(P_-)$, $R(P'_+)$ and $R(P'_-)$.

1.3 Fourier Transform and the Inverse Fourier Transform on Sectors

In order to apply the results of Sect. 1.2 to the operator D_γ on $L^p(\gamma)$, we need the inverse Fourier transform of the functions in $H^\infty(S^0_\mu)$. For $0 < \mu < \pi/2$, the sets $S^0_{\mu,+}$, $S^0_{\mu,-}$ and S^0_μ are open sets defined in Definition 1.2.1. We also define the following sets:

Definition 1.3.1 Define the open sets $C^0_\mu = C^0_{\mu,+} \cap C^0_{\mu,-}$, where

$$C^0_{\mu,+} = \{z \in \mathbb{C}, -\mu < \arg(z) < \pi + \mu\}, \ C^0_{\mu,-} = -C^0_{\mu,+}$$

as shown in Figs. 1.3 and 1.4.

We use ρ_θ to denote the ray $\{se^{i\theta}, 0 < s < \infty\}$. For $b \in H^\infty(S^0_{\mu,+})$ and $z \in C^0_{\mu,+}$, define

$$G(b)(z) = \phi(z) = (2\pi)^{-1} \int_{\rho_\theta} e^{-iz\xi} b(\xi) d\xi,$$

Fig. 1.3 $C^0_{w,+}$

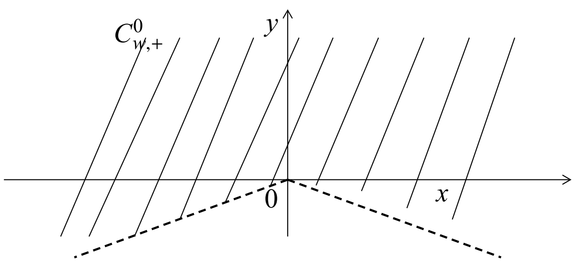


Fig. 1.4 $C^0_{w,-}$

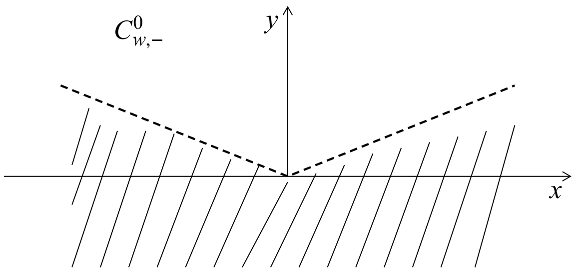
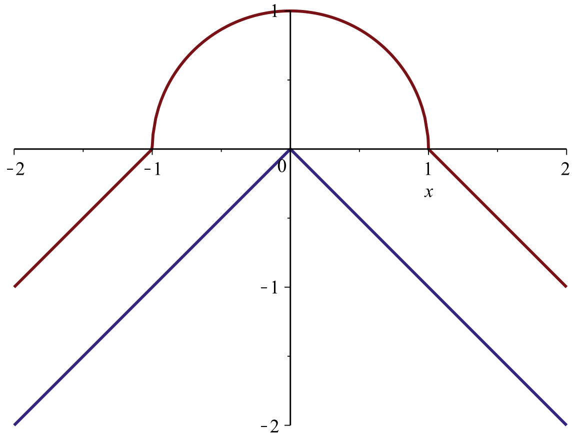


Fig. 1.5 $\delta(z)$



where $-\mu < -\theta < \arg(z) < \pi - \theta < \pi + \mu$. Because b is bounded and holomorphic in $S^0_{\mu,+}$, it is clear that the definition of b is independent of the choice of θ. When $z \in S^0_{\mu,+}$, define

$$G_1(b)(z) = \phi_1(z) = \int_{\delta(z)} \phi(\xi) d\xi,$$

where the integral is along a contour $\delta(z)$ from $-z$ to z in $C^0_{\mu,+}$, see Fig. 1.5.

Let $\mathcal{S}(\mathbb{R})$ denote the Schwartz class of rapidly decreasing functions.

Theorem 1.3.1 *Let $b \in H^\infty(S^0_{\mu,+})$, $\phi = G(b)$ and $\phi_1 = G_1(b)$. Then*

(i) *ϕ is a holomorphic function on $C^0_{\mu,+}$ which satisfies*

$$|\phi(z)| \leqslant \{2\pi \, dist(z, \rho_{-\mu} \cup \rho_{\pi+\mu})\}^{-1} \|b\|_\infty.$$

(ii) *ϕ_1 is a holomorphic function on $S^0_{\mu,+}$ which satisfies $\phi_1'(z) = \phi(z) + \phi(-z)$ and belongs to $H^\infty(S^0_{\nu,+})$ for all $\nu < \mu$.*

(iii) *For all functions $u \in \mathcal{S}(\mathbb{R})$,*

$$(2\pi)^{-1}\int_0^\infty b(\xi)\hat{u}(-\xi)d\xi = \lim_{\alpha\to 0+}\int_{\mathbb{R}}\phi(x+i\alpha)u(x)dx$$
$$= \lim_{\epsilon\to 0}\Big(\int_{|z|\geqslant\epsilon}\phi(x)u(x)dx + \phi_1(\epsilon)u(0)\Big).$$

Proof Choose a ray ρ_θ as suggested such that the integrand is decreasing at ∞ exponentially. Then (i) and (ii) can be proved directly. For $\alpha > 0$, let $b_\alpha(\xi) = e^{-\alpha\xi}b(\xi)$. Then for all $z \in C^0_{\mu,+}$, $G(b_\alpha)(z) = \phi(z + i\alpha)$. Specially, for $x > 0$,

$$\phi(x+i\alpha) = G(b_\alpha)(x) = (2\pi)^{-1}\int_{\rho_\theta} e^{iz\xi}b_\alpha(\xi)d\xi$$
$$= (2\pi)^{-1}\int_0^\infty e^{iz\xi}b_\alpha(\xi)d\xi.$$

Hence $\phi(x+i\alpha) = (\check{b_\alpha})(x)$. Similarly, we can prove the case $x < 0$. By Parseval's formula, we can obtain that for $u \in \mathcal{S}(\mathbb{R})$,

$$(2\pi)^{-1}\int_0^\infty b_\alpha(\xi)\hat{u}(-\xi)d\xi = \int_{\mathbb{R}}\phi(x+i\alpha)u(x)dx$$

and

$$(2\pi)^{-1}\int_0^\infty b(\xi)\hat{u}(-\xi)d\xi = \lim_{\alpha\to 0+}\int_{\mathbb{R}}\phi(x+i\alpha)u(x)dx.$$

Finally, we prove the last equality in (iii). Let $\epsilon > 0$.

$$(2\pi)^{-1}\int_0^\infty b(\xi)\hat{u}(-\xi)d\xi$$
$$= \lim_{\alpha\to 0+}\Big[\int_{|x|\geqslant\epsilon}\phi(x+i\alpha)u(x)dx + \int_{|x|\leqslant\epsilon}\phi(x+i\alpha)u(0)dx$$
$$+ \int_{|x|\leqslant\epsilon}\phi(x+i\alpha)(u(x)-u(0))dx\Big]$$
$$= \int_{|x|\geqslant\epsilon}\phi(x)u(x)dx + \phi_1(\epsilon)u(0) + \lim_{\alpha\to 0+}\int_{|x|\leqslant\epsilon}\phi(x+i\alpha)(u(x)-u(0))dx.$$

Then for $u \in \mathcal{S}(\mathbb{R})$, we can get

$$\lim_{\epsilon\to 0}\lim_{\alpha\to 0+}\int_{|x|\leqslant\epsilon}|\phi(x+i\alpha)(u(x)-u(0))|dx$$
$$\leqslant C\lim_{\epsilon\to 0}\lim_{\alpha\to 0+}\int_{|x|\leqslant\epsilon}(x^2+\alpha^2)^{-1/2}|u(x)-u(0)|dx$$

$$\leqslant C \lim_{\epsilon\to 0}\lim_{\alpha\to 0+}\int_{|x|\leqslant\epsilon}|x|^{-1}|u(x)-u(0)|dx$$
$$=0,$$

which implies

$$(2\pi)^{-1}\int_0^\infty b(\xi)\hat{u}(-\xi)d\xi = \lim_{\epsilon\to 0}\Big[\int_{|x|\geqslant\epsilon}\phi(x)u(x)dx+\phi_1(\epsilon)u(0)\Big].$$

□

For $b\in H^\infty(S^0_{\mu,-})$ and $z\in C^0_{\mu,-}$, the inverse Fourier transform is defined by

$$G(b)(z)=\phi(z)=\frac{-1}{2\pi}\int_{\rho_\theta}e^{iz\xi}b(\xi)d\xi,$$

where $\pi-\mu<-\theta<\arg(z)<\pi-\theta<\mu$ and $\rho_\theta=\{se^{i\theta}:0<s<\infty\}$. Because b is a holomorphic function in $S^0_{\mu,-}$, it is obvious that the definition of $G(b)$ is independent of the choice of θ. Moreover, for $z\in S^0_{\mu,+}$, define

$$G_1(b)(z)=\phi_1(z)=\int_{\delta(z)}\phi(\lambda)d\lambda,$$

where the integral is along a contour from $-z$ to z in $C^0_{\mu,-}$. If we replace $S^0_{\mu,+}$ and $C^0_{\mu,+}$ by $S^0_{\mu,-}$ and $C^0_{\mu,-}$, respectively, and replace $\lim_{\alpha\to 0+}$ by $\lim_{\alpha\to 0-}$, Theorem 1.3.1 still holds. Now we consider a bounded holomorphic function $b\in H^\infty(S^0_\mu)$. Set $b=b_+ + b_-$, where $b_\pm\in H^\infty(S^0_{\mu,\pm})$. Define $G(b)=G(b_+)+G(b_-)$ and $G_1(b)=G_1(b_+)+G_1(b_-)$. The following result can be deduced from Theorem 1.3.1.

Theorem 1.3.2 *Let $b\in H^\infty(S^0_\mu)$, $\phi=G(b)$ and $\phi_1=G_1(b)$. Then*

(1) *ϕ is holomorphic on S^0_μ and satisfies*

$$|\phi(z)|\leqslant\{2\pi\,\mathrm{dist}(z,p_{-\mu}\cup p_{\pi+\mu})\}^{-1}\|b\|_\infty.$$

(2) *ϕ_1 is a holomorphic function on $S^0_{\mu,+}$ which satisfies $\phi_1'(z)=\phi(z)+\phi(-z)$ and belongs to $H^\infty(S^0_\nu)$ for all $\nu<\mu$.*

(3) *For all $u\in\mathcal{S}(\mathbb{R})$,*

$$(2\pi)^{-1}\int_{-\infty}^\infty b(\xi)\hat{u}(-\xi)d\xi=\lim_{\epsilon\to 0}\Big(\int_{|x|\geqslant\epsilon}\phi(x)u(x)dx+\phi_1(\epsilon)u(0)\Big).$$

Theorem 1.3.2 indicates that any bounded holomorphic function b in $H^\infty(S^0_\nu)$ can be regarded as the Fourier transform of the function ϕ satisfying $|\phi(z)|\leqslant C/|z|$. Similar to the classical Fourier theory, the converse of the above theorem holds.

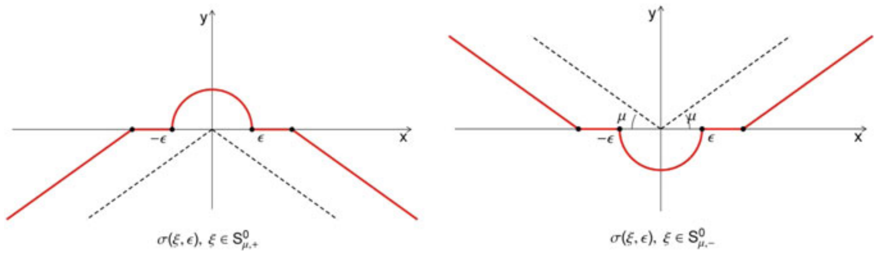


Fig. 1.6 $\sigma(\xi, \epsilon)$

Theorem 1.3.3 *Given* $0 < \nu < \mu < \pi/2$. *Suppose that* ϕ *and* ϕ_1 *are holomorphic functions on* S^0_μ *and* $S^0_{\mu,+}$ *which satisfy* $z\phi(z)$ *and* $\phi_1(z)$ *are bounded, and* $\phi_1'(z) = \phi(z) + \phi(-z)$ *for all* $z \in S^0_{\mu,+}$. *Then there exists unique function* $b \in H^\infty(S^0_\nu)$ *such that* $\phi = G(b)$ *and* $\phi_1 = G_1(b)$. *Moreover, for some constant* $C_{\mu,\nu}$ *which depends only on* μ *and* ν,

$$\|b\|_\infty \leqslant C_{\mu,\nu} \sup\Big\{|z\phi(z)| : z \in S^0_{\mu,+}\Big\} + \sup\Big\{|\phi_1(z)| : z \in S^0_\mu\Big\}.$$

Proof By the fact that $\phi_1(-z) = -\phi_1(z)$, we extend ϕ_1 to the whole S^0_μ. For $\xi \in S^0_\mu$, define

$$\begin{aligned} b(\xi) &= \lim_{\epsilon \to 0} \Big(\int_{\sigma(\xi,\epsilon)} e^{-i\xi z}\phi(z)dz + \phi_1(\epsilon)\Big) \\ &= \lim_{\epsilon\to 0}\Big[\int_{\sigma(\xi,\epsilon)} e^{-i\xi z}\Big(\phi_0(z) + \frac{i\xi}{2}\phi_1(z)\Big)dz\Big], \end{aligned}$$

where $\phi_0(z) = \frac{1}{2}(\phi(z) - \phi(-z))$, $\sigma(\xi, \epsilon)$ is shown in Fig. 1.6, where the interval $(-\epsilon, \epsilon)$ is omitted. For large $z \in \sigma(\xi, \epsilon)$, $\mathrm{Im}(z\xi) \leqslant \kappa$, $\kappa < 0$ (when $\xi \in S^0_{\mu,-}$, the curve $\sigma(\xi, \epsilon)$ is the conjugate of the one shown), see Fig. 1.6.

We can obtain the following properties:

(a) If $\nu < \mu$, then $b \in H^\infty(S^0_\nu)$.
(b) If $\xi \in \mathbb{R}$, $\xi \neq 0$, then

$$b(\xi) = \lim_{\epsilon\to 0, N\to\infty}\Big[\int_{\epsilon\leqslant|x|\leqslant N} e^{-i\xi x}\Big(\phi_0(x) + \frac{i\xi}{2}\phi_1(x)\Big)dx\Big].$$

(c)

$$\sup_{\xi,\epsilon,N}\Big|\int_{\epsilon\leqslant|x|\leqslant N} e^{-i\xi x}\Big[\phi_0(x) + \frac{i\xi}{2}\phi_1(x)\Big]dx\Big| < \infty.$$

(d) The function b, ϕ and ϕ_1 satisfy (3) of Theorem 1.3.2.
(e) We can deduce that $\phi = G(b)$ and $\phi_1 = G_1(b)$.

Now we prove (c) for $\xi > 0$, and the others can be dealt with similarly. At first, assume that $\xi^{-1} \leqslant \epsilon$. Let $0 < \alpha < \mu$ and let $C(N)$, $C(\epsilon)$ and δ be the contours shown (Each consists of two pieces).

$$
\begin{aligned}
&\Big|\int_{\epsilon\leqslant|x|\leqslant N} e^{-i\xi x}\Big[\phi_0(x)+\frac{i\xi}{2}\phi_1(x)\Big]dx\Big| \\
&\leqslant C_1\Big[\int_{C(N)} |e^{-i\xi z}|(N^{-1}+\xi)|dz| + \int_{C(\epsilon)} |e^{-i\xi z}|(\epsilon^{-1}+\xi)|dz| \\
&\quad + \int_{[\epsilon,N]} e^{-\xi r\sin\alpha}(r^{-1}+\xi)dr\Big] \\
&\leqslant C_2\Big[\int_0^{\alpha} e^{-\xi N\sin\theta}(1+\xi N)d\theta + \int_0^{\alpha} e^{-\xi\epsilon\sin\theta}(1+\xi\epsilon)d\theta + \int_0^{\infty} e^{-s}ds\Big] \\
&\leqslant C_3.
\end{aligned}
$$

Now suppose $\epsilon < \xi^{-1} < N$. We replace the integral by one on the contour shown. Similarly, we can obtain

$$
\begin{aligned}
\Big|\int_{\epsilon}^{1/\xi} e^{-i\xi x}(\phi_0(x)+\frac{i\xi}{2}\phi_1(x))dx\Big| &\leqslant C_1\int_0^{1/\xi}(\sin\xi x)(x^{-1}+\xi)dx \\
&= C_1\int_0^1(\sin t)(t^{-1}+1)dt \\
&= C_2.
\end{aligned}
$$

Finally, when $N \leqslant \xi^{-1}$, we only need a bound of the second type. □

Remark 1.3.1 If the restriction of ϕ on $\mathbb{R}$, $\phi|_{\mathbb{R}}$, is a good function, i.e., $\phi|_{\mathbb{R}} \in L^2(\mathbb{R}) \cap L^1_{loc}(\mathbb{R})$, then $b|_{\mathbb{R}}$ is the Fourier transform of $\phi|_{\mathbb{R}}$. For $z \in S^0_{\omega,+}$, $\lim_{\epsilon\to 0}\phi_1(\epsilon z) = 0$. Also (3) of Theorem 1.3.2 is equivalent to the classical Parseval formula.

Now we give several well-known examples.

(a) If

$$
b(\xi) = \begin{cases} \chi_+(\xi), & \xi \in S^0_{\mu,+}; \\ 0, & \xi \in S^0_{\mu,-}, \end{cases}
$$

then $\phi(z) = i(2\pi z)^{-1},\ \phi_1(z) = 1/2.$

(b) If

$$
b(\xi) = \begin{cases} \mathrm{sgn}\xi, & \xi \in S^0_{\mu,+}; \\ -1, & \xi \in S^0_{\mu,-}, \end{cases}
$$

then $\phi(z) = i(\pi z)^{-1},\ \phi_1(z) = 0.$

(c) If $b(\xi) = \chi_+(\xi)\xi^{is}$, $s \in \mathbb{R}$, then

$$
\phi(z) = i(2\pi)^{-1}e^{-\pi s/2}\Gamma(1+is)z^{-1-is},
$$

$$\phi_1(z) = (2\pi s)^{-1} e^{-\pi s/2} \Gamma(1+is)(z^{is} - z^{-is}).$$

(d) If $b(\xi) = \chi_+(\xi)e^{-t\xi}$, $t > 0$, then

$$\phi(z) = i(2\pi)^{-1}(z+it)^{-1}, \ \lim_{|z|\to 0} \phi_1(z) = 0.$$

(e) If $b(\xi) = \chi_+(\xi)t\xi e^{-t\xi}$, $t > 0$, then

$$\phi(z) = -(2\pi)^{-1}t(z+it)^{-2}, \ \phi_1(z) = (2\pi)^{-1}2zt(z^2+t^2)^{-1}.$$

In (e), the function ϕ is absolutely integrable on $\mathbb{R}$ and $\lim_{\epsilon\to 0} \phi_1(\epsilon) = 0$. This assertion is true for the following class of functions.

$$\Psi(S^0_\mu) = \left\{\psi \in H^\infty(S^0_\mu), \ \text{ for some } c_s > 0, s > 0, |\psi(z)| \leqslant \frac{c_s|z|^s}{(1+|z|^{2s})}\right\}.$$

Theorem 1.3.4 *Let $\psi \in \Psi(S^0_\mu)$ and $\phi = G(\psi)$. Then for any $\nu < \mu$, there exist $s > 0$ and $c_\nu > 0$ such that*

$$|\phi(z)| \leqslant c_\nu \min\{|z|^{-1+s}, |z|^{-1-s}\}, \ z \in S^0_\nu.$$

Therefore, $\lim_{|z|\to 0} G_1(\psi)(z) = 0$ uniformly on S^0_ν.

Proof Let $z = |z|e^{i\theta_0}$ and p_θ be the integral contour in the definition of $G(\phi)$. At first we assume that $z \in S^0_\mu$ and $|z| \geqslant 1$. For any $s \in (-1, 1)$, because $\psi \in \Psi(S^0_\mu)$, then $|\psi(z)| \leqslant C|z|^s$. This gives for $z \in S^0_\nu$,

$$\begin{aligned} |\phi(z)| = |G(\psi)(z)| &\leqslant c\int_0^\infty e^{-|z|\sin(\theta+\theta_0)t} t^s dt \\ &\leqslant c(|z|\sin(\theta+\theta_0))^{-1-s} \\ &\leqslant c(\operatorname{dist}(z, \mathbb{C}\backslash S^0_\mu))^{-1-s} \\ &\leqslant c|z|^{-1-s}, \\ &= c\min\left\{|z|^{-1+s}, |z|^{-1-s}\right\}. \end{aligned}$$

Similarly, when $|z| < 1$, we have $|\psi(z)| \leqslant |z|^{-s}$. Hence when $s \in (-1, 1)$, for $z \in S^0_\nu$, we can get

$$\begin{aligned} |\phi(z)| = |G(\psi)(z)| &\leqslant c\int_0^\infty e^{-|z|\sin(\theta+\theta_0)t} t^{-s} dt \\ &\leqslant c(|z|\sin(\theta+\theta_0))^{-1+s} \\ &\leqslant c(\operatorname{dist}(z, \mathbb{C}\backslash S^0_\mu))^{-1+s} \end{aligned}$$

$$= c \min \left\{ |z|^{-1+s}, |z|^{-1-s} \right\}.$$

At last, when $|z| \to 0$, we estimate the decay property of the function $G_1(\psi)$. Without loss of generality, we assume that $|z| < 1$. From the definition of $G_1(\psi)$, taking

$$\delta(z) = \{\xi = re^{i\theta} : 0 < r \leqslant |z|, \ \arg z \leqslant \theta \leqslant \ \arg z + \pi\},$$

we deduce that

$$\begin{aligned} |G_1(\psi)(z)| &\leqslant \int_{\delta(z)} |\phi(\xi)| d\xi \\ &\leqslant \int_0^{|z|} \min\{r^{-1+s}, r^{-1-s}\} r \\ &\leqslant C|z|^s. \end{aligned}$$

□

1.4 Convolution Singular Integral Operators on the Lipschitz Curves

In this section, applying the H^∞-functional calculus of the differential operator D_γ on the Lipschitz curve γ, we prove the L^p-boundedness of the convolution singular integral operators on γ. Roughly speaking, the main idea is to prove the operator D_γ satisfies the reverse quadratic estimate by the square integral estimate obtain by Kenig [5]. Hence for $b \in H^\infty(S^0_\mu)$, by Theorem 1.2.4, we can see that the operator $b(D_\gamma)$ is bounded on $L^p(\gamma)$. Then we use the Fourier transform on the sectors to prove that if the kernel satisfies certain conditions, then the convolution singular integral operator on γ can be represented as the H^∞-functional calculus $b(D_\gamma)$, see Theorems 1.3.3, 1.4.1, 1.4.2 and 1.4.3 for the details.

Let $D = \frac{1}{i}\frac{d}{dx}$. The differential operator D_γ on the Lipschitz curve γ is defined as $D_\gamma = (1 + iA(x))^{-1}D$. Let $(D_\gamma, -D_\gamma)$ be a dual pair of type ω operators in $(L^p(\gamma), L^{p'}(\gamma))$, $1 \leqslant p \leqslant \infty$, or $\langle C_0(\gamma), L^1(\gamma)\rangle$. When $1 < p < \infty$, D_γ is a one-one operator on $L^p(\gamma)$ and has a dense domain and a dense range.

Our first aim is to represent $\psi(D_\gamma)$ as an integral operator for $\psi \in \Psi(S^0_\mu)$, $\mu > \omega$. Notice that

$$\psi(D_\gamma) = (2\pi)^{-1} \int_\delta (D_\gamma - \lambda I)^{-1} \psi(\lambda) d\lambda,$$

where δ consists of the rays ρ_θ, $-\rho_\theta$, $\rho_{\theta+\pi}$ and $-\rho_{\theta+\pi}$, $\omega < \theta < \mu$. We also notice that for almost all $z \in \gamma$,

$$(D_\gamma - \lambda I)^{-1}u(z) = R_\lambda u(z) = \begin{cases} i\displaystyle\int_{\gamma^-(z)} e^{i\lambda(z-\xi)}u(\xi)d\xi, \ z \in \rho_\theta \cup \rho_{-\theta+\pi}, \\ -i\displaystyle\int_{\gamma^+(z)} e^{i\lambda(z-\xi)}u(\xi)d\xi, \ z \in \rho_{-\theta} \cup \rho_{\theta+\pi}. \end{cases}$$

Hence, for $u \in L^p(\gamma)$, $1 \leqslant p \leqslant \infty$, or $C_0(\gamma)$, and $z \in \gamma$,

$$\begin{aligned} \psi(D_\gamma)u(z) &= \frac{1}{2\pi}\Big[\int_{\gamma^-(z)}\int_{\rho_\theta} - \int_{\gamma^-(z)}\int_{\rho_{-\theta+\pi}} \\ &\quad + \int_{\gamma^+(z)}\int_{\rho_{-\theta}} - \int_{\gamma^+(z)}\int_{\rho_{\theta+\pi}}\Big]e^{i\lambda(z-\xi)}\psi(\lambda)u(\xi)d\lambda \\ &= \int_\gamma \phi(z-\xi)u(\xi)d\xi, \end{aligned}$$

where $\phi = G(\psi)$ and we have used the estimate obtained in Theorem 1.3.4 when changing the order of the integrals. We thus can obtain

$$\psi(D_\gamma)u = \phi * u.$$

Specially, for $\tau > 0$, define Ψ_r as

$$\Psi_r(z) = \begin{cases} \tau z e^{-\tau z}, \ z \in S^0_{\mu,+}, \\ 0, \ z \in S^0_{\mu,-}. \end{cases}$$

Then by Example (e) in Sect. 1.3,

$$\Psi_\tau(D_\gamma)u(z) = -(2\pi)^{-1}\tau\int_\gamma (z + i\tau - \xi)^{-2}u(\xi)d\xi.$$

We will prove that the operator D_γ in $L^2(\gamma)$ satisfies the reverse quadratic estimate with respect to $\Psi = \Psi_1$. We choose $C_c(\gamma)$, the space of compactly supported functions on γ, as a dense linear subspace. Denote by $L^2(\gamma)_+$ the space of the functions

$$u_+(z) = \int_\alpha^\beta \psi_+(\tau D_\gamma)f(\tau)(z)\frac{d\tau}{\tau},$$

where f denotes a Borel function from $[\alpha, \beta]$ to $C_c(\lambda)$, $\psi_+ \in \Psi(S^0_{\mu,+})$, $\Psi(S^0_{\mu,-})$, $\mu > \omega$. We will prove that for some constant q_+ and all $U_+ \in L^2(\gamma)_+$,

$$R_+(\Psi): \ \|u_+\| \leqslant q_+\Big(\int_0^\infty \|\psi_\tau(D_\gamma)u_+\|^2\frac{d\tau}{\tau}\Big)^{1/2}.$$

We have seen that

$$\psi_+(\tau D_\gamma) f_\tau(z) = F_\tau(z) = \int_\gamma \phi_\tau(z-\xi) f_\tau(\xi) d\xi,$$

where $\phi_\tau(z) = \tau^{-1}\phi(\tau z)$, and $\phi = G(\psi_+)$ is holomorphic on $C^0_{\mu,+}$. By Theorem 1.3.4, for $\omega < \nu < \mu$ and $0 < \sigma < 1$, on $C^0_{\nu,+}$, the function satisfies

$$|z\phi(z)| \leqslant \frac{C|z|^\sigma}{1+|z|^{2\sigma}}.$$

In fact, via the above formula, F_τ can be understood as not only being defined on γ but also being defined on the open set Ω_+ over γ, that is,

$$U(z) = \int_\alpha^\beta F_\tau(z)\frac{d\tau}{\tau}.$$

The following results can be proved easily:

(i) U is holomorphic on Ω_+,
(ii) U is continuous on $\Omega_+ \cup \gamma$ and equals to u_+ on γ,
(iii) U satisfies
$$|U(z)| \leqslant C|z|^{-1},\ |U'(z)| \leqslant C|z|^{-2}.$$

Hence, for $z \in \gamma$ and $t > 0$,

$$U'(z+it) = (2\pi i)^{-1}\int_\gamma (z+it-\xi)^{-2} u_+(\xi) d\xi = it^{-1}\psi_t(D_\gamma)u_+(z).$$

So the desired reverse quadratic estimate is

$$R_+(\Psi):\ \|u_+\|_2 \leqslant 2\pi q_+\Big(\int_0^\infty \int_\gamma t|U'(z+it)|^2|dz|dt\Big)^{1/2},$$

where u_+ and U satisfy the above properties (i), (ii), (iii). Now we prove the estimate $R_+(\Psi)$. For $0 < \alpha < \pi/2$ and $z = x+iy$ a complex number, $\Gamma_\alpha(z)$ will denote the open angle with axis in the vertical direction, wertex z, opening α and pointing upwards, i.e.

$$\Gamma_\alpha(z) = \{\omega = a+ib \in \mathbb{C}, b > y, |x-a| < [\tan\alpha](b-y)\}.$$

Obviously, if $0 < \alpha < [\arctan 1/M]$, then for some $\epsilon > 0$, $\Gamma_{\alpha+\epsilon} \subset \Omega_+$ for all $z \in \gamma$.

Assume that $0 < \alpha < \arctan 1/M$, and that U is defined in Ω_+, if $z \in \Lambda$, We define the non-tangential maximal function as

$$(M_\alpha U)(z) = \sup_{\xi\in\Gamma_\alpha(z)} |U(\xi)|.$$

For the functions U defined on Ω_+, we define the following Littlewood-Paley type g-function on γ. Assume that U is differentiable in Ω_+, for $z = t + i\eta(t) \in \gamma$, define

$$g(U)(z) = \Big(\int_0^\infty t\Big|U'(z+it)\Big|^2 dt\Big)^{1/2}.$$

Then by [5, Theorem 3.17], we can obtain there exists a constant c such that

$$\|M_\alpha U\|_{L^2(\gamma)} \le c\|g(U)\|_{L^2(\gamma)}.$$

On the other hand, on γ, the boundary value of U coincides with u_+. This gives

$$\|u_+\|_{L^2(\gamma)} \le \|M_\alpha U\|_{L^2(\gamma)} \le c\|g(U)\|_{L^2(\gamma)},$$

which, together with the change of variables, implies that

$$\begin{aligned}
\|u_+\|_{L^2(\gamma)} &\le c\|g(U)\|_{L^2(\gamma)}\\
&= \Big(\int_\gamma |g(U)(z)|^2|dz|\Big)^{1/2}\\
&= \Big[\int_\gamma \Big(\int_0^\infty t\Big|U'(z+it)\Big|^2 dt\Big)|dz|\Big]^{1/2}\\
&= \Big[\int_0^\infty t\Big(\int_\gamma \Big|U'(z+it)\Big|^2|dz|\Big)dt\Big]^{1/2}\\
&= \Big[\int_0^\infty t\Big(\int_\gamma \Big|t^{-1}\psi_t(D_\gamma)u_+(z)\Big|^2|dz|\Big)dt\Big]^{1/2}\\
&= \Big(\int_0^\infty \Big\|\psi_t(D_\gamma)u_+\Big\|^2_{L^2(\gamma)}\frac{dt}{t}\Big)^{1/2}
\end{aligned}$$

We can see that D_γ satisfies the reverse quadratic estimate $R_+(\Psi)$. Similarly, D_γ satisfies $R_-(\Psi)$. Denote by D'_γ the duality of D_γ. Because $D'_\gamma = -D_\gamma$, D'_γ also satisfies $R_-(\Psi)$ and $R_+(\Psi)$. Hence, the assumptions of Theorem 1.2.4 are satisfied. For the case of L^p, the following result holds.

Theorem 1.4.1 *For any $\mu > \omega = \tan^{-1} N$ and $p \in (1, \infty)$, there exists a constant $c_{\mu,p}$ such that for all $b \in H^\infty(S^0_\mu)$ and $u \in L^p(\gamma)$,*

$$\|b(D_\gamma)u\|_p \leqslant c_{\mu,p}\|b\|_\infty\|u\|_p.$$

Proof We have proved that there exists a constant c_μ such that for all $b \in \Psi(S^0_\mu)$ and $u \in L^2(\gamma)$,

$$\|b(D_\gamma)u\|_2 \leqslant c_\mu\|b\|_\infty\|u\|_2.$$

For all $z \in S^0_\nu$,

$$b(D_\gamma)u(z) = \int_\gamma \phi(z-\xi)u(\xi)d\xi,$$

where $\phi = G(b)$. By Theorem 1.3.2, the function ϕ satisfies $|\phi(z)| \leqslant \kappa_\nu \|b\|_\infty |z|^{-1}$ for all $z \in S^0_\nu$ where $\omega < \nu < \mu$, κ_ν depends on ν. Hence for all non-zero $z \in S_\omega$ and some constant κ, $|\phi'(z)| \leqslant \kappa \|b\|_\infty |z|^{-2}$. Notice that the Lipschitz curve satisfies the doubling measure condition. By the theory of Calderón-Zygmund operators ([6]), when $1 < p < \infty$, there exists a constant $c_{\mu,p}$ such that for all $b \in \Psi(S^0_\mu)$ and $u \in L^p(\gamma)$,

$$\|b(D_\gamma)u\|_p \leqslant c_{\mu,p}\|b\|_\infty \|u\|_p.$$

By Lemma 1.2.5, we know that the above estimate holds for all $b \in H^\infty(S^0_\mu)$. □

Next, applying the results of Sect. 1.3, we give an precise representation of the operator $b(D_\gamma)$ in $L^p(\gamma)$.

Theorem 1.4.2 *Assume $b \in H^\infty(S^0_\mu)$, $\mu > \omega$. Let $\phi_\pm$ be a holomorphic function on $C^0_{\mu,\pm}$ satisfying $\phi_\pm(z) = G(\chi_\pm b)(z)$. Let ϕ and ϕ_1 be holomorphic functions on S^0_μ and $S^0_{\mu,+}$ which are defined by*

$$\begin{cases} \phi(z) = G(b)(z) = \phi_+(z) + \phi_-(z), \\ \phi_1(z) = G_1(b)(z). \end{cases}$$

If $u \in L^p(\gamma)$, then for almost all $z \in \gamma$,

$$\begin{aligned} b(D_\gamma)u(z) &= \lim_{\alpha\to 0+} \int_\gamma \Big[\phi_+(z-\xi+i\alpha) + \phi_-(z-\xi-i\alpha)\Big]u(\xi)d\xi \\ &= \lim_{\epsilon\to 0+} \Big[\int_{|z-\xi|\geqslant\epsilon} \phi(z-\xi)u(\xi)d\xi + \phi_1(\epsilon \underline{t}(z))\Big], \end{aligned}$$

where for $z \in \gamma$, $\underline{t}(z)$ denotes the unit tangent vector at z.

Proof We prove this theorem for the case $b \in H^\infty(S^0_{\mu,+})$, and the case $b \in H^\infty(S^0_{\mu,-})$ can be dealt with similarly.

Suppose that $u \in L^p(\gamma)$ and $b \in H^\infty(S^0_{\mu,+})$. Let $\phi = G(b)$ and $\phi_1 = G_1(b)$. For $\alpha > 0$ and $s > 0$, define the following functions in $H^\infty(S^0_{\mu,+})$ and $\Psi(S^0_{\mu,+})$, respectively,

$$b_\alpha(\xi) = b(\xi)e^{-\alpha\xi} \in H^\infty(S^0_{\mu,+})$$

and

$$b_{\alpha,s}(\xi) = \xi^s(1+\xi)^{-2s}b_\alpha(\xi) \in \Psi(S^0_{\mu,+}).$$

Notice that the functions $b_{\alpha,s}$ and b_α are uniformly bounded on $S^0_{\mu,+}$, and for any fixed α, on every set with the form

$$\left\{z \in S^0_{\mu,+},\ 0 < \delta \leqslant |z| \leqslant \Delta < \infty\right\},$$

when $s \to 0$, $b_{\alpha,s}$ converges to b_α uniformly. Hence by Lemma 1.2.5 and Theorem 1.4.1, as $s \to 0$,

$$\|b_{\alpha,s}(D_\gamma)u - b_\alpha(D_\gamma)u\|_p \to 0.$$

Set $\phi_{\alpha,s} = G(b_{\alpha,s})$ and $\phi_\alpha = G(b_\alpha)$. Notice that, for almost all $z \in \gamma$,

$$b_{\alpha,s}(D_\gamma)u(z) = \int_\gamma \phi_{\alpha,s}(z-\xi)u(\xi)d\xi.$$

Therefore we can get

$$\begin{aligned}|\phi_{\alpha,s}(z-\xi)| &\leqslant (2\pi)^{-1}\int_{\rho_\theta} |e^{i(z-\xi)\lambda} b_{\alpha,s}(\lambda)||d\lambda| \\ &\leqslant C(\alpha + |z-\xi|)^{-1}\end{aligned}$$

and when $s \to 0$, $\phi_{\alpha,s}$ converges to ϕ_α in the pointwise sense. By the Lebesgue dominated convergence theorem,

$$b_\alpha(D_\gamma)u(z) = \int_\gamma \phi_\alpha(z-\xi)u(\xi)d\xi = \int_\gamma \phi(z-\xi+i\alpha)u(\xi)d\xi \ a.e.$$

Using Lemma 1.2.5 and Theorem 1.4.1 again, we can obtain that, in $L^p(\gamma)$,

$$b(D_\gamma)u(z) = \lim_{\alpha\to 0+}\int_\gamma \phi(z-\xi+1\alpha)u(\xi)d\xi.$$

To prove the second equality, we first assume that u is a compactly supported Lipschitz function on γ. We will prove

$$\left|b(D_\gamma)u(z) - \int_{|z-\xi|>\delta}\phi(z-\xi)u(\xi)d\xi - u(z)\int_{C(z,\delta)}\int_\gamma \phi(z-\xi)d\xi\right| \leqslant C_\epsilon \|u'\|_\infty,$$

where $C(z,\epsilon) = \{\xi \in \mathbb{C},\ |\xi - z| = \epsilon, \text{Im}\xi > g(\text{Re}\xi)\}$.

In fact, take a subsequence $\{t_n\} \to 0$ in the first equality. We can assume that the first equality converges in the pointwise sense. Hence, for any fixed $\epsilon > 0$, we have $b(D_\gamma)u(z) = J_1 + J_2 + J_3$, where

$$J_1 = \int_{|z-\xi|>\epsilon}\phi(z-\xi)u(\xi)d\xi,$$

$$|J_2| \leqslant \int_{|z-\xi|\geqslant\epsilon} |\phi(z-\xi)||u(\xi)-u(z)||d\xi| \leqslant C_\epsilon \|u'\|_\infty,$$

$$J_3 = \lim_{t_n \to 0} u(z) \int_{C(z,\epsilon)} \phi(z-\xi+it_n)d\xi = u(z) \int_{C(z,\epsilon)} \phi(z-\xi)d\xi.$$

Note the fact that

$$\lim_{\epsilon\to 0}\Big[\int_{C(z,\epsilon)} \int_\gamma \phi(z-\xi)d\xi - \phi_1(\epsilon \underline{t}(z))\Big] = 0,$$

where for almost all $z \in \gamma$, $\underline{t}(z)$ denotes the unit tangent vector on γ. We can get the second equality for Lipschitz functions with compact support on γ.

To extend the second equality to $u \in L^p(\gamma)$, similar to the convolution singular integral operators on $\mathbb{R}^n$, we need the following maximal estimate:

Lemma 1.4.1 *Set*

$$T_\epsilon u(z) = \int_{|z-\xi|>\epsilon} \phi(z-\xi)u(\xi)d\xi$$

and

$$T^* u(z) = \sup_{\epsilon>0} |T_\epsilon u(z)|, z \in \gamma.$$

Then

$$\|T^* u\|_p \leqslant C_p \|u\|_p, \ 1 < p < \infty.$$ □

Denote by $S^0_{\mu,\pm}$ the sector $\Big\{z \in \mathbb{C} : |\arg(\pm z)| < \mu\Big\}$. Set $S^0_\mu = S^0_{\mu,+} \cup S^0_{\mu,-}$. The main result of this section is as follows.

Theorem 1.4.3 *Given* $\tan^{-1}(N) < \mu < \pi/2$ *and* $1 < p < \infty$. *Let* ϕ *and* ϕ_1 *be holomorphic functions defined on* S^0_μ *and* $S^0_{\mu,+}$ *such that* $z\phi(z)$ *and* $\phi_1(z)$ *are bounded. If for all* $z \in S^0_{\mu,+}$,

$$\phi_1'(z) = \phi(z) + \phi(-z),$$

then for $u \in L^p(\gamma)$ *and* $z \in \gamma$ *a.e., we can define the following bounded linear operator* T *on* $L^p(\gamma)$:

$$(Tu)(z) = \lim_{\epsilon\to 0+}\Big[\int_{|z-\zeta|\geqslant\epsilon} \phi(z-\zeta)u(\zeta)d\zeta + \phi_1(\epsilon\underline{t}(z))\Big],$$

where $\underline{t}(z)$ *is the unit tangent vector at* $z \in \gamma$. *Moreover,*

$$\|Tu\|_p \leqslant C_{N,\mu,p}\Big[\sup\Big\{|z\phi(z)| : z \in S^0_\mu\Big\} + \sup\Big\{|\phi_1(z)| : z \in S^0_{\mu,+}\Big\}\Big]\|u\|_p.$$

Here $C_{N,\mu,p}$ *denotes the constant which depends only on* N, μ *and* p.

Proof In fact, let $B = b(D_\gamma)$. This theorem is a direct corollary of Theorems 1.3.3, 1.4.1 and 1.4.2. □

1.5 L^p-Fourier Multipliers on Lipschitz Curves

In this section, let γ be a Lipschitz graph defined as

$$\gamma = \gamma_g =: \Big\{x + ig(x) \in \mathbb{C},\ x \in \mathbb{R}\Big\}.$$

where g is a Lipschitz function. Assume that $\|g'\|_\infty \leqslant M < \infty$ and $\|g\|_\infty \leqslant M < \infty$. We will discuss a class of L^p-Fourier multipliers on γ. We first introduce the following class of Bancah spaces.

Definition 1.5.1 Let $-\infty < \beta < \infty$.

(1) If a Lebesgue measurable function $w : (-\infty, \infty) \to \mathbb{C}$ satisfies

$$\|w\|_{C_\beta} = \Big(\int_{-\infty}^{\infty} |w(\xi)|^2 \exp(2\beta|\xi|)d\xi\Big)^{1/2} < \infty,$$

we call $w \in C_\beta$.

(2) For $w \in C_\beta$, if $w', w'' \in C_\beta$, we call $w \in C_\beta^2$ and the norm is defined as

$$\|w\|_{C_\beta^2} =: \Big\{\|w\|_{C_\beta}^2 + \|w'\|_{C_\beta}^2 + \|w''\|_{C_\beta}^2\Big\}.$$

We regard C_β^2 as a test space, its dual is denoted by $(C_\beta^2)'$. Define the dual relation as

$$\langle w,\ v\rangle = \int_{-\infty}^{\infty} w(\xi)v(\xi)d\xi, w \in C_{-\beta}, v \in C_\beta^2.$$

At first, $|\langle w,\ v\rangle| \leqslant \|w\|_{C_{-\beta}}\|v\|_{C_\beta^2}$. Secondly, $\langle w,\ v\rangle = 0$ for all $v \in C_\beta^2$ if and only if $w = 0$. Hence $C_{-\beta} \hookrightarrow (C_\beta^2)'$. If $\alpha < \beta$, then $C_\beta \subset C_\alpha$ and $C_\beta^2 \subset C_\alpha^2$. This embedding is continuous and dense. Therefore $(C_\alpha^2)' \subset (C_\beta^2)'$.

We will use the Fourier transform and the inverse Fourier transform concepts on γ which are defined as follows. If $u \in L^1(\gamma)$, define

$$\hat{u}(\xi) = \int_\gamma e^{iz\xi}u(z)dz.$$

Then $\hat{u}$ is a continuous function and satisfies

$$|\hat{u}(\xi)| \leqslant e^{|\xi|M}\|u\|_1,$$

so $\hat{u} \in C_{-\beta}$, $\beta > M$ and $\|\hat{u}\|_{C_{-\beta}} \leqslant (\beta - M)^{-1/2}\|u\|_1$.

For $\beta > M$ and $w \in C_\beta$, on the strip

$$X_\beta = \{\xi \in \mathbb{C},\ |\mathrm{Im}\xi| < \beta\},$$

define the holomorphic function $\check{w}$ as

$$\check{w}(\zeta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\zeta\xi} w(\xi)d\xi.$$

If $\check{w} = 0$, then $w = 0$. Let

$$(C_\beta)^\vee(\gamma) = \left\{\check{w}|_\gamma,\ w \in C_\beta\right\} \text{ and } (C_\beta^2)^\vee(\gamma) = \left\{\check{w}|_\gamma,\ w \in C_\beta^2\right\}.$$

The norms of the above spaces are defined as $\|\check{w}\|_{(C_\beta)^\vee(\gamma)} = \|w\|_{C_\beta}$ and $\|\check{w}\|_{(C_\beta^2)^\vee(\gamma)} = \|w\|_{C_\beta^2}$.

Theorem 1.5.1 *(i) A holomorphic function f on X_β belongs to $(C_\beta)^\vee$ if and only if*

$$\sup_{|y|<\beta}\int |f(x+iy)|^2 dx < \infty.$$

Moreover

$$\frac{1}{2}\|w\|_{C_\beta} \leqslant \sqrt{2\pi}\sup_{|y|<\beta}\left(\int |\check{w}(x+iy)|^2 dx\right)^{1/2} \leqslant \|w\|_{C_\beta}.$$

(ii) If $w \in C_\beta$ and $|\mathrm{Im}z| \leqslant M$, then

$$|\check{w}(z)| \leqslant \frac{1}{2\pi}(\beta - M)^{-1}\|w\|_{C_\beta}$$

and

$$\sup_{|y|\leqslant M}|\check{w}(x+iy)| \to 0, |x| \to \infty,$$

so $\check{w}\ |_\gamma \in C_0(\gamma)$. Hence $(C_\beta)^\vee(\gamma)$ is embedded in $C_0(\gamma)$ continuously.

(iii) If $w, v \in C_\beta$, then

$$\int_\gamma \check{w}(z)\check{v}(z)dz = \frac{1}{2\pi}\int_{-\infty}^{\infty} w(\xi)v(-\xi)d\xi.$$

(iv) A holomorphic function f on X_β belongs to $(C_\beta^2)^\vee$ if and only if

$$\sup_{|y|<\beta}\int |(1+x^2)f(x+iy)|^2 dx < \infty.$$

Moreover, there exists $c_\beta > 0$ *such that*

$$\frac{1}{c_\beta}\|w\|_{C^2_\beta} \leqslant \sup_{|y|<\beta}\left(\int |(1+x^2)\check{w}(x+iy)|^2 dx\right)^{1/2} \leqslant c_\beta \|w\|_{C^2_\beta}.$$

(v) *For all* $p \in [1,\infty]$, $(C^2_\beta)^\vee(\gamma)$ *is embedded in* $L^p(\gamma)$ *continuously.*

Proof The second part of (ii) can be obtained via the Cauchy integral formula on the rectangle with vertices $(1 \pm \frac{1}{2})\mathrm{Re} z \pm \frac{i}{2}(M+\beta)$. By (ii), applying Cauchy's theorem, we can prove that

$$\int_\gamma \check{w}(z)\check{v}(z)dz = \int_{\mathbb{R}} \check{w}(x)\check{v}(x)dx,$$

so (iii) can be deduced from the so-called Parseval identity. It is easy to see that the functions f with the standard properties are of the form $f = \check{w}$, where $w \in C_\beta$ or C^2_β. To prove (v), we first prove that if $w \in C^2_\beta$ and $|y| \leqslant M$, then for some constant c, $(1+x^2)|\check{w}(x+iy)| \leqslant c\|w\|_{C^2_\beta}$. □

Remark 1.5.1 Let $\mathcal{A}(\gamma)$ be the space of all functions f which are holomorphic for

$$\min_{z\in\gamma}|g(z)| - \epsilon < \mathrm{Im} z < \max_{z\in\gamma}|g(z)| + \epsilon$$

and satisfy

$$\int |f(x+iy)|^2 dx < C_\epsilon.$$

Then $\bigcup_{\beta>M}(C_\beta)^\vee(\gamma) = \mathcal{A}(\gamma)$.

For the arguments of approximation, we shall use the maximal function M_γ defined as follows. For a locally integrable function u on γ, define

$$M_\gamma u(z) = \sup_{\rho>0}\rho^{-1}\int_{B(z,\rho)}|u(\xi)||d\xi|,$$

where $z \in \gamma$ and $B(z,\rho) = \{\xi \in \gamma,\ |\xi - z| < \rho\}$. The follows results can be obtained by the usual methods. Noticing that $\|g'\|_\infty \leqslant N$, we have

Proposition 1.5.1 *For* $1 < p \leqslant \infty$, *there exists constants* $c_{p,N}$ *and* c_N *such that*

$$\|M_\gamma u\|_p \leqslant c_{p,N}\|u\|_p,\ u \in L^p(\gamma)$$

and

$$\lambda\mu(\{z \in \gamma,\ M_\gamma u(z) > \lambda\}) \leqslant c_N\|u\|_1,$$

where μ *denotes the measure introduced by the arc-length.*

Proposition 1.5.2 *Suppose that u is a locally integrable function on γ, $\phi * u$ and $\psi * u$ are well-defined, where ψ is a decreasing function in $L^1(0,\infty)$ such that*

$$|\phi(z)| \leqslant \psi(|x|),\ z = x + iy \in \Gamma,$$

where $\Gamma = \{z - \xi,\ z, \xi \in \gamma\}$. Then for all $z \in \gamma$,

$$|\phi * u(z)| \leqslant c_N \|\psi\|_1 M_\gamma u(z).$$

The above result can be proved directly for the functions ψ with the form $\psi(\xi) = \sum_k a_k \chi_k(\xi)$, where χ_k is the characteristic function of a ball centered at 0. For general ψ, we use the sequence of such functions to approximate ψ. We prove that, in an appropriate sense, the function u is the limit of the sequence $\{\phi_n * u\}$, where $\phi_n(z) = n\phi(nz)$ and ϕ is a holomorphic function defined on all of $\mathbb{C}$ which satisfies

$$\int_{-\infty}^{\infty} \phi(x)dx = 1 \tag{1.5}$$

and for some constant c,

$$|\phi(z)| \leqslant \frac{c}{1+x^2},\ z = x + iy \in S^0_\mu, \tag{1.6}$$

where $\tan\mu > N$.

For $N < \pi/4$, take $\phi(z) = \exp(-z^2)$. For $\pi/4 \leqslant N < \pi/2$, there exists a function satisfying two conditions mentioned above. The following example was constructed by McIntosh-Qian [7].

Lemma 1.5.1 *([7, Section 8]) Let $0 < \mu < \pi/2$. There exists an entire holomorphic function ϕ satisfying (1.5) and (1.6).*

Proof At first, let f be a holomorphic function on the upper-half plane which is defined as

$$f(z) = (i + z)^{-2}\exp((-iz)^\lambda),$$

where λ satisfies $\pi\lambda/2 < \pi/2 - \mu$. The following conclusions can be verified easily.

(i) Let δ denote the curve $\{z,\ |\arg z - \pi/2| = \pi/2\lambda\}$. For all $z \in \delta$, $|f(z)| = |i + z|^{-2}$.
(ii) When $y \to +\infty$, $|f(iy)| \to \infty$.

Define a function

$$G(z) := \begin{cases} \frac{1}{2\pi i}\int_\delta \frac{1}{z-\zeta} f(\zeta)d\zeta, & z \text{ below } \delta, \\ \frac{1}{2\pi i}\int_\delta \frac{1}{z-\zeta} f(\zeta)d\zeta + f(z), & z \text{ above } \delta. \end{cases} \tag{1.7}$$

The function G can be extended to $\mathbb{C}$ continuously, so it is an entire holomorphic function. It is bounded below δ and unbounded above δ. Define ϕ as $\phi(z) = \kappa G'(z)\overline{G'(\overline{z})}$, where is κ a normalizing factor. Then ϕ satisfies (1.5) and (1.6).

□

We call a sequence $\{\phi_n\}$ constructed as above an identity sequence. The sequence $\{\phi_n\}$ satisfy the following properties. Let $\psi_n(s) = n(1+n^2s^2)^{-1}$, $s > 0$. Then

(1) for every n,

$$|\phi_n(z)| \leqslant c\psi_n(|x|),\ z = x + iy \in S^0_\mu. \tag{1.8}$$

(2) For every n,

$$\int_0^\infty \psi_n(s)ds = \frac{1}{2}\pi. \tag{1.9}$$

(3) For all $\delta > 0$,

$$\int_\delta^\infty \psi_n(s)ds \to 0,\ n \to 0. \tag{1.10}$$

(4) For every n and every $\xi \in \gamma$,

$$\int_\gamma \phi_n(z-\xi)dz = 1. \tag{1.11}$$

The following two theorems give the ways in which $\phi_n * u$ converges to u.

Theorem 1.5.2 *Let $\{\phi_n\}$ be an identity sequence. Then*

(i) for $1 < p \leqslant \infty$, there exists a constant $c_{p,N}$ such that

$$\|\sup_n |\phi_n * u|\|_p \leqslant c_{p,N}\|u\|_p,\ \ u \in L^p(\gamma).$$

(ii) If $u \in L^p(\gamma)$, $1 \leqslant p \leqslant \infty$, then for almost all $z \in \gamma$,

$$\lim_{n\to\infty} (\phi_n * u)(z) = u(z).$$

(iii) If $u \in L^p(\gamma)$, $1 \leqslant p < \infty$, then

$$\lim_{n\to\infty} \|(\phi_n * u) - u\|_p = 0.$$

(iv) If $u \in C_0(\gamma)$, then

$$\lim_{n\to\infty} \|(\phi_n * u) - u\|_\infty = 0.$$

Proof Part (i) is a corollary of the former two propositions. Next assume that $u \in C_0(\gamma)$. It can be deduced from (1.11) that

$$\begin{aligned}(\phi * u)(z) - u(z) &= \int_{|\zeta - z| < \delta} \phi_n(z-\zeta)(u(\zeta) - u(z))d\zeta \\ &+ \int_{|\zeta - z| \geqslant \delta} \phi_n(z-\zeta)(u(\zeta) - u(z))d\zeta \\ &= I_1 + I_2.\end{aligned}$$

Let $\epsilon > 0$. Take δ small enough such that for all ζ on γ which satisfies $|\zeta - z| < \delta$, $|u(\zeta) - u(z)| < \epsilon$. Hence, by (1.8) and (1.9),

$$I_1 \leqslant \epsilon \int_\gamma |\phi_n(z-\zeta)||d\zeta| \leqslant c\epsilon\pi\sqrt{1+N^2}.$$

Applying (1.10), we can get $I_2 \leqslant \epsilon$ for all sufficiently large n. Hence

$$|\phi_n * u(z) - u(z)| \leqslant \epsilon(1 + c\pi\sqrt{1+N^2}).$$

Therefore (iv) holds as part (ii) does in the case when $u \in C_0(\gamma)$.

For $u \in L^p(\gamma)$, $1 \leqslant p < \infty$ and any $\delta > 0$, there exists the decomposition $u = v + w$, where $v \in C_0(\gamma)$ and $\|w\|_p < \delta$. Hence, by the former propositions,

$$\begin{aligned}&\mu\Big(\Big\{z \in \gamma,\ \overline{\lim_{n\to\infty}}|\phi_n * u(z) - u(z)| > \kappa\Big\}\Big) \\ &= \mu\Big(\Big\{z \in \gamma,\ \overline{\lim_{n\to\infty}}|\phi_n * w(z) - w(z)| > \kappa\Big\}\Big) \\ &\leqslant \mu\Big(\Big\{z \in \gamma,\ \overline{\lim_{n\to\infty}}|\phi_n * w(z)| > \kappa/2\Big\}\Big) + \mu\Big(\Big\{z \in \gamma,\ \overline{\lim_{n\to\infty}}|w(z)| > \kappa/2\Big\}\Big) \\ &\leqslant \mu\Big(\Big\{z \in \gamma,\ M_\gamma w(z) > \kappa/2\Big\}\Big) + \mu\Big(\Big\{z \in \gamma,\ \overline{\lim_{n\to\infty}}|w(z)| > \kappa/2\Big\}\Big) \\ &\leqslant c\kappa^{-p}\|w\|_p^p \leqslant c\kappa^{-p}\delta^p.\end{aligned}$$

Let $\delta \to 0$ first and then $\kappa \to 0$. We can get

$$\mu\Big(\Big\{z \in \gamma,\ \overline{\lim_{n\to\infty}}|\phi_n * u(z) - u(z)| > 0\Big\}\Big) = 0.$$

This implies Part (ii) holds for the case $1 \leqslant p < \infty$. The case $p = \infty$ can be reduced to the case $p = 1$ by a localization argument.

Now we prove (iii). For $u \in L^p(\gamma)$, define $U \in L^p(\mathbb{R})$ by $U(x) = u(x + ig(x))$. Then

$$\begin{aligned}\|(\phi_n * u) - u\|_p &= \Big\| \int_\gamma \phi_n(\cdot - \zeta)u(\zeta) - u(\cdot)d\zeta \Big\|_{L^p(\gamma)} \\ &\leqslant c \Big\| \int_{\mathbb{R}} \psi_n(|x-y|)|U(x) - U(y)|dy \Big\|_{L^p(dx)}\end{aligned}$$

$$
\begin{aligned}
&= c\left\| \int_{\mathbb{R}} \psi_n(|s|)|U(x) - U(x - s/n)|ds \right\|_{L^p(dx)} \\
&\leqslant c \int_{\mathbb{R}} \psi_n(|s|) \Big\| |U(x) - U(x - s/n)| \Big\|_{L^p(dx)} ds \\
&= c \int_{\mathbb{R}} \psi(|s|)\Delta(U,\ s/n)ds,
\end{aligned}
$$

where

$$
\Delta(U,\ s/n) = \Big\| |U(x) - U(x - s/n)| \Big\|_{L^p(dx)}.
$$

Notice that when $n \to \infty$, $\Delta(U,\ s/n) \to 0$ and $\Delta(U,\ s/n) \leqslant 2\|U\|_p$. By the Lebesgue dominated convergence theorem, when $n \to \infty$, the last integral tends to 0. □

Let ϕ satisfy (1.5) and (1.6). Define an identity sequence $\{\phi_n\}$ by $\phi_n(z) = n\phi(nz)$. Let $\Phi_n = \hat{\phi}_n$ and $\Phi = \hat{\phi}$. Then $\Phi_n(\xi) = \Phi(n^{-1}\xi)$, where Φ is continuous and satisfies $\Phi(0) = 1$. When $\xi > 0$,

$$
\begin{aligned}
|\Phi(\xi)| &= \left| \int_{-\infty}^{\infty} e^{-i\xi x}\phi(x)dx \right| = \left| \int_{-\infty}^{\infty} e^{-i\xi(x - i\lambda)}\phi(x - i\lambda)dx \right| \\
&\leqslant e^{-\lambda|\xi|} \int_{-\infty}^{\infty} |\phi(x - i\lambda)|dx = c_\lambda e^{-\lambda|\xi|}.
\end{aligned}
$$

When $\xi < 0$, in the above estimate, replace λ by $-\lambda$. Hence for every $\lambda > 0$, there exists c_λ such that

$$
|\Phi(\xi)| \leqslant c_\lambda e^{-\lambda|\xi|},\ -\infty < \xi < \infty. \tag{1.12}
$$

Therefore for all n and all $\beta > 0$, $\Phi_n \in C_\beta$. Clearly, on any compact subset of $(-\infty, \infty)$, $\Phi_n \to 1$ uniformly.

Theorem 1.5.3 *Let $\{\phi_n\}$ be an identity sequence and let $\Phi_n = \hat{\phi}_n$. Suppose that $\beta \geqslant \alpha \geqslant M$.*

(1) If $w \in C_\alpha$, then $\phi_n * \check{w} = (\Phi_n w)^\vee \in (C_\beta)^\vee(\gamma)$ and $\Phi_n w \to w$ in C_α.
(2) If $u \in L^1(\gamma)$, then $\phi_n * u \in (C_\beta)^\vee(\gamma)$ and $\Phi_n \hat{u} \to \hat{u}$ in $C_{-\beta}$.

Proof We first prove (i). Taking $\lambda = 2n(\beta - \alpha)$ in (1.12), we can see that $|\Phi(\xi)| \leqslant c_\lambda \exp(-2(\beta - \alpha)|\xi|)$. Then

$$
\begin{aligned}
\|\Phi_n w\|_{C_\beta}^2 &= \int_{-\infty}^{\infty} |\Phi_n(\xi)w(\xi)|^2 \exp(2\beta|\xi|)d\xi \\
&\leqslant \int_{-\infty}^{\infty} |w(\xi)|^2 \exp(2\beta|\xi|) \exp(-2(\beta - \alpha)|\xi|)d\xi \\
&\leqslant \int_{-\infty}^{\infty} |w(\xi)|^2 \exp(2\alpha|\xi|)d\xi \\
&\leqslant \|w\|_{C_\alpha}^2.
\end{aligned}
$$

Hence $\phi_n * \check{w} \in (C_\beta)^\vee(\gamma)$. Also, at each point, $\Phi_n \to 1$,

$$\begin{aligned}\|\Phi_n w - w\|_{C_\alpha}^2 &= \int_{-\infty}^{\infty} |\Phi_n(\xi)w(\xi) - w(\xi)|^2 \exp(2\alpha|\xi|)d\xi \\ &= \int_{-\infty}^{\infty} |\Phi_n(\xi) - 1|^2 |w(\xi)|^2 \exp(2\alpha|\xi|)d\xi \to 0, n \to \infty.\end{aligned}$$

Now we prove (ii). Because $u \in L^1(\gamma)$, $|\hat{u}(\xi)| \leqslant \exp(|\xi|M)\|u\|_1$. Similarly, we can get $\Phi_n\hat{u} \in C_\beta$ and $\Phi_n\hat{u} \to \hat{u}$ in $C_{-\beta}$. Also

$$\begin{aligned}\phi_n * u(z) &= \int_\gamma \phi_n(z-\zeta)u(\zeta)d\zeta \\ &= \int_\gamma \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi_n(\xi)e^{i\xi(z-\zeta)}d\xi u(\zeta)d\zeta \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_\gamma e^{-i\xi\zeta}u(\zeta)d\zeta e^{i\xi z}\Phi_n(\xi)d\xi \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{u}(\xi)\Phi_n(\xi)e^{\xi z}d\xi \\ &= (\Phi_n\hat{u})^\vee(z).\end{aligned}$$

□

We state several density results and a version of Parseval's formula.

Theorem 1.5.4 *Let $\beta > \alpha$ and $1 \leqslant p < \infty$.*

(i) The following inclusions are all dense:

$$\begin{cases}(C_\beta^2)^\vee(\gamma) \subset L^1 \cap L^p \cap (C_\beta)^\vee(\gamma) \subset L^p \cap (C_\beta)^\vee(\gamma) \subset (C_\beta)^\vee(\gamma); \\ (C_\beta^2)^\vee(\gamma) \subset (C_\alpha^2)^\vee(\gamma); \\ L^1 \cap L^p \cap (C_\beta)^\vee(\gamma) \subset L^1 \cap L^p(\gamma) \subset L^p(\gamma); \\ L^p \cap (C_\beta)^\vee(\gamma) \subset L^p \cap (C_\alpha)^\vee(\gamma) \subset L^p(\gamma); \\ (C_\beta)^\vee(\gamma) \subset (C_\alpha)^\vee(\gamma).\end{cases} \tag{1.13}$$

(ii) In the above inclusions, $L^p(\gamma)$ can be replaced by $C_0(\gamma)$.

(iii) For $u \in L^1(\gamma)$ and $w \in C_\beta$,

$$\int_\gamma u(z)\check{w}(z)dz = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{u}(\xi)w(-\xi)d\xi.$$

(iv) If $u \in L^1(\gamma)$ and $\check{u} = 0$, then $u = 0$.

Proof (i) We prove the inclusions in first line. Let $u \in (C_\beta)^\vee(\gamma)$. For any $\epsilon > 0$, define $u_\epsilon(z) = (1 + \epsilon^2 z^2)^{-1}u(z)$. By (i) and (iv) of Theorem 1.5.1, $u_\epsilon \in (C_\beta^2)^\vee(\gamma)$.

In $L^p(\gamma)$ and $C_0(\gamma)$, $u_\epsilon \to u$. Hence these inclusions are dense. In addition, we can see that $(C^2_\beta)^\vee(\gamma)$ is dense in $(C^2_\alpha)^\vee(\gamma)$, and $(C_\beta)^\vee(\gamma)$ is dense in $(C_\alpha)^\vee(\gamma)$. This proves the second and the fifth inclusions.

Let $u \in L^1 \cap L^p(\gamma)$. Then $\phi_n * u \in (C_\beta)^\vee(\gamma)$ and $\phi_n * u$ belongs to $L^1(\gamma)$ and $L^p(\gamma)$. In $L^1(\gamma)$ and $L^p(\gamma)$, $\phi_n * u \to u$. we get $L^1 \cap L^p \cap (C_\beta)^\vee(\gamma)$ is dense in $L^1 \cap L^p(\gamma)$, and is dense in $L^p(\gamma)$. This proves the third inclusion. Similarly, we can prove $L^p \cap (C_\alpha)^\vee(\gamma)$ is dense in $L^p(\gamma)$, and $L^p \cap (C_\beta)^\vee(\gamma)$ is dense in $L^p \cap (C_\alpha)^\vee(\gamma)$.

(ii). In the proof of (i), replacing $L^p(\gamma)$ by $C_0(\gamma)$, we can use the same method to verify that the conclusion of (i) is valid for $C_0(\gamma)$.

(iii) Let $u \in L^1(\gamma)$ and $w \in C_\beta$. Then $\phi_n * u = (\Phi_n \hat{u})^\vee \in (\beta)^\vee(\gamma)$. By (iii) of Theorem 1.5.1,

$$\int_\gamma (\phi_n * u)(z)\check{w}(z)dz = \frac{1}{2\pi}\int_{-\infty}^{\infty} (\Phi_n \hat{u})(\xi)w(-\xi)d\xi.$$

Because $\phi_n * u \to u$ in $L^1(\gamma)$ and $\Phi_n \hat{u} \to \hat{u}$ in $C_{-\beta}(\gamma)$, (iii) holds.

At last, by (iii) and the fact that $(C^2_\beta)^\vee(\gamma)$ is dense in $L^1(\gamma)$, we can obtain (iv). □

From Theorem 1.5.1, we can see that the following definition of the Fourier transform coincides with the definition for $u \in L^1(\gamma)$.

For some $1 \leqslant p \leqslant \infty$, $u \in L^p(\gamma)$, define $\hat{u} \in (C^2_\beta)'$ by

$$\langle \hat{u},\ w_- \rangle = \int_\gamma u(z)\check{w}(z)dz,$$

where $w \in C^2_\beta$ and $w_-(\xi) = w(-\xi)$.

It should be pointed out that this definition is independent of the choice of β, $\beta > M$, and the mapping $\mathcal{F} : L^p(\gamma) \to (C^2_\beta)'$ satisfying $\mathcal{F}(u) = \hat{u}$ is continuous and one-one. If the above Fourier transform and the inverse Fourier transform are well-defined, two transforms are inverse operations.

Theorem 1.5.5 *Let $u \in L^p(\gamma)$, $p \in [1, \infty]$ and let $w \in C_\beta,\ \ \beta > M$. Then $u = \check{w}$ if and only if $w = \hat{u}$.*

Proof Let $u \in L^p(\gamma)$, $1 \leqslant p \leqslant \infty$ and $w \in C_\beta$. By (iii) of Theorem 1.5.1, we can see that for all $v \in C^2_\beta$,

$$\int_\gamma \check{w}(z)\check{v}(z)dz = \frac{1}{2\pi}\int_{-\infty}^{\infty} w(\xi)v(-\xi)d\xi = \frac{1}{2\pi}\langle w,\ v_- \rangle.$$

Assume that $\check{w} = u \in L^p(\gamma)$. Then for any $v \in C^2_\beta$,

$$\int_\gamma \check{w}(z)\check{v}(z)dz = \int_\gamma u(z)\check{v}(z)dz = \frac{1}{2\pi}\langle \hat{u},\ v_- \rangle.$$

Hence $\hat{u} = w$. On the other hand, assume that $\hat{u} = w \in C_\beta$. Then for all $v \in C_\beta^2$,

$$\int_\gamma \check{w}(z)\check{v}(z)dz = \frac{1}{2\pi}\langle w,\ v_-\rangle = \int_\gamma u(z)\check{v}(z)dz.$$

Specially, for an identity sequence $\{\phi_n\}$,

$$\int_\gamma \phi_n(\zeta - z)\check{w}(z)dz = \int_\gamma \phi_n(\zeta - z)u(z)dz.$$

Taking limits for all $\zeta \in \gamma$, we can get $\check{w} = u$.

□

Now we introduce Fourier multipliers. Let $1 \leqslant p \leqslant \infty$ and take $\beta < M$.

Definition 1.5.2 Let $b \in L^\infty(-\infty, \infty)$. If a L^∞-function b satisfies

$$\|b\|_{M_p(\gamma)} =: \sup\left\{\|(b\hat{u})^\vee\|_{L^p(\gamma)},\ u \in L^p(\gamma) \cap (C_\beta)^\vee(\gamma),\ \|u\|_p = 1\right\} < \infty,$$

we call b a $L^p(\gamma)$-Fourier multiplier, denoted by $b \in M_p(\gamma)$.

When $1 \leqslant p \leqslant \infty$ and $b \in M_p(\gamma)$, there exists a unique L^p-bounded linear operator B defined on the dense subspace $L^p(\gamma) \cap (C_\beta)^\vee(\gamma)$:

$$Bu = (b\hat{u})^\vee.$$

When $p = \infty$ and $b \in M_p(\gamma)$, we can define a unique bounded linear operator B on $C_0(\gamma)$ similarly. If b_1 and b_2 are L^p-Fourier multipliers, the corresponding operators are denoted by B_1 and B_2, respectively, then $b_1 b_2$ is also a L^p-Fourier multiplier with the corresponding operator denoted by $B_1 B_2$. The function 1 also belongs to $M_p(\gamma)$ with the corresponding operator I.

The reason for using C_β to define L^p-Fourier multipliers is that if $w \in C_\beta$ and $b \in L^\infty(-\infty, \infty)$, then $bw \in C_\beta$. By (i) of Theorem 1.5.4, when $\beta > M$, the definition of bw is independent of the choice of β.

Proposition 1.5.3 *Let* $b \in L^\infty(-\infty, \infty)$. *Then*

$$\|b\|_{M_p(\gamma)} = \sup\left\{\|(bw)^\vee\|_{L^p(\gamma)},\ w \in C_\beta^2,\ \|\check{w}\|_{L^p(\gamma)} = 1\right\}.$$

When the right-hand side of the above equality is finite, b is a L^p*-Fourier multiplier.*

Proof Assume that the right-hand side of the above equality is finite. Let $u \in L^p(\gamma) \cap (C_\beta)^\vee(\gamma)$. By (i) of Theorem 1.5.4, there is a sequence $\{w_n\} \subset C_\beta^2$ such that $\check{w}_n \to u$ in $L^p(\gamma) \cap (C_\beta)^\vee(\gamma)$. Then the sequence $\{\check{w}_n\}$ is a Cauchy sequence in $L^p(\gamma)$. By the assumption, $\{(bw_n)^\vee\}$ is also a Cauchy sequence in $L^p(\gamma)$. Hence there exists $v \in L^p(\gamma)$ such that $(bw_n)^\vee \to v$ in $L^p(\gamma)$. Therefore, $bw_n \to \hat{v}$ in $(C_\beta^2)'$.

On the other hand, in C_β, $w_n \to \hat{u}$, so in C_β and $(C_\beta^2)'$, $bw_n \to b\hat{u}$. Finally, we can obtain that $\hat{v} = b\hat{u}$ and $(bw_n)^\vee \to (b\hat{u})^\vee$ in $L^p(\gamma)$. □

Proposition 1.5.4 *Let $1 \leqslant p \leqslant \infty$ and $p' = (1 - p^{-1})^{-1}$. Then $b \in M_p(\gamma)$ if and only if $b_- \in M_{p'}(\gamma)$, where $b_-(\xi) = b(-\xi)$, and $\|b\|_{M_p(\gamma)} = \|b_-\|_{M_{p'}(\gamma)}$. Denote by B and B_- the operators corresponding to b and b_-, respectively. Then B and B_- are dual operators in the sense that $\langle Bu, v\rangle = \langle u, B_- v\rangle$ for all u and v. Hence they have the same spectra $\sigma(B) = \sigma(B_-)$.*

Proof In (iii) of Theorem 1.5.4, we use the Parseval formula twice, and the proof is completed. □

We also need the following lemma.

Lemma 1.5.2 *Denote by $L_{loc}(-\infty, \infty)$ the Fréchet space of all locally integrable functions on $(-\infty, \infty)$. Let $1 \leqslant p \leqslant 2$.*

(i) If $u \in L^p(\gamma)$, then $\hat{u} \in L_{loc}(-\infty, \infty)$, and the mapping $u \to \hat{u}$ is continuous from $L^p(\gamma)$ to $L_{loc}(-\infty, \infty)$.
(ii) If $u \in L^p(\gamma)$, then $\widehat{Bu} = b\hat{u}$.

Proof Let θ be a C^2-function on $(-\infty, \infty)$ with support in $[-1-\epsilon,\ 1+\epsilon]$ which equals to 1 in a neighborhood of $[-1, 1]$. For $s \geqslant 1$, define $\theta_s(\xi) = \theta(\xi/s)$. Then $\check{\theta}_s$ is an entire function and satisfies for some c_ϵ,

$$(1 + |\zeta|^2)|\check{\theta}_s(\zeta)| \leqslant c_\epsilon |\mathrm{Im}\zeta|^{-1}(e^{(1+\epsilon)s|\mathrm{Im}\zeta|} - 1).$$

Hence for all z satisfying $|\mathrm{Im}\zeta| \leqslant M$, $|\check{\theta}_s(\zeta)| \leqslant f_s(|\zeta|)$, where f_s is a L^1-function with the norm satisfying $\|f_s\|_1 \leqslant c_s M^{-1}\exp((1+\epsilon)sM)$. By Young's inequality, there exists a constant $c_{p,s}$ such that

$$\|(\theta_s w)^\vee\|_{L^p(\mathbb{R})} \leqslant c_{p,s}\|\check{w}\|_{L^p(\gamma)}.$$

Then by Titchmarsh's restriction theorem, we obtain that for all $w \in C_\beta$,

$$\|\theta_s w\|_{p'} \leqslant c_p\|(\theta_s w)^\vee\|_{L^p(\mathbb{R})} \leqslant c_{p,s}\|\check{w}\|_{L^p(\gamma)}.$$

Because $L^p \cap (C_\beta)^\vee(\gamma)$ is dense in $L^p(\gamma)$, we can prove that for any $u \in L^p(\gamma)$, the Fourier transform $\hat{u}$ is equivalent to a locally integrable function. Hence (ii) holds. □

It is well-known that if b is a Lebesgue measurable function on $(-\infty, \infty)$ which satisfies $b\hat{u} = \widehat{(Bu)}$ for all $u \in L^1 \cap L^p(\gamma)$ and a bounded linear operator B, then $b \in L^\infty(-\infty, \infty)$ and $\|b\|_\infty \leqslant \|B\|$. For L^p-Fourier multipliers on the Lipschitz curve γ, we can prove a similar result.

Theorem 1.5.6 *Suppose $1 \leqslant p \leqslant \infty$ and let $b \in M_p(\gamma)$.*

(i) The spectrum $\sigma(B)$ of the operator B corresponding to the function b satisfies

$$\sigma \supset \text{ess} - \text{range}(b).$$

(ii) $\|b\|_\infty \leqslant \|b\|_{M_p(\gamma)}$.
(iii) $M_p(\gamma)$ *is complete, and so is a Banach algebra.*

Proof (i) We first suppose $1 \leqslant p \leqslant 2$. Let $\mathcal{B}(\lambda, p)$ and $\overline{\mathcal{B}}(\lambda, p)$ denote the open ball and the closed ball centered at λ with radius ρ, respectively. Assume that $\lambda \notin \sigma(B)$. There exist κ and $\rho > 0$ such that for all $\mu \in \overline{\mathcal{B}}(\lambda, p)$, $(B - \mu I)$ is invertible and satisfies $\|(B - \mu I)^{-1}\| \leqslant \kappa$. Let θ_s be the function used in Lemma 1.5.2, where $\epsilon < 1$. For $u \in L^p(\gamma)$, define the operator $F_{s,\mu}$ as

$$F_{s,\mu}(u) = \check{\theta}_{2s} * (B - \mu I)^{-1}(\check{\theta}_s * u).$$

Then

$$\begin{aligned}\|F_{s,\mu}(u)\|_{L^p(\mathbb{R})} &\leqslant c_{p,2s}\|(B - \mu I)^{-1}(\check{\theta}_s * u)\|_{L^p(\gamma)}\\ &\leqslant \kappa c_{p,2s}\|\check{\theta}_s * u\|_{L^p(\gamma)} \leqslant \kappa c_{p,s} c_{p,2s}\|u\|_{L^p(\mathbb{R})}.\end{aligned}$$

We can see that, for $u \in L^p(\gamma)$,

$$(b - \mu)\widehat{(F_{s,\mu})} = \theta_s \hat{u}$$

and $\|F_{s,\mu}\| \leqslant c_s\kappa$, where c_s depends on s and is independent of μ. Hence we can get

$$\|\theta_s/(b - \mu)\|_\infty \leqslant \|F_{s,\mu}\| \leqslant c_s\kappa.$$

So the measure of the set

$$\{b(\xi), -s \leqslant \xi \leqslant s\} \cap \mathcal{B}(\mu,\ (c_s\kappa)^{-1})$$

is zero. On covering $\overline{\mathcal{B}}(\lambda, \rho)$ with finitely many balls of the form $\mathcal{B}(\mu,\ (c_s\kappa)^{-1})$, we can see that the measure of the set

$$\{b(\xi), -s \leqslant \xi \leqslant s\} \cap \mathcal{B}(\lambda,\ \rho)$$

is also zero. Taking a sequence $\{s\}$ which tends to infinity, we know

$$\mathcal{B}(\lambda,\ \rho) \cap \text{ ess-range } (b) = \emptyset.$$

This implies that ess $-$ range$(b) \subset \sigma(B)$.

For the case $2 < p \leqslant \infty$, we can use the duality and Lemma 1.5.4.

Now we prove (ii). It is obvious that

$$\begin{aligned}\|b\|_\infty &= \sup\{|\lambda|,\ \lambda \in \text{ ess-range}(b)\}\\ &\leqslant \sup\{|\lambda|,\ \lambda \in \sigma(B)\} \leqslant \|B\| = \|b\|_{M_p(\gamma)}.\end{aligned}$$

Now we prove (iii). It is sufficient to verify that $M_p(\gamma)$ is complete. Let $\{b_n\}$ be a Cauchy sequence in $M_p(\gamma)$ and let B_n be an operator corresponding to b_n on $L^p(\gamma)$. Then B_n converges to an operator B on $L^p(\gamma)$ in the operator norm. By (ii), b_n tends to $b \in L^\infty(-\infty, \infty)$ in L^∞. So it can be deduced immediately that $b \in M_p(\gamma)$, $b_n \to b$ in $M_p(\gamma)$, and the operator corresponding to b is B. □

1.6 Remarks

Remark 1.6.1 In [8], A. McIntosh and T. Qian give another proof for the main results of Sects. 1.3 and 1.4. Precisely, let γ be a Lipschitz curve: $\gamma(x) = x + iA(x)$, where $\tan^{-1}\|A'\|_\infty < \omega < \pi/2$. The set S^0_ω is defined by

$$\{z \in \mathbb{C} :\ |\arg z| < \omega \text{ or } |\arg(-z)| < \omega\}.$$

They obtain the following result. Suppose that ϕ is holomorphic in S^0_ω and satisfies $|\phi(z)| \leqslant C/|z|$. Let $C_c(\gamma)$ be the class of all continuous functions with compact supports. For the operator

$$Tf(z) = \int \phi(z-\zeta)f(\zeta)d\zeta,\ \ f \in C_c(\gamma), z \notin \operatorname{supp} f,$$

the following conclusions are equivalent:

(i) T can be extended to a bounded operator on $L^2(\gamma)$;
(ii) There exists a function $\phi_1 \in H^\infty(S^0_\omega)$ such that for any $z \in S^0_\omega$, $\phi'(z) = \phi(z) + \phi(-z)$.

Remark 1.6.2 If

$$\begin{cases} \phi(z-\zeta) = i\pi^{-1}(z-\zeta)^{-1}, \\ \phi_1 = 0, \end{cases}$$

the above operator is the well-known singular Cauchy integral operator on γ. If N is small enough, the L^p-boundedness of the operator is obtained by C. P. Calderón in [9]. R. Coifman, A. McIntosh and Y. Meyer prove that the L^p-boundedness for arbitrary constant N ([10]). From then on, other mathematicians give several different proofs. For example, in [11], R. Coifman and Y. Meyer obtain that the L^p-boundedness of singular Cauchy integrals can be used to prove the L^p-boundedness of other convolution singular integral operators. We refer the reader to [5, 12] for further information. As a summary of this chapter, we give the following theorem. This theorem indicates that any convolution type Calderón-Zygmund operator on the Lipschitz curve γ can be regarded as a special case of Theorem 1.4.3.

Theorem 1.6.1 *For some $\mu > \omega$, let ϕ be a holomorphic function on S^0_μ such that for a constant c and all $z \in S^0_\mu$, $|\phi(z)| \leqslant c|z|^{-1}$. Suppose S is a bounded linear functional on $L^2(\gamma)$ and*

$$(Su)(z) = \int_\gamma \phi(z-\xi)u(\xi)d\xi.$$

for all compactly supported continuous functions u and $z \in \gamma$ outside of the support of u. Then there exists $b \in H^\infty(S^0_\theta)$, $\omega, \theta < \mu$ and $\alpha \in \mathbb{C}$ such that $S = b(D_\gamma) + \alpha I$. Specially, if $\omega < \nu < \mu$, there exists a bounded holomorphic function $\phi_1 = G_1(b)$ on S^0_ν such that for all $z \in S^0_\mu$, $\phi'(z) = \phi(z) + \phi(-z)$.

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Chapter 2
Singular Integral Operators on Closed Lipschitz Curves

In Chap. 1, we state a theory of convolution singular integral operators and Fourier multipliers on infinite Lipschitz curves. A natural question is whether there exists an analogy on closed Lipschitz curves. In this chapter, we establish such a theory for starlike Lipschitz curves. A curve is called a starlike Lipschitz curve if the curve has the following parameterization: $\widetilde{\gamma} = \{\exp(iz) : z \in \gamma\}$, where

$$\gamma = \Big\{x + ig(x) : g' \in L^\infty([-\pi, \pi]), g(-\pi) = g(\pi)\Big\}.$$

It can be proved that the starlike Lipschitz curves defined using such parameterization are the same as those defined as star-shaped and Lipschitz in the ordinary sense.

In the same pattern as in the infinite Lipschitz graph case, we can define Fourier series of L^2 functions on γ. The question can now be specified into the following two:

The first, what kind of holomorphic kernels give rise to L^2-bounded operators on starlike Lipschitz curves γ?

The second, is there a corresponding Fourier multiplier theory? In other words, what complex number sequences act as L^p-bounded Fourier multipliers on the curves?

It should be pointed out that these questions are not trivial even for the case $p = 2$, as the Plancherel theorem does not hold in this case. However, on the other hand, the case $p = 2$ is essential, as the boundedness for $1 < p < \infty$ can be deduced from the L^2 theory using the standard Calderón-Zygmund techniques.

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2.1 Preliminaries

Let γ be a Lipschitz curve defined on the interval $[-\pi, \pi]$ with the parameterization

$$\gamma(x) = x + ig(x),\ \ g : [-\pi, \pi] \to \mathbb{R},$$

where $\mathbb{R}$ denotes the real number field, $g(-\pi) = g(\pi)$, $g' \in L^\infty([-\pi, \pi])$ with $\|g'\|_\infty = N$. Denote by $p\gamma$ the 2π-periodic extension of γ to $-\infty < x < \infty$, and by $\widetilde{\gamma}$ the closed curve

$$\widetilde{\gamma} = \Big\{ \exp(iz) : z \in \gamma \Big\} = \Big\{ \exp(i(x + ig(x))) : -\pi \leqslant x \leqslant \pi \Big\}.$$

We call $\widetilde{\gamma}$ the starlike Lipschitz curve associated with γ.

We use f, F and $\widetilde{F}$ to denote the functions defined on $p\gamma$, γ and $\widetilde{\gamma}$, respectively. For $\widetilde{F} \in L^2(\widetilde{\gamma})$, the nth coefficient of $\widetilde{F}$ on $\widetilde{\gamma}$ is defined as

$$\widehat{\widetilde{F}}_{\widetilde{\gamma}}(n) = \frac{1}{2\pi i} \int_{\widetilde{\gamma}} z^{-n} \widetilde{F}(z) \frac{dz}{z}.$$

In the case of no confusion, we will sometimes suppress the subscript and write $\widehat{\widetilde{F}}(n)$.

Set

$$\sigma = \exp(-\max g(x)), \tau = \exp(-\min g(x)).$$

We consider the following dense subclass of $L^2(\widetilde{\gamma})$:

$$\mathcal{A}(\widetilde{\gamma}) = \Big\{ \widetilde{F}(z) : \widetilde{F}(z) \text{ is holomorphic in } \sigma - \eta < |z| < \tau + \eta \text{ for some } \eta > 0 \Big\}.$$

Without loss of generality, we assume that $\min g(x) < 0$ and $\max g(x) > 0$. In the case, the domains of the functions in $\mathcal{A}(\widetilde{\gamma})$ contain the unit circle $\mathbb{T}$, and by Cauchy's theorem, we know $\widehat{\widetilde{F}}_{\widetilde{\gamma}}(n) = \widehat{\widetilde{F}}_{\mathbb{T}}(n)$. If $\widetilde{F}$ and $\widetilde{G}$ belong to $\mathcal{A}(\widetilde{\gamma})$, by the Laurent series, we can obtain the inverse Fourier transform formula

$$\widetilde{F}(z) = \sum_{n=-\infty}^{\infty} \widehat{\widetilde{F}}_{\widetilde{\gamma}}(n) z^n, \tag{2.1}$$

where z is in the annulus where $\widetilde{F}$ is defined. We apply Cauchy's theorem to get the Parseval identity

$$\frac{1}{2\pi i} \int_{\widetilde{\gamma}} \widetilde{F}(z) \widetilde{G}(z) \frac{dz}{z} = \sum_{n=-\infty}^{\infty} \widehat{\widetilde{F}}_{\widetilde{\gamma}}(n) \widehat{\widetilde{G}}_{\widetilde{\gamma}}(-n). \tag{2.2}$$

Similar to Chap. 1, we will use the following half and double sectors on the complex plane $\mathbb{C}$. For $\omega \in (0, \pi/2]$, define the sets

$$S^0_{\omega,+} = \Big\{z \in \mathbb{C} : \ |\arg(z)| < \omega, z \neq 0\Big\},$$

$$S^0_{\omega,-} = -S^0_{\omega,+}, \ S^0_{\omega} = S^0_{\omega,+} \cup S^0_{\omega,-},$$

and

$$C^0_{\omega,+} = S^0_{\omega} \cup \Big\{z \in \mathbb{C} : \ \mathrm{Im}(z) > 0\Big\},$$

$$C^0_{\omega,-} = S^0_{\omega} \cup \Big\{z \in \mathbb{C} : \ \mathrm{Im}(z) < 0\Big\},$$

where $S^0_{\omega,\pm}$, S^0_{ω}, $C^0_{\omega,\pm}$ and C^0_{ω} are shown in Figs. 1.2, 1.3 and 1.4. Let X be one of the sets defined above. Denote by

$$X(\pi) = X \cap \Big\{z \in \mathbb{C} : \ |\mathrm{Re}(z)| \leqslant \pi\Big\}$$

the truncated set and by

$$pX(\pi) = \bigcup_{k=-\infty}^{\infty} \Big\{X(\pi) + 2k\pi\Big\}$$

the periodic set associated with the truncated set. The graphs of $S^0_{\omega,\pm}(\pi)$, $S^0_{\omega}(\pi)$ and $C^0_{\omega,\pm}(\pi)$ are shown in Figs. 2.1, 2.2 and 2.3.

(1) The figures of the sets $S^0_{\omega,+}$ and $S^0_{\omega,-}$ are as follows:

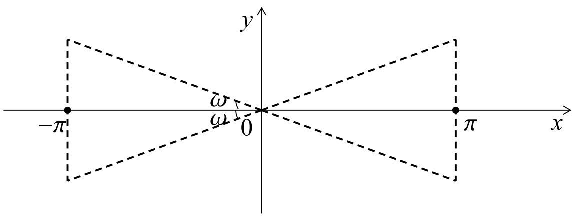

Fig. 2.1 $S^0_{\omega,-}(\pi) \cup S^0_{\omega,+}(\pi)$

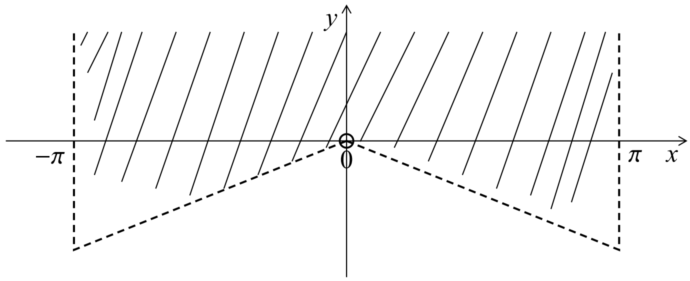

Fig. 2.2 $C^0_{\omega,+}(\pi)$

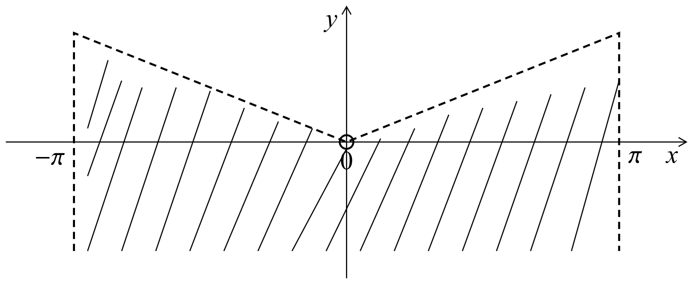

Fig. 2.3 $C^0_{\omega,-}(\pi)$

(2) The sets $C^0_{\omega,+}(\pi)$ and $C^0_{\omega,-}(\pi)$ are shown in the following figures:
We also use the sets of the form $\exp(iO) = \{\exp(iz) : \ z \in O\}$, where O is the truncated set defined above. Let Q be a double or half sector defined above. $H^\infty(Q)$ denotes the function space

$$\Big\{ f : Q \to \mathbb{C} : f \text{ is bounded and holomorphic in } Q \Big\}.$$

If no confusion occurs, we write $\|\cdot\|_\infty$ as $\|\cdot\|_{H^\infty(Q)}$.

Let $b \in H^\infty(S^0_\omega)$, $\omega \in (0, \pi/2]$. Then b can be divided into two parts: $b = b^+ + b^-$, where

$$\begin{cases} b^+ = b\chi_{\{z:\ \mathrm{Re}(z)>0\}}, \\ b^- = b\chi_{\{z:\ \mathrm{Re}(z)<0\}}. \end{cases} \tag{2.3}$$

Hence, $b^\pm \in H^\infty(S^0_{\omega,\pm})$.

In each of the following statements, the symbol "$\pm$" should be read as either all "$+$", or all "$-$". The following transform has been used in Sect. 1.3:

$$G^\pm(b^\pm)(z) = \phi^\pm(z) = \frac{1}{2\pi} \int_{\rho^\pm_\theta} \exp(iz\zeta) b(\zeta) d\zeta, \ z \in C^0_{\omega,\pm},$$

where $\rho_\theta^\pm$ denotes the ray $s\exp(i\theta), 0 < s < \infty, \theta$ is a constant which depends on $z \in C^0_{\omega,\pm}$ and satisfies $\rho_\theta^\pm \subset S^0_{\omega,\pm}$. Also

$$G_1^\pm(b^\pm)(z) = \phi_1^\pm(z) = \int_{\delta^\pm(z)} \phi^\pm(\zeta)d\zeta,\ z \in S^0_{\omega,\pm},$$

where the integral is along any path from $-z$ to z in $C^0_{\omega,\pm}$.

In what follows, we denote by c_0, c_1, C the fixed constants, and by $C_{\omega,\mu}$ the constants which depend on ω, μ and so on. These constants may vary from one occurrence to another. For $b \in H^\infty(S^0_\omega)$, using the decomposition $b = b^+ + b^-$ and Theorem 1.3.2, and letting

$$\phi = \phi^+ + \phi^-, \quad \phi_1 = \phi_1^+ + \phi_1^-,$$

we can see that the following two theorems are the main results obtained in Sect. 1.3. We reformulate them for the sake of convenience.

Theorem 2.1.1 *Let $\omega \in (0, \pi/2]$ and $b \in H^\infty(S^0_\omega)$. Then there exists a pair of holomorphic functions (ϕ, ϕ_1) defined in S^0_ω and $S^0_{\omega,+}$ such that for any $\mu \in (0, \omega)$,*

(i) $|\phi(z)| \leqslant C_{\omega,\mu}\|b\|_\infty/|z|,\ z \in S^0_\mu$;
(ii) *$\phi_1 \in H^\infty(S^0_{\mu,+})$, $\|\phi_1\|_{H^\infty(S^0_{\mu,+})} \leqslant C_{\omega,\mu}\|b\|_\infty$, and $\phi_1'(z) = \phi(z) + \phi(-z),\ z \in S^0_{\omega,+}$;*
(iii) *for all $f \in \mathcal{S}(\mathbb{R})$*

$$(2\pi)^{-1}\int_{-\infty}^{\infty} b(\zeta)\hat{f}(-\zeta)d\zeta = \lim_{\epsilon\to 0}\left\{\int_{|x|\geqslant\epsilon} \phi(x)f(x)dx + \phi_1(\epsilon)f(0)\right\}.$$

Theorem 2.1.2 *Let $\omega \in (0, \pi/2]$ and $b \in H^\infty(S^0_\omega)$. There exists a pair of holomorphic functions (ϕ, ϕ_1) defined in S^0_ω and $S^0_{\omega,+}$ satisfying*

(i) *there exists a constant c_0 such that*

$$|\phi(z)| \leqslant \frac{c_0}{|z|},\ z \in S^0_\omega;$$

(ii) *there exists a constant c_1 such that $\|\phi_1\|_{H^\infty(S^0_{\omega,+})} < c_1$, and*

$$\phi_1'(z) = \phi(z) + \phi(-z),\ z \in S^0_{\omega,+}.$$

Then for any $\mu \in (0, \omega)$, there exists a unique function $b \in H^\infty(S^0_\mu)$ such that

$$\|b\|_{H^\infty(S^0_\mu)} \leqslant C_{\omega,\mu}(c_0 + c_1),$$

and the function pair determined by b according to Theorem 2.1.1 is identical to (ϕ, ϕ_1). Moreover, for all complex numbers $\xi \in S^0_\omega$, the function b is given by

$$b(\xi) = \lim_{\epsilon\to 0}\lim_{N\to\infty}\left(\int_{\epsilon<|x|<N}\exp(-i\xi x)\phi(x)dx + \phi_1(\epsilon)\right).$$

2.2 Fourier Transforms Between S^0_ω and $pS^0_\omega(\pi)$

Theorem 2.2.1 *Let $\omega \in (0, \pi/2]$ and $b \in H^\infty(S^0_\omega)$, and let (ϕ, ϕ_1) be the function pair associated with b in the pattern of Theorem 1.3.2. Then there exists a pair of holomorphic functions (Φ, Φ_1) defined in $S^0_\omega(\pi)$ and $S^0_{\omega,+}(\pi)$, respectively, satisfying, for every $\mu \in (0, \omega)$,*

(i) *Φ can be holomorphically and periodically extended to $pS^0_\omega(\pi)$ and*

$$|\Phi(z)| \leqslant \frac{C_{\omega,\mu}\|b\|_\infty}{|z|},\ z \in S^0_\mu(\pi).$$

Moreover, $\Phi(z) = \phi(z) + \phi_0(z),\ z \in S^0_\mu(\pi)$, where ϕ_0 is a bounded holomorphic function in $S^0_\mu(\pi)$;

(ii) *$\Phi_1 \in H^\infty(S^0_{\mu,+}(\pi))$, $\|\Phi_1\|_{H^\infty(S^0_{\mu,+})} \leqslant C_{\omega,\mu}\|b\|_\infty$, and*

$$\Phi_1'(z) = \Phi(z) + \Phi(-z),\ z \in S^0_\omega(\pi);$$

(iii) *Φ and Φ_1 are uniquely determined (modulo constants) by the Parseval formula. Precisely, for any continuous 2π-periodic function F defined on $\mathbb{R}$,*

$$2\pi\sum_{n=-\infty}^{\infty} b(n)\widehat{F}(-n) = \lim_{\epsilon\to 0}\left(\int_{\epsilon\leqslant|x|\leqslant\pi}\Phi(x)F(x)dx + \Phi_1(\epsilon)F(0)\right),$$

where $\widehat{F}(n)$ denotes the nth Fourier coefficient of F, and $b(0) = \frac{1}{2\pi}\Phi_1(\pi)$.

Proof By the Poisson summation formula, we define Φ as

$$\Phi(z) = 2\pi\sum_{k=-\infty}^{\infty}\phi(z + 2k\pi), z \in pS^0_\omega(\pi), \tag{2.4}$$

where the summation takes the following sense: there is a subsequence $\{n_l\}$ of $\{n\}$ such that for all $z \in S^0_\omega(\pi)$, when $l \to \infty$, the partial sum locally uniformly converges to a 2π-periodic and holomorphic function satisfying the assertion (i). In the sequel, we call such sequences applicable sequences. Moreover, we shall show that the limit functions defined through different applicable sequences differ from one another by constants which are bounded by $c\|b\|_\infty$.

We use the following decomposition

$$\sum_{k=-n}^{n} \phi(z+2k\pi) = \phi(z) + \sum_{k\neq 0}^{\pm n}(\phi(z+2k\pi) - \phi(2k\pi)) + \sum_{k=1}^{n} \phi_1'(2k\pi)$$
$$= \phi(z) + \sum_1 + \sum_2 .$$

We will prove that the series $\sum_1$ locally uniformly converges to a bounded holomorphic function in $S^0_\mu(\pi)$, and some subsequence of the partial sums of $\sum_2$ converges to a constant dominated by $C_\mu \|b\|_\infty$.

By (i) of Theorem 2.1.1, Cauchy's theorem and the fact that ϕ is a holomorphic function, we can deduce the estimate:

$$|\phi'(z)| \leqslant \frac{C_\mu}{|z|^2},\ z \in S^0_\mu,$$

so the convergence of $\sum_1$ is proved. For $\sum_2$, we use the mean value theorem for the integrals to get

$$\sum_{k=1}^{n} \phi_1'(2k\pi) = \int_{2\pi}^{2(n+1)\pi} \phi_1'(r)dr + \sum_{k=1}^{n}\Big[\phi_1'(2k\pi) - \mathrm{Re}(\phi_1'(\xi_k)) - i\mathrm{Im}(\phi_1'(\eta_k))\Big]$$
$$= \phi_1(2(n+1)\pi) - \phi_1(2\pi)$$
$$+ \sum_{k=1}^{n}\Big[\phi_1'(2k\pi) - \mathrm{Re}(\phi_1'(\xi_k)) - i\mathrm{Im}(\phi_1'(\eta_k))\Big],$$

where $\xi_k, \eta_k \in (2k\pi, 2(k+1)\pi)$. By the estimate of ϕ', the above series converges absolutely. It can be deduced from the boundedness of ϕ_1 that there exists an applicable subsequence $\{n_l\}$ such that $\phi_1(2(n_l+1)\pi)$ converges to a constant c_0. Therefore, we have

$$\frac{1}{2\pi}\Phi(z) = \phi(z) + \sum_{k\neq 0}\Big[\phi(z+2k\pi) - \phi(2k\pi)\Big] + \lim_{l\to\infty}\sum_{n=1}^{n_l} \phi_1'(2n\pi)$$
$$= \phi(z) + \phi_0(z) + c_0,$$

where ϕ_0 is a bounded holomorphic function in $S^0_\mu(\pi)$, c_0 is a constant depending on the subsequence $\{n_l\}$ chosen. At the same time, Φ can be extended holomorphically to $pS^0_\omega(\pi)$, and the different Φ's associated with different applicable sequences may differ from one another by constants dominated by $c\|b\|_\infty$.

Now we prove (ii) and (iii). We use the decomposition $b = b^+ + b^-$ given in (2.3). Define

$$b^{\pm,\alpha}(z) = \exp(\mp\alpha z)b^\pm(z),\ \alpha > 0.$$

Let $\phi^{\pm}$ and $\phi^{\pm,\alpha}$ be the functions associated with $b^{\pm}$ and $b^{\pm,\alpha}$, respectively. By Remark 1.3.1, $\phi^{\pm,\alpha}(\cdot) = \phi^{\pm}(\cdot \pm i\alpha)$, and the latter are inverse Fourier transforms of $b^{\pm,\alpha}$. Now we define the corresponding periodic functions $\Phi^{\pm,\alpha}$ and holomorphic functions $\Phi^{\pm}$ in $pC^0_{\omega,\pm}(\pi)$, respectively, which satisfy the size condition in the assertion (i). It is to be noted that for all $\Phi^{\pm,\alpha}$, we choose the same applicable sequence $\{n_l\}$ as we have chosen for $\Phi^{\pm}$. By the estimate in (i) of Theorem 2.1.1 and the fact that ϕ is holomorphic, we can prove that when $\alpha \to 0$, $\sum_1$ is locally uniformly and absolutely convergent. Let

$$\frac{1}{2\pi}\Phi^{\pm,\alpha}(z) = \phi^{\pm,\alpha}(z) + \phi_0^{\pm,\alpha}(z) + c_0^{\pm,\alpha}$$

and

$$\frac{1}{2\pi}\Phi^{\pm}(z) = \phi^{\pm}(z) + \phi_0^{\pm}(z) + c_0^{\pm},$$

where $\phi_0^{\pm,\alpha}$ and $\phi_0^{\pm}$ are holomorphic and uniformly bounded in $C^0_{\mu,\pm}(\pi)$. Since the convergence as $n_l \to \infty$ is uniform for $\alpha \to 0$, we can change the order of taking limits $n_l \to \infty$ and $\alpha \to 0$, and conclude that $\phi^{\pm,\alpha}$, $\phi_0^{\pm,\alpha}$ and $c_0^{\pm}$ are convergent locally uniformly in $C^0_{\omega,\pm}(\pi)$. Hence,

$$\lim_{\alpha\to 0}\Phi^{\pm,\alpha}(z) = \Phi^{\pm}(z).$$

Notice that for fixed α, $\Phi^{\pm,\alpha} \in L^{\infty}([-\pi,\pi])$, and when $n_l \to \infty$, the series which define $\Phi^{\pm,\alpha}$ converges uniformly in $x \in [-\pi,\pi]$. For all non-zero real ξ in the sense of (3) in Theorem 2.1.2, we have

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi}\exp(-i\xi x)\Phi^{\pm,\alpha}(x)dx &= \int_{-\pi}^{\pi}\exp(-i\xi x)\lim_{l\to\infty}\sum_{k=-n_l}^{n_l}\phi^{\pm,\alpha}(x+2k\pi)dx\\ &= \int_{-\infty}^{\infty}\exp(-i\xi x)\phi^{\pm,\alpha}(x)dx = b^{\pm,\alpha}(\xi).\end{aligned}$$

In particular, $\{b^{\pm,\alpha}(n)\}$, $n \neq 0$, are the standard Fourier coefficients of $\Phi^{\pm,\alpha}$. If F is any smooth periodic function on $[-\pi,\pi]$, then Parseval's identity holds:

$$2\pi\sum_{n=-\infty}^{\infty} b^{\pm,\alpha}(n)\widehat{F}(-n) = \int_{-\pi}^{\pi}\Phi^{\pm,\alpha}(x)F(x)dx,$$

where

$$b^{\pm,\alpha}(0) = (2\pi)^{-1}\int_{-\pi}^{\pi}\Phi^{\pm}(x \pm i\alpha)dx.$$

Let $\epsilon > 0$. Since $\widehat{F}(n)$ decays rapidly as $n \to \pm\infty$, on letting $\alpha \to 0+$, we have

$$2\pi \sum_{n=-\infty}^{\infty} b^{\pm}(n)\widehat{F}(-n) = \lim_{\alpha\to 0+} \left\{ \int_{[-\pi,\pi]\backslash(-\epsilon,\epsilon)} \Phi^{\pm}(x \pm i\alpha)F(x)dx \right. + \int_{|x|\leqslant\epsilon} \Phi^{\pm}(x \pm i\alpha)(F(x) - F(0))dx \left. + \int_{|x|\leqslant\epsilon} \Phi^{\pm}(x \pm i\alpha)F(0)dx \right\}.$$

Then we can get

$$\lim_{\alpha\to 0+} \int_{[-\pi,\pi]\backslash(-\epsilon,\epsilon)} \Phi^{\pm}(x + i\alpha)F(x)dx = \int_{[-\pi,\pi]\backslash(-\epsilon,\epsilon)} \Phi^{\pm}(x)F(x)dx$$

and

$$\limsup_{\alpha\to 0+} \int_{|x|\leqslant\epsilon} |\Phi^{\pm}(x + i\alpha)| \cdot |F(x) - F(0)|dx \leqslant \limsup_{\alpha\to 0} \int_{|x|\leqslant\epsilon} \frac{1}{|x|} \cdot |x|dx \leqslant C\epsilon.$$

Define

$$\Phi_1^{\pm}(z) = \int_{\delta^{\pm}(z)} \Phi^{\pm}(\eta)d\eta,$$

where $\delta^{\pm}(z)$ is a path from $-z$ to z in $C^0_{\omega,\pm}(\pi)$. Hence for $\Phi_1^{\pm}$, (ii) holds and

$$\lim_{\alpha\to 0+} \int_{|x|\leqslant\epsilon} \Phi^{\pm}(x \pm i\alpha)F(0)dx = \Phi_1^{\pm}(\epsilon)F(0).$$

This gives Parseval's identity associated with $b^{\pm}$:

$$2\pi \sum_{n=-\infty}^{\infty} b^{\pm}(n)\widehat{F}(-n) = \lim_{\epsilon\to 0} \left(\int_{[-\pi,\pi]\backslash(-\epsilon,\epsilon)} \Phi^{\pm}(x)F(x)dx + \Phi_1^{\pm}(\epsilon)F(0) \right),$$

where $b^{\pm}(0) = \frac{1}{2\pi}\Phi_1^{\pm}(\pi)$. Note that if we replace $\Phi^{\pm}$ by $\Phi^{\pm} + c^{\pm}$ in the above formulas, then, correspondingly, we need to replace $b^{\pm}(0)$ by $b^{\pm}(0) + c^{\pm}$ in order to make the formulas still hold. Since $\Phi = \Phi^+ + \Phi^-$, on letting $\Phi_1 = \Phi_1^+ + \Phi_1^-$, we see that (ii) and (iii) hold. This completes the proof. □

Remark 2.2.1 When we prove Parseval's identity related to $b \in H^\infty(S^0_\omega)$, the value of b at the origin is naturally involved. For the sake of convenience, we take $b(0) = \frac{1}{2\pi}\Phi_1(\pi)$ in consistency with the formula as shown in the theorem. The proof of the theorem indicates that adding a constant to Φ does not change the Fourier coefficients $\widehat{\Phi}(n) = b(n)$, $n \neq 0$, but we should add the same constant to $b(0)$.

Theorem 2.2.2 *Let* $\omega \in (0, \pi/2]$ *and* (Φ, Φ_1) *be a pair of holomorphic functions defined on* $pS^0_\omega(\pi)$ *and* $S^0_{\omega,+}(\pi)$, *respectively, satisfying*

(i) *Φ is 2π-periodic, and there exists a constant c_0 such that*

$$|\Phi(z)| \leqslant \frac{c_0}{|z|},\ z \in S^0_\omega(\pi);$$

(ii) *there exists a constant c_1 such that $\|\Phi_1\|_{H^\infty(S^0_{\omega,+}(\pi))} < c_1$, and*

$$\Phi_1'(z) = \Phi(z) + \Phi(-z),\ z \in S^0_{\omega,+}(\pi).$$

Then for any $\mu \in (0, \omega)$, there exists a function b^μ such that $b^\mu \in H^\infty(S^0_\mu)$ and

$$\|b^\mu\|_{H^\infty(S^0_\mu)} \leqslant C_\mu(c_0 + c_1).$$

By Theorem 2.2.1, the function pair determined by b^μ is identical with (Φ, Φ_1) (modulo constants). Moreover, $b^\mu = b^{\mu,+} + b^{\mu,-}$,

$$b^{\mu,\pm}(\eta) = \frac{1}{2\pi}\lim_{\epsilon\to 0}\left(\int_{A^\pm(\epsilon,\theta,|\eta|^{-1})} \exp(-i\eta z)\Phi(z)dz + \Phi_1(\epsilon)\right),\ \eta \in S^0_{\mu,\pm}, \quad (2.5)$$

where $\theta = (\mu+\omega)/2$, $A^\pm(\epsilon,\theta,\varrho) = l(\epsilon,\varrho) \cup c^\pm(\theta,\varrho) \cup \Lambda^\pm(\theta,\varrho)$. Here when $\varrho \leqslant \pi$,

$$\begin{aligned} l^\pm(\epsilon,\varrho) &= \Big\{z = x + iy:\ y = 0, \epsilon \leqslant \pm x \leqslant \varrho\Big\},\\ c^\pm(\theta,\varrho) &= \Big\{z = \varrho\exp(i\alpha):\ \alpha \textit{ from } \pi \pm \theta \textit{ to } \pi,\ \textit{and then from } 0 \textit{ to } \mp\theta\Big\},\\ \Lambda^\pm(\theta,\varrho) &= \Big\{z \in C^0_{\omega,\pm}(\pi): z = r\exp(i(\pi\pm\theta)), r \textit{ from } \pi\sec\theta \textit{ to } \varrho,\\ &\qquad \textit{and } z = r\exp(\mp i\theta), r \textit{ from } \varrho \textit{ to } \pi\sec\theta\Big\}, \end{aligned}$$

when $\varrho > \pi$,

$$l(\epsilon,\varrho) = l^\pm(\epsilon,\pi),\ c^\pm(\theta,\varrho) = c^\pm(\theta,\pi),\ \Lambda^\pm(\theta,\varrho) = \Lambda^\pm(\theta,\pi).$$

Proof The integral is along the path $A^\pm(\epsilon,\theta,|\eta|^{-1})$, see Figs. 2.4 and 2.5.

Fix $\mu \in (0,\omega)$ and write b^μ as b in the rest of the proof. For all $\epsilon \in (0,\pi)$ and $\eta \in S^0_\omega \cup \{0\}$, define $b_\epsilon(\eta) = b^+_\epsilon(\eta) + b^-_\epsilon(\eta)$, where $b^\pm_\epsilon$ are the functions in the definition of $b^\pm$ in the theorem before taking the limit as $\epsilon \to 0$. We see that for all ϵ, $b_\epsilon(0) = \frac{1}{2\pi}\Phi_1(\pi)$.

For $|\eta|^{-1} \leqslant \pi$, applying the estimate in Theorem 1.3.3, we can prove that $b_\epsilon(\eta)$ is uniformly bounded, and $\lim_{\epsilon\to 0+} b_\epsilon(\eta) = b(\eta)$ exists.

If $|\eta|^{-1} > \pi$, for the integral over the contour $l(\epsilon,\pi)$, we use the same argument as to the integral over $l(\epsilon,|\eta|^{-1})$ for the case $|\eta|^{-1} \leqslant \pi$. To estimate the integrals

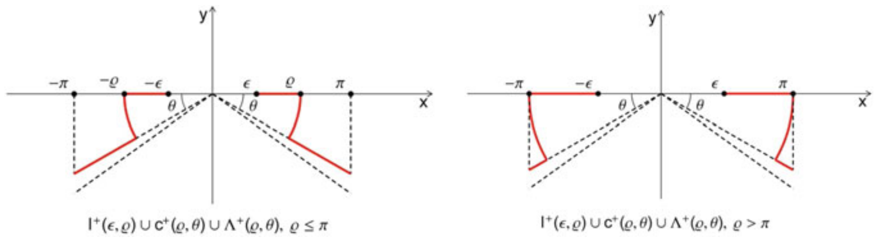

Fig. 2.4 $l^+(\epsilon, \varrho) \cup c^+(\varrho, \theta) \cup \Lambda^+(\varrho, \theta)$

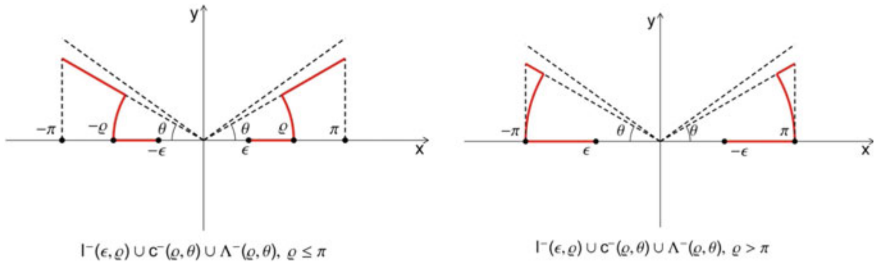

Fig. 2.5 $l^-(\epsilon, \varrho) \cup c^-(\varrho, \theta) \cup \Lambda^-(\varrho, \theta)$

over $c^\pm(\theta, \pi)$ and $\Lambda^\pm(\theta, \pi)$, we use Cauchy's theorem to change the contour of integration and so to integrate over the set

$$\Big\{z = x + iy : x = -\pi,\ y \text{ from } -(\pm\pi)\tan\theta \text{ to } 0,\ \text{ and } x = \pi,\ y \text{ from } 0 \text{ to } -(\pm\pi\tan\theta)\Big\}.$$

However, by the condition $\pm\mathrm{Re}(z) > 0$, it is easy to prove that the integral over the above contour is bounded. Then b is well-defined and bounded.

Let F be any 2π-periodic continuous function on $[-\pi, \pi]$. Expanding F in a Fourier series and using the definition of b_ϵ, we have

$$2\pi \sum_{n=-\infty}^{\infty} b_\epsilon(n)\widehat{F}_{[-\pi,\pi](-n)} = \int_{\epsilon<|x|\leqslant\pi} \Phi(x)F(x)dx + \Phi_1(\epsilon)F(0).$$

On letting $\epsilon \to 0$, we get

$$2\pi \sum_{n=-\infty}^{\infty} b(n)\widehat{F}_{[-\pi,\pi]}(-n) = \lim_{\epsilon\to 0}\left(\int_{\epsilon<|x|\leqslant\pi} \Phi(x)F(x)dx + \Phi_1(\epsilon)F(0)\right).$$

Denote by $(G(b), G_1(b))$ a pair of holomorphic functions associated to b in the pattern of Theorem 2.2.1. It can be deduced from Parseval's identity that

$$\lim_{\epsilon\to 0}\left(\int_{\epsilon<|x|<\pi}(G(b)(x)-\Phi(x))F(x)dx+[G_1(\epsilon)-\Phi_1(\epsilon)]F(0)\right)$$
$$=2\pi\Big(b_1(0)-b(0)\Big)\widehat{F}_{[-\pi,\pi]}(0),$$

where $b_1(0)$ is the function associated with $(G(b), G_1(b))$ in Parseval's identity of Theorem 2.2.1. By Theorem 2.2.1, we can add any constant to $G(b)$ and accordingly adjust the value of $b_1(0)$ such that (iii) of Theorem 2.2.1 holds. In particular, we can take a constant such that $b_1(0)-b(0)=0$. The right hand side of the last displayed equality then becomes 0. Using an approximation to identity $\{F_n\}$ with the property $F_n(0)=0$, we conclude that $G(b)(x)=\Phi(x)$ for $x\neq 0$. Because of analyticity, we know for all $z\in S^0_\omega(\pi)$, $G(b)(z)=\Phi(z)$. For $G_1(b)$, using (iii) of Theorem 2.2.1 together with the assumption (ii) of Φ_1, we can get $\Phi_1'=G_1'(b)$ and Φ_1-G_1 is a constant. Then by the use of

$$\lim_{\epsilon\to 0}\Big[G_1(b)(\epsilon)-\Phi_1(\epsilon)\Big]=0,$$

we have $\Phi_1=G_1(b)$. The uniqueness of b can be proved similarly. □

2.3 Singular Integrals on Starlike Lipschitz Curves

The results obtained in Sect. 2.2 can be applied to study the relation between the singular integral operators defined on periodic Lipschitz curves in Sect. 2.1 and the Fourier multipliers. Taking the change of variable $z\to\exp(iz)$ and substituting $\widetilde{\Phi}=\Phi\circ(\frac{1}{i}\ln)$ and $\widetilde{\Phi_1}=\Phi_1\circ(\frac{1}{i}\ln)$ in Theorems 2.2.1 and 2.2.2, we obtain the following theorem.

Theorem 2.3.1 *Let $\omega\in(0,\pi/2]$ and $b\in H^\infty(S^0_\omega)$. There exists a pair of functions $(\widetilde{\Phi},\widetilde{\Phi_1})$ such that $\widetilde{\Phi}$ and $\widetilde{\Phi_1}$ are holomorphic in $\exp(iS^0_\omega(\pi))$ and $\exp(iS^0_{\omega,+}(\pi))$, respectively. Moreover, for any $\mu\in(0,\omega)$,*

(i) $|\widetilde{\Phi}(z)|\leqslant C_{\omega,\mu}\|b\|_\infty/|1-z|,\ z\in\exp(iS^0_\mu(\pi))$;

(ii) $\widetilde{\Phi}_1\in H^\infty(\exp(iS^0_\mu(\pi))),\ \|\widetilde{\Phi}_1\|_{H^\infty(\exp(iS^0_\mu(\pi)))}<C_{\omega,\mu}\|b\|_\infty$ *and*
$$\widetilde{\Phi}'_1(z)=\frac{1}{iz}\Big(\widetilde{\Phi}(z)+\widetilde{\Phi}(z^{-1})\Big),\ z\in\exp(iS^0_{\omega,+}(\pi));$$

(iii) *For all continuous functions $\widetilde{F}$ defined on $\mathbb{T}$,*

$$2\pi\sum_{n=-\infty}^{\infty}b(n)\widehat{\widetilde{F}}_{\mathbb{T}}(-n)=\lim_{\epsilon\to 0}\left(\int_{|\ln z|>\epsilon,z\in\mathbb{T}}\widetilde{\Phi}(z)\widetilde{F}(z)\frac{dz}{z}+\widetilde{\Phi}_1(\exp(i\epsilon))\widetilde{F}(1)\right).$$

where $\widehat{\widetilde{F}}_{\mathbb{T}}(n)$ is nth Fourier coefficient of $\widetilde{F}$, and $b(0)=\frac{1}{2\pi}\widetilde{\Phi}_1(\exp(i\pi))$.

Theorem 2.3.2 *Let $\omega \in (0, \pi/2]$ and $(\widetilde{\Phi}, \widetilde{\Phi}_1)$ be a pair of functions defined on* $\exp(iS^0_\omega(\pi))$ *and* $\exp(iS^0_{\omega,+}(\pi))$*, respectively, satisfying*

(i) *there exists a constant c_0 such that*

$$|\widetilde{\Phi}(z)| \leqslant \frac{c_0}{|1-z|},\ z \in \exp(iS^0_\omega(\pi));$$

(ii) *there exists a constant c_1 such that $\|\widetilde{\Phi}_1\|_{H^\infty(\exp(iS^0_{\omega,+}(\pi)))} < c_1$, and*

$$\widetilde{\Phi}'_1(z) = \frac{1}{iz}\Big(\widetilde{\Phi}(z) + \widetilde{\Phi}(z^{-1})\Big),\ z \in \exp(iS^0_{\omega,+}(\pi)).$$

Then for any $\mu \in (0, \omega)$, there exists a function b^μ in $H^\infty(S^0_\mu)$ such that

$$\|b^\mu\|_{H^\infty(S^0_\mu)} \leqslant C_\mu(c_0 + c_1).$$

The function pair determined by b^μ according to Theorem 2.3.1 equals to $(\widetilde{\Phi}, \widetilde{\Phi}_1)$ (modulo constants). Moreover, $b^\mu = b^{\mu,+} + b^{\mu,-}$,

$$b^\pm(\eta) = \frac{1}{2\pi}\lim_{\epsilon\to 0}\left(\int_{-i\ln z\in A^\pm(\epsilon,\theta,\varrho)} z^{-\eta}\frac{dz}{z} + \widetilde{\Phi}_1(\exp(i\epsilon))\right),\ \eta \in S^0_{\mu,\pm},$$

where $A^\pm(\epsilon, \theta, \varrho)$ is the path defined in Theorem 2.2.2, and

$$\widetilde{\Phi}_1(\exp(i\epsilon)) = \int_{l(\epsilon)} \widetilde{\Phi}(\exp(iz))dz,$$

where $l(\epsilon)$ is any path from $-\epsilon$ to ϵ lying in $C^0_{\omega,\pm}$.

The following corollaries are in terms of holomorphic extension of series with positive and negative powers and can be deduced from Theorems 2.3.1 and 2.3.2 immediately.

Corollary 2.3.1 *Let $\{b_n\}_{n=\pm1}^{\pm\infty} \in l^\infty$, $\widetilde{\Phi}(z) = \sum_{n=\pm1}^{\pm\infty} b_n z^n$, $|z^{\pm1}| < 1$, and $\omega \in (0, \pi/2)$. If there exists $\delta > 0$ such that $\omega + \delta \leqslant \pi/2$, and there exists a function $b \in H^\infty(S^0_{\omega+\delta,\pm})$ such that for all $\pm n = \pm1, \pm2, \ldots,$ $b(n) = b_n$, then the function $\widetilde{\Phi}$ can be extended holomorphically to the domain* $\exp(iC^0_{\omega+\delta,\pm}(\pi))$*. Moreover, we get*

$$|\widetilde{\Phi}(z)| \leqslant \frac{C_{\omega,\delta}}{|1-z|},\ z \in \exp(iC^0_{\omega,\pm}(\pi)).$$

Corollary 2.3.2 *Let $\omega \in (0, \pi/2)$ and let $\widetilde{\Phi}$ be a holomorphic function satisfying*

$$|\widetilde{\Phi}(z)| \leqslant \frac{C}{|1-z|},\ z \in \exp(iC^0_{\omega,\pm}(\pi)).$$

Then for any $\mu \in (0, \omega)$*, there exists a function* b^{μ} *such that* $b^{\mu} \in H^{\infty}(S^0_{\mu,\pm})$ *and* $\widetilde{\Phi}(z) = \sum_{n=\pm 1}^{\pm\infty} b_n z^n$*. Moreover,* $b^{\mu} = b^{\mu,+} + b^{\mu,-}$*, and for* $\eta \in S^0_{\mu,\pm}$*,*

$$b^{\mu,\pm}(\eta) = \frac{1}{2\pi} \lim_{\epsilon \to 0+} \left(\int_{-i \ln z \in A^{\pm}(\epsilon,\theta,\varrho)} \exp(-i\eta z)\widetilde{\Phi}(\exp(iz))dz + \widetilde{\Phi}_1(\exp(i\epsilon)) \right),$$

where $A^{\pm}(\epsilon, \theta, \varrho)$ *is defined by Theorem 2.2.2, and*

$$\widetilde{\Phi}_1(\exp(i\epsilon)) = \int_{l(\epsilon)} \widetilde{\Phi}(\exp(iz))dz,$$

where $l(\epsilon)$ *is any path from* $-\epsilon$ *to* ϵ *in* $C^0_{\omega,\pm}$*.*

Remark 2.3.1 As indicated in Corollary 2.3.2, the mapping $\widetilde{\Phi} \to b$ satisfying $\widetilde{\Phi}(z) = \sum b(n)z^n$ is not single-valued. In fact, if $\mu_1 \neq \mu_2$, then both b^{μ_1} and b^{μ_2} satisfy the requirement. In general, $b^{\mu_1} \neq b^{\mu_2}$. This can be verified by using $\widetilde{\Phi}(z) = z^n, n \in \mathbb{Z}^+$.

Corollary 2.3.3 *For any* $\omega \in (0, \pi/2)$*, there does not exist any function b such that* $b \in H^{\infty}(S^0_{\omega,+})$ *and satisfies* $b(n) = 1$ *for* $n = 2^k, k = 1, 2, \ldots,$ *and* $b(n) = 0$ *for the other positive integers.*

Proof Consider the function

$$\widetilde{\Phi}(z) = z + z^2 + z^{2^2} + \cdots + z^{2^k} + \cdots$$

It is well known that $\widetilde{\Phi}$ does not have any holomorphic extension across any interval on the unit circle, and according to Corollary 2.3.1, it is not induced by a function b in $H^{\infty}(S^0_{\omega,+})$. □

For the functions b and $\widetilde{F}$ defined in Theorem 2.3.1, by the Laurent series theory, the series

$$\sum_{n=-\infty}^{\infty} b(n)\widehat{\widetilde{F}}_{\mathbb{T}}(n)z^n$$

locally uniformly converges to a holomorphic function in the annulus on which $\widetilde{F}$ is defined. Noticing that $\widehat{\widetilde{F}}_{\mathbb{T}}(n) = \widehat{\widetilde{F}}_{\widetilde{\gamma}}(n)$, we can define an operator $\widetilde{M}_b : \mathcal{A}(\widetilde{\Gamma}) \longrightarrow \mathcal{A}(\widetilde{\gamma})$ as

$$\widetilde{M}_b(\widetilde{F})(z) = 2\pi \sum_{n=-\infty}^{\infty} b(n)\widehat{\widetilde{F}}_{\widetilde{\gamma}}(n)z^n.$$

On the other hand, for the function pair $(\widetilde{\Phi}, \widetilde{\Phi}_1)$ occurring in Theorem 2.3.2, there holds

$$T_{(\widetilde{\Phi},\widetilde{\Phi}_1)}\widetilde{F}(z) = \lim_{\epsilon\to 0}\left(\int_{\pi\geqslant|Re(i^{-1}\ln(\eta z^{-1}))|>\epsilon,\eta\in\widetilde{\gamma}} \widetilde{\Phi}(z\eta^{-1})\widetilde{F}(\eta)\frac{d\eta}{\eta} + \widetilde{\Phi}_1(\exp(i\epsilon t(z)))\widetilde{F}(z)\right),$$

where $t(z)$ is the unit tangent vector of γ at z in $S^0_{\omega,+}(\pi)$. We have the following theorem:

Theorem 2.3.3 *Let* $\omega \in (\arctan N,\ \pi/2]$, $b \in H^\infty(S^0_\omega)$ *and let* $(\widetilde{\Phi},\ \widetilde{\Phi}_1)$ *be the pair of functions corresponding to b in the pattern of Theorem 2.3.1. Then the following conclusions hold.*

(i) $T_{(\widetilde{\Phi},\widetilde{\Phi}_1)}$ *is a well-defined operator from* $\mathcal{A}(\widetilde{\gamma})$ *to* $\mathcal{A}(\widetilde{\gamma})$, *and in the sense of modulo constants,*

$$T_{(\widetilde{\Phi},\widetilde{\Phi}_1)} = \widetilde{M}_b.$$

(ii) $\widetilde{M}_b$ *can be extended to a bounded operator on* $L^2(\widetilde{\gamma})$, *and the norm is dominated by* $c\|b\|_\infty$.

Proof (i) For any $\alpha > 0$, define $b_z^{\pm,\alpha}(\xi) = -z^{-\xi}b^{\pm,\alpha}(-\xi)$, where $b^{\pm,\alpha}$ is the function defined in Theorem 2.2.1. Let $(\widetilde{\Phi}_z^{\pm,\alpha}, (\widetilde{\Phi}_z^{\pm,\alpha})_1)$ be the pair of functions corresponding to b in the pattern of Theorem 2.3.1. By (iii) of Theorem 2.3.1 and Cauchy's theorem, we have

$$\begin{aligned}\widetilde{M}_{b_z^{\pm,\alpha}}\widetilde{F}(z) &= 2\pi\sum_{n=-\infty}^{\infty} b_z^{\pm,\alpha}(n)\widehat{\widetilde{F}}_{\widetilde{\gamma}}(n)\\ &= 2\pi\sum_{n=-\infty}^{\infty} b_z^{\pm,\alpha}(n)\widehat{\widetilde{F}}_{\mathbb{T}}(n)\\ &= \int_{\mathbb{T}}\widetilde{\Phi}_z^{\pm,\alpha}(\eta^{-1})\widetilde{F}(\eta)\frac{d\eta}{\eta}\\ &= \int_{\widetilde{\gamma}}\widetilde{\Phi}_z^{\pm,\alpha}(\eta^{-1})\widetilde{F}(\eta)\frac{d\eta}{\eta}.\end{aligned}$$

Similar to the proof of Theorem 2.2.1, taking the limit $\alpha \to 0$ and noticing that

$$\widetilde{\Phi}_z^{\pm}(\eta^{-1}) = \widetilde{\Phi}^{\pm}(z\eta^{-1}),$$

we can get the desired equality for $b^\pm$ and b.

(ii) Now we prove the boundedness of the following operator:

$$T_{(\Phi,\Phi_1)}F(z)= \lim_{\epsilon\to 0}\left\{\int_{\epsilon<|\mathrm{Re}(z-\eta)|\leqslant\pi}\Phi(z-\eta)F(\eta)d\eta + \Phi_1(\epsilon t(z))F(z)\right\},\ F\in\mathcal{A}(\gamma),$$

where $t(z)$ is the unit tangent vector of γ at z in $S^0_{\omega,+}(\pi)$. Here $\mathcal{A}(\gamma)$ denotes the class of all 2π-periodic holomorphic functions satisfying: $F \in \mathcal{A}(\gamma)$ if and only if $\widetilde{F} = F \circ (i^{-1}\ln) \in \mathcal{A}(\widetilde{\gamma})$. By the decomposition of (i) of Theorem 2.2.1, we have

$$
\begin{aligned}
T_{(\Phi,\Phi_1)}F(z) = \lim_{\epsilon_n\to 0}\Big\{ &\int_{\pi\geqslant|\mathrm{Re}(z-\eta)|>\epsilon_n} \phi(z-\eta)F(\eta)d\eta \\
&+ \int_{\pi\geqslant|Re(z-\eta)|>\epsilon_n} \phi_0(z-\eta)F(\eta)d\eta\Big\} \\
&+ c_1\int_{-\pi}^{\pi} F(\eta)d\eta + c_2F(z),
\end{aligned}
$$

where $\epsilon_n \to 0$ is a subsequence of the sequence $\epsilon \to 0$, c_1 and c_2 are constants.

The second and the third integrals are dominated by the L^2-norm of F, while the first integral is dominated by

$$
\sup_{\epsilon>0}\left|\int_{|\mathrm{Re}(z-\eta)|>\epsilon} \phi(z-\eta)F_1(\eta)d\eta\right| + c\mathcal{M}F_1(z),\ \mathrm{Re}(z)\in[-\pi,\pi],
$$

where for $|\mathrm{Re}(\eta)| \leqslant 2\pi$, $F_1(\eta) = F(\eta)$; otherwise $F_1(\eta) = 0$. $\mathcal{M}F_1$ is the Hardy-Littlewood maximal function of F_1 on the curve . By the boundedness of the operators introduced by (ϕ,ϕ_1) and that of $\mathcal{M}$, we obtain the desired boundedness. □

Theorem 2.3.4 *Let ϕ be a holomorphic function satisfying $|\phi(z)| \leq C/|z|$ on S^0_ω. Assume that $\gamma = x + iA(x)$ is a Lipschitz curve, $\|A'\|_\infty < \tan\omega$. If there exists a $L^2(\gamma)$-bounded operator T such that*

$$
T(f)(z) = \int_\gamma \phi(z-\zeta)f(\zeta)d\zeta,\ \forall f\in C_c(\gamma), z\notin \ supp\ f,
$$

where $C_c(\gamma)$ denotes the class of continuous functions with compact support on γ, then there exists a function $\phi_1 \in H^\infty(S^0_\omega)$ such that $\phi_1' = \phi(z) + \phi(-z)$, $z \in S^0_\omega$.

Proof Because T is bounded on $L^2(\gamma)$, the formula for T can be extended to

$$
T(f)(z) = \int_\gamma \phi(z-\zeta)f(\zeta)d\zeta,
$$

where $f = \chi_Q$, Q is any finite interval on γ and $z \notin \overline{Q}$. Define a new family of functions $\phi_\epsilon = \phi\chi_{\{z\in\mathbb{C}:\ |z|>\epsilon\}}$ and the corresponding operators:

$$
T_\epsilon(f)(z) = \int_\gamma \phi_\epsilon(z-\zeta)f(\zeta)d\zeta.
$$

By a standard argument, we can get the operator norm $\|T_\epsilon\|_{L^2(\gamma)\to L^2(\gamma)}$ is uniformly bounded. This implies that for any interval Q on γ and any ϵ, we have uniformly:

$$
\int_Q |T_\epsilon\chi_Q||d\zeta| < c|Q|. \tag{2.6}
$$

For $z \in \gamma$, denote by γ_z the curve $\gamma - z$. Because γ is a Lipschitz curve, then γ_z is also a Lipschitz curve passing the origin. We also write $Q_{z,\eta} = \{\zeta \in \gamma_z : |\zeta| < \eta\}$. Fix $z_0 \in \gamma$. For $z_1 \in Q_{z_0,\eta/2}$, we will prove the following estimate:

$$\left|T_\epsilon(\chi_{Q_{z_0,\eta}})(z_1) - \int_{\zeta\in\gamma_0,\epsilon<|\zeta|<\eta} \phi(\zeta)d\zeta\right| \leq C < \infty, \tag{2.7}$$

where C is a constant independent of ϵ, η and $z_1 \in Q_{z_0,\eta/2}$.

In fact, denote by $\gamma^{\pm}(z_0, \eta)$ the right and the left endpoints of $Q_{z_0,\eta}$. Define

$$S_1 =: \{\zeta \in \gamma_{z_1}, \ \text{ from } z_1 + \gamma^+(z_0, \eta) \text{ to } z_1 + \gamma^-(z_0, \eta), \ |\zeta| > \epsilon\}$$

and

$$S_2 =: \{\zeta \in \gamma_0, \ \text{ from } \gamma^-(z_0, \eta) \text{ to } \gamma^+(z_0, \eta), |\zeta| > \epsilon\}.$$

We have

$$\begin{aligned} &T_\epsilon \chi_{Q_{z_0,\eta}}(z_1) - \int_{\zeta\in\gamma_0,\epsilon<|\zeta|<\eta} \phi(\zeta)d\zeta \\ &= \int_{S_1} \phi(\zeta)d\zeta + \int_{S_2} \phi(\zeta)d\zeta. \end{aligned}$$

Using the Cauchy theorem, we can reduce the above integrals to the integrals along circles of radius η and ϵ and along the directions of radius within $\{z \in \mathbb{C} : \eta \leqslant |z| \leqslant 3\eta/2\}$. Then from the condition $|\phi(z)| \leq C/|z|$, we can conclude (2.7).

From (2.7), we have

$$\left|\int_{\zeta\in\gamma_{z_0},\epsilon<|\zeta|<\eta} \phi(\zeta)d\zeta\right| \leqslant C + |T\chi_{Q_{z_0,\eta}}|.$$

Taking average to both sides of this inequality w.r.t. $z_1 \in Q_{z_0,\eta/2}$ and using (2.6), we obtain for any $0 < \epsilon < \eta < \infty$,

$$\left|\int_{\zeta\in\gamma_{z_0},\epsilon<|\zeta|<\eta} \phi(\zeta)d\zeta\right| \leqslant C. \tag{2.8}$$

From the Cauchy theorem, the condition $|\phi(z)| \leqslant C/|z|$ and the inequality (2.8), we have

$$\left|\left(\int_{l^-(z_1^-,z_2^-)} + \int_{l^+(z_1^+,z_2^+)}\right)\phi(\zeta)d\zeta\right| \leqslant C,$$

where $z_1^{\pm}, z_2^{\pm} \in S^0_{\omega,\pm}$, and $l^-(z_1^-, z_2^-)$ is a contour lying in $S^0_{\omega,-}$ from z_1^- to z_2^-, $l^+(z_1^+, z_2^+)$ is a contour lying in $S^0_{\omega,+}$ from z_2^+ to z_1^+, and $|z_1^-| = |z_1^+|$, $|z_2^-| = |z_2^+|$. For $z \in S^0_{\omega,\pm}$, we let

$$\phi_1(z) = \frac{1}{2}\Big(\int_{l^-(-1,\mp z)} \phi(\zeta)d\zeta + \int_{l^+(1,\pm z)} \phi(\zeta)d\zeta \Big).$$

Now it is easy to check that $\phi_1 \in H^\infty(S^0_\omega)$ and for $z \in S^0_\omega$,

$$\phi'(z) = \frac{1}{2}\Big(\phi(z) + \phi(-z)\Big).$$

□

We state the following theorem.

Theorem 2.3.5 *Assume that* $\omega \in (\arctan N,\ \pi/2]$. *Let* $\widetilde{\Phi}$ *be holomorphic in* $\exp(iS^0_\omega(\pi))$ *and satisfy (i) of Theorem 2.3.2 with respect to* ω. *If* T *is a bounded operator on* $L^2(\widetilde{\gamma})$ *and for all* $\widetilde{F}$ *belonging to the class of continuous functions* $C_0(\widetilde{\gamma})$,

$$T(\widetilde{F})(z) = \int_{\widetilde{\gamma}} \widetilde{\Phi}(z\xi^{-1})\widetilde{F}(\xi)\frac{d\xi}{\xi},\ z \notin\ supp\ (\widetilde{F}),$$

then there exists a unique function $\widetilde{\Phi}_1 \in H^\infty(\exp(iS^0_{\mu,+}))$, $\mu \in (0,\omega)$ *such that for* $\widetilde{F} \in C_0(\widetilde{\gamma})$,

$$\widetilde{\Phi}'_1(z) = \frac{1}{iz}\Big(\widetilde{\Phi}(z) + \widetilde{\Phi}(z^{-1})\Big),\ z \in \exp(iS^0_{\omega,+}(\pi))$$

and

$$T(\widetilde{F}) = T_{(\widetilde{\Phi},\widetilde{\Phi}_1)}(\widetilde{F}).$$

Proof On $S^0_{\omega,+}(\pi)$, we define the function ϕ as

$$\phi(\eta) =: \widetilde{\Phi}(e^{i\eta}).$$

Then, on the one hand, we can get for $\eta \in S^0_{\omega,+}(\pi)$,

$$|\phi(\eta)| \leqslant |\widetilde{\Phi}(e^{i\eta})| \leqslant \frac{C}{|1-e^{i\eta}|} \leqslant \frac{C}{|\eta|}.$$

On the other hand, let $f(z) = \widetilde{F}(e^{iz})$. For $z \notin$ supp $(\widetilde{F})$ and $\xi \in \widetilde{\gamma}$, without loss of generality, we can write $z = e^{i\eta}$ and $\xi = e^{iw}$, where $z \notin$ supp f and $w \in \gamma$. If the operator

$$T(\widetilde{F})(z) = \int_{\widetilde{\gamma}} \widetilde{\Phi}(z\xi^{-1})\widetilde{F}(\xi)\frac{d\xi}{\xi},\ z \notin \text{ supp } (\widetilde{F}),$$

is bounded on $L^2(\widetilde{\gamma})$, then by change of variables, we can see that

$$T(\widetilde{F})(z)\ =\ \int_\gamma \widetilde{\Phi}(e^{i(\eta-w)})\widetilde{F}(e^{iw})\frac{de^{iw}}{e^{iw}}$$

$$= i \int_\gamma \phi(\eta - w) f(w) dw$$
$$=: T_\phi(f),$$

which implies that the operator T_ϕ is also bounded on $L^2(\gamma)$. By Theorem 2.3.4, there exists a function $\phi_1 \in H^\infty(S^0_\omega)$ such that $\phi_1'(\eta) = \phi(\eta) + \phi(-\eta)$, $\eta \in S^0_\omega$. For $z = e^{i\eta} \in \exp(iS^0_{\omega,+}(\pi))$ with $\eta \in S^0_{\omega,+}(\pi)$, define

$$\widetilde{\Phi}_1(z) = \widetilde{\Phi}_1(e^{i\eta}) =: \phi_1(\eta).$$

Then

$$\begin{aligned}
\widetilde{\Phi}_1'(z) &= \frac{d\eta}{dz}\frac{d}{d\eta}(\widetilde{\Phi}_1(e^{i\eta}))\\
&= \frac{1}{ie^{i\eta}}\frac{d}{d\eta}(\phi_1(\eta))\\
&= \frac{1}{ie^{i\eta}}[\phi(\eta) + \phi(-\eta)]\\
&= \frac{1}{ie^{i\eta}}\Big[\widetilde{\Phi}(e^{i\eta}) + \widetilde{\Phi}(e^{-i\eta})\Big]\\
&= \frac{1}{iz}\Big[\widetilde{\Phi}(z) + \widetilde{\Phi}(z^{-1})\Big].
\end{aligned}$$

This completes the proof of Theorem 2.3.5. □

2.4 Holomorphic H^∞-Functional Calculus on Starlike Lipschitz Curves

The purpose of this section is to clarify that the theory of holomorphic H^∞-functional calculus on infinite Lipschitz curves established by A. McIntosh in [1] can also be established in the case of closed curves. Precisely, we study the relations between the operator classes $\widetilde{M}_b$, $T_{(\widetilde{\Phi},\widetilde{\Phi}_1)}$ and the holomorphic H^∞-functional calculus, see also [2, 3] for further information.

For the functions $\widetilde{F} \in \mathcal{A}(\widetilde{\gamma})$, we define the differential operator $\frac{d}{dz}\mid_{\widetilde{\gamma}}$ as

$$\frac{d}{dz}\mid_{\widetilde{\gamma}} \widetilde{F}(z) = \lim_{h\to 0,\ z+h\in\widetilde{\gamma}} \frac{\widetilde{F}(z+h) - \widetilde{F}(z)}{h}, z \in \widetilde{\gamma}.$$

For $1 < p < \infty$, $\langle L^p(\widetilde{\gamma}),\ L^{p'}(\widetilde{\gamma})\rangle$ is the dual of Banach spaces defined as follows:

$$\langle \widetilde{F},\ \widetilde{G}\rangle = \int_{\widetilde{\gamma}} \widetilde{F}(z)\widetilde{G}(z)dz,$$

where $p' = (1 - p^{-1})^{-1}$. Now by duality, we define $D_{\widetilde{\gamma},p}$ as the closed operator with the largest domain in $L^p(\widetilde{\gamma})$ which satisfies

$$\left\langle D_{\widetilde{\gamma},p}\widetilde{F},\ \widetilde{G}\right\rangle = \left\langle \widetilde{F},\ -z\frac{d}{dz}\mid_{\widetilde{\gamma}} \widetilde{G}\right\rangle$$

for all $\widetilde{F}$ and $\widetilde{G}$ in $\mathcal{A}(\widetilde{\gamma})$.

Let $\omega \in (\arctan N,\ \pi/2]$ and $\lambda \notin S^0_\omega$. It is easy to prove $D_{\widetilde{\gamma},p}$ is the surface Dirac operator on $\widetilde{\gamma}$ and $\frac{1}{2\pi}\widetilde{\Phi}_\lambda$ is the functions defined below.

Let $\lambda \notin S^0_\omega$. Then on any starlike Lipschitz curve , $b(z) = \frac{1}{z-\lambda}$ corresponds to the resolvent of the surface Dirac operator. If $\mathrm{Im}(\lambda) > 0$, by (1.1) and (1.2), we have

$$\phi_\lambda(z) = \begin{cases} i\exp(i\lambda z), & \mathrm{Re}(z) > 0, \\ 0, & \mathrm{Re}(z) < 0. \end{cases}$$

If $\mathrm{Im}(\lambda) < 0$, then we have

$$\phi_\lambda(z) = \begin{cases} 0, & \mathrm{Re}(z) > 0, \\ i\exp(i\lambda z), & \mathrm{Re}(z) < 0. \end{cases}$$

It is easy to prove that for every case, ϕ_λ belongs to $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Hence for the two cases, we can use the remark made after Theorem 2.1.2.

For $\mathrm{Im}(\lambda) > 0$, we can deduce from the definition that

$$\Phi_\lambda(z) = \begin{cases} \frac{i\exp(i\lambda(z+2\pi))}{1-\exp(i\lambda 2\pi)}, & \text{if } -\pi < \mathrm{Re}(z) < 0, \\ \frac{i\exp(i\lambda z)}{1-\exp(i\lambda 2\pi)}, & \text{if } 0 < \mathrm{Re}(z) < \pi. \end{cases}$$

For $\mathrm{Im}(\lambda) < 0$,

$$\Phi_\lambda(z) = \begin{cases} \frac{-i\exp(i\lambda(z-2\pi))}{1-\exp(-i\lambda 2\pi)}, & \text{if } 0 < \mathrm{Re}(z) < \pi, \\ \frac{-i\exp(i\lambda z)}{1-\exp(-i\lambda 2\pi)}, & \text{if } -\pi < \mathrm{Re}(z) < 0. \end{cases}$$

For $\mathrm{Im}(\lambda) > 0$,

$$\widetilde{\Phi}_\lambda(z) = \begin{cases} \frac{i\exp(i\lambda 2\pi)z^\lambda}{1-\exp(i\lambda 2\pi)}, & \text{if } -\pi < \mathrm{Re}(\frac{\ln z}{i}) < 0, \\ \frac{iz^\lambda}{1-\exp(i\lambda 2\pi)}, & \text{if } 0 < \mathrm{Re}(\frac{\ln z}{i}) < \pi. \end{cases}$$

For $\mathrm{Im}(\lambda) < 0$,

$$\widetilde{\Phi}_\lambda(z) = \begin{cases} \frac{-i\exp(-i\lambda 2\pi)z^\lambda}{1-\exp(-i\lambda 2\pi)}, & \text{if } 0 < \mathrm{Re}(\frac{\ln z}{i}) < \pi, \\ \frac{-iz^\lambda}{1-\exp(-i\lambda 2\pi)}, & \text{if } -\pi < \mathrm{Re}(\frac{\ln z}{i}) < 0. \end{cases}$$

We can verify that $D_{\widetilde{\gamma},p}$ is the surface Dirac operator on $\widetilde{\gamma}$, and in the sense of Theorem 2.3.3, the function $\frac{1}{2\pi}\widetilde{\Phi}_\lambda$ is the convolution kernel of the resolvent operator $(D_{\widetilde{\gamma},p}-\lambda)^{-1}$. Moreover,

$$\|(D_{\widetilde{\gamma},p}-\lambda)^{-1}\| \leqslant \|\frac{1}{2\pi}\widetilde{\Phi}_\lambda\| \leqslant \sum_{n=-\infty}^{\infty}\|\phi_\lambda(\cdot+2\pi n)\|_{L^1(\gamma)}$$
$$= \|\phi_\lambda\|_{L^1(p\gamma)} \leqslant \sqrt{1+N^2}\{\mathrm{dist}(\lambda, S^0_\omega)\}^{-1}.$$

The above estimate implies that $D_{\widetilde{\gamma},p}$ is a type ω operator. For the H^∞ functions b with good decay properties at both 0 and ∞, we can define $b(D_{\widetilde{\gamma},p})$ via spectral integrals as follows:

$$b(D_{\widetilde{\gamma},p}) = \frac{1}{2\pi}\int_\delta b(\eta)(D_{\widetilde{\gamma},p}-\eta I)^{-1}d\eta.$$

Here δ is a path consisting of four rays:

$$\begin{cases} \{s\exp(-i\theta):\ s \text{ from } \infty \text{ to } 0\}; \\ \{s\exp(i\theta):\ s \text{ from } 0 \text{ to } \infty\}; \\ \{s\exp(-i(\pi-\theta)):\ s \text{ from } \infty \text{ to } 0\}; \\ \{s\exp(i(\pi+\theta)):\ s \text{ from } 0 \text{ to } \infty\}, \end{cases}$$

where $\arctan N < \delta < \omega$.

By the above estimates, it is easy to prove that any $b(D_{\widetilde{\gamma},p})$ is a bounded operator, and

$$b(D_{\widetilde{\gamma},p}) = \widetilde{M}_b = \widetilde{T}(\widetilde{\Phi}, 0).$$

Taking limits of the sequences of Calderón-Zygmund operators, we can extend the definition of $b(D_{\widetilde{\gamma},p})$ to all functions in $H^\infty(S^0_\omega)$, and prove that

$$b(D_{\widetilde{\gamma},p}) = \widetilde{M}_b = \widetilde{T}(\widetilde{\Phi}, \widetilde{\Phi}_1).$$

Alternative proofs of the boundedness of the operators can be found in [2] by G. Gaudry, T. Qian and S. Wang. In addition, when $b_1, b_2 \in H^\infty(S^0_\omega)$ and $\alpha_1,\ \alpha_2$ are complex numbers,

$$\|b(D_{\widetilde{\gamma},p})\| \leqslant C_\omega\|b\|_\infty,$$

$$(b_1b_2)(D_{\widetilde{\gamma},p}) = b_1(D_{\widetilde{\gamma},p})b_2(D_{\widetilde{\gamma},p})$$

and

$$(\alpha_1b_1+\alpha_2b_2)(D_{\widetilde{\gamma},p}) = \alpha_1b_1(D_{\widetilde{\gamma},p}) + \alpha_2b_2(D_{\widetilde{\gamma},p}).$$

Below we shall not restrict ourselves to H^∞-multipliers. It should be pointed out that all the results and methods of the Fourier multiplier theory for infinite Lipschitz curves can be adapted to the present case. The main difference is that the function class $\mathcal{A}(\widetilde{\gamma})$ has even better properties. When we deal with the kernels on γ, we refer to its corresponding kernel on $p\gamma$ via the Poisson summation formula. The following theorem can be proved via the corresponding Schur lemma, and we omit the proofs.

For $b = \{b_n\}_{n=-\infty}^{\infty} \in l^\infty$, define

$$\|b\|_{M_p(\widetilde{\gamma})} = \sup\left\{ \left\| \sum b_n \widehat{\widetilde{F}}(n) z^n \right\|_{L^p(\widetilde{\gamma})} : \ \|\widetilde{F}\|_{L^p(\widetilde{\gamma})} \leqslant 1 \right\},$$

and

$$M_p(\widetilde{\gamma}) = \left\{ b : \ \|b\|_{M_p(\widetilde{\gamma})} < \infty \right\}.$$

We call the functions b in $M_p(\widetilde{\gamma})$ the $L^p(\widetilde{\gamma})$−Fourier multipliers.

Theorem 2.4.1 *Let $\widetilde{\Phi}$ be a holomorphic function defined on a simple connected open neighborhood of the set*

$$\widetilde{\gamma} - \widetilde{\gamma} = \left\{ z - \xi : \ z, \xi \in \widetilde{\gamma} \right\}$$

satisfying $|\widetilde{\Phi}(r\exp(i\theta))| \leqslant \psi(\exp(i\theta))$, where $\int_{-\pi}^{\pi} \psi(\exp(i\theta))d\theta < \infty$. Then

$$b = (\widehat{\widetilde{\Phi}}(n))_{n=-\infty}^{\infty} \in M_p(\widetilde{\gamma}), \ 1 < p < \infty,$$

and the corresponding convolution operator $T_{\widetilde{\Phi}}$ can be represented as

$$T_{\widetilde{\Phi}} \widetilde{F}(z) = \int_{\widetilde{\gamma}} \widetilde{\Phi}(z\eta^{-1}) \widetilde{F}(\eta) \frac{d\eta}{\eta}, \ \widetilde{F} \in \mathcal{A}(\widetilde{\gamma}).$$

Let $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ be two curves of the type under consideration. Define

$$M_p(\widetilde{\gamma}_1, \widetilde{\gamma}_2) = \left\{ b \in l^\infty : \ \|b\|_{M_p(\widetilde{\gamma}_1, \widetilde{\gamma}_2)} < \infty \right\},$$

where

$$\|b\|_{M_p(\widetilde{\gamma}_1, \widetilde{\gamma}_2)} = \sup\left\{ \frac{\| \sum b_n \widehat{\widetilde{F}}(n) z^n \|_{L^p(\widetilde{\gamma}_2)}}{\|\widetilde{F}\|_{L^p(\widetilde{\gamma}_1)}} : \ \widetilde{F} \in \mathcal{A}(\widetilde{\gamma}_1) \cap \mathcal{A}(\widetilde{\gamma}_2) \right\}.$$

If $\widetilde{\gamma}_3$ is the third such curve, and $b_1 \in M_p(\widetilde{\gamma}_1, \widetilde{\gamma}_2)$, $b_2 \in M_p(\widetilde{\gamma}_2, \widetilde{\gamma}_3)$, then $b_2 b_1 \in M_p(\widetilde{\gamma}_1, \widetilde{\gamma}_3)$, and

$$\|b_2 b_1\|_{M_p(\widetilde{\gamma}_1, \widetilde{\gamma}_3)} \leqslant \|b_2\|_{M_p(\widetilde{\gamma}_2, \widetilde{\gamma}_3)} \|b_1\|_{M_p(\widetilde{\gamma}_1, \widetilde{\gamma}_2)}.$$

Theorem 2.4.2 *Let* $b \in l^\infty$ *and* $f_\beta(n) = b(n)\exp(2\beta|n|)$. *If for some* $\beta > M = \max A(x)$, $f_\beta \in M_p(\mathbb{T})$, *where* $\mathbb{T}$ *is the unit circle and* $1 < p < \infty$, *then* $b \in M_p(\widetilde{\gamma})$ *and*

$$\|b\|_{M_p(\widetilde{\gamma})} \leqslant (2\pi\beta)^2(\beta^2 - M^2)^{-1}(1 + N^2)^{1/2}\|f_\beta\|_{M_p(\mathbb{T})}.$$

For flat curves γ, it is obvious that $\|b\|_{M_2(\widetilde{\gamma})} \leqslant C_{\widetilde{\gamma}}\|b\|_\infty$. But the following example indicates that in general case, this fact may not hold.

Take $\gamma(x) = x + iA(x)$ to be a piece of the Lipschitz curve defined on $[-\pi, \pi]$ with $g(0) > 0$ and $m = \min g(x) < 0$. For any integer S, let b_S be a l^∞-sequence satisfying $b_S(n) = 1$ for $n \leqslant S$ and $b_S(n) = 0$ otherwise. Using $F(z) = \frac{1}{1-\exp(iz)}$ as the test function, we can prove that for any $\epsilon > 0$,

$$\|b_S\|_{M_2(\widetilde{\gamma})} \geqslant C_\epsilon \exp(-S(m + \epsilon)).$$

2.5 Remarks

Remark 2.5.1 We can obtain the following generalizations of Theorems 2.3.1 and 2.3.2. Let γ be a closed starlike Lipschitz curve. Suppose that the multiplier b satisfies $|b(z)| \leqslant C|z \pm 1|^s$ in any $S_{\mu,\pm}$, $0 < \mu < w$. Then it can be proved that $\phi(z) = \sum_{n=\pm 1}^{\pm\infty} b(n)z^n$ satisfies

$$|\phi(z)| \leqslant \frac{C_\mu}{|1 - z|^{1+s}}, z \in C_{\mu,\pm}, 0 < \mu < w. \tag{2.9}$$

Conversely, if the holomorphic function ϕ satisfies the estimate (2.9), then there exists a function b such that $|b(z)| \leqslant C|z \pm 1|^s$ and $\phi(z) = \sum_{n=\pm 1}^{\pm\infty} b(n)z^n$, see Sect. 7.1 for the details.

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Chapter 3
Clifford Analysis, Dirac Operator and the Fourier Transform

In this chapter, we state basic knowledge, notations and terminologies in Clifford analysis and some related results. These preliminaries will be used to establish the theory of convolution singular integrals and Fourier multipliers on Lipschitz surfaces. In Sect. 3.1, we give a brief survey on basics of Clifford analysis. In Sect. 3.2, we state the monogenic functions on sectors introduced by Li, McIntosh, Qian [1]. Section 3.3 is devoted to the Fourier transform theory on sectors established by [1]. Section 3.4 is based on the Möbius covarian of iterated Dirac operators by Peeter and Qian in [2]. In Sect. 3.5, we give a generalization of the Fueter theorem in the setting of Clifford algebras [3]. In Chaps. 6 and 7, this generalization will be used to estimate the kernels of holomorphic Fourier multiplier operators on closed Lipschitz surfaces.

3.1 Preliminaries on Clifford Analysis

In this section, n and M denote the positive integers, L equals to 0 or $n+1$, and $M \geqslant \max\{n, L\}$. The real 2^M-dimensional Clifford algebra $\mathbb{R}_{(M)}$ or the complex 2^M-dimensional Clifford algebra $\mathbb{C}_{(M)}$ has basis vectors e_S, where S is any subset in $\{1, 2, \ldots, M\}$. Under the identifications

$$\begin{cases} e_0 = e_{\emptyset}, \\ e_j = e_{\{j\}},\ 1 \leqslant j \leqslant M, \end{cases}$$

the multiplication of basis vectors satisfies

$$\begin{cases} e_0 = 1,\ e_j^2 = -e_0 = -1,\ 1 \leqslant j \leqslant M,\ e_0 = 1; \\ e_j e_k = -e_k e_j = e_{\{j,k\}},\ 1 \leqslant j < k \leqslant M; \\ e_{j_1} e_{j_2} \ldots e_{j_s} = e_S,\ 1 \leqslant j_1 < j_2 < \cdots < j_s \leqslant M \text{ and } S = \{j_1, j_2, \ldots, j_s\}. \end{cases}$$

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Let $u = \sum_S u_S e_S$ and $v = \sum_T v_T e_T$ be two elements in $\mathbb{R}_{(M)}$ (or $\mathbb{C}_{(M)}$). Then the product of u and v can be expressed as

$$uv = \sum_{S,T} u_S v_T e_S e_T,$$

where $u_S, v_T \in \mathbb{R}$ (or $\mathbb{C}$), $u_\emptyset e_\emptyset$ is usually written as $u_0 e_0$ or u_0, and is called the scalar part of u.

By identifying the standard basis vectors $e_1,\ e_2, \ldots, e_n$ of $\mathbb{R}^n$ with their counterparts in $\mathbb{R}_{(M)}$ or $\mathbb{C}_{(M)}$, we embed the vector space $\mathbb{R}^{n+1}$ in the Clifford algebras $\mathbb{R}_{(M)}$ and $\mathbb{C}_{(M)}$. There are two usual methods to embed $\mathbb{R}^{n+1}$ in Clifford algebras. We treat them together by denoting standard basis vectors of $\mathbb{R}^{n+1}$ by $e_1, e_2, \ldots, e_n, e_L$, and identifying e_L with either e_0 or e_{n+1}.

On $\mathbb{R}_{(M)}$ and $\mathbb{C}_{(M)}$, we use the Euclidean norm $|u| = (\sum_S |u_S|^2)^{1/2}$. For a constant C depending only on M, $|uv| \leqslant C|u||v|$. If $u \in \mathbb{R}^{n+1}$, then we can take the constant C as 1. If $u \in \mathbb{C}^{n+1}$, the constant C is taken as $\sqrt{2}$.

We write $x \in \mathbb{R}^{n+1}$ as $x = \underline{x} + x_L e_L$, where $\underline{x} \in \mathbb{R}^n$ and $x_L \in \mathbb{R}$. We also write the Clifford conjugate of x as $\bar{x} = -\underline{x} + x_L \bar{e}_L$, where $\bar{e}_L e_L = 1$. Then

$$\bar{x}x = x\bar{x} = \sum_{j=1}^{n} x_j^2 + x_L^2 = |x|^2.$$

The Clifford algebras $\mathbb{R}_{(0)}$, $\mathbb{R}_{(1)}$ and $\mathbb{R}_{(2)}$ are the real numbers, the complex numbers and the quaternions, respectively. An important property of the three algebras is that every non-zero element has an inverse. Although this is not true in general, but every non-zero element $x = \underline{x} + x_L e_L$ in $\mathbb{R}^{n+1}$ has an inverse x^{-1} in $\mathbb{R}_{(M)}$. In fact, $x^{-1} = |x|^{-2}\bar{x} \in \mathbb{R}^{n+1} \subset \mathbb{R}_{(M)}$.

For the sake of convenience, we recall some basic knowledge in Clifford analysis. The differential operator

$$D = \underline{D} + \frac{\partial}{\partial x_L} e_L, \text{ where } \underline{D} = \sum_{k=1}^{n} \frac{\partial}{\partial x_k} e_k,$$

acts on C^1 functions $f = \sum_S f_S e_S$ of $n + 1$ variables to give

$$Df = \sum_{k=1}^{n} \frac{\partial f_S}{\partial x_k} e_k e_S + \frac{\partial f_S}{\partial x_L} e_L e_S$$

and

$$fD = \sum_{k=1}^{n} \frac{\partial f_S}{\partial x_k} e_S e_k + \frac{\partial f_S}{\partial x_L} e_S e_L.$$

Let f be a C^1 function defined on an open subset of $\mathbb{R}^{n+1}$ with values in $\mathbb{R}_{(M)}$ or $\mathbb{C}_{(M)}$. If $Df = 0$, then f is called a left-monogenic function. If $fD = 0$, then f is called a right-monogenic function. If f is both left-monogenic and right-monogenic, we call f a monogenic function. For the left-monogenic function and the right-monogenic function, each component is harmonic. It is easy to prove that for fixed ζ, the function $e(x,\zeta)$ is a left-monogenic and right-monogenic function of variable x because

$$\begin{aligned}\frac{\partial}{\partial x_L} e_L e(x,\zeta) &= -e_L i \zeta e_L e(x,\zeta)\\ &= -e_L i \overline{e_L} \zeta e(x,\zeta)\\ &= -i\zeta e(x,\zeta) = -\underline{D} e(x,\zeta).\end{aligned}$$

Define a function E on $\mathbb{R}^{n+1} \setminus \{0\}$ as

$$k(x) = \frac{1}{\sigma_n}\frac{\overline{x}}{|x|^{n+1}},\ x \neq 0,$$

where σ_n is the volume of the unit n-sphere in $\mathbb{R}^{n+1}$. In Clifford analysis, for the above function E, the corresponding Cauchy integral formula holds.

Theorem 3.1.1 *Let Ω be a bounded open subset of $\mathbb{R}^{n+1}$ with the Lipschitz boundary $\partial\Omega$ and the exterior unit normal $\mathfrak{n}(y)$ defined for almost all $y \in \partial\Omega$. Assume that f is a left-monogenic function on the neighborhood of $\Omega \cup \partial\Omega$ and g is a right-monogenic function on the neighborhood of $\Omega \cup \partial\Omega$. Then*

$$\text{(i)}\ \int_\Sigma g(y)\mathfrak{n}(y) f(y) dS_y = 0;$$

$$\text{(ii)}\ \int_{\partial\Omega} g(y)\mathfrak{n}(y) E(x-y) dS_y = \begin{cases} g(x), & x \in \Omega,\\ 0, & x \notin \Omega \cup \partial\Omega;\end{cases}$$

$$\text{(iii)}\ \int_{\partial\Omega} E(x-y)\mathfrak{n}(y) f(y) dS_y = \begin{cases} f(x), & x \in \Omega,\\ 0, & x \notin \Omega \cup \partial\Omega.\end{cases}$$

Proof (i) is a direct corollary of Gauss's divergence theorem, while (ii) and (iii) can be deduced from (i) and the following identity which is easily verified:

$$\int_{|y-x|=r} \mathfrak{n}(y) E(y-x) dS_y = \int_{|y-x|=r} E(y-x)\mathfrak{n}(y) dS_y = 1,\ r > 0.$$

□

We also need the following result.

Theorem 3.1.2 *Let f be a right-monogenic function on $\mathbb{R}^{n+1} \setminus \{0\}$ and satisfy $|f(x)| \leqslant C/|x|^n$ for $x \in \mathbb{R}^{n+1} \setminus \{0\}$. Then for some constant $c \in \mathbb{C}_{(n)}$, $f(x) = c\overline{x}/|x|^{n+1}$.*

For $\xi \in \mathbb{R}^n$, $\xi \neq 0$, define

$$\chi_{\pm}(\xi) = (1 \pm i\xi e_L |\xi|^{-1})/2$$

such that $\chi_+(\xi) + \chi_-(\xi) = 1$. By $(i\xi e_L)^2 = |\xi|^2$, we get

$$\begin{cases} \chi_+(\xi)^2 = \chi_+(\xi), \\ \chi_-(\xi)^2 = \chi_-(\xi), \\ \chi_+(\xi)\chi_-(\xi) = 0 = \chi_-(\xi)\chi_+(\xi). \end{cases}$$

Moreover, $i\xi e_L = |\xi|\chi_+(\xi) - |\xi|\chi_-(\xi)$, and in fact, for any polynomial $P(\lambda) = \sum a_k \lambda_k$ in one variable with scalar coefficients, we have

$$P(i\xi e_L) = \sum_k a_k (i\xi e_L)^k = P(|\xi|)\chi_+(\xi) + P(-|\xi|)\chi_-(\xi).$$

Hence, the polynomial p in m variables defined by $p(\xi) = P(i\xi e_L)$ satisfies $p(0) = P(0)$ and

$$p(\xi) = P(i\xi e_L) = P(|\xi|)\chi_+(\xi) + P(-|\xi|)\chi_-(\xi), \ \xi \neq 0.$$

It is natural to associate every function B of one real variable with a function b of n real variables. Precisely, if $|\xi|$ and $-|\xi|$ are in the domain of B,

$$b(\xi) = B(i\xi e_L) = B(|\xi|)\chi_+(\xi) + B(-|\xi|)\chi_-(\xi).$$

When 0 is in the domain of B, $b(\underline{0}) = B(0)$, where $\underline{0}$ denotes the 0-vector in $\mathbb{R}^n$.

We repeat this procedure for holomorphic functions of complex variables. At first for $\zeta = \xi + i\eta \in \mathbb{C}^n$, define

$$|\zeta|_{\mathbb{C}}^2 = \sum_{j=1} \zeta_j^2 = |\xi|^2 - |\eta|^2 + 2i\langle \xi, \eta \rangle,$$

where $\xi, \eta \in \mathbb{R}^n$, and note that $(i\zeta e_L)^2 = |\zeta|_{\mathbb{C}}^2$. Hence, we extend $|\xi|^2$ holomorphically to $\mathbb{C}^n$. When $|\zeta|_{\mathbb{C}}^2 \neq 0$, take $\pm|\zeta|_{\mathbb{C}}$ as its two square roots and define

$$\chi_{\pm}(\zeta) = \frac{1}{2}\left(1 \pm \frac{i\zeta e_L}{|\zeta|_{\mathbb{C}}}\right)$$

such that

$$\begin{cases} \chi_+(\zeta) + \chi_-(\zeta) = 1, \\ \chi_+(\zeta)^2 = \chi_+(\zeta), \\ \chi_-(\zeta)^2 = \chi_-(\zeta), \\ \chi_+(\zeta)\chi_-(\zeta) = 0 = \chi_-(\zeta)\chi_+(\zeta) \\ i\zeta e_L = |\zeta|_{\mathbb{C}}\chi_+(\xi) - |\zeta|_{\mathbb{C}}\chi_-(\xi). \end{cases}$$

Let $P(\lambda) = \sum a_k \lambda_k$ be a polynomial in one variable with complex coefficients, the corresponding polynomial in n variables is defined by

$$p(\zeta) = P(i\zeta e_L) = \sum_k a_k (i\zeta e_L)^k$$

and satisfies if $|\zeta|^2_{\mathbb{C}} \neq 0$, then

$$\begin{aligned} p(\zeta) &= P(i\zeta e_L) = P(|\zeta|_{\mathbb{C}})\chi_+(\zeta) + P(-|\zeta|_{\mathbb{C}})\chi_-(\zeta) \\ &= \frac{1}{2}\Big(P(|\zeta|_{\mathbb{C}}) + P(-|\zeta|_{\mathbb{C}})\Big) + \frac{1}{2}\frac{\Big(P(|\zeta|_{\mathbb{C}}) - P(-|\zeta|_{\mathbb{C}})\Big)i\zeta e_L}{|\zeta|_{\mathbb{C}}}; \end{aligned}$$

if $|\zeta|^2_{\mathbb{C}} = 0$, then

$$p(\zeta) = P(0) + P'(0)i\zeta e_L.$$

Let B be any holomorphic function in one variable defined on the open subset S in $\mathbb{C}$ and let b be the holomorphic Clifford-valued function in n variables. For all $\zeta \in \mathbb{C}^n$, $\{\pm|\zeta|_{\mathbb{C}}\} \subset S$. The correspondence between B and b can be defined as follows naturally. If $|\zeta|^2_{\mathbb{C}} \neq 0$, then

$$\begin{aligned} b(\zeta) &= B(i\zeta e_L) = B(|\zeta|_{\mathbb{C}})\chi_+(\zeta) + B(-|\zeta|_{\mathbb{C}})\chi_-(\zeta) \\ &= \frac{1}{2}\Big(B(|\zeta|_{\mathbb{C}}) + B(-|\zeta|_{\mathbb{C}})\Big) + \frac{1}{2}\frac{\Big(B(|\zeta|_{\mathbb{C}}) - B(-|\zeta|_{\mathbb{C}})\Big)i\zeta e_L}{|\zeta|_{\mathbb{C}}}. \end{aligned}$$

If $|\zeta|^2_{\mathbb{C}} = 0$, then

$$b(\zeta) = B(0) + B'(0)i\zeta e_L.$$

The reason that the above correspondence is natural because not only b is the desired polynomial when B is a polynomial, but also the mapping from B to b is an algebra homomorphism. In other words, if F is another holomorphic function defined on S and $c_1, c_2 \in \mathbb{C}$, then

$$(c_1 F + c_2 B)(i\zeta e_L) = c_1 F(i\zeta e_L) + c_2 B(i\zeta e_L)$$

and

$$(FB)(i\zeta e_L) = F(i\zeta e_L)B(i\zeta e_L).$$

We give an example. For any real number t, define the holomorphic function $E_t(\lambda) = e^{-t\lambda}$ with variable $\lambda \in \mathbb{C}$. The corresponding function in n variables is defined as follows. If $|\zeta|_{\mathbb{C}}^2 \neq 0$,

$$\begin{aligned} e(te_L,\ \zeta) &= E_t(i\zeta e_L) = e^{-t|\zeta|_{\mathbb{C}}}\chi_+(\zeta) + e^{t|\zeta|_{\mathbb{C}}}\chi_-(\zeta) \\ &= \cosh(t|\zeta|_{\mathbb{C}}) - \sinh(t|\zeta|_{\mathbb{C}})|\zeta|_{\mathbb{C}}^{-1} i\zeta e_L. \end{aligned}$$

If $|\zeta|_{\mathbb{C}}^2 = 0$,

$$e(te_L, \zeta) = 1 - ti\zeta e_L.$$

Then

$$e(te_L, \zeta)e(se_L, \zeta) = e((t+s)e_L, \zeta)$$

and $e(te_L, -\zeta) = e(-te_L, \zeta)$. In addition,

$$\frac{d}{dt}e(te_L, \zeta) = -i\zeta e_L e(te_L, \zeta) = -e(te_L, \zeta)i\zeta e_L.$$

For any complex number α, another example is the function defined by $R_\alpha(\lambda) = (\lambda - \alpha)^{-1}$, $\lambda \neq \alpha$. Then

$$R_\alpha(i\zeta e_L) = (i\zeta e_L - \alpha)^{-1} = (i\zeta e_L + \alpha)(|\zeta|_{\mathbb{C}}^2 - \alpha^2)^{-1},\ \ |\zeta|_{\mathbb{C}}^2 \neq \alpha^2.$$

From now on, although we assume that $|\zeta|_{\mathbb{C}}^2 \notin (-\infty, 0]$ and $\mathrm{Re}|\zeta|_{\mathbb{C}} > 0$, it has been unimportant which sign we assign to each square root of $|\zeta|_{\mathbb{C}}^2$. Now we prove some estimates.

Theorem 3.1.3 *Let $\zeta = \xi + i\eta \in \mathbb{C}^n$, where $\xi, \eta \in \mathbb{R}^n$, and assume that $|\zeta|_{\mathbb{C}}^2 \notin (-\infty, 0]$. Let*

$$\theta = \tan^{-1}\left(\frac{|\eta|}{\mathrm{Re}|\zeta|_{\mathbb{C}}}\right) \in [0,\ \pi/2).$$

Then

(a) $0 < \mathrm{Re}|\zeta|_{\mathbb{C}} \leqslant |\xi| \leqslant \sec\theta \mathrm{Re}|\zeta|_{\mathbb{C}}$,
(b) $\mathrm{Re}|\zeta|_{\mathbb{C}} \leqslant ||\zeta|_{\mathbb{C}}| \leqslant \sec\theta \mathrm{Re}|\zeta|_{\mathbb{C}} \leqslant |\zeta| \leqslant (1 + 2\tan^2\theta)^{1/2}\mathrm{Re}|\zeta|_{\mathbb{C}}$,
(c) $-\theta \leqslant \arg|\zeta|_{\mathbb{C}} \leqslant \theta$,
(d) $|\chi_\pm(\zeta)| \leqslant \sec\theta/\sqrt{2}$.

Proof It is easy to prove

$$||\zeta|_{\mathbb{C}}|^2 = ||\zeta|_{\mathbb{C}}^2| = \left((|\xi|^2 - |\eta|^2)^2 + 4\langle\xi,\ \eta\rangle^2\right)^{1/2} \leqslant |\xi|^2 + |\eta|^2 = |\zeta|^2.$$

Hence

$$\mathrm{Re}|\zeta|_{\mathbb{C}} \leqslant ||\zeta|_{\mathbb{C}}| \leqslant |\zeta|. \tag{3.1}$$

Taking the real part of the identity

$$-(\xi + i\eta)^2 = -\zeta^2 = |\zeta|_{\mathbb{C}}^2 = \left(\mathrm{Re}|\zeta|_{\mathbb{C}} + i\mathrm{Im}|\zeta|_{\mathbb{C}}\right)^2,$$

we obtain

$$|\xi|^2 - |\eta|^2 = (\mathrm{Re}|\zeta|_{\mathbb{C}})^2 - (\mathrm{Im}|\zeta|_{\mathbb{C}})^2 \tag{3.2}$$

or

$$2|\xi|^2 - |\zeta|^2 = 2(\mathrm{Re}|\zeta|_{\mathbb{C}})^2 - ||\zeta|_{\mathbb{C}}|^2.$$

Therefore, by (3.1), we get $\mathrm{Re}|\zeta|_{\mathbb{C}} \leqslant |\xi|$. In addition, by (3.2), we have

$$|\xi|^2 \leqslant |\eta|^2 + (\mathrm{Re}|\zeta|_{\mathbb{C}})^2 = (\tan^2\theta + 1)(\mathrm{Re}|\zeta|_{\mathbb{C}}^2).$$

This means $|\xi| \leqslant \sec\theta \mathrm{Re}|\zeta|_{\mathbb{C}}$ and (a) is proved.

Another corollary of (3.2) is

$$|\zeta|^2 = 2|\eta|^2 + (\mathrm{Re}|\zeta|_{\mathbb{C}})^2 - (\mathrm{Im}|\zeta|_{\mathbb{C}})^2 \leqslant (1 + 2\tan^2\theta)(\mathrm{Re}|\zeta|_{\mathbb{C}}^2),$$

which implies (b).

(c) is a direct corollary of the inequality $||\zeta|_{\mathbb{C}}| \leqslant \sec\theta \mathrm{Re}|\zeta|_{\mathbb{C}}$, and (d) can be deduced from $|\zeta| \leqslant (1 + 2\tan^2\theta)^{1/2}||\zeta|_{\mathbb{C}}|$. □

Define

$$S_\mu^0 = \left\{\zeta = \xi + i\eta \in \mathbb{C}^n : \ |\zeta|_{\mathbb{C}}^2 \notin (-\infty, 0] \text{ and } |\eta| < \mathrm{Re}(|\zeta|_{\mathbb{C}}) \tan\mu\right\}.$$

By (c) of Theorem 3.1.3, we know that $|\zeta|_{\mathbb{C}} \in S_{\mu,+}^0(\mathbb{C})$ and $-|\zeta|_{\mathbb{C}} \in S_{\mu,-}^0$ when $\zeta \in S_\mu^0(\mathbb{C}^m)$. So for any holomorphic function B defined on $S_\mu^0(\mathbb{C}) = S_{\mu,+}^0(\mathbb{C}) \cup S_{\mu,-}^0$, the corresponding holomorphic function b in n variables:

$$b(\zeta) = B(i\zeta e_L) = B(|\zeta|_{\mathbb{C}})\chi_+(\zeta) + B(-|\zeta|_{\mathbb{C}})\chi_-(\zeta)$$

is defined on $S_\mu^0(\mathbb{C}^n)$. In addition, by (d) of Theorem 3.1.3, if B is bounded, then

$$\|b\|_\infty \leqslant \sqrt{2}\sec\mu \|B\|_\infty.$$

Let

$$H^\infty(S_\mu^0(\mathbb{C}^n)) = H^\infty(S_\mu^0(\mathbb{C}^n), \mathbb{C}_{(M)})$$

be the Banach space of bounded Clifford-valued holomorphic functions on $S_\mu^0(\mathbb{C}^n)$. We have the following result.

Theorem 3.1.4 *The mapping $B \to b$ defined above is a one-one bounded algebra homomorphism from $H^\infty(S_\mu^0(\mathbb{C}))$ to $H^\infty(S_\mu^0(\mathbb{C}^n))$.*

Proof We only need to prove the mapping is one-one. In fact, this point can be deduced from the following formula, and the reverse result from b to B still holds:

$$B(\lambda) = \frac{2}{\sigma_{n-1}} \int_{|\xi|=1} b(\lambda\xi)\chi_{\pm}(\xi)d\xi,\ \lambda \in S^0_{\mu,\pm}(\mathbb{C}),$$

where σ_{n-1} is the volume of the unit $(n-1)$-sphere in $\mathbb{R}^n$. □

So far we have considered Clifford-valued holomorphic functions of n complex variables. What is called Clifford analysis is the study of monogenic functions of $n+1$ real variables. In the next section, we will relate these two concepts via the Fourier transform. We need to introduce the following generalized exponential function:

$$\begin{aligned} e(x,\zeta) &= e(\underline{x} + x_L e_L, \zeta) \\ &= e^{i\langle \underline{x},\ \zeta\rangle} e(x_L e_L, \zeta) \\ &= e^{i\langle \underline{x},\ \zeta\rangle}(e^{-x_L|\zeta|_{\mathbb{C}}}\chi_+(\zeta) + e^{x_L|\zeta|_{\mathbb{C}}}\chi_-(\zeta)). \end{aligned}$$

For any $x = \underline{x} + x_L e_L \in \mathbb{R}^{n+1}$, this function is holomorphic on $\zeta \in \mathbb{C}^n$ and is a left-monogenic function of $x \in \mathbb{R}^{n+1}$ for any $\zeta \in \mathbb{C}^n$. This function satisfies

$$\begin{cases} e(x,\zeta)e(y,\zeta) = e(x+y,\zeta), \\ e(x,-\zeta) = e(-x,\zeta). \end{cases}$$

Specially, when $\underline{x} \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^n$, $e(\underline{x},\ \xi) = e^{i\langle \underline{x},\ \xi\rangle}$, i.e., the usual exponential function in the Fourier theory. Moreover, for any $\zeta \in \mathbb{C}^n$, $e(x,\zeta)\overline{e_L}$ is also a right-monogenic function of $x \in \mathbb{R}^{n+1}$. We point out that

$$e(x,\zeta) = \exp i(\langle \underline{x}, \zeta\rangle - x_L\zeta e_L) = \sum_{k=0} \frac{1}{k!}(i(\langle \underline{x}, \zeta\rangle - x_L\zeta e_L))^k.$$

3.2 Monogenic Functions on Sectors

On the Lipschitz surface, to establish the relation between holomorphic multipliers and the functional calculus of Dirac operator, Li, McIntosh, Qian [1] introduced the monogenic functions on sectors. In this section, we will give a detailed statement for the function classes $K(S_{N_\mu})$, $K(C^+_{N_\mu})$ and $K(C^-_{N_\mu})$ which will be used in Sect. 3.3 and Chap. 5 below.

We start by specifying some sets of unit vectors in

$$\mathbb{R}^{n+1}_+ = \left\{x = \underline{x} + x_L e_L \in \mathbb{R}^{n+1} : \ x_L > 0\right\}.$$

For these unit vectors, we use the metric $\angle(\mathfrak{n}, y) = \cos^{-1}\langle \mathfrak{n}, y\rangle$.

Let N be a compact set of unit vectors in $\mathbb{R}^{n+1}_+$ which contains e_L and let

$$\mu_N = \sup_{\mathfrak{n} \in N} \angle(\mathfrak{n}, e_L).$$

Then $0 \leqslant \mu_N < \pi/2$. For $0 < \mu < \pi/2 - \mu_N$, define the open neighborhood N_μ of N in the unit sphere by

$$N_\mu = \left\{ y \in \mathbb{R}^{n+1}_+ : \ |y| = 1, \ \angle(y, \mathfrak{n}) < \mu \text{ for some } \mathfrak{n} \in N \right\}.$$

For every unit vector n, let C_n^+ be the open half space

$$C_\mathfrak{n}^+ = \left\{ x \in \mathbb{R}^{n+1} : \ \langle x, \mathfrak{n} \rangle > 0 \right\},$$

and define the open cones in $\mathbb{R}^{n+1}$ as follows. Let

$$\begin{cases} C_{N_\mu}^+ = \bigcup \{ C_n^+ : \ \mathfrak{n} \in N_\mu \}, \\ C_{N_\mu}^- = -C_{N_\mu}^+, \\ S_{N_\mu} = C_{N_\mu}^+ \cap C_{N_\mu}^-. \end{cases}$$

Definition 3.2.1 $K(C_{N_\mu}^+)$ is defined as the Banach space of all right monogenic functions Φ from $C_{N_\mu}^+$ to $\mathbb{C}_{(M)}$ satisfying

$$\|\Phi\|_{K(C_{N_\mu}^+)} = \frac{1}{2}\sigma_n \sup_{x \in C_{N_\mu}^+} |x|^n |\Phi(x)| < \infty.$$

Similarly, we can define $K(C_{N_\mu}^-)$.

Definition 3.2.2 Define $K(S_{N_\mu})$ as the Banach space of all function pairs $(\Phi, \underline{\Phi})$, where Φ is a right-monogenic function from S_{N_μ} to $\mathbb{C}_{(M)}$, and $\underline{\Phi}$ is continuous on $(0, +\infty)e_L$ such that $(\Phi, \underline{\Phi})$ satisfies

$$\underline{\Phi}(Re_L) - \underline{\Phi}(re_L) = \int_{r \leqslant |\underline{x}| \leqslant R} \Phi(\underline{x}) d\underline{x} e_L,$$

and

$$\|(\Phi, \underline{\Phi})\|_{K(S_{N_\mu})} = \frac{1}{2}\sigma \sup_{x \in S_{N_\mu}} |x|^n |\Phi(x)| + \sup_{r > 0} |\underline{\Phi}(re_L)| < +\infty.$$

Notice that $\underline{\Phi}$ is determined by Φ up to an additive constant, and

$$\underline{\Phi}'(re_L) = \int_{|\underline{x}| = r} \Phi(\underline{x}) d\underline{x} e_L.$$

In addition, $\underline{\Phi}$ has a unique and continuous extension to the cone

$$T_{N_\mu} = \left\{ y = \underline{y} + y_L e_L \in \mathbb{R}^{n+1}_+ \,:\; y^\perp \subset S_{N_\mu} \right\}.$$

This extension satisfies

$$\underline{\Phi}(y) - \underline{\Phi}(z) = \int_{A(y,z)} f(x)\mathfrak{n}(x) dS_x,$$

where $A(y, z)$ is a smoothly oriented n-manifold in S_{N_μ} jointing the $(m - 1)$-sphere

$$S_y = \left\{ x \in \mathbb{R}^{n+1} \,:\; \langle x, y \rangle = 0 \text{ and } |x| = |y| \right\}$$

to the $(n - 1)$-sphere S_x, in which case, for all $y \in T_{N_\mu}$,

$$|\underline{\Phi}(y)| \leqslant \|(\Phi, \underline{\Phi})\|_{K(S_{N_\mu})}.$$

If N is rotationally symmetric, i.e.,

$$N = \left\{ \mathfrak{n} = \underline{\mathfrak{n}} + \mathfrak{n}_L e_L \in \mathbb{R}^{n+1}_+ \,:\; |\mathfrak{n}| = 1, \mathfrak{n}_L \geqslant |\underline{\mathfrak{n}}| \cot \omega \right\},$$

we use the symbol

$$T^0_\mu = T_{N_{\mu-\omega}} = \left\{ y = \underline{y} + y_L e_L \in \mathbb{R}^{n+1} \,:\; y_L > |x| \cot \mu \right\}.$$

Now we state the relationship between these spaces. Here $H_{y,\pm}$ denote the hemispheres

$$H_{y,\pm} = \left\{ x \in \mathbb{R}^{n+1} \,:\; \pm\langle x, y \rangle \geqslant 0 \text{ and } |x| = |y| \right\}.$$

with the boundaries S_y.

Theorem 3.2.1 *(i) Let $\Phi_\pm \in K(C^\pm_{N_\mu})$. Define the function $\underline{\Phi_\pm}$ on T_{N_μ} as*

$$\underline{\Phi_\pm}(y) = \pm \int_{H_{y,\pm}} \Phi_\pm(x)\mathfrak{n}(x) dS_x,\;\; y \in T_{N_\mu},$$

where $\mathfrak{n}(x) = x/|x|$ is the normal to the hemisphere $H_{y,\pm}$. Then

$$(\Phi, \underline{\Phi}) = (\Phi_+ + \Phi_-,\;\; \underline{\Phi_+} + \underline{\Phi_-}) \in K(S_{N_\mu})$$

and

$$\|(\Phi, \underline{\Phi})\|_{K(S_{N_\mu})} \leqslant \|\Phi_+\|_{K(C^+_{N_\mu})} + \|\Phi_-\|_{K(C^-_{N_\mu})}.$$

(ii) Conversely, assume that $(\Phi, \underline{\Phi}) \in K(S_{N_\mu})$. There exists unique functions $\Phi_\pm \in K(C^\pm_{N_\mu})$ satisfying $\Phi = \Phi_+ + \Phi_-$ and $\underline{\Phi} = \underline{\Phi_+} + \underline{\Phi_-}$. For all $n \in N_\mu$ and $x \in C^\pm_n \subset C^\pm_{N_\mu}$,

$$\Phi_{\pm}(x) = \pm \lim_{\epsilon \to 0} \left(\int_{\langle y, \mathfrak{n} \rangle = 0,\ |y| \geqslant \epsilon} \Phi(y)\mathfrak{n}(x)E(x-y)dS_y + \underline{\Phi}(\epsilon e_L)k(x) \right),$$

where $E(x) = \overline{x}/\sigma|x|^{n+1}$, *and* $(\Phi, \underline{\Phi})$ *satisfies*

$$\|\Phi_{\pm}\|_{K(C^{\pm}_{N_\mu})} \leqslant c\|(\Phi, \underline{\Phi})\|_{K(S_{N_\mu})},$$

where c *only depends on* μ_N, μ *and the dimension* n.

Proof (i) In order to prove

$$\underline{\Phi_{\pm}}(y) - \underline{\Phi_{\pm}}(z) = \int_{A(y,z)} \Phi_{\pm}(x)\mathfrak{n}(x)dS_x,$$

we apply Cauchy's theorem to the right monogenic functions $\Phi_{\pm}$. The bound is straightforward.

(ii) This is a direct corollary of the results of [4]. In other words, there exists a natural isomorphism:

$$K(S_{N_\mu}) \simeq K(C^{+}_{N_\mu}) \oplus K(C^{-}_{N_\mu}).$$

We also need the closed linear subspaces $M(C^{\pm}_{N_\mu})$ of $K(C^{\pm}_{N_\mu})$. The functions in $M(C^{\pm}_{N_\mu})$ are both left monogenic and right monogenic. The subspaces $M(S_{N_\mu})$ of $K(S_{N_\mu})$ satisfying

$$M(S_{N_\mu}) \simeq M(C^{+}_{N_\mu}) \oplus M(C^{-}_{N_\mu})$$

are

$$M(S_{N_\mu}) = \left\{(\Phi, \underline{\Phi}) \in K(S_{N_\mu}) :\ \Phi \text{ is left monogenic and satisfies (3.3)}\right\},$$

where for $r > 0$,

$$\int_{|\underline{y}|=r} \langle \underline{y}, \underline{x} \rangle r^{-1}(e_L\Phi(\underline{y})\underline{y} - \underline{y}\Phi(\underline{y}e_L))dS_y + \underline{x}\underline{\Phi}(re_L) - e_L\underline{\Phi}(re_L)\underline{x}e_L = 0. \quad (3.3)$$

It is easy to see that

(i) the value of the integral is independent of r,
(ii) if $\Phi \in M(C^{\pm}_{N_\mu})$, the integral equals to 0.

We only need to prove that when $(\Phi, \underline{\Phi}) \in M(S_{N_\mu})$, the function $\Phi_{\pm}$ defined in (ii) of Theorem 3.2.1 is left monogenic. We omit the details and refer to [4]. □

Now we consider convolutions. Assume that $\Phi \in K(C^{+}_{N_\mu})$, $\Psi \in M(C^{+}_{N_\mu})$ and $x \in C^{+}_{\mathfrak{n}} \subset C^{+}_{N_\mu}$. Define $(\Phi * \Psi)$ as

$$\begin{aligned}(\Phi * \Psi)(x) &= \int\limits_{\langle y, \mathfrak{n}\rangle=\delta} \Phi(x-y)\mathfrak{n}(y)\Psi(y)dS_y \\ &= \int\limits_{\langle y,\ \mathfrak{n}\rangle=0,\ |y|\geqslant\epsilon} \Phi(x-y)\mathfrak{n}(y)\Psi(y)dS_y + \underline{\Phi}(\epsilon e_L)\Psi(x),\end{aligned}$$

where $0 < \delta < \langle x,\ \mathfrak{n}\rangle$. By Cauchy's theorem and the assumptions that Φ is right monogenic and Ψ is left monogenic, we can deduce that the integral is independent of the choice of the surfaces. On the other hand, because Ψ is right monogenic, $\Phi * \Psi$ is right monogenic. In fact, we can see from the following Theorem 3.3.1 that for all $\nu < \mu$,

$$\|\Phi * \Psi\|_{K(C^+_{N_\nu})} \leqslant c_{\nu,\mu}\|\Phi\|_{K(C^+_{N_\mu})}\|\Psi\|_{K(C^+_{N_\mu})}.$$

Moreover, if $\Psi_1 \in M(C^+_{N_\mu})$, then $\Psi * \Psi_1$ is both left monogenic and right monogenic, and

$$\Phi * (\Psi * \Psi_1) = (\Phi * \Psi) * \Psi_1.$$

For the functions defined on $C^-_{N_\mu}$, we have a similar result.

If $(\Phi, \underline{\Phi}) \in K(S_{N_\mu})$ and $(\Psi, \underline{\Psi}) \in M(S_{N_\mu})$, define

$$(\Phi, \underline{\Phi}) * (\Psi, \underline{\Psi}) \in M(S_{N_\mu}) = (\Phi_+ * \Psi_+ + \Phi_- * \Psi_-,\ \underline{\Phi_+ * \Psi_+} + \underline{\Phi_- * \Psi_-}).$$

Hence we can get for all $\nu < \mu$,

$$\|(\Phi, \underline{\Phi}) * (\Psi, \underline{\Psi})\|_{K(S^0_{\nu,+})} \leqslant C_{\nu,\mu}\|(\Phi, \underline{\Phi})\|_{K(S^0_{\mu,+})}\|(\Psi, \underline{\Psi})\|_{K(S^0_{\mu,+})}.$$

Let K^+_N be the linear space of all functions Φ on $\mathbb{R}^n \setminus \{0\}$ which can be extended monogenically to $\Phi \in K(C^+_{N_\mu})$ for some $\mu > 0$. Similarly, we define K^-_N, K_N, M^+_N, M^-_N and M_N, such that

$$K_N \simeq K^+_N \oplus K^-_N$$

and

$$M_N \simeq M^+_N \oplus M^-_N,$$

while M_N, M^+_N and M^-_N are all convolution algebras. The functions which belong to both K^+_N and K^-_N are of the form $\Phi(\underline{x}) = ck(\underline{x})$ for some $c \in \mathbb{C}_{(M)}$, where

$$E(\underline{x}) = \frac{1}{\sigma_n}\frac{-\underline{x}}{|\underline{x}|^{n+1}},\ \underline{x} \in \mathbb{R}^n \setminus \{0\},$$

with the monogenic extension

$$k(x) = \frac{1}{\sigma_n}\frac{\overline{x}}{|x|^{n+1}}.$$

The embedding of K_N^+ into K_N is defined as $ck \in K_N^+ \to (ck,\ c/2) \in K_N$, while the embedding K_N^- in K_N is defined as $ck \in K_N^- \to (ck,\ -c/2) \in K_N$.

3.3 Fourier Transforms on the Sectors

The section is devoted to the Fourier transform $\mathcal{F}_\pm(\Phi)$ of the function $\Phi \in K_N^\pm$ and the Fourier transform $\mathcal{F}(\Phi, \underline{\Phi})$ of $(\Phi, \underline{\Phi})$ introduced by Li, McIntosh, Qian [1]. We will prove these transforms are bounded holomorphic functions defined on the cones in $\mathbb{C}^m$. We also prove that $\mathcal{F}_+, \mathcal{F}_-$ and $\mathcal{F}$ are algebra homomorphism from the convolution algebras M_N^+, M_N^- and M_N to holomorphic functions.

We first associate with every unit vector $\mathfrak{n} = \underline{\mathfrak{n}} + \mathfrak{n}_L e_L \in \mathbb{R}^{n+1}$ satisfying $\mathfrak{n}_L > 0$, a real n-dimensional surface $\mathfrak{n}(\mathbf{C}^m)$ in $\mathbb{C}^n$, defined as follows.

$$\begin{aligned}\mathfrak{n}(\mathbb{C}^n) &= \left\{\zeta = \xi + i\eta \in \mathbb{C}^n : \ \xi \neq 0 \text{ and } \mathfrak{n}_L \eta = (\mathfrak{n}_L^2|\xi|^2 + \langle x, \underline{\mathfrak{n}}\rangle^2)^{1/2}\underline{\mathfrak{n}}\right\} \\ &= \left\{\zeta = \xi + i\eta \in \mathbb{C}^n : \ |\zeta|_{\mathbb{C}}^2 \notin (-\infty, 0] \text{ and } \mathfrak{n}_L \eta = \mathrm{Re}(|\zeta|_{\mathbb{C}})\underline{\mathfrak{n}}\right\} \\ &= \left\{\zeta = \xi + i\eta \in \mathbb{C}^n : \ |\zeta|_{\mathbb{C}}^2 \notin (-\infty, 0] \text{ for some } \kappa > 0, \eta + \mathrm{Re}(|\zeta|_{\mathbb{C}})e_L = \kappa\mathfrak{n}\right\},\end{aligned}$$

where

$$|\zeta|_{\mathbb{C}}^2 = \sum_{j=1}^n \zeta_j^2 = |\xi|^2 - |\eta|^2 + 2i\langle x, \eta\rangle.$$

The surfaces associated with distinct unit vectors are disjoint. In particular, $e_L(\mathbb{C}^n) = \mathbb{R}^n \setminus \{0\}$. On these surfaces, $|\xi|$, $|\zeta|$, $\mathrm{Re}(|\zeta|_{\mathbb{C}})$ and $||\zeta|_{\mathbb{C}}|$ are all equivalent. In fact, by Theorem 3.1.3,

$$\mathrm{Re}|\zeta|_{\mathbb{C}} \leqslant |\xi| \leqslant (\mathfrak{n}_L)^{-1}\mathrm{Re}|\zeta|_{\mathbb{C}},$$

and for all $\zeta \in \mathfrak{n}(\mathbb{C}^n)$,

$$\mathrm{Re}|\zeta|_{\mathbb{C}} \leqslant ||\zeta|_{\mathbb{C}}| \leqslant (\mathfrak{n}_L)^{-1}\mathrm{Re}|\zeta|_{\mathbb{C}} \leqslant |\zeta| \leqslant (n_L)^{-1}(1 + |\underline{\mathfrak{n}}|^2)^{1/2}\mathrm{Re}|\zeta|_{\mathbb{C}}.$$

Further, the parametrization used in the first definition of $\mathfrak{n}(\mathbb{C}^n)$ is smooth, with

$$\left|\det\left(\frac{\partial \zeta_j}{\partial \xi_k}\right)\right| \leqslant \frac{1}{\mathfrak{n}_L},\ \xi \neq 0.$$

To prove this, without loss of generality, we can assume that $\mathfrak{n} = \mathfrak{n}_1 e_1 + \mathfrak{n}_L e_L$. So

$$\zeta = \xi + i\frac{\mathfrak{n}_1}{\mathfrak{n}_L}(|\xi|^2\mathfrak{n}_L^2 + \xi_1^2\mathfrak{n}_1^2)^{1/2}e_1.$$

Then if $j \geqslant 2$, $\partial\zeta_j/\partial\xi_k = \delta_{jk}$ and

$$\frac{\partial\zeta_1}{\partial\xi_k} = \delta_{1k} + \frac{i\mathfrak{n}_1\xi_k(\mathfrak{n}_L^2 + \delta_{1k}\mathfrak{n}_1^2)}{\mathfrak{n}_L(|\xi|^2\mathfrak{n}_L^2 + \xi_1^2\mathfrak{n}_1^2)^{1/2}}.$$

Hence when $k \geqslant 2$,

$$\left|\frac{\partial\zeta_1}{\partial\xi_1}\right| \leqslant \frac{1}{\mathfrak{n}_L} \text{ and } \left|\frac{\partial\zeta_1}{\partial\xi_k}\right| \leqslant \mathfrak{n}_1.$$

The estimate for the Jacobian follows.

For the open set N_μ of the unit vectors defined above, we define the open cones $N_\mu(\mathbb{C}^n)$ in $\mathbb{C}^n$ as follows:

$$\begin{aligned} N_\mu(\mathbb{C}^n) &= \bigcup_{\mathfrak{n}\in N_\mu} \mathfrak{n}(\mathbb{C}^n) \\ &= \Big\{\zeta = \xi + i\eta \in \mathbb{C}^n :\ |\zeta|_{\mathbb{C}}^2 \notin (-\infty, 0] \text{ for some } \kappa > 0 \text{ and } \mathfrak{n} \in N_\mu, \\ &\qquad \eta + \mathrm{Re}(|\zeta|_{\mathbb{C}})e_L = \kappa\mathfrak{n}\Big\}. \end{aligned}$$

Because $N_\mu(\mathbb{C}^n) \subset S^0_{\mu_N+\mu}(\mathbb{C}^n)$, the estimates of Theorem 3.1.3 all hold, where $\theta = \mu_N + \mu$.

When N is rotationally symmetric, namely

$$N = \Big\{\mathfrak{n} = \underline{\mathfrak{n}} + n_L e_L \in \mathbb{R}^{n+1}_+ :\ |\mathfrak{n}| = 1, \mathfrak{n}_L \geqslant |\underline{\mathfrak{n}}| \cot w\Big\},$$

we have $S^0_\mu(\mathbb{C}^n) = N_{\mu-w}(\mathbb{C}^n)$. We let the functions take their values in the complex Clifford algebra $\mathbb{C}_{(M)}$, so for example $H_\infty(N_\mu(\mathbb{C}^n))$ denotes the Banach space of all bounded holomorphic functions from $N_\mu(\mathbb{C}^n)$ to $\mathbb{C}_{(M)}$ with the norm defined as

$$\|b\|_\infty = \sup\Big\{|b(\zeta)| :\ \zeta \in N_\mu(\mathbb{C}^n)\Big\}.$$

The exponential functions are defined as

$$e(x, \zeta) = e_+(x, \zeta) + e_-(x, \zeta),$$

where

$$e_+(x, \zeta) = e^{i\langle \underline{x},\, \zeta\rangle} e^{-x_L|\zeta|_{\mathbb{C}}} \chi_+(\zeta)$$

and

$$e_-(x, \zeta) = e^{i\langle \underline{x},\, \zeta\rangle} e^{x_L|\zeta|_{\mathbb{C}}} \chi_-(\zeta).$$

For fixed ζ, these functions are entire left monogenic functions of $x \in \mathbb{R}^{n+1}$. For fixed x, these functions are holomorphic functions of $\zeta \in N_\mu(\mathbb{C}^n)$ which satisfy

$$|e_+(x,\zeta)| = e^{-\langle \underline{x},\eta\rangle - x_L \mathrm{Re}|\zeta|_{\mathbb{C}}}|\chi_+(\zeta)| \leqslant \frac{\sec(\mu_N+\mu)}{\sqrt{2}} e^{-\langle x,\ \mathrm{n}\rangle \mathrm{Re}|\zeta|_{\mathbb{C}}/\mathrm{n}_L},\ \zeta \in \mathfrak{n}(\mathbb{C}^n)$$

and

$$|e_-(x,\zeta)| = e^{-\langle \underline{x},\ \eta\rangle + x_L \mathrm{Re}|\zeta|_{\mathbb{C}}}|\chi_-(\zeta)| \leqslant \frac{\sec(\mu_N+\mu)}{\sqrt{2}} e^{-\langle x,\ \mathrm{n}\rangle \mathrm{Re}|\zeta|_{\mathbb{C}}/\mathrm{n}_L},\ \zeta \in \overline{\mathfrak{n}}(\mathbb{C}^n).$$

Let

$$H_\infty^{\pm}(N_\mu(\mathbb{C}^n)) = \Big\{ b \in H_\infty(N_\mu(\mathbb{C}^n)) :\ b\chi_\pm = b \Big\}.$$

Then any function $b \in H_\infty(N_\mu(\mathbb{C}^n))$ can be uniquely decomposed as

$$b = b_+ + b_-,\ \text{where } b_\pm = b\chi_\pm \in H_\infty^{\pm}(N_\mu(\mathbb{C}^n)).$$

$H_\infty^{\pm}(N_\mu(\mathbb{C}^n))$ is the closed linear subspace of $H_\infty(N_\mu(\mathbb{C}^n))$. Actually, because for all $b \in H_\infty(N_\mu(\mathbb{C}^n))$,

$$\|b\chi_\pm\|_\infty \leqslant \sqrt{2}\|b\|_\infty \|\chi_\pm\|_\infty \leqslant \sec(\mu_N+\mu)\|b\|_\infty,$$

then

$$H_\infty(N_\mu(\mathbb{C}^n)) = H_\infty^{+}(N_\mu(\mathbb{C}^n)) \oplus H_\infty^{-}(N_\mu(\mathbb{C}^n)).$$

We also introduce the subalgebra

$$\mathcal{A}(N_\mu(\mathbb{C}^n)) = \Big\{ b \in H_\infty(N_\mu(\mathbb{C}^m)) :\ \zeta e_L b(\zeta) = b(\zeta)\zeta e_L \text{ for all } \zeta \Big\}.$$

Similarly, we can define $\mathcal{A}^{\pm}(N_\mu(\mathbb{C}^n))$. Notice that if $b \in \mathcal{A}(N_\mu(\mathbb{C}^n))$, then $b_\pm = b\chi_\pm \in \mathcal{A}^{\pm}(N_\mu(\mathbb{C}^n))$ such that

$$\mathcal{A}(N_\mu(\mathbb{C}^n)) = \mathcal{A}^{+}(N_\mu(\mathbb{C}^n)) \oplus \mathcal{A}^{-}(N_\mu(\mathbb{C}^n)).$$

Particular functions b belonging to $\mathcal{A}(N_\mu(\mathbb{C}^n))$ are those of the form

$$b(\zeta) = B(i\zeta e_L) = B(|\zeta|_{\mathbb{C}})\chi_+(\zeta) + B(-|\zeta|_{\mathbb{C}})\chi_-(\zeta),$$

where $B \in H_\infty(S^0_{\mu_N+\mu}(\mathbb{C}))$. All scalar-valued holomorphic functions in $H_\infty(N_\mu(\mathbb{C}^n))$ belong to $\mathcal{A}(N_\mu(\mathbb{C}^n))$. One of the simplest examples is $r_k(\zeta) = i\zeta_k/|\zeta|_{\mathbb{C}},\ k = 1, 2, \ldots, n$.

Let H_N^+ be the algebra of all functions b on $\mathbb{R}^n \setminus \{0\}$ which can be holomorphically extended to $b \in H_\infty^+(N_\mu(\mathbb{C}^n))$ for some $\mu > 0$. Let H_N^- denote the algebra of all functions b on $\mathbb{R}^n \setminus \{0\}$ which can be holomorphically extended to $b \in H^-_\infty(\overline{N}_\mu(\mathbb{C}^n))$, where $\overline{N} = \{\overline{\mathfrak{n}} \in \mathbb{R}^{n+1} : \mathfrak{n} \in N\}$. Then $H^+_N \cap H^-_N = \{0\}$.

Define H_N as $H_N = H^+_N + H^-_N$. Then $H_N = H^+_N \oplus H^-_N$. Let $\mathcal{A}^+_N$, $\mathcal{A}^-_N$ and $\mathcal{A}_N$ be the spaces of the functions in H^+_N, H^-_N and H_N satisfying $\xi e_L b(\xi) = b(\xi)\xi e_L$, $\xi \neq 0$. Then

$$\mathcal{A} = \mathcal{A}^+_N \oplus \mathcal{A}^-_N.$$

If we assume that N is connected, these holomorphic extensions are unique. In fact we assume that the compact set N of unit vectors in $\mathbb{R}^{n+1}_+$ satisfies a stronger condition:N are starlike about e_L, that is, if $\mathfrak{n} \in N$ and $0 \leqslant \tau \leqslant 1$, then

$$\frac{(\tau\mathfrak{n} + (1 - \tau e_L))}{|\tau\mathfrak{n} + (1 - \tau e_L)|} \in N.$$

Under this case, the open set N_μ is also starlike about e_L and $N_\mu(\mathbb{C}^n)$ is the connected open subset in $\mathbb{C}^n$.

Theorem 3.3.1 *Let N be a compact set of unit vectors in $\mathbb{R}^{n+1}_+$ and starlike about e_L. For any $(\Phi, \underline{\Phi}) \in K_N$, there exists a unique function $b \in H_N$ such that for all u in the Schwarz space $\mathcal{S}(\mathbb{R}^n)$, we have the Parseval identity*

$$\begin{aligned}(2\pi)^{-n} \int_{\mathbb{R}^n} b(\xi)\hat{u}(-\xi)d\xi &= \lim_{\alpha \longrightarrow 0+} \int_{\mathbb{R}^n} \Phi(\underline{x} + \alpha e_L)e_L u(\underline{x})d\underline{x} \\ &= \lim_{\varepsilon \to 0} \Big(\int_{|\underline{x}| \geqslant \varepsilon} \Phi(\underline{x})e_L u(\underline{x})d\underline{x} + \underline{\Phi}(\varepsilon e_L)u(\underline{0})\Big).\end{aligned} \tag{3.4}$$

Hence $b\overline{e_L}$ is the Fourier transform of the distributions of $(\Phi, \underline{\Phi})$. We write $b = \mathcal{F}(\Phi, \underline{\Phi})e_L$.

The Fourier transform $\mathcal{F}$ is linear and satisfies the following properties.

(i) *$\mathcal{F}$ is one-one from K_N to H_N. In other words, for any $b \in H_N$, there exists unique functions $(\Phi, \underline{\Phi}) \in K_N$ such that $b = \mathcal{F}(\Phi, \underline{\Phi})e_L$. Actually, if $b = b_+ + b_-$ and $b_\pm = b\chi_\pm \in H^\pm_N$, then*

$$(\Phi, \underline{\Phi}) = (\Phi_+, \underline{\Phi_+}) + (\Phi_-, \underline{\Phi_-}),$$

where $\Phi_\pm = \mathcal{G}_\pm(b_{\pm\overline{e_L}}) \in K^\pm_N$. We write $(\Phi, \underline{\Phi}) = \mathcal{G}(b\overline{e_L})$ and call $\mathcal{G}$ the inverse Fourier transform.

(ii) *If $0 < \nu < \mu \leqslant \pi/2 - \mu_N$ and $(\Phi, \underline{\Phi}) \in K(S_{N_\mu})$, then $b_+ \in H^+_\infty(N_\nu(\mathbb{C}^n))$, $b_- \in H^-_\infty(\overline{N}_\nu(\mathbb{C}^m))$ and*

$$\|b_\pm\|_\infty \leqslant c_\nu \|(\Phi, \underline{\Phi})\|_{K(S_{N_\mu})},$$

where the constant c_ν only depends on ν.

(iii) *If $0 < \nu < \mu \leqslant \pi/2 - \mu_N$, $b_+ \in H^+_\infty(N_\mu(\mathbb{C}^n))$ and $b_- \in H^-_\infty(\overline{N}_\mu(\mathbb{C}^n))$, then $(\Phi, \underline{\Phi}) \in K(S_{N_\nu})$ and*

$$\|(\Phi, \underline{\Phi})\|_{K(S_{N_\nu})} \leqslant c_\nu(\|b_+\|_\infty + \|b_-\|_\infty),$$

where the constant c_ν depends on ν.

(iv) $(\Phi, \underline{\Phi}) \in M_N$ *if and only if* $b \in \mathcal{A}_N$.

(v) *If* $(\Phi, \underline{\Phi}) \in K_N$, $(\Psi, \underline{\Psi}) \in M_N$, $b = \mathcal{F}(\Phi, \underline{\Phi})e_L$ *and* $f = \mathcal{F}(\Psi, \underline{\Psi})e_L$, *then*

$$bf = \mathcal{F}((\Phi, \underline{\Phi}) * (\Psi, \underline{\Psi}))e_L.$$

(vi) *The mapping* $(\Phi, \underline{\Phi}) \longmapsto b$ *is an algebra homomorphism from the convolution algebra* M_N *to the function algebra* $\mathcal{A}_N$.

(vii) *If* $(\Phi, \underline{\Phi})$, $(\Psi, \underline{\Psi}) \in K_N$, $b = \mathcal{F}(\Phi, \underline{\Phi})e_L$, $f = \mathcal{F}(\Psi, \underline{\Psi})e_L$ *and if* $f = pb$, *where* p *is a polynomial in* n *variables with values in* $\mathbb{C}_{(M)}$, *then*

$$\Psi = p\Big(-i\frac{\partial}{\partial x_1}, \ldots, -i\frac{\partial}{\partial x_n}\Big)\Phi.$$

(viii) *Let* $0 < \nu < \mu \leqslant \pi/2 - \mu_N$, $s > -n$. *If* b_+ *(or* b_-*) can be holomorphically extended to a bounded function for some* c_s *and all* $\zeta \in N_\mu(\mathbb{C}^n)$ *(correspondingly,* $\zeta \in \overline{N}_\mu(\mathbb{C}^n)$*), which satisfies* $|b_\pm(\zeta)| \leqslant c_s|\zeta|^s$, *then there exists* $c_{s,\nu}$ *such that for all* $x \in C^+_{N_\nu}$,

$$|\Phi(x)| \leqslant c_{s,\nu}|x|^{-n-s};$$

For all $y \in T_{N_\mu}$,

$$|\underline{\Phi}(y)| \leqslant c_{s,\nu}|y|^{-s}.$$

In particular, when $-n < s < 0$, *we have* $\lim_{y\to 0}\underline{\Phi}(y) = 0$.

Proof Without loss of generality, we only verify (i)-(viii) for the case $C^+_{N_\mu}$, $N_\mu(\mathbb{C}^n)$, K^+_N, M^+_N, $H^+_\infty(N_\mu(\mathbb{C}^n))$, H^+_N, $\mathcal{A}^+_N$ and $\mathcal{F}_+$. The case $C^-_{N_\mu}$, $\overline{N}_\mu(\mathbb{C}^n)$, K^-_N, M^-_N, $H^-_\infty(\overline{N}_\mu(\mathbb{C}^n))$, H^-_N, $\mathcal{A}^-_N$ and $\mathcal{F}_-$ can be dealt with similarly. In the proof, the constant c may depend on μ_N, μ and the dimension n, and may vary from line to line. We denote by c_ν a constant if the constant only depends on ν. Let $\Phi \in K(C^+_{N_\mu})$. Either form of the Parseval identity uniquely determines b on $\mathbb{R}^n$. Because $N_\mu(\mathbb{C}^n)$ is a connected open set, Parseval's identity also determines b on $N_\mu(\mathbb{C}^n)$.

We construct b as follows. For $\alpha > 0$, define $\Phi_\alpha(x) = \Phi(x + \alpha e_L)$, $x + \alpha e_L \in C^+_{N_\mu}$. We have

$$\begin{aligned}\|\Phi_\alpha\|_{K(C^+_{N_\mu})} &= \frac{1}{2}\sigma_n \sup\Big\{|x|^n|\Phi(x+\alpha e_L)| : x \in C^+_{N_\mu}\Big\}\\ &\leqslant \sup\Big\{|y|^n|\Phi(y)| : y \in C^+_{N_\nu} + \alpha e_L\Big\} \leqslant \|\Phi\|_{K(C^+_{N_\mu})}.\end{aligned}$$

For

$$\zeta \in \mathfrak{n}(\mathbb{C}^n) \subset N_\nu(\mathbb{C}^n) \subset N_\mu(\mathbb{C}^n), \ \nu < \mu,$$

define

$$b_\alpha(\zeta) = \int_\sigma \Phi_\alpha(x)\mathfrak{n}(x)e_+(-x, \zeta)dS_x,$$

where the surface σ is defined as

$$\sigma = \Big\{x \in \mathbb{R}^{n+1} : \ \langle x, \mathfrak{n}\rangle = -|x|\sin(\mu - \nu)\Big\}.$$

Note that the function in the integral is continuous and exponentially decreasing at infinity. As usual, $\mathfrak{n}(x)$ denotes the normal of σ and $\mathfrak{n}_L(x) > 0$. In fact, for $x \in \sigma$,

$$\begin{aligned}|e_+(-x, \zeta)| &\leqslant \frac{\sec(\mu_N + \mu)}{\sqrt{2}} e^{\langle x, \ \mathfrak{n}\rangle \mathrm{Re}|\zeta|_{\mathbb{C}/\mathfrak{n}_L}} \\ &\leqslant \frac{\sec(\mu_N + \mu)}{\sqrt{2}} e^{-|x||\xi|\sin\theta},\end{aligned}$$

where $\theta = \mu - \nu$.

By this fact and Cauchy's theorem for monogenic functions, noticing that Φ_α is right monogenic and $e_+(-x, \zeta)$ is left monogenic in x, we can see that the definition of $b_\alpha(\zeta)$ is independent of the choice of the surfaces σ. So $b_\alpha(\zeta)$ depends on $\zeta \in N_\mu(\mathbb{C}^n)$ holomorphically. Hence for all $\alpha, \beta > 0$,

$$\begin{aligned}b_\alpha(\zeta)e^{\alpha|\zeta|_{\mathbb{C}}} &= \int_\sigma \Phi(x + \alpha e_L)\mathfrak{n}(x)e_+(-(x + \alpha e_L), \ \zeta)dS_x \\ &= \int_\sigma \Phi(x + \beta e_L)\mathfrak{n}(x)e_+(-(x + \beta e_L), \ \zeta)dS_x \\ &= b_\beta(\zeta)e^{\beta|\zeta|_{\mathbb{C}}}.\end{aligned}$$

Then we define b as the holomorphic function on $N_\mu(\mathbb{C}^n)$ which satisfies the following condition:

$$b(z) = b_\alpha(\zeta)e^{\alpha|\zeta|_{\mathbb{C}}} \ \forall\, \alpha > 0.$$

We shall prove that for all $z \in N_\mu(\mathbb{C}^n)$,

$$|b_\alpha(\zeta)| \leqslant c_\nu \|\Phi\|_{K(C^+_{N_\mu})}, \tag{3.5}$$

where c_ν is independent of α and

$$(2\pi)^{-n}\int_{\mathbb{R}^n} b_\alpha(\xi)\hat{u}(-\xi)d\xi = \int_{\mathbb{R}^n} \Phi(\underline{x} + \alpha e_L)u(\underline{x})d\underline{x}. \tag{3.6}$$

As the estimate in (ii), the first form of Parseval's identity (3.4) can be deduced as a corollary.

We prove (3.5). Let

$$\zeta \in n(\mathbb{C}^n) \subset N_\nu(\mathbb{C}^n) \subset N_\mu(\mathbb{C}^n)$$

and $\theta = \mu - \nu$. Changing the surface in the integral by Cauchy's theorem, we can get

$$b_\alpha(\zeta) = \left(\int\limits_{\sigma(0,0,|\zeta|^{-1})} + \int\limits_{\tau(\theta,|\zeta|^{-1})} + \int\limits_{\sigma(\theta,|\zeta|^{-1},\infty)} \right) \Phi_\alpha(x)\mathfrak{n}(x)e_+(-x,\zeta)dS_x,$$

where

$$\sigma(\theta, r, R) = \left\{x \in \mathbb{R}^{n+1} : \langle x, \mathfrak{n}\rangle = |x| \sin\theta,\ r \leqslant |x| \leqslant R\right\},$$
$$\tau(\theta, R) = \left\{x \in \mathbb{R}^{n+1} : |x| = R, 0 \geqslant \langle x, \mathfrak{n}\rangle \geqslant -R\sin\theta\right\}.$$

We need the following estimates.

(i) For $R \leqslant |\zeta|^{-1}$,

$$\begin{aligned}
&\Big| \int\limits_{\sigma(0,0,R)} \Phi_\alpha(x)\mathfrak{n}(x)e_+(-x,\zeta)dS_x \Big| \qquad (3.7)\\
&\leqslant c\Big| \int\limits_{\sigma(0,0,R)} \Phi_\alpha(x)\mathfrak{n}(x)e(-x,\zeta)dS_x \Big|\\
&\leqslant c\Big| \int\limits_{\sigma(0,0,R)} \Phi_\alpha(x)\mathfrak{n}(x)[e(-x,\zeta)-1]dS_x \Big| + c\Big| \int\limits_{\langle x,\mathfrak{n}\rangle \geqslant 0, |x|=R} \Phi_\alpha(x)\mathfrak{n}(x)dS_x \Big|\\
&\leqslant c\|\Phi_\alpha\|_{K(C^+_{N_\mu})}\Big(\sup\left\{|\nabla_y e(-y,\zeta)| : y \in \sigma(0,0,R)\right\} \int_{\sigma(0,0,R)} |x|^{1-n} dS_x + 1\Big)\\
&\leqslant c\|\Phi_\alpha\|_{K(C^+_{N_\mu})}(R|\zeta|+1) \leqslant c\|\Phi_\alpha\|_{K(C^+_{N_\mu})}.
\end{aligned}$$

(ii) $R \geqslant |\zeta|^{-1}$,

$$\begin{aligned}
\Big| \int_{\tau(\theta,R)} \Phi_\alpha(x)\mathfrak{n}(x)e_+(-x,\zeta)dS_x \Big| &\leqslant c\|\Phi_\alpha\|_{K(C^+_{N_\mu})} R^{-n} \int_{\tau(\theta,R)} e^{\langle x,\mathfrak{n}\rangle \mathrm{Re}|\zeta|_{\mathbb{C}}/\mathfrak{n}_L} dS_x\\
&= c\|\Phi_\alpha\|_{K(C^+_{N_\mu})} \int_{\tau(\theta,1)} e^{\langle x,\mathfrak{n}\rangle R\ \mathrm{Re}|\zeta|_{\mathbb{C}}/\mathfrak{n}_L} dS_x \qquad (3.8)\\
&= c\|\Phi_\alpha\|_{K(C^+_{N_\mu})} \int_{-\theta}^{0} e^{R\ \mathrm{Re}|\zeta|_{\mathbb{C}} \sin\Phi/\mathfrak{n}_L} d\Phi
\end{aligned}$$

$$\leqslant \frac{c}{R|\zeta|}\|\Phi_\alpha\|_{K(C^+_{N_\mu})}$$
$$\leqslant c\|\Phi_\alpha\|_{K(C^+_{N_\mu})}.$$

(iii) $R \geqslant |\zeta|^{-1}$,

$$\begin{aligned}
&\left|\int_{\sigma(\theta,R,\infty)} \Phi_\alpha(x)\mathfrak{n}(x)e_+(-x,\zeta)dS_x\right| \\
&\leqslant c\|\Phi_\alpha\|_{K(C^+_{N_\mu})}\int_{\sigma(\theta,R,\infty)} |x|^{-n}e^{\langle x,\ \mathfrak{n}\rangle \mathrm{Re}|\zeta|_{\mathbb{C}}/\mathfrak{n}_L}dS_x \qquad (3.9)\\
&= c\|\Phi_\alpha\|_{K(C^+_{N_\mu})}\int_R^\infty s^{-1}e^{-s\sin\theta\mathrm{Re}|\zeta|_{\mathbb{C}}/\mathfrak{n}_L}dS_x \\
&\leqslant \frac{c_\nu}{R|\zeta|}\|\Phi_\alpha\|_{K(C^+_{N_\mu})}.
\end{aligned}$$

In the above estimates (i)–(iii), taking $R = |\zeta|^{-1}$, we can use the representation of b to obtain (3.5).

Now we prove (3.6). For $\zeta \in \mathbb{R}^n$, we define $b_{\alpha,N}$ as

$$b_{\alpha,N}(\xi) = \int_{|\underline{x}|\leqslant N} \Phi_\alpha(\underline{x})e_L e^{i\langle \underline{x},\ \xi\rangle}d\underline{x}.$$

Then it can be deduced from Parseval's identity that for $u \in \mathcal{S}(\mathbb{R}^n)$,

$$(2\pi)^{-n}\int_{\mathbb{R}^n} b_{\alpha,N}(\xi)\hat{u}(-\xi)d\xi = \int_{|\underline{x}|\leqslant N} \Phi(\underline{x}+\alpha e_L)u(\underline{x})d\underline{x}.$$

We will prove

$$\text{for all } \xi \in \mathbb{R}^n \text{ and } N > 0,\ |b_{\alpha,N}(\xi)| \leqslant c\|\Phi\|_{K(C^+_{N_\mu})}, \qquad (3.10)$$

$$\text{for any } \xi \in \mathbb{R}^n,\quad \text{when } N \to \infty,\ b_{\alpha,N}(\xi)\chi_+(\xi) \to b_\alpha(\xi) \qquad (3.11)$$

and

$$\text{for any } \xi \in \mathbb{R}^n,\quad \text{when } N \to \infty,\ b_{\alpha,N}(\xi)\chi_-(\xi) \to 0_\alpha(\xi). \qquad (3.12)$$

Then (3.6) can be deduced from the above estimates and the Lebesgue dominate convergence theorem.

To prove (3.10) and (3.11), letting $\mathfrak{n} = e_L$ in the definitions of σ, $\sigma(\theta,\tau,R)$ and $\tau(\theta,R)$, we use the estimates (3.7)–(3.9).

At first, when $|\xi|^{-1} \leqslant N$, we prove (3.10). Taking $0 < \theta < \mu$ and using Cauchy's theorem, we have

$$b_{\alpha,N}(\xi)\chi_+(\xi) = \left(\int_{\sigma(0,0,|\xi|^{-1})} + \int_{\tau(\theta,|\xi|^{-1})} + \int_{\sigma(\theta,|\xi|^{-1},N)} - \int_{\tau(\theta,N)} \right) \Phi_\alpha(x)\mathfrak{n}(x)e_+(-x,\xi)dS_x.$$

So we can apply (3.7)–(3.9) to prove that $b_{\alpha,N}(\xi)\chi_+(\xi)$ is uniformly bounded for ξ and N. On the other hand, similar to the proof of (3.8), we get

$$|b_{\alpha,N}(\xi)\chi_-(\xi)| \leqslant \frac{c}{N|\xi|}\|\Phi\|_{K(C^+_{N\mu})} \leqslant c\|\Phi_\alpha\|_{K(C^+_{N\mu})}.$$

When $|\xi|^{-1} \geqslant N$, to prove (3.10), only (3.7) is needed. To prove (3.11), fix $\xi \in \mathbb{R}^m$, $\xi \neq 0$, and apply Cauchy's theorem to write

$$b_\alpha(\xi) - b_{\alpha,N}(\xi)\chi_+(\xi) = \left(\int_{\tau(\theta,N)} + \int_{\sigma(\theta,N,\infty)} \right) \Phi_\alpha(x)\mathfrak{n}(x)e_+(-x,\xi)dS_x.$$

So, by (3.8) and (3.9), as $N \to 0$,

$$\left| b_\alpha(\xi) - b_{\alpha,N}(\xi)\chi_+(\xi) \right| \leqslant \frac{c}{N|\xi|}\|\Phi_\alpha\|_{K(C^+_{N\mu})} \to 0.$$

Moreover, (3.12) follows from the estimate given above.

As noted previously, the first version of Parseval's identity (3.4) holds. The next aim is to prove the second version of (3.4). Let $\varepsilon > 0$. Then

$$\begin{aligned}
(2\pi)^{-n}\int_{\mathbb{R}^n} b(\xi)\hat{u}(-\xi)d\xi &= \lim_{\alpha\to 0+}\left(\int_{|\underline{x}|\geqslant\varepsilon} \Phi_\alpha(\underline{x}+\alpha e_L)e_L u(\underline{x})d\underline{x} + \int_{|\underline{x}|\leqslant\varepsilon} \Phi_\alpha(\underline{x}+\alpha e_L)e_L u(\underline{0})d\underline{x} \right.\\
&\quad \left. + \int_{|\underline{x}|\leqslant\varepsilon} \Phi_\alpha(\underline{x}+\alpha e_L)e_L(u(\underline{x})-u(\underline{0}))d\underline{x} \right)\\
&= \int_{|\underline{x}|\geqslant\varepsilon} \Phi(\underline{x})e_L u(\underline{x})d\underline{x} + \underline{\Phi}(\varepsilon)u(\underline{0})\\
&\quad + \lim_{\alpha\to 0+}\left(\int_{|\underline{x}|\leqslant\varepsilon} \Phi_\alpha(\underline{x}+\alpha e_L)e_L(u(\underline{x})-u(\underline{0}))d\underline{x} \right),
\end{aligned}$$

where in the second integral we have used Cauchy's theorem.

Now when $u \in \mathcal{S}(\mathbb{R}^n)$,

$$\begin{aligned}&\overline{\lim_{\varepsilon\to 0}}\lim_{\alpha\to 0+}\left(\int\limits_{|\underline{x}|\leqslant\varepsilon}\left|\Phi(\underline{x}+\alpha e_L)e_L(u(\underline{x})-u(\underline{0}))\right|d\underline{x}\right)\\&\leqslant\overline{\lim_{\varepsilon\to 0}}\lim_{\alpha\to 0+}\left(C\int\limits_{|\underline{x}|\leqslant\varepsilon}|\underline{x}+\alpha e_L|^{-n}|u(\underline{x})-u(\underline{0})|d\underline{x}\right)\\&\leqslant\overline{\lim_{\varepsilon\to 0}}\left(C\int\limits_{|\underline{x}|\leqslant\varepsilon}|\underline{x}|^{-n}|u(\underline{x})-u(\underline{0})|d\underline{x}\right)=0,\end{aligned}$$

so

$$(2\pi)^{-n}\int_{\mathbb{R}^n}b(\xi)\hat{u}(-\xi)d\xi=\lim_{\varepsilon 0}\left(\int\limits_{|\underline{x}|\geqslant\varepsilon}\Phi(\underline{x})e_Lu(\underline{x})d\underline{x}+\underline{\Phi}(\varepsilon)u(\underline{0})\right).$$

This gives (ii).

We prove (i) and (iii). It is easy to verify that $\mathcal{F}_+$ is one-one. By constructing the inverse Fourier transform $\mathcal{G}_+$, we prove the mapping is onto H^+_N.

Consider the function $b\in H^+_\infty(N_\mu(\mathbb{C}^n))$. For $\mathfrak{n}\in N_\mu$ and

$$x=\underline{x}+x_Le_L\in C^+_{\mathfrak{n}}\subset C^+_{N_\mu},$$

define

$$\begin{aligned}\Phi_n(x)&=(2\pi)^{-n}\int\limits_{\mathfrak{n}(\mathbb{C}^n)}b(\zeta)e(x,\zeta)d\zeta_1\wedge d\zeta_2\wedge\cdots\wedge d\zeta_n\overline{e_L}\\&=(2\pi)^{-n}\int\limits_{\mathfrak{n}(\mathbb{C}^n)}b(\zeta)e_+(x,\zeta)d\zeta_1\wedge d\zeta_2\wedge\cdots\wedge d\zeta_n\overline{e_L},\end{aligned}$$

where in the last equality we have used the facts that $e(x,\zeta)=e_+(x,\zeta)+e_-(x,\zeta)$ and $\langle b(\zeta),e_-(x,\zeta)\rangle=0$ for $b\in H^+_\infty(N_\mu(\mathbb{C}^n))$. On the surface $\mathfrak{n}(\mathbb{C}^n)$, the function in the integral is exponentially decreasing at infinity. In fact, when $\zeta\in\mathfrak{n}(\mathbb{C}^n)$, then

$$|e^{i\langle\mathbf{x},\zeta\rangle}e^{-x_L|\zeta|_{\mathbb{C}}}|\leqslant ce^{-\langle x,\mathfrak{n}\rangle\mathrm{Re}|\zeta|_{\mathbb{C}}/\mathfrak{n}_L}$$

and $\langle x,\mathfrak{n}\rangle>0$. Moreover, $e(x,\zeta)\overline{e_L}$ is right monogenic and Φ_n is a right monogenic function on $C^+_{\mathfrak{n}}$ satisfying

$$|\Phi_{\mathfrak{n}}(x)|\leqslant\frac{c\|b\|_\infty}{\langle x,\mathfrak{n}\rangle^n},$$

where c only depends on μ_N and μ.

Moreover the integrand depends holomorphically on the single complex variable $z = \langle \zeta, \underline{\mathfrak{n}} \rangle$. So by the starlike nature of N_μ and Cauchy's theorem in the z-plane, we find that for all $x \in C_{\mathfrak{n}}^+$ satisfying $x_L > 0$, $\Phi_{\mathfrak{n}}(x) = \Phi_{e_L}(x)$. Hence there exists unique right monogenic function Φ on $C_{N_\mu}^+$ which coincides with $\Phi_{\mathfrak{n}}(x)$ on $C_{\mathfrak{n}}^+$. We call Φ the Fourier transform of $b\overline{e_L}$ and denote $\Phi = \mathcal{G}_+(b\overline{e_L})$. The above estimates for $\Phi_{\mathfrak{n}}$ indicate that for all $\nu < \mu$, $\Phi \in K(C_{N_\nu}^+)$ and

$$\|\Phi\|_{K(C_{N_\nu}^+)} \leqslant c_\nu \|b\|_\infty.$$

For the special case $x_L = 0$ and all $\zeta \in N_\mu(\mathbb{C}^n)$,

$$|b(\zeta)| \leqslant \frac{c}{1 + |\zeta|^{n+1}}.$$

Then by Cauchy's theorem, we can change the surface of integration to obtain

$$\mathcal{G}_+(b\overline{e_L})(\underline{x}) = \Phi(\underline{x}) = (2\pi)^{-n} \int_{\mathbb{R}^n} b(\xi) e^{i\langle \underline{x},\ \xi \rangle} d\xi \overline{e_L} = \check{b}(\underline{x})\overline{e_L},$$

which is the usual inverse Fourier transform of $b\overline{e_L}$.

We prove that b and $\Phi = \mathcal{G}_+(b\overline{e_L})$ satisfy Parseval's identity (3.4). Hence we can deduce that $\mathcal{G}_+$ is the inverse of the Fourier transform $\mathcal{F}_+$, and complete the proofs of (i) and (iii).

For $\alpha > 0$, let $b_\alpha(\zeta) = b(\zeta) e^{-\alpha|\zeta|_{\mathbb{C}}}$. Then for $\underline{x} \in \mathbb{R}^n$,

$$\Phi(\underline{x} + \alpha e_L) = \mathcal{G}_+(b\overline{e_L})(\underline{x} + \alpha e_L) = \mathcal{G}_+(b_\alpha \overline{e_L})(\underline{x}) = (b_\alpha)\check{}(\underline{x})\overline{e_L}.$$

By the usual Parseval's identity, we can obtain

$$(2\pi)^{-n} \int_{\mathbb{R}^n} b_\alpha(\xi)\hat{u}(-\xi) d\xi = \int_{\mathbb{R}^n} \Phi(\underline{x} + \alpha e_L) e_L u(\underline{x}) d\underline{x},$$

and for all $u \in \mathcal{S}(\mathbb{R}^n)$,

$$(2\pi)^{-n} \int_{\mathbb{R}^n} b(\xi)\hat{u}(-\xi) d\xi = \lim_{\alpha \to 0+} \int_{\mathbb{R}^n} \Phi(\underline{x} + \alpha e_L) e_L u(\underline{x}) d\underline{x}.$$

Now we prove (iv). Take $\Phi \in K(C_{N_\mu}^+)$. Then Φ is left monogenic (and is also right monogenic) if and only if for all $x \in C_{N_\mu}^+$,

$$\underline{D} e_L \Phi(x) = (\Phi e_L)\underline{D}(x),$$

where the both sides all equal to $-\partial \Phi / \partial x_L(x)$.

Let $b\overline{e_L} = \mathcal{F}_+(\Phi)$ and define b_α as above. Using twice Parseval's identity for b_α, we can see that for all $u \in \mathcal{S}(\mathbb{R}^n)$,

$$(2\pi)^{-n}\int_{\mathbb{R}^n} \xi e_L b_\alpha(\xi)\hat{u}(-\xi)d\xi = -i\int_{\mathbb{R}^n}(\underline{D}e_L\Phi)(\underline{x}+\alpha e_L)e_L u(\underline{x})d\underline{x}$$

and

$$(2\pi)^{-n}\int_{\mathbb{R}^n} b_\alpha(\xi)\xi e_L\hat{u}(-\xi)d\xi = -i\int_{\mathbb{R}^n}(\Phi e_L\underline{D})(\underline{x}+\alpha e_L)e_L u(\underline{x})d\underline{x}.$$

Hence $\Phi \in M(C^+_{N_\mu})$ if and only if for all $u \in \mathcal{S}(\mathbb{R}^n)$, $\underline{D}e_L\Phi(x) = (\Phi e_L)\underline{D}(x)$. So the above equality holds if and only if

$$\underline{D}e_L\Phi(\underline{x}+\alpha e_L) = (\Phi e_L)\underline{D}(\underline{x}+\alpha e_L) \text{ for all } x \in \mathbb{R}^n \setminus \{0\}.$$

The above equality is equivalent to $\xi e_L b_\alpha(\xi) = b_\alpha(\xi)\xi e_L$. This equation is equivalent to $\zeta e_L b(\zeta) = b(\zeta)\zeta e_L$ for all $z \in N_\mu(\mathbb{C}^n)$. This proves (iv).

The remaining part can be proved in a similar way with the estimates in (viii) requiring a modification of the proof of (iii). □

Denote by $\mathcal{G}_- : H^-_N \to K^-_N$ the inverse of $\mathcal{F}_-$. We call $\mathcal{F}_-$ the Fourier transform and $\mathcal{G}_-$ the inverse Fourier transform.

Remark 3.3.1 When $N = \overline{N}$, $b_+ \in H^+_\infty(N_\mu(\mathbb{C}^n))$ and $b_- \in H^-_\infty(\overline{N}_\mu(\mathbb{C}^n))$ if and only if

$$b \in H_\infty(N_\mu(\mathbb{C}^n)).$$

Let $B \in H^\infty(S^0_\mu(\mathbb{C}))$, where $0 < \mu < \pi/2$. We have seen that B is associated with the function $b \in H^\infty(S^0_\mu(\mathbb{C}^n))$ defined as

$$b(\zeta) = B(i\zeta e_L) = B(|\zeta|_{\mathbb{C}})\chi_+(\zeta) + B(-|\zeta|_{\mathbb{C}})\chi_-(\zeta).$$

In fact,

$$b \in \mathcal{A}(S^0_\mu(\mathbb{C}^n)) = \left\{b \in H^\infty(S^0_\mu(\mathbb{C}^n)) :\ \zeta e_L b(\zeta) = b(\zeta)\zeta e_L \text{ for all } \zeta\right\},$$

and the mapping $B \longmapsto b$ is a one-one algebra homomorphism from $H^\infty(S^0_\mu(\mathbb{C}))$ to $\mathcal{A}(S^0_\mu(\mathbb{C}^n))$. Recall that

$$C^0_{\mu,+}(\mathbb{C}) = \left\{Z = X + iY \in \mathbb{C} :\ Z \neq 0, Y > -|X|\tan\mu\right\},$$

$$C^0_{\mu,-}(\mathbb{C}) = -C^0_{\mu,+}(\mathbb{C}),$$

$$S^0_{\mu,+}(\mathbb{C}) = \left\{\lambda \in \mathbb{C} :\ \lambda \neq 0, |\arg(\lambda)| < \mu\right\},$$

$$S^0_{\mu,-}(\mathbb{C}) = -S^0_{\mu,+}(\mathbb{C}),$$

$$C_{\mu,+}^{0}=\Big\{x=\underline{x}+x_L e_L\in\mathbb{R}^{n+1}:\ x_L>-|\underline{x}|\tan\mu\Big\},$$

$$C_{\mu,-}^{0}=-C_{\mu,+}^{0},\quad S_{\mu}^{0}=C_{\mu,+}^{0}\cap C_{\mu,-}^{0},$$

$$T_{\mu}^{0}=\Big\{y=\underline{y}+y_L e_L\in\mathbb{R}^{n+1}:y_L>|\underline{y}|\cot\mu\Big\},$$

$$S_{\mu}^{0}(\mathbb{C}^n)=\Big\{\zeta=\xi+i\eta\in\mathbb{C}^n:\ |\zeta|_{\mathbb{C}}^2\notin(-\infty,0]\text{ and }|\eta|<\mathrm{Re}(|\zeta|_{\mathbb{C}})\tan\mu\Big\}.$$

We find the inverse Fourier transform of b in terms of the inverse Fourier transform of B. We first assume that $B\in H^{\infty}(S_{\mu,+}^{0}(\mathbb{C}))$. In this case, the inverse Fourier transform of B, $\Phi=\mathcal{G}(B)$, is a complex-valued holomorphic function defined on $C_{\mu,+}^{0}(\mathbb{C})$. Specially, when $\mathrm{Im}(Z)>0$,

$$\Phi(Z)=\frac{1}{2\pi}\int_0^{\infty}B(r)e^{irZ}dr.$$

When $x_L>0$,

$$\begin{aligned}
\Phi(x)&=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}B(|\xi|)e_+(x,\xi)d\xi\overline{e_L}\\
&=\frac{1}{2(2\pi)^n}\int_{\mathbb{R}^n}(\overline{e_L}+\frac{i\xi}{|\xi|})B(|\xi|)e^{-x_L|\xi|}e^{i\langle\underline{x},\ \xi\rangle}d\xi\\
&=\frac{1}{2(2\pi)^n}\int_{S^{m-1}}(\overline{e_L}+i\tau)\int_0^{\infty}B(r)e^{-x_Lr}e^{i\langle\underline{x},\ \tau\rangle r}r^{n-1}drdS_\tau \qquad (3.13)\\
&=\frac{1}{2(2\pi i)^{n-1}}\int_{\mathbb{S}^{n-1}}(\overline{e_L}+i\tau)\Phi^{(n-1)}(\langle\underline{x},\ \tau\rangle+ix_L)dS_\tau\\
&=\frac{1}{2(2\pi i)^{n-1}}\int_{\mathbb{S}^{n-1}}(\overline{e_L}+i\langle\underline{x},\ \tau\rangle\underline{x}|\underline{x}|^{-2})\Phi^{(n-1)}(\langle\underline{x},\ \tau\rangle+ix_L)dS_\tau\\
&=\frac{\sigma_{n-2}}{2(2\pi i)^{n-1}}\int_{-1}^{1}(1-t^2)^{(n-3)/2}\Big(\overline{e_L}+\frac{it\underline{x}}{|\underline{x}|}\Big)\Phi^{(n-1)}(|\underline{x}|t+ix_L)dt,
\end{aligned}$$

where $\Phi^{(n-1)}$ is the $(n-2)$th derivative of Φ. On $C_{\mu,+}^{0}$, Φ extends to a left and right monogenic function. For all $\nu<\mu$, this function belongs to $M(C_{\nu,+}^{0})$.

For $B\in H^{\infty}(S_{\mu,-}^{0}(\mathbb{C}))$, $\Phi=\mathcal{G}(B)$ and

$$b(\zeta)=B(i\zeta e_L)=B(-|\zeta|_{\mathbb{C}})\chi_-(\zeta).$$

Then $b\in H_{\infty}^{-}(S_{\mu}^{0}(\mathbb{C}^n))$. Hence we can construct $\Phi=\mathcal{G}_-(b\overline{e_L})$. We see that if $x_L<0$,

$$\begin{aligned}\Phi(x) &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^m} B(-|\xi|)e_-(x,\xi)d\xi\overline{e_L}\\ &= \frac{1}{2(2\pi)^n}\int_{\mathbb{R}^n}(\overline{e_L}-\frac{i\xi}{|\xi|})B(-|\xi|)e^{-x_L|\xi|}e^{i\langle \underline{x},\ \xi\rangle}d\xi\\ &= \frac{1}{2(2\pi)^n}\int_{S^{n-1}}(\overline{e_L}-i\tau)\int_0^{+\infty}B(-r)e^{x_Lr}e^{i\langle \underline{x},\ \tau\rangle r}r^{n-1}drdS_\tau\\ &= \frac{(-1)^{n-1}}{2(2\pi)^n}\int_{S^{n-1}}(\overline{e_L}+i\tau)\int_{-\infty}^0 B(-r)e^{-x_Lr}e^{i\langle \underline{x},\ \tau\rangle r}r^{n-1}drdS_\tau\\ &= \frac{1}{2(-2\pi i)^{n-1}}\int_{S^{n-1}}(\overline{e_L}+i\tau)\Phi^{(n-1)}(\langle \underline{x},\ \tau\rangle + ix_L)dS_\tau\\ &= \frac{\sigma_{n-2}}{2(-2\pi i)^{n-1}}\int_{-1}^1(1-t^2)^{(n-3)/2}\left[\overline{e_L}+\frac{it\underline{x}}{|\underline{x}|}\right]\Phi^{(n-1)}(|\underline{x}|t+ix_L)dt.\end{aligned}$$

When $B\in H_\infty(S^0_\mu(\mathbb{C}))$, write $B = B_+ + B_-$, where $B_+ = B\chi_{\mathrm{Re}>0}\in H_\infty(S^0_{\mu,+}(\mathbb{C}))$ and $B_- = B\chi_{\mathrm{Re}<0}\in H_\infty(S^0_{\mu,-}(\mathbb{C}))$. Then $b = b_+ + b_-$, where $b_\pm$ is the function with respect to $B_\pm$. We can use this decomposition to relate the inverse Fourier transform $\mathcal{G}(b\overline{e_L}) = (\Phi,\ \underline{\Phi})$ of $b\overline{e_L}$ to the inverse Fourier transform $\mathcal{G}(B) = (\Phi,\ \Phi_1)$ of B.

In the end of this section, we give two examples to make the reader to understand the relation between $(\Phi(z),\ \Phi_1(z))$ and (b, B), and between $(\Phi(z),\ \Phi_1(z))$ and $(\Phi(x), \underline{\Phi}(y))$, see [1] for more examples.

Example 3.3.1 As usual,

$$E(x) = \frac{1}{\sigma_n}\frac{\overline{x}}{|x|^{n+1}}.$$

(i) $(\Phi(z),\ \Phi_1(z)) = (0, 1)$, $B(\lambda) = 1$, $b(\zeta) = 1$;
(ii) $(\Phi(z),\ \Phi_1(z)) = (\frac{i}{2\pi z}, \frac{1}{2})$, $B(\lambda) = \chi_{\mathrm{Re}>0}$, $b(\zeta) = \chi_+(\zeta)$;
(iii) $(\Phi(z),\ \Phi_1(z)) = (\frac{i}{2\pi z}, \frac{1}{2})$, $B(\lambda) = \chi_{\mathrm{Re}<0}$, $b(\zeta) = \chi_-(\zeta)$;
(iv) $(\Phi(z),\ \Phi_1(z)) = (\frac{i}{\pi z}, 0)$, $B(\lambda) = \mathrm{sgn}(\lambda)$, $b(\zeta) = \frac{i\zeta e_L}{|\zeta|_{\mathbb{C}}}$;

The above example describes the relation between the function pair $(\Phi(z),\ \Phi_1(z))$ and the function pair $(\Phi(x), \underline{\Phi}(y))$.

Example 3.3.2 (i) Let $(\Phi(z),\ \Phi_1(z)) = \left(\frac{1}{(z+it)}, -i\pi + \log(\frac{z+it}{z-it})\right)(t>0)$. Then

$$(\Phi(x), \underline{\Phi}(y)) = (k(x+te_L),\ \underline{\Phi}(y)),\ \lim_{y\to 0}\underline{\Phi}(y) = 0.$$

(ii) Let $(\Phi(z),\ \Phi_1(z)) = \frac{t}{2\pi}\left(\frac{-1}{(z+it)^2},\ \frac{2z}{z^2+t^2}\right)(t>0)$. Then

$$(\Phi(x), \underline{\Phi}(y)) = \Big(-t\frac{\partial k}{\partial t}(x+te_L),\ \underline{\phi}(y)\Big), \lim_{y\to 0}\underline{\Phi}(y) = 0.$$

(iii) Let $(\Phi(z),\ \Phi_1(z)) = \Gamma(1+is)\Big(\frac{i}{2\pi}e^{-\pi s/2}z^{-1-is},\ (\pi s)^{-1}\sinh(\pi s/2)z^{-is}\Big)$. Then

$$(\Phi(x), \underline{\Phi}(y)) = \left(\frac{-1}{\Gamma(1-is)}\int_0^\infty t^{-is}\frac{\partial k}{\partial t}(x+te_L)dt,\ \underline{\Phi}_s(y)\right),$$

where the function $\underline{\Phi}_s$ is represented as

$$\underline{\Phi}_s(r\mathfrak{n}) = \frac{r^{-is}}{\Gamma(1-is)}\int_0^\infty t^{is-1}F(n,\mathfrak{n}_L,\tau)d\tau\overline{e_L}\mathfrak{n},$$

where $r>0$, $|\mathfrak{n}|=1$, and F is real-valued and satisfies

$$|F(n,\mathfrak{n}_L,t)| \leqslant c(n,\mathfrak{n}_L)\frac{t^n}{(1+t)^{n+1}}.$$

In particular, if $\mathfrak{n}=e_L$, then

$$\underline{\Phi}_s(re_L) = \frac{\sigma_{n-1}r^{-is}}{\Gamma(1-is)}\int_0^\infty \frac{t^{n+is-1}}{(1+t^2)^{(n+1)/2}}dt,\ r>0.$$

(To prove this, first show that the function $\underline{\Phi}$ in the preceding row has the form $\underline{\Phi}(r\mathfrak{n}) = F(n,\mathfrak{n}_L,r/t)\overline{e_L}\mathfrak{n}$.)

The functions Φ_1 and $\underline{\Phi}$ are really only of interest near zero, and when they tend to 0, these functions do not enter into Parseval's identity or the convolution formulae. It has been proved in Chap. 1 that if $|B(\lambda)| \leqslant c_s|\lambda|^s$ holds for all $\lambda \in S^0_{\mu,+}(\mathbb{C})$ and some $s<0$, then when $z\to 0$ $(z\in S_{\nu,+}(\mathbb{C}),\ \nu<\mu)$, $\Phi_1(z)\to 0$, and for all $\zeta \in S^0_\mu(\mathbb{C})$, $|b(\zeta)| \leqslant c_s|\zeta|^s$. Hence by (viii) of Theorem 3.3.1, we conclude that $y\to 0$ $(y\in T^0_\nu,\ \nu<\mu)$, $\underline{\Phi}(y)\to 0$. Therefore for $|B(\lambda)|\leqslant c_s|\lambda|^s,\ s<0$, there is no need to find Φ_1 and $\underline{\Phi}$.

Let us turn our attention to the function $B = B_+ = B\chi_{\mathrm{Re}>0}$, and substitute the corresponding values of Φ and Φ in (3.13). Using the fact that

$$(\overline{e_L}+i\tau)(a+ib)^k = (\overline{e_L}+i\tau)(a-be_L\tau)^k$$

for $\tau\in\mathbb{S}^{n-1}$ and $a,b\in\mathbb{R}$, we obtain

$$\begin{aligned}\frac{\overline{x}}{\sigma_n|x|^{n+1}} &= \frac{1}{2(2\pi i)^{n-1}}\int_{\mathbb{S}^{n-1}}(\overline{e_L}+i\tau)\frac{i}{2\pi}\frac{(-1)^{n-1}(n-1)!}{(\langle \underline{x},\ \tau\rangle + ix_L)^n}dS_\tau \\ &= \frac{(n-1)!}{2}\left(\frac{i}{2\pi}\right)^n\int_{\mathbb{S}^{n-1}}(\overline{e_L}+i\tau)(\langle\underline{x},\ \tau\rangle - x_Le_L\tau)^{-n}dS_\tau,\end{aligned}$$

where $x_L > 0$. If we take the real part of the right hand side, the above result is the plane wave decomposition of the Cauchy kernel obtained by Sommen in [5]. For the function $B = \chi_{\mathrm{Re}<0}$, we obtain

$$\frac{\overline{x}}{\sigma_n |x|^{n+1}} = \frac{-(n-1)!}{2}\Big(\frac{-i}{2\pi}\Big)^n \int_{\mathbb{S}^{n-1}} (\overline{e_L} + i\tau)(\langle \underline{x},\ \tau\rangle - x_L e_L \tau)^{-n} dS_\tau,$$

where $x_L > 0$. This coincides with Sommen's formula, see Ryan [6] for the details.

3.4 Möbius Covariance of Iterated Dirac Operators

In this section, we deduce the fundamental solutions of Dirac operators

$$\underline{D}^l = \sum_{i=1}^n e_i \frac{\partial}{\partial x_i}$$

and

$$D^l = \Big(\frac{\partial}{\partial x_0} + \underline{D}\Big)^l$$

in the setting of Clifford algebras. In [2], Peeter and Qian obtained the Möbius covariance of iterated Dirac operators.

For $\alpha > 0$, define the operator $\underline{D}^{-\alpha}$ as

$$\underline{D}^{-\alpha} f(\underline{x}) = c_n \int_{\mathbb{R}^n} e^{-i\langle \underline{x},\ \underline{\xi}\rangle} (i\underline{\xi})^{-\alpha} \hat{f}(\underline{\xi}) d\underline{\xi},$$

where $(i\underline{\xi})^{-\alpha}$ is defined by

$$(i\underline{\xi})^{-\alpha} = |\underline{\xi}|^{-\alpha}\chi_+(\underline{\xi}) + (-|\underline{\xi}|)^{-\alpha}\chi_-(\underline{\xi})$$

and

$$\chi_\pm(\underline{\xi}) = \frac{1}{2}\Big(1 \pm i\underline{\xi}/|\underline{\xi}|\Big).$$

Hence if $\alpha = l$ is a positive integer, we have

$$(i\underline{\xi})^{-l} = \begin{cases} 1/|\underline{\xi}|^l, & \text{if } l \text{ is even,} \\ i\underline{\xi}/|\underline{\xi}|^{l+1}, & \text{if } l \text{ odd .} \end{cases}$$

Therefore

$$\underline{D}^{-\alpha} f(\underline{x}) = \frac{c_n}{2}\Big[\int_{\mathbb{R}^n} e^{-i\langle \underline{x},\,\underline{\xi}\rangle}|\underline{\xi}|^{-\alpha}\hat{f}(\underline{\xi})d\underline{\xi} + \underline{D}\int_{\mathbb{R}^n} e^{-i\langle \underline{x},\,\underline{\xi}\rangle}|\underline{\xi}|^{-\alpha-1}\hat{f}(\underline{\xi})d\underline{\xi}$$
$$+\int_{\mathbb{R}^n} e^{-i\langle \underline{x},\,\underline{\xi}\rangle}(-|\underline{\xi}|)^{-\alpha}\hat{f}(\underline{\xi})d\underline{\xi} + \underline{D}\int_{\mathbb{R}^n} e^{-i\langle \underline{x},\,\underline{\xi}\rangle}(-|\underline{\xi}|)^{-\alpha-1}\hat{f}(\underline{\xi})d\underline{\xi}\Big].$$

If $0 < \alpha, \alpha + 1 < n$, by the formula

$$\left(\frac{1}{|\underline{\xi}|^{\beta}}\right)^{\vee} = c_{n,\beta}\frac{1}{|\underline{x}|^{n-\beta}},$$

we can deduce

$$\underline{D}^{-\alpha} f(\underline{x}) = K_{n,\alpha} * f(\underline{x}),$$

where

$$K_{n,\alpha}(\underline{x}) = c_{n,\alpha}(1 + e^{-i\alpha\pi})\frac{1}{|\underline{x}|^{n-\alpha}} + d_{n,\alpha}(1 - e^{-i\alpha\pi})\underline{D}\left(\frac{1}{|\underline{x}|^{n-\alpha-1}}\right).$$

For general $\alpha > 0$, by the same method, we can get

$$K_{n,\alpha}(\underline{x}) = c_{n,\alpha}(1 + e^{-i\alpha\pi})G_{n,\alpha}(\underline{x}) + d_{n,\alpha}(1 - e^{-i\alpha\pi})\underline{D}G_{n,\alpha+1}(\underline{x}),$$

where $G_{n,\beta}$ is the fundamental solution of the operator $|\underline{D}|^{\beta}$ with the symbol $|\underline{\xi}|^{\beta}$. Then for odd n,

$$K_{n,l}(\underline{x}) = \begin{cases} c_{n,l}\frac{\underline{x}}{|\underline{x}|^{n-l+1}}, & \text{if } l \text{ is odd;} \\ c_{n,l}\frac{1}{|\underline{x}|^{n-l}}, & \text{if } l \text{ is even.} \end{cases} \tag{3.14}$$

For the even n,

$$K_{n,l}(\underline{x}) = \begin{cases} c_{n,l}\frac{\underline{x}}{|\underline{x}|^{n-l+1}}, & \text{if } l \text{ is odd and } l < n; \\ c_{n,l}\frac{1}{|\underline{x}|^{n-l}}, & \text{if } l \text{ is odd and } l < n; \\ (c_{n,l}\log|\underline{x}| + d_{n,l})\frac{\underline{x}}{|\underline{x}|^{n-l+1}}, & \text{if } l \text{ odd and } l > n; \\ (c_{n,l}\log|\underline{x}| + d_{n,l})\frac{1}{|\underline{x}|^{n-l}}, & \text{if } l \text{ is even and } l > n; \end{cases} \tag{3.15}$$

Now we consider the fundamental solutions of the operators $D^l, l \in \mathbb{Z}_+$. Write $D_0 = \frac{\partial}{\partial x_0}$. Then

$$D^{-l} = (D_0 + \underline{D})^{-l} = (D_0 - \underline{D})^l(D_0^2 - \underline{D}^2)^{-l}.$$

By the Fourier transform, the symbol of $(D_0^2 - \underline{D}^2)^{-l}$ is $|\underline{\xi}|^{-2l}$. For $0 < 2l < n + 1$, the inverse Fourier transform of $|\underline{\xi}|^{-2l}$ is $c_{n,l}|x|^{-(n+1-2l)}$. This indicates that the kernel of the operator D^{-l} is

$$L_{n,l}(x) = c_{n,l}(D_0 - \underline{D})^l \left(\frac{1}{|x|^{n+1-2l}} \right),\ 0 < 2l < n+1.$$

A direct computation gives

$$L_{n,l}(x) = c_{n,l} \frac{x_0^{l-1}\overline{x}}{|x|^{n+1}},\ l \in \mathbb{Z}_+. \tag{3.16}$$

For any $x = x_0 + x_1 e_1 + \cdots + x_n e_n$, we write $x = x_0 + \underline{x}$ and $\underline{x} = x_1 e_1 + \cdots + x_n e_n \in \mathbb{R}^n$. We define two elementary operations

$$(e_{i_1} \cdots e_{i_l})^* =: e_{i_l} \cdots e_{i_1},$$

$$(e_{i_1} \cdots e_{i_l})' = (-1)^l (e_{i_1} \cdots e_{i_l}).$$

Let Γ_n be the multiplicative group of all elements in the Clifford algebra which can be written as products of non-zero vectors in $\mathbb{R}^n$. For any $a, b \in \Gamma_n \cup \{0\}$, $\overline{a}a = |a|^2$ and $|ab| = |a| \cdot |b|$. If $a \in \Gamma_n$, then $a = \prod_{j=1}^{M(a)} a_j$, where $a_j \in \mathbb{R}^n$. Generally speaking, such a representation and $M(a)$ are not unique. Denote by $m(a)$ the minimum of $M(a)$ over all such representations. If $a \in \mathbb{R} \setminus \{0\}$, we set $m(a) = 0$. Hence, $m(\underline{x}) = 1$, and for $a \in \Gamma_n$, $aa^* = a^*a = (-1)^{m(a)}|a|^2$. We call a group to be a Möbius group if this group consists of orientation preserving transforms acting in the Euclidean spaces. All Möbius transforms from $\mathbb{R}^n \cup \{\infty\}$ to $\mathbb{R}^n \cup \{\infty\}$ can be represented as

$$\phi(\underline{x}) = (a\underline{x} + b)(c\underline{x} + d)^{-1},$$

where $a, b, c, d \in \Gamma \cup \{0\}$ and

$$ad^* - bc^* \in \mathbb{R} \setminus \{0\},\ a^*c, cd^*, d^*b, ba^* \in \mathbb{R}^n.$$

In addition, under 2×2 block matrix multiplication, the identification between the ϕ's and Clifford matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ gives a homomorphism. For simplicity, we take $ad^* - bc^* = 1$ to normalize the Möbius transform. We consider the following multipliers:

$$T_l(\phi) f(\underline{x}) = J_{l,\phi} \cdot f(\phi(\underline{x})),$$

where for $l \in \mathbb{Z}$,

$$J_{l,\phi}(\underline{x}) = \begin{cases} \frac{(c\underline{x}+d)^*}{|c\underline{x}+d|^{n-l+1}},\ l \text{ is odd}, \\ \frac{1}{|c\underline{x}+d|^{n-l}},\ l \text{ is even}. \end{cases} \tag{3.17}$$

We will use the closed relation between $K_{n,l}$, $\underline{D}^l$ and the conformal weights $J_{l,\phi}$ to prove the following result.

Theorem 3.4.1 *For $l \in \mathbb{Z}_+$, the iterated Dirac operator $\underline{D}^l$ intertwines the representations T_l, T_{-l} of the Möbius transform group, that is, for $c \neq 0$,*

$$\underline{D}^l(T_l f) = \begin{cases} (-1)^{m(c)+1} T_{-l}(\underline{D}^l f), l \text{ is odd}; \\ T_{-l}(\underline{D}^l f), l \text{ is even}. \end{cases} \tag{3.18}$$

If $c = 0$, then $d \neq 0$ and the factor $(-1)^{m(c)+1}$ in the last formula should be replaced by $(-1)^{m(d)}$.

Proof We only prove the case $c \neq 0$. The case $c = 0$ can be dealt with similarly and is easier, so we omit the proof. We only need to prove

$$(T_l f) = \begin{cases} (-1)^{m(c)+1} \underline{D}^{-l} T_{-l}(\underline{D}^l f), l \text{ is odd}; \\ \underline{D}^{-l} T_{-l}(\underline{D}^l f), l \text{ is even}. \end{cases} \tag{3.19}$$

At first, we assume that n is odd or n is even and $l < n$. Denote by ψ the inverse of ϕ. If $\underline{y} = \phi(\underline{x}) = (a\underline{x} + b)(c\underline{x} + d)^{-1} \in \mathbb{R}^n$, then $\underline{y}(c\underline{x} + d) = a\underline{x} + b$. Hence we have $\underline{x} = \psi(\underline{y}) = (\underline{y}c - a)^{-1}(-\underline{y}d + b)$. Let $z = z(\underline{y}) = \underline{y} - a$ and $A = b - ac^{-1}d$. We can get

$$\underline{x} = z^{-1}A - c^{-1}d. \tag{3.20}$$

On the other hand, because $\underline{x} = \underline{x}^*$, $\underline{y} = \underline{y}^*$, (3.20) is equivalent to

$$\underline{x} = A^*(z^*)^{-1} - d^*(c^*)^{-1}. \tag{3.21}$$

By the Möbius transform and the formula (3.20), we deduce from $c \neq 0$ that $A \neq 0$ and

$$\begin{aligned} \underline{D}^{-l}(T_{-l}(\underline{D}^l f))(\psi(\underline{x})) &= c_{n,l} \int K_{n,l}(\psi(\underline{x}) - \underline{y}) \cdot J_{-l,\phi}(\underline{y})(\underline{D}^l f)(\phi(\underline{y}))d\underline{y} \\ &= c_{n,l} \int K_{n,l}(\psi(\underline{x}) - \psi(\underline{y})) \cdot J_{-l,\phi}(\psi(\underline{y}))(\underline{D}^l f)(\underline{y}) \left| \frac{d\psi(\underline{y})}{d\underline{y}} \right| d\underline{y}, \end{aligned} \tag{3.22}$$

where $|d\psi(\underline{y})/d\underline{y}|$ is the Jacobian matrix. Noticing that $\underline{x} = \psi(\underline{y})$ is also a Möbius transform, by the formula (2.4) in [7] and the condition $ad^* - bc^* = 1$, we can obtain the Jacobian matrix equals to $|z(y)|^{-2n}$. By equalities (3.15), (3.17) and

$$\begin{aligned} \psi(\underline{x}) - \psi(\underline{y}) &= (z^{-1}(\underline{x}) - z^{-1}(\underline{y}))A, \\ z^{-1}(\underline{x}) - z^{-1}(\underline{y}) &= -z^{-1}(\underline{x})(z(\underline{x}) - z(\underline{y}))z^{-1}(\underline{y}) \\ z(\underline{x}) - z(\underline{y}) &= (\underline{x} - \underline{y})c, \end{aligned}$$

we can deduce that (3.22) is equivalent to

$$-c_{n,l}\frac{z^{-1}(\underline{x})}{|z^{-1}(\underline{x})|^{n-l+1}}\int\frac{(\underline{x}-\underline{y})}{|\underline{x}-\underline{y}|^{n-l+1}}\frac{c}{|c|^{n-l+1}}\frac{z^{-1}(\underline{y})}{|z^{-1}(\underline{y})|^{n-l+1}}\frac{A}{|A|^{n-l+1}}$$
$$\cdot\frac{A^*}{|A|^{n-l+1}}\frac{(z^{-1}(\underline{y}))^*}{|z^{-1}(\underline{y})|^{n-l+1}}\frac{c^*}{|c|^{n-l+1}}(\underline{D}^l f)(\underline{y})\frac{1}{|z(\underline{y})|^{2n}}d\underline{y}$$
$$=c_{n,l}\frac{1}{|A|^{2n}}\frac{(-1)^{m(c)}}{|c|^{2n}}\frac{z^{-1}(\underline{x})}{|z^{-1}(\underline{x})|^{n-l+1}}\int\frac{(\underline{x}-\underline{y})}{|\underline{x}-\underline{y}|^{n-l+1}}(\underline{D}^l f)(\underline{y})(\underline{y})$$
$$=\frac{1}{|A|^{2n}}\frac{(-1)^{m(c)}}{|c|^{2n}}\frac{z^{-1}(\underline{x})}{|z^{-1}(\underline{x})|^{n-l+1}}\int K_{n,l}(\underline{x}-\underline{y})(\underline{D}^l f)(\underline{y})(\underline{y})$$
$$=\frac{1}{|A|^{2n}}\frac{(-1)^{m(c)}}{|c|^{2n}}\frac{z^{-1}(\underline{x})}{|z^{-1}(\underline{x})|^{n-l+1}}f(\underline{x}),$$

where in the above estimate we have used $m(z^{-1}A)=1$. Replacing $\underline{x}$ by $\phi(\underline{x})$ and noticing that $(\underline{x}+d^*(c^*)^{-1})=z^{-1}(\phi(\underline{x}))A$, we get

$$\underline{D}^{-l}(T_{-l}(\underline{D}^l f))(\underline{x})=\frac{(-1)^{m(c)}cA^*}{|cA|^{n+l+1}}\frac{(cx+d)^*}{|cx+d|^{n-l+1}}f(\phi(\underline{x})).$$

By $bc^*=ad^*-1$ and $c^{-1}d\in\mathbb{R}^n$, we can deduce that $b=-(c^*)^{-1}+ac^{-1}d$ and $A=-(c^*)^{-1}$, which gives (3.19).

The case for even l can be proved similarly. The only difference is that we should use the formulas (3.14), (3.15) and (3.17) for the case l even. Now we consider the case $l\geqslant n$, where n is even. Similar to the case l being odd, it can be deduced from (3.15) that

$$\underline{D}^{-l}(T_{-l}(\underline{D}^l f))(\psi(\underline{x}))$$
$$=\frac{1}{|A|^{2n}}\frac{(-1)^{m(c)}}{|c|^{2n}}\frac{z^{-1}(\underline{x})}{|z^{-1}(\underline{x})|^{n-l+1}}\int\Big[(-c_{n,l})\log|z(\underline{x})|+(c_{n,l}\log|\underline{x}-\underline{y}|+d_{n,l})$$
$$+c_{n,l}\log|c|+(-c_{n,l}\log|z(\underline{y})|)\Big]\frac{\underline{x}-\underline{y}}{|\underline{x}-\underline{y}|^{n-l+1}}(\underline{D}^l f)(\underline{y})d\underline{y}$$
$$=\sum_{i=1}^{4}I_i.$$

When n is even and l is odd satisfying $l\geqslant n$, $(\underline{x}-\underline{y})/|\underline{x}-\underline{y}|^{n-l+1}=\pm(\underline{x}-\underline{y})^{l-n}$, $I_1=I_3=0$. For I_2, by the property of fundamental solution, we can deduce

$$I_2=\frac{1}{|A|^{2n}}\frac{(-1)^{m(c)}}{|c|^{2n}}\frac{z^{-1}(\underline{x})}{|z^{-1}(\underline{x})|^{n-l+1}}f(\underline{x}).$$

We shall prove $I_4=0$. In fact, because

$$\frac{\underline{x}-\underline{y}}{|\underline{x}-\underline{y}|^{n-l+1}} = \pm[(\underline{x}-ac^{-1})-(\underline{y}-ac^{-1})]^{l-n} = \sum_{k+j=l-n} h_{kj}(\underline{x}-ac^{-1})^k(\underline{y}-ac^{-1})^j,$$

by integration by parts, we have

$$\begin{aligned}
I_4 &= -c_{n,l}\sum_{k+j=n-l} h_{kj}(\underline{x}-ac^{-1})^k \int (\log|\underline{y}-ac^{-1}|+\log|c|)(\underline{y}-ac^{-1})^j(\underline{D}^l f)((\underline{y}))d\underline{y}\\
&= -c_{n,l}\sum_{k+j=n-l} h_{kj}(\underline{x}-ac^{-1})^k \int \log|\underline{y}-ac^{-1}|(\underline{y}-ac^{-1})^j(\underline{D}^l f)((\underline{y}))d\underline{y}\\
&= -c_{n,l}\sum_{k+j=n-l,\, j<l-n} h_{kj}(\underline{x}-ac^{-1})^k \int (\log|\underline{y}-ac^{-1}|+\log|c|)(\underline{y}-ac^{-1})^j(\underline{D}^l f)((\underline{y}))d\underline{y}\\
&\quad -c_{n,l}h_{0,l-n}\int \log|\underline{y}-ac^{-1}|(\underline{y}-ac^{-1})^{l-n}(\underline{D}^l f)((\underline{y}))d\underline{y}\\
&= \pm h_{0,l-n}\int c_{n,l}(\log|\underline{y}-ac^{-1}|+d_{n,l})\frac{(\underline{y}-ac^{-1})}{|\underline{y}-ac^{-1}|^{n-l+1}}(\underline{D}^l f)((\underline{y}))d\underline{y}\\
&= \pm h_{0,l-n} f(ac^{-1})\\
&= 0,
\end{aligned}$$

where in the last step we have used the following fact: the function $f\circ\phi$ is compactly supported and hence

$$f(ac^{-1}) = f\circ\phi\circ\psi(ac^{-1}) = f\circ\phi(\infty) = 0.$$

As above, we still obtain

$$\underline{D}^{-l}(T_{-l}(\underline{D}^l f))(\psi(\underline{x})) = \frac{1}{|A|^{2n}}\frac{(-1)^{m(c)}}{|c|^{2n}}\frac{z^{-1}(\underline{x})}{|z^{-1}(\underline{x})|^{n-l+1}} f(\underline{x}).$$

Replacing $\underline{x}$ by $\phi(\underline{x})$, we get (3.19) for the case l odd and $l \geqslant n$ with n even. The case l even can be obtained similarly. □

Now we consider the following question: for the operator D^l, if we have the similar conformal covariance. In fact, if we replace $\mathbb{R}^n$ in the Möbius transform, the identification relation and the certain Clifford matrices by $\mathbb{R}^n_1$, respectively, all conclusions still hold. Now let ϕ denote a Möbius transform from $\mathbb{R}^n_1\cup\{\infty\}$ to $\mathbb{R}^n_1\cup\{\infty\}$ and let g be a fixed function from $\mathbb{R}^n_1\cup\{\infty\}$ to $\mathbb{R}^n_1\cup\{\infty\}$. Define

$$g(x) = \frac{x^*}{|x|^{n+1}},\ x = x_0+\underline{x}.$$

Define the representations

$$S_1(\phi)f(x) := L_{n,1}((cx+d)^*)f(\phi(x))$$

and

$$S_{-1}(\phi)f(x) := g(cx+d)f(\phi(x)).$$

We have the following result.

Theorem 3.4.2

$$D(S_1 f) = S_{-1}(Df). \tag{3.23}$$

Proof By the fundamental solution of the operator D and (3.16), replacing $\underline{x}$ and $\underline{y}$ in Theorem 3.4.1 by x and y, respectively, we have

$$\begin{aligned}
D^{-1}(S_{-1}(Df))(\psi(x)) &= \int L_{n,1}(\psi(x)-\psi(y))g(cz^{-1}(y)A)(Df)(y)\frac{1}{|z(y)|^{2(n+1)}}dy \\
&= c_{n,1}\frac{-\overline{z^{-1}(x)}}{|z^{-1}(x)|}\int \frac{\overline{(x-y)cz^{-1}(y)A}}{|x-y|^{n+1}|c|^{n+1}|z^{-1}(y)|^{n+1}|A|^{n+1}} \\
&\quad \times \frac{A^*(z^{-1}(y))^*c^*}{|A|^{n+3}|z^{-1}(y)|^{n+3}|c|^{n+3}}\frac{1}{|z(y)|^{2(n+1)}}dy \\
&= \frac{-1}{|Ac^*|^{2n+2}}\frac{-\overline{z^{-1}(x)}}{|z^{-1}(x)|}\int L_{n,1}(x-y)(Df)(y)dy \\
&= \frac{-1}{|Ac^*|^{2n+2}}\frac{-\overline{z^{-1}(x)}}{|z^{-1}(x)|}f(x).
\end{aligned}$$

Replacing x by $\phi(x)$ and using $Ac^* = -1$, we get (3.23). □

3.5 The Fueter Theorem

In this section, we elaborate Qian's work on the generalization of Fueter's mapping theorem, see [3]. We shall work in $\mathbb{R}^{n+1}$, the real-linear span of $\{e_0, e_1, \ldots, e_n\}$, where e_0 is identical with 1 and $e_i e_j + e_j e_i = -2\delta_{ij}$. $\mathbb{R}^{n+1}$ is embedded into Clifford algebra $\mathbb{R}_1^n$ generated by $e_1, \ldots e_n$. The elements in $\mathbb{R}^{n+1}$ are represented as $x = x_0 + \underline{x}$, where $x_0 \in \mathbb{R}$ and $\underline{x} = x_1 e_1 + \cdots + x_n e_n$ with $x_j \in \mathbb{R}$. If $x \neq 0$, there exists an inverse x^{-1}: $x^{-1} = \frac{\overline{x}}{|x|^2}$, where $\overline{x} = x_0 - \underline{x}$. We will study the $\mathbb{R}^{n+1}$-valued and Clifford-valued functions, and the left and right monogeneity introduced by the Dirac operator

$$D = \frac{\partial}{\partial x_0} + e_1\frac{\partial}{\partial x_1} + \cdots + e_n\frac{\partial}{\partial_n}.$$

The Kelvin inversion of a function f is $I(f)(x) = E(x)f(x^{-1})$. The symbols $\mathbb{Z}$ and $\mathbb{Z}^+$ denote the sets of all integers and positive integers, respectively.

For a function f on $\mathbb{R}^{n+1}$, the Fourier transform of f is defined by

$$\mathcal{F}(f)(\xi) = \int_{\mathbb{R}^{n+1}} e^{2\pi i\langle x,\ \xi\rangle} f(x)dx.$$

A useful result associated with Fourier transform is

$$\mathcal{F}\Big(\frac{P_k(\cdot)}{|\cdot|^{k+n+1-\alpha}}\Big)(\xi) = \gamma_{k,\alpha}\frac{P_k(\xi)}{|\xi|^{k+\alpha}}, \tag{3.24}$$

where $0 < \alpha < n+1$, $k \in \mathbb{Z}^+$, P_k is the homogeneous harmonic polynomial of degree k, and

$$\gamma_{k,\alpha} = i^k \pi^{(n+1)/2-\alpha} \frac{\Gamma(k/2+\alpha/2)}{\Gamma(k/2+(n+1)/2-\alpha/2)},$$

(Γ denotes the usual Gamma function).

For a function g, the inverse Fourier transform $\mathcal{R}(g)$ is defined as

$$\mathcal{R}(g)(x) = \int_{\mathbb{R}^{n+1}} e^{-2\pi i\langle x,\ \xi\rangle} g(\xi) d\xi.$$

The Fourier transform of a function in the Schwartz class still belongs to the Schwartz class. In this case, the Fourier inversion formula holds: $\mathcal{R}\mathcal{F}(f) = f$. In the sequel, the Fourier transform and the inverse Fourier transform will be used in the distributional sense.

For the function g defined on $\mathbb{R}^{n+1}$, we can introduce the Fourier multiplier M_g as $M_g f = \mathcal{R}(g\mathcal{F}f)$. It is easy to prove that the Fourier multiplier induced by $-4\pi^2|\xi|^2$ is identical to the Laplace operator.

Let f^0 be a complex-valued function defined on an open set O in the upper-half complex plane. Write $f^0 = u + iv$, where u and v are real-valued. For $x \in \overrightarrow{O}$, set

$$\overrightarrow{f}^0(x) = u(x_0, |\underline{x}|) + \frac{\underline{x}}{|\underline{x}|} v(x_0, |\underline{x}|),$$

where

$$\overrightarrow{O} = \Big\{x \in \mathbb{R}^{n+1} \ : \ (x_0, |\underline{x}|) \in O\Big\}.$$

$\overrightarrow{f}^0$ is called the function induced from f^0, and $\overrightarrow{O}$ is call the set induced from O.

We shall work with the functions of the form

$$g(x) = p(x_0, |\underline{x}|) + i\frac{\underline{x}}{|\underline{x}|} q(x_0, |\underline{x}|),$$

where p and q are real-valued. We call p and q the real part and the imaginary part of g, respectively.

The concepts of intrinsic functions and intrinsic sets naturally fit to our theory. On the complex plane $\mathbb{C}$, if an open set is symmetric with respect to the real axis, then the set is called an intrinsic set. If a function is defined on an intrinsic set and satisfies $\overline{f^0(z)} = f^0(\overline{z})$ within its domain, then the function is called an intrinsic function. For $f^0 = u + iv$, the above condition is equivalent to requiring that u is even respect to the second variable, and v is odd respect to the second variable. In particular, $v(x_0, 0) = 0$, i.e., if the domain f^0 is restricted on the real axis, then f^0 is real-valued.

Denote by τ the mapping

$$\tau(f^0) = \Delta^{(n-1)/2}\overrightarrow{f}^0,$$

where f^0 is any holomorphic intrinsic function and the differential operation is in the distributional sense. For the sake of convenience, outside the intrinsic set $\overrightarrow{O}$, we take $\overrightarrow{f}^0 = 0$.

Note that for odd $n \in \mathbb{Z}^+$, the operator $\Delta^{(n-1)/2}$ is a pointwise differential operator, while for even $n \in \mathbb{Z}^+$, $\Delta^{(n-1)/2}$ is the Fourier multiplier induced by $(2\pi i|\xi|)^{n-1}$ mapping some functions to the distributions. If b is a complex-valued function defined on an intrinsic set, then

$$\begin{cases} g^0(z) = \frac{1}{2}\Big[b(z) + \overline{b}(\overline{z})\Big], \\ b^0(z) = \frac{1}{2i}\Big[b(z) - \overline{b}(\overline{z})\Big] \end{cases}$$

both are intrinsic functions defined on the same set, and $b = g^0 + ib^0$.

The above observation enables us to extend the domain of τ to the sets of the complex-valued functions b on the intrinsic set. These functions b may not be intrinsic functions. For such a function b, we define

$$\tau(b) = \tau(g^0) + i\tau(b^0).$$

The mapping τ extended in such a way is linear under addition and real-scalar multiplication. In the sequel, for the mapping τ, we only need to consider the holomorphic intrinsic functions. For intrinsic functions, the coefficients of their Laurent series expansions in annuli centered at real points in their domains are all real. Hence we only need to consider the functions $\tau((\cdot)^{-k}),\ k \in \mathbb{Z}$. For $k \in \mathbb{Z}^+$, define

$$P^{(-k)} = \tau((\cdot)^{-k}), \quad P^{(k-1)} = I(P^{(-k)}).$$

We have the following result.

Theorem 3.5.1 *Let $k \in \mathbb{Z}^+$. Then*

(i) *$P^{(-k)}$ and $P^{(k-1)}$ are monogenic functions;*
(ii) *$P^{(-k)}$ is homogeneous of degree $(n + 1 - k)$ and $P^{(k-1)}$ is homogeneous of degree $(k - 1)$;*
(iii) *If n is odd, then $P^{(k-1)} = \tau((\cdot)^{n+k-2})$.*

Proof (i) By the Fourier transform and the following relation:

$$\overrightarrow{(\cdot)^{-k}}(x) = \left(\frac{\overline{x}}{|x|^2}\right)^k = \frac{(-1)^{k-1}}{(k-1)!}\left(\frac{\partial}{\partial x_0}\right)^{k-1}\left(\frac{\overline{x}}{|x|^2}\right),$$

we get

$$\begin{aligned} P^{(-k)}(x) &= \tau((\cdot)^{-k})(x) \\ &= \frac{(-1)^{k-1}}{(k-1)!}\left(\frac{\partial}{\partial x_0}\right)^{k-1} \mathcal{RF}\left(\Delta^{(n-1)/2}\frac{\overline{x}}{|x|^2}\right) \\ &= \frac{(-1)^{k-1}}{(k-1)!}\left(\frac{\partial}{\partial x_0}\right)^{k-1} \mathcal{R}\left(\gamma_{1,n}(2\pi i|\xi|)^{n-1}\frac{\overline{\xi}}{|\xi|^{n+1}}\right) \\ &= \frac{(-1)^{k-1}}{(k-1)!}\left(\frac{\partial}{\partial x_0}\right)^{k-1} \gamma_{1,n}^2(2\pi i)^{n-1}\frac{\overline{x}}{|x|^{n+1}} \\ &= \frac{(-1)^{k-1}}{(k-1)!}\kappa_n\left(\frac{\partial}{\partial x_0}\right)^{k-1} E(x), \end{aligned} \tag{3.25}$$

where

$$\kappa_n = (2\pi i)^{n-1}\gamma_{1,n}^2 = (2i)^{n-1}\Gamma^2((n+1)/2).$$

This means that for $k \in \mathbb{Z}^+$, $P^{(-k)}$ is monogenic. The monogeneity of $P^{(k)}$ can be deduced by the property of the Kelvin inversion, or the result of Bojarski, see [2].

The conclusion (ii) can be obtained by the expression of $P^{(-k)}$ and the property of the Kelvin inversion.

(iii) Let $n = 2m+1$. We have

$$\kappa_n = (-1)^m 2^{2m}(m!)^2 = (-1)^m((2m)!!)^2.$$

We use the mathematical induction. The case $k = 1$ reduces to verifying $\Delta^m(x^{2m}) = (-1)^m(2m)!!$. We need the following lemma.

Lemma 3.5.1 *Let $f^0(z) = u(x_0, y) + iv(x_0, y)$ be a holomorphic function defined on an open set U in the upper-half complex plane. Write $u_0 = u$, $v_0 = v$, and for $s \in \mathbb{Z}^+$, write*

$$u_s = 2s\frac{\partial u_{s-1}}{\partial y}\frac{1}{y}$$

and

$$v_s = 2s\left(\frac{\partial v_{s-1}}{\partial y}\frac{1}{y} - \frac{v_{s-1}}{y^2}\right) = 2s\frac{\partial}{\partial y}\frac{v_{s-1}}{y}.$$

Then

$$\Delta^s \overrightarrow{f}^0(x) = u_s(x_0, |\underline{x}|) + \frac{\underline{x}}{|\underline{x}|}v_s(x_0, |\underline{x}|),\ x_0 + i|\underline{x}| \in U.$$

This lemma can be proved using mathematical induction via a computation of $\Delta(u_{s-1} + iv_{s-1})$ invoking the following relation proved in [8]:

$$\frac{\partial u_{s-1}}{\partial x_0} = \frac{\partial v_{s-1}}{\partial y} + 2(s-1)\frac{v_{s-1}}{y}, \quad \frac{\partial u_{s-1}}{\partial y} = -\frac{\partial v_{s-1}}{\partial x_0}.$$

We will frequently use the formula given in [8]: for any function $f^0 = u + iv$ and $r \in \mathbb{Z}^+$,

$$(\overrightarrow{f}^0)^r(x) = \sum_{l=0}^{[r/2]}(-1)^l C_r^{2l} u^{r-2l} v^{2l} + \frac{\underline{x}}{|\underline{x}|}\sum_{l=0}^{[r/2]}(-1)^l C_r^{2l+1} u^{r-2l-1} v^{2l+1}, \quad (3.26)$$

where C_r^l are binomial coefficients with the convention that $C_r^l = 0$ for $l > r$, and $[s]$ denotes the largest integer that does not exceed s.

For $f^0(z) = z$, using the formula (3.26), by $r = 2m$ and Lemma 3.5.1, we can obtain $\Delta^m(x^{2m}) = (-1)^m((2m)!!)^2$, which proves the case $k = 1$. Now assume that $P^{(k)} = \tau((\cdot)^{n+k-1})$. We need to prove $P^{(k+1)} = \tau((\cdot)^{n+k})$. This is equivalent to proving

$$\frac{-1}{k+1}\frac{\partial}{\partial x_0}(I(\Delta^m((\cdot)^{2m+k}))) = I(\Delta^m((\cdot)^{2m+k+1})), \quad (3.27)$$

where $k \in \mathbb{Z}^+$ or $k = 0$.

By (3.26) and Lemma 3.5.1, we have

$$\begin{aligned}
&\Delta((\cdot)^{2m+k})(x)\\
&= (2m)!!\Big[\sum_{l=0}^{m+[k/2]}(-1)^l C_{2m+k}^{2l}(2l)(2l-2)\cdots(2l-2m+2)x_0^{2m+k-2l}y^{2l-2m}\\
&+\frac{\underline{x}}{y}\sum_{l=0}^{m+[k/2]}(-1)^l C_{2m+k}^{2l+1}(2l)(2l-2)\cdots(2l-2m+2)x_0^{2m+k-2l-1}y^{2l+1-2m}\Big],
\end{aligned}$$

where we take $y = |\underline{x}|$.

By the Kelvin inversion, we replace x_0, y and $\underline{x}/y$ by $x_0|x|^{-2}$, $y|x|^{-2}$ and $-\underline{x}/y$, respectively. It follows that the above becomes

$$\begin{aligned}
&(2m)!!\frac{\overline{x}}{|x|^{n+2k+1}}\Big[\sum_{l=0}^{m+[k/2]}(-1)^l C_{2m+k}^{2l} \quad (3.28)\\
&(2l)(2l-2)\cdots(2l-2m+2)x_0^{2m+k-2l}y^{2l-2m} + \frac{\underline{x}}{y}\sum_{l=0}^{m+[k/2]}(-1)^{l+1}C_{2m+k}^{2l+1}\\
&\times(2l)(2l-2)\cdots(2l-2m+2)x_0^{2m+k-2l-1}y^{2l+1-2m}\Big].
\end{aligned}$$

Applying the differential operator $[-1/(k+1)]\partial/\partial x_0$ to (3.28), we have

$$\frac{-(2m)!!}{k+1}E(x)\frac{1}{|x|^{2k+2}}\Big\{\Big(-(n+2k)x_0+\frac{\underline{x}}{y}y\Big)[\cdots]+(x_0^2+y^2)\frac{\partial}{\partial x_0}[\cdots]\Big\}, \tag{3.29}$$

where $[\cdots]$ is as $[\cdots]$ in (3.28).

Now we have

$$\begin{aligned}
&\Big(-(n+2k)x_0+\frac{\underline{x}}{y}y\Big)[\cdots]\\
&=\Bigg\{\sum_{l=0}^{m+[k/2]}(-1)^{l+1}C_{2m+k}^{2l}(n+2k)(2l)(2l-2)\cdots(2l-2m+2)x_0^{2m+k-2l}y^{2l-2m}\\
&+\sum_{l=0}^{m+[k/2]}(-1)^{l}C_{2m+k}^{2l+1}(2l)(2l-2)\cdots(2l-2m+2)x_0^{2m+k-2l-1}y^{2l+1-2m}\Bigg\}\\
&+\frac{\underline{x}}{y}\Bigg\{\sum_{l=0}^{m+[k/2]}(-1)^{l+1}C_{2m+k}^{2l+1}(n+2k)(2l)(2l-2)\cdots(2l-2m+2)x_0^{2m+k-2l}y^{2l+1-2m}\\
&+\sum_{l=0}^{m+[k/2]}(-1)^{l}C_{2m+k}^{2l}(2l)(2l-2)\cdots(2l-2m+2)x_0^{2m+k-2l}y^{2l+1-2m}\Bigg\}
\end{aligned}$$

and

$$\begin{aligned}
&(x_0^2+y^2)\frac{\partial}{\partial x_0}[\cdots]\\
&=\Bigg\{\sum_{l=0}^{m+[k/2]}(-1)^{l}C_{2m+k}^{2l}(2l)(2l-2)\cdots(2l-2m+2)(2m+k-2l)\\
&\quad\times\ (x_0^{2m+k-2l+1}y^{2l-2m}+x_0^{2m+k-2l-1}y^{2l-2m+2})\Bigg\}\\
&+\frac{\underline{x}}{y}\Bigg\{\sum_{l=0}^{m+[k/2]}(-1)^{l+1}C_{2m+k}^{2l+1}(2l)(2l-2)\cdots(2l-2m+2)(2m+k-2l-1)\\
&\quad\times(x_0^{2m+k-2l}y^{2l+1-2m}+x_0^{2m+k-2l-2}y^{2l+1-2m+2})\Bigg\}.
\end{aligned}$$

By comparing the coefficients of a general nomial $x_0^{2m+k+1-2l}y^{2l-2m}$ in the real part of (3.29) with those in the real part of

$$I(\Delta^m((\cdot)^{2m+k+1}))(x)=E(x)(\Delta^m((\cdot)^{2m+k+1}))(x^{-1}),$$

the latter being of the expression (3.28) but with $k+1$ in place of k, we are reduced to verifying

$$\begin{aligned}&-2l(n+2k)C_{2m+k}^{2l}+(2m-2l)C_{2m+k}^{2l-1}\\&+2l(2m+k-2l)C_{2m+k}^{2l}+(2m-2l)(2m+k-2l+2)C_{2m+k}^{2l-2}\\&=-(k+1)2lC_{2m+k+1}^{2l}.\end{aligned} \tag{3.30}$$

By $(s-l)C_s^l=(l+1)C_s^{l+1}$, the second and fourth entries on the left hand side of (3.30) add up to

$$2l(2m-2l)C_{2m+k}^{2l-1}, \tag{3.31}$$

while the first and third to

$$\begin{aligned}-2l(2l+k+1)C_{2m+k}^{2l}&=[-4l^2-2l(k+1)]C_{2m+k}^{2l}\\&=-2l(2m+k-2l+1)C_{2m+k}^{2l-1}-2l(k+1)C_{2m+k}^{2l}.\end{aligned} \tag{3.32}$$

Combining (3.31) with the right hand side of (3.32) and using $C_s^l+C_s^{l-1}=C_{s+1}^l$, we get (3.30). Similarly, we can prove that the imaginary part of (3.29) is equivalent to the imaginary part of $I(\Delta^m((\cdot)^{2m+k+1}))$. This proves (iii). □

In [9], Kou, Qian and Sommen obtained the following generalization of Theorem 3.5.1. For any $x=x_0+\underline{x}\in\mathbb{R}^n$, let P_k be a homogeneous polynomial of $\underline{x}$ of degree k and satisfy

$$\underline{\partial}P_k(\underline{x})=0.$$

We consider the following question: if

$$D\Delta^{k+(n-1)/2}\Big(\Big(u(x_0,\underline{x})+\frac{\underline{x}}{|\underline{x}|}v(x_0,\underline{x})P_k(\underline{x})\Big)\Big)=0.$$

We first prove that if $l\in\mathbb{Z}$, the function

$$\Delta^{k+(n-1)/2}\Big((x_0+\underline{x})^lP_k(\underline{x})\Big) \tag{3.33}$$

is still a left monogenic function.

At first, we assume that l is negative. By a simple computation, we can see that

$$(x_0+\underline{x})^{-l}=\left(\frac{\overline{x}}{|x|^2}\right)^l=\frac{(-1)^{l-1}}{(l-1)!}\left(\frac{\partial}{\partial x_0}\right)^l\left(\frac{\overline{x}}{|x|^2}\right),\ l=1,2,\ldots.$$

Hence we only need to prove

$$\Delta^{k+(n-1)/2}\left(\frac{\overline{x}}{|x|^2}P_k(\underline{x})\right)$$

is left monogenic.

Lemma 3.5.2 *$Q_{k+1}(x)=\overline{x}P_k(\underline{x})$ is harmonic and homogeneous of degree $k+1$.*

Proof By definition, it can be verified directly that

$$\left(\frac{\partial}{\partial x_0}\right)^2 Q_{k+1}(x) = 0.$$

Using Leibniz's formula for second derivative, we can get

$$\left(\frac{\partial}{\partial x_i}\right)^2 Q_{k+1}(x) = 2\left(\frac{\partial}{\partial x_i}\right)(\overline{x})\left(\frac{\partial}{\partial x_i}\right) P_k(\underline{x}) + \overline{x}\left(\frac{\partial}{\partial x_i}\right) P_k(\underline{x})^2.$$

This implies that

$$\Delta Q_{k+1}(x) = -2\underline{\partial} P_k(\underline{x}) + \overline{x}\underline{\Delta} P_k(\underline{x}) = 0.$$

□

In the proof of Theorem 3.5.1, we use the following Bochner type formula: in the sense of tempered distributional sense,

$$\left(\frac{Q_j(\cdot)}{|\cdot|^{j+(n+1)-\alpha}}\right)^{\wedge}(\xi) = \gamma_{j,\alpha}\frac{Q_j(\cdot)(\xi)}{|\xi|^{j+\alpha}},\ j \in \mathbb{Z}_+, 0 < \alpha < n+1, \tag{3.34}$$

where Q_j is a harmonic homogeneous polynomial of degree j, and

$$\gamma_{j,\alpha} = i^j \pi^{(n+1)/2-\alpha}\frac{\Gamma(j/2+\alpha/2)}{\Gamma(j/2+(n+1)/2-\alpha/2)}.$$

By the Fourier transform, (3.34) is equivalent to the following equality: for any Schwartz function ϕ on $\mathbb{R}^n_1$ and $j \in \mathbb{Z}_+, 0 < \alpha < n+1$,

$$\int_{\mathbb{R}^n_1} \frac{Q_j(x)}{|x|^{j+(n+1)-\alpha}}\hat{\phi}(x)dx = i^j \pi^{(n+1)/2-\alpha}\frac{\Gamma(j/2+\alpha/2)}{\Gamma(j/2+(n+1)/2-\alpha/2)}\int_{\mathbb{R}^n_1}\frac{Q_j(x)}{|x|^{j+\alpha}}\phi(x)dx.$$

Now we will generalize the above formula to the case $\mathrm{Re}(\alpha) > -j$ and $j \in \mathbb{Z}_+$.

Lemma 3.5.3 *Let* $-j < \beta, \alpha < (n+1)+j, \alpha+\beta = n+1$ *and* $j \in \mathbb{Z}_+$. *For any Schwartz function* ϕ *on* $\mathbb{R}^n_1$, *we have*

$$\pi^{\beta/2}\Gamma\left(\frac{j+\beta}{2}\right)\int_{\mathbb{R}^n_1}\frac{Q_j(x)}{|x|^{j+\beta}}\hat{\phi}(x)dx = i^j\pi^{\alpha/2}\Gamma\left(\frac{j+\alpha}{2}\right)\int_{\mathbb{R}^n_1}\frac{Q_j(x)}{|x|^{j+\alpha}}\phi(x)dx. \tag{3.35}$$

Proof For $0 < \alpha < n+1$, both sides of (3.34) are holomorphic. For $j \geqslant 1$, by the orthogonality of the spherical harmonic polynomials, there follows

$$\int_{\mathbb{R}_1^n} \frac{Q_j(x)}{|x|^{j+\beta}} \hat{\phi}(x)dx = \lim_{\epsilon\to 0+} \frac{Q_j(x)}{|x|^{j+(n+1)-\alpha}} \left(\hat{\phi}(x) - \hat{\phi}(0) - \frac{1}{(j-1)!} \left(\sum_{i=0}^{n} x_i \frac{\partial}{\partial x_i} \right)^{j-1} \hat{\phi}(0) \right) dx$$
$$+ \int_{|x|>1} \frac{Q_j(x)}{|x|^{j+(n+1)-\alpha}} \hat{\phi}(x)dx,$$

that can be extended to all complex numbers α with $\mathrm{Re}(\alpha) > -j$ holomorphically. Similarly, the right hand side of (3.34) can also be extended holomorphically to all complex numbers α such that $\mathrm{Re}(\alpha) > -j$. □

Proposition 3.5.1 *Let $l \in \mathbb{Z}_+$, where $n+1$ is odd and k is non-negative. Then the functions*

$$\Delta^{k+(n-1)/2} \left(\left(\frac{\overline{x}}{|x|^2} \right)^l P_k(\underline{x}) \right),\ l \in \mathbb{Z}_+$$

are all left monogenic.

Proof In Lemma 3.5.3, letting $\alpha = 2 - j$, we have

$$\lim_{\epsilon\to 0+} \int_{|x|>\epsilon} \frac{Q_j(x)}{|x|^{j+(n+1)+j-2}} \hat{\phi}(x)dx = \frac{i^j \pi^{(n+1)/2+(j-2)}}{\Gamma((n+1)/2+j-1)} \int_{\mathbb{R}_1^n} \frac{Q_j(x)}{|x|^2} \phi(x)dx.$$

Replacing ϕ by $\Delta^{k+(n+1)/2}\phi$ and j by $k+1$, we get

$$\lim_{\epsilon\to 0+} \int_{|x|>\epsilon} \frac{Q_{k+1}(x)}{|x|^{(n+1)+k}} |x|^{2k+(n-1)} \hat{\phi}(x)dx = \beta_k \int_{\mathbb{R}_1^n} \Delta^{k+(n-1)/2} \left(\frac{Q_{k+1}(x)}{|x|^2} \right) \phi(x)dx,$$

where

$$\beta_k = 2^{1-n-2k} i^{2-n-k} \pi^{-k-(n-1)/2} \frac{1}{\Gamma((n+1)/2+k)}.$$

Hence we obtain

$$\int_{\mathbb{R}_1^n} \frac{Q_{k+1}(x)}{|x|^2} \hat{\phi}(x)dx = \beta_k \int_{\mathbb{R}_1^n} \Delta^{k+(n-1)/2} \left(\frac{Q_{k+1}(x)}{|x|^2} \right) \phi(x)dx.$$

Replacing Q_{k+1} by $\overline{x} P_k(\underline{x})$, we have

$$\int_{\mathbb{R}_1^n} \frac{Q_{k+1}(x)}{|x|^2} \hat{\phi}(x)dx = \int_{\mathbb{R}_1^n} \left(\frac{\bar{\cdot}}{|x|^2} P_k(\underline{\cdot}) \right)^{\wedge} (x)\phi(x)dx$$
$$= \gamma_{1,n}^{-1} \int_{\mathbb{R}_1^n} E * (P_k(\underline{\partial})\delta)(x)\phi(x)dx,$$

where $E(x) = \frac{\overline{x}}{|x|^{n+1}} = \gamma_{1,n}(\frac{\bar{\cdot}}{|\cdot|^2})^{\wedge}(x)$ is the Cauchy kernel on $\mathbb{R}_1^n$, δ is the Dirac function. Therefore,

$$\Delta^{k+(n-1)/2}\left(\frac{\overline{x}}{|x|^2}P_k(\underline{x})\right)=\gamma_{1,n}^{-1}\beta_k^{-1}E*(P_k(\underline{\partial})\delta)(x)=\gamma_{1,1}\beta_k^{-1}EP_k(\underline{\partial})(x).$$

This implies

$$\Delta^{k+(n-1)/2}\left(\frac{\overline{x}}{|x|^2}P_k(\underline{x})\right)$$

is left monogenic. In addition,

$$\begin{aligned}\Delta^{k+(n-1)/2}\left(\left(x_0+\underline{x}\right)^{-l}P_k(\underline{x})\right)&=\Delta^{k+(n-1)/2}\left(\left(\frac{\overline{x}}{|x|^2}\right)^{l}P_k(\underline{x})\right)\qquad(3.36)\\&=\Delta^{k+(n-1)/2}\left(\frac{(-1)^{l-1}}{(l-1)!}\left(\frac{\partial}{\partial x_0}\right)^{l-1}\left(\frac{\overline{x}}{|x|^2}\right)P_k(\underline{x})\right)\\&=\gamma_{1,1}\beta_k^{-1}\left(\frac{\partial}{\partial x_0}\right)^{l-1}EP_k(\underline{\partial})(x).\end{aligned}$$

So

$$\Delta^{k+(n-1)/2}\left(\left(x_0+\underline{x}\right)^{-l}P_k(\underline{x})\right),\ l\in\mathbb{Z}_+,$$

are all left monogenic. □

Now for the case $l\geqslant 0$, we prove the function

$$\Delta^{k+(n-1)/2}\left[(x_0+\underline{x})^l P_k(\underline{x})\right]\qquad(3.37)$$

is left monogenic function. We first discuss the fundamental solution of the operator $D\Delta^{k+(n-1)/2}$. Below we assume that $2s=2k+(n-1)$. Hence $2s$ may be even or odd. It is even if and only if $n+1$ is even.

Lemma 3.5.4 *The operator $D|D|^{2s}$ in $\mathbb{R}_1^n$ has a fundamental solution of the same form as those in the above list for $\underline{\partial}^{2s+1}$ in $\mathbb{R}^{n+1}$, except that the term $\underline{x}$ in the latter is replaced by $\overline{x}$.*

Proof We divide the proof into two cases based on the parity of $2s$.

(i) Case 1: $2s$ is even. The Fourier multiplier corresponding to the fundamental solution of $D|D|^{2s}$ is

$$c_{n,k}\frac{1}{\xi}\frac{1}{|\xi|^{2s}}=c_{n,k}\frac{\overline{\xi}}{|s|^{2s+2}},$$

where $c_{n,k}$ is a constant depending on n and k. A fundamental solution of $|D|^{2s+2}$ is a radial function and is the same as the one in the above list for $\underline{\partial}^{2s+2}$ in $\mathbb{R}^{n+1}$. We denote the fundamental solution by $K(x)$. By (3.14) and (3.15), when $n+1$ is even and $2s+2<n+1$,

$$K(x) = \frac{1}{|x|^{n-2s-1}}.$$

When $n+1$ is even and $2s+2 \geqslant n+1$,

$$K(x) = (c \log |x| + d) \frac{1}{|x|^{n-2s-1}}.$$

Then $\overline{D}K$ is a fundamental solution of $D|D|^{2s}$. Hence the function $\overline{D}K$ can be represented as follows. When $n+1$ is even and $2s+2 < n+1$,

$$K(x) = \frac{\overline{x}}{|x|^{n-2s+1}}.$$

When $n+1$ is even and $2s+2 \geqslant n+1$,

$$K(x) = (c \log |x| + d) \frac{\overline{x}}{|x|^{n-2s+1}}.$$

(ii) Case 2: $2s$ is odd. At first, because $\xi\overline{\xi} = |\xi|^2$, $\frac{1}{\xi}\frac{1}{|\xi|^{2s}} = \frac{1}{|\xi|}\frac{\overline{\xi}}{|\xi|^{2s+1}}$. Also, the Fourier multiplier corresponding to a fundamental solution of $D|D|^{2s-1}$ is $\frac{\overline{\xi}}{|\xi|^{2s+1}}$. By (3.14) and (3.15), when $n+1$ is odd, the fundamental solution is $\overline{x}/|x|^{n-2s+2}$. Because the Fourier transform of $1/|\xi|$ is the Riesz potential $1/|x|^n$, then in the tempered distributional sense, the fundamental solution of $D|D|^{2s}$ can be represented via convolution:

$$\frac{1}{|\cdot|^n} * \frac{\overline{(\cdot)}}{|\cdot|^{n-2s+2}}.$$

It is easy to see that the convolution is a locally integrable function away from the origin. In fact, after being applied a certain times Laplace operator, the above distribution becomes a locally integrable function away from the origin. Secondly, as a distribution, the convolution is homogeneous of degree $2s-n$. To show this, letting M and N denote the distributions induced by $\frac{1}{|x|^n}$ and $\frac{\overline{x}}{|x|^{n-2s+2}}$, respectively, then for any Schwartz function ϕ, we have

$$\langle M * N(x),\ \phi(x/\delta)\rangle = \delta^{(n+1)+(2s-n)}\langle M * N(x),\ \phi(x)\rangle.$$

Write $\tau_\delta f(x) = f(\delta x)$. By the homogeneous properties of M and N, we know

$$\begin{aligned}\Big\langle M * N(x),\ \phi(x/\delta)\Big\rangle &= \Big\langle M * N,\ \tau_{\delta^{-1}}\phi(x)\Big\rangle \\ &= \Big\langle N(x),\ M * (\tau_{\delta^{-1}}\phi)(x)\Big\rangle \\ &= \delta\Big\langle N(x),\ \tau_{\delta^{-1}}M * \phi(x)\Big\rangle\end{aligned}$$

$$= \delta^{1+2s}\Big\langle N,\ M * \phi\Big\rangle$$
$$= \delta^{1+2s}\Big\langle M * N,\ \phi\Big\rangle.$$

Let ρ denote the rotation about the origin in $\mathbb{R}_1^n$. The representation matrix of ρ is (ρ_{ij}) and the operation of ρ on x is denoted by $\rho^{-1}x$. The operation of ρ on the functions is denoted by $\rho(f)(x) = f(\rho^{-1}x)$. Because M is scalar-valued and N is vector-valued, the function $M * N$ is vector-valued and homogeneous with degree $2s - n$. Write this vector-valued function as $K(x) = M * N(x)$. Then we can get

$$\begin{aligned}\Big\langle \rho\overline{K(x)},\ \phi(x)\Big\rangle &= \Big\langle \overline{K(x)},\ \rho^{-1}\phi(x)\Big\rangle \\ &= \Big\langle \overline{N(x)},\ M * \rho^{-1}\phi(x)\Big\rangle \\ &= \Big\langle \overline{N(\rho^{-1}x)},\ M * \phi(x)\Big\rangle \\ &= \Big\langle (\rho_{ij})\overline{(N(x))},\ M * \phi(x)\Big\rangle \\ &= (\rho_{ij})\Big\langle \overline{K(x)},\ \phi(x)\Big\rangle \\ &= \Big\langle (\rho_{ij})\overline{(K(x))},\ \phi(x)\Big\rangle,\end{aligned}$$

that is, $\overline{K}(\rho^{-1}x) = \rho(\overline{K(x)})$. Applying the lemma obtained in [10, Chap. 3, Sect. 1.2] to $\overline{K(x)}/|x|^{2s-n}$, we get $\overline{K(x)}/|x|^{2s-n} = Cx/|x|$, Hence

$$M * N(x) = \frac{C\overline{x}}{|x|^{n-2s+1}}.$$

□

We prove when $l \in \mathbb{Z}_+$, the function

$$\Delta^{k+(n-1)/2}\Big(\Big(x_0 + \underline{x}\Big)^{-l} P_k(\underline{x})\Big)$$

is left monogenic. We need the intertwining relation for the operator.

Lemma 3.5.5 *Let n be any positive integer. Then for $s = k + (n-1)/2$ and any infinitely differentiable function g in $\mathbb{R}_1^n \setminus \{0\}$, we have*

$$(D\Delta^s)\Big(\frac{\overline{x}}{|x|^{(n+1)-2s}} g(x^{-1})\Big) = \alpha_{n,s}\frac{x}{|x|^{(n+1)+2s+2}}(D\Delta^s)(g)(x^{-1}), \qquad (3.38)$$

where $\alpha_{n,s}$ is a constant depending on n and s.

Proof Write $L = D\Delta^s = D|D|^{2s}$. Because $n+1$ is odd, by Case 2 of Lemma 3.5.4, the fundamental solution of L is $G(x) = \frac{C\overline{x}}{|x|^{n-2s+1}}$. We have

$$\begin{aligned}
&L^{-1}\left(\frac{(\cdot)}{|(\cdot)|^{(n+1)+2s+2}}(Lg)((\cdot)^{-1})\right)(x^{-1})\\
&=\int_{\mathbb{R}_1^n} G(x^{-1}-y^{-1})\frac{y^{-1}}{|y^{-1}|^{(n+1)+2s+2}}\frac{1}{|y|^{2n+2}}(Lg)(y)dy\\
&=\frac{C\overline{x^{-1}}}{|x^{-1}|^{n-2s+1}}\int_{\mathbb{R}_1^n}\frac{-\overline{(x-y)}}{|x-y|^{n-2s+1}}\frac{\overline{y^{-1}}}{|y^{-1}|^{n-2s+1}}\\
&\quad\times\frac{y^{-1}}{|y^{-1}|^{(n+1)+2s+2}}\frac{1}{|y|^{2n+2}}(Lg)(y)dy\\
&=\frac{C\overline{x^{-1}}}{|x^{-1}|^{n-2s+1}}\int_{\mathbb{R}_1^n}\frac{\overline{x-y}}{|x-y|^{n-2s+1}}(Lg)(y)dy\\
&=\frac{C\overline{x^{-1}}}{|x^{-1}|^{n-2s+1}}g(x).
\end{aligned}$$

Then we can deduce that

$$L\left(\frac{\overline{(\cdot)}}{|\cdot|^{(n+1)-2s}}g((\cdot)^{-1})\right)(x)=C\frac{x}{|x|^{(n+1)+2s+2}}(Lg)(x^{-1}).$$

□

In Lemma 3.5.5, take $g(x)=\left(\frac{\overline{x}}{|x|^2}\right)^l P_k(\underline{x}),\ \ l\in\mathbb{Z}_+$. Because $g(x^{-1})=(-1)^k x^l|x|^{-2k}P_k(\underline{x})$, we have

$$\left(D\Delta^{k+(n-1)/2}\right)\left((-1)^k x^{l-1}P_k(\underline{x})\right)=\alpha_{n,s}\frac{x}{|x|^{2n+2k+2}}\left(D\Delta^{k+(n-1)/2}\right)\left(\left(\frac{\bar{\cdot}}{|\cdot|^2}\right)^l P_k(\underline{\cdot})\right)(x^{-1}). \tag{3.39}$$

By Proposition 3.5.1, we can see that the right hand side of (3.39) is zero and conclude that

$$\left(D\Delta^{k+(n-1)/2}\right)\left((x_0+\underline{x})^{l-1}P_k(\underline{x})\right)=0,\ l\in\mathbb{Z}_+.$$

Based on the following preliminary lemma, we give a generalization of Theorem 3.5.1.

Theorem 3.5.2 *Let f be a holomorphic function defined on an open set B in the upper half complex plane. Define the set*

$$\overrightarrow{B}=\left\{x=x_0+\underline{x}\in\mathbb{R}_1^n,\ (x_0,|\underline{x}|)\in B\right\}.$$

(i) Let $P_k(\underline{x})$ be left-monogenic and homogeneous of degree k. If $k+(n-1)/2$ is a non-negative integer, then in the set $\overrightarrow{B}$, the function

$$\Delta^{k+(n-1)/2}[f(x_0+\underline{x})P_k(\underline{x})]$$

is left monogenic.

(ii) If $(n-1)/2$ *is odd and* k *is a non-negative integer and* $P_k(\underline{x})$ *is monogenic and homogeneous of degree* k*, then in the set* $\overrightarrow{B}$*, the function*

$$\Delta^{k+(n-1)/2}[f(x_0+\underline{x})P_k(\underline{x})]$$

is left monogenic.

Proof We only need to prove that if the function

$$\Delta^{k+(n-1)/2}((x_0+\underline{x})^l P_k(\underline{x})),\ l\in\mathbb{Z},$$

is monogenic, then the function

$$\Delta^{k+(n-1)/2}(f(x_0+\underline{x})P_k(\underline{x}))$$

is also monogenic. Through a translation, we may assume that the function f is holomorphic in a disc centered at the origin of the complex plane. Further, we define the holomorphic function

$$\begin{cases} g(z)=\frac{1}{2}[f(z)+\overline{f}(\overline{z})], \\ h(z)=\frac{1}{2i}[f(z)-\overline{f}(\overline{z})]. \end{cases}$$

It is easy to see that $f(z)=g(z)+ih(z)$. Then we can further assume that the Taylor series expansion of f is of real coefficients. We will prove:

(i) the series $\sum_{l=-\infty}^{-1} c_l z^l$ and

$$\sum_{l=-\infty}^{-1} c_l \Delta^{k+(n-1)/2}[(x_0+\underline{x})^l P_k(\underline{x})]$$

have the same convergence radius;

(ii) the series $\sum_{l=0}^{\infty} c_l z^l$ and

$$\sum_{l=0}^{\infty} c_l \Delta^{k+(n-1)/2}[(x_0+\underline{x})^l P_k(\underline{x})]$$

have the same convergence radius.

For (i), it can be deduced from (3.36) and (3.25) that

$$|\Delta^{k+(n-1)/2}[(x_0+\underline{x})^l P_k(\underline{x})]| \leqslant C(1+|l|)^{n+2k}\frac{1}{|x|^{n+k+|l|-1}},$$

which implies that the two series in (i) have the same convergence radius.

At last, we prove (ii). For this case, n is even. Because $\Delta^s = |D|^{-1}\Delta^{k+n/2}$, the fundamental solution of Δ^s can be represented as the convolution of Riesz potential $1/|x|^n$ and the fundamental solution of $\Delta^{k+n/2}$. Under the case that the spatial dimension is odd, the fundamental solution of $\Delta^{k+n/2}$ is $C/|x|^{(n+1)-2s-1}$, where C is a constant depending on n and k. By Lemma 3.5.4, the fundamental solution of Δ^s can be represented as $C/|x|^{n+1-2s}$. Then applying Lemma 3.5.5, we can get

$$(\Delta^s)\left(\frac{1}{|x|^{(n+1)-2s}}g(x^{-1})\right) = \frac{C}{|x|^{(n+1)+2s+2}}(\Delta^s)(g)(x^{-1}).$$

Let $g(x) = \left(\frac{\overline{x}}{|x|^2}\right)^l P_k(\underline{x})$. Then $g(x^{-1}) = (-1)^k x^l P_k(\underline{x})$. Replacing s by $s+1$, we have

$$\Delta^{k+1+(n-1)/2}\Big((-1)^k x^l P_k(\underline{x})\Big) = \frac{C}{|x|^{2n+2k+2}}\Delta^{(k+1)+(n-1)/2}\left(\left(\frac{\overline{x}}{|x|^2}\right)^l P_k(\underline{x})\right)(x^{-1}).$$

By the Newton potential and (3.36), in the sense of distributions,

$$\Delta^{k+1+(n-1)/2}\Big(x^l P_k(\underline{x})\Big) = \frac{C}{(l-1)!}\int_{\mathbb{R}_1^n}\frac{1}{|x-y|^{n-1}}\frac{1}{|y|^{2n+2k-2}}\partial_0^{l-1}\Delta E P_k(\underline{x})(y^{-1})dy.$$

By Lemma 3.5.4,

$$|\Delta^{k+(n-1)/2}[(x_0+\underline{x})^l P_k(\underline{x})]| \leqslant C(1+|l|)^{n+2k}|x|^{l-k-n+1}.$$

□

3.6 Remarks

Remark 3.6.1 The idea of Theorem 3.5.1 is to investigate the similarity between the Clifford analysis and the complex analysis of single variable. Via the correspondence $z^k \to P^{(k)}$, some similarity has been obtained in [11].

The quaternionic space does not coincide with our result for $n = 3$. The quaternion forms a complete algebra, and the latter is not a complete algebra. Fueter's theorem implies that τ maps a holomorphic function of one variable to a regular function of variables in the quaternionic space. M. Sce generalized Fueter's result and proved that if n is odd, then τ maps the holomorphic functions defined on the subset in the upper-half complex plane to the monogenic functions. Theorem 3.5.1 (iii) indicates that if n is odd, the result obtained by the Kelvin inversion coincides with the result for $f^0(z) = z^k,\ k \in \mathbb{Z}$ obtained by Sce.

However, for even n, the method of using the the differential operator $\Delta^{(n-1)/2}$ introduced by Fueter and Sce is not valid. By the Fourier multiplier transform, the results of Fueter and Sce can be extended to the case of the power function with negative index, that is, $f^0(z) = z^k$, $-k \in \mathbb{Z}^+$; while for the power function with non-negative index, this method is not directly valid.

Remark 3.6.2 There is the following generalization of the result in Sect. 3.5. In [12], F. Sommen proved that if $n + 1$ is a positive even integer, P_k is any homogeneous polynomial in $\underline{x}$ of degree k, and is left monogenic for the Dirac $\underline{D}$: $\underline{D}P_k(\underline{x}) = 0$, then

$$D\Delta^{k+(n-1)/2}\left(\left(u(x_0, \underline{x}) + \frac{\underline{x}}{|\underline{x}|}v(x_0, \underline{x})P_k(\underline{x})\right)\right) = 0.$$

It is readily seen that the above result is a special case of Theorem 3.5.2.

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Chapter 4
Convolution Singular Integral Operators on Lipschitz Surfaces

As the high-dimensional generalization of the boundedness of singular integrals on Lipschitz curves, the $L^p(\Sigma)$-boundedness of the Cauchy-type integral operators on the Lipschitz surfaces Σ is a meaningful question. The increase of the dimensions means that we need to apply a new method to solve the above question. In 1994, C. Li, A. McIntosh and S. Semmes embedded $\mathbb{R}^{n+1}$ into Clifford algebra $\mathbb{R}_{(n)}$ and considered the class of holomorphic functions on the sectors $S_{w,\pm}$, see [1]. They proved that if the function ϕ belongs to $K(S_{w,\pm})$, then the singular integral operator T_ϕ with the kernel ϕ on Lipschitz surface is bounded on $L^p(\Sigma)$.

In [2], G. Gaudry, R. Long and T. Qian applied Clifford-valued martingales to prove the same result as is proved in [1], that i.e., the L^2-boundedness of the Cauchy integral operators on Lipschitz surfaces [2]. The authors of [2] then indicated how to prove the Clifford $T(b)$ theory. The idea of the proof is similar to that of [3], but there is some difference. We define a suitable sequence of atomic σ-fields on $\mathbb{R}^n$. Because Clifford algebra is non-commutative, it is necessary to associate each atom with a pair of Clifford-valued Haar functions. Hence, the appropriate Haar system is in fact a system of pairs of Clifford-valued functions. We only use the martingale technique to prove the L^2-norm equivalence between the function f and its Littlewood–Paley function $S(f)$.

4.1 Clifford-Valued Martingales

We first state some backgrounds of the martingales and the Littlewood–Paley estimate of Clifford-valued functions. Let X be a set and $\mathcal{B}$ be a σ-field in X. Assume that ν is a non-negative measure on $\mathcal{B}$ and $\{\mathcal{F}_m\}_{m=-\infty}^{\infty}$ is a non-decreasing family of σ-field in X satisfying

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(i) $\bigcup_{m=-\infty}^{\infty} \mathcal{F}_m$ generates $\mathcal{B}$;

(ii) $\bigcap_{m=-\infty}^{\infty} \mathcal{F}_m = \{\emptyset,\ X\}$;

(iii) the measure ν is $\sigma-$finite on $\mathcal{B}$ and on each $\mathcal{F}_m$.

Let $\mathcal{F}$ be a sub-$\sigma-$field of $\mathcal{B}$ such that ν is σ-finite on $\mathcal{F}$. Because $(X,\ \mathcal{F})$ is $\sigma-$finite, X can be written as $X = \bigcup_j U_j$, where $U_j \in \mathcal{F}$ and $\nu(U_j) < +\infty$. If f is a locally integrable scalar-valued function on $(X, \mathcal{B}, \nu)$, i.e., a function whose integral is finite on every set of finite ν-measure, its conditional expectation $\widetilde{E}(f \mid \mathcal{F})$ is well-defined. On each U_j, $\widetilde{E}(f \mid \mathcal{F})$ equals to the conditional expectation of $f\ |_{U_j}$ with respect to $(\mathcal{F}\ |_{U_j},\ \nu\ |_{U_i})$. If A is any set in $\mathcal{F}$ with finite $\nu-$measure, then

$$\int_A \widetilde{E}(f \mid \mathcal{F}) d\nu = \int_A f d\nu. \tag{4.1}$$

If f is integrable, then (4.1) also holds for any $A \in \mathcal{F}$, whether of finite $\nu-$measure or not.

Let $\mathbb{R}_{(n)}$ denote the Clifford algebra generated by $\{e_0, e_1, \ldots, e_n\}$. The definition of the conditional expectation can be extended to locally integrable $\mathbb{R}_{(n)}$-valued functions. In fact, if $f = \sum_S f_S e_S$, then

$$\widetilde{E}(f \mid \mathcal{F}) = \sum_S \widetilde{E}(f_S \mid \mathcal{F}) e_S.$$

The characteristic martingale property (4.1) holds also for $\mathbb{R}_{(n)}$-valued functions f.

We denote by $L^p(\mathcal{F},\ d\nu;\ \mathbb{R}_{(n)})$ or simply $L^p(d\nu;\ \mathbb{R}_{(n)})$, $1 \leqslant p \leqslant \infty$, the Lebesgue spaces of all $\mathbb{R}_{(n)}$-vauled $\mathcal{F}-$ measurable functions on X. The space $L^1_{\text{loc}}(d\nu; \mathbb{R}_{(n)})$ has the obvious interpretation.

Assume that ψ is a fixed L^∞ function on X with values in $\mathbb{R}^{1+n}$.

Definition 4.1.1 Suppose that $\widetilde{E}(\psi \mid \mathcal{F}) \notin 0$ a.e., and let $f \in L^1_{\text{loc}}(d\nu;\ \mathbb{R}_{(n)})$. Then the left and the right conditional expectations E^l and E^r of f respect to $\mathcal{F}$ are given by the following formulas

$$E^l(f) = E^l(f \mid \mathcal{F}) = \widetilde{E}(\psi \mid \mathcal{F})^{-1} \widetilde{E}(\psi f \mid \mathcal{F}) \tag{4.2}$$

and

$$E^r(f) = E^r(f \mid \mathcal{F}) = \widetilde{E}(f\psi \mid \mathcal{F}) \widetilde{E}(\psi \mid \mathcal{F})^{-1}. \tag{4.3}$$

The left conditional expectation of f respect to $\mathcal{F}_m$ is denoted by $E^l(f \mid \mathcal{F}_m)$ or $E^l_m(f)$, and the right conditional expectation of f respect to $\mathcal{F}_m$ is denoted by $E^r(f \mid \mathcal{F}_m)$ or $E^r_m(f)$.

The mapping properties of E^l and E^r are good only under further assumptions on the function ψ.

Proposition 4.1.1 *Let $1 \leqslant p \leqslant \infty$. The operators E^l and E^r are bounded on L^p if there exists a constant $c_0 > 0$ such that for x a.e.,*

$$c_0^{-1} \leqslant |\widetilde{E}(\psi \mid \mathcal{F})(x)| \leqslant c_0. \tag{4.4}$$

Proof This theorem can be proved via modifying the corresponding argument in [4]. □

If a function $\psi \in L^\infty(X; \mathbb{R}^{1+n})$ and satisfies (4.4), we call this function pseudo-accretive with respect to $\mathcal{F}$. Now we assume that for a general $\mathcal{F}$, the condition (4.4) holds, and for all $\mathcal{F}_m$, the constant in (4.4) is independent of n. That being so, it follows that, if $f \in L^1_{\mathrm{loc}}(d\nu; \mathbb{R}_{(n)})$, then $E^l(f)$ and $E^r(f)$ are locally integrable. The main elementary properties of E^l and E^r are as follows:

Proposition 4.1.2 (a) *If $g \in L^\infty(\mathcal{F}, d\nu; \mathbb{R}_{(n)})$, then $E^l(fg) = E^l(f)g$. Similarly, the right conditional expectation E^r commutes with the multiplication on the left by g.*

(b) $E^l(1) = E^r(1) = 1$.

(c) *If $f \in L^1_{\mathrm{loc}}(d\nu; \mathbb{R}_{(n)})$ and A is of finite measure (or $f \in L^1(d\nu; \mathbb{R}_{(n)})$ and A is $\mathcal{F}$-measurable), then*

$$\int_A \psi E^l(f) d\nu = \int_A \psi f d\nu, \tag{4.5}$$

$$\int_A E^r(f)\psi d\nu = \int_A f\psi d\nu. \tag{4.6}$$

(d) *For $m \leqslant \kappa$, we have*

$$E_m(E_\kappa(f)) = E_m(f), \tag{4.7}$$

where E_m denotes the left (or right) conditional expectation with respect to $\mathcal{F}_m$.

(e) *Set $\Delta^l_m = E^l_m - E^l_{m-1}$, $\Delta^r_m = E^r_m - E^r_{m-1}$, and*

$$\langle f,\ g\rangle_\psi = \int f\psi g d\nu.$$

We have for all $m \neq \kappa$ and $f, g \in L^2(d\nu; \mathbb{R}_{(n)})$,

$$\langle \Delta^r_m f,\ \Delta^l_\kappa g\rangle_\psi = 0.$$

Proof (a) and (b) are obvious. To prove (c), assume that $A \in \mathcal{F}$. Because $E^l f$ and A is $\mathcal{F}-$measurable,

$$\int\limits_A \psi E^l f d\nu = \int\limits_X \chi_A \psi E^l f d\nu = \int\limits_X \widetilde{E}(\chi_A \psi E^l f) d\nu = \int\limits_X \chi_A \widetilde{E}(\psi) E^l f d\nu = \int\limits_A \psi f d\nu.$$

For E^r, we can give a similar proof and so is omitted.

The conclusion (d) can be proved as follows. For example, for the left conditional expectation,

$$\begin{aligned} E^l_m(E^l_\kappa(f)) &= \widetilde{E}_m(\psi)^{-1}\widetilde{E}_m(\psi\widetilde{E}_\kappa(\phi)^{-1}\widetilde{E}_\kappa(\psi f)) \\ &= \widetilde{E}_m(\psi)^{-1}\widetilde{E}_m(\widetilde{E}_\kappa[\psi\widetilde{E}_\kappa(\psi)^{-1}\widetilde{E}_\kappa(\psi f)]) \\ &= \widetilde{E}_m(\psi)^{-1}\widetilde{E}_m(\psi f) = E^l_m(f). \end{aligned}$$

The proof for the right conditional expectation is similar.

At last, we prove (e). For $n > \kappa$,

$$\begin{aligned} \langle \Delta^r_m,\ \Delta^l_\kappa g\rangle_\psi &= \int \Delta^r_m f\psi\Delta^l_\kappa g d\nu \\ &= \int \widetilde{E}_{m-1}(\Delta^r_m f\psi\Delta^l_\kappa g)d\nu \\ &= \int \widetilde{E}_{m-1}(\Delta^r_m f\psi)\Delta^l_\kappa g d\nu \\ &= \int \widetilde{E}_{m-1}(\Delta^r_m f\psi)\widetilde{E}_{m-1}(\psi)^{-1}\widetilde{E}_{m-1}(\psi)\Delta^l_\kappa g d\nu \\ &= \int \widetilde{E}^r_{m-1}(\Delta^r_m f)\widetilde{E}_{m-1}(\psi)\Delta^l_\kappa g d\nu = 0, \end{aligned}$$

where in the last step we have used (4.7). The proof for $\kappa > n$ is similar. □

Definition 4.1.2 Let $f \in L^1_{\text{loc}}(d\nu; \mathbb{R}_{(n)})$. The left martingale with respect to $\{\mathcal{F}_m\}_{m=-\infty}^{\infty}$ generated by f is the sequence $\{f^l_m\}_{m=-\infty}^{\infty} = \{E^l_m(f)\}_{m=-\infty}^{\infty}$. If the limit $f^l_{-\infty} = \lim_{m\to-\infty} E^l_m(f)$ exists a.e., the left-Littlewood–Paley square function $S^l(f)$ is defined by

$$S^l(f) = \Big(|f^l_{-\infty}|^2 + \sum_{m=-\infty}^{\infty} |\Delta^l_m f|^2\Big)^{1/2}.$$

The right martingale and the right-Littlewood–Paley square function can be defined similarly. If $f \in \bigcup_{1\leqslant p<\infty} L^p(d\nu; \mathbb{R}_{(n)})$ and $\nu(X) = +\infty$, then $f^l_{-\infty} = 0$.

If $f \in L^1_{\text{loc}}(d\nu; \mathbb{R}_{(n)})$, then the BMO-norm of f is defined as

$$\|f\|_{BMO} = \sup_m \|\widetilde{E}_m(|f - \widetilde{E}_{m-1}f|^2)\|^{1/2}_\infty. \tag{4.8}$$

We need the following facts: if $\psi \in L^\infty(d\nu; \mathbb{R}^{1+n})$ then $\psi \in BMO$ and for every m,

$$\widetilde{E}_m\Big(\sum_{k=m}^{\infty} |\widetilde{\Delta}_k(\psi)|^2\Big) \leqslant C\|\psi\|^2_{BMO} \leqslant C\|\psi\|^2_\infty. \tag{4.9}$$

By the John–Nirenberg inequality, the right hand side of (4.8) is equivalent to

$$\sup_m \left\| \widetilde{E}_m\Big(\Big|f - \widetilde{E}_m(f)\Big|\Big)\right\|_\infty,$$

see [5, 6] for the proof.

The following Littlewood–Paley result is one of the essential ingredients of this chapter. We use C to denote a constant which may vary from line to line.

Lemma 4.1.1 *There exists a constant* $c > 0$ *depending only on* c_0 *and* d *such that for all* $f \in L^2_{\mathrm{loc}}(d\nu; \mathbb{R}_{(n)})$,

$$c^{-1}\|S(f)\|_{L^2} \leqslant \|f\|_{L^2} \leqslant c\|S(f)\|_{L^2}, \tag{4.10}$$

where S *denotes* S^l *or* S^r.

Proof We only consider the case of left martingales and the case of right martingales can be dealt with similarly. Fix m_0. Consider the sequence $\{\mathcal{F}_m\}_{n\geqslant m_0}$ and the corresponding square function:

$$\Big(\sum_{m\geqslant m_0+1} |\Delta^l_m f|^2\Big)^{1/2}.$$

If $n \geqslant n_0 + 1$, we have

$$\begin{aligned}
\Delta^l_m f &= \widetilde{E}(\psi \mid \mathcal{F}_m)^{-1}\widetilde{E}(\psi f \mid \mathcal{F}_m) - \widetilde{E}(\psi \mid \mathcal{F}_{m-1})^{-1}\widetilde{E}(\psi f \mid \mathcal{F}_{m-1})\\
&= \Big[\widetilde{E}(\psi \mid \mathcal{F}_m)^{-1} - \widetilde{E}(\psi \mid \mathcal{F}_{m-1})^{-1}\Big]\widetilde{E}(\psi f \mid \mathcal{F}_m)\\
&\quad +\widetilde{E}(\psi \mid \mathcal{F}_{m-1})^{-1}\Big[\widetilde{E}(\psi f \mid \mathcal{F}_m)^{-1} - \widetilde{E}(\psi f \mid \mathcal{F}_{m-1})^{-1}\Big].
\end{aligned} \tag{4.11}$$

Hence by (4.4),

$$|\delta^l_m(f)|^2 \leqslant C\Big(|\widetilde{\Delta}_m(\psi)|^2|\widetilde{E}(\psi f \mid \mathcal{F}_m)|^2 + |\widetilde{\Delta}_m(\psi f)|^2\Big). \tag{4.12}$$

Because ν is σ-finite on $\mathcal{F}_{m_0}$, we can write $X = \bigcup_{j=1}^{\infty} U_j$, where $U_1 \subseteq U_2 \subseteq \cdots$, and the set $U_j \subset \mathcal{F}_{m_0}$ that has a finite measure. Fix $M \geqslant 1$. Then by (4.12) and the standard Littlewood–Paley estimate, we get

$$\begin{aligned}
&\int_{U_M} \sum_{m\geqslant m_0+1} |\Delta^l_m f|^2 \qquad (4.13)\\
&\leqslant C\Big(\int_{U_M} \sum_{m\geqslant m_0+1} |\widetilde{E}_m(\psi f \mid \mathcal{F}_m)|^2|\widetilde{\Delta}_n\psi|^2 d\nu + \int_{U_M} \sum_{m\geqslant m_0+1} |\widetilde{\Delta}_m(\psi f)|^2 d\nu\Big)
\end{aligned}$$

$$\leqslant C\Big(\int\limits_{U_M}\sum_{m\geqslant m_0+1}|\widetilde{E}^*_m(\psi f)|^2|\widetilde{\Delta}_m(\psi)|^2d\nu+\int\limits_X|\psi f|^2d\nu\Big)$$
$$\leqslant C\Big(\int\limits_{U_M}\sum_{m\geqslant m_0+1}|\widetilde{E}^*_m(\psi f)|^2|\widetilde{\Delta}_m(\psi)|^2d\nu+\int\limits_X|f|^2d\nu\Big),$$

where

$$\widetilde{E}^*_m(f)=\sup_{m_0+1\leqslant j\leqslant m}\Big|\widetilde{E}(f\mid\mathcal{F}_j)\Big|.$$

For $m\geqslant m_0+1$, let $T_m=\sum_{k=m}^{\infty}|\widetilde{\Delta}_k\psi|^2$ and set $T_{m_0}=0$. If $N>m_0$, we have

$$\begin{aligned}\sum_{m=m_0+1}^{N}|\widetilde{E}^*_m(\psi f)|^2|\widetilde{\Delta}_m(\psi)|^2&=\sum_{m=m_0+1}^{N}|\widetilde{E}^*_m(\psi f)|^2(T_m-T_{m+1})\\&=\sum_{m=m_0}^{N-1}T_{m+1}\Big[|\widetilde{E}^*_{m+1}(\psi f)|^2-|\widetilde{E}^*_m(\psi f)|^2\Big]-|\widetilde{E}^*(\psi f)|^2T_{N+1}.\end{aligned}$$

It can be deduced from (4.9) and (4.14) that

$$\begin{aligned}&\int\limits_{U_M}\sum_{m\geqslant m_0+1}|\widetilde{E}^*_m(\psi f)|^2|\widetilde{\Delta}_m(\psi)|^2d\nu \qquad (4.14)\\&\leqslant\int\limits_{U_M}\sum_{m=n_0}^{\infty}\Big(\sum_{k=m+1}^{\infty}|\widetilde{\Delta}_k(\psi)|^2\Big)\Big[|\widetilde{E}^*_{m+1}(\psi f)|^2-|\widetilde{E}^*_n(\psi f)|^2\Big]d\nu\\&\leqslant\int\limits_{U_M}\sum_{m=m_0}^{\infty}\widetilde{E}_{m+1}\Big(\sum_{k=m+1}^{\infty}|\widetilde{\Delta}_k(\psi)|^2\Big)\Big[|\widetilde{E}^*_{m+1}(\psi f)|^2-|\widetilde{E}^*_m(\psi f)|^2\Big]d\nu\\&\leqslant\|\psi\|^2_{BMO}\int\limits_{U_M}|\psi f|^{*2}d\nu\\&\leqslant C\|\psi\|^2_{\infty}\int\limits_{U_M}|f|^2d\nu.\end{aligned}$$

In the last step, we have used the $L^2(U_M)$-boundedness of the maximal function. The constant is independent of M or m_0.

By (4.13) and (4.14), we can obtain

$$\int\limits_{U_M}\sum_{m\geqslant m_0+1}|\Delta^l_m f|^2d\nu\leqslant C\int\limits_{U_M}|f|^2d\nu. \qquad (4.15)$$

In (4.15), letting $M\to\infty$ and then letting $m_0\to-\infty$, we can conclude that the inequality on the left hand side of (4.10).

To prove the inequality on the left hand side of (4.10) we need the following facts. If $g \in L^2(d\nu; \mathbb{R}_{(n)})$, then

(a) $\lim\limits_{m\to+\infty} E^l_m g = g = \lim\limits_{m\to+\infty} E^r_m g$ in the sense of L^2.

(b) $\lim\limits_{m\to-\infty} E^l_m g = 0 = \lim\limits_{m\to-\infty} E^r_m g$ in the sense of L^2-.

(c) $g = \sum\limits_{m=-\infty}^{\infty} \Delta^l_m g = \sum\limits_{m=-\infty}^{\infty} \Delta^r_m g.$

These facts can be proved in the same way as the corresponding scalar-valued results in [5, Chap. 5]. Of course, the condition (4.4) is crucial in the proofs.

Suppose that $f, g \in L^2(d\nu; \mathbb{R}_{(n)})$. By (4.4) and the right hand inequality in (4.10), we can get

$$\begin{aligned} \Big|\int_X f\psi g d\nu\Big| &= \Big|\int_X \Big(\sum_{m=-\infty}^{\infty} \Delta^r_m g\Big)\psi\Big(\sum_{\kappa=-\infty}^{\infty} \Delta^l_\kappa f\Big)d\nu\Big| \qquad (4.16)\\ &= \Big|\int_X \Big(\sum_{m=-\infty}^{\infty} \Delta^r_m g\psi\Delta^l_n f\Big)d\nu\Big| \\ &\leqslant C\|S^r g\|_2\|S^l(f)\|_2. \end{aligned}$$

In (4.16), taking supremum over all g satisfying $\|g\|_2 \leqslant 1$ and using again the condition (4.4), we complete the proof. □

We now construct a special example, and the associated Haar functions are appropriate to the analysis of the Cauchy integral. Let $X = \mathbb{R}^n$ and $\mathcal{B}$ be the Borel $\sigma-$field. Assume that $d\nu$ is the Lebesgue measure, also denoted by dx. The Lebesgue measure of a measurable set U is denoted by $|U|$. Let $\mathcal{F}_0$ be the $\sigma-$ field generated by the family $\mathcal{J}_0$ of cubes with side length 1 whose corners lie at the points of the integer lattice.

Let I be any cube in $\mathcal{J}_0$. Divide I equally by the hyperplane that bisects the edges parallel to the x_1-axis, and let $\mathcal{J}_1$ denote the family of dyadic-quasi-cubes so produced. Let $\mathcal{F}_1$ be the $\sigma-$ generated by $\mathcal{J}_1$. Now subdivide each dyadic-quasi-cube by the hyperplane that bisects the edges parallel to the x_2-axis, and let $\mathcal{F}_2$ be the $\sigma-$field generated by the new family of dyadic-quasi-cubes.

Continue in this manner, at each stage bisecting each dyadic-quasi-cube of the previous family by the hyperplane perpendicular to the next coordinate axis. This produces the sequence $\{\mathcal{F}_m\}_{m=0}^{\infty}$. For $m < 0$, the $\sigma-$ field $\mathcal{F}_m$ are produced by the reverse procedure to the one just described-successive doubling in the coordinate directions. Note that each dyadic-quasi-cube in $\mathcal{F}_{kn}$, $k \in \mathbb{Z}$, i.e., atom, is actually a standard dyadic cube of side length 2^{-k}.

At last, let $\mathcal{J} = \bigcup\limits_{m=-\infty}^{\infty} \mathcal{J}_m$. Note that any $I \in \mathcal{J}$ is a dyadic-quasi-cube, say $I \in \mathcal{J}_{m-1}$, and so can be written as $I = I_1 \cup I_2$, where I_1 and I_2 are dyadic-quasi-cubes in $\mathcal{J}_m$.

From now on, we only discuss the left martingale. Hence we simplify the notation by writing E_m, Δ_m, f_m etc. in place of E_m^l, Δ_m^l, f_m^l etc. We still assume that the function $\psi \in L^\infty(X : \mathbb{R}^{1+n}) = L^\infty(\mathbb{R}^n; \mathbb{R}^{1+n})$ satisfies (4.4), but corresponds to the particular sequence $\{\mathcal{F}_m\}_{-\infty}^{\infty}$ in the $\sigma-$field. The following lemma is an essential ingredient of this chapter.

Lemma 4.1.2 *For any $I \in \mathcal{J}_{m-1}$, where $I = I_1 \cup I_2$ with $I_1, I_2 \in \mathcal{J}_m$, there exist a pair of $\mathbb{R}_{(n)}$-valued functions α_I and β_I on $\mathbb{R}^n$ and a positive constant C such that*

(i)

$$\alpha_I = a_1\chi_{I_1} + a_2\chi_{I_2},\ a_j \in \mathbb{R}_{(n)},$$
$$\beta_I = b_1\chi_{I_1} + b_2\chi_{I_2},\ b_j \in \mathbb{R}_{(n)};$$

(ii) *For all $f \in L^1_{\text{loc}}(\mathbb{R}^n; \mathbb{R}_{(n)})$,*

$$\Delta_m f(x) = \alpha_I(x)\langle\beta_I,\ f\rangle_\psi,\ x \in I;$$

(iii) $C^{-1}|I|^{-1/2} \leqslant |\alpha_I(x)| \leqslant C|I|^{-1/2}$, *and for all $x \in I$,* $C^{-1}|I|^{-1/2} \leqslant |\beta_I(x)| \leqslant C|I|^{-1/2}$;

(iv)

$$\int \psi\alpha_I dx = \int \beta_I\psi dx = 0.$$

Proof Define α_I and β_I as in (i). We need to choose a_1, a_2, b_1 and b_2 such that (ii)–(iv) hold.

We consider (ii). Because $\mathcal{F}_m$ and $\mathcal{F}_{m-1}$ are atoms, on I, we have

$$\widetilde{E}_{m-1}f = \Big(\frac{1}{|I|}\int_I f(y)dy\Big)\chi_I.$$

For $\widetilde{E}_m(f)$, a similar formula holds. Let

$$u = \int_I \psi(t)dt,\ u_j = \int_{I_j}\psi(t)dt,\ \ j = 1, 2.$$

Then on I,

$$\begin{aligned}\Delta_m f &= \widetilde{E}(\psi \mid \mathcal{F}_m)^{-1}\widetilde{E}(\psi f \mid \mathcal{F}_m) - \widetilde{E}(\psi \mid \mathcal{F}_{m-1})^{-1}\widetilde{E}(\psi f \mid \mathcal{F}_{m-1})\\ &= u_1^{-1}\Big(\int_{I_1}\psi f dx\Big)\chi_{I_1} + u_2^{-1}\Big(\int_{I_2}\psi f dx\Big)\chi_{I_2}\\ &\quad -u^{-1}\Big(\int_{I_1}\psi f dx + \int_{I_2}\psi f dx\Big)(\chi_{I_1} + \chi_{I_2})\end{aligned}$$

$$= \left((u_1^{-1} - u^{-1}) \int_{I_1} \psi f dx - u^{-1} \int_{I_2} \psi f dx \right) \chi_{I_1}$$
$$+ \left((u_2^{-1} - u^{-1}) \int_{I_2} \psi f dx - u^{-1} \int_{I_1} \psi f dx \right) \chi_{I_2}.$$

On the other hand,

$$\alpha_I \langle \beta_I,\ f \rangle_\psi = \left(a_1 b_1 \int_{I_1} \psi f dx + a_1 b_2 \int_{I_2} \psi f dx \right) \chi_{I_1}$$
$$+ \left(a_2 b_2 \int_{I_2} \psi f dx + a_2 b_1 \int_{I_1} \psi f dx \right) \chi_{I_2}.$$

Comparing the last two expressions, we choose $a_i, b_i, i = 1, 2$, such that

$$a_1 b_1 = u_1^{-1} - u^{-1},\ a_2 b_2 = u_2^{-1} - u^{-1},\ a_1 b_2 = -u^{-1} = a_2 b_1.$$

Letting $u = u_1 + u_2$ and applying the equality

$$a^{-1} - b^{-1} = a^{-1}(b - a)b^{-1} = b^{-1}(b - a)a^{-1}, \tag{4.17}$$

we can see that the above equation has a concise expression:

$$a_1 b_1 = u^{-1} u_2 u_1^{-1},\ a_2 b_2 = u^{-1} u_1 u_2^{-1},\ a_1 b_2 = -u^{-1}, a_2 b_1 = -u^{-1}. \tag{4.18}$$

The solutions of (4.18) can be represented as

$$a_1 = u^{-1} u_2 c,\ a_2 = -u^{-1} u_1 c,\ b_1 = c^{-1} u_1^{-1},\ b_2 = -c^{-1} u_2^{-1}, \tag{4.19}$$

where c is any invertible element in $\mathbb{R}_{(n)}$. We want to choose c such that (iii) holds. In fact, by (i) and (4.19), it is obvious that if c is taken to be $|I|^{-1/2}$, then (iii) holds.

At last, we verify (iv). By (i) and (4.19), we can get

$$\begin{aligned} \int \psi \alpha_I dx &= \int \psi (a_1 \chi_{I_1} + a_2 \chi_{I_2}) dx \\ &= u_1 a_1 + u_2 a_2 \\ &= (u_1 u^{-1} u_2 - u_2 u^{-1} u_1) c \\ &= u_1 u^{-1} (u - u_1) c - (u - u_1) u^{-1} u_1 c = 0. \end{aligned}$$

We can deduce from (4.19) that $\int \beta_I \psi dx = 0$. □

4.2 Martingale Type $T(b)$ Theorem

In this section, we prove the boundedness of Cauchy singular integral operators via the Clifford martingale. The main result is as follows. We suppress the fact that the Cauchy singular integral is a principal value by writing our operators in terms of ordinary integrals. The principal values are to be interpreted as the ones obtained by projecting the Euclidean balls in Σ onto $\mathbb{R}^n$ and integrating over their complements.

Theorem 4.2.1 *If Σ is a Lipschitz graph, then the Cauchy singular integral operator is bounded from $L^2(\Sigma; \mathbb{R}_{(n)})$ to $L^2(\Sigma; \mathbb{R}_{(n)})$.*

Let $\phi(v) = A(v)e_0 + v$ ($v \in \mathbb{R}^n$) be the coordinate system on Σ defined by A. The unit normal of Σ is

$$n(\phi(v)) = (e_0 - \nabla A(v))\sqrt{1 + |\nabla A(v)|^2}.$$

For these coordinates, we have

$$\begin{aligned} T_\Sigma h(\phi(u)) &= \int_{\mathbb{R}^n} \frac{\overline{\phi(v) - \phi(u)}}{|\phi(v) - \phi(u)|^{1+n}} \mathfrak{n}(\phi(v))h(\phi(v))\sqrt{1 + |\nabla A(v)|^2}dv \\ &= \int_{\mathbb{R}^n} \frac{\overline{\phi(v) - \phi(u)}}{|\phi(v) - \phi(u)|^{1+n}} \psi(v)h(\phi(v))dv, \end{aligned}$$

where $\psi(v) = e_0 - \nabla A(v)$. Because $|\nabla A(v)| \leqslant C$, we can see that T_Σ is bounded on $L^2(\Sigma; \mathbb{R}_{(n)})$ if and only if the operator

$$T: f \mapsto \int_{\mathbb{R}^n} \frac{\overline{\phi(v) - \phi(u)}}{|\phi(v) - \phi(u)|^{1+n}} f(v)dv \tag{4.20}$$

is bounded from $L^2(\mathbb{R}^n; \mathbb{R}_{(n)})$ to $L^2(\mathbb{R}^n; \mathbb{R}_{(n)})$.

Notice that if I is a dyadic-quasi-cube, then the principal value integral

$$T(\psi\chi_I)(u) = \text{p.v.} \int_{\mathbb{R}^n} \frac{\overline{\phi(v) - \phi(u)}}{|\phi(v) - \phi(u)|^{1+n}} \psi(v)\chi_I(v)dv$$

exists and defines a locally integrable function. The existence and the local integrability of $T(\psi\chi_I)(u)$ on $\mathbb{R}^n \setminus I$ are straightforward. Moreover, in $\mathbb{R}^n \setminus I$, the singularity of $T(\psi\chi_I)(u)$ is $O(\log(\text{dist}(u, \partial I)))$ as $u \to \partial I$. To deal with the case $u \in I$, we only need to consider

$$T_\Sigma F(x) = \text{p.v.} \int_\Sigma \frac{\overline{y - x}}{|y - x|^{1+n}} \mathfrak{n}(y)F(y)d\sigma(y),$$

where F vanishes outside $\phi(I)$ and satisfies a uniform Lischitz condition. Write

$$\begin{aligned} T_\Sigma F(x) = \text{p.v.} \int_\Sigma \int_\Sigma \frac{\overline{y - x}}{|y - x|^{1+n}} \mathfrak{n}(y)\Big[F(y) - F(x)\Big]d\sigma(y) \\ + \int_\Sigma \frac{\overline{y - x}}{|y - x|^{1+n}} \mathfrak{n}(y)F(x)d\sigma(y). \end{aligned}$$

The Lipschitz condition of F gives an appropriate control on the first integral. By Cauchy's theorem, the monogenicity and cancellation properties of the kernel $\overline{(y-x)}/|y-x|^{1+n}$, we obtain a suitable control on the second integral.

We write the operator in (4.20) as

$$Tf(u) = \int_{\mathbb{R}^n} K(u, v) f(v) dv.$$

In the following lemma, we give some elementary properties of the kernel K.

Lemma 4.2.1 *For all x, x', y such that $x \neq y$ and $|x - x'| < 1/2|x - y|$, the kernel K satisfies*

$$|K(x, y)| \leqslant \frac{C}{|x-y|^n}, \; x \neq y, \tag{4.21}$$

$$|K(x, y) - K(x', y)| \leqslant C\frac{|x - x'|}{|x-y|^{1+n}}, \tag{4.22}$$

and

$$|K(y, x) - K(y, x')| \leqslant C\frac{|x - x'|}{|x-y|^{1+n}}. \tag{4.23}$$

Let $\mathcal{S}$ denote the span over $\mathbb{R}_{(n)}$ of the set of all characteristic functions of dyadic-quasi-cubes. The space $\mathcal{S}\psi$ of pointwise products with the function ψ is a left-linear space over $\mathbb{A}_d$. By use of the idea of [7], we can define $T\psi$ as a Clifford left functional on the subspace $(\mathcal{S}\psi)_0$ of $\mathcal{S}\psi$. The space $(\mathcal{S}\psi)_0$ consists of the functions having integral 0: fix $g\psi \in (\mathcal{S}\psi)_0$ and choose N large enough such that the ball B_N of radius N centered at 0 contains the support of g. Then we define

$$\begin{aligned} T\psi(g\psi) &= T(\psi\chi_{B_N})(g\psi) + \iint g(x)\psi(x)\Big[K(x, y) - K(0, y)\Big]\Big[1 - \chi_{B_N}(y)\Big]\psi(y)dxdy \\ &= I_N^{(1)} + I_N^{(2)}. \end{aligned}$$

By (4.22) and (4.23), this definition is meaningful. An important fact is that

$$\langle \beta_J, \; T\psi\rangle_\psi = T\psi(\beta_J\psi) = 0. \tag{4.24}$$

This can be proved as follows.

(a) When $N \to \infty$, $I_N^{(2)} \to 0$.
(b) By the monogenicity of the Cauchy kernel, using Cauchy's theorem, we can prove that $\lim_{N\to\infty} T(\psi\chi_{B_N})(x)$ exists and is independent of $x \in \text{supp}\beta_J$.

Because the integral of $\beta_J\psi$ is 0, we can conclude that

$$\lim_{N\to\infty} T(\psi\chi_{B_N})(\beta_J\psi) = 0.$$

In establishing (b), one works on the surface Σ.

We note that, if T^t is the operator $f \mapsto \int f(y)K(y,x)dy$, then for all dyadic-quasi-cubes I and J,

$$\langle T^t(\chi_I\psi),\ \chi_J\rangle_\psi = \langle \chi_I,\ T(\psi\chi_J)\rangle_\psi.$$

Similar to T, we have

$$\langle T^t\psi,\ \beta_J\rangle_\psi = T^t\psi(\psi\beta_J) = 0. \tag{4.25}$$

By Lemma 4.1.2, if $f \in L^2(\mathbb{R}^n; \mathbb{R}_{(n)})$, we get

$$f = \sum_{m=-\infty}^{\infty} \Delta_m f = \sum_I \alpha_I \langle \beta_I,\ f\rangle_\psi$$

and

$$\begin{aligned} T(\psi f) &= \sum_{J\in\mathcal{J}} T(\psi\alpha_J)\langle\beta_J,\ f\rangle_\psi \\ &= \sum_{J,I} \alpha_I\langle\beta_I,\ T(\psi\alpha_J)\rangle_\psi\langle\beta_J,\ f\rangle_\psi \\ &= \sum_I \alpha_I \sum_J \langle\beta_I,\ T(\psi\alpha_J)\rangle_\psi\langle\beta_J,\ f\rangle_\psi. \end{aligned}$$

Let $u_{IJ} = \langle\beta_I,\ T(\psi\alpha_J)\rangle_\psi$. By Lemmas 4.1.1 and 4.1.2, we only need to prove the linear transform defined by the matrix (u_{IJ}) on $l^2(\mathcal{J}; \mathbb{R}_{(n)})$ is bounded. We need the following Schur lemma.

Lemma 4.2.2 (Schur) *Assume that there exist a family of positive numbers (ω_I) and a constant C such that*

$$\sum_J |\omega_J u_{IJ}| \leqslant C\omega_I,\ \ I \in \mathcal{J}, \tag{4.26}$$

and

$$\sum_I |\omega_I u_{IJ}| \leqslant C\omega_J,\ \ I \in \mathcal{J}. \tag{4.27}$$

Then the matrix (u_{IJ}) defines a bounded operator on $l^2(\mathcal{J}; \mathbb{R}_{(n)})$.

Proof This is a natural modification of the proof of the scalar version. □

Now we state some facts associated with the estimate of $|\langle\beta_I,\ T(\psi\alpha_J)\rangle_\psi|$. Assume that I and J are atoms in $\mathcal{F}_m$ and $\mathcal{F}_\kappa$, and assume that $m \geqslant \kappa$. If the atom $A \in \mathcal{F}_n$ is not contained in J (or J^c) but a part of its boundary is in common with the boundary of J, then A is said to be contiguous to J (or contiguous to J^c). If the atoms of A are in the same $\sigma-$field as I and are contiguous to J, we denote the union of J and such atoms by $I + J$. Specially, $2J$ denotes the union of J with all of atoms in $\mathcal{F}_m$ which are contiguous to J. The bottom-left corner x_J of J is the vertex of J having minimal coordinates.

Lemma 4.2.3 *Let I and J be atoms of $\mathcal{F}_m$ and $\mathcal{F}_\kappa$, respectively, and $m \geqslant \kappa$. There exists a constant C, independent of κ and m, such that if $I \subseteq 2J \backslash J$, then*

$$\int_{I\times J} \frac{dxdy}{|x-y|^n} \leqslant C|I|\Big(\log\frac{|J|}{|I|}+1\Big).$$

Proof We can prove this lemma via a simple calculation and we omit the details. □

Lemma 4.2.4 *Let I and J be atoms in $\bigcup_{j=-\infty}^{\infty} \mathcal{F}_j$. Then*

(i) *for all $x \notin 2J$,*

$$|T(\psi\alpha_J)| \leqslant C|J|^{1/2+1/n}|x-x_J|^{-1-d}; \tag{4.28}$$

(ii) *if $I \subseteq (2J)^c$, then*

$$|\langle \beta_I,\ T(\psi\alpha_J)\rangle_\psi| \leqslant C|I|^{-1/2}|J|^{1/2+1/n}\int_I |x-x_J|^{-1-n}dx; \tag{4.29}$$

(iii) *for all $x \notin J$,*

$$|T(\psi\alpha_J)(x)| \leqslant C|J|^{-1/2}\int_J |x-y|^{-n}dy;$$

(iv) *if $I \subseteq 2J\backslash J$, then*

$$|\langle \beta_I,\ T(\psi\alpha_J)\rangle_\psi| \leqslant C\frac{|I|^{1/2}}{|J|^{1/2}}\Big(\log\frac{|J|}{|I|}+1\Big).$$

(In the above (i)–(iv), the constant C is independent of I and J).

Proof The assertion (i) can be proved by the canceling property of Haar functions. Hence

$$\begin{aligned} T(\psi\alpha_J) &= \int K(x,y)\psi(y)\alpha_J(y)dy \\ &= \int_J [K(x,y)-K(x,x_J)]\psi(y)\alpha_J(y)dy. \end{aligned}$$

So we can deduce from (4.23) that if $x \notin 2J$, then

$$
\begin{aligned}
|T(\psi\alpha_J)(x)| &\leqslant C|J|^{-1/2}\int_J \frac{|y-x_J|}{|x-x_J|^{1+n}}dy \\
&\leqslant C|J|^{1/2}|x-x_J|^{-1-n}\sup_{y\in J}|y-x_J| \\
&\leqslant C|J|^{1/2+1/d}\frac{1}{|x-x_J|^{1+n}}.
\end{aligned}
$$

To prove (ii), we can use (i) and (iii) of Lemma 4.1.2. The assertion (iii) follows from (4.21). The assertion (iv) is clear from (iii) and Lemma 4.2.3. □

We divide the estimate of

$$\sum_I |I|^t|\langle\beta_I,\ T(\psi\alpha_J)\rangle_\psi|$$

into three parts, each with a number of separate cases based on the relative size and disposition of the atoms I and J.

Case 1. The sum with respect to atoms I larger that J.
Fix $J\in\mathcal{F}_\kappa$ and consider the set $2J$. Let x_J be the bottom-left corner of J. Consider $I\in\mathcal{F}_m, m<\kappa$.

(a) If I lies outside $2J$, by (ii) of Lemma 4.2.4 and (iii) of Lemma 4.1.2, we have

$$|\langle\beta_I,\ T(\psi\alpha_J)\rangle_\psi| \leqslant C|I|^{-1/2}|J|^{1/2+1/n}\int_I |x-x_J|^{-1-n}dx.$$

Hence, in this case, if $t<1/2$, the estimate for the Schur sum is

$$
\begin{aligned}
&\sum_{I\in\bigcup_{m<\kappa}\mathcal{F}_m, I\subseteq(2J)^c} |I|^t|\langle\beta_I,\ T(\psi\alpha_J)\rangle_\psi| \\
&\leqslant C\sum_{j=1}^{\infty}(2^j|J|)^{t-1/2}\sum_{I\in\mathcal{F}_{\kappa-j},\ I\subseteq(2J)^c}|J|^{1/2+1/n}\int_I|x-x_J|^{-1-n}dx \\
&\leqslant C\sum_{j=1}^{\infty}2^{j(t-1/2)}|J|^{t+1/d}\int_{(2J)^c}|x-x_J|^{-1-n}dx \\
&\leqslant C\sum_{j=1}^{\infty}2^{j(t-1/2)}|J|^t \\
&\leqslant C|J|^t.
\end{aligned}
$$

(b) For a fixed $m<\kappa$, the dyadic-quasi-cubes which meet $2J$ are of two kinds: those that lie in $2J\backslash J$, and one that contains J. If I lies in $2J\backslash J$, then because the ration of the measures of I and J is bounded above and away from 0 and independent of I and J, by (iv) Lemma 4.2.4, we know

$$|I|^t|\langle\beta_I,\ T(\psi\alpha_J)\rangle_\psi| \leqslant C\frac{|I|^{t+1/2}}{|J|^{1/2}}\Big(\log\frac{|J|}{|I|}+1\Big) \leqslant C|J|^t.$$

Because the number of such terms is bounded and is independent of I and J, the corresponding part of the Schur sum is $O(|J|^t)$.

If I contains J and is larger than J, the I can be written as $I = I_1 \cup I_2$, where I_1 and I_2 are atoms in $\mathcal{F}_{m+1}$. Assume that $J \subseteq I_1$ and write $\beta_I = \beta_1\chi_{I_1} + \beta_2\chi_{I_2}$. Then similar to (4.24) and (4.25), we can get

$$\Big\langle\beta_1\chi_{I_1},\ T(\psi\alpha_J)\Big\rangle_\psi = -\Big\langle\beta_1\chi_{I_1^c},\ T(\psi\alpha_J)\Big\rangle_\psi.$$

Now I_1^c contains part of the region $2J\backslash J$. We can use (i) of Lemma 4.2.4 on this region. In particular,

$$\begin{aligned}
|\langle\beta_1\chi_{I_1},\ T(\psi\alpha_J)\rangle_\psi| &= \Big|\beta_1\int_{I_1^c}\psi(x)T(\psi\alpha_J)(x)dx\Big| \qquad (4.30)\\
&\leqslant C|\beta_1|\Big(\int_{2J\backslash J}|T(\psi\alpha_J)(x)|dx + \int_{(2J)^c}|T(\psi\alpha_J)(x)|dx\Big)\\
&\leqslant C|I|^{-1/2}|J|^{-1/2}\int_{2J\backslash J}dx\int_J|x-y|^{-n}dy\\
&+ C|I|^{-1/2}|J|^{1/2+1/n}\int_{(2J)^c}|x-x_J|^{-1-n}dx\\
&\leqslant C\Big\{|I|^{-1/2}|J|^{-1/2} + |I|^{-1/2}|J|^{1/2}\Big\} \leqslant C\frac{|J|^{1/2}}{|I|^{1/2}},
\end{aligned}$$

where in the second-last step we have used Lemma 4.2.3. As for $\langle\beta_2\chi_{I_2},\ T(\psi\alpha_J)\rangle_\psi$, we have I_2 is disjoint with J, so we can obtain an estimate similar to that of (4.30).

The estimate for the Schur sum of the dyadic-quasi-cubes satisfying $I \supseteq J$ is

$$\sum_{I\in\bigcup_{m<\kappa}\mathcal{F}_m,\ I\supseteq J}|I|^t|\langle\beta_I,\ T(\psi\alpha_J)\rangle_\psi| \leqslant C\sum_{k=1}^{\infty}(2^k|J|)^{t-1/2}|J|^{1/2} \leqslant C|J|^t,$$

where $t < 1/2$.

Case 2. The sum with respect to atoms I smaller than J.
For this case, we deal with the atoms $J \in \mathcal{F}_\kappa$ and $I \in \mathcal{F}_m$ with $m > \kappa$.

(a) If I lies outside $2J$, then J lies outside $2I$. Hence we apply (i) of Lemma 4.2.4 to T^t and get

$$|T^t(\beta_I\psi)(x)| \leqslant \frac{C|I|^{1/2+1/n}}{|x-x_I|^{1+n}},$$

which implies that

$$|\langle T^t(\beta_I\psi),\ \alpha_J\rangle_\psi| \leqslant C|I|^{1/2+1/n}|J|^{-1/2}\int_J \frac{dx}{|x-x_I|^{1+n}}$$
$$\leqslant C|I|^{1/2+1/n}|J|^{1/2}\frac{1}{|x-x_J|^{1+n}}$$
$$\leqslant C|I|^{1/n-1/2}|J|^{1/2}\int_I \frac{dx}{|x-x_J|^{1+n}},$$

where in the middle step we have used the fact that $I\subseteq(2J)^c$. The estimate of the corresponding Schur sum is

$$\sum_{I\in\bigcup_{m>\kappa}\mathcal{F}_m,\ I\cap 2J=\emptyset}|I|^{t+1/n-1/2}|J|^{1/2}\int_I\frac{dx}{|x-x_J|^{1+n}}$$
$$\leqslant C\sum_{j=1}^{\infty}(2^{-j}|J|)^{t+1/n-1/2}|J|^{1/2}\int_{(2J)^c}\frac{dx}{|x-x_J|^{1+n}}$$
$$\leqslant C\sum_{j=1}^{\infty}(2^{-j})^{t+1/n-1/2}|J|^t\leqslant C|J|^t,$$

where $t>1/2-1/n$.
(b) If $I\cap J=\emptyset$ and $I\subseteq 2J\setminus(I+J)$, then $J\subseteq(2I)^c$. So for T^t, we can use (ii) of Lemma 4.2.4 to obtain

$$|\langle\beta_I,\ T(\psi\alpha_J)\rangle_\psi| = C\left|\left\langle T^t(\beta_I\psi),\ \alpha_J\right\rangle_\psi\right| \tag{4.31}$$
$$\leqslant C|J|^{-1/2}|I|^{1/2+1/n}\int_J\frac{dx}{|x-x_I|^{1+n}}.$$

Let $d(x,\ J)$ denote the distance of the point x from J. The atom I may have unequal side length. Let $l(I)$ be the smallest side length. We can deduce from (4.31) that

$$|\langle\beta_I,\ T(\psi\alpha_J)\rangle_\psi| \leqslant C|J|^{-1/2}|I|^{1/2+1/n}\frac{1}{d(x_I,\ J)} \tag{4.32}$$
$$\leqslant C|J|^{-1/2}|I|^{1/2+1/n}|I|^{-1}\int_I\frac{dx}{d(x,\ J)+l(I)}.$$

Denote by L the maximal side length of J and by l the minimal side length of J, respectively. Then $L\leqslant 2l$ and $l^n\leqslant|J|\leqslant 2^nl^n$. The smallest side length of the dyadic-quasi-cubes $I\in\mathcal{F}_{\kappa+j}$ is $l(I)\geqslant l/2^{k/n+1}$. It follows from (4.32) that the estimate of the relevant part of the Schur sum is: if $t>1/2-1/n$, then

$$\sum_{I\in\bigcup_{m>\kappa},\ I\subseteq 2J\setminus(I+J)} |I|^t|\langle\beta_I,\ T(\psi\alpha_J)\rangle_\psi|$$

$$\leqslant \sum_{j=1}^{\infty}(2^{-j}|J|)^{t+1/n-1/2}|J|^{-1/2}\int_{2J\setminus(I+J)}\frac{dx}{d(x,J)+2^{-j/n-1}l}$$

$$\leqslant C\sum_{j=1}^{\infty}(2^{-j}|J|)^{t+1/n-1/2}|J|^{-1/2}\int_0^{3L}dx_1\cdots\int_0^{3L}dx_{d-1}\int_{2^{-j/n-1}l}^{2l}\frac{du}{u+2^{-j/n-1}l}$$

$$\leqslant C\sum_{j=1}^{\infty}(2^{-j}|J|)^{t+1/n-1/2}|J|^{-1/2}|J|^{(n-1)/n}\log\Big(\frac{2l+2^{-j/d-1}l}{2^{-j/n}l}\Big)$$

$$\leqslant C\sum_{j=1}^{\infty}(2^{-j})^{t+1/n-1/2}\frac{j}{n}|J|^t\leqslant C|J|^t.$$

(c) If $I\subseteq(I+J)\setminus J$, we have $I\subseteq 2J\setminus J$. By (iv) of Lemma 4.2.4,

$$|\langle\beta_I,\ T(\psi\alpha_J)\rangle_\psi|\leqslant C\frac{|I|^{1/2}}{|J|^{1/2}}\Big(\log\frac{|J|}{|I|}+1\Big). \tag{4.33}$$

In the region $(I+J)\setminus J$, there exist $O(L^{d-1}/(2^{-j/n-1}l)^{n-1})$ atoms which belong to $\mathcal{F}_m$. In other words, there exist $O(2^{j(1-1/n)})$ atoms. By (4.33), if $t>1/2-1/n$, the corresponding estimate of the Suchr sum is

$$C\sum_{j=1}^{\infty}(2^{-j}|J|)^{t+1/2}|J|^{-1/2}j2^{j(1-1/n)}=C|J|^t\sum_{j=1}^{\infty}j(2^{-j})^{t-1/2+1/n}\leqslant C|J|^t.$$

(d) If $I\subseteq J$ and L is contiguous to J^c, we write $J=J_1+J_2$, where J_1 and J_2 are atoms in $\mathcal{F}_{m+1}$. Let $\alpha_J=\alpha_1\chi_{J_1}+\alpha_2\chi_{J_2}$, and assume that $I\subseteq J_1$.

We first consider the atoms $I\subseteq J_1$ which are contiguous to J_1^c. We have

$$\begin{aligned}\Big|\Big\langle\beta_I,\ T(\psi\alpha_1\chi_{J_1})\Big\rangle_\psi\Big| &= \Big|\Big\langle\beta_I,\ T(\psi\alpha_1\chi_{J_1^c})\Big\rangle_\psi\Big|\\ &= \Big|\Big\langle T(\beta_1\psi),\ \alpha_I\chi_{J_1^c}\Big\rangle_\psi\Big|\\ &\leqslant \Big|\int_{J_1^c\cap 2I}T^t(\beta_I\psi)(x)\alpha_1dx\Big|+\Big|\int_{J_1^c\setminus 2I}T^t(\beta_I\psi)(x)\alpha_1dx\Big|.\end{aligned}$$

Hence by Lemma 4.2.3, applying (i) of Lemma 4.2.4 to $T^t(\beta_I\psi)$, we get

$$\begin{aligned}\left|\left\langle\beta_I,\ T(\psi\alpha_1\chi_{J_1})\right\rangle_\psi\right| &\leqslant C|I|^{-1/2}|J|^{-1/2}\int_{2I\setminus I}dx\int_I\frac{dy}{|x-y|^n}\\ &\quad+C|I|^{1/2+1/n}\int_{(2I)^c}\frac{dx}{|x-x_I|^{1+n}}\\ &\leqslant C|I|^{-1/2}|J|^{-1/2}|I|\log\left(\frac{|I|}{|I|}+1\right)+C|I|^{1/2}|J|^{-1/2}\\ &\leqslant C\frac{|I|^{1/2}}{|J|^{1/2}}.\end{aligned}\tag{4.34}$$

Because $J_2\subseteq J_1^c$, we can use an estimate similar to (4.34) to obtain

$$\left|\left\langle\beta_I,\ T(\psi\alpha_2\chi_{J_2})\right\rangle_\psi\right|\leqslant C\frac{|I|^{1/2}}{|J|^{1/2}}.\tag{4.35}$$

In $\mathcal{F}_{\kappa+j}$, there exist $O(2^{j(1-1/n)})$ atoms that are contiguous to J_1^c. It follows from (4.34) and (4.35) that for the atoms which are contiguous to J_1^c, the corresponding estimate of the Schur sum is

$$C\sum_{j=1}^{\infty}(2^{-j}|J|)^{t+1/2}|J|^{-1/2}|J|^{-1/2}2^{j(1-1/n)}=C|J|^t\sum_{j=1}^{\infty}(2^{-j})^{t-1/2+1/n}\leqslant C|J|^t,$$

where $t>1/2-1/n$.

(e) If $I\subseteq J$ and I is disjoint with J_1^c, similar to (i) of Lemma 4.2.4, we have

$$\begin{aligned}\left|\left\langle\beta_I,\ T(\psi\alpha_1\chi_{J_1})\right\rangle_\psi\right| &=\left|\int T^t(\beta_I\psi)(x)\psi(x)\alpha_1\chi_{J_1^c}(x)dx\right|\\ &\leqslant C|J|^{-1/2}\int_{J_1^c}|T^t(\beta_I\psi)(x)|dx\\ &\leqslant C|I|^{1/2+1/n}|J|^{-1/2}\int_{J_1^c}\frac{dx}{|x-x_I|^{1+n}}\\ &\leqslant C|I|^{1/2+1/n}|J|^{-1/2}\frac{1}{d(x,\ J_1^c)}.\end{aligned}$$

For $|\langle\beta_I,\ T(\phi\alpha_2\chi_{J_2})\rangle_\psi|$, a similar estimate holds. So the corresponding estimate of the Schur sum is

$$\begin{aligned}&\sum_{j=1}^{\infty}(2^{-j}|J|)^{t+1/2+1/n}|J|^{-1/2}\sum_{j=1}^{2^{j(1-1/n)}}\frac{1}{j2^{-j/n}}\\ &\leqslant C\sum_{j=1}^{\infty}(2^{-j}|J|)^{t+1/2+1/n}|J|^{-1/2-1/n}2^j\log(2^{j(1-1/n)})\end{aligned}$$

$$\leqslant C \sum_{j=1}^{\infty} k(2^{-j})^{t-1/2+1/n} |J|^t$$
$$\leqslant C|J|^t,$$

where $t > 1/2 - 1/n$.

Case 3. Atoms of the same size.
Here we only need to estimate the term $\langle \beta_I,\ T(\psi \alpha_I)\rangle_\psi$ since the arguments for Case 1 can be used to estimate the other parts of the Schur sum.

By Lemma 4.1.2, it suffices to prove that for all dyadic-quasi-cubes I,

$$|\langle \chi_I,\ T(\psi \chi_I)\rangle_\psi| \leqslant C|I|.$$

For this, we need to use the monogenicity of the Cauchy kernel. So we pass from T back to T_Σ. The coordinate mapping is $\phi(v) = A(v)e_0 + v$. For small $\epsilon > 0$ and $x = \phi(u)(u \in I)$, consider

$$\int_{|x-y|>\epsilon} \frac{\overline{y-x}}{|y-x|^{1+n}} \mathfrak{n}(y) \chi_{\phi(I)}(y) d\sigma(y). \tag{4.36}$$

Let P_z be the tangent hyperplane Σ to at x. Set $a(u) = \mathrm{dist}(u,\ \partial\phi I)$ and $b = b(x) = \mathrm{dist}(x,\ \partial\phi(I))$. Write (4.36) as $I_1 + I_2$, where

$$I_1 := \int_{b>|x-y|>\epsilon} \frac{\overline{y-x}}{|y-x|^{1+n}} \mathfrak{n}(y) \chi_{\phi(I)}(y) d\sigma(y)$$

and

$$I_2 := \int_{|x-y|>b} \frac{\overline{y-x}}{|y-x|^{1+n}} \mathfrak{n}(y) \chi_{\phi(I)}(y) d\sigma(y).$$

Then

$$|I| \leqslant C \log\Big(\frac{C|I|^{1/n}}{a(u)}\Big).$$

By Cauchy's theorem, we write

$$\begin{aligned} I_1 = & \int_{S_b} \frac{\overline{y-x}}{|y-x|^{1+n}} \mathfrak{n}(y) \chi_{\phi(I)}(y) d\sigma(y) + \int_{S_\epsilon} \frac{\overline{y-x}}{|y-x|^{1+n}} \mathfrak{n}(y) \chi_{\phi(I)}(y) d\sigma(y) \\ & + \int_{x,\ y\in P_x,\ b>|x-y|>\epsilon} \frac{\overline{y-x}}{|y-x|^{1+n}} \mathfrak{n}(y) \chi_{\phi(I)}(y) d\sigma(y), \end{aligned} \tag{4.37}$$

where S_b and S_ϵ are the portions of the sphere of radii b and ϵ, respectively, that lie between Σ and P_x. Because the kernel is anti-systemic and the integrals on S_b and S_ϵ are dominated by a constant, independent of x, ϵ and b, then the third integral in (4.37) is 0. Hence

$$|\langle \chi_I,\ T(\psi\chi_I)\rangle_\psi| \leqslant C|I| + C\int_I \log\Big(\frac{|I|^{1/n}}{a(u)}\Big)du \leqslant C|I|.$$

Assume that b_1 and b_2 are two pseudoaccretive functions. The space $b_1L^2(\mathbb{R}^n;\mathbb{R}_{(n)})$ is defined as the set of all products of the form $b_1 f,\ f \in L^2(\mathbb{R}^n;\mathbb{R}_{(n)})$. Similarly, we can define $L^2(\mathbb{R}^n;\mathbb{R}_{(n)})b_2$. These spaces are isomorphic to $L^2(\mathbb{R}^n;\mathbb{R}_{(n)})$. Let $\mathcal{S}$ denote the space of finite linear combinations over $\mathbb{R}_{(n)}$ of characteristic functions of dyadic-quasi-cubes. Then $b_1\mathcal{S}$ is dense in $b_1L^2(\mathbb{R}_{(n)})$. Denote by $(\mathcal{S}b_2)^*$ the space of all Clifford left linear functionals on $\mathcal{S}b_2$ with values in $\mathbb{R}_{(n)}$. Similarly, $(b_1\mathcal{S})^*$ denotes the space of all Clifford right linear functionals on $b_1\mathcal{S}$.

Let T be a Clifford right linear mapping from $b_1\mathcal{S}$ to $(\mathcal{S}b_2)^*$ and let $\Delta = \{(x, y) : x = y\}$. We call T a standard Calderón-Zygmund operator, if there exists a C^∞ function K in $\mathbb{R}^n \times \mathbb{R}^n\backslash\Delta$ with values in $\mathbb{R}_{(n)}$ satisfying:

(i) for $x \neq y$,

$$|K(x,\ y)| \leqslant C\frac{1}{|x-y|^n}; \tag{4.38}$$

(ii) there exist a constant δ such that for $0 < \delta \leqslant 1$ and $|y - y_0| < |y - x|/2$,

$$|K(x, y) - K(x, y_0)| + |K(y, x) - K(y_0, x)| \leqslant C\frac{|y-y_0|^\delta}{|x-y|^{n+\delta}}; \tag{4.39}$$

(iii) for all $f, g \in \mathcal{S}$ having disjoint supports,

$$T(b_1 f)(gb_2) = \iint g(x)b_2(x)K(x, y)b_1(y)f(y)dxdy. \tag{4.40}$$

In conformity with (4.40), we write

$$T(b_1 f)(gb_2) = \langle g,\ T(b_1 f)\rangle_{b_2}.$$

If T^t is a left linear mapping from $\mathcal{S}b_2$ to $(b_1\mathcal{S})^*$ such that for all $f, g \in \mathcal{S}$,

$$\langle g,\ T(b_1 f)\rangle_{b_2} = \langle T^t(gb_2),\ f\rangle_{b_1}$$

and T is associated with the kernel K, then T^t is associated with the kernel $K(y, x)$ in the sense that

$$T^t(gb_2)(b_1 f) = \int\Big(\int g(x)b_2(x)K(x, y)dx\Big)b_1(y)f(y)dy.$$

If there exists a constant C such that for all dyadic-quasi-cubes Q,

$$|T(b_1\chi_Q)(\chi_Q b_2)| \leqslant C|Q|,$$

We say that T is weakly bounded with respect to b_1 and b_2. This definition is formally different from the usual one in [7, 8], in which the test functions are taken to be smooth. However, the two definitions are equivalent.

If $h \in L^\infty(\mathbb{R}^n; \mathbb{R}^{1+n})$, then Th can be defined as a linear functional on the subspace $(\mathcal{S}b_2)_0$ of $\mathcal{S}b_2$ consisting of functions having integral 0. In the next theorem, we say $T(b_1) \in BMO$ if there exists a locally integrable BMO function ϕ such that for all $g \in (\mathcal{S}b_2)_0$, $\langle g,\ T(b_1)\rangle_{b_2} = \langle g,\ \phi\rangle_{b_2}$. A similar interpretation applies to $T^t(b_2)$. For the sequence of $\sigma-$fields, the space BMO is the one defined in (4.8).

Theorem 4.2.2 ($T(b)$ theorem) *Let T and T^t ba as above and T be associated with the standard Calderón-Zygmund kernel K. Then T is extendible to a bounded linear operator from $b_1 L^2(\mathbb{R}^n; \mathbb{R}_{(n)})$ to $L^2(\mathbb{R}^n; \mathbb{R}_{(n)})b_2$ if and only if*

(i) *$T(b_1)$, $T^t(b_2) \in BMO$;*

(ii) *T is weakly bounded for b_1 and b_2.*

Proof The necessity of the conditions (i) and (ii) was proved in the classical case by [9–11]. Their proof adapted to the more general Clifford algebra setting.

To prove the sufficiency, we first deal with the case $T(b_1) = T^t(b_2) = 0$. For every pair of pesduoaccretive functions b_1 and b_2, we associate a Haar basis and denote the respective pair-base by $\{(\alpha_I^{(1)},\ \beta_I^{(1)})\}_{I\in\mathcal{J}}$ and $\{(\alpha_I^{(2)},\ \beta_I^{(2)})\}_{I\in\mathcal{J}}$. Formally, we have the following expansion

$$T(b_1 f) = \sum_{I,J} \alpha_I^{(2)} \left\langle \beta_I^{(2)},\ Tb_1\alpha_J^{(1)} \right\rangle_{b_2} \left\langle \beta_J^{(1)},\ f \right\rangle_{b_1}.$$

Let

$$u_{IJ} = \left\langle \beta_I^{(2)},\ Tb_1\alpha_J^{(1)} \right\rangle_{b_2}.$$

It suffices to prove for a suitable number t, when ω_I is taken to be $|I|^t$, the conditions of Lemma 4.2.2 are satisfied.

Because $T(b_1) = T^t(b_2) = 0$ and the kernel with respect to T satisfies (4.38) and (4.39), for the present more general operator T, the statement and the proof of Lemma 4.2.4 still hold. Because of the assumption that $T(b_1) = T^t(b_2) = 0$, we find that the estimates for Case 1 and Case 2 go through unchanged. The estimate of the part of the Schur sum corresponding to Case 3 holds by virtue of the weak boundedness assumption.

The general case: $T(b_1), T^t(b_2) \in BMO$. Let $T(b_1) = \phi_1$ and $T^t(b_2) = \phi_2$. We define

$$U_i f = \sum_{k=-\infty}^{\infty} \Delta_k^{(j)}(\phi_i) E_{k-1}^{(i)}(b_i^{-1} f),\ i, j = 1, 2, i \neq j, \tag{4.41}$$

where $E_k^{(i)}$ and $\Delta_k^{(i)}$ are the left conditional expectation operator and the left martingale difference with respect to the pseudoaccretive function b_i. It is obvious that $U_i b_i = \phi_i$, $i = 1, 2$. The kernel K_i of the operator U_i is given by the expression

$$K_i(x,\ y) = \sum_{k=-\infty}^{\infty} \sum_{I \in \mathcal{J}_{k-1}} \chi_I(x)\alpha_I^{(j)}(x)\Big\langle \beta_I^{(j)},\ \phi_i \Big\rangle_{b_j} \left(\int_I b_i \right)^{-1} \chi_I(y). \tag{4.42}$$

By (4.42), it is easy to verify

$$\Delta_m^{(i)} U_i f = \Delta_m^{(j)}(\phi_i) E_{m-1}^{(i)}(b_i^{-1} f).$$

We claim

$$\|S^{(i)}(U_i f)\|_2 \leqslant C\|f\|_2, \tag{4.43}$$

where $S^{(i)}$ denotes the Littlewood–Paley square function with respect to b_i. Hence U_i is bounded on L^2. To prove (4.43), note that

$$\begin{aligned}
&\|S^{(i)}(U_i f)\|_2^2 \qquad\qquad (4.44)\\
&= \int \sum_k |\Delta_k^{(j)}(\phi_i) E_{k-1}^{(i)}(b_i^{-1} f)|^2 dx \\
&\leqslant C \int \sum_k |\Delta_k^{(j)}(\phi_i)|^2 \Big(E_{k-1}^{(i)*}(b_i^{-1} f) \Big)^2 dx \\
&\leqslant C \int \sum_{k=-\infty}^{\infty} \widetilde{E}_{k-1} \Big(\sum_{m=k}^{\infty} |\Delta_m^{(j)}(\phi_i)|^2 \Big) \Big[\Big(E_{k-1}^{(i)*}(b_i^{-1} f) \Big)^2 - \Big(E_{k-2}^{(i)*}(b_i^{-1} f) \Big)^2 \Big] dx,
\end{aligned}$$

where $E_k^{(i)*} g = \sup_{m \leqslant k} |E_m^{(i)} g|$. Now, for every k,

$$\widetilde{E}_{k-1} \Big(\sum_{m=k}^{\infty} |\Delta_m^{(j)}(\phi_i)|^2 \Big) \leqslant C\|\phi_i\|_{BMO}^2. \tag{4.45}$$

This is because, if $I \in \mathcal{J}_{k-1}$, then we can restrict σ−field $\{\mathcal{F}_m\}_{m=k-1}^{\infty}$ to I and deduce that on I,

$$\begin{aligned}
\widetilde{E}_{k-1} \Big(\sum_{m=k}^{\infty} |\Delta_m^{(j)}(\phi_i)|^2 \Big) &= \frac{1}{|I|} \int_I \sum_{m=k}^{\infty} |\Delta_m^{(j)}(\phi_i)|^2 dx \\
&= \frac{1}{|I|} \int_I \sum_{m=k}^{\infty} |\Delta_m^{(j)}(\phi_i - E_{k-1}^{(j)}(\phi_i)|^2 dx \\
&= \frac{C}{|I|^{(j)}} \int_I \Big| \phi_i - \frac{1}{|I|^{(j)}} \int_I b_j \phi_i \Big|^2 dx \\
&= \frac{C}{|I|} \int_I \Big| \phi_i - \frac{1}{|I|} \int_I \phi_i dy + \frac{1}{|I|^{(j)}} \int_I b_j \Big(\phi_i - \frac{1}{|I|} \int_I \phi_i dz \Big) dx \Big|^2 \\
&\leqslant C\|\phi_i\|_{BMO}^2,
\end{aligned}$$

where we have used the fact that $|I^{(j)}| = \int_I b_j dx$. This gives (4.45). Returning to (4.43), we have

$$\|S^{(i)}(U_i f)\|_2^2 \leqslant C\|\phi_i\|_{BMO}^2 \int (Mf(x))^2 dx \leqslant C\|f\|_2^2,$$

where Mf denotes the usual Hardy–Littlewood maximal function. This proves (4.43).

By Lemma 4.1.1, U_i is bounded on L^2. The operator U_i^t is still bounded on L^2. If $i \neq j$, because $\int b_j \alpha_I^{(j)} dx = 0$,

$$\begin{aligned}\langle U_i^t(b_j),\ f\rangle_{b_i} &= \langle b_j,\ U_i(b_i f)\rangle \\ &= \sum_{k=-\infty}^{\infty} \sum_{I \in \mathcal{J}_{k-1}} \Big(\int b_j \alpha_I^{(j)}\Big)\langle \beta_I^{(j)},\ \phi_i\rangle_{b_j}\Big(\int_I b_i\Big)^{-1}\Big(\int_I b_i f\Big) \\ &= 0.\end{aligned}$$

Hence if $i \neq j$, $U_i^t(b_j) = 0$. Letting $R = t - U_1 - U_2^t$, we have

$$R(b_1) = R^t(b_2) = 0. \tag{4.46}$$

The operator R is also weakly bounded. Applying the method of Theorem 4.2.1, we wish to show that R and T are bounded on L^2. This effectively reduces to checking that the operator R and R^t satisfy the same kind of conditions as those given in Lemma 4.2.4. The proofs of (iii) and (iv) of Lemma 4.2.4 use only the property (4.21) of the kernel K. Consider the kernels associated with the operators U_1 and U_2^t. For $i = 1, 2$, they are given by (4.42). Now for fixed $x \neq y$, and k, there exists at most one $I \in \mathcal{J}_{k-1}$, denoted by I_{k-1}, such that the summand in (4.42) is nonzero. For such a term,

$$|x - y| \leqslant C2^{-k}, \tag{4.47}$$

where C is independent of x, y and k. Let k_0 be the largest integer such that (4.47) holds. By (4.47), the sum in (4.42) is then, in norm, at most

$$\begin{aligned}&C \sum_{k=-\infty}^{k_0} |I_{k-1}|^{-1/2} \frac{1}{|I_{k-1}|} \int_{I_{k-1}} |\beta_{I_{k-1}}^{(j)}(y) b_j(y)||\phi_i - (\phi_i)_{I_{k-1}}| dy \\ &\leqslant C\|\phi_i\|_{BMO} \sum_{k=-\infty}^{k_0} |I_{k-1}|^{-1} \\ &\leqslant C\|\phi_i\|_{BMO} \sum_{k=-\infty}^{k_0} 2^{nk} \\ &\leqslant C\|\phi_i\|_{BMO} 2^{nk_0} \\ &\leqslant C\|\phi_i\|_{BMO} |x - y|^{-n}.\end{aligned}$$

As to (i) and (ii) of Lemma 4.2.4, we note that, if J is a dyadic-quasi-cube with $x \notin 2J$, then

$$U_1(b_1\alpha_J^{(1)})(x) = \sum_{-\infty}^{\infty} \sum_{I \in \mathcal{J}_{k-1}} \alpha_I^{(2)}(x)\chi_I(x)\langle \beta_I^{(2)},\ \phi_1\rangle_{b_2}\left(\int_I b_1\right)^{-1}\left(\int_I b_1\alpha_J^{(1)}\right) = 0.$$

In fact, the last factor in a term of the double summation is nonzero only when $I \subseteq J$. But because $x \notin 2J$, then $\chi_I(x) = 0$. So this term is 0. A similar argument applies to U_2^t. Hence, (i)–(iv) of Lemma 4.2.4 hold for the operator R. The operator R^t can be dealt with similarly. Assume that $R(b_1) = R^t(b_2) = 0$. With some appropriate modifications, the proof of Theorem 4.2.1 applies to the operator R. □

4.3 Clifford Martingale $\Phi-$Equivalence Between $S(f)$ and f^*

In Sect. 4.2, the L^2-norm equivalence between a Clifford martingale and its square function plays an important role in the proof of the main results. The L^2-boundedness of the maximal function f^* indicates the L^2 equivalence between f^* and its square function. The later mentioned result is associated with $\Phi(t) = t^2$. In this section, we will generalize this result to more general functions Φ.

Let $(\Omega, \mathcal{F}, \nu)$ be a nonnegative $\sigma-$finite space and let ϕ be a bounded Clifford-valued measurable function. Consider the Clifford-valued measure $d\mu = \phi\nu$. The martingales are with respect to $d\mu$ and a family of $\{\mathcal{F}_m\}_{-\infty}^{\infty}$ of sub-σ-field satisfying

$$\{\mathcal{F}_m\}_{-\infty}^{\infty} \text{ nondecreasing}, \mathcal{F} = \cup\mathcal{F}_m,\ \cap\mathcal{F}_m = \emptyset, \tag{4.48}$$

and

$$(\Omega, \mathcal{F}_m, \nu) \text{ complete}, \sigma - \text{finite } \ \forall\, m. \tag{4.49}$$

Let $e_1, \cdots, e_n$ be the basic vectors of $\mathbb{R}^n$ satisfying

$$e^2 = -1,\ e_ie_j = -e_je_i,\ i \neq j,\ i, j = 1, 2, \ldots, n, \tag{4.50}$$

and $\mathbb{R}_{(n)}$ be the Clifford algebra on 2^n-dimensional real number field generated by the increasingly ordered subset e_A, $\{1, \cdots, n\}$, where $e_A = e_{j_1}\cdots e_{j_l}$, $A = \{j_1, \ldots, j_l\}$, $1 \leqslant l \leqslant n$, $e_\emptyset = e_0 = 1$. We will use the following norm in $\mathbb{R}_{(n)}$:

$$|\lambda| = \left(\sum_A \lambda_A^2\right)^{1/2},\ \lambda = \sum_A \lambda_A e_A. \tag{4.51}$$

For this norm, we have the following relation:

$$|\lambda\mu| \leqslant k|\lambda||\mu| \ \forall\ \lambda, \mu \in \mathbb{R}_{(n)}, \tag{4.52}$$

where k is a constant which depends only on the dimension m. When at least one of λ and μ, say λ, is of the form $\lambda = \sum_{i=0}^{d} \lambda_i e_i$, i.e., a vector in $\mathbb{R}^{n+1} \subset \mathbb{R}_{(n)}$, we have

$$k^{-1}|\lambda||\mu| \leqslant |\lambda|. \tag{4.53}$$

For a martingale $f = (f_m)_{-\infty}^{\infty}$, the maximal square function is defined as

$$f_m^* = \sup_{k \leqslant m} |f_k|, \ \ f^* = f_\infty^*. \tag{4.54}$$

For $1 \leqslant p \leqslant \infty$, $f = \{f_m\}_{-\infty}^{\infty}$ is called bounded on L^p if

$$\|f\|_p = \sup_m \|f_m\|_p < \infty. \tag{4.55}$$

In the next proposition, we prove the boundedness of the maximal operator f^*.

Proposition 4.3.1 *Let $1 < p \leqslant \infty$. The maximal operator "$*$" is (p, p) type and weak $(1, 1)$ type. For $1 < p \leqslant \infty$, every L^p-bounded martingale $f = \{f_m\}_{-\infty}^{\infty}$ is generated by some function $f \in L^p(\nu)$ which satisfies $\|f\|_p \approx \sup_m \|f_m\|_p$.*

Proof Let $f = \{f_m\}_{-\infty}^{\infty}$ be a martingale. On the one hand,

$$f_m = E(f_{m+1} \mid \mathcal{F}_m) = \widetilde{E}(\phi \mid \mathcal{F}_m)^{-1}\widetilde{E}(\phi f_{m+1} \mid \mathcal{F}_m).$$

On the other hand,

$$\begin{aligned} f_m &= E(f_{n+2} \mid \mathcal{F}_m) = \widetilde{E}(\phi \mid \mathcal{F}_m)^{-1}\widetilde{E}(\phi f_{m+2} \mid \mathcal{F}_m) \\ &= \widetilde{E}(\phi \mid \mathcal{F}_m)^{-1}\widetilde{E}(\widetilde{E}(\phi f_{m+2} \mid \mathcal{F}_{m+1}) \mid \mathcal{F}_m). \end{aligned}$$

The above estimates give

$$\widetilde{E}(\phi f_{m+1}) = \widetilde{E}(\widetilde{E}(\phi f_{n+2} \mid \mathcal{F}_m) \mid \mathcal{F}_m).$$

Hence $\{\widetilde{E}(\phi f_{m+1})\}_{-\infty}^{\infty}$ is the martingale with respect to $(\Omega, \mathcal{F}, \nu, \{\mathcal{F}_m\}_{-\infty}^{\infty})$. We can deduce from the expression of f_m that the following relation holds:

$$\widetilde{E}(\phi f_{m+1} \mid \mathcal{F}_m) = \widetilde{E}(\phi \mid \mathcal{F}_m) f_m.$$

Then it is L^p-bounded. Moreover, we have

$$\sup_n \|f_n\|_p \approx \sup_m \|\widetilde{E}(\phi f_{m+1} \mid \mathcal{F}_m)\|_p,$$
$$f^* \approx \sup_m |\widetilde{E}(\phi f_{m+1} \mid \mathcal{F}_m)|.$$

Because of the result in the classical case, $*$ is (p, p) type and weak $(1, 1)$ type. For $1 < p \leqslant \infty$ and any integer $M > 0$, we decompose $\Omega = \cup_k \Omega_k$, where $\Omega_k \in \mathcal{F}_{-M}$ and $|\Omega_k| < \infty$. Because for any k, $\{\widetilde{E}(\phi f_{m+1} \mid \mathcal{F}_m)\chi_{\Omega_k}\}_{n \geqslant -M}$ is a classical martingale, we can obtain some $\phi f \in L^p(\Omega_k, \nu)$ such that on Ω_k,

$$\widetilde{E}(\phi f_{m+1} \mid \mathcal{F}_m) = \widetilde{E}(\phi f \mid \mathcal{F}_m),\ n \geqslant -M.$$

Therefore, for $n \geqslant -M$,

$$\begin{aligned} f_m &= \widetilde{E}(\phi \mid \mathcal{F}_m)^{-1}\widetilde{E}(\phi f_{n+1} \mid \mathcal{F}_m) \\ &= \widetilde{E}(\phi \mid \mathcal{F}_m)^{-1}\widetilde{E}(\phi f \mid \mathcal{F}_m) \\ &= E(f \mid \mathcal{F}_m). \end{aligned}$$

Letting $M \to \infty$, we can see that $f_m = E(f \mid \mathcal{F}_m)\ \forall\, n$. Moreover, we have

$$\|f\chi_{\Omega_k}\|_p \leqslant C \sup_n \|f_m \chi_{\Omega_k}\|_p$$

and

$$\|f\|_p \leqslant C \sup_m \|f_m\|_p.$$

In addition, $\sup_m \|f_m\|_p \leqslant C\|f\|_p$ and $\|f\|_p \approx \sup_m \|f_m\|_p$. □

By Proposition 4.3.1, we can identify a L^p-bounded martingale with the function that generalizes the martingale as follows

$$f = \{f_m\}_{-\infty}^{\infty} = \{E(f \mid \mathcal{F}_m)\ \forall\, m\}_{-\infty}^{\infty}.$$

Proposition 4.3.2 *Let* $1 \leqslant p \leqslant \infty$ *and* $f = \{f_m\}_{-\infty}^{\infty}$ *be a* L^p*-bounded martingale. Then*

$$\lim_{m \to \infty} f_m = f,\ 1 < p \leqslant \infty, \tag{4.56}$$

where f *is the* L^p*-function which generates* $\{f_m\}_{-\infty}^{\infty}$ *in Proposition 4.3.1, and for* $p = 1$*, the following limits exists:*

$$\lim_{m \to \infty} f_m \text{ exists },\ p = 1 \tag{4.57}$$

and

$$\lim_{m\to-\infty} f_m = 0, 1 \leqslant p < \infty. \tag{4.58}$$

Proof Let $\Omega = \cup\Omega_k$, where $\Omega_k \in \mathcal{F}_0$ with $|\Omega_k| < \infty\ \forall\, k$. Then $\{\widetilde{E}(\phi \mid \mathcal{F}_m)\chi_{\Omega_k}\}_{m>0}$ and $\{\widetilde{E}(\phi f_{m+1} \mid \mathcal{F}_m)\chi_{\Omega_k}\}_{m>0}$ are L^p-bounded martingales with respect to $(\Omega_k, \mathcal{F} \cap \Omega_k, \{\mathcal{F}_m \cap \Omega_k\}_{m\geqslant 0})$, and have their respective limits:

$$\begin{cases} \text{on every } \Omega_k, \ \lim\limits_{m\to\infty} \widetilde{E}(\phi \mid \mathcal{F}_m) = \phi \text{ a.e.;} \\ \text{on every } \Omega_k, \ \text{for some } g, \lim_{m\to\infty} \widetilde{E}(\phi f_{m+1} \mid \mathcal{F}_m) = \phi g \text{ a.e.;} \\ \text{for } 1 < p \leqslant \infty, g = f. \end{cases}$$

The last two limits imply that (4.56) and (4.57) hold. Now we prove (4.58). Write $\theta(\omega) = \overline{\lim}_{m\to-\infty}|f_m|$. Then $\theta(\omega) \leqslant f^*(\omega)$ and $\theta(\omega)$ are $\cap\mathcal{F}_m$ measurable. This means that $\theta(\omega) = a \geqslant 0$ a.e. Because $*$ is weak (p, p) type, for $1 \leqslant p < \infty$, we have

$$|\{\theta(\omega) > \lambda\}|_\nu \leqslant |\{f^* > \lambda\}|_\nu \leqslant \left(\frac{C}{\lambda}\|f\|_p\right)^p \ \forall\, \lambda > 0.$$

Hence $a = 0$. This gives (4.58). □

Let Φ be a nondecreasing and continuous function from $\mathbb{R}^+$ to $\mathbb{R}^+$ satisfying $\Phi(0) = 0$ with the moderate growth condition

$$\Phi(2u) \leqslant C_1\phi(u), u > 0. \tag{4.59}$$

We begin to establish the $\Phi-$equivalence between $S(f)$ and f^*, where f is the martingale such that for any m,

$$|\Delta_m f| \leqslant D_{m-1}, \tag{4.60}$$

where $D = \{D_m\}$ is a nonnegative nondecreasing and adapted process to $\{\mathcal{F}_m\}$. We only consider the case $\{\mathcal{F}_m\}_{m\geqslant 0}$.

Theorem 4.3.1 *Let $f = \{f_m\}_{m\geqslant 0}$ be a $l-$martingale or a $r-$martingale satisfying (4.60). Then*

$$\int_\Omega \Phi(S(f))d\nu \leqslant C \int_\Omega \Phi(f^* + D_\infty)d\nu \tag{4.61}$$

and

$$\int_\Omega \Phi(f^*)d\nu \leqslant C \int_\Omega \Phi(S(f) + D_\infty)d\nu, \tag{4.62}$$

where the involved constants depend only on C_0 and C_1.

Proof We shall use the stopping time argument and the good $\lambda-$ inequality. Let α be any real number larger than 1, $\beta > 0$ to be determined and λ be any level. Notice that

$$|f_m| \leqslant |f_{m-1}| + |\Delta_m f| \leqslant f^*_{m-1} + D_{m-1} = \rho_{m-1}.$$

Define the stopping time $\tau = \inf\{m : \ \rho_m > \beta\lambda\}$ and the associated stopping martingale

$$f^{(\tau)} = \{f^{(\tau)}_m\}_{m\geqslant 0} = \{f_{\min\{m,\tau\}}\}_{m\geqslant 0}.$$

Then we have

$$\{\tau < \infty\} = \{\rho_\infty > \beta\lambda\}, \ \ f^{(\tau)*} = \sup_m |f_{\min\{m,\tau\}}| \leqslant f^*_\tau \leqslant \rho_{\tau-1} \leqslant \beta\lambda.$$

Now consider the adapted process $\{S_m(f^{(\tau)}) > \lambda\}$ and define the stopping time

$$T = \inf\{m : \ S_m(f^{(\tau)}) > \lambda\}.$$

Then we have

$$\{T < \infty\} = \{S(f^{(\tau)}) > \lambda\}, \ S_{T-1}(f^{(\tau)}) \leqslant \lambda.$$

Hence

$$\begin{aligned}\{S(f) > \alpha\lambda\} &\subset \{\tau < \infty\} \cup \{\tau = \infty, \ S_\tau(f)^2 > \alpha^2\lambda^2\} \\ &\subset \{\tau < \infty\} \cup \{S(f^{(\tau)})^2 - S_{T-1}(f^\tau)^2 > (\alpha^2 - 1)\lambda^2\}\end{aligned}$$

and

$$\begin{aligned}&\widetilde{E}(\chi_{S(f^{(\tau)})^2 - S_{T-1}(f^\tau)^2 > (\alpha^2-1)\lambda^2} \mid \mathcal{F}_T) \\ &\leqslant \frac{1}{(\alpha^2-1)\lambda^2}\widetilde{E}(S(f^{(\tau)})^2 - S_{T-1}(f^\tau)^2 \mid \mathcal{F}_T).\end{aligned}$$

Now we consider a new underlying space $(\Omega, \mathcal{F}, \nu, \{\mathcal{J}_m\}_{m\geqslant 0})$ with $J_m = \mathcal{F}_{T+m}$, and the martingale

$$g = \{g_m\}_{m\geqslant 0} \text{ such that } g_m = f^{(\tau)}_{T+m} - f^{(\tau)}_{T-1}.$$

Then we have

$$\Delta_m g = f^{(\tau)}_{T+m} - f^{(\tau)}_{T-1} - (f^{(\tau)}_{T+m-1} - f^{(\tau)}_{T-1}) = \Delta_{T+m} f^{(\tau)}$$

and

$$\begin{aligned}S(g)^2 &= \sum_{m=0}^{\infty} |\Delta_m g|^2 = \sum_{m=0}^{\infty} |\Delta_{T+m} f^{(\tau)}|^2 \\ &= \sum_{k=T}^{\infty} |\Delta_k f^{(\tau)}|^2 = S(f^{(\tau)})^2 - S_{T-1}(f^{(\tau)})^2.\end{aligned}$$

By Lemma 4.1.1, we get

$$\begin{aligned}\widetilde{E}(S(f^{(\tau)})^2 - S_{T-1}(f^{(\tau)})^2 \mid \mathcal{F}_T) &= \widetilde{E}(S(g)^2 \mid \mathcal{J}_\prime) \\ &\leqslant C\widetilde{E}(|g|^2 \mid \mathcal{J}_0) \\ &= C\widetilde{E}(|f^{(\tau)} - f^{(\tau)}_{T-1}| \mid \mathcal{F}_T) \\ &\leqslant C\beta^2\lambda^2.\end{aligned}$$

Now, because $\{S(f^\tau) > \alpha\lambda\} \subset \{T \leqslant \infty\}$, we obtain

$$\begin{aligned}|\{S(f^{(\tau)}) > \alpha\lambda\}|_\nu &\leqslant \int_{\{T<\infty\}} \chi_{\{S(f^{(\tau)})>\alpha\lambda\}} d\nu \\ &= \int_{\{T<\infty\}} \widetilde{E}(\chi_{\{S(f^{(\tau)})>\alpha\lambda\}} \mid \mathcal{F}_T) d\nu \\ &\leqslant \int_{\{T<\infty\}} \widetilde{E}(\chi_{\{S(f^{(\tau)})^2 - S_{T-1}(f^{(\tau)})^2 > (\alpha^2-1)\lambda^2\}} \mid \mathcal{F}_T) d\nu \\ &\leqslant \frac{C\beta^2}{\alpha^2-1} |\{S(f^{(\tau)}) > \lambda\}|_\nu \leqslant \frac{C\beta^2}{\alpha^2-1} |\{S(f) > \lambda\}|_\nu,\end{aligned}$$

and hence

$$|\{S(f) > \alpha\lambda\}|_\nu \leqslant |\{\rho_\infty > \beta\lambda\}|_\nu + \frac{C\beta^2}{\alpha^2-1} |\{S(f) > \lambda\}|_\nu,$$

which is the desired good λ inequality for the couple $(S(f),\ f^* + D_\infty)$. The one for the couple $(f^*,\ S(f) + D_\infty)$ is similar. From them, we obtain (4.61) and (4.62). □

We can get rid of D_∞ in the following cases:

(i) Φ is convex;
(ii) $(\Omega, \mathcal{F}, \nu, \{\mathcal{F}_m\}_{-\infty}^{\infty})$ is regular in some sense.

For simplicity, we only consider the simplest regularity, i.e., the dyadic type one: each $\mathcal{F}_m$ is atomic, whose atom $I^{(m)} = I_1^{(m+1)} + I_2^{(m+1)}$ satisfies $||I_1^{(m+1)}|_\mu| = ||I_2^{(m+1)}|_\mu|$. A little more general regularity is applicable to our case. We have

Theorem 4.3.2 *Under the additional condition (i) on Φ or (ii) on* $(\Omega, \mathcal{F}, \nu, \{\mathcal{F}_m\}_{-\infty}^{\infty})$*, we have*

$$\int_\Omega \Phi(S(f)) d\nu \approx \int_\Omega \Phi(f^*) d\nu,$$

where in the above equivalence, all the constants only depend on C_0 *and* C_1.

Proof We first consider $\{\mathcal{F}_m\}_{m\geqslant 0}$. Davis' decomposition holds in such case: every Clifford martingale $f = \{f_m\}_{m\geqslant 0}$ can be decomposed into a sum of two martingales: $g = \{g_m\}_{m\geqslant 0}$ and $h = \{h_m\}_{m\geqslant 0}$ satisfying

$$|\Delta_m g| \leqslant 4d^*_{m-1},\ d^* = \sup_{k\leqslant m} |d_k|,\ d_k = \Delta_k f, \tag{4.63}$$

and

$$\int_\Omega \Phi(\sum_{m=0}^{\infty} |\Delta_m h|) d\nu \leqslant C \int_\Omega \Phi(d^*) d\nu\ \forall\ \text{convex } \Phi. \tag{4.64}$$

Now for $f = \{f_m\}_{m\geqslant 0}$, we have

$$\begin{aligned}\int_\Omega \Phi(S(f)) d\nu &\leqslant C \int_\Omega \Phi(S(g)) d\nu + C \int_\Omega \Phi(S(h)) d\nu \\ &\leqslant C \int_\Omega \Phi(g^*) + C \int_\Omega \Phi(d^*) + C \int_\Omega \Phi(\sum_{m=0}^{\infty} |\Delta_m h|) d\nu \\ &\leqslant C \int_\Omega \Phi(f^*) d\nu.\end{aligned}$$

The proof for the reverse inequality is similar. Next we consider the dyadic type case. We claim that in such case, (4.60) holds for any martingale $f = \{f_m\}_{-\infty}^{\infty}$ and suitably defined $D = \{D_m\}$. In fact,

$$D_{m-1} \mid_{I^{m-1}} = \sup_{k\leqslant m} \max(|\Delta_k f| \mid_{I_1^{(k)}},\ |\Delta_k f| \mid_{I_2^{(k)}})$$

is a nonnegative, nondecreasing and adapted process such that

$$|\Delta_m f| \leqslant D_{m-1}$$

and

$$D_\infty \leqslant C \min(f^*,\ S(f)).$$

Only the last assertion needs to be verified. In fact,

$$\int_{I^{(k-1)}} \Delta_k f d\mu = 0$$

implies

$$\int_{I_1^{(k-1)}} \Delta_k f d\mu = -\int_{I_2^{(k-1)}} \Delta_k f d\mu.$$

This gives

$$\Delta_k f \mid_{I_1^{(k)}} |I_1^{(k)}|_\mu = -\Delta_k f \mid_{I_2^{(k)}} |I_2^{(k)}|_\mu$$

or

$$\frac{|\Delta_k f\ |_{I_1^{(k)}}\ |}{|\Delta_k f\ |_{I_2^{(k)}}\ |} = \frac{||I_2^{(k)}|_\mu|}{||I_1^{(k)}|_\mu|}.$$

Therefore, on $I^{(k-1)}$,

$$\max(|\Delta_k f|\ |_{I_1^{(k)}},\ |\Delta_k f|\ |_{I_2^{(k)}}) \leqslant C|\Delta_k f|$$

and

$$D_\infty \leqslant C \sup_k |\Delta_k f| \leqslant C \min\{S(f),\ f^*\}.$$

□

4.4 Remarks

Remark 4.4.1 Another method to prove the boundedness of Calderón-Zygmund operators lays on the multi-resolution technique developed by R. Coifman, Y. Meyer etc. That method is usually called the fast algorithm of Calderón-Zygmund. The basic idea is to decompose the kernel of the Calderón-Zygmund operator T under consideration by wavelet basis and then represent T as a linear combination of quasi-annular operators. Then applying the smoothness and the canceling condition of the regular wavelets, we estimate the coefficients of the kernel and obtain that the L^2−norms of the quasi-annular operators have a good rate of decay. This implies the L^2-boundedness of Calderón-Zygmund operators. In 1994, similar to the result on $\mathbb{R}^n$, using Clifford-valued regular wavelets, M. Mitrea obtained the L^2-boundedness of singular integral operators on Lipschitz surface, see [12].

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Chapter 5
Holomorphic Fourier Multipliers on Infinite Lipschitz Surfaces

It is well-known that there exists a one-one correspondence between the classical convolution singular integral operators and the Fourier multiplier operators on the Euclidean spaces $\mathbb{R}^n$. Because Plancherel's identity involving the Fourier transform is invalid on Lipschitz surfaces Σ, the relation between singular Cauchy integral operators and Fourier multipliers on Σ is an open problem for a long time. In 1994, by the aid of Clifford analysis, Li, McIntosh and Qian [1] introduced a class of holomorphic Fourier multipliers $H(S^c_{\omega,\pm})$ on Lipschitz surfaces. In [1], based on the idea of the functional calculus of the Dirac operator, the authors proved the following result: for $\phi \in K(S_{\omega,\pm})$, there exists a holomorphic function $b \in H(S^c_{\omega,\pm})$ such that on the Lipschitz surface, any singular integral operator T_ϕ with the convolution kernel ϕ corresponds to a Fourier multiplier operator M_b, where b is the Fourier transform of the kernel ϕ. In this chapter, we will elaborate on the theory established by the above three authors.

5.1 Singular Convolution Integrals on Infinite Lipschitz Surfaces

Let Σ denote a Lipschitz surface consisting of the points $x = \underline{x} + g(\underline{x})e_L \in \mathbb{R}^{n+1}$, where $\underline{x} \in \mathbb{R}^n$, and g is a real-valued Lipschitz function satisfying

$$\|\nabla g\|_\infty = \sup_{x\in\mathbb{R}^n} \left(\sum_{j=1}^{n} \left| \frac{\partial g}{\partial x_j} \right|^2 \right)^{1/2} \leqslant \tan\omega < \infty,$$

where $0 \leqslant \omega < \pi/2$.

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The unit normal $\mathfrak{n}(x) \in \mathbb{R}_+^{n+1}$ is defined at almost all $x \in \Sigma$. Take N to be the compact set of unit vectors in $\mathbb{R}_+^{n+1}$ which is starlike about e_L, $\mu_N \leqslant \omega$ and contains $\mathfrak{n}(x)$ for almost all $x \in \Sigma$.

Let χ be a finite-dimensional left module on $\mathbb{C}_{(M)}$. If $1 \leqslant p < \infty$, then $L^p(\Sigma)$ is the space of the equivalent classes of functions $u : \Sigma \to \chi$ is measurable with respect to

$$dS_{\underline{x}} = \sqrt{1 + |\nabla g(\underline{x})|^2} d\underline{x}$$

and

$$\|u\|_p = \left(\int_\Sigma |u(\underline{x})|^p dS_{\underline{x}} \right)^{1/p} < +\infty.$$

In the rest of this section, fix Σ, N and χ. Assume that $1 < p < \infty$. As usual, $\mathcal{L}(L^p(\Sigma))$ denotes the Banach algebra of bounded linear operators on $L^p(\Sigma)$. The following theorems are generalizations of the main results of [2].

Theorem 5.1.1 *Let* $1 < p < \infty$.

(i) *If* $\Phi \in K_N^+$ *or* K_N^-, *then there exists* $T_\Phi \in \mathcal{L}(L^p(\Sigma))$ *defined by*

$$\begin{aligned}(T_\Phi u)(x) &= \lim_{\delta \to 0+} \int_\Sigma \Phi(x \pm \delta e_L - y)\mathfrak{n}(y)u(y)dS_y \\ &= \lim_{\epsilon \to} \left(\int_{|x-y| \geqslant \epsilon, y \in \Sigma} \Phi(x - y)\mathfrak{n}(y)u(y)dS_y + \underline{\Phi}(\epsilon \mathfrak{n}(x))u(x) \right)\end{aligned}$$

for all $u \in L^p(\Sigma)$ *and almost all* $x \in \Sigma$. *Moreover, if* $\Phi \in K(C_{N_\mu}^\pm)$ *for all* $0 < \mu \leqslant \pi/2 - \omega$, *then there exists a constant* $C_{\omega,\mu,p}$ *depending only on* ω, μ *and* p *such that*

$$\|T_\Phi u\|_p \leqslant C_{\omega,\mu,p} \|\Phi\|_{K(C_{N_\mu}^+)} \|u\|_p.$$

(ii) *If* $(\Phi, \underline{\Phi}) \in K_N$, *for all* $u \in L^p(\Sigma)$ *and almost all* $x \in \Sigma$, *there exists* $T_{(\Phi,\underline{\Phi})} \in \mathcal{L}(L^p(\Sigma))$ *defined as*

$$(T_{(\Phi,\underline{\Phi})} u)(x) = \lim_{\epsilon \to 0} \left(\int_{|x-y| \geqslant \epsilon, y \in \Sigma} \Phi(x - y)n(y)u(y)dS_y + \underline{\Phi}(\epsilon \mathfrak{n}(x))u(x) \right).$$

Moreover, if $(\Phi, \underline{\Phi}) \in K(S_{N_\mu})$ *for* $0 < \mu \leqslant \pi/2 - \omega$, *then there exists some constant* $C_{\omega,\mu,p}$ *depending only on* ω, μ *and* p *such that*

$$\|T_{(\Phi,\underline{\Phi})} u\|_p \leqslant C_{\omega,\mu,p} \|(\Phi, \underline{\Phi})\|_{K(S_{N_\mu})} \|u\|_p.$$

For Φ_+ and Φ_-,

$$T_{(\Phi,\underline{\Phi})} = T_{\Phi_+} + T_{\Phi_-}.$$

Note that (ii) can be deduced from (i) and Theorem 3.2.1 immediately.

We point out that the spaces K_N^+, K_N^- and K_N are not convolution algebras, while the subspaces M_N^+, M_N^- and M_N are convolution algebras.

Theorem 5.1.2 *The mapping from $\Phi \in M_N^{\pm}$ to $T_\Phi \in \mathcal{L}(L^p(\Sigma))$ and the mapping from $(\Phi, \underline{\Phi}) \in M_N$ to $T_{(\Phi,\underline{\Phi})} \in \mathcal{L}(L^p(\Sigma))$ are algebra homomorphisms.*

Let

$$E(x) = \frac{\overline{x}}{\sigma_n |x|^{n+1}}, \ x \neq 0.$$

Then the function E belongs to M_N^+ and M_N^-. When we consider the function E in M_N^+, we denote by E_+ the function E. When we consider E in M_N^-, we write E as E_-. In addition,

$$(2E, 0) = (E_+, 1/2) + (E_-, 1/2) \in M_N.$$

The corresponding bounded linear operator on $L^p(\Sigma)$ is

$$C_\Sigma = T_{(2E,0)}, \ P_+ = T_{E_+} \text{ and } P_- = -T_{E_-}.$$

By Theorem 5.1.1, we know that for all $u \in L^p(\Sigma)$ and almost all $x \in \Sigma$, these operators are defined as

$$(P_\pm u)(x) = \pm \lim_{\delta \to 0+} \int_\Sigma E(x \pm \delta e_L - y)\mathfrak{n}(y)u(y)dS_y$$

and

$$(C_\Sigma u)(x) = 2 \lim_{\epsilon \to 0} \int_{|x-y| \geqslant \epsilon, y \in \Sigma} E(x - y)\mathfrak{n}(y)u(y)dS_y.$$

The starting point of this section is the boundedness of the operator C_Σ. By Theorems 5.1.1 and 5.1.2, we can deduce the following properties.

Theorem 5.1.3 *Let $\Phi_\pm \in M_N^{\pm}$. Cauchy integral operators P_+, P_- and C_Σ are bounded linear operators on $L^p(\Sigma)$ and satisfy the following identity.*

(1) $P_+ + P_- = I$, $P_+ - P_- = C_\Sigma$ (Plemelj's formula).

(2)
$$P_+T_{\Phi_+} = T_{\Phi_+}P_+ = T_{\Phi_+}, \quad P_-T_{\Phi_+} = T_{\Phi_+}P_- = 0,$$
$$P_-T_{\Phi_-} = T_{\Phi_-}P_- = T_{\Phi_-}, \quad P_+T_{\Phi_-} = T_{\Phi_-}P_+ = 0.$$

(3) $P_+^2 = P_+$, $P_-^2 = P_-$, $P_+P_- = P_-P_+ = 0$, $C_\Sigma^2 = I$;

(4) $T_{\Phi_+}T_{\Phi_-} = T_{\Phi_-}T_{\Phi_+} = 0$.

Defining Hardy spaces $L^{p,\pm}(\Sigma)$ as the images of the projections $P_\pm$, there follows

$$L^p(\Sigma) = L^{p,+}(\Sigma) \oplus L^{p,-}(\Sigma).$$

The operator T_{Φ_+} maps $L^p(\Sigma)$ to $L^{p,+}(\Sigma)$ and is zero on $L^{p,-}(\Sigma)$, while the operator T_{Φ_-} maps $L^p(\Sigma)$ to $L^{p,-}(\Sigma)$ and is identity with 0 on $L^{p,+}(\Sigma)$. Hence we can define $T_{\Phi_\pm} \in \mathcal{L}(L^{p,\pm}(\Sigma))$ such that

$$T_{(\Phi,\underline{\Phi})} = T_{\Phi_+} \oplus T_{\Phi_-},$$

where $(\Phi, \underline{\Phi})$ is related to Φ_+ and Φ_- as in Theorem 3.2.1.

At the end of Sect. 3.3, we used the Fourier theory to prove $(2E_j,\ 0) \in K_N$, where

$$E_j(x) = -\frac{x_j}{(\sigma_n |x|^{n+1})},\ x \in \mathbb{R}^n \setminus \{0\},\ j = 1, 2, \ldots, n.$$

Then the operators $R_{j,\Sigma} = T_{2E_j}$, $j = 1, 2, \ldots, n$, are bounded on $L^p(\Sigma)$. These operators can be regarded as Riesz transforms on Σ. The L^p-boundedness of the operators $R_{j,\Sigma}$ is one of the motivations to establish the Fourier theory on Σ. Because $R_{j,\Sigma}$ is not merely the jth component of C_Σ, the boundedness of these operators is not a direct consequence of the boundedness of the Cauchy integral operator $C_\Sigma := \sum e_j R_{j,\Sigma}$.

Theorem 5.1.4 *The Riesz transforms $R_{j,\Sigma}$ are bounded linear operators on $L^p(\Sigma)$ which satisfy*

$$R_{j,\Sigma} R_{k,\Sigma} = R_{k,\Sigma} R_{j,\Sigma}, \quad \sum e_j R_{j,\Sigma} = C_\Sigma \text{ and } \sum (R_{j,\Sigma})^2 = -I.$$

The following results are corollaries of Theorems 5.1.1 and 5.1.2. When $\Phi \in K_N^+$ and $\delta > 0$, $\Phi_\delta \in K_N^+$ is defined as $\Phi_\delta(x) = \Phi(x + \delta e_L)$. In particular, $E_\delta \in M_N^+$, where

$$E_\delta(x) = E_{+\delta}(x) = E_+(x + \delta e_L).$$

If p is a polynomial of m variables with values in $\mathbb{C}_{(M)}$, then $p(-i\underline{D})E_\delta \in K_N^+$, where

$$p(-i\underline{D})E_\delta(x) = p(-i\frac{\partial}{\partial x_1},\ -i\frac{\partial}{\partial x_2}, \cdots, -i\frac{\partial}{\partial x_n})E_+(x + \delta e_L).$$

Theorem 5.1.5 *Let $\alpha > 0$ and $\delta > 0$.*

(i) *If $\Phi \in K_N^+$, then $\Phi * E_\delta = \Phi_\delta \in K_N^+$, and $T_\Phi T_{E_\delta} = T_{\Phi_\delta}$.*

(ii) *If $\Phi \in M_N^+$, then $E_\delta * \Phi = \Phi_\delta \in M_N^+$, and $T_{E_\delta} T_\Phi = T_{\Phi_\delta}$.*

(iii) *$E_\alpha * E_\delta = E_{\alpha+\delta} \in M_N^+$, and $T_{E_\alpha} T_{E_\delta} = T_{E_{\alpha+\delta}}$.*

Assume that p and q are two polynomials, where p satisfies $p(\xi)\xi e_L = \xi e_L p(\xi)$. Then $p(-i\underline{D})E_\delta \in M_N^+$ and

(iv) *$E_\alpha * p(-i\underline{D})E_\delta = p(-i\underline{D})E_{\alpha+\delta} \in M_N^+$ and*

$$T_{E_\alpha} T_{p(-i\underline{D})E_\delta} = T_{p(-i\underline{D})E_{\alpha+\delta}}.$$

(v) $q(-i\underline{D})E_\alpha * p(-i\underline{D})E_\delta = (qp)(-i\underline{D})E_{\alpha+\delta} \in K_N^+$, *and*

$$T_{q(-i\underline{D})E_\alpha} T_{p(-i\underline{D})E_\delta} = T_{(qp)(-i\underline{D})E_{\alpha+\delta}}.$$

Let Ω_+ be an open subset of $\mathbb{R}^{n+1}$ above Σ, that is,

$$\Omega_+ = \Big\{X \in \mathbb{R}^{n+1} : \ X = x + \delta e_L, \ x \in \Sigma, \delta > 0\Big\}.$$

For $u \in L^p(\Sigma)$, let $C_\Sigma^+ u$ be the left monogenic function on Ω_+ defined by

$$(C_\Sigma^+ u)(X) = \int_\Sigma E(X-y)\mathfrak{n}(y)u(y)dS_y, \ X \in \Omega_+.$$

Then for almost all $x \in \Sigma$, when $\delta \to 0+$,

$$(C_\Sigma^+ u)(x + \delta e_L) = T_{E_\delta} u(x) \to P_+ u(x).$$

The limit exists in the sense of L^p (see [2]). In other words, as $\delta \to 0+$,

$$\|T_{E_\delta} u - P_+ u\|_p \to 0.$$

Although the limit need not always exist as X approaches Σ, we can differentiate $(C_\Sigma^+ u)(X)$ before taking the limit. Generally, given any polynomial p of n variables with values in $\mathbb{C}_{(M)}$. Although the limit may not exist as X approaches Σ, we can construct

$$p(-i\underline{D})(C_\Sigma^+ u)(X) = p\Big(-i\frac{\partial}{\partial X_1}, \ -i\frac{\partial}{\partial X_2}, \cdots, -i\frac{\partial}{\partial X_n}\Big)(C_\Sigma^+)(X).$$

If the limit exists in $L^p(\Sigma)$, we define $p(-i\underline{D}_\Sigma)u(x)$ as the limit of

$$p(-i\underline{D})(C_\Sigma^+ u)(x + \delta e_L) = T_{p(-i\underline{D})E_\delta} u(x)$$

as $\delta \to 0+$.

Precisely, define $p(-i\underline{D}_\Sigma)$ as the linear transformation from $\mathcal{D}^+(p(-i\underline{D}_\Sigma)) \subset L^{p,+}(\Sigma)$ to $L^p(\Sigma)$:

$$\mathcal{D}^+(p(-i\underline{D}_\Sigma)) = \Big\{u \in L^{p,+}(\Sigma) : \ T_{p(-i\underline{D})E_\delta} u \to w \in L^p(\Sigma)\Big\}$$

and $p(-i\underline{D}_\Sigma)u = w$.

If for some $v \in L^{p,+}(\Sigma)$, $u = T_{k_\delta} v$, then u is the restriction of the left monogenic function U to Σ, where

$$U(X) = (C_\Sigma^+ v)(X + \alpha e_L),\ \ X + \alpha e_L \in \Omega_+.$$

Such a function u belongs to $\mathcal{D}^+(p(-i\underline{D}_\Sigma))$ and

$$p(-i\underline{D}_\Sigma)u = (p(-i\underline{D})U)\mid_\Sigma .$$

Specially, we consider the functions

$$q_k(x) = i\xi_k,\ \ k = 1, 2, \ldots, n$$

and

$$q(\xi) = i\xi e_L = \sum_{k=1}^{n} i\xi_k e_k e_L.$$

By use of these functions, we define the operators $D_{k,\Sigma} = q_k(-i\underline{D}_\Sigma)$ and $\underline{D}_\Sigma e_L = q(-i\underline{D}_\Sigma)$ such that for the function u mentioned above,

$$\underline{D}_\Sigma e_L u = (\underline{D} e_L U)\mid_\Sigma$$

and

$$D_{k,\Sigma} u = \frac{\partial U}{\partial X_k},\ \ k = 1, 2, \ldots, n.$$

When Σ has a parametric representation: $x = \underline{s} + g(\underline{s})e_L$, these functions can be represented as the parameter $\underline{s}$. We obtain that for all functions u such that $u = T_{E_\alpha} v$,

$$D_{k,\Sigma} u(\underline{s} + g(\underline{s})e_L) = \Big(\frac{\partial}{\partial s_k} + \frac{\partial g}{\partial s_k}(e_L - \underline{D}g)^{-1}\underline{D}_{\mathbf{s}}\Big)u(\underline{s} + g(\underline{s})e_L);$$

and

$$\begin{aligned}\underline{D}_\Sigma e_L u(\underline{s} + g(\underline{s})e_L) &= \sum_{k=1}^{m} e_k e_L D_{k,\Sigma} u(\underline{s} + g(\underline{s})e_L)\\ &= (e_L - \underline{D}g)^{-1}\underline{D}_{\underline{s}} u(\underline{s} + g(\underline{s})e_L),\end{aligned}$$

where $v \in L^{p,+}(\Sigma)$. In the following theorem, we will see that this representation of $\underline{D}_\Sigma$ is valid for any function u in its domain. From the following theorem, we also conclude that these operators are closed linear operators on $L^{p,+}(\Sigma)$. In the next two sections, we will study how to represent the convolution operators in Theorem 5.1.1 as bounded holomorphic functions of $(D_{k,\Sigma})$ and $\underline{D}_\Sigma$. We still assume that $1 < p < \infty$.

Theorem 5.1.6 *Let p be a polynomial of n variables with values in $\mathbb{C}_{(M)}$. Then $p(-i\underline{D}_\Sigma)$ is a linear transformation from $L^{p,+}(\Sigma)$ to $L^p(\Sigma)$, where its domain $\mathcal{D}^+(p(-i\underline{D}_\Sigma))$ is dense in $L^{p,+}(\Sigma)$.*

If $p(\xi)\xi e_L = \xi e_L p(\xi)$, then $p(-i\underline{D}_\Sigma)u \in L^{p,+}(\Sigma)$ for all $u \in \mathcal{D}^+(p(-i\underline{D}_\Sigma))$, and actually $p(-i\underline{D}_\Sigma)$ is a closed linear operator on $L^{p,+}(\Sigma)$.

Suppose that p and q are two polynomials such that p satisfies $p(\xi)\xi e_L = \xi e_L p(\xi)$. Let $u \in \mathcal{D}^+(p(-i\underline{D}_\Sigma))$. Then $p(-i\underline{D}_\Sigma)u \in \mathcal{D}^+(q(-i\underline{D}_\Sigma))$ if and only if $u \in \mathcal{D}^+((qp)(-i\underline{D}_\Sigma))$. In this case,

$$q(-i\underline{D}_\Sigma)p(-i\underline{D}_\Sigma)u = (qp)(-i\underline{D}_\Sigma)u.$$

Proof Because any function $u \in L^{p,+}(\Sigma)$ is the limit of $T_{E_\alpha}u \in \mathcal{D}^+(p(-i\underline{D}_\Sigma))$ as $\alpha \to 0$, the domain $\mathcal{D}^+(p(-i\underline{D}_\Sigma))$ is dense in $L^{p,+}(\Sigma)$.

Assume that

$$p(\xi)\xi e_L = \xi e_L p(\xi).$$

Let $u \in \mathcal{D}^+(p(-i\underline{D}_\Sigma))$. In Theorem 5.1.5, we can see that when $\delta > 0$, $p(-i\underline{D})E_\delta \in M_N^+$. When $\alpha > 0$,

$$T_{E_\alpha}T_{p(-i\underline{D})E_\delta} = T_{p(-i\underline{D})E_{\alpha+\delta}}u.$$

Letting $\delta \to 0$ and $\alpha \to 0$, we get

$$T_{E_\alpha}p(-i\underline{D}_\Sigma)u = T_{p(-i\underline{D})E_\alpha}u$$

and

$$p(-i\underline{D}_\Sigma)u = P_+p(-i\underline{D}_\Sigma)u \in L^{p,+}(\Sigma),$$

respectively. To prove $p(-i\underline{D}_\Sigma)$ is closed in $L^{p,+}(\Sigma)$, choose the sequence $\{v_m\}$ in $\mathcal{D}^+(p(-i\underline{D}_\Sigma))$ such that $v_m \to v \in L^{p,+}(\Sigma)$ and $p(-i\underline{D})v_m \to w \in L^{p,+}(\Sigma)$. We need to prove $v \in \mathcal{D}^+(p(-i\underline{D}_\Sigma))$ and $p(-i\underline{D}_\Sigma)v = w$. For any $\alpha > 0$,

$$T_{E_\alpha}p(-i\underline{D}_\Sigma)v_m \to T_{k_\alpha}w$$

and

$$T_{E_\alpha}p(-i\underline{D}_\Sigma)v_m = T_{p(-i\underline{D})E_\alpha}v_m \to T_{p(-i\underline{D})E_\alpha}v$$

such that $T_{p(-i\underline{D})E_\alpha}v = T_{k_\alpha}w$. Hence

$$T_{p(-i\underline{D})E_\alpha}v = T_{E_\alpha}w \to w \text{ as } \alpha \to 0.$$

We obtain $v \in \mathcal{D}(p(-i\underline{D}_\Sigma))$ and $p(-i\underline{D}_\Sigma)v = w$.

By Theorem 5.1.5, we get

$$T_{q(-i\underline{D})E_\alpha}T_{p(-i\underline{D})E_\delta} = T_{(qp)(-i\underline{D})E_{\alpha+\delta}}u.$$

Hence, letting $\delta \to 0$, we can obtain

$$T_{q(-i\underline{D})E_\alpha}p(-i\underline{D}_\Sigma)u = T_{(qp)(-i\underline{D}_\Sigma)E_\alpha}u.$$

Letting $\alpha \to 0$, we can see that $p(-i\underline{D}_\Sigma)u \in \mathcal{D}^+(q(-i\underline{D}_\Sigma))$ if and only if $u \in \mathcal{D}^+((qp)(-i\underline{D}_\Sigma))$. In this case,

$$q(-i\underline{D}_\Sigma)p(-i\underline{D}_\Sigma)u = (qp)(-i\underline{D}_\Sigma)u. \qquad \square$$

Similarly, we can define the linear transformation $p(-i\underline{D}_\Sigma)$ from the domain $\mathcal{D}^-(p(-i\underline{D}_\Sigma)) \subset L^{p,-}(\Sigma)$ to $L^p(\Sigma)$.

At last, we define the linear operator $p(-i\underline{D}_\Sigma)$ on $L^p(\Sigma)$ as

$$p(-i\underline{D}_\Sigma)u = p(-i\underline{D}_\Sigma)P_+u + p(-i\underline{D}_\Sigma)P_-u$$

with the dense domain

$$\begin{aligned}\mathcal{D}(p(-i\underline{D}_\Sigma)) &= \mathcal{D}^+(p(-i\underline{D}_\Sigma)) \oplus \mathcal{D}^-(p(-i\underline{D}_\Sigma))\\ &\subset L^{p,+}(\Sigma) \oplus L^{p,-}(\Sigma) = L^p(\Sigma).\end{aligned}$$

Theorem 5.1.7 *If $L^{p,+}(\Sigma)$ is replaced by $L^p(\Sigma)$ and $\mathcal{D}^+(p(-i\underline{D}_\Sigma))$ is replaced by $\mathcal{D}(p(-i\underline{D}_\Sigma))$, Theorem 5.1.6 still holds.*

Assume that U is a left monogenic function on the strip $\Sigma + (-t,t)e_L$. The function u_α defined by

$$u_\alpha(x) = U(x + \alpha e_L), \ x \in \Sigma, \ \alpha \in (-t,t),$$

is uniformly bounded on $L^p(\Sigma)$. Let $u = u_0 = U \mid_\Sigma$. Then by the remark following the definition of $p(-i\underline{D}_\Sigma)$ in $L^{p,+}(\Sigma)$, $P_+u = T_{k_\alpha}P_+u_{-\alpha}$ and the similar result for P_-u, we can conclude that for any polynomial p,

$$p(-i\underline{D}_\Sigma) = (p(-i\underline{D})U) \mid_\Sigma .$$

Specially, for such a left monogenic function U, when $u = U \mid_\Sigma$,

$$\underline{D}_\Sigma e_L u = (\underline{D}e_L U) \mid_\Sigma \text{ and } D_{k,\Sigma}u = \frac{\partial U}{\partial X_k} \mid_\Sigma, \ k = 1, 2, \ldots, n.$$

5.2 H^∞-Functional Calculus of Functions of n Variables

Let $(\Phi, \underline{\Phi}) \in K(S_{N_\mu})$. We can regard $b = \mathcal{F}(\Phi, \underline{\Phi})e_L$ as the Fourier multiplier corresponding to the bounded linear operator $T_{(\Phi,\underline{\Phi})}$. We also regard the mapping from $b \in H_N$ to $T_{(\Phi,\underline{\Phi})} \in \mathcal{L}(L^p(\Sigma))$ as the bounded H^∞-functional calculus of

$$-i\underline{D}_\Sigma = \sum_{k=1}^{n} = ie_k D_{k,\Sigma}.$$

Write

$$T_{(\Phi,\underline{\Phi})} = b(-i\underline{D}_\Sigma) = b(-iD_{1,\Sigma},\ -iD_{2,\Sigma}, \cdots, -iD_{n,\Sigma}).$$

We introduce an algebra $\mathcal{P}_N$ which is larger than H_N. $\mathcal{P}_N$ consists of all functions b from $\mathbb{R}^n \setminus \{0\}$ to $\mathbb{C}_{(M)}$ such that $b_+ = b\chi_+$ can be extended holomorphically to $\overline{N_\mu}(\mathbb{C}^n)$. For s and $c \geqslant 0$, this extension satisfies

$$|b_\pm(\zeta)| \leqslant c(1 + |\zeta|^s).$$

For such a $b \in \mathcal{P}_N$, the functions $b_{+\delta}$ and $b_{-\delta}$ belong to H_N^+ and H_N^-, respectively, where $b_{+\delta}(\zeta) = b_+(\zeta)e^{-\delta|\zeta|_{\mathbb{C}}}$ and $b_{-\delta}(\zeta) = b_-(\zeta)e^{-\delta|\zeta|_{\mathbb{C}}}$ for $\delta > 0$. Hence

$$\Phi_{\pm\delta} = \mathcal{G}_\pm(b_{\pm\delta}\overline{e_L}) \in K_N^+.$$

Define $b(-i\underline{D}_\Sigma)$ to be the linear operator with the domain

$$\mathcal{D}(b(-i\underline{D}_\Sigma)) = \Big\{u \in L^p(\Sigma) :\ T_{\Phi_{\pm\delta}} \to w_\pm \in L^p(\Sigma) \text{ as } \delta \to 0\Big\}$$

in $L^p(\Sigma)$ by

$$b(-i\underline{D}_\Sigma)u = w_+ + w_-.$$

From the following theorem, we know that the above definition is meaningful.

Theorem 5.2.1 *Assume that* $1 < p < \infty$. *Let* $b \in \mathcal{P}_N$.

(i) *If* $b \in H_N$, *then* $b(-i\underline{D}_\Sigma) = T_{(\Phi,\underline{\Phi})} \in \mathcal{L}(L^p(\Sigma))$, *where* $(\Phi, \underline{\Phi})e_L = \mathcal{G}(b)$. *Specially,*

$$1(-i\underline{D}_\Sigma) = I, \quad \chi_\pm(-i\underline{D}_\Sigma) = P_\pm,$$
$$(r_je_L)(-i\underline{D}_\Sigma) = R_{j,\Sigma},$$
$$r(-i\underline{D}_\Sigma) = C_\Sigma = \sum e_jR_{j,\Sigma},$$

where $r(\xi) = i\xi|\xi|^{-1}e_L$.

(ii) *If* $b_+ = b\chi_+ \in H_\infty^+(N_\mu(\mathbb{C}^n))$ *and* $b_- = b\chi_- \in H_\infty^-(\overline{N_\mu}(\mathbb{C}^n))$ *for* $0 < \mu \leqslant \pi/2 - \omega$, *then the following inequality*

$$\|b(-i\underline{D}_\Sigma)u\|_p \leqslant C_{\omega,\mu,p}(\|b_+\|_\infty + \|b_-\|_\infty)\|u\|_p$$

holds for the constant $C_{\omega,\mu,p}$ *depending only on* ω, μ, p *(and the dimension* n*).*

(iii) *If* b *is a polynomial of* n *variables, then the definition of the domain of* $b(-i\underline{D}_\Sigma)$ *coincides with that given in Sect. 5.1.*

(iv) *The domain of* $b(-i\underline{D}_\Sigma)$, $\mathcal{D}(b(-i\underline{D}_\Sigma))$, *is dense in* $L^p(\Sigma)$.

(v) *If* $b(\xi)\xi e_L = \xi e_L b(\xi)$ *for all* $\xi \in \mathbb{R}^n \setminus \{0\}$, *then* $b(-i\underline{D}_\Sigma)$ *is a closed linear operator in* $L^p(\Sigma)$.

(vi) *If* $u \in \mathcal{D}(b(-i\underline{D}_\Sigma))$, $f \in \mathcal{P}_N$ *and* $c \in \mathbb{C}_{(M)}$, *then* $u \in \mathcal{D}(f(-i\underline{D}_\Sigma))$ *if and only if* $u \in \mathcal{D}((cb+f)(-i\underline{D}_\Sigma))$ *and* $cb(-i\underline{D}_\Sigma)u + f(-i\underline{D}_\Sigma)u = (cb+f)(-i\underline{D}_\Sigma)u$.

(vii) *If for all* $\xi \in \mathbb{R}^n \setminus \{0\}$, $b(\xi)\xi e_L = \xi e_L b(\xi)$, $u \in \mathcal{D}(b(-i\underline{D}_\Sigma))$ *and* $f \in \mathcal{P}_N$, *then* $b(-i\underline{D}_\Sigma)u \in \mathcal{D}(f(-i\underline{D}_\Sigma))$ *if and only if* $u \in \mathcal{D}((fb)(-i\underline{D}_\Sigma))$, *and*

$$f(-i\underline{D}_\Sigma)b(-i\underline{D}_\Sigma)u = (fb)(-i\underline{D}_\Sigma)u.$$

Proof For $b \in H_N$, let $b_+ = b\chi_+$ and $\Phi_+ = \mathcal{G}_+(b_+\overline{e_L})$. We have

$$\Phi_{+\delta}(x) = \mathcal{G}_+(b_{+\delta}\overline{e_L})(x) = \Phi_+(x + \delta e_L).$$

Hence, for all $u \in L^p(\Sigma)$, $T_{\Phi_{+\delta}}u \to T_{\Phi_+}u$ in $L^p(\Sigma)$ as $\delta \to 0$. So $u \in \mathcal{D}(b(-i\underline{D}_\Sigma))$ and

$$b(-i\underline{D}_\Sigma)u = T_{\Phi_+}u = T_{\Phi_+}u + T_{\Phi_-}u = T_{(\Phi,\underline{\Phi})}.$$

The estimates in (ii) can be deduced from (iii) of Theorem 3.3.1 and (ii) of Theorem 5.1.1.

To prove (iii), we use the equality

$$\mathcal{F}_\pm(p(-i\underline{D}_\Sigma)k_{\pm\delta})e_L = p_{\pm\delta},$$

which is deduced from (vi) of Theorem 3.3.1. The rest of the proof is similar to that of Theorem 5.1.6. □

Now we give some applications. We consider the following boundary values problem of the harmonic functions.

$$\begin{cases} \Delta U(X) = \displaystyle\sum_{k=1}^{n} \frac{\partial^2 U}{\partial X_k^2}(X) + \frac{\partial^2 U}{\partial x_L^2}(X) = 0, \ X \in \Omega_+, \\ \left.\left(\displaystyle\sum_{k=1}^{n} \beta_k \frac{\partial U}{\partial X_k} + \beta_L \frac{\partial U}{\partial x_L}\right)\right|_\Sigma = w \in L^p(\Sigma, \mathbb{C}), \end{cases}$$

where β_k, $k = 1, 2, \ldots, n$, $\beta_L \in \mathbb{C}$ and $2 \leqslant p < \infty$.

For the special cases $\beta_L = 1$ and $\beta_k = 0$, $k = 1, 2, \ldots, n$, the solution of this problem is

$$U(\underline{X}) = U(\underline{X} + X_L e_L) = -\int_{X_L}^{\infty} (C_{\Sigma_0}^+ v)(\underline{X} + te_L)dt,$$

where $v = (P_{+0})^{-1}w \in L^p(\Sigma)$. Here $C^+_{\Sigma_0}$ denotes the scalar part of Cauchy integral C^+_Σ :

$$(C^+_{\Sigma_0}v)(\underline{X}) = \int_\Sigma \langle\overline{k(\underline{X}-y)}, \mathfrak{n}(y)\rangle v(y)dS_y, \ \underline{X} \in \Omega_+,$$

that is, the double-layer potential operator on Σ, and $P_{+0} = \frac{1}{2}(I + C_{\Sigma_0})$, where C_{Σ_0} is the singular double-layer potential operator on Σ. The invertibility of P_{+0} in $L^p(\Sigma, \mathbb{C})$ was proved by Verchota [3].

For the general case that β_k and β_L are complex numbers, we assume that for some $\kappa > 0$,

$$|\langle\beta, \ \mathfrak{n} + it\rangle| \geqslant \kappa \text{ for } \mathfrak{n} \in N \text{ and } t \in \mathbb{R}^{n+1} \text{ such that } |t| = 1 \text{ and } \langle\mathfrak{n}, \ t\rangle = 0, \quad (5.1)$$

where $\beta = \sum \beta_k e_k + \beta_L e_L$. (This is the weakest condition on β under which we can expect to solve the boundary values problem, because if Σ is smooth in a neighborhood of a point $x \in \Sigma$, then the covering condition of Agmon, Douglis, Nirenberg for this problem is that there does not exist a unit tangent vector t to Σ at x satisfying

$$\langle\beta, \ \mathfrak{n}(x) + it\rangle = 0,$$

where $\mathfrak{n}(x)$ is the unit normal to Σ at x.)

We can deduce from (5.1) that for all $\zeta \in N(\mathbb{C}^m)$,

$$|\langle\beta, \ |\zeta|_{\mathbb{C}} e_L - i\zeta\rangle| \geqslant \kappa||\zeta|_{\mathbb{C}}|. \quad (5.2)$$

There exists a holomorphic function b defined by

$$b(\zeta) = \frac{|\zeta|_{\mathbb{C}}}{\langle\beta, \ |\zeta|_{\mathbb{C}} e_L - i\zeta\rangle}$$

which is bounded by κ^{-1} on $N(\mathbb{C}^n)$. In fact, for some sufficiently small μ, this function is bounded by $2\kappa^{-1}$ on $N_\mu(\mathbb{C}^n)$. In order to deduce (5.1) from (5.2), we take $\zeta \in N(\mathbb{C}^n)$: there exist $\mathfrak{n} \in N$ and $c > 0$ such that $\eta + \mathrm{Re}(|\zeta|_{\mathbb{C}})e_L = c\mathfrak{n}$. By (5.1) and the choice of $\mathfrak{n}$ and $t = c^{-1}(-\xi + \mathrm{Im}(|\zeta|_{\mathbb{C}})e_L)$, we can obtain the desired result.

Hence $b(-i\underline{D}_\Sigma)$ is a bounded linear operator on $L^p(\Sigma, \mathbb{C}_{(M)})$. Notice the equality

$$\Big(\sum_{k=1}^n \beta_k\zeta_k - \beta_L\zeta e_L\Big)b(\zeta)\chi_+(\zeta) = -\zeta e_L\chi_+(\zeta).$$

We can prove directly that the solution of the boundary values problem is

$$U(\underline{X}) = U(\underline{X} + X_L e_L) = -\int_{X_L}^\infty (C^+_\Sigma b(-i\underline{D}_\Sigma)v)_0(\underline{X} + te_L)dt,$$

where $\underline{X} \in \Omega_+$ and $v = (P_{+0})^{-1}w \in L^p(\Sigma, \mathbb{C})$.

Moreover, if $x \in \Sigma$ and $\delta > 0$,

$$(C_\Sigma^+ b(-i\underline{D}_\Sigma)v)_0(x + \delta e_L) = (T_{\Phi_\delta} v)_0(x)$$

where $\Phi = \mathcal{G}_+(b\chi_+\overline{e_L}) \in M_N^+$. All integrals can be represented as

$$(C_\Sigma^+ b(-i\underline{D}_\Sigma)v)_0(\underline{X}) = \int_\Sigma \langle \overline{\Phi(\underline{X} - y)},\ \mathfrak{n}(y)\rangle v(y) dS_y,$$

where $\underline{X} \in \Omega_+$.

We point out that the Fourier theory established in Sect. 3.3 has been used to prove the assumption (5.1) implies $\Phi \in M_N^+$, and that $T_\Phi \in \mathcal{L}(L^p(\Sigma, \mathbb{C}_{(M)}))$. Now we give a covering lemma. Especially, this lemma can be used to prove that other reasonable definitions of $b(-i\mathbf{D}_\Sigma)$ could also lead to the same operator as ours. We still assume that $1 < p < \infty$.

Lemma 5.2.1 (Covering lemma) *Suppose that* $0 < \mu \leqslant \pi/2 - \omega$. *Let*

$$b_{(\alpha)} = b_{(\alpha)+} + b_{(\alpha)-},$$

where $b_{(\alpha)+}$ *is a uniformly bounded net of functions in* $H_\infty^+(N_\mu(\mathbb{C}^n))$ *which converges to a function* $b_+ \in H_\infty^+(N_\mu(\mathbb{C}^n))$ *uniformly on each set with the form*

$$\left\{\zeta \in N_\mu(\mathbb{C}^n) :\ 0 < \delta \leqslant |\zeta| \leqslant \Delta < \infty\right\},$$

and $b_{(\alpha)-}$ *is a uniformly bounded net of functions in* $H_N^-(\overline{N}_\mu(\mathbb{C}^n))$ *which converges to* $b_- \in H_\infty^-(\overline{N}_\mu(\mathbb{C}^n))$ *in a similar way. Let* $b = b_+ + b_-$. *Then for any* $u \in L^p(\Sigma)$, $b_{(\alpha)}(-i\underline{D}_\Sigma)u$ *converges to* $b(-i\underline{D}_\Sigma)u$. *Hence*

$$\|b(-i\underline{D}_\Sigma)\| \leqslant \sup_\alpha \|b_{(\alpha)}(-i\underline{D}_\Sigma)\|.$$

Proof In fact, by the definition, we can directly deduce that

$$\Phi_{(\alpha)\pm} = \mathcal{G}_\pm(b_{(\alpha)\pm}\overline{e_L})$$

converges to $\Phi_\pm = \mathcal{G}_\pm(b_\pm\overline{e_L})$. Hence we obtain that for every $u \in L^p(\Sigma)$,

$$b_{(\alpha)}(-i\underline{D}_\Sigma)u = T_{\Phi_{(\alpha)+}}u + T_{\Phi_{(\alpha)-}}u$$

converges to

$$T_{\Phi_+}u + T_{\Phi_-}u = b(-i\underline{D}_\Sigma)u.$$

□

The following is an easy corollary. We state it for functions defined on sets of the form $S^0_\mu(\mathbb{C}^n)$, rather than on the general sets $N_\mu(\mathbb{C}^n)$ and $\overline{N}_\mu(\mathbb{C}^n)$.

Theorem 5.2.2 *Let b be a holomorphic function which satisfies $|b(\zeta)| \leqslant c(1+|\zeta|^d)$ on $S^0_\mu(\mathbb{C}^n)$ for some $\mu \in (\omega, \pi/2)$, d and $c \geqslant 0$. Assume that*

(i) for all $\xi \in \mathbb{R}^n$, $b(\xi)\xi e_L = \xi e_L b(\xi)$;
(ii) for all $\zeta \in S^0_\mu(\mathbb{C}^n)$, $b(\zeta)$ has an inverse $b(\zeta)^{-1} \in \mathbb{C}_{(M)}$;
(iii) there exists $s \geqslant 0$ such that

$$|b(\zeta)^{-1}| \leqslant c(|\zeta|^s + |\zeta|^{-s}),\ \zeta \in S^0_\mu(\mathbb{C}^n).$$

Then the operator $b(-i\underline{D}_\Sigma)$ is one-one and has dense range $\mathcal{R}(b(-i\underline{D}_\Sigma))$ in $L^p(\Sigma)$.

Proof Define the sequence $\{F_m\}$ as

$$F_m(\lambda) = (m\lambda)^s(i+m\lambda)^{-s}(\chi_{\mathrm{Re}>0}(\lambda)e^{-\lambda/m} + \chi_{\mathrm{Re}<0}(\lambda)e^{\lambda/m}),$$

where $\lambda \in S^0_\mu(\mathbb{C})$, $m = 1, 2, \ldots$. Then the sequence $\{F_m\}$ is uniformly bounded and converges to 1 on any set of the form

$$\left\{\lambda \in S^0_\mu(\mathbb{C}) :\ 0 < \delta \leqslant |\lambda| \leqslant \Delta < \infty\right\}.$$

For any n, define $\{f_m\} \subset H_\infty(S^0_\mu(\mathbb{C}^n))$ as

$$f_m(\zeta) = F_m(|\zeta|_{\mathbb{C}})\chi_+(\zeta) + F_m(-|\zeta|_{\mathbb{C}})\chi_-(\zeta).$$

Then the sequence $\{f_m\}$ is uniformly bounded and uniformly converges to 1 on every set of the form

$$\left\{\zeta \in S^0_\mu(\mathbb{C}^n) :\ 0 < \delta \leqslant |\zeta| \leqslant \Delta < \infty\right\}.$$

Let

$$g_m = f_m b^{-1} \in H_\infty(S^0_\mu(\mathbb{C}^n))$$

and

$$h_m = b^{-1} f_m \in H_\infty(S^0_\mu(\mathbb{C}^n)).$$

such that $f_m = g_m b = b h_m$.

Assume that $u \in \mathcal{D}(b(-i\underline{D}_\Sigma))$ and $b(-i\underline{D}_\Sigma)u = 0$. By (vii) of Theorem 5.2.1, we know

$$f_{(n)}(-i\underline{D}_\Sigma)u = g_{(n)}(-i\underline{D}_\Sigma)b(-i\underline{D}_\Sigma)u = 0.$$

By Lemma 5.2.1, $f_m(-i\underline{D}_\Sigma)u$ converges to u. So, $u = 0$. We obtain that $b(-i\underline{D}_\Sigma)$ is a one-one operator.

Let $w \in L^p(\Sigma)$. Then

$$f_m(-i\underline{D}_\Sigma)w = b(-i\underline{D}_\Sigma)h_m(-i\underline{D}_\Sigma)w \in \mathcal{R}(b(-i\underline{D}_\Sigma))$$

and

$$\lim_{n\to\infty} f_m(-i\underline{D}_\Sigma)w = w.$$

We obtain that $\mathcal{R}(b(-i\underline{D}_\Sigma))$ is dense in $L^p(\Sigma)$. □

5.3 H^∞-Functional Calculus of Functions of One Variable

We turn our attention to functions b as holomorphic functions of one complex variable. For any holomorphic function B defined on $S^0_\mu(\mathbb{C})$, where $\omega < \mu \leqslant \pi/2$, there exists a function b defined on $S^0_\mu(\mathbb{C}^n)$

$$b(\zeta) = B(i\zeta e_L) = B(|\zeta|_{\mathbb{C}})\chi_+(\zeta) + B(-|\zeta|_{\mathbb{C}})\chi_-(\zeta).$$

When $b(-i\underline{D}_\Sigma)$ itself is defined, we can naturally define the operator $B(\underline{D}_\Sigma e_L)$ as $B(\underline{D}_\Sigma e_L) = b(-i\underline{D}_\Sigma)$.

It follows from Theorems 3.1.4 and 5.2.1 that the mapping $B \to B(\underline{D}_\Sigma e_L)$ from $H_\infty(S^0_\mu(\mathbb{C}))$ to $\mathcal{L}(L^p(\Sigma))$ is a bounded algebra homomorphism.

We point out that the usually used condition $b(\zeta)\zeta e_L = \zeta e_L b(\zeta)$ is satisfied by the functions b of the form $b(\zeta) = B(i\zeta e_L)$.

Let H_ω be the linear space consisting of the following functions B on $\mathbb{R}\setminus\{0\}$: for some $\mu > \omega$, the function B can be extended to $B \in H_\infty(S^0_\mu(\mathbb{C}))$. Let $\mathcal{P}_\omega$ be the linear space consisting of the following functions B on $\mathbb{R}\setminus\{0\}$: for some $\mu > \omega$, these functions B can be extended holomorphically to $S^0_\mu(\mathbb{C})$ and on $S^0_\mu(\mathbb{C})$,

$$|B(\zeta)| \leqslant c(1 + |\zeta|^s).$$

for some s and $c \geqslant 0$.

Theorem 5.3.1 *Assume that* $1 < p < \infty$. *Let* $B \in \mathcal{P}_\omega$.

(i) *The operator* $B(\underline{D}_\Sigma e_L)$ *is a closed linear operator in* $L^p(\Sigma)$ *and its domain* $\mathcal{D}_\Sigma(B(\underline{D}e_L))$ *is dense in* $L^p(\Sigma)$.

(ii) *If* $B \in H_\omega$, *then*

$$B(\underline{D}_\Sigma e_L) = T_{(\Phi,\underline{\Phi})} \in \mathcal{L}(L^p(\Sigma)),$$

where $\mathcal{F}(\Phi, \underline{\Phi})e_L = b$ *and* $b(\xi) = B(i\xi e_L)$. *Specially,*

$$\begin{cases} 1(\underline{D}_\Sigma e_L) = I, \\ \chi_{\mathrm{Re}>0}(\underline{D}_\Sigma e_L) = P_+, \\ \chi_{\mathrm{Re}<0}(\underline{D}_\Sigma e_L) = P_-, \\ \mathrm{sgn}(\underline{D}_\Sigma e_L) = C_\Sigma. \end{cases}$$

(iii) *If $B \in H_\infty(S^0_\mu(\mathbb{C}))$ and $\omega < \mu < \pi/2$, there exists a constant $C_{\omega,\mu,p}$ depending only on ω, μ, p and the dimension n such that*

$$\|B(\underline{D}_\Sigma e_L)u\|_p \leqslant C_{\omega,\mu,p}\|B\|_\infty\|u\|_p,\ u \in L^p(\Sigma).$$

(iv) *If $u \in \mathcal{D}(B(\underline{D}_\Sigma e_L))$, $F \in \mathcal{P}_\omega$ and $c \in \mathbb{C}$, then $u \in \mathcal{D}(F(\underline{D}_\Sigma e_L))$ if and only if $u \in \mathcal{D}((cB + F)(\underline{D}_\Sigma e_L))$, in which case,*

$$cB(\underline{D}_\Sigma e_L)u + F(\underline{D}_\Sigma e_L)u = (cB + F)(\underline{D}_\Sigma e_L)u.$$

(v) *If $u \in \mathcal{D}(B(\underline{D}_\Sigma e_L))$ and $F \in \mathcal{P}_\omega$, then $B(\underline{D}_\Sigma e_L)u \in \mathcal{D}(F(\underline{D}_\Sigma e_L))$ if and only if $u \in \mathcal{D}((FB)(\underline{D}_\Sigma e_L))$, in which case,*

$$F(\underline{D}_\Sigma e_L)B(\underline{D}_\Sigma e_L)u = (FB)(\underline{D}_\Sigma e_L)u.$$

(vi) *The complex spectrum $\sigma(B(\underline{D}_\Sigma e_L))$ is a subset of*

$$\bigcap\left\{(B(S^0_\mu(\mathbb{C}))^{cl} : \mu > \omega\right\}.$$

In fact, for all $u \in L^p(\Sigma)$,

$$\|(B(\underline{D}_\Sigma e_L) - \alpha I)^{-1}u\|_p \leqslant C_{\omega,\mu,p}\frac{\|u\|_p}{\mathrm{dist}\{\alpha, B(S^0_\mu(\mathbb{C}))\}}.$$

(vii) *Assume that there exist $\mu \in (\omega, \pi/2)$, $s \geqslant 0$ and $c > 0$ such that*

$$|B(\lambda)| \geqslant c|\lambda|^s(1 + |\lambda|^{2s})^{-1},\ \lambda \in S^0_\mu(\mathbb{C}).$$

Then the operator $B(\underline{D}_\Sigma e_L)$ is one-one and has a dense range $\mathcal{R}(B(\underline{D}_\Sigma e_L))$ in $L^p(\Sigma)$.

Proof The first five parts are immediate corollaries of Theorem 5.2.1. To prove (vi), let α be a complex number such that for some $\mu > \omega$,

$$d = \mathrm{dist}\left\{\alpha, B(S^0_\mu(\mathbb{C}))\right\} > 0.$$

Then

$$F = (B - \alpha)^{-1} \in H_\infty(S_\mu^0(\mathbb{C}))$$

and $\|F\|_\infty \leqslant d^{-1}$. Hence, by (ii) and (iii), for all $u \in L^p(\Sigma)$,

$$F(\underline{D}_\Sigma e_L) \in \mathcal{L}(L^p(\Sigma))$$

and

$$\|F(\underline{D}_\Sigma e_L)u\|_p \leqslant C_{\omega,\mu,p} d^{-1} \|u\|_p.$$

Then by (iv) and (v), for all $u \in L^p(\Sigma)$,

$$(B(\underline{D}_\Sigma e_L) - \alpha I)F(\underline{D}_\Sigma e_L)u = u$$

and for all $u \in \mathcal{D}(B(\underline{D}_\Sigma e_L))$,

$$F(\underline{D}_\Sigma e_L)(B(\underline{D}_\Sigma e_L) - \alpha I)u = u.$$

Therefore

$$(B(\underline{D}_\Sigma e_L) - \alpha I)^{-1} = F(\underline{D}_\Sigma e_L).$$

This proves (vi).

(vii) is a corollary of Theorem 5.2.2. □

The closed linear operator $\underline{D}_\Sigma e_L$ on $L^p(\Sigma)$ is defined by $\underline{D}_\Sigma e_L = B(\underline{D}_\Sigma e_L)$ for $B(\lambda) = \lambda$. It follows from (vi) of Theorem 5.3.1 that the spectrum $\sigma(\underline{D}_\Sigma e_L)$ is a subset of the set

$$S_\omega(\mathbb{C}) = S_{\omega+}(\mathbb{C}) \cup S_{\omega-}(\mathbb{C}),$$

where

$$S_{\omega\pm}(\mathbb{C}) = \left\{\lambda \in \mathbb{C} : \lambda = 0 \text{ or } |\arg(\pm\lambda)| \leqslant \omega\right\}.$$

Moreover, for all $\mu > \omega$, there exists $c_{\omega,\mu,p}$ such that for all $\alpha \notin S_\mu(\mathbb{C})$ and all $u \in L^p(\Sigma)$,

$$\|(\underline{D}_\Sigma e_L - \alpha)^{-1} u\|_p \leqslant c_{\omega,\mu,p} |\alpha|^{-1} \|u\|_p.$$

In other words, $\underline{D}_\Sigma e_L$ is a type ω operator on $L^p(\Sigma)$. In fact, we can deduce from (vii) that $\underline{D}_\Sigma e_L$ is a one-one type ω operator on $L^p(\Sigma)$ and has dense domain $\mathcal{D}(\underline{D}_\Sigma e_L)$ and dense range $\mathcal{R}(\underline{D}_\Sigma e_L)$ in $L^p(\Sigma)$.

We can see that the restriction of $\underline{D}_\Sigma e_L$ on $L^{p,\pm}(\Sigma)$ is a closed linear operator on $L^{p,\pm}(\Sigma)$ with spectra in $S_{\omega\pm}(\mathbb{C})$. In fact, $\mp \underline{D}_\Sigma e_L$ are the infinitesimal generators of the holomorphic C_0-semigroup $u \mapsto T_{k_{\pm\alpha}} u, \alpha > 0$, in $L^{p,\pm}(\Sigma)$.

The next theorem indicates that the resolvents and polynomials of $\underline{D}_\Sigma e_L$ are equal to their counterparts $B(\underline{D}_\Sigma e_L)$. Hence we can regard the mapping $B \mapsto B(\underline{D}_\Sigma e_L)$ reasonably as the functional calculus of the single operator $\underline{D}_\Sigma e_L$. The mapping defined in Sect. 5.2:

$$b \mapsto b(-i\underline{D}_\Sigma) = b(-iD_{1,\sigma}, -iD_{2,\Sigma}, \ldots, -iD_{n,\Sigma})$$

can be regarded as the functional calculus of the m commuting operators $-iD_{k,\Sigma}$, $k = 1, 2, \ldots, n$. For $L = 0$, $\underline{D}_\Sigma e_L$ is the operator considered by Murry and McIntosh [4, 5].

Theorem 5.3.2 *Assume that* $1 < p < \infty$.

(i) *If* $\alpha \notin S_\omega(\mathbb{C})$, *define* $R_\alpha(\lambda) = (\lambda - \alpha)^{-1}$, *where* $R_\alpha(i\zeta e_L) = (i\zeta e_L - \alpha)^{-1}$. *Then*
$$R_\alpha(\underline{D}_\Sigma e_L) = (\underline{D}_\Sigma e_L - \alpha I)^{-1} \in \mathcal{L}(L^p(\Sigma)).$$

(ii) *For a positive integer* k, *define* $S_k(\lambda) = \lambda^k$ *such that* $S_k(i\zeta e_L) = (i\zeta e_L)^k$. *Then* $\mathcal{D}(S_k(\underline{D}_\Sigma e_L)) = \mathcal{D}((\underline{D}e_L)^k)$ *and for all* $u \in \mathcal{D}((\underline{D}_\Sigma e_L)^k)$, $S_k(\underline{D}_\Sigma e_L)u = (\underline{D}_\Sigma e_L)^k u$.

(iii) *Given a polynomial of one variable with complex values* $P(\lambda) = \sum_{k=0}^{d} a_k \lambda^k$ *and* $a_d \neq 0$. *Define*
$$P(\underline{D}_\Sigma e_L)u = \sum a_k (\underline{D}_\Sigma e_L)^k u, \ u \in \mathcal{D}(P(\underline{D}_\Sigma e_L)) = \mathcal{D}((\underline{D}_\Sigma e_L)^d).$$
Then $\mathcal{D}(P(\underline{D}_\Sigma e_L)) = \mathcal{D}((\underline{D}_\Sigma e_L)^d)$, *and for all* $u \in \mathcal{D}(\underline{D}_\Sigma e_L)$, $P(\underline{D}_\Sigma e_L)u = (\underline{D}_\Sigma e_L)^d u$.

(iv) *If* Σ *has a parametric representation:* $x = \underline{s} + g(\underline{s})e_L$, $s \in \mathbb{R}^n$, *then*
$$\mathcal{D}(\underline{D}e_L) = W_p^1(\Sigma) = \left\{u \in L^p(\Sigma) : \frac{\partial}{\partial s_j} u(\underline{s} + g(\underline{)}e_L) \in L^p(\mathbb{R}^n, d\underline{s}),\ j = 1, 2, \ldots, n\right\}$$
and
$$(\underline{D}_\Sigma e_L u)(\underline{s} + g(\underline{s})e_L) = (e_L - \underline{D}g)^{-1}\underline{D}_{\underline{s}} u(\underline{s} + g(\underline{s})e_L),\ u \in W_p^1(\Sigma).$$

Proof The proofs of (i)–(iii) require repeated use of parts (iv) and (v) of Theorem 5.3.1 (see the proof of (vi) of Theorem 5.3.1).

To prove (iv), let $\underline{A}_\Sigma$ be a closed linear operator with domain $W_p^1(\Sigma)$ in $L^p(\Sigma)$. For all $u \in W_p^1(\Sigma)$, define $\underline{A}_\Sigma$ by

$$(\underline{A}_\Sigma u)(\underline{s} + g(\underline{s})e_L) = (e_L - \underline{D}g)^{-1}\underline{D}_s u(\underline{s} + g(\underline{s})e_L),$$

and $\underline{A}_\Sigma - iI$ is one-one, see [5]. In fact, we can see that $\underline{A}_\Sigma$ is a type ω operator.

For fixed $u \in \mathcal{D}(\underline{D}_\Sigma e_L)$, write $u = u_+ + u_-$, where $u_\pm = P_\pm u$. For $\delta > 0$, let $u_{+\delta} = T_{k+\delta} u_+$. In Sect. 5.1, we see that for $u_{+\delta} \in \mathcal{D}(\underline{D}_\Sigma e_L)$, $u_{+\delta} \to u_+$ and

$\underline{D}_\Sigma e_L u_{+\delta} \to \underline{D}_\Sigma e_L u_+$ as $\delta \to 0$. In addition, $u_{+\delta} \in W_p^1(\Sigma)$. In Sect. 5.1, we have known that $\underline{D}_\Sigma e_L u_{+\delta} = \underline{A}_\Sigma u_{+\delta}$. The fact that the operator $\underline{A}_\Sigma$ is closed indicates that $u_+ \in \mathcal{D}(\underline{A}_\Sigma)$ and $\underline{D}_\Sigma e_L u_+ = \underline{A}_\Sigma u_+$. In a similar way, we can deal with u_- and find that $u \in \mathcal{D}(\underline{A}_\Sigma)$ and $\underline{D}_\Sigma e_L u = \underline{A}_\Sigma u$. By the facts that $(\underline{A}_\Sigma - iI)$ is one-one and $(\underline{D}_\Sigma e_L - iI)$ maps onto $L^p(\Sigma)$, we conclude that $\mathcal{D}(\underline{A}_\Sigma)$ can be no larger than $\mathcal{D}(\underline{D}_\Sigma e_L)$. This completes the proof. □

For $B \in H_\omega$, and indeed for $B \in \mathcal{P}_\omega$, the operator $B(\underline{D}_\Sigma e_L)$ coincides with the one obtained using the holomorphic functional calculus in [6–9]. This is derived from Theorem 5.3.2, Lemma 5.2.1 in Sect. 5.2 and the convergence lemma of this operator. We omit the details and give the following result: the boundedness of the algebra homomorphism $B \mapsto B(\underline{D}_\Sigma e_L)$ is equivalent to the following fact: $\underline{D}_\Sigma$ satisfies the square function estimate in $L^p(\Sigma)$.

For $p = 2$, a special consequence is the square function estimate:

$$\|u\|_2 \leqslant C\Big(\int_0^\infty \|\Psi_+(t\underline{D}_\Sigma e_L)u\|_2 \frac{dt}{t}\Big)^{1/2}, \; u \in L^{2,+}(\Sigma),$$

where $\Psi_+(\lambda) = \chi_{\mathrm{Re}>0}(\lambda)\lambda e^\lambda$. In other words, let $U = C_\Sigma^+ u$ denote the left monogenic extension of u to Ω_+. Then we have

$$\begin{aligned}\|u\|_2 &\leqslant C\Big(\iint_{\Omega_+} |(DU)(X)|^2 \mathrm{dist}\{X,\ \Sigma\} dX\Big)^{1/2}\\ &\leqslant C\Big(\iint_{\Omega_+} \Big(\sum_{k=1}^m \Big|\frac{\partial U}{\partial X_k}(X)\Big|^2 + \Big|\frac{\partial U}{\partial X_L}(X)\Big|^2\Big)\mathrm{dist}\{X,\ \Sigma\} dX\Big)^{1/2},\end{aligned}$$

where $u \in L^{2,+}(\Sigma)$. We refer to [2, Theorem 4.1] for the details.

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Chapter 6
Bounded Holomorphic Fourier Multipliers on Closed Lipschitz Surfaces

On the infinite Lipschitz graph, the theory of singular integrals has been established in [1–6]. In [7, 8], the authors discussed the singular integrals and Fourier multipliers for the case of starlike Lipschitz curves on the complex plane. The cases of $n-$tours and their Lipschitz disturbance are studied in [9, 10]. In 1998 and 2001, by a generalization of Fueter's theorem, T. Qian established the theory of bounded holomorphic Fourier multipliers and the relation with singular integrals on Lipschitz surfaces in the setting of quaternionic space and Clifford algebras with general dimension, respectively. Fueter's theorem and its generalizations seem to be the unique method to deal with singular integral operator algebras in the sphere contexts. In this chapter, we systematically elucidate the results obtained by Qian [11–13]. Denote by $\mathbb{R}_1^n$ and $\mathbb{R}^n$ the linear subspaces of $\mathbb{R}_{(n)}$ spanned by $\{e_0, e_1, \ldots, e_n\}$ and by $\{e_1, e_2, \ldots, e_n\}$, respectively.

6.1 Monomial Functions in $\mathbb{R}_1^n$

The concept of intrinsic functions naturally fits to the theory. A set in the complex plane $\mathbb{C}$ is called intrinsic if it is symmetric with respect to the real axis. For a function f^0, if the domain of f^0 is an intrinsic set and $f^0(z) = f^0(\overline{z})$ in the domain, then we call this function an intrinsic function. For a set in $\mathbb{R}_1^n$, if it does not change under the rotations in $\mathbb{R}_1^n$ which keep the e_0-axis unchanged, then the set is said to be an intrinsic set in $\mathbb{R}_1^n$. If O is a set in the complex plane, then

$$\overline{O} = \left\{x \in \mathbb{R}_1^n : (x_0, |\underline{x}| \in O)\right\}$$

is called a set induced by O. It is obvious that an induced set is always an intrinsic set in $\mathbb{R}_1^n$. The functions of the form $\sum c_k(z-a_k)^k$, $k \in \mathbb{Z}$, $a_k, c_k \in \mathbb{R}$, are intrinsic functions. If $f^0 = u + iv$, where u and v are real-valued, then f^0 is an intrinsic function if and only if $u(x, -y) = u(x, y)$ and $v(x, -y) = -v(x, y)$ in

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the domain of f^0. Specially, $v(x,0)=0$, that is, f^0 is real-valued if its domain is restricted on the real line.

If $f^0(z)=u(x,y)+iv(x,y)$ is an intrinsic function defined on an intrinsic set $U\subset\mathbb{C}$, we can induce a function $\overrightarrow{f^0}$ defined on the intrinsic set $\overrightarrow{U}$ from f^0 as follows.

$$\overrightarrow{f^0}(x)=u(x_0,|\underline{x}|)+\frac{\underline{x}}{|\underline{x}|}v(x_0,|\underline{x}|),\ x\in\overrightarrow{U}.$$

The function $\overrightarrow{f^0}$ is called the induced function from f^0.

We first assume that f^0 is the function of the form z^k, $k\in\mathbb{Z}$, and denote by τ the mapping

$$\tau(f^0)=\kappa_n^{-1}\Delta^{(n-1)/2}\overrightarrow{f^0},$$

where $\Delta=D\overline{D}$, $\overline{D}=D_0-\underline{D}$ and $\kappa_n=(2i)^{n-1}\Gamma^2((n+1)/2)$ is the normalizing constant such that $\tau((\cdot)^{-1})=E$.

The operator $\Delta^{(n-1)/2}$ is defined by the Fourier multiplier transform on tempered distributions $M:\mathcal{S}'\to\mathcal{S}'$ with the corresponding multiplier $m(\xi)=(2\pi i|\xi|)^{n-1}$. The Fourier multiplier operator with respect to m is expressed as

$$Mf=\mathcal{R}(m\mathcal{F}f),$$

where

$$\mathcal{F}f(\xi)=\int_{\mathbb{R}_1^n}e^{2\pi i\langle x,\,\xi\rangle}f(x)dx$$

and

$$\mathcal{R}h(x)=\int_{\mathbb{R}_1^n}e^{-2\pi i\langle\xi,\,x\rangle}h(\xi)d\xi.$$

The monomial functions in $\mathbb{R}_1^n$ are defined to be

$$P^{(-k)}=\tau((\cdot)^{-k}),\quad P^{(k-1)}=I(P^{(-k)}),\quad k\in\mathbb{Z}^+.$$

If it is necessary to emphasize the dimension n, we write the sequence $P^{(k)}$ defined in $\mathbb{R}_1^n$ as $P_n^{(k)}$. We have

Proposition 6.1.1 *Let $k\in\mathbb{Z}^+$. Then*

(i) $P^{(-1)}=E$;
(ii) $P^{(-k)}(x)=\frac{(-1)^{k-1}}{(k-1)!}(\partial/\partial x_0)^{k-1}E(x)$;
(iii) $P^{(-k)}$ *and* $P^{(k-1)}$ *are monogenic;*
(iv) $P^{(-k)}$ *is homogeneous of degree* $(-n+1-k)$ *and* $P^{(k-1)}$ *is homogeneous of degree* $(k-1)$*;*

(v)

$$c_n P_{n-1}^{(-k)}(x_0 + x_1e_1 + \cdots + x_{n-1}e_{n-1}) = \int_{-\infty}^{\infty} P_n^{(-k)}(x)dx_n,$$

where $c_n = \int_{-\infty}^{\infty}(1+t^2)^{-((n+1)/2)}dt$;

(vi) $P^{(-k)} = I(P^{(k-1)})$;

(vii) *If* n *is odd, then* $P^{(k-1)} = \tau((\cdot)^{n+k+2})$.

Proof Using the Fourier transform result on rational homogeneous functions with harmonic numerators (see [14]) and the relation

$$\overrightarrow{(\cdot)^k}(x) = \left(\frac{\overline{x}}{|x|^2}\right) = \frac{(-1)^{k-1}}{(k-1)!}\left(\frac{\partial}{\partial x_0}\right)^{k-1}\left(\frac{\overline{x}}{|x|^2}\right),$$

we obtain

$$\begin{aligned}
P^{(-k)}(x) = \tau((\cdot)^{-k})(x) &= \kappa_n^{-1}\frac{(-1)^{k-1}}{(k-1)!}\left(\frac{\partial}{\partial x_0}\right)^{k-1}M\left(\frac{\overline{(\cdot)}}{|\cdot|^2}\right)\\
&= \kappa_n^{-1}\frac{(-1)^{k-1}}{(k-1)!}\left(\frac{\partial}{\partial x_0}\right)^{k-1}\mathcal{R}\left(\gamma_{1,n}(2\pi i|\xi|)^{n-1}\frac{\overline{\xi}}{|\xi|^{1+n}}\right)\\
&= \kappa_n^{-1}\frac{(-1)^{k-1}}{(k-1)!}\left(\frac{\partial}{\partial x_0}\right)^{k-1}\gamma_{1,n}^2(2\pi i)^{n-1}\frac{\overline{x}}{|x|^{1+n}}\\
&= \kappa_n^{-1}\frac{(-1)^{k-1}}{(k-1)!}\kappa_n\left(\frac{\partial}{\partial x_0}\right)^{k-1}E(x),
\end{aligned}$$

where we have let $\kappa_n = (2\pi i)^{n-1}\gamma_{1,n}^2 = (2i)^{n-1}\Gamma^2((n+1)/2)$. This implies that for all $k \in \mathbb{Z}^+$, $P^{(-k)}$ is monogenic. The monogeneity of $P^{(k-1)}$ and the homogeneity of $P^{(-k)}$ and $P^{(k-1)}$ can be deduced from the expression and the properties of the Kelvin inversion. This proves (i)–(iv). By a direct computation, we can get

$$c_n P_{n-1}^{(-1)}(x_0 + x_1e_1 + \cdots + x_{n-1}e_{n-1}) = \int_{-\infty}^{\infty} P_n^{(-1)}(x)dx_n. \tag{6.1}$$

Then (v) can be proved by (i), (ii) and the above equality (6.1). The assertion (vi) follows from $I^2 = I$. □

Remark 6.1.1 It follows from the definition of the monomial function and the properties given in Proposition 6.1.1 that there exists a generalization of the Fueter theorem in the setting of the quaternionic space. If $f^0(z) = u(x,y) + iv(x,y)$ is defined holomorphically on an open set O on the upper half complex plane, then the function $\Delta(\overrightarrow{f^0}(q))$ is regular with respect to $q \in O$, where Δ is the Laplace operator with respect to the variables q_0, q_1, q_2, q_3. In 1957, Sce generalized this result to the setting $\mathbb{R}_1^n$ for the case of odd n. In around 1997, T. Qian extended Fueter's and Sce's results to $\mathbb{R}_1^n$, where n can be either odd or even integers. The assertions

(iii) and (vii) of Proposition 6.1.1 are identical to Sce's result for the functions z^k, $k \in \mathbb{Z}$. In particular, (vii) of Proposition 6.1.1 indicates that if n is odd, then $P^{(k-1)}$ can be defined via either the operator τ or the Kelvin inversion.

By (ii) of Lemma 6.1.1, we can get

Proposition 6.1.2 *For $k \in \mathbb{Z}^+$, the monomial functions satisfy*

$$|P^{(-k)}(x)| \leqslant C_n k^n |x|^{-(n+k-1)}, \ |x| > 1, \tag{6.2}$$

and

$$|P^{(k)}(x)| \leqslant C_n k^n |x|^k, \ |x| < 1. \tag{6.3}$$

We have the following corollary.

Corollary 6.1.1

$$E(x-1) = P^{(-1)}(x) + P^{(-2)}(x) + \cdots + P^{(-k)}(x) + \cdots, \quad |x| > 1, \tag{6.4}$$

and

$$E(1-x) = P^{(0)}(x) + P^{(-1)}(x) + \cdots + P^{(k)}(x) + \cdots, \quad |x| < 1. \tag{6.5}$$

Proof The equality (6.4) can be obtained by the Taylor expansion of $E(x-1)$ and (6.2). Then the equality (6.5) follows from (6.3) and the relation:

$$I(E(\cdot - 1))(x) = E(x)E(x^{-1} - 1) = E(1-x).$$

Notice that $\tau(\frac{1}{z-1}) = E(x-1)$. Applying the mapping τ term by term to the series

$$\frac{1}{z-1} = \frac{1}{z} + \frac{1}{z^2} + \cdots \frac{1}{z^k} + \cdots, \ |z| > 1,$$

we can obtain (6.4). The equality (6.5) can be deduced similarly from

$$\frac{1}{1-z} = 1 + z + z^2 + \cdots + z^k + \cdots, \ |z| < 1. \qquad \square$$

The series of the form $\sum\limits_{k=-\infty}^{\infty} c_k(z-a)^k$, $c_k, a \in \mathbb{C}$, is a Laurent series at a. If $c_k = 0$ for all $k < 0$, the series is a power series or a Taylor series. If $c_k = 0$ for all $k \geqslant 0$, then the series is called a principal series. For $a, c_k \in \mathbb{R}$, the series

$$\begin{cases} \phi(x) = \sum c_k P^{(k)}(x - a e_0), \\ f^0(z) = \sum c_k (z-a)^k \end{cases}$$

is said to be associated to each other and the relation is denoted by $\phi = \Upsilon f^0$. This notation is also valid for a pair of functions defined through the associated series. We define the function $f^0 = \sum c_k(z-a)^k$ to be the holomorphic extension with the largest connected domain from the function originally defined through the power series in its convergence disk. We call this domain a holomorphic domain. The same convention applies to the principal series. Adopting this convention, the series

$$\sum_{k=1}^{\infty} z^k + \sum_{k=-\infty}^{-1} -z^k = -1 + \frac{2}{1-z}$$

defines a holomorphic function in $\mathbb{C}\backslash\{1\}$. If we replace holomorphic by monogenic, this convention also applies to functions defined through $\sum c_k P^{(k)}(x - ae_0)$. An example is that

$$\sum_{k=1}^{\infty} P^{(k)}(x) + \sum_{-\infty}^{-1} -P^{(k)}(x)$$

defines a function which is monogenic everywhere except $x = 1$. By (6.4) and (6.5), we can see that the above function is $2E(1-x)$ and

$$\Upsilon(-1 + \frac{2}{1-z}) = 2E(1-x).$$

For the non-intrinsic series, we have the following proposition.

Proposition 6.1.3 *If the function f^0 is defined on an intrinsic set, then the functions*

$$\begin{cases} g^0(z) = \dfrac{1}{2}\Big(f^0(z) + \overline{f^0(\overline{z})}\Big), \\ h^0(z) = \dfrac{1}{2i}\Big(f^0(z) - \overline{f^0(\overline{z})}\Big) \end{cases}$$

are intrinsic functions defined on the same intrinsic set, and $f^0 = g^0 + ih^0$.

This proposition indicates extending Υ by

$$\Upsilon(f^0) = \Upsilon(g^0) + i\Upsilon(h^0).$$

The functions f^0 and $\Upsilon(f^0)$ are said to be associated with each other. In this manner, we can see that for $a \in \mathbb{R}$ and $c_k \in \mathbb{C}$,

$$f^0(z) = \sum_{-\infty}^{\infty} c_k(z-a)^k = g^0 + ih^0,$$

where $g^0(z) = \sum_{-\infty}^{\infty} \mathrm{Re}(c_k)(z-a)^k$ and $h^0(z) = \sum_{-\infty}^{\infty} \mathrm{Im}(c_k)(z-a)^k$. We note that $\sum_{-\infty}^{\infty} c_k P^{(k)}(x - ae_0)$ is associated with f^0.

Below we give a corollary of Lemma 6.1.2.

Proposition 6.1.4 *Let $a \in \mathbb{R}$ and $c_k \in \mathbb{C}$. If the series $\sum_{k=\pm 1}^{\pm\infty} c_k(z-a)^k$ is absolutely convergent in $|(z-a)^{\pm 1}| < r$, then the series $\sum_{k=\pm 1}^{\pm\infty} c_k P^{(k)}(x - ae_0)$ is absolutely convergent in $|(x - ae_0)^{\pm 1}| < r$.*

By Lemma 6.1.4, the mapping τ can be extended to the Laurent series. Note that if f^0 represents a principal series, then $\tau(f^0) = \Upsilon(f^0)$. If $f^0 = \sum_{k=0}^{\infty} c_k(z-a)^k$ represents a power series and the dimension n is odd, then

$$\tau\Big(\sum_{k=0}^{\infty} c_k(z-a)^k\Big) = \sum_{k=-n-1}^{\infty} c_k P^{(k-n+1)}(x - ae_0)$$

exhibits a shift of coefficients. Since we always use the Kelvin inversion to reduce power series to principal series, for the sake of convenience, we will use the correspondence Υ rather than τ.

In the following, we call the series of the form $\sum c_k(z-a)^k, a, c_k \in \mathbb{R}$, an intrinsic series. If n is odd, there is a direct relation between the holomorphic domain of an intrinsic series in the complex plane and the monogenic domain of its associated series in $\mathbb{R}_1^n$.

Proposition 6.1.5 *Let $\sum c_k(z-a)^k$ be an intrinsic series whose holomorphic domain is an open intrinsic set O. Then for n odd, in $\mathbb{R}_1^n$, the associated series $\sum c_k P^{(k)}(x - ae_0)$ can be monogenically extended to the intrinsic set $\overrightarrow{O}$.*

Proof Write $n = 2m + 1$. We first consider the case of the principal series. Let $f^0 = \sum_{k=-\infty}^{-1} c_k(z-a)^k$ be an intrinsic principal series with the convergence disc $B(a, \delta) \subset \mathbb{C}$. For $x \in B(ae_0, \delta) \subset \mathbb{R}_1^n$, we have

$$\begin{aligned}
\Upsilon(f^0)(x) &= \sum_{-\infty}^{-1} c_k P^{(k)}(x - ae_0) \\
&= \kappa_n \sum_{k=-\infty}^{-1} c_k \Delta^m \overrightarrow{(\cdot - a)^k}(x) \\
&= \kappa_n \Delta^m \Big(\overrightarrow{\sum_{k=-\infty}^{-1} c_k(\cdot - a)^k}(x)\Big) \\
&= \kappa_n \Delta^m (\overrightarrow{f^0}),
\end{aligned}$$

where the change of order of differentiation and summation is justified by Proposition 6.1.4. Because f^0 can be extended holomorphically to O, when n is odd, applying Sce's result on the pointwise monogeneity (see [15]), we can extend the function $\Upsilon(f^0)$ monogenically to $\overrightarrow{O}$.

Now let f^0 be an intrinsic power series defined holomorphically in an open intrinsic set O. Denote by I^c the Kelvin inversion on the complex plane. We obtain that $I^c(f^0)$ is an intrinsic principal series defined holomorphically in the intrinsic set

$$O^{-1} = \{z \in \mathbb{C} : z^{-1} \in O\}.$$

Then the assertion for power series follows from what is proved for principal series together with the relations $(I^c)^2 = I$ and $\overrightarrow{O}^{-1} = \overrightarrow{O^{-1}}$.

The assertion for the Laurent series follows from what have been proved for the principal series and the power series. This completes the proof. □

For $\omega \in (0, \pi/2)$, write

$$S^c_{\omega,\pm} = \left\{z \in \mathbb{C} : |\arg(\pm z)| < \omega\right\},$$

where the angle $\arg(z)$ of z takes values in $(-\pi, \pi]$, see Fig. 1.2. Let

$$\begin{aligned}
S^c_{\omega,\pm}(\pi) &= \left\{z \in \mathbb{C} : |\mathrm{Re}(z)| < \pi,\ z \in S^c_{\omega,\pm}\right\},\\
S^c_{\omega} &= S^c_{\omega,+} \cup S^c_{\omega,-},\\
S^c_{\omega}(\pi) &= S^c_{\omega,+}(\pi) \cup S^c_{\omega,-}(\pi),\\
W^c_{\omega,\pm}(\pi) &= \left\{z \in \mathbb{C} : |\mathrm{Re}(z)| < \pi \text{ and } \pm \mathrm{Im}(z) > 0\right\} \cup S^c_{\omega}(\pi),\\
H^c_{\omega,\pm} &= \left\{z = \exp(i\eta) \in \mathbb{C} : \eta \in W^c_{\omega,\pm}(\pi)\right\},
\end{aligned}$$

and

$$H^c_{\omega} = H^c_{\omega,+} \cap H^c_{\omega,-}.$$

These sets are illustrated as follows (Figs. 6.1 and 6.2).

(1) The figures of the sets $S^c_{\omega,+}$ and $S^c_{\omega,-}$ are as follows:
(2) The sets $W^c_{\omega,+}(\pi)$ and $W^c_{\omega,+}(\pi)$ are "W" and "M" shaped regions, respectively, see the following figures (Figs. 6.3, 6.4 and 6.5):
(3) The set $H^c_{\omega,+}$ is a heart-shaped region, and the complement of $H^c_{\omega,-}$ is a heart-shaped region, see the following figures (Figs. 6.6, 6.7 and 6.8):

With the obvious meaning, we shall sometimes write $H^c_{\omega,\pm} = e^{iW^c_{\omega,\pm}(\pi)}$. We also need the following function spaces.

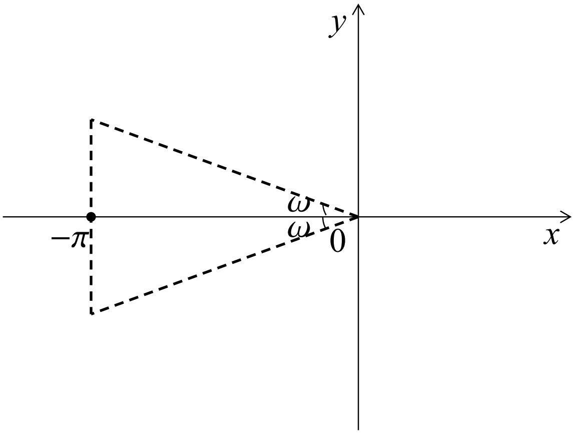

Fig. 6.1 $S^c_{\omega,-}(\pi)$

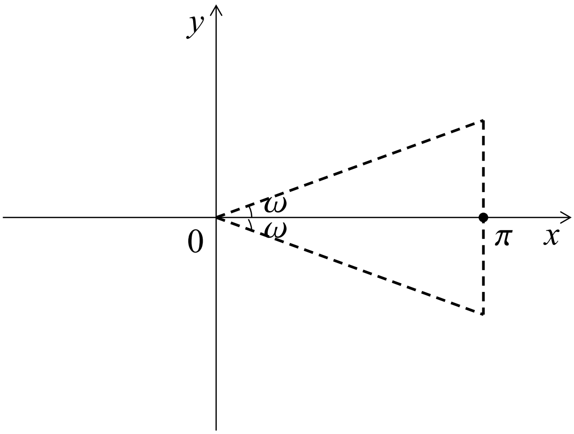

Fig. 6.2 $S^c_{\omega,+}(\pi)$

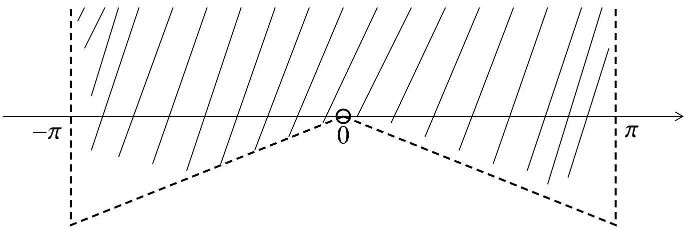

Fig. 6.3 $W^c_{\omega,+}(\pi)$

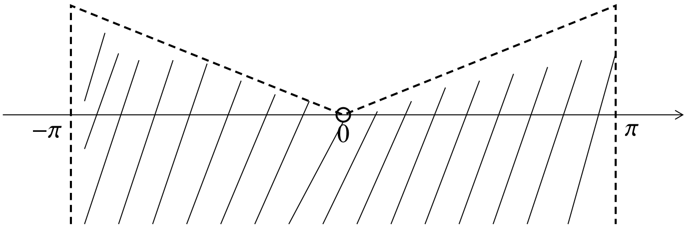

Fig. 6.4 $W^c_{\omega,-}(\pi)$

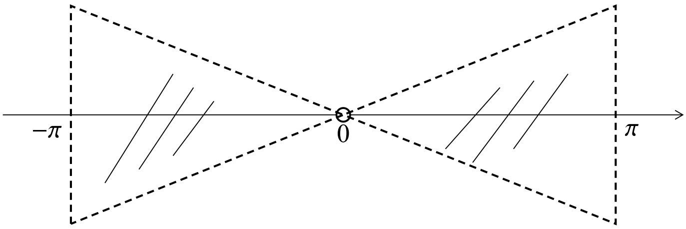

Fig. 6.5 $W^c_{\omega}(\pi)$

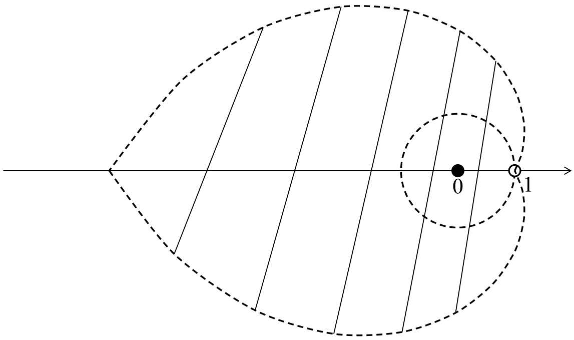

Fig. 6.6 $H^c_{\omega,+}$

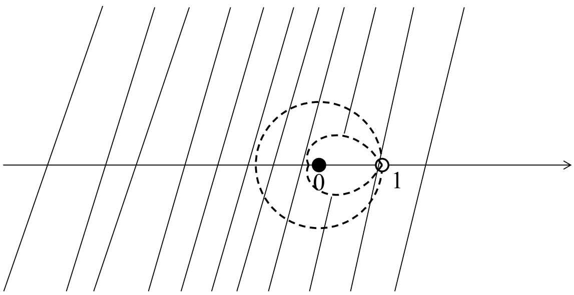

Fig. 6.7 $H^c_{\omega,-}$

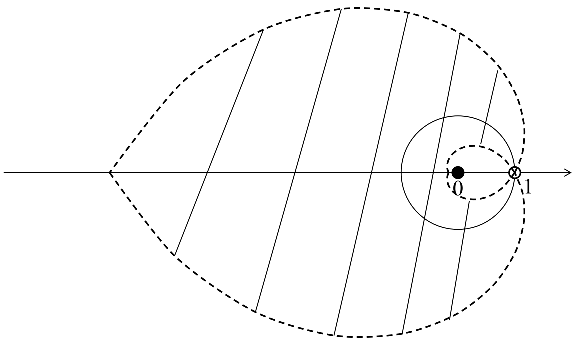

Fig. 6.8 H^c_ω

$$K(H^c_{\omega,\pm}) = \Big\{\phi^0 :\ H^c_{\omega,\pm} \to \mathbb{C}, \phi^0 \text{ is holomorphic and in every } H^c_{\mu,\pm},\ 0 < \mu < \omega,$$
$$|\phi^0(z)| \leqslant C_\mu/|1-z|\Big\},$$
$$K(H^c_\omega) = \Big\{\phi^0 :\ H^c_\omega \to \mathbb{C},\ \phi^0 = \phi^{0,+} + \phi^{0,-}, \phi^{0,\pm} \in K^s(H^c_{\omega,\pm})\Big\},$$
$$H^\infty(S^c_{\omega,\pm}) = \Big\{b :\ S^c_{\omega,\pm} \to \mathbb{C},\ b \text{ is holomorphic and in every } S^c_{\mu,\pm},\ 0 < \mu < \omega, |b(z)| \leqslant C_\mu\Big\}$$

and

$$H^\infty(S^c_\omega) = \big\{b :\ S^c_\omega \to \mathbb{C},\ b_\pm = b\chi_{\{z\in\mathbb{C}:\pm Rez>0\}} \in H^\infty(S^c_{\omega,\pm})\big\}.$$

Remark 6.1.2 The above sets and function spaces fit into the theory for closed curves and surfaces. From the point of the view of complex analysis, Khavinson in [16] shows interests to those sets and related holomorphic functions. The theory on the infinite Lipschitz graph is established in [1–6]. In [7, 8], the authors discussed the case of starlike Lipschitz curves in the complex plane. In [11], the author studied the case of the Lipschitz surface in the quaternionic space. $H^\infty(S^c_\omega)$ is the space of Fourier multipliers. $K(H^c_{\omega,\pm})$ and $K(H^c_\omega)$ are spaces of kernels of singular integrals. On the Fourier multiplier side, this is consistent with the fact that the closure of S^c_ω contains the spectrum of the surface Dirac operator on Lipschitz curves or Lipschitz surfaces whose Lipschitz constants are less than $\tan(\omega)$. On the singular integral side, in the complex plane for instance, we consider the singular integral operator of the form

$$\int_\gamma \phi(z\eta^{-1}) f(\eta)\frac{d\eta}{\eta},\ z \in \gamma,$$

on a starlike Lipschitz curve with the Lipschitz constant less than $\tan\omega$. It is easy to prove that the condition $z, \eta \in \gamma$ means $z\eta^{-1} \in H^c_\omega$ for $\omega > \arctan N$. This requires that our kernel functions ought to be defined in H^c_ω.

In $\mathbb{R}^n_1$, we will be working on heart-shaped regions or their complements

$$H_{\omega,\pm} = \left\{x \in \mathbb{R}^n_1 :\ \frac{(\pm \ln|x|)}{\arg(e_0, x)} < \tan\omega\right\} = \overrightarrow{H^c_{\omega,\pm}},$$

and

$$H_\omega = H_{\omega,+} \cap H_{\omega,-} = \overrightarrow{H_\omega^c},$$

that is,

$$H_\omega = \left\{x \in \mathbb{R}_1^n : \frac{(\ln|x|)}{\arg(e_0, x)} < \tan\omega\right\}.$$

Remark 6.1.3 The reason for using these sets on surfaces is the same as that described in Remark 6.1.2 for starlike Lipschitz curves. Precisely, our object of study is singular convolution integrals with kernels defined in H_ω on starlike Lipschitz surfaces. The definition of H_ω is inspired by the following observation for the complex plane case. It is easy to show that a starlike Lipschitz curve has the parameterisation $\gamma = \gamma(x) = e^{i(x+iA(x))}$, where $A = A(x)$ is a $2\pi-$periodic Lipschitz function. Assume that the Lipschitz constant of γ is less than $\tan\omega$. Then for $z = \exp i(x + iA(x))$ and $\eta = \exp i(y + iA(y))$, we have

$$z\eta^{-1} = \exp i((x - y) + i(A(x) - A(y)).$$

This implies that

$$\frac{|\ln|z\eta^{-1}||}{\arg(z\eta^{-1}, 1)} = \frac{|A(x) - A(y)|}{|x - y|} < \tan\omega.$$

Let $\mathbb{C}_{(n)}$ denote the complex Clifford algebra generated by $\{e_1, e_2, \ldots, e_n\}$. In $\mathbb{R}_1^n$ we use the following function spaces

$$K(H_{\omega,\pm}) = \Big\{\phi : H_{\omega,\pm} \to \mathbb{C}_{(n)} : \ \phi \text{ is monogenic and satisfies}$$
$$|\phi(x)| \leqslant C_\mu/|1 - x|^n, \ x \in H_{\mu,\pm}, 0 < \mu < \omega\Big\}$$

and

$$K(H_\omega) = \Big\{\phi : H_\omega \to \mathbb{C}_{(n)} : \ \phi = \phi^+ + \phi^-, \phi^\pm \in K(H_{\omega,\pm})\Big\}.$$

Lemma 6.1.1 *Assume that $b \in H^\infty(S_{\omega,-}^c)$. For the multiplier defined by $\phi^0(z) = \sum_{k=1}^{\infty} b(-k)z^{-k}$, its jth derivation satisfies*

$$|(\phi^0)^{(j)}(z)| \leqslant \frac{C}{|1 - z|^{j+1}},$$

where $z \in H_{\mu,-}^c$, $0 < \mu < \omega$, and j is any positive integer.

Proof Without loss of generality, for $b \in H^\infty(S^c_{\omega,-})$, we can assume that $|b(-k)| \leqslant C_\mu$. For $\phi^0(z) = \sum_{k=1}^{\infty} b(-k)z^{-k}$, by Theorem 2.3.2,

$$|\phi^0(z)| \leqslant \frac{C}{|1-z|}.$$

Take a circle $C(z, r)$ centered at z with radius r. By Cauchy's formula, we can get

$$\left|(\phi^0)^{(j)}(z)\right| \leqslant \frac{C_j}{2\pi} \int_{C(z,r)} \frac{|\phi^0(\xi)|}{|z-\xi|^{j+1}} |d\xi|.$$

Let $r = \frac{1}{2}|1-z|$. Then $\xi \in C(z, r)$ implies that

$$|1-\xi| \geqslant |1-z| - |z-\xi| = |1-z| - \frac{1}{2}|1-z| = \frac{1}{2}|1-z|.$$

Hence we get

$$\left|(\phi^0)^{(j)}(z)\right| \leqslant \frac{2j!C_\mu}{\delta^j(\mu)} \frac{1}{|1-z|^{j+2}} |1-z| \leqslant \frac{C_{\mu,j}}{|1-z|^{j+1}}.$$

This completes the proof of Lemma 6.1.1. □

Lemma 3.5.1 is a powerful tool in the proof of the main result of this section. Based on this lemma, we can estimate the multipliers in $K(H_{\omega,\pm})$ by induction. Before stating the main result of this section, we first give an auxiliary lemma.

Lemma 6.1.2

$$\sum_{k=0}^{\infty} \frac{(j+4l+2k+2)\cdots(2k+2)}{2^k} = 2^{j+4l+3}\left(\frac{j+4l+2}{2}\right)! \tag{6.6}$$

Proof Denote by s the sum of the series. Then $s/2$ is the sum of the series obtained by multiplying the original series, term by term, by $1/2$. We get

$$s = 2(j+4l+2)\sum_{k=0}^{\infty} \frac{(j+4l+2k)\cdots(2k+2)}{2^k}.$$

Repeating this procedure up to $\frac{1}{2}(j+4l+2)$ times, we obtain

$$s = 2^{(j+4l+2)/2}(j+4l+2)!!2 = 2^{j+4l+3}\left(\frac{j+4l+2}{2}\right)!.$$

This completes the proof. □

The main result of this section is as follows.

Theorem 6.1.1 *If* $b \in H^\infty(S^c_{\omega,\pm})$ *and* $\phi(x) = \sum_{k=\pm 1}^{\pm\infty} b(k)P^{(k)}(x)$, *then* $\phi \in K(H_{\omega,\pm})$.

Proof We will divide the proof into two cases: n odd and n even.

The case of n **odd**. Let $n = 2m + 1$. By Proposition 6.1.3, we are reduced to proving the theorem for b in $H^{\omega,r}(S^c_{\omega,\pm})$, where

$$H^{\infty,r}(S^c_{\omega,\pm}) = \left\{ b \in H^\infty(S^c_{\omega,\pm}) : \ b\mid_{\mathbb{R}\cap S^c_{\omega,\pm}} \text{ is real-valued} \right\}.$$

In fact, in the decomposition $b = g^0 + ih^0$, g^0 and h^0 belong to $H^{\infty,r}(S^c_{\omega,\pm})$ and are dominated by the bound of b. We first consider the case "$-$", and next use the Kelvin inversion to conclude the case "+".

Now we assume that $b \in H^{\infty,r}(S^c_{\omega,-})$ and consider

$$\phi(x) = \sum_{k=1}^{\infty} b(-k)P^{(-k)}(x) = \Delta^m \phi^0(x_0, |\underline{x}|),$$

where $\phi^0(z) = \sum_{k=1}^{\infty} b(-k)z^{-k}$. By Theorem 2.3.1, we know that $\phi^0 \in K(H^c_{\omega,-})$. By Cauchy's formula, we further deduce that for $z \in H^c_{\mu,-}$, $0 < \mu < \omega$,

$$|(\phi^0)^{(j)}(z)| \leqslant \frac{2j!C_\mu}{\delta^j(\mu)} \frac{1}{|1-z|^{1+j}}, \ j \in \mathbb{Z}^+ \cup \{0\},$$

where C_μ is the constant in the definition of $K(H^c_{\omega,-})$ and $\delta(\mu) = \min\{1/2, \tan(\omega - \mu)\}$.

Proposition 6.1.5 indicates that ϕ is a monogenic function in $H_{\omega,-}$. We only need to prove

$$|\phi(x)| \leqslant \frac{C_\mu}{|1-x|^n}, \ x \in H_{\mu,-} = \overrightarrow{H^c_{\mu,-}}, \ 0 < \mu < \omega.$$

To proceed, we only need to consider the points $x \approx 1$ in the region $H_{\omega,-}$. We shall deal with two cases.

Case 1: $|\underline{x}| > (\delta(\mu)/2^{m+1/2})|1-x|$. By Lemma 3.5.1, this reduces to study u_l and v_l in the region $H^c_{\omega,-}$ with the conditions that $z \approx 1$ and $|t| \approx |1-z|$. We shall later substitute $z = s + it$, $s = x_0$, $t = |\underline{x}|$. We see that $u = u_0$, $v = v_0$ and $1/t$ are all of the magnitude $1/|1-z|$. If we take derivative with respect to t to each of them, we can reduce the power by one in the magnitude and thus get the magnitude $1/|1-z|^2$. To obtain u_1, starting from u_0, we first take the derivative and then divide the quantity by t, leading to the magnitude $1/|1-z|^3$. Repeating this procedure to m times to get u_m, we obtain $1/|1-z|^{2m+1} = 1/|1-z|^n$. The estimate for v_m is similar.

Case 2: $|\underline{x}| \leqslant (\delta(\mu)/2^{m+1/2})|1-x|$. Points in $H_{\omega,-}$ satisfying $x \approx 1$ and $x_0 \leqslant 1$, belong to Case 1. Then we assume that $x_0 > 1$. By Lemma 7.2.2, we need to prove that for any $0 < \mu < \omega$,

$$|u_m(s,t)| + |v_m(s,t)| \leqslant \frac{C_{\mu,m}}{|1-z|^n},\ z = s+it \in H^c_{\omega,-}.$$

We first discuss $u_l,\ 0 \leqslant l \leqslant m$. The proof will involve partial derivatives of u_l with respect to the second argument. We claim that for $z = s+it \approx 1$, $s > 1$, $z \in H^c_{\mu,-}$, $\delta = \delta(\mu)$ and $|t| \leqslant (\delta/2^{m+1/2})|1-z|$, the following hold

(i) u_l is even with respect to its second argument;
(ii) for any integer $0 \leqslant j < \infty$,

$$\left|\frac{\partial^j}{\partial t^j} u_l(s,t)\right| \leqslant \frac{C_\mu C_l 2^{lj}(j+4l)!}{\delta^{2l+j}} \frac{1}{|1-z|^{2l+j+1}},\ j \text{ is even},$$

and

$$\left|\frac{\partial^j}{\partial t^j} u_l(s,t)\right| \leqslant \frac{C_\mu C_l 2^{lj}(j+5l)!}{\delta^{2l+j}} \frac{1}{|1-z|^{2l+j+1}},\ j \text{ is odd}.$$

We use the mathematical induction to l. For $l=0$, the assertion are from the corresponding properties of ϕ^0.

We assume that (i) and (ii) hold for the indices l: $0 \leqslant l \leqslant m-1$. We will verify that they remain to hold for the next index $l+1$.

Because (i) holds for the index l, by the definition of u_{l+1}, we get that (i) also holds for $l+1$.

Now we prove that (ii) holds for $l+1$. Because $u_l(s,t)$ is even with respect to t, then $\partial u_l/\partial t$ is odd with respect to t. This implies that $(\partial u_l/\partial t)(s,0) = 0$. Similarly, we can prove that for $k \in \mathbb{Z}^+ \cup \{0\}$,

$$((\partial^{2k+1} u_l)/(\partial t^{2k+1}))(s,0) = 0.$$

For small t, the Taylor expansion of $(\partial u_l/\partial t)(s,t)$ at $t=0$ is

$$u_{l+1}(s,t) = \frac{2(l+1)}{t}\frac{\partial u_l}{\partial t}(s,t) = 2(l+1)\sum_{t=0}^{\infty} \frac{\partial^{2k+2} u_l/\partial t^{2k+2}(s,0)}{(2k+1)!} t^{2k}.$$

For j even, we take derivatives with respect to t up to j times and obtain that

$$\frac{\partial^j}{\partial t^l} u_{l+1}(s,t) = 2(l+1)\sum_{k=j/2}^{\infty} \frac{\partial^{2k+2} u_l}{\partial t^{2k+2}}(s,0) \frac{(2k)(2k-1)\cdots(2k-j+1)}{(2k+1)!} t^{2k-j}.$$

Using the induction hypothesis (ii) for the index l and changing the index k to $j/2+k$, we have

$$\begin{aligned}\left|\frac{\partial^j}{\partial t^j}u_{l+1}(s,t)\right| &\leqslant 2(l+1)\frac{C_\mu C_l 2^{l(j+2)}}{\delta^{2(l+1)+j}|1-z|^{2(l+1)+j+1}}\\ &\quad\times\sum_{k=0}^{\infty}\frac{(j+4l+2k+2)!2^{2kl}}{(j+k+1)!}\\ &\quad\times(j+2k)\cdots(2k+1)\left(\frac{t}{\delta|1-z|}\right)^{2k}\\ &\leqslant 2(l+1)\frac{C_\mu C_l 2^{l(j+2)}}{\delta^{2(l+1)+j}|1-z|^{2(l+1)+j+1}}\\ &\leqslant\sum_{k=0}^{\infty}\frac{(j+4l+2k+2)\cdots(2k+2)}{2^k},\end{aligned}$$

where we have used the condition $t/\delta|1-z| \leqslant 1/2^{m+1/2}$.

Now we evaluate the last series. To simplify the expression of the constant C_l, we use the following weaker estimate derived from Lemma 6.1.2:

$$\sum_{k=0}^{\infty}\frac{(j+4l+2k+2)\cdots(2k+2)}{2^k}\leqslant 2^{j+4l-1}(j+4l+1)!. \tag{6.7}$$

The last estimate gives rise to the desired estimate for $|(\partial^j/\partial t^j)u_{l+1}(s,t)|$ with $C_l = 2^{3l(l-1)}l!$.

For the case that j is odd, by a similar method, we obtain

$$\begin{aligned}\left|\frac{\partial^j}{\partial t^j}u_{l+1}(s,t)\right| &\leqslant 2(l+1)\frac{C_\mu C_l 2^{l(j+2)}}{\delta^{2(l+1)+j}|1-z|^{2(l+1)+j+1}}\frac{t}{\delta|1-z|}\\ &\quad\times\sum_{k=0}^{\infty}\frac{(j+5l+2k+3)!2^{2kl}}{(j+2k+2)!}\\ &\quad\times(j+2k+1)\cdots(2k+3)\left(\frac{t}{\delta|1-z|}\right)^{2k}\\ &\leqslant 2(l+1)\frac{C_\mu C_l 2^{l(j+2)}}{\delta^{2(l+1)+j}|1-z|^{2(l+1)+j+1}}\frac{1}{2^{m+1/2}}\\ &\quad\times\sum_{k=0}^{\infty}\frac{(j+5l+2k+3)\cdots(2k+3)}{2^k}\\ &\leqslant\frac{C_\mu C_{l+1}2^{(l+1)j}(j+5(l+1))!}{\delta^{2(l+1)+j}}\frac{1}{|1-z|^{2(l+1)+j+1}},\end{aligned}$$

where C_l is an appropriate constant. Let $l = m$ and $j = 0$. We obtain the desired estimate for u_m.

Now we study v_m and still consider the two cases: $|\underline{x}| > (\delta(\mu)/2^{m+1/2})|1-x|$ and $|\underline{x}| \leqslant (\delta(\mu)/2^{m+1/2})|1-x|$. The first case can be dealt with by the method used for u_m. For the second case, we will prove: for $0 \leqslant l \leqslant m$, $z = s + it \approx 1$, $s > 1$, $z \in H^c_{\mu,-}$, $0 < \mu < \omega$, and $|t| \leqslant (\delta/2^{m+1/2})|1 - z|$,

(i) v_l is odd with respect to the second argument;
(ii) for any integer $0 \leqslant j < \infty$

$$\left|\frac{\partial^j}{\partial t^j} v_l(s,t)\right| \leqslant \frac{C_\mu C_l 2^{lj}(j+5l)!}{\delta^{2l+j}} \frac{1}{|1-z|^{2l+j+1}}, \ j \text{ is even,}$$

and

$$\left|\frac{\partial^j}{\partial t^j} v_l(s,t)\right| \leqslant \frac{C_\mu C_l 2^{lj}(j+4l)!}{\delta^{2l+j}} \frac{1}{|1-z|^{2l+j+1}}, \ j \text{ is odd.}$$

We use the mathematical induction and the proof is similar to that for μ_l.

For $l = 0$, (i) and (ii) follow from the corresponding properties of ϕ^0.

Now we assume that (i) and (ii) hold for the index l: $0 \leqslant l \leqslant m-1$. We will prove (i) and (ii) also hold for the index $l+1$.

For $l+1$, we can use the definition of v_{l+1} and the assertion (i) for l to prove that the assertion (i) holds for $l+1$.

Now we prove that (ii) holds for $l+1$. Because $v_l(s,t)$ is odd with respect to t, for $k \in \mathbb{Z}^+ \cup \{0\}$, $(\partial^{2k} v_l(s,0))/\partial t^{2k} = 0$ and we can see that its Taylor expansion in t at $t = 0$ is

$$v_l(s,t) = \sum_{k=0}^{\infty} \frac{(\partial^{2k+1} v_l/\partial t^{2k+1})(s,0)}{(2k+1)!} t^{2k+1}.$$

Hence,

$$t\frac{\partial v_l(s,t)}{\partial t} = \sum_{k=0}^{\infty} \frac{(\partial^{2k+1} v_l/\partial t^{2k+1})(s,0)}{(2k)!} t^{2k+1}$$

and

$$v_{l+1}(s,t) = 2(l+1)\frac{t\frac{\partial v_l}{\partial t} - v_l}{t^2} = 2(l+1)\sum_{k=0}^{\infty} \frac{2k+2}{(2k+3)!} \frac{\partial^{2k+3} v_l(s,0)}{\partial t^{2k+3}} t^{2k+1}.$$

Taking derivative with respect to t up to j times and discussing the two cases that j is even and odd, we can apply a similar method as used above to get the desired estimate for $l+1$. In the estimate of $|\frac{\partial^j}{\partial t^j} v_l(s,t)|$, taking $l = m$ and $j = 0$, we get the desired estimate for v_m.

Now we consider the case "+". Assume that $b \in H^{\infty,r}(S^c_{\omega,+})$ and $\psi(x) = \sum_{i=1}^{\infty} b(i) P^{(i)}(x)$. The Kelvin inversion implies that

$$I(\psi)(x) = \sum_{i=-1}^{-\infty} b'(i) P^{(i-1)}(x),$$

where $b'(z) = b(-z) \in H^{\infty,r}(S^c_{\omega,-})$. Because $I(\psi) = \tau(\psi^0)$, where

$$\psi^0(z) = \sum_{i=-1}^{-\infty} b'(i) z^{i-1} = \frac{1}{z} \sum_{i=-1}^{-\infty} b'(i) z^i \in H^c_{\omega,-},$$

the conclusions for the above considered case "−" all apply to $I(\psi)$. Using the relation

$$\psi = I^2(\psi) = E(x) I(\psi)(x^{-1})$$

and the fact that $x \in H_{\nu,+}$ if and only if $x^{-1} \in H_{\nu,-}$, we can get for $x \in H_{\nu,+}$,

$$\begin{aligned} |\psi(x)| = |E(x) I(\psi)(x^{-1})| &\leqslant \frac{1}{|x|^n} \frac{C_\nu}{|1 - x^{-1}|^n} \\ &= \frac{C_\nu}{|1 - x|^n}. \end{aligned}$$

This proves the case $b \in H^{\infty,r}(S^c_{\omega,+})$ and the proof for n being odd is complete.

The case of n even. The same argument reduces the case "+" to the case "−". Let $b \in H^{\infty,r}(S^c_{\omega,-})$ and consider

$$\phi(x) = \sum_{k=1}^{\infty} b(-k) P_n^{(-k)}(x).$$

Now $n + 1$ is odd and so the conclusions obtained in the first part applies to $n + 1$. We deduce from (v) of Lemma 6.1.1 that

$$c_{n+1} \phi(x) = \int_{-\infty}^{\infty} \sum_{k=1}^{\infty} b(-k) P_{n+1}^{(-k)}(x + x_{n+1} e_{n+1}) dx_{n+1},$$

where the function ϕ is defined on $H_{\omega,-}$ monogenically. Here $H_{\omega,-}$ is the intersection of $\mathbb{R}_1^n$ and the corresponding $H_{\omega,-}$ set in $\mathbb{R}_1^{n+1}$. Based on the right order of decaying of $P_{n+1}^{(k)}$, we can change the order of integration and summation to obtain for $x \in H_{\nu,-}$,

$$|c_{n+1} \phi(x)| \leqslant C_\nu \int_{-\infty}^{\infty} \frac{1}{|1 - (x + x_{n+1} e_{n+1})|^{n+1}} dx_{n+1} \leqslant \frac{C_\nu}{|1 - x|^n}.$$

□

The following corollary can be deduced from Theorem 6.1.1 immediately.

Corollary 6.1.2 *Let $b \in H^\infty(S^c_\omega)$ and $\phi(x) = \sum_{i=-\infty}^{\infty} b(i) P^{(i)}(x)$. Then $\phi \in K(H_\omega)$.*

6.2 Bounded Holomorphic Fourier Multipliers

If a surface Σ is n-dimensional and starlike about the origin, and there exists a constant $M < \infty$ such that for $x, x' \in \Sigma$,

$$\frac{\left| \ln |x^{-1}x'| \right|}{\arg(x, x')} \leqslant M,$$

then we call Σ a starlike Lipschitz surface. The minimum value of M is called the Lipschitz constant of Σ, denoted by $N = \mathrm{Lip}(\Sigma)$.

Because locally

$$\begin{aligned} \ln |x^{-1}x'| &= \ln(1 + (|x^{-1}x'| - 1)) \approx (|x^{-1}x'| - 1) \\ &\approx |x^{-1}|(|x'| - |x|) \approx (|x'| - |x|), \end{aligned}$$

the above defined sense of Lipschitz is consistent with the standard sense.

Let $s \in \mathbb{S}_{\mathbb{R}_1^n}$. We consider the mapping $r_s : x \to sxs^{-1}$, $x \in \mathbb{R}_1^n$. Although r_s does not preserve $\mathbb{R}_1^n$, it satisfies the following properties.

Lemma 6.2.1 *For any $x,\ y \in \mathbb{R}_1^n$, the following properties hold:*

(i) $|r_s(y^{-1}x)| = |y^{-1}x|$. *More generally, r_s preserves norms of the elements in $\mathbb{R}_{(n)}$ which can be expressed as a product of vectors;*

(ii) $\langle r_s(x),\ r_s(y)\rangle = \langle x,\ y\rangle$;

(iii) $\arg(r_s(x),\ r_s(y)) = \arg(x,\ y)$;

(iv) $(r_s(y))^{-1} r_s(x) = r_s(y^{-1}x)$;

(v) *there exists a vector $s \in \mathbb{S}_{\mathbb{R}_1^n}$ such that $r_s(y^{-1}x) = |y|^{-1}\tilde{x}$, where $\tilde{x} \in \mathbb{R}_1^n$. In addition, $|x - y| = |y||e_0 - \tilde{x}|$ and $\arg(y,\ x) = \arg(|y|e_0, \tilde{x})$;*

(vi) *for the same s as in (v), we have $r_s(E(y)) = E(y)$.*

Proof (i) is a direct corollary of the property of $|x|$. By the relation between the inner product and the norms in $\mathbb{C}_{(n)}$, we can deduce (ii) from (i). (iii) can be deduced from (i) and (ii). (iv) is trivial.

To prove (v), we introduce a new basic vector e' such that

$$\begin{cases} (e')^2 = 1 \\ e'e_i = -e_i e',\ i = 1, 2, \cdots, n. \end{cases}$$

Let $f_0 = e'$, $f_i = e_i f_0$, $i = 1, \ldots, n$. We have

$$f_i^2 = 1,\ f_i f_j = -f_j f_i,\ 0 \leqslant i, j \leqslant n,\ i \neq j.$$

So $\{f_j\}_{j=0}^n$ forms a basis of type $(n+1, 0)$. It is a basis of

$$\mathbb{R}^{n+1} = \mathbb{R}^{n+1,0} = \Big\{x_0 f_0 + \cdots + x_n f_n : \ x_j \in \mathbb{R}, \ j = 0, 1, \ldots, n\Big\}.$$

By the property of the Clifford group in $\mathbb{R}^{n+1}$, we can choose $s \in \mathbb{R}_1^n$ such that the mapping $(\cdot) \to (sf_0)(\cdot)(sf_0)^{-1}$ on $\mathbb{R}^{n+1}$ maps yf_0 to $f_0|y|$. The same mapping maps xf_0 to $f_0\tilde{x}$, where $\tilde{x} \in \mathbb{R}_1^n$. Hence we have

$$\begin{aligned} r_s(y^{-1}x) &= [(sf_0)(yf_0)(sf_0)^{-1}]^{-1}[(sf_0)(xf_0)(sf_0)^{-1}] \\ &= (f_0|y|)^{-1}(f_0\tilde{x}) = |y|^{-1}\tilde{x}. \end{aligned}$$

Because the mappings induced by the elements in the Clifford group preserve the distance between vectors, we obtain

$$|x - y| = |yf_0 - xf_0| = |f_0|y| - f_0\tilde{x}| = ||y|e_0 - \tilde{x}|.$$

Owing to (iii),

$$\arg(y, \ x) = \arg(r_s(y), \ r_s(x)) = \arg(f_0|y|, \ f_0\tilde{x}) = \arg(|y|e_0, \ \tilde{x}).$$

(vi) can be proved as

$$r_s(E(y)) = \frac{1}{|y|^{n-1}} s(y^{-1}e_0)s^{-1} = \frac{1}{|y|^{n-1}}(|y|^{-1}f_0)(f_0 e_0 \tilde{e_0}),$$

where

$$\tilde{e_0} = (sf_0)(f_0)(sf_0)^{-1} = sf_0 s^{-1} = f_0 \frac{\overline{y}}{|y|},$$

and the last inequality is deduced from $(sf_0)(yf_0)(sf_0)^{-1} = f_0|y|$. By the expression of $\tilde{e_0}$, we get $r_s(E(y)) = E(y)$. □

Remark 6.2.1 We explain how the sets H_ω are related to the starlike Lipschitz surface. Lemma 6.2.1 indicates that if we choose an appropriate $s \in \mathbb{S}_{\mathbb{R}_1^n}$, then

$$\ln(|x^{-1}x'|) = \ln|r_s(x^{-1}x')| = \ln||x|^{-1}\tilde{x}|.$$

On the other hand,

$$\arg(x, \ x') = \arg(|x|e_0, \ \tilde{x}) = \arg(e_0, |x|^{-1}\tilde{x}).$$

Hence, if x and x' belong to the starlike Lipschitz surface and the Lipschitz constant is N, then

$$\frac{|\ln|x^{-1}x'||}{\arg(x, \ x')} = \frac{|\ln||x|^{-1}\tilde{x}||}{\arg(1, \ |x|^{-1}\tilde{x})} \leqslant N.$$

This implies that $|x|^{-1}\tilde{x} \in H_\omega$ for any $\omega \in (\arctan(N),\ \pi/2)$.

We will work on a fixed starlike Lipschitz surface with the Lipschitz constant N. We assume that $\omega \in (\arctan(N),\ \pi/2)$. Write

$$\rho = \min\{|x| :\ x \in \Sigma\} \text{ and } \iota = \max\{|x| :\ x \in \Sigma\}.$$

Without loss of generality, we assume $\rho < 1 < \iota$.

Write $L^2(\Sigma) = L^2(\Sigma,\ d\sigma)$, where $d\sigma$ is the surface area measure. The norm of $f \in L^2(\Sigma)$ is denoted by $\|f\|$.

Coifman–McIntosh–Meyer [17] proved that on any Lipschitz surface Σ, the Cauchy integral operator

$$C_\Sigma f(x) = p.v. \frac{1}{\Omega_n} \int_\Sigma E(x-y)\mathfrak{n}(y) f(y) d\sigma(y)$$

can be extended to a bounded operator on $L^2(\Sigma)$, where $\mathfrak{n}(y)$ is the outward normal of Σ at $y \in \Sigma$ and Ω_n is the area of the n-dimensional unit sphere $S_{\mathbb{R}^n_1}$.

We will use the following test function space $\mathcal{A}$

$$\mathcal{A} = \left\{ f :\ \text{ for some } s > 0,\ f(x) \text{ is left monogenic in } \rho - s < |x| < \iota + s. \right\}$$

We have the following result.

Proposition 6.2.1 *The class $\mathcal{A}$ is dense in $L^2(\Sigma)$.*

Proof For $f, g \in L^2(\Sigma)$, define the bilinear form:

$$\langle f,\ g \rangle = \int_\Sigma f \overline{g} d\sigma.$$

It is easy to prove that for any fixed $x \in \mathbb{R}^n_1$,

$$\langle f,\ f \rangle = \|f\|^2,\ \overline{\langle f,\ g \rangle} = \langle g,\ f \rangle,\ \langle xf,\ g \rangle = x\langle f,\ g \rangle.$$

If $\langle f,\ g \rangle = 0$, we call f is orthogonal to g. Assume that $\mathcal{A}$ is not dense in $L^2(\Sigma)$. Because the bilinear form $\langle \cdot,\ \cdot \rangle$ satisfies the conditions of the inner product on $\mathbb{R}^n_1$, the basic Hilbert space methods are adaptable to this case. Specially, there exists a nonzero function in $L^2(\Sigma)$ which is orthogonal to all functions in $\mathcal{A}$ and so in particular to $E(\cdot - x')$, where x' lines outside the annuls: $\rho - s < |x| < \tau + s$.

Hence we can get

$$\langle E(\cdot - x'),\ g \rangle = \int_\Sigma E(x - x')\mathfrak{n}(x) h(x) d\sigma(x) = 0, \tag{6.8}$$

where $h(x) = \overline{\mathfrak{n}(x)g(x)}$ is a function in $L^2(\Sigma)$. Because the integral in (6.8) is absolutely convergent, by the regular continuation, it would remain valid for all $x' \notin \Sigma$.

Let x be a point on Σ, $y' = rx$ and $y^* = r^{-1}x$. As a corollary of the main results of [17], on the Lipschitz surface, for almost all $x \in \Sigma$, we have

$$0 = h(x) = \lim_{r \to 1-} \frac{1}{2\pi^2} \int_\Sigma \Big[E(y - y') - E(y - y^*)\Big]\mathfrak{n}(y)h(y)d\sigma(y).$$

Hence $g(x) = 0$ for almost all $x \in \Sigma$. This is a contradiction and the proof is complete. □

Now suppose $f \in \mathcal{A}$. In the annulus where f is defined, we have the Laurent series expansion:

$$f(x) = \sum_{k=0}^{\infty} P_k(f)(x) + \sum_{k=0}^{\infty} Q_k(f)(x),$$

where for $k \in \mathbb{Z}^+ \cup \{0\}$, $P_k(f)$ belongs to the finite dimensional right module M_k of k-homogeneous left monogenic functions in $\mathbb{R}_1^n$, and $Q_k(f)$ belongs to the finite-dimensional right module of $-(k+n)-$homogeneous left-monogenic functions in $\mathbb{R}_1^n \setminus \{0\}$. The spaces M_k and M_{-k} are the eigenspaces of the left-spherical Dirac operator. The mappings

$$\begin{cases} P_k : \ f \to P_k(f), \\ Q_k : \ f \to Q_k(f) \end{cases}$$

are projections on M_k and $M_{-(k+n)}$, respectively. If f is a $k-$homogeneous spherical harmonic, $k \geqslant 1$, then $f = f^+ + f^-$, where $f^+ \in M_k$ and $f^- \in M_{-k+1-n}$. It is a remarkable fact that the spaces $M_k, k = -1, -2, \ldots, -n+1$, do not exist (see [18]).

Formally, we consider the Fourier multiplier operator induced by a bounded sequence $\{b_k\}$:

$$M_{\{b_k\}}f(x) = \sum_{k=0}^{\infty} b_k P_k(f)(x) + \sum_{k=0}^{\infty} b_{-k-1} Q_k(f)(x).$$

It is easy to see that $M_{\{b_k\}} : \mathcal{A} \to \mathcal{A}$ is a linear operator. The question is whether $M_{(b_k)}$ can be extended to a bounded operator on $L^2(\Sigma)$. If Σ is a sphere, then by Plancherel's theorem, the boundedness of the operator $M_{\{b_k\}}$ follows from the fact that $\{b_k\}$ is a bounded sequence. If Σ is a starlike Lipschitz surface, then the boundedness condition is not sufficient.

In order to prove the boundedness of the operator $M_{(b(k))}$, we need to employ the singular integral convolution expression of $M_{(b(k))}$. We first give the integral type expression of the projections P_k and Q_k.

In the annulus where f is defined, we have

$$P_k(f)(x) = \frac{1}{\Omega_n} \int_\Sigma |y^{-1}x|^k C^+_{n+1,k}(\xi, \eta) E(y)\mathfrak{n}(y) f(y) d\sigma(y)$$

and

$$Q_k(f)(x) = \frac{1}{\Omega_n} \int_\Sigma |y^{-1}x|^{-k-n} C^-_{n+1,k}(\xi, \eta) E(y)\mathfrak{n}(y) f(y) d\sigma(y),$$

where $x = |x|\xi$, $y = |y|\eta$,

$$C^+_{n+1,k}(\xi,\ \eta) = \frac{1}{1-n}\Big[-(n+k-1) C^{(n-1)/2}_{k+1}(\langle \xi,\ \eta\rangle) \\ +(1-n) C^{(n+1)/2}_{k-1}(\langle \xi,\ \eta\rangle)(\langle \xi,\ \eta\rangle - \overline{\xi}\eta)\Big]$$

and

$$C^-_{n+1,k}(\xi,\ \eta) = \frac{1}{n-1}\Big[(k+1) C^{(n-1)/2}_{k+1}(\langle \xi,\ \eta\rangle) \\ +(1-n) C^{(n+1)/2}_{k-1}(\langle \eta,\ \xi\rangle)(\langle \eta,\ \xi\rangle - \overline{\eta}\xi)\Big],$$

where C^ν_k is the Gegenbauer polynomial of degree k with respect to ν. Because

$$\langle \xi,\ \eta\rangle = \frac{\langle y^{-1}x,\ 1\rangle}{|y^{-1}x|},\ \overline{\eta}\xi = \frac{y^{-1}x}{|y^{-1}x|} \text{ and } \overline{\xi}\eta = \left(\frac{y^{-1}x}{|y^{-1}x|}\right)^{-1}, \tag{6.9}$$

we conclude that $C^\pm_{n+1,k}$ are the functions of $y^{-1}x$. For $k \in \mathbb{Z}^+ \cup \{0\}$, define

$$\widetilde{P}^{(k)}(y^{-1}x) = |y^{-1}x|^k C^+_{n+1,k}(\xi,\ \eta)$$

and

$$\widetilde{P}^{(-k-1)}(y^{-1}x) = |y^{-1}x|^{-k-n} C^-_{n+1,k}(\xi,\ \eta).$$

We can see that $\widetilde{P}^{(k)}$ and $\widetilde{P}^{(-k-1)}$ are defined on the two-forms $\mathbb{R}^n_1 \times \mathbb{R}^n_1$, and $\widetilde{P}^{(k)}(y^{-1}x)E(y)$ and $\widetilde{P}^{(-k-1)}(y^{-1}x)E(y)$ are monogenic functions in variables x and y. Particularly, if $y = 1$, comparing the Taylor expansion and the Laurent expansion of $E(x-1)$ and $E(1-x)$, by (6.4) and (6.5), we conclude that the above two functions reduce to $P^{(k)}(x)$ and $P^{(-k-1)}(x)$. Because the inner product and the vector product can be extended to $\mathbb{R}_{(n)} \times \mathbb{R}_{(n)}$, the domains of $\widetilde{P}^{(k)}$ and $\widetilde{P}^{(-k-1)}$ may be extended to $\mathbb{R}_{(n)} \times \mathbb{R}_{(n)}$.

Using the above notations, for $k \in \mathbb{Z}^+ \cup \{0\}$ and $f \in \mathcal{A}$, we have

$$P_k(f)(x) = \frac{1}{\Omega_n} \int_\Sigma \widetilde{P}^{(k)}(y^{-1}x) E(y)\mathfrak{n}(y) f(y) d\sigma(y)$$

and

$$Q_k(f)(x) = \frac{1}{\Omega_n} \int_{\Sigma} \widetilde{P}^{(-k-1)}(y^{-1}x)E(y)\mathfrak{n}(y)f(y)d\sigma(y).$$

Hence we get

$$f(x) = \sum_{k=-\infty}^{\infty} \frac{1}{\Omega_n} \int_{\Sigma} \widetilde{P}^{(k)}(y^{-1}x)E(y)\mathfrak{n}(y)f(y)d\sigma(y).$$

Remark 6.2.2 The above result coincides with the convolution integral expression of the projection operator in the complex plane. Actually, if f^0 is a holomorphic function on the annulus $\rho - s < |z| < \iota + s$ in $\mathbb{C}$ and σ is a starlike Lipschitz curve in this annulus, the Laurent series of f^0 is

$$f^0(z) = \sum_{k=-\infty}^{\infty} \frac{1}{2\pi} \int_{\sigma} (\eta^{-1}z)^k f^0(\eta)\frac{d\eta}{\eta}.$$

In each of these contexts, using the natural multiplicative structure of the underlying space, we write the projection operators as convolution integral operators. For the case $\mathbb{R}^n_1$, the difference with the previous ones is that now the kernel functions are defined in the two-forms in $\mathbb{R}^n_1 \times \mathbb{R}^n_1$.

The functions $\widetilde{P}^{(k)}$, $k \in \mathbb{Z}$, defined above satisfy the following properties.

Proposition 6.2.2 *For any $s \in \mathbb{S}_{\mathbb{R}^n_1}$, we have*

$$\widetilde{P}^{(k)}(r_s(y^{-1}x)) = r_s(\widetilde{P}^{(k)}(y^{-1}x)).$$

Proof This proposition follows from (i), (ii), (iv) of Lemma 6.2.1 and the fact that r_s is the identity on scalars. □

We call

$$\widetilde{\phi}(y^{-1}x) = \sum_{-\infty}^{\infty} b_k \widetilde{P}^{(k)}(y^{-1}x)$$

the kernel function associated with the multiplier operator $M_{\{b_k\}}$.

Proposition 6.2.3 *Let $\omega \in (\arctan(N), \pi/2)$ and $b \in H^{\infty}(S^c_{\omega})$. The kernel function $\widetilde{\phi}(y^{-1}x)E(y)$ associated with the sequence $\{b(k)\}$ given above is monogenic in an open neighborhood of $\Sigma \times \Sigma \setminus \{(x, y) : x = y\}$. In addition, in this neighborhood,*

$$|\widetilde{\phi}(y^{-1}x)| \leqslant \frac{C}{|1 - y^{-1}x|^n}.$$

Proof We first consider the left monogeneity with respect to x. Similar to Lemma 6.2.1, choosing $s \in \mathbb{S}_{\mathbb{R}_1^n}$ and applying the mapping r_s term by term to the entries of the series $\widetilde{\phi}(y^{-1}x)E(y)$, we can use the relation $I = r_{s^{-1}}r_s$ and Lemma 6.2.1 to deduce that

$$\widetilde{\phi}(y^{-1}x)E(y) = r_{s^{-1}}(\widetilde{\phi}(|y|^{-1}\tilde{x})E(y)).$$

Set

$$D_{\tilde{x}} = (\partial/\partial\tilde{x}_0)e_0 + (\partial/\partial\tilde{x}_1)e_1 + \cdots + (\partial/\partial\tilde{x}_n)e_n,$$

where every $\tilde{x}_k$ is the linear combination of x_i and the components x. The coefficients of the combination are determined by the chosen $s \in \mathbb{S}_{\mathbb{R}_1^n}$ based on the relation

$$(sf_0)(sf_0)^{-1} = f_0\tilde{x}.$$

Because $\tilde{x} = s^{-1}xs^{-1}$, we have

$$Ds^{-1}E(\tilde{x}) = Ds^{-1}(s\tilde{x}s/|x|^{n+1}) = 0.$$

Hence, $Ds^{-1} = p(s)D_{\tilde{x}}$, where $p(s)$ is a rational function in $\mathbb{S}$. Because

$$\begin{aligned} D(\widetilde{\phi}(y^{-1}x)E(y)) &= (Ds^{-1})(\phi(|y|^{-1}\tilde{x})E(y))s \\ &= (p(s)D_{\tilde{x}})(\phi(|y|^{-1}\tilde{x})E(y)s, \end{aligned}$$

we invoke Theorem 6.1.1 and Remark 6.2.1 to obtain for any fixed $y', x' \in \Sigma$ and $x' \notin y'$, $\widetilde{\phi}(y^{-1}x)E(y)$ is left monogenic in a neighborhood U of x', where $y' \in U$. Then $\widetilde{\phi}(y^{-1}x)$ satisfies the desired estimate, where the constant C in this estimate depends on the size of the neighborhood.

Now we consider the right monogeneity of $\widetilde{\phi}(y^{-1}x)E(y)$ with respect to x. It follows from

$$E(y)E(1 - xy^{-1}) = E(x - y) = E(1 - y^{-1}x)E(y)$$

that

$$E(y)\widetilde{P}^{(k)}(xy^{-1}) = \widetilde{P}^{(k)}(y^{-1}x)E(y).$$

We can obtain

$$E(y)\widetilde{\phi}(xy^{-1}) = \widetilde{\phi}(y^{-1}x)E(y).$$

Hence, similarly, we replace $\widetilde{\phi}(y^{-1}x)E(y)$ by $E(y)\widetilde{\phi}(xy^{-1})$ to get that the function is right monogenic with respect to x.

Now we prove that the function $\widetilde{\phi}(y^{-1}x)E(y)$ is monogenic with respect to y. We claim that this function is also of the form $\widetilde{\psi}(x^{-1}y)E(x)$, where $\widetilde{\psi}$ is function like $\widetilde{\phi}$ associated with a certain bounded holomorphic function. To prove this, we can see that

$$C^-_{n+1,k}(\xi,\ \eta)\overline{\eta} = C^+_{n+1,k}(\eta,\ \xi)\overline{\xi}$$

implies

$$\widetilde{P}^{(k)}(y^{-1}x)E(y) = \widetilde{P}^{(-k-1)}(x^{-1}y)E(x).$$

If $\widetilde{\phi}$ is defined through $b \in H^\infty(S^c_\omega)$ by $\widetilde{\phi}(x) = \sum' b(k)P^{(k)}(x)$, then $\widetilde{\psi}(y) = \sum_{k\neq -1} b'(k)P^{(k)}(y)$, where $b'(z) = b(-z-1)$. The function b' is similar to b. The proof of Theorem 6.1.1 can be modified to show that the function $\widetilde{\psi}$ enjoys the same properties as $\widetilde{\phi}$ does. The monogeneity in y follows from the conclusions established in the early part of the proof. This completes the proof. □

Similar to the above proposition, we obtain

Proposition 6.2.4 *Let $\omega \in (\arctan(N),\ \pi/2)$ and b be a holomorphic function in S^c_ω which is bounded near the origin and satisfies $|b(z)| \leqslant C_\mu |z|$ at ∞ in S^c_μ, $0 < \mu < \omega$. Then the kernel function $\widetilde{\phi}(y^{-1}x)E(y)$ associated with $\{b(k)\}$ is monogenic with respect to x and y in the neighborhood of $\Sigma \times \{\Sigma \setminus \{x = y\}\}$. Moreover,*

$$|\widetilde{\phi}(y^{-1}x)| \leqslant \frac{C}{|1-y^{-1}x|^{n+1}}.$$

For $b \in H^\infty(S^c_\omega)$, we write briefly $M_{(b(k))} = M_b$, that is,

$$M_b f(x) = \sum_{k=1}^{\infty} b(k)P_k(f)(x) + \sum_{k=1}^{\infty} b(-k)Q_{k-1}(f)(x).$$

For $x \in \Sigma$, $r \approx 1$ and $r < 1$, we consider the function

$$\begin{aligned} M^r_b f(x) &= \sum_{k=1}^{\infty} b(k)P_k(f)(rx) + \sum_{k=1}^{\infty} b(-k)Q_{k-1}(f)(r^{-1}x) \\ &= P^r(x) + Q^r(x),\ \ \rho - s < |x| < \iota + s. \end{aligned}$$

By the convolution expression of the projections, we have

$$\begin{aligned} P^r(x) &= \sum_{k=1}^{\infty} b(k)\frac{1}{\Omega_n}\int_\Sigma \widetilde{P}^{(k)}(y^{-1}rx)E(y)\mathfrak{n}(y)f(y)d\sigma(y) \\ &= \frac{1}{\Omega_n}\int_\Sigma \Big(\sum_{k=1}^{\infty} b(k)\widetilde{P}^{(k)}(y^{-1}rx)\Big)E(y)\mathfrak{n}(y)f(y)d\sigma(y) \\ &= \frac{1}{\Omega_n}\int_\Sigma \widetilde{\phi}^+(y^{-1}rx)E(y)\mathfrak{n}(y)f(y)d\sigma(y), \end{aligned}$$

where $\widetilde{\phi}^+ = \sum_{k=1}^{\infty} b(k)\widetilde{P}^{(k)}$. Similarly, we have

$$Q^r(x) = \frac{1}{\Omega_n}\int_\Sigma \widetilde{\phi}^-(y^{-1}r^{-1}x)E(y)\mathfrak{n}(y)f(y)d\sigma(y),$$

where $\widetilde{\phi}^- = \sum_{k=-\infty}^{-1} b(k)\widetilde{P}^{(k)}$.

Because the series that defines $M_b^r f$ converges uniformly as $r \to 1-$, we can change the orders of the summation and the limit to obtain

$$M_b f(x) = \lim_{r\to 1-}\frac{1}{\Omega_n}\int_\Sigma \left(\widetilde{\phi}^+(y^{-1}rx) + \widetilde{\phi}^{-1}(y^{-1}r^{-1}x)\right)E(y)\mathfrak{n}(y)f(y)d\sigma(y).$$

For the Fourier multiplier operator M_b defined in this section, the following Plemelj type formula holds.

Theorem 6.2.1 *If $b \in H^\infty(S^c_\omega)$, then for $f \in \mathcal{A}$ and $x \in \Sigma$,*

$$\begin{aligned} M_b f(x) &= \lim_{r\to 1-}\frac{1}{\Omega_n}\int_\Sigma \left(\widetilde{\phi}^+(y^{-1}rx) + \widetilde{\phi}^{-1}(y^{-1}r^{-1}x)\right)E(y)\mathfrak{n}(y)f(y)d\sigma(y) \\ &= \lim_{\epsilon\to 0}\frac{1}{\Omega_n}\left\{\int_{|y-x|>\epsilon, y\in\Sigma}\widetilde{\phi}(y^{-1}x)E(y)\mathfrak{n}(y)f(y)d\sigma(y) + \widetilde{\phi}^1(\epsilon,x)f(x)\right\}, \end{aligned}$$

where $\widetilde{\phi} = \widetilde{\phi}^+ + \widetilde{\phi}^-$ is the function associated with b as specified in Corollary 6.1.2. $\widetilde{\phi}^1 = \widetilde{\phi}^{+,1} + \widetilde{\phi}^{-,1}$, where

$$\widetilde{\phi}^{\pm,1}(\epsilon,x) = \int_{S(\epsilon,x,\pm)}\widetilde{\phi}^{\pm}(y^{-1}x)E(y)\mathfrak{n}(y)d\sigma(y).$$

Here $S(\epsilon,x,\pm)$ is the part of the sphere $|x-y| = \epsilon$ inside or outside Σ, depending on $\pm$ taking $+$ or $-$.

Proof By the decompositions $\widetilde{\phi} = \widetilde{\phi}^+ + \widetilde{\phi}^-$ and $\widetilde{\phi}^1 = \widetilde{\phi}^{+,1} + \widetilde{\phi}^{-,1}$, we only need to prove the equality for the "+" half. The "−" half can be dealt with similarly. For a fixed constant $\epsilon > 0$, the integral can be decomposed as

$$\begin{aligned} \lim_{r\to 1-}&\left\{\int_{|y-x|>\epsilon, y\in\Sigma}\widetilde{\phi}^+(y^{-1}rx)E(y)\mathfrak{n}(y)f(y)d\sigma(y)\right. \\ &+\left.\int_{|y-x|\leqslant\epsilon, y\in\Sigma}\widetilde{\phi}^+(y^{-1}rx)E(y)\mathfrak{n}(y)f(y)d\sigma(y)\right\}. \end{aligned}$$

As $r \to 1-$, the first part tends to

$$\int_{|y-x|>\epsilon, y\in\Sigma}\widetilde{\phi}^+(y^{-1}x)E(y)\mathfrak{n}(y)f(y)d\sigma(y).$$

The second part can be further decomposed as

$$\int_{|y-x|\leqslant\epsilon, y\in\Sigma} \widetilde{\phi}^{+}(y^{-1}rx)E(y)\mathfrak{n}(y)(f(y)-f(x))d\sigma(y)$$
$$+\int_{|y-x|\leqslant\epsilon, y\in\Sigma} \widetilde{\phi}^{+}(y^{-1}rx)E(y)\mathfrak{n}(y)d\sigma(y)f(x).$$

As $\epsilon \to 0$ and $r \to 1-$, the first integral tends to zero. By Cauchy's theorem, for a fixed ϵ, the second integral tends to $\widetilde{\phi}^{+,1}(\epsilon, x)f(x)$ as $r \to 1-$. This completes the proof of Theorem 6.2.1. □

Next, we state some knowledge of Hardy spaces of monogenic functions and the geometry about Lipschitz surfaces.

Let Δ and Δ^c be the bounded and unbounded connected components of $\mathbb{R}^n_1 \setminus \Sigma$. For $\alpha > 0$, define the non-tangential approach regions $\Lambda_\alpha(x)$ and $\Lambda^c_\alpha(x)$ to a point $x \in \Sigma$ as

$$\Lambda_\alpha(x) = \Lambda_\alpha(x, \Delta) = \Big\{x \in \Delta : \ |y-x| < (1+\alpha)\mathrm{dist}(y, \Sigma)\Big\}$$

and

$$\Lambda^c_\alpha(x) = \Lambda_\alpha(x, \Delta^c) = \Big\{x \in \Delta^c : \ |y-x| < (1+\alpha)\mathrm{dist}(y, \Sigma)\Big\}.$$

Similar to the complex variable case considered in [19, 20], it is easy to prove that there exists a positive constant α_0 depending only on the Lipschitz constant of Σ such that for all $0 < \alpha < \alpha_0$ and all $x \in \Sigma$, there hold $\Lambda_\alpha(x) \subset \Delta$ and $\Lambda^c_\alpha(x) \subset \Delta^c$. The following argument is independent of the choice of $\alpha \in (0, \alpha_0)$. In this section, we choose and fix α.

Let f be a function defined in Δ. The interior non-tangential maximal function $N_\alpha(f)$ is defined as

$$N_\alpha(f)(x) = \sup\Big\{|f(y)| : \ y \in \Lambda_\alpha(x)\Big\}, \ x \in \Sigma.$$

The exterior non-tangential maximal function $N^c_\alpha(f)$ can be defined similarly. For $0 < p < \infty$, the left-Hardy space $H^p(\Delta)$ is defined as the set of all left-monogenic functions f in Δ satisfying $N_\alpha(f) \in L^p(\Sigma)$. If $f \in H^p(\Delta)$, the norm $\|f\|_{H^p(\Delta)}$ is defined as the L^p-norm of $N_\alpha(f)$ on Σ.

Except that the functions in $H^{p_0}(\Delta^c)$ are assumed to vanish at ∞, the spaces $H^p(\Delta^c)$ can be defined similarly. Similar to the Hardy space of monogenic functions in [21], we can prove the following result.

Proposition 6.2.5 *If $f \in H^p(\Delta)$, $p > 1$, then the non-tangential limit of f, i.e.,*

$$\lim_{y\to x, y\in\Lambda_\alpha(x)} f(y)$$

exists on Σ a.e. We still use f to denote the limit function and get

$$c_{N,p}\|f\|_{H^p(\Delta)} \leqslant \|f\|_{L^p(\Sigma)} \leqslant C_{N,p}\|f\|_{H^p(\Delta)},$$

where $c_{N,p}$, $C_{N,p}$ depend on p and the Lipschitz constant N.

In other words, for $p > 1$, the $H^p(\Delta)$ norm of a function is equivalent to the L^p-norm of its non-tangential boundary limit on the boundary. For the functions in the Hardy space associated with Δ^c, a similar relation holds.

In polar coordinate, The Dirac operator D can be represented as

$$D = \zeta\partial_r - \frac{1}{r}\partial_\zeta = \zeta\Big(\partial_r - \frac{1}{r}\Gamma_\zeta\Big),$$

where Γ_ζ is a first order differential operator depending only on the angular coordinate known as the special Dirac operator. It is well known that

$$\Gamma_\zeta f(\zeta) = kf(\zeta),\ \ f \in M_k, \tag{6.10}$$

where M_k, $k \neq -1, -2, \ldots, -n+1$, is the subspace of homogeneous left monogenic functions of degree k. For $f \in \mathcal{A}$, we define $\Gamma_\zeta(f\mid_{\mathbb{S}_{\mathbb{R}_1^n}})$. The definition of Γ_ζ can be extended to $\Gamma_\zeta : \mathcal{A} \to \mathcal{A}$.

Similar to the results of Lipschitz graphs studied in [21] (see also [20]), we can get the following norm equivalence of high order g-functions of $f \in H^2(\Delta)$. For $f \in H^2(\Delta^c)$, a similar result holds.

Proposition 6.2.6 *Suppose that $f \in H^2(\Delta)$. Then the norm $\|f\|_{H^2(\Delta)}$ is equivalent to the norm*

$$\Big(\int_0^1\int_\Sigma |(\Gamma_\xi^j f)(sx)|^2(1-s)^{2j-1}d\sigma(x)\frac{ds}{s}\Big)^{1/2},\ \ j = 1, 2.$$

The following result is equivalent to the CMcM theorem [17] on Σ.

Proposition 6.2.7 *Assume that $f \in L^2(\Sigma)$. Then there are $f^+ \in H^2(\Delta)$ and $f^- \in H^2(\Delta^2)$ such that their non-tangential boundary limits which are still denoted by f^+ and f^- exist in $L^2(\Sigma)$, and $f = f^+ + f^-$. The mappings $f \to f^\pm$ are continuous on $L^2(\Sigma)$.*

It is easy to see that if $f \in \mathcal{A}$, then the natural decomposition of f into its power series part and principal series part is identical to the decomposition given in Proposition 6.2.7.

Denote by Σ_s, $0 < s < 1$, the surface $\{sx : x \in \Sigma\}$.

Lemma 6.2.2 *Let $x_0 \in \Sigma$, $0 < s < 1$, and $x = sx_0$. Then there exists a constant C_Σ such that*

$$|1 - y^{-1}x| \geqslant C_\Sigma\Big\{(1-\sqrt{s})^2 + \theta^2\Big\}^{1/2},\ \ y \in \Sigma_{\sqrt{s}},$$

where $\theta = \arg(x, y)$.

Proof The assertion is equivalent with

$$|y - x| \geqslant C_\Sigma \sqrt{s}\left\{(1-\sqrt{s})^2 + \theta^2\right\}^{1/2}, \ y \in \Sigma_{\sqrt{s}}.$$

Let $x_0 = r_0\xi$, $y = r\eta$ and $x_1 = \sqrt{s}x_0 \in \Sigma_{\sqrt{s}}$, where $\xi, \eta \in \mathbb{S}_{\mathbb{R}_1^n}$. A direct computation gives

$$\begin{aligned} |y-x|^2 &= r^2\left[(1-\beta)^2 + 4\beta\sin^2\tfrac{\theta}{2}\right] \\ &\geqslant C_\Sigma s[(1-\beta)^2 + \beta\theta^2], \end{aligned} \tag{6.11}$$

where $\beta = sr_0/r$.

If s is small, then β is small and $1-\beta$ has a positive lower bound. Since the right hand side of the desired inequality is bounded from above, then it is dominated by a constant multiple of $1-\beta$. We thus obtain the desired estimate.

Now assume that s is close to, but less than 1. In this case, β has a positive lower bound. We further divide the proof into two subcases. Write $r_1 = |x_1| = \sqrt{s}r_0$.

(i) $r_1/r \leqslant s^{-1/4}$. In this case, $\beta \leqslant s^{1/4}$ and hence $1-\beta \geqslant 1 - s^{1/4} > C(1-\sqrt{s})$. The desired estimate then follows.

(ii) $r_1/r > s^{-1/4}$. In this case,

$$\ln(s^{-1/4}) < \ln(r_1/r) \leqslant N\theta,$$

where we have used the fact that $\Sigma_{\sqrt{s}}$ is Lipschitz, and the Lipschitz constant is N. Thus

$$\theta > \frac{-1}{4N}\ln s \geqslant \frac{1}{4N}(1-\sqrt{s}).$$

Hence

$$\theta > \frac{1}{2}\theta + \frac{1}{8N}(1-\sqrt{s}).$$

Substituting (6.11) and ignoring the entry related to $1-\beta$, we obtain the desired estimate. □

For the main result of this section, we give the following preliminary lemma.

Lemma 6.2.3 *If* $\nu \in (\arctan(N), \ \omega)$, *then for* $y \in \Sigma$ *and* $x \in \Delta$

$$|\Gamma_\zeta(\widetilde{\phi}(y^{-1}x))| \leqslant \frac{C_\nu}{|1 - y^{-1}x|^{n+1}}.$$

Proof In the expansion

$$\widetilde{\phi}(y^{-1}x)E(y) = \sum_{k=1}^{\infty} b(k)\widetilde{P}^{(k)}(y^{-1}x)E(y),$$

substituting

$$\widetilde{P}^{(k)}(y^{-1}x)E(y) = \sum_{|\alpha|=k} V_{\underline{\alpha}}(x)W_{\underline{\alpha}}(y),$$

where $V_{\underline{\alpha}} \in M_k$, $W_{\underline{\alpha}} \in M_{-n-k}$, and applying Γ_ζ with respect to x to the series, we get

$$\Gamma_\zeta(\widetilde{\phi}(y^{-1}x))E(y) = \sum_{k=1}^{\infty} kb(k)\widetilde{P}^{(k)}(y^{-1}x)E(y).$$

The series on the right hand side is associated with the multiplier $b'(z) = zb(z)$. By Proposition 6.2.4, we can obtain the proof of the lemma. □

As the main result of this section, we prove

Theorem 6.2.2 *Let* $\omega \in (\arctan(N), \pi/2)$. *If* $b \in H^\infty(S^c_\omega)$, *then with the convention* $b(0) = 0$, *the above defined* $M_{\{b(k)\}}$ *can be extended to a bounded operator from* $L^2(\Sigma)$ *to* $L^2(\Sigma)$. *Moreover,*

$$\|M_{\{b(k)\}}\|_{L^2(\Sigma)\to L^2(\Sigma)} \leqslant C_\nu \|b\|_{L^\infty(S^c_\nu)}, \ \arctan(N) < \nu < \omega.$$

Proof Let $f \in \mathcal{A}$. By the decomposition of f defined in Proposition 6.2.7, we have $f = f^+ + f^-$, where $f^+ \in H^2(\Delta)$, $f^- \in H^2(\Delta^c)$ and $\|f^\pm\|_{L^2(\Sigma)} \leqslant C_N\|f\|_{L^2(\Sigma)}$. We can get $M_b f = M_{b^+}f^+ + M_{b^-}f^-$, where

$$M_{b^\pm}f^\pm(x) = \lim_{r\to 1-}\int_\Sigma \widetilde{\phi}^\pm(r^{\pm 1}y^{-1}x)E(y)\mathfrak{n}(y)f(y)d\sigma(y), \ x \in \Sigma.$$

Using

$$M_{b^\pm}f^\pm(x) = \int_\Sigma \widetilde{\phi}^\pm(y^{-1}x)E(y)\mathfrak{n}(y)f(y)d\sigma(y)$$

for $x \in \Delta$ and $x \in \Delta^c$, respectively, $M_{b^\pm}f^\pm$ can be left-monogenically extended to Δ and Δ^c.

By Proposition 6.2.5, it suffices to prove

$$\|M_{b^\pm}f^\pm\|_{H^2} \leqslant C_N\|f^\pm\|_{H^2}.$$

We only prove the inequality for "+". The case "−" can be dealt with similarly. For simplicity, in the rest of the proof, we suppress the superscript "+". By the Taylor expansions of f and $M_b f$, we can prove that Γ_ζ commutes with M_b. To prove this, because the Fourier expansions of the functions in $\mathcal{A}$ has a fast decay, we can change the order of taking differentiation Γ_ζ and the infinite summation. As a consequence, for $x \in \Delta$, we have

$$\Gamma_\zeta M_b f(x) = \frac{1}{\Omega_n}\int_\Sigma \widetilde{\phi}(y^{-1}x)E(y)\mathfrak{n}(y)\Gamma_\zeta f(y)d\sigma(y).$$

By Lemma 6.2.3, we can also obtain

$$\Gamma_\zeta^2 M_b f(x) = \frac{1}{\Omega_n}\int_\Sigma \Gamma_\zeta \widetilde{\phi}(y^{-1}x)E(y)\mathfrak{n}(y)\Gamma_\zeta f(y)d\sigma(y).$$

For $x \in \Sigma$, changing contour in the integral expression of $\Gamma_\zeta^2 M_b f$ and using Lemmas 6.2.2 and 6.2.3, we have

$$\begin{aligned}
|\Gamma_\zeta^2 M_b f(x)| &\leqslant C\Big(\int_{\Sigma_{\sqrt{s}}} |\Gamma_\zeta(\phi(y^{-1}x))|\frac{d\sigma(y)}{|y|^n}\Big)^{1/2} \\
&\quad \times\Big(\int_{\Sigma_{\sqrt{s}}} |\Gamma_\zeta(\phi(y^{-1}x))||\Gamma_\zeta f(y)|^2\frac{d\sigma(y)}{|y|^n}\Big)^{1/2} \\
&\leqslant C\Big(\int_{\Sigma_{\sqrt{s}}} \frac{1}{|1-y^{-1}x|^{n+1}}\frac{d\sigma(y)}{|y|^n}\Big)^{1/2} \\
&\quad \times\Big(\int_{\Sigma_{\sqrt{s}}} \frac{1}{|1-y^{-1}x|^{n+1}}|\Gamma_\zeta f(y)|^2\frac{d\sigma(y)}{|y|^n}\Big)^{1/2} \\
&\leqslant C\Big(\int_{\Sigma} \frac{1}{[(1-\sqrt{s})^2+\theta_0^2]^{(n+1)/2}}d\sigma(y)\Big)^{1/2} \\
&\quad \times\Big(\int_{\Sigma} \frac{1}{[(1-\sqrt{s})^2+\theta_0^2]^{(n+1)/2}}|\Gamma_\zeta f(\sqrt{s}y)|^2 d\sigma(y)\Big)^{1/2},
\end{aligned}$$

where θ_0 is the angle between $x \in \Sigma_{\sqrt{s}}$ and $y \in \Sigma$.

Because

$$\begin{aligned}
\int_\Sigma \frac{1}{[(1-\sqrt{s})^2+\theta_0^2]^{(n+1)/2}}d\sigma(y) &\leqslant C\int_0^\pi \frac{\sin^{n-1}\theta_0}{[(1-\sqrt{s})^2+\theta_0^2]^{(n+1)/2}}d\theta_0 \\
&\leqslant C\int_0^\pi \frac{\theta_0^{n-1}}{[(1-\sqrt{s})^2+\theta_0^2]^{(n+1)/2}}d\theta_0 \\
&= \frac{C}{1-\sqrt{s}},
\end{aligned}$$

using Proposition 6.2.6 for $j = 1, 2$, we get

$$\begin{aligned}
\|M_b f\|_{H^2(\Delta)}^2 &\approx \int_0^1\int_\Sigma |\Gamma_\zeta^2(M_b f)(sx)|^2(1-s)^3 d\sigma(x)\frac{ds}{s} \\
&\leqslant C\int_0^1\int_\Sigma \frac{1}{1-\sqrt{s}}\Big(\int_\Sigma \frac{1}{[(1-\sqrt{s})^2+\theta_0^2]^{(n+1)/2}} \\
&\quad \times|\Gamma_\zeta f(\sqrt{s}y)|^2 d\sigma(y)\Big)(1-\sqrt{s})^3 d\sigma(x)\frac{ds}{s} \\
&\leqslant C\int_0^1\int_\Sigma |\Gamma_\zeta f(\sqrt{s}y)|^2
\end{aligned}$$

$$
\begin{aligned}
&\times\Big(\int_{\Sigma}\frac{1-\sqrt{s}}{[(1-\sqrt{s})^2+\theta_0^2]^{(n+1)/2}}d\sigma(x)\Big)(1-\sqrt{s})d\sigma(y)\frac{ds}{s}\\
&\leqslant C\int_0^1\int_{\Sigma}|\Gamma_\zeta f(\sqrt{s}y)|^2(1-\sqrt{s})d\sigma(y)\frac{ds}{s}\\
&\leqslant C\int_0^1\int_{\Sigma}|\Gamma_\zeta f(sy)|^2(1-s)d\sigma(y)\frac{ds}{s}\\
&\approx \|f\|^2_{H^2(\Delta)}.
\end{aligned}
$$

The bound of the norm $\|M_b\|$ can be deduced from the proof of Lemma 6.2.3. This completes the proof of Theorem 6.2.2. □

Remark 6.2.3 The boundedness result is identical with that of CMcM's theorem. Because the surface has a doubling measure, by the standard Calderón-Zygmund technique, the L^p-boundedness, $1 < p < \infty$, as well as the weak (1, 1) boundedness can be deduced from the L^2−boundedness.

Remark 6.2.4 As in the standard cases, the Hilbert transforms on the unit sphere and on the starlike Lipschitz surfaces are well defined but not the one with the Fourier multiplier $b(z) = -i\text{sgn}(z)$, where $\text{sgn}(z)$ is the signum function that takes the value $+1$ for $\text{Re}z > 0$ and the value -1 for $\text{Re}z < 0$. The associated singular integral is given by the kernel

$$
\begin{aligned}
\frac{1}{\Omega_n}\widetilde{\phi}(y^{-1}x)E(y) &= \frac{1}{\Omega_n}\sum_{k=-\infty}^{\infty}-i\text{sgn}(k)\widetilde{P}^k(y^{-1}x)E(y)\\
&= -\frac{2i}{\Omega_n}E(1-y^{-1}x)E(y) = -\frac{2i}{\Omega_n}E(y-x).
\end{aligned}
$$

When $y = 1$, the above reduces to

$$
-\frac{2i}{\Omega_n}E(1-x) = \frac{1}{\Omega_n}\Upsilon\Big(\sum_{k=-\infty}^{\infty}-i\text{sgn}(k)z^k\Big).
$$

For Hilbert transforms on spheres, we refer to Sect. 6.5.

6.3 Holomorphic Functional Calculus of the Spherical Dirac Operator

In this section, we will illuminate that the class of bounded operators M_b in Sect. 6.1 constitutes a functional calculus of Γ_ζ and is identical to the Cauchy–Dunford bounded holomorphic functional calculus of Γ_ζ. The operators M_b satisfy the following properties and thus the class $\{M_b,\ b \in H^\infty(S^c_\omega)\}$ is called a bounded holomorphic functional calculus.

Let $N = \mathrm{Lip}(\Sigma)$, $\arctan(N) < \omega < \pi/2$, $1 < p_0 < \infty$, $b, b_1, b_2 \in H^\infty(S^c_\omega)$, and $\alpha_1, \alpha_2 \in \mathbb{C}$. Then

$$\|M_b\|_{L^{p_0}(\Sigma)\to L^{p_0}(\Sigma)} \leqslant C_{p_0,\nu}\|b\|_{L^\infty(S^c_\nu)},\ \arctan(N) < \nu < \omega;$$

$$M_{b_1 b_2} = M_{b_1} \circ M_{b_2};$$

$$M_{\alpha_1 b_1 + \alpha_2 b_2} = \alpha_1 M_{b_1} + \alpha_2 M_{b_2}.$$

The first conclusion follows from Remark 6.2.3. The second and the third conclusions can be deduced from the Laurent expansion of test functions.

Denote the resolvent operator of Γ_ζ at $\lambda \in \mathbb{C}$ by

$$R(\lambda,\ \Gamma_\zeta) = (\lambda I - \Gamma_\zeta)^{-1}.$$

We prove that for non-real λ,

$$R(\lambda,\ \Gamma_\zeta) = M_{1/(\lambda-(\cdot))}.$$

In fact, by the relation (6.10), the Fourier multiplier $\lambda - k$ is associated with $\lambda I - \Gamma_\zeta$, and the Fourier multiplier $(\lambda - k)^{-1}$ is associated with $R(\lambda,\ \Gamma_\zeta)$. The property of the functional calculus in relation to the boundedness then asserts that for $1 < p < \infty$,

$$\|R(\lambda,\ \Gamma_\zeta)\|_{L^p(\Sigma)\to L^p(\Sigma)} \leqslant \frac{C_\nu}{|\lambda|},\ \lambda \notin S^c_\nu.$$

By this estimate, for $b \in S^c_\omega$ with good decays at both zero and the infinity, the Cauchy–Dunford integral

$$b(\Gamma_\zeta)f = \frac{1}{2\pi i}\int_\Pi b(\lambda)R(\lambda,\ \Gamma_\zeta)d\lambda f$$

defines a bounded operator, where Π is the path consisting of four rays: $L_1 \cup L_2 \cup L_3 \cup L_4$, where

$$L_1 = \Big\{s\exp(i\theta) :\ s \text{ is from } \infty \text{ to } 0\Big\},$$

$$L_2 = \Big\{s\exp(-i\theta) :\ s \text{ is from } 0 \text{ to } \infty\Big\},$$

$$L_3 = \Big\{s\exp(i(\pi+\theta)) :\ s \text{ is from } \infty \text{ to } 0\Big\}$$

and

$$L_4 = \Big\{ s \exp(i(\pi - \theta)) : \; s \text{ is from } 0 \text{ to } \infty \Big\},$$

and arctan $N < \theta < \omega$. In the sense of the convergence lemma obtained by McIntosh [22], the functions b of this sort form a dense subclass of $H^\infty(S^c_\omega)$. By this lemma, we can extend the definition given by the Cauchy–Dunford integral and define a functional calculus $b(\Gamma_\zeta)$ on general functions $b \in H^\infty(S^c_\omega)$.

Now we prove $b(\Gamma_\zeta) = M_b$. Assume that b has good decays at both zero and the infinity and $f \in \mathcal{A}$. Then changing the order of the integral and the summation in the following equality, we can get

$$\begin{aligned}
b(\Gamma_\zeta) &= \frac{1}{2\pi} \int_\Pi b(\lambda) R(\lambda, \; \Gamma_\zeta) d\lambda f(x) \\
&= \frac{1}{2\pi i} \int_\Pi b(\lambda) \sum_{k=-\infty}^{\infty} (\lambda - k)^{-1} \\
&\quad \times \frac{1}{\Omega_n} \int_\Sigma \widetilde{P}^{(k)}(y^{-1}x) E(y) \mathfrak{n}(y) f(y) d\sigma(y) d\lambda \\
&= \sum_k \Big(\frac{1}{2\pi i} \int_\Pi b(\lambda)(\lambda - k)^{-1} d\lambda \Big) \\
&\quad \times \frac{1}{\Omega_n} \int_\Sigma \widetilde{P}^{(k)}(y^{-1}x) E(y) \mathfrak{n}(y) f(y) d\sigma(y) \\
&= \sum_k b(k) \frac{1}{\Omega_n} \int_\Sigma \widetilde{P}^{(k)}(y^{-1}x) E(y) \mathfrak{n}(y) f(y) d\sigma(y) \\
&= M_b f(y).
\end{aligned}$$

As defined in Proposition 6.2.7, we denote by $P^\pm$ the projection operators $P^\pm f = f^\pm$. It follows from the estimate of the resolvent $R(\lambda, \; \Gamma_\zeta)$ that $\Gamma_\zeta P^\pm$ is a type ω operator (see [22]).

The operators $\Gamma_\zeta P^\pm$ and Γ_ζ are identical to their dual operators on $L^2(\Sigma)$ in the dual pair $(L^2(\Sigma), \; L^2(\Sigma))$ under the bilinear pairing

$$\langle\langle f, \; g \rangle\rangle = \frac{1}{\Omega_n} \int_\Sigma f(x) \mathfrak{n}(x) g(x) d\sigma(x).$$

That is

$$\langle\langle \Gamma_\zeta P^\pm f, \; g \rangle\rangle = \langle\langle f, \; \Gamma_\zeta P^\pm g \rangle\rangle$$

and

$$\langle\langle \Gamma_\zeta f, \; g \rangle\rangle = \langle\langle f, \; \Gamma_\zeta g \rangle\rangle.$$

These can be easily derived from Parseval's identity

$$\sum_{k=0}^{\infty}\sum_{|\underline{\beta}|=k}\lambda_{\underline{\beta}}\lambda'_{\underline{\beta}}+\mu_{\underline{\beta}}\mu'_{\underline{\beta}}=\frac{1}{\Omega_n}\int_{\mathbb{S}_{\mathbb{R}^n_1}} f(x)\mathfrak{n}(x)g(x)d\sigma(x),$$

and (6.10). For the Banach dual pair $(L^p(\Sigma), L^{p'}(\Sigma))$, $1<p<\infty$, $1/p+1/p'=1$, analogous conclusions also hold.

6.4 The Analogous Theory in $\mathbb{R}^n$

In this section, we state briefly how to establish an analogous theory on the symmetric Euclidean space

$$\mathbb{R}^n=\left\{\underline{x}=x_1e_1+\cdots+x_ne_n,\ x_i\in\mathbb{R}\right\}.$$

In $\mathbb{R}^n$, the Cauchy kernel is $\underline{E}(\underline{x})=\overline{x}/|\underline{x}|^n$ and the Dirac operator is

$$\underline{D}=(\partial/\partial x_1)e_1+\cdots+(\partial/\partial x_n)e_n.$$

We also have Cauchy's theorem and Cauchy's formula. Corresponding to (6.4), we have

$$\underline{E}(\underline{x}-e_1)=\underline{P}^{(-1)}(\underline{x})+\underline{P}^{(-2)}(\underline{x})+\cdots+\underline{P}^{(-k)}(\underline{x})+\cdots,\ |\underline{x}|>1. \quad (6.12)$$

Based on the relation

$$\underline{E}(\underline{x}-\underline{y})=\frac{\overline{\underline{x}}-\overline{\underline{y}}}{|\overline{\underline{x}}-\overline{\underline{y}}|^n}=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(k-1)!}\langle\underline{y},\ \nabla_{\underline{x}}\rangle^{k-1}\frac{\overline{\underline{x}}}{|\overline{\underline{x}}|^n}, \quad (6.13)$$

where $\nabla_{\underline{x}}=(\partial/\partial x_1,\cdots,\partial/\partial x_n)$, letting $\underline{y}=e_1$, we get

$$\underline{P}^{(-k)}(\underline{x})=\frac{(-1)^{k-1}}{(k-1)!}\left(\frac{\partial}{\partial x_1}\right)^{k-1}\underline{E}(\underline{x}).$$

From the Taylor series theory, we know that the general entries of the infinite series (6.13) are monogenic with respect to $\underline{D}$ in both $\underline{x}$ and $\underline{y}$. So $\underline{P}^{(-k)}(\underline{x})$ is monogenic. Define

$$\underline{P}^{(k-1)}=\underline{I}(\underline{P}^{(-k)}),\ k\in\mathbb{Z}^+,$$

where $\underline{I}$ is the Kelvin inversion: $\underline{I}(f)(\underline{x})=\underline{E}f(\underline{x}^{-1})$. The properties of the Kelvin inversion indicate that $\underline{P}^{(k-1)}$ is monogenic. It can be verified that when we replace $P^{(k)}$ by $\underline{P}^{(k)}$, x by $\underline{x}$ and n by $n-1$, Proposition 6.1.1 still holds.

In $\mathbb{R}^n$, there are corresponding objects like the heart shaped regions $H_{\omega,\pm}$, that is,

$$\underline{H}_{\omega,\pm} = \left\{\underline{x} \in \mathbb{R}^n : \frac{(\pm \ln |e_1\underline{x}|)}{\arg(e_1, \underline{x})} < \tan\omega\right\},$$

and

$$\underline{H}_\omega = \underline{H}_{\omega,+} \cap \underline{H}_{\omega,-}.$$

In fact,

$$\underline{H}_\omega = \left\{\underline{x} \in \mathbb{R}^n : \frac{|\ln |e_1\underline{x}||}{\arg(e_1, \underline{x})} < \tan\omega\right\}.$$

We use the following function spaces

$$K(\underline{H}_{\omega,\pm}) = \left\{\underline{\phi} : \underline{H}_{\omega,\pm} \to \mathbb{C}_{(n)} : \phi \text{ is monogenic and satisfies} \right.$$
$$\left. |\underline{\phi}(\underline{x})| \leqslant \frac{C_\mu}{|e_1 - \underline{x}|^{n-1}},\ 0 < \mu < \omega\right\},$$

and

$$K(\underline{H}_{\omega,\pm}) = \left\{\underline{\phi} : \underline{H}^c_\omega \to \mathbb{C}_{(n)} : \underline{\phi} = \underline{\phi}^+ + \underline{\phi}^-,\ \underline{\phi}^\pm \in K(\underline{H}_{\omega,\pm})\right\}.$$

Similar to Theorem 6.1.1, the following is a technical result.

Theorem 6.4.1 *If* $b \in H^\infty(S^c_{\omega,\pm})$ *and* $\underline{\phi}(\underline{x}) = \sum_{k=\pm 1}^{\pm\infty} b(k)\underline{P}^{(k)}(\underline{x})$, *then* $\underline{\phi} \in K(\underline{H}_{\omega,\pm})$.

Proof As in the proof of Theorem 6.1.1, the case $b \in H^\infty(S^c_{\omega,\pm})$ is reduced to the case $b \in H^{\infty,r}(S^c_{\omega,\pm})$, and the case $b \in H^{\infty,r}(S^c_{\omega,+})$ is reduced to the case $b \in H^{\infty,r}(S^c_{\omega,-})$.

Let $b \in H^{\infty,r}(S^c_{\omega,-})$. We have

$$\begin{aligned}
\underline{\phi}(\underline{x}) &= \sum_{k=1}^{\infty} b(-k)\underline{P}^{(-k)}(\underline{x}) = \sum_{k=1}^{\infty} b(-k)\frac{(-1)^{k-1}}{(k-1)!}\left(\frac{\partial}{\partial x_1}\right)^{k-1}\underline{E}(\underline{x})\\
&= -e_1\sum_{k=1}^{\infty} b(-k)\frac{(-1)^{k-1}}{(k-1)!}\left(\frac{\partial}{\partial x_1}\right)^{k-1}\left(\frac{x_1 - x_2 g_1 - \cdots - x_n g_{n-1}}{|x_1 + x_2 g_1 + \cdots + x_n g_{n-1}|^n}\right)\\
&= -e_1\sum_{k=1}^{\infty} b(-k)\frac{(-1)^{k-1}}{(k-1)!}\left(\frac{\partial}{\partial x_1}\right)^{k-1} E(\tilde{x})\\
&= -e_1\tilde{\phi}(\tilde{x}),
\end{aligned}$$

where $g_i = e_{i+1}e_1^{-1}$, $i = 1, 2, \ldots, n-1$ are the basic vectors like $e_1, e_2, \ldots, e_{n-1}$, and $\tilde{x} = x_1 + x_2 g_1 + \cdots x_n g_{n-1}$ is a vector in $\mathbb{R}^{n-1}_1$. We can also get

$$\underline{D} = \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} g_1 + \cdots + \frac{\partial}{\partial x_n} g_{n-1}\right) e_1 = \tilde{D} e_1,$$

where $\tilde{D}$ is the Dirac operator in $\mathbb{R}_1^{n-1}$. Hence, if $\tilde{\phi}$ is left monogenic with respect to $\tilde{D}$ in $\mathbb{R}_1^{n-1}$, we obtain that $\underline{\phi}$ is left monogenic with respect to $\underline{D}$ in $\mathbb{R}^n$. If we replace e_1 by 1, the heart shaped regions $\underline{H}_{\omega,\pm}$ are identical to those in $\mathbb{R}_1^{n-1}$. The desired left monogeneity and the estimate follow from Theorem 6.1.1. The right-monogeneity can be proved similarly, with the only difference that e_1 is factorized out of $\underline{E}(\underline{x})$ from the right hand side, and of $\underline{D}$ from the left hand side, and define $g_i = e_1^{-1} e_{i+1}$. The proof is complete. □

Let $\underline{\Sigma}$ be a surface in $\mathbb{R}^n$. If $\underline{\Sigma}$ is $(n-1)$-dimensional, starlike about the origin and there exists a constant $M < \infty$ such that for $\underline{x}, \underline{x}' \in \underline{\Sigma}$,

$$\frac{|\ln |\underline{x}^{-1}\underline{x}'||}{\arg(\underline{x},\ \underline{x}')} \leqslant M,$$

we call this surface a starlike Lipschitz surface. The minimum value of M is called the Lipschitz constant of $\underline{\Sigma}$ denoted by $\underline{N} = \mathrm{Lip}(\underline{\Sigma})$.

We use the following class of functions

$$\mathbb{A} = \left\{ f :\ f \text{ is left monogenic in } \underline{\rho} - s < |\underline{x}| < \underline{\iota} + s \text{ for some } s > 0 \right\},$$

where $\underline{\rho} = \inf\left\{|\underline{x}| :\ \underline{x} \in \underline{\Sigma}\right\}$ and $\underline{\iota} = \sup\left\{|\underline{x}| :\ \underline{x} \in \underline{\Sigma}\right\}$. By CMcM's Theorem, we can deduce that $\mathcal{A}$ is dense in $L^2(\underline{\Sigma})$.

If $f \in \mathcal{A}$,

$$f(\underline{x}) = \sum_{k=0}^{\infty} \underline{P}_k(f)(\underline{x}) + \sum_{k=0}^{\infty} \underline{Q}_k(f)(\underline{x}),$$

where for $k \in \mathbb{Z}^+ \cup \{0\}$, $\underline{P}_k(f)$ belongs to the finite-dimensional right module $\underline{M}_k$ of $k-$homogeneous left monogenic functions in $\mathbb{R}^n$, and $\underline{Q}_k(f)$ belongs to the finite-dimensional right module of $-(k+n-1)-$homogeneous left monogenic functions in $\mathbb{R}^n \setminus \{0\}$. The spaces $\underline{M}_k$ and $\underline{M}_{-k-n+1}$ are eigenspaces of the spherical left-Dirac operator $\underline{\Gamma}_\zeta$, where the operator is defined by

$$\underline{M}_{b(k)} f(\underline{x}) = \sum_{k=0}^{\infty} b_k \underline{P}_k(f)(\underline{x}) + \sum_{k=0}^{\infty} b_{-k-1} \underline{Q}_k(f)(\underline{x}).$$

In the present case, we can obtain a singular integral expression similar to that in Theorem 6.2.1. There exists also an analogous theory of Hardy space H^2. Based on these, applying Theorem 6.4.1, we can prove the following result with the method of Theorem 6.2.2.

Theorem 6.4.2 *Let* $\omega \in (\arctan(\underline{N}), \pi/2)$. *If* $b \in H^\infty(S^c_\omega)$ *and without loss of generality, assume that* $b(0) = 0$. *The above defined* $\underline{M}_b = \underline{M}_{(b(k))}$ *extends to a bounded operator from* $L^2(\underline{\Sigma})$ *to* $L^2(\underline{\Sigma})$. *Moreover,*

$$\|\underline{M}_{(b(k))}\|_{L^2(\underline{\Sigma})\to L^2(\underline{\Sigma})} \leqslant C_\nu \|b\|_{L^\infty(S^c_\nu)}, \ \arctan(\underline{N}) < \nu < \omega.$$

Remark 6.4.1 By Theorem 6.4.2, we can prove that the Fourier multiplier class $\underline{M}_b$ is identical to a certain class of singular integral operators (see Theorem 6.2.1). By the method in Sect. 6.3, we can also prove that such class is identical to the Cauchy–Dunford bounded holomorphic functional calculus of the spherical Dirac operator $\Gamma_{\underline{\zeta}}$.

6.5 Hilbert Transforms on the Sphere and Lipschitz Surfaces

Let Ω be a bounded connected Lipschitz domain in $\mathbb{R}^n$ with the Lipschitz constant less than or equal to M. By this, it means that Ω is a bounded and connected open set whose boundary $\partial\Omega$ is denoted by Σ in the sequel. The boundary Σ may be covered by a finite number of balls in each of which the piece of the boundary of the domain under a suitable rotation and translation can become locally a piece of Lipschitz graph with a Lipschitz constant less than or equal to M. We further assume that the complement $(\overline{\Omega})^c$ is unbounded and connected. Alternatively, we may assume that Ω is an open connected domain above a Lipschitz graph Σ. In both cases, the whole space $\mathbb{R}^n$ is divided into two parts by Σ, that is, $\Omega^+ = \Omega$ and $\Omega^- = \mathbb{R}^n \backslash (\Sigma \cup \Omega)$.

For a scalar-valued function f in $L^p(\Sigma)$, $1 < p < \infty$, define the Cauchy integral $C^\pm_\Sigma f$ as

$$C^\pm_\Sigma f(\underline{x}) = C^\pm f(\underline{x}) = \frac{1}{\sigma_{n-1}} \int_\Sigma E(\underline{y} - \underline{x})\mathfrak{n}^\pm(\underline{y}) f(\underline{y}) d\sigma(\underline{y}), \ x \in \Omega^\pm, \tag{6.14}$$

where $d\sigma(\underline{y})$ is the area measure of the surface and $\mathfrak{n}^\pm(\underline{y})$ are the outward-pointing and inward-pointing normals of the surface $\Omega^\pm$ at the point $\underline{y} \in \Sigma$, σ_{n-1} is the surface area of the $(n-1)$-dimensional unit sphere. Notice that

$$\underline{x}\underline{y} = -\langle \underline{x}, \ \underline{y} \rangle + \underline{x} \wedge \underline{y}. \tag{6.15}$$

By the relation (6.15), the above Cauchy integral becomes

$$\begin{aligned} C^\pm_\Sigma f(\underline{x}) = & \frac{1}{\sigma_n} \int_\Sigma \langle E(\underline{x} - \underline{y}), \ \mathfrak{n}^\pm(\underline{y}) \rangle f(\underline{y}) d\sigma(\underline{y}) \\ & + \frac{1}{\sigma_n} \int_\Sigma E(\underline{x} - \underline{y}) \wedge \mathfrak{n}^\pm(\underline{y}) f(\underline{y}) d\sigma(\underline{y}), \ \underline{x} \in \Omega^\pm. \end{aligned} \tag{6.16}$$

Plemelj's formula ensures that the non-tangential boundary values $\mathbb{C}^{\pm}f(\underline{x})$ of $C^{\pm}f(\underline{x})$ exist and equal to

$$\mathbb{C}^{\pm}f(\underline{x}) = \frac{1}{2}\Big[f(\underline{x}) \pm \mathbb{C}f(\underline{x})\Big], \text{ a.e.} \underline{x} \in \Sigma, \tag{6.17}$$

where the operator denoted by $\mathbb{C}$ is the principal value Cauchy singular integral operator defined by

$$\mathbb{C}f(\underline{x}) = \frac{2}{\sigma_{n-1}} \lim_{\varepsilon \to 0+} \int_{|\underline{y}-\underline{x}|>\varepsilon,\ \underline{y}\in\Sigma} E(\underline{y}-\underline{x})\mathfrak{n}^{+}(\underline{y})f(\underline{y})d\sigma(\underline{y}), \quad \text{a.e. } \underline{x} \in \Sigma. \tag{6.18}$$

In the last integral, using "p.v." to replace $\lim\limits_{\varepsilon \to 0+}$ and decomposing the integral into the scalar-valued part and the two-form part, we have

$$\begin{aligned} \mathbb{C}f(\underline{x}) &= \frac{2}{\sigma_{n-1}}\text{p.v.}\int_{\Sigma} E(\underline{y}-\underline{x})\mathfrak{n}^{+}(\underline{y})f(\underline{y})d\sigma(\underline{y}) \\ &= \frac{2}{\sigma_{n-1}}\text{p.v.}\int_{\Sigma} \langle E(\underline{y}-\underline{x}),\ \mathfrak{n}^{+}(\underline{y})\rangle f(\underline{y})d\sigma(\underline{y}) \\ &\quad + \frac{2}{\sigma_{n-1}}\text{p.v.}\int_{\Sigma} E(\underline{y}-\underline{x}) \wedge \mathfrak{n}^{+}(\underline{y})f(\underline{y})d\sigma(\underline{y}),\ \text{a.e. } \underline{x} \in \Sigma. \end{aligned} \tag{6.19}$$

By Coifman–McIntosh–Meyer's Theorem [17], the operator $\mathbb{C}$ is bounded on L^p, $1 < p < \infty$. Hence the operator $\mathbb{C}^{\pm}$ is the projection satisfying the property $(\mathbb{C}^{\pm})^2 = \mathbb{C}^{\pm}$. As a corollary, $\mathbb{C}$ itself is a reflection operator, that is, $\mathbb{C}^2 = I$, where I is the identity operator. Because the boundary value f is assumed to be scalar-valued, the Cauchy integrals $C^{\pm}f$ and the boundary values $\mathbb{C}^{\pm}f$ are all para-bivector-valued. In the complex plane and $\mathbb{R}^n_1$, the operators mentioned above are para-vector-valued. (In the complex plane, the boundary data are assumed to be real-valued.)

There exists a second reflection operator N representing the Clifford conjugation, namely,

$$Nf(\underline{x}) = \overline{f(\underline{x})}.$$

The operator N will be used to the boundary values of the para-bivector-valued Cauchy integrals. The corresponding projections are $N^{+} : f \to \text{Sc}[f]$ and $N^{-} : f \to \text{NSc}[f]$. With the four pairs of the combinations $(\mathbb{C},\ \mathbb{C}^{\pm})$ and $(N,\ N^{\pm})$ of reflections and projections, we can formulate the corresponding transmission problems. With a minor modifications dealing with the infinity, the operator theory of $\mathbb{C}^{-}$ is similar to that of $\mathbb{C}^{+}$. Below we mainly deal with the operator $\mathbb{C}^{+}$.

We write

$$\mathbb{C}^{+}f = \frac{1}{2}(I + \mathbb{C})f = u + v,$$

where the scalar-part of the para-bivector-valued $\mathbb{C}^{+}f$ is u and denote $u = \text{Sc}[\mathbb{C}^{+}f]$. The two-form part of $\mathbb{C}^{+}f$ is v: $v = \text{NSc}[\mathbb{C}^{+}f]$. The above relation gives

$$u = \frac{1}{2}(I + N^+\mathbb{C})f \text{ and } v = \frac{1}{2}N^-\mathbb{C}f.$$

Hence, formally,

$$f = 2(I + N^+\mathbb{C})^{-1}u.$$

We define the mapping from u to v as the inner Hilbert transform denoted by H^+. We have

$$\begin{aligned} v &= H^+u \\ &= \frac{1}{2}N^-\mathbb{C}f \\ &= N^-\mathbb{C}(I + N^+\mathbb{C})^{-1}u. \end{aligned}$$

In the above formulation, in order to deduce the extension and the L^p-boundedness of the Hilbert transform from u to v, we need to require the topological isomorphism property from u to f in the L^p space of the boundary. In other words, the double layer potential part $N^+\mathbb{C}$ should be comparatively smaller than the identity operator I. If the curves and surfaces are smooth, this requirement is satisfied for $1 < p < \infty$. However, for general Lipschitz curves and surfaces Σ, this is met only for $p_0 < p < \infty$, where the index $p_0 \in [1, \infty)$ depends on the Lipschitz constant of Σ. However, for $1 < p \leqslant p_0$, the closure of the graph u in $L^p(\Sigma)$ forms a proper subspace of $L^p(\Sigma)$, see [23] and the related references.

Similarly, define

$$v = \frac{1}{2}N^-\mathbb{C}f = N^-\mathbb{C}(I - N^+\mathbb{C})^{-1}u$$

as the outer Hilbert transform H^-u. To give a proper meaning of the above definition, as well as H^+ and H^-, we claim that the existence of the inverse operator $(I \pm N^+\mathbb{C})^{-1}$ under certain conditions. Before we deal with the general theory, we check several interesting examples.

Example 6.5.1 Consider $\Sigma = \mathbb{R}^n$, where Ω is the upper half space

$$\mathbb{R}^n_{1,+} = \left\{x = x_0 + \underline{x} : \ x_0 > 0, \ \underline{x} \in \mathbb{R}^n\right\}.$$

For the case (6.14),

$$\mathfrak{n}^{\pm}(\underline{y}) = \mp e_0 = \pm 1$$

and the scalar part of

$$E(\underline{y} - \underline{x}) = \frac{\overline{\underline{y} - \underline{x}}}{|\underline{y} - \underline{x}|^{n+1}}$$

is zero, while f and $d\sigma(\underline{y})$ are scalar-valued. Hence, $N^+\mathbb{C}f = 0$. As consequence, $f = 2u$ and $v = \mathbb{C}u$. For the outer Hilbert transform, a similar conclusion holds. Therefore, the outer and inner Hilbert transforms coincide with the singular Cauchy integral transform.

Example 6.5.2 Let $\Omega = \mathbb{D}$ be the unit disc in the complex plane. Let f be real-valued. We have

$$\begin{aligned} N^+\mathbb{C}f(e^{i\xi}) &= \text{p.v.Re}\Big[\frac{1}{\pi i}\int_0^{2\pi} \frac{f(e^{it})}{e^{it}-e^{i\xi}}d(e^{it})\Big] \\ &= \text{p.v.}\frac{1}{\pi}\int_0^{2\pi} \text{Re}\Big[\frac{e^{it}}{e^{it}-e^{i\xi}}\Big]f(e^{it})dt. \end{aligned}$$

A direct computation gives

$$\text{Re}\Big[\frac{e^{it}}{e^{it}-e^{i\xi}}\Big] = \frac{1}{2}.$$

Hence,

$$N^+\mathbb{C}f(e^{i\xi}) = f_0 = I_0 f,$$

where f_0 denotes the average of $f(e^{it})$ on $[0, 2\pi]$ and I_0 is the operator which maps f to f_0. We notice that the norm of the operator $\|I_0\| = 1$. Simple computation gives

$$(I + I_0)^{-1}u = -\frac{u_0}{2} + u,$$

where $u_0 = I_0 u$.

Let

$$\widetilde{H}f(e^{i\theta}) = N^-\mathbb{C}f(e^{i\theta}).$$

We have

$$\widetilde{H}f = \text{p.v.}\frac{i}{2\pi}\int_0^{2\pi} \cot\Big(\frac{\theta - t}{2}\Big)f(e^{it})dt$$

that annihilates constants. Therefore,

$$\begin{aligned} H^+u &= N^-\mathbb{C}(I + I_0)^{-1}u \\ &= \widetilde{H}(-\frac{u_0}{2} + u) \\ &= \widetilde{H}u. \end{aligned}$$

Similarly, $(I - I_0)^{-1}$ is defined on the closed subspace:

$$\Big\{u \in L^2 : \int_0^{2\pi} u(e^{it})dt = 0\Big\}$$

and satisfies $(I - I_0)^{-1}u = u$, and

$$H^- u = N^- \mathbb{C}(I - I_0)^{-1} u = \widetilde{H} u.$$

Notice that in this example, the inverse operator $(I - N^+\mathbb{C})^{-1}$ does not exist in L^p, but it exists in the proper and closed subspace L_0^p of L^p:

$$L_0^p(\partial \mathbb{D}) = \left\{ f \in L^p(\partial \mathbb{D}),\ \int_0^{2\pi} f(e^{it})dt = 0 \right\}.$$

Example 6.5.3 Consider $\Omega = B^n$, $\Sigma = S^{n-1}$, $n > 2$. For high-dimensional spheres, the double-layer potential $N^+\mathbb{C}$ is replaced by a non-trivial operator. The inner and outer Hilbert transforms are distinguished. On the sphere, a direct computation indicates that the double-layer potential becomes

$$\langle E(\underline{x} - \underline{y},\ \mathfrak{n}^+(\underline{y}))\rangle = \frac{1}{2} \frac{1}{|\underline{x} - \underline{y}|^{n-2}}. \tag{6.20}$$

Hence, by (6.19),

$$N^+\mathbb{C} f(\underline{x}) = \frac{1}{\sigma_{n-1}} \int_{\mathbb{S}^{n-1}} \frac{f(\underline{y})}{|\underline{x} - \underline{y}|^{n-2}} d\sigma(\underline{y}).$$

Proposition 6.5.1 *On the sphere, the double-layer potential operator $N^+\mathbb{C}$ is L^p-bounded, $1 \leqslant p \leqslant \infty$, and the norm of the operator equals to 1.*

Proof See [24] for the proof. □

What is important is the order of the singularity of double-layer potential rather than the bound of the operator. On the sphere, the existence and the boundedness in L^p of the inverse operator are guaranteed by the Fredholm theory. Generally, we can prove that if Σ is C^∞, then

$$|\langle E(\underline{x} - \underline{y}),\ \mathfrak{n}^+(\underline{y})\rangle| \leqslant \frac{C}{|\underline{x} - \underline{y}|^{n-2}}.$$

This estimate coincides with (6.20). If Σ is $C^{1,\alpha}$, $0 < \alpha < 1$, then

$$|\langle E(\underline{x} - \underline{y}),\ \mathfrak{n}^+(\underline{y})\rangle| \leqslant \frac{C}{|\underline{x} - \underline{y}|^{n-1-\alpha}}. \tag{6.21}$$

In the above cases, the operator $N^+\mathbb{C}$ is compact. We can use the Fredholm theory to prove $(I \pm N^+\mathbb{C})^{-1}$ exists and for $1 < p < \infty$, this operator is bounded in L^p, see

[23] and [25]. However, there exists an essential difficulty to use the Fredholm theory to C^1 domains and Lipschitz domains. In fact, for this case, we can only obtain the following estimate of the kernel of the double-layer potential

$$|\langle E(\underline{x}-\underline{y}),\ \mathfrak{n}^+(\underline{y})\rangle| \leqslant \frac{C}{|\underline{x}-\underline{y}|^{n-1}}.$$

This estimate implies that this kernel has the same singularity as the kernel of the singular Cauchy integral operator on the surface. Fabes–Jodeit–Riviére [26] proved that $N^+\mathbb{C}$ was compact for C^1 domain. For the Lipschitz domain, the operator $N^+\mathbb{C}$ is not necessarily compact in $L^p(\Sigma)$. We need a new method to prove the invertibility of $I + N^+\mathbb{C}$. The case for $p = 2$ was solved in [25], Dahlberg and Kenig [23] gave the optimal range of p, see [23, 27]. The existence and boundedness of $(I \pm N^+\mathbb{C})^{-1}$ are corollaries of those results [24].

Theorem 6.5.1 *For a Lipschitz domain $\Omega \subset \mathbb{R}^n$, $n > 2$, the inner and outer Hilbert transforms both exist and are bounded from $L^p(\Sigma)$ to $L^p(\Sigma)$, where $2-\varepsilon < p < \infty$, $\varepsilon \in (0, 1]$ depends on the Lipschitz constant of Ω (For the C^1 domain one may take $\varepsilon = 1$). In addition, for $u \in L^p(\Sigma)$,*

$$H^{\pm}u(\underline{x}) = \frac{2}{\sigma_{n-1}}\text{p.v.}\int_{\Sigma}\Big[E(\underline{y}-\underline{x}) \wedge \mathfrak{n}^{\pm}(\underline{y})\Big](I \pm N^+\mathbb{C})^{-1}u(\underline{y})d\sigma(\underline{y}). \quad (6.22)$$

Now we consider existence of explicit formulas of the Cauchy-type Poisson and conjugate Poisson kernels on Lipschitz surfaces. Then we deduce explicit formulas of the kernel of the high-dimensional Hilbert transform. Generally, it is difficult to derive such formulas. The explicit formulas and the related issues for the sphere case are studied in [13, 28–30].

Let U be a scalar-valued harmonic function in Ω. If a harmonic function V satisfies

(i) $\mathrm{Sc}[V] = 0$,
(ii) $\underline{D}(U + V) = 0$,

then V is said to be a harmonic conjugate of U. By this definition, if $n > 2$, the harmonic conjugate of a given harmonic function is not unique even modulo Clifford constants. In fact, there exists a non-constant harmonic function V satisfying (i) but $\underline{D}(V) = 0$. To eliminate this situation, we introduce

Definition 6.5.1 If V is a harmonic conjugate of U and there exists a scalar-valued boundary data f such that

$$U + V = C^+_{\Sigma} f,$$

then V is called a Cauchy-type harmonic conjugate, or a canonical harmonic conjugate of U in Ω.

It is easy to see that if we choose f as the Dirac function δ at $\underline{y}$ on the boundary, then we can get

$$\frac{1}{\sigma_{n-1}} E(\underline{y}-\underline{x}) \wedge \mathfrak{n}^{\pm}(\underline{y})$$

is the Cauchy-type harmonic conjugate of

$$\frac{1}{\sigma_{n-1}} \langle E(\underline{y}-\underline{x}),\ \mathfrak{n}^{\pm}(\underline{y})\rangle.$$

Notice that for a scalar-valued harmonic function, there exist more than one harmonic conjugates that differ by non-constant functions. On the other hand, Cauchy-type harmonic conjugate is unique. We have the following result (see [24]).

Theorem 6.5.2 *Assume that Ω is a Lipschitz domain and the scope of p satisfies the conditions as in Theorem 6.5.1. Let U be a scalar-valued harmonic function whose non-tangential maximal function*

$$u^*(\underline{x}) = \sup_{\underline{y}\in\Gamma_\alpha(\underline{x})} |U(\underline{y})|$$

belongs to $L^p(\Sigma)$, where $\Gamma_\alpha(\underline{x})$ is the truncated cone of opening α whose axis is perpendicular to the tangent plane of Σ at $\underline{x} \in \Sigma$. Then there exists unique Cauchy-type harmonic conjugate of U.

Now we discuss the Schwarz kernel and the associated Cauchy-type Poisson and conjugate Poisson kernels. We will seek for the integral representation of the operator S^+ such that $S^+u = C^+f$, where $u = \mathrm{Sc}[\mathbb{C}^+f]$. This is equivalent to find out the kernel $S^+(\underline{x}, \underline{y})$ such that for $\underline{x} \in \Omega$,

$$\begin{aligned} C^+f(\underline{x}) &= \frac{1}{\sigma_{n-1}} \int_\Sigma E(\underline{y}-\underline{x})\mathfrak{n}^+(\underline{y}) f(\underline{y}) d\sigma(\underline{y}) \\ &= \int_\Sigma S^+(\underline{y}, \underline{x}) u(\underline{y}) d\sigma(\underline{y}) \\ &= S^+u(\underline{x}). \end{aligned} \tag{6.23}$$

If the kernel $S^+(\underline{x}, \underline{y})$ exists, then it is called the inner Schwarz kernel. The functions $P^+(\underline{x}, \underline{y})$ and $Q^+(\underline{x}, \underline{y})$ are called the inner Poisson kernel and the conjugate inner Poisson kernel, respectively, where

$$P^+ = \mathrm{Sc}[S^+], \quad Q^+ = \mathrm{NSc}[S^+].$$

By P^+ and Q^+, we can see that for $\underline{x} \in \Omega$,

$$\begin{aligned} U^+(\underline{x}) &= \int_\Sigma P^+(\underline{x}, \underline{y}) u(\underline{y}) d\sigma(\underline{y}) \\ &= \frac{-2}{\sigma_{n-1}} \int_\Sigma \langle E(\underline{y}-\underline{x}), \mathfrak{n}^+(\underline{y})\rangle (I + N^+\mathbb{C})^{-1} u(\underline{y}) d\sigma(\underline{y}) \end{aligned} \tag{6.24}$$

and

$$V^+(\underline{x}) = \int_\Sigma Q^+(\underline{x}, \underline{y})u(\underline{y})d\sigma(\underline{y}) \tag{6.25}$$
$$= \frac{-2}{\sigma_{m-1}} \int_\Sigma E(\underline{y} - \underline{x}) \wedge \mathfrak{n}^+(\underline{y})(I + N^+\mathbb{C})^{-1}u(\underline{y})d\sigma(\underline{y})$$

are the unique solutions of the Dirichlet problem inside Ω with boundary data u and H^+u, respectively. Actually, $V^+(\underline{x})$ is the Cauchy-type harmonic conjugate of $U^+(\underline{x})$. The functions P^+ and Q^+ are the unique harmonic representations of the Dirac function δ_Σ and of its $H^+\delta_\Sigma$, where in the sense of distributions, $H^+\delta_\Sigma$ can be represented as the principal value singular kernel of the inner Hilbert transform. As for Lipschitz domains, the existence of the Poisson kernel can be deduced from the theory of Green's function. For Lipschitz curves and surfaces, by the Plemelj formula, the non-tangential boundary limit of the harmonic function $V^+(\underline{x}')$ is $H^+f(\underline{x})$. In other words, for almost all $\underline{x} \in \Sigma$,

$$\lim_{\underline{x}' \to \underline{x}} V^+(\underline{x}') = H^+u(\underline{x})$$
$$= \text{p.v.} \int_\Sigma Q^+(\underline{x}, \underline{y})u(\underline{y})d\sigma(\underline{y}) \tag{6.26}$$
$$= \frac{2}{\sigma_{n-1}} \text{p.v.} \int_\Sigma \Big[E(\underline{y} - \underline{x}) \wedge \mathfrak{n}^+(\underline{y})\Big](I + N^+\mathbb{C})^{-1}u(\underline{y})d\sigma(\underline{y}).$$

Because

$$V^+(\underline{x}') = \int_\Sigma P^+(\underline{x}', \underline{y})H^+f(\underline{y})d\underline{y},$$

we have for $\underline{x} \in \Sigma' \subset \Omega, \ \underline{y} \in \Sigma$,

$$Q^+(\underline{x}', \underline{y}) = H^+_{\Sigma'}P^+(\underline{x}', \underline{y}),$$

where the Hilbert transform is with respect to an admissible Lipschitz surface Σ'.

The outer Cauchy integral C^- is with respect to the outer Poisson kernel and the conjugate outer Poisson kernel, and therefore the outer Schwartz kernel. The theory for them are parallel.

The inner Poisson kernel on the sphere is well known. There are a number of methods to derive this kernel. The harmonic conjugate of the inner Poisson kernel was first obtained by Brackx et al. Those authors gave the explicit formula of the kernel in the integral form. They further obtained a finite form of the formulas inductively with the space dimension (see [31]). As a result, their methods yield the Cauchy harmonic conjugates. Below we present a different approach based on the double-layer potential.

Theorem 6.5.3 *On the unit sphere, the inner Poisson kernel and its Cauchy-type harmonic conjugate are*

$$P(\underline{x}, \underline{\omega}) = \frac{1}{\sigma_{n-1}} \frac{1 - |\underline{x}|^2}{|\underline{x} - \underline{\omega}|^n} \tag{6.27}$$

and, for $0 < r < 1$,

$$Q(\underline{x}, \underline{\omega}) = \frac{1}{\sigma_{n-1}} \left(\frac{2}{|\underline{x} - \underline{\omega}|^n} - \frac{n-2}{r^{n-1}} \int_0^r \frac{\rho^{n-2}}{|\rho\underline{\xi} - \underline{\omega}|^n} d\rho \right) \underline{x} \wedge \underline{\omega}. \tag{6.28}$$

Proof By (6.17) and (6.19), we have

$$\mathbb{C}^{\pm} f = \frac{1}{2}\Big(f \pm \mathrm{Sc}[\mathbb{C} f]\Big) \pm \frac{1}{2}\mathrm{NSc}[\mathbb{C} f]. \tag{6.29}$$

For the inner Poisson kernel and its harmonic conjugate, we will work with the case "+" in the above formula. Comparing the formula for this case with (6.16) and in view of the harmonic extensions of the scalar part and the non-scalar part of the boundary values to inside part of the unit sphere, we get for $|\underline{x}| < 1$ and $|\underline{\omega}| < 1$,

$$\frac{1}{\sigma_{n-1}} \frac{\langle \underline{\omega} - \underline{x},\ \underline{\omega} \rangle}{|\underline{\omega} - \underline{x}|^n} = \frac{1}{2} P(\underline{x}, \underline{\omega}) + \frac{1}{2} S(\underline{x}, \underline{\omega}) \tag{6.30}$$

and

$$\frac{1}{\sigma_{n-1}} \frac{(\underline{x} - \underline{\omega}) \wedge \underline{\omega}}{|\underline{x} - \underline{\omega}|^n} = \frac{1}{2} Q(\underline{x}, \underline{\omega}) + \frac{1}{2} \widetilde{S}(\underline{x}, \underline{\omega}),$$

where $\widetilde{S}(\underline{x}, \underline{\omega})$ is the Cauchy-type harmonic conjugate of $S(\underline{x}, \underline{\omega})$. Then we can immediately obtain that the formula of $P(\underline{x}, \underline{\omega})$. To obtain the formula of $Q(\underline{x}, \underline{\omega})$, we need to compute $\widetilde{S}(\underline{x}, \underline{\omega})$ which can be derived from the following lemma by taking $r < 1$. □

Lemma 6.5.1 *For* $r = |\underline{x}| < 1$,

$$\begin{aligned} &\sum_{k=0}^{\infty} r^k \frac{n-2}{n+k-2} P^{(k)}(\underline{\omega}^{-1} \underline{\xi}) \\ &= \frac{n-2}{r^{n-2}} \int_0^r \rho^{n-3} E(\underline{\omega} - \rho\underline{\xi}) \underline{\omega} d\rho \\ &= \frac{1}{|\underline{x} - \underline{\omega}|^{n-2}} + \frac{n-2}{r^{n-1}} \Big(\int_0^r \frac{\rho^{n-2}}{|\rho\underline{\xi} - \underline{\omega}|^n} d\rho \Big) \underline{x} \wedge \underline{\omega}. \end{aligned} \tag{6.31}$$

For $r = |\underline{x}| > 1$,

$$\sum_{k=1}^{\infty} \frac{n-2}{k} \frac{1}{r^{n-2+k}} P^{(-k)}(\underline{\omega}^{-1}\underline{\xi}) \tag{6.32}$$
$$= \frac{n-2}{r^{n-2}} \int_r^{\infty} \rho^{n-3} E(\underline{\omega} - \rho\underline{\xi})\overline{\underline{\omega}} d\rho$$
$$= \frac{1}{|\underline{x} - \underline{\omega}|^{n-2}} - \frac{1}{r^{n-2}} - \frac{n-2}{r^{n-1}} \left(\int_r^{\infty} \frac{\rho^{n-2}}{|\rho\underline{\xi} - \underline{\omega}|^n} d\rho \right) \underline{x} \wedge \underline{\omega}.$$

When $r < 1$, we refer the reader to the work of Brackx et al. The proof for $r > 1$ is given in [32].

Similarly, we have (see [32]).

Theorem 6.5.4 *On the unit sphere, the outer Poisson kernel and its Cauchy-type harmonic conjugate are*

$$P^{-}(\underline{x}, \underline{\omega}) = \frac{1}{\sigma_{n-1}} \frac{|\underline{x}|^2 - 1}{|\underline{x} - \underline{\omega}|^n} \tag{6.33}$$

and

$$Q^{-}(\underline{x}, \underline{\omega}) = \frac{1}{\sigma_{n-1}} \left(-\frac{2}{|\underline{x} - \underline{\omega}|^n} + \frac{n-2}{r^{n-1}} \int_0^r \frac{\rho^{n-2}}{|\rho\underline{\xi} - \underline{\omega}|^n} d\rho \right) \underline{x} \wedge \underline{\omega}, \ r > 1, \underline{\xi} \neq \underline{\omega}.$$

For $f \in L^2(S^{n-1})$, $\underline{y} = \underline{\omega} \in \mathbb{S}^{n-1}$ and $\underline{x} = r\underline{\xi}$ with $0 \leqslant r < 1$, we have

$$C^{+} f(\underline{x}) = \sum_{k=0}^{\infty} \frac{|\underline{x}|^k}{\sigma_{n-1}} \int_{\mathbb{S}^{n-1}} C_{n,k}^{+}(\underline{\xi}, \underline{\omega}) f(\underline{\omega}) d\sigma(\underline{\omega}), \tag{6.34}$$

where

$$C_{n,k}^{+}(\underline{\xi}, \underline{\omega}) = \frac{n+k-2}{n-2} C_k^{(n-2)/2}(\langle \underline{\xi}, \underline{\omega} \rangle) + C_{k-1}^{n/2}(\langle \underline{\xi}, \underline{\omega} \rangle) \underline{\xi} \wedge \underline{\omega}, \tag{6.35}$$

and $C_{-1}^{n/2}(\langle \underline{\xi}, \underline{\omega} \rangle) = 0$. In fact, by (6.9), the right hand side of (6.35) is a function of $\underline{\omega}^{-1}\underline{x}$. Hence we can write

$$P^{(k)}(\underline{\omega}^{-1}\underline{x}) = r^k C_{n,k}^{+}(\underline{\xi}, \underline{\omega}), \ k = 0, 1, 2, \ldots.$$

Therefore,

$$C^{+} f(\underline{x}) = \sum_{k=0}^{\infty} \frac{1}{\sigma_{n-1}} \int_{\mathbb{S}^{n-1}} P^{(k)}(\underline{\omega}^{-1}\underline{x}) f(\underline{\omega}) d\sigma(\omega). \tag{6.36}$$

Similar to (6.36), we have

$$C^- f(\underline{x}) = \sum_{k=-1}^{-\infty} \frac{|\underline{x}|^{-n+2-k}}{\sigma_{n-1}} \int_{\mathbb{S}^{n-1}} C^-_{n,|k|-1}(\underline{\xi}, \underline{\omega}) f(\underline{\omega}) d\sigma(\underline{\omega}) \tag{6.37}$$

$$= \sum_{k=-1}^{-\infty} \frac{1}{\sigma_{n-1}} \int_{\mathbb{S}^{n-1}} P^{(k)}(\underline{\omega}^{-1}\underline{x}) f(\underline{\omega}) d\sigma(\underline{\omega}),$$

where

$$C^-_{n,|k|-1}(\underline{\xi}, \underline{\omega}) = \frac{|k|}{n-2} C^{(n-2)/2}_{|k|}(\langle \underline{\xi}, \underline{\omega} \rangle) - C^{n/2}_{|k|-1}(\langle \underline{\xi}, \underline{\omega} \rangle) \underline{\xi} \wedge \underline{\omega}. \tag{6.38}$$

Set

$$P^{(k)}(\underline{\omega}^{-1}\underline{x}) = r^{-n+2-k} C^-_{n,|k|-1}(\underline{\xi}, \underline{\omega}), \ k = -1, -2, \ldots.$$

For the Fourier-Laplace expansion in the sense of L^2, there holds

$$f(\underline{\xi}) = \sum_{k=-\infty}^{\infty} \frac{1}{\sigma_{n-1}} \int_{\mathbb{S}^{m-1}} P^{(k)}(\underline{\omega}^{-1}\underline{\xi}) f(\underline{\omega}) d\sigma(\underline{\omega}). \tag{6.39}$$

This result indicates that the series

$$\mathrm{Sc}\Big[\sum_{k=-\infty}^{\infty} \frac{1}{\sigma_{n-1}} P^{(k)}(\underline{\omega}^{-1}\underline{\xi}) \Big]$$

plays the role of the Dirac-δ function.

Theorem 6.5.5 *The Able sum expansions of the inner Poisson kernel and its Cauchy-type harmonic conjugate are*

$$P^+(\underline{x}, \underline{\omega}) = \frac{1}{\sigma_{n-1}} \sum_{-\infty}^{\infty} r^{|k|} P^{(k)}(\underline{\omega}^{-1}\underline{\xi}), \ \underline{x} = r\underline{\xi}, r < 1, \tag{6.40}$$

and

$$Q^+(r\underline{\xi}, \omega) \tag{6.41}$$

$$= \frac{1}{\sigma_{n-1}} \Big[\sum_{k=1}^{\infty} \frac{k}{n+k-2} r^k P^{(k)}(\underline{\omega}^{-1}\underline{\xi}) - \sum_{k=-\infty}^{-1} r^{|k|} P^{(k)}(\underline{\omega}^{-1}\underline{\xi}) \Big], \ r < 1.$$

Proof We write

$$A^+(r) = \frac{1}{\sigma_{n-1}} \Big[\sum_{k=0}^{\infty} r^k P^{(k)}(\underline{\omega}^{-1}\underline{\xi}) \Big]. \tag{6.42}$$

From the above arguments, we can see that

$$A^+(r) = \frac{1}{2}P^+(r\underline{\xi}, \underline{\omega}) + \frac{1}{2}S^+(r\underline{\xi}, \underline{\omega}) + \frac{1}{2}\widetilde{S}^+(r\underline{\xi}, \underline{\omega}) + \frac{1}{2}Q^+(r\underline{\xi}, \underline{\omega}), \ r < 1, \tag{6.43}$$

where the last three entries are the harmonic representation of a half of the Cauchy singular integral of f, that is, $(1/2)\mathbb{C}f$. Similarly,

$$\begin{aligned} A^-(r) &= \frac{1}{\sigma_{n-1}}\left[\sum_{-\infty}^{-1} P^{(k)}(\underline{\omega}^{-1}\underline{x})\right] \\ &= \frac{1}{2}P^-(r\underline{\xi}, \underline{\omega}) - \frac{1}{2}S^-(r\underline{\xi}, \underline{\omega}) \\ &\quad -\frac{1}{2}\widetilde{S}^-(r\underline{\xi}, \underline{\omega}) - \frac{1}{2}Q^-(r\underline{\xi}, \underline{\omega}), \ r > 1. \end{aligned} \tag{6.44}$$

If the Kelvin inversion of A^- is denoted by $\mathbb{K}(A^-)$, it is easy to verify that $\mathbb{K}(A^-)$ satisfies the following relation:

$$\mathbb{K}(A^-)(r) = \frac{1}{2}P^+(r\underline{\xi}, \underline{\omega}) - \frac{1}{2}S^+(r\underline{\xi}, \underline{\omega}) - \frac{1}{2}\widetilde{S}^+(r\underline{\xi}, \underline{\omega}) - \frac{1}{2}Q^+(r\underline{\xi}, \underline{\omega}), \ r < 1.$$

Hence we obtain

$$P^+(\underline{x}, \underline{\omega}) = A^+(r) + \mathbb{K}(A^-)(r).$$

In the first equality of (6.44), using the Kelvin inversion term by term for the series expansion of A^- and applying (6.42), we get the Abel sum expansion of the Poisson kernel (6.40).

Next we deduce the Abel sum formula of the conjugate Poisson kernel $Q^+(\underline{x}, \underline{\omega})$. In fact, by (6.31) of Lemma 6.5.1, we can see that all entries in (6.43) are of the Abel sum form except for $(1/2)Q^+(r\underline{\xi}, \underline{\omega})$. Therefore,

$$\begin{aligned} \frac{1}{2}Q^+(r\underline{\xi}, \underline{\omega}) &= A^+(r) - \frac{1}{2}P^+(r\underline{\xi}, \underline{\omega}) - \frac{1}{2}\frac{1}{\sigma_{n-1}}\left[\sum_{k=0}^{\infty} r^k \frac{n-2}{n+k-2} P^{(k)}(\underline{\omega}^{-1}\underline{\xi})\right] \\ &= \frac{1}{\sigma_{n-1}}\Big[\sum_{k=0}^{\infty} r^k P^{(k)}(\underline{\omega}^{-1}\underline{\xi}) - \frac{1}{2}\sum_{-\infty}^{\infty} r^{|k|} P^{(k)}(\underline{\omega}^{-1}\underline{\xi}) \\ &\quad -\frac{1}{2}\sum_{k=0}^{\infty} r^k \frac{n-2}{n+k-2} P^{(k)}(\underline{\omega}^{-1}\underline{\xi})\Big] \\ &= \frac{1}{\sigma_{n-1}}\left[\frac{1}{2}\sum_{k=0}^{\infty}\frac{k}{m+k-2} r^k P^{(k)}(\underline{\omega}^{-1}\underline{\xi}) - \frac{1}{2}\sum_{k=-\infty}^{-1} r^{|k|} P^{(k)}(\underline{\omega}^{-1}\underline{\xi})\right]. \end{aligned}$$

Hence we get (6.41). This completes the proof.

Theorem 6.5.6 *The Abel sum expansions of the outer Poisson kernel and its canonical harmonic conjugate are*

$$P^-(\underline{x},\underline{\omega}) = \frac{1}{\sigma_{n-1}}\sum_{-\infty}^{\infty} r^{-|k|-n+2}P^{(k)}(\underline{\omega}^{-1}\underline{\xi}),\ \underline{x} = r\underline{\xi}, r > 1, \tag{6.45}$$

and

$$\begin{aligned} Q^-(\underline{x},\underline{\omega}) = \frac{1}{\sigma_{n-1}}\Big[\sum_{k=1}^{\infty}\frac{1}{r^{n+k-2}}P^{(k)}(\underline{\omega}^{-1}\underline{\xi}) \\ - \sum_{k=-\infty}^{-1}\frac{n+|k|-2}{|k|}\frac{1}{r^{n+|k|-2}}P^{(k)}(\underline{\omega}^{-1}\underline{\xi})\Big] - \widetilde{N}(r\underline{\xi},\underline{\omega}), \end{aligned} \tag{6.46}$$

where $\widetilde{N}$ *is the canonical harmonic conjugate of the double-layer potential* N *outside the unit sphere, where*

$$N(r\underline{\xi}) = \frac{1}{\sigma_{n-1}}\frac{1}{r^{n-2}}$$

and

$$\widetilde{N}(r\underline{\xi},\underline{\omega}) = \frac{1}{\sigma_{n-1}}\frac{n-2}{r^{n-2}}\int_0^{\infty}\frac{\rho^{n-2}}{|\rho\underline{\xi}-\underline{\omega}|^n}d\rho\underline{\xi}\wedge\underline{\omega},\ \text{a.e.} r > 1. \tag{6.47}$$

□

6.6 Remarks

Remark 6.6.1 Let $\mathbb{Q}$ and $\mathbb{Q}^c$ denote the algebras of Hamilton's quaternions over $\mathbb{R}$, the real number field, and $\mathbb{C}$, the complex number field, respectively, with the usual canonical basis, i_0, i_1, i_2, i_3 (i_0 being the identity of $\mathbb{Q}$ which will hence forth be identified with the identity 1 of $\mathbb{R}$), where

$$i_1 i_2 = -i_2 i_1 = i_3,\ i_2 i_3 = -i_3 i_2 = i_1,\ i_3 i_1 = -i_1 i_3 = i_2,$$

and $i_1^2 = i_2^2 = i_3^2 = -1$. A general quaternion is of the form $q = \sum_{l=0}^{3} q_l i_l = q_0 + \underline{q}$, $q_l \in \mathbb{R}$ or $q_l \in \mathbb{C}$, depending on $q \in \mathbb{Q}$ or $q \in \mathbb{Q}^c$, respectively, where q_0 and $\underline{q} = q_1 i_1 + q_2 i_2 + q_3 i_3$ are called the real and imaginary part of q, respectively. In [11], T. Qian established a singular integral theory on star-shaped Lipschitz surfaces in the setting of $\mathbb{Q}$. The results in [11] provides explicit formulas to obtain singular integral kernels from Fourier multipliers and vice versa. Moreover, macroscopically speaking, the theory proves identifications between the following three forms: Fourier multipliers, singular integrals and Cauchy–Dunford's integrals for functional calculi on both the unit sphere and star-shaped Lipschitz surfaces.

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Chapter 7
The Fractional Fourier Multipliers on Lipschitz Curves and Surfaces

The main contents of this chapter are based on some new developments on the holomorphic Fourier multipliers which are obtained by the two authors in recent years, see the author's paper joint with Leong [1] and the joint work [2]. In the above chapters, we state the convolution singular integral operators and the related bounded holomorphic Fourier multipliers on the finite and infinite Lipschitz curves and surfaces. Let $S^c_{\mu,\pm}$ and S^c_{μ} be the regions defined in Sect. 1.1. The multiplier b belongs to the class $H^\infty(S^c_{\mu,\pm})$ defined as

$$H^\infty(S^c_\mu) = \Big\{b:\ S^c_\mu \to \mathbb{C}:\ b_\pm = b\chi_{\{z\in\mathbb{C}:\ \pm\mathrm{Re}z>0\}} \in H^\infty(S^c_{\mu,\pm})\Big\},$$

where $H^\infty(S^c_{\mu,\pm})$ is defined as the set of all holomorphic function b satisfying $|b(z)| \leqslant C_\nu$ in any $S^c_{\nu,\pm},\ 0<\nu<\mu$. A natural question is that whether we can establish the corresponding theory of Fourier multiplier operators if b is dominated by a polynomial?

On the other hand, in new progress of Clifford analysis studies, there exist some examples which can not be included in the theory of singular operator on the Lipschitz graph. We give the following example.

Example 7.0.1 In [3, 4], in order to investigate the so-called Photogenic-Dirac equation which have the singular-valued functional solution, D. Eelbode introduce the Photogenic-Cauchy transform C^α_P on the unit sphere in $\mathbb{R}^n$. To give the definition of this transform, we state some backgrounds on this topic.

Let $\mathbb{R}^{1,n}$ be the real orthogonal space with the orthogonal basis $B_{1,n}(\varepsilon, e_j) = \{\varepsilon, e_1, \ldots, e_n\}$ endowed with the quadratic form

$$Q_{1,n}(T, \underline{X}) = T^2 - \sum_{j=1}^{n} X_j^2 = T^2 - R^2,$$

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where we take

$$R = |\underline{X}| = \left(\sum_{j=1}^{n} X_j^2\right)^{1/2}.$$

The orthogonal space $\mathbb{R}^{1,n}$ is called the m-dimensional space-time, n denotes the spatial dimension. The space-time Clifford algebra $\mathbb{R}_{1,n}$ is generated by the following multiplication rules: for all $1 \leqslant i, j \leqslant n$, $e_i e_j + e_j e_i = -2\delta_{ij}$. For all i and $\varepsilon^2 = 1$, $e_i\varepsilon + \varepsilon e_i = 0$. The vectors in $\mathbb{R}^{1,m}$, i.e., $(m+1)$-tuples $(T, \underline{X})$ or space-time vectors is identified with the 1-vectors in $\mathbb{R}_{1,n}$ under the canonical mapping

$$(T, \underline{X}) = (T, X_1, \ldots, X_n) \longmapsto \varepsilon T + \underline{X} \in \mathbb{R}_{1,n}.$$

The Dirac operator on $\mathbb{R}^{1,n}$ is given by the vector derivative

$$D(T, \underline{X})_{1,n} = \varepsilon\partial_T - \sum_{j=1}^{n} e_j \partial_{X_j},$$

which factorizes the wave operator $\Box_n = \partial_T^2 - \Delta_n$ on $\mathbb{R}^{1,n}$ as

$$\Box_n = \left(\varepsilon\partial_T - \sum_{j=1}^{n} e_j \partial_{X_j}\right)^2.$$

For $\alpha + n \geqslant 0$ and $\underline{\omega} \in \mathbb{S}^{n-1}$, we consider the following Photogenic-Dirac equation

$$(\varepsilon\partial_T - \partial_{\underline{X}})\mathcal{F}_{\alpha,\underline{\omega}}(T, \underline{X}) = T^{\alpha+n-1}\delta(T\underline{\omega} - \underline{X})$$

and take the transformation:

$$\lambda = T \text{ and } \underline{x} = \frac{\underline{X}}{T} = r\underline{\xi} \in B_n(1),$$

where $B_n(1)$ is the unit sphere in $\mathbb{R}^n$ and $|\xi| = 1$. In [3], D. Eelbode proved that

$$\begin{aligned}\mathcal{F}_\alpha(\underline{x}, \underline{\omega}) = &(2\alpha + n + 1)c(\alpha, n)(\varepsilon + \underline{x})\frac{(1 - r^2)^{\alpha+(n-1)/2}}{(1- < \underline{x}, \underline{\omega} >)^{\alpha+n}} \\ &+ (\alpha + n)c(\alpha, n)(\varepsilon + \underline{\omega})\frac{(1 - r^2)^{\alpha+(n+1)/2}}{(1- < \underline{x}, \underline{\omega} >)^{\alpha+n+1}},\end{aligned}$$

where $c(\alpha, n)$ is the constant depending on α and n. In addition, let $f(\underline{\omega})$ be any function defined on the sphere $\mathbb{S}^{n-1}$. For all $\underline{x} \in B_n(1)$, the Photogenic-Cauchy transform of f $C_P^\alpha[f](\underline{x})$ is defined by

$$C_P^\alpha[f](\underline{x}) = \frac{1}{\Omega_n}\int_{\mathbb{S}^{m-1}} \mathcal{F}_\alpha(\underline{x}, \underline{\omega})\underline{\omega} f(\omega)d\omega,$$

where Ω_n is the surface area of the sphere $\mathbb{S}^{n-1}$.

If we apply this transform C_P^α to the inner and outer spherical monogenic polynomials P_k and Q_k on $\mathbb{R}^n \setminus \{0\}$ and let $r \to 1-$, we can obtain the boundary values $C_P^\alpha[P_k] \uparrow$ and $C_P^\alpha[Q_k] \uparrow$ as follows:

$$C_P^\alpha[P_k] \uparrow (\underline{\xi}) = \frac{\Gamma(n/2-1/2)}{8\pi^{n/1-1/2}} \frac{(\alpha+n+k)\{(\alpha+n+k-1)+(k-\alpha)\underline{\xi}\varepsilon\}P_k(\underline{\xi})}{(\alpha+n/2+1/2)(\alpha+n/2-1/2)},$$
$$C_P^\alpha[Q_k] \uparrow (\underline{\xi}) = \frac{\Gamma(n/2-1/2)}{8\pi^{n/1-1/2}} \frac{(1+\alpha-k)\{(\alpha-k)+(\alpha+n+k-1)\underline{\xi}\varepsilon\}Q_k(\underline{\xi})}{(\alpha+n/2+1/2)(\alpha+n/2-1/2)}.$$

It is obvious that the occurrence of

$$k^2P_k(\underline{\xi}),\ kP_k(\underline{\xi}),\ k^2Q_k(\underline{\xi}),\ kQ_k(\underline{\xi})$$

indicates that for $f \in L^2(\mathbb{S}^{n-1})$, the boundary value $C_P^\alpha[f] \uparrow$ does not belong to $L^2(\mathbb{S}^{n-1})$. Hence, in order to obtain the boundedness of this operator, we need to restrict f into a space smaller that $L^2(\mathbb{S}^{n-1})$. In [3], the author replaced $L^2(\mathbb{S}^{n-1})$ by a special Sobolev space and obtained the boundedness of $C_P^\alpha[f] \uparrow$. Based on the above result, in this chapter, we consider the Fourier multiplier b satisfying $|b(\xi)| \leqslant C|\xi+1|^s$ in some region for $s \neq 0$ and study the boundedness of the integral operators associated with these multipliers.

Remark 7.0.2 Particularly, if we take some special b_k in the definition of the Fourier multiplier (see Definition 7.3.2 and the remark below), we can see that the multiplier operator becomes the boundary value of the Cauchy transform on the hyperbolic sphere which was studied in [3, 4].

Compared with the Photogenic-Cauchy transform in Example 7.0.1, there exist two difficulties for the study of Fourier multipliers:

(1) The kernel $\mathcal{F}_\alpha(\underline{x}, \underline{\omega})$ of the Cauchy transform C_P^α can be derived from the fundamental solution of the wave operator $\Box_n$, while the kernel of the Fourier multiplier does not have an explicit expression.
(2) On the unit sphere in $\mathbb{R}^n$, the Plancherel theorem holds. After obtaining the decomposition of $C_P^\alpha(f)$ with respect to the spherical harmonics, the author of [3] can deduce easily that if f belongs to some Sobolev spaces, the function $C_P^\alpha(f)$ belongs to $L^2(\mathbb{S}^{n-1})$. However, in the case of Lipschitz surfaces, there is no corresponding Plancherel theorem, and the method of [3] is invalid.

To overcome the above difficulties, we use the Fueter theorem to estimate the kernel of the multiplier operator. We prove that the kernel of the Fourier multiplier operator has a decay with the form of a polynomial of degree $-(n+s)$. The proof is similar to that of Chap. 6 but with some modifications. When we deal with the case $s < 0$, the function $|x|^s$ is unbounded in the domain $H_{\omega,+}$. After getting the estimate of the kernel on $H_{\omega,-}$, we can not use the Kelvin inversion to obtain the corresponding estimate on $H_{\omega,+}$, see Theorem 7.2.2 for details.

7.1 The Fractional Fourier Multipliers on Lipschitz Curves

In this section, we generalize the results in Chaps. 1 and 2 to the following cases: $|b_n| \leqslant Cn^s$, $-\infty < s < \infty$. Such result corresponds to the fractional integrations and differentials on the closed Lipschitz curve and has a closed relation with the boundary value problem on Lipschitz domains.

We still use the following sets in the complex plane $\mathbb{C}$. For $\omega \in (0, \pi/2]$, write

$$S_{\omega,\pm} = \left\{ z \in \mathbb{C} : \ |\arg(\pm z)| < \omega \right\}$$

as the sets defined in Definition 1.2.1. Define the sets

$$W_{\omega,\pm} = \left\{ z \in \mathbb{Z} : \ |\mathrm{Re}(z)| \leqslant \pi \text{ and } \mathrm{Im}(\pm z) > 0 \right\} \cup S_{\omega},$$

see the following graph (Figs. 7.1 and 7.2):

The periodization of $W_{\omega,\pm}$ is the following heart shaped regions:

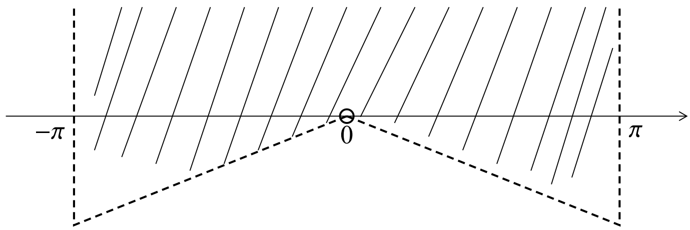


Fig. 7.1 $W_{\omega,+}$

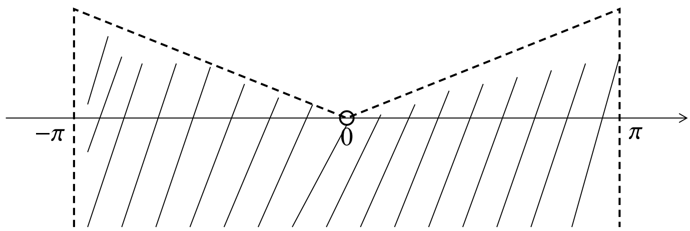


Fig. 7.2 $W_{\omega,-}$

Fig. 7.3 $C_{\omega,+}$

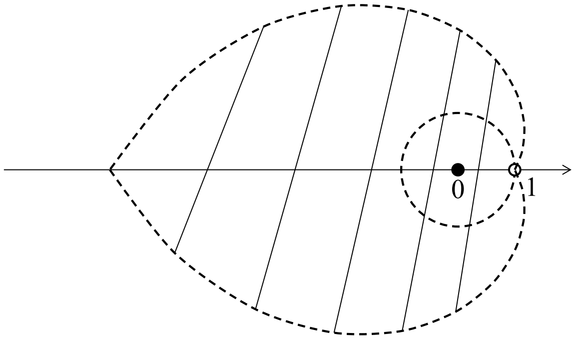
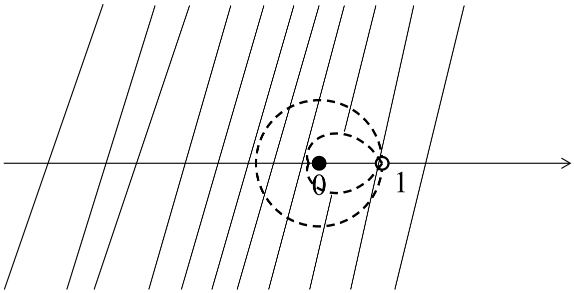


Fig. 7.4 $C_{\omega,-}$



$$C_{\omega,\pm} = \Big\{ z = \exp(i\eta) \in \mathbb{C} : \ \eta \in W_{\omega,\pm} \Big\},$$

which are shown in the following figure (Figs. 7.3 and 7.4):

Define

$$S_\omega = S_{\omega,+} \cup S_{\omega,-},$$

$$W_\omega = W_{\omega,+} \cap W_{\omega,-},$$

and

$$C_\omega = C_{\omega,+} \cap C_{\omega,-}.$$

Let O be a set in the complex plane. If $rz \in O$ for $z \in O$ and all $0 < r \leqslant 1$, we call O the inner starlike region with the pole zero. If $rz \in O$ for $z \in O$ and all $1 \leqslant r < \infty$, we call O the outer starlike region with the pole zero. For $\omega \in (0, \pi/2]$, $C_{\omega,+}$ is heart-shaped and inner starlike with the pole zero, while $C_{\omega,-}$ can be regarded as the complement of a heart shaped region and an outer starlike region with pole zero.

The following function spaces defined on the sectors will be used in the rest of this section. For $-\infty < s < \infty$,

$$H^s(S_{\omega,\pm}) = \Big\{b: \ S_{\omega,\pm} \to \mathbb{C} \mid b \text{ is holomorphic and satisfies}$$
$$|b(z)| \leqslant C_\mu |z \pm 1|^s \text{ in every } \ S_{\mu,\pm}, 0 < \mu < \omega\Big\}.$$

For $s = -1, -2, \ldots$, we will also use another class of function spaces.

$$H^s_{\ln}(S_{\omega,\pm}) = \Big\{b: \ S_{\omega,\pm} \to \mathbb{C} \mid b \text{ are holomorphic and satisfies}$$
$$|b(z)| \leqslant C_\mu |z \pm 2|^s \ln |z \pm 2| \text{ in every } \ S_{\mu,\pm}, 0 < \mu < \omega\Big\}.$$

On the double sectors, we can define the corresponding function spaces. For $-\infty < s < \infty$,

$$H^s(S_\omega) = \Big\{b : S_\omega \to \mathbb{C} \mid b_\pm \in H^s(S_{\omega,\pm}), \text{ where } \ b_\pm = b\chi_{\{z\in\mathbb{C}, \pm\mathrm{Re}z>0\}}\Big\}$$

and

$$H^s_{\ln}(S_\omega) = \Big\{b : S_\omega \to \mathbb{C} \mid b_\pm \in H^s_{\ln}(S_{\omega,\pm}), \text{ where } \ b_\pm = b\chi_{\{z\in\mathbb{C}, \pm\mathrm{Re}z>0\}}\Big\},$$

where χ_E denotes the characteristic function of the set E.

Hence, the function spaces $H^s(S_\omega)$ and $H^s_{\ln}(S_\omega)$ defined above consist of the functions on sectors which are bounded near zero and dominated by $C_\mu|z|^s$ and $C_\mu|z|^s \ln|z|$ at infinity in any smaller sectors than those in which the functions are holomorphically defined.

If a function defined by the Laurent series converges to a holomorphic function in a region, then this function is called holomorphically defined. In this case, by the Abel theorem, the power series part is holomorphically defined in the related inner starlike region with the pole zero. The negative power series part is holomorphically defined in the related outer starlike region with the pole zero.

For $s > -1$, define

$$K^s(C_{\omega,\pm}) = \Big\{\phi : C_{\omega,\pm} \to \mathbb{C} \mid \phi \text{ is holomorphic and satisfies}$$
$$|\phi(z)| \leqslant \frac{C_\mu}{|1-z|^{1+s}} \text{ in any } C_{\mu,\pm}, 0 < \mu < \omega\Big\}$$

and

$$K^s(C_\omega) = \Big\{\phi : C_\omega \to \mathbb{C} \mid \phi \text{ is holomorphic and satisfies}$$
$$|\phi(z)| \leqslant \frac{C_\mu}{|1-z|^{1+s}} \text{ in any } C_{\mu,\pm}, 0 < \mu < \omega\Big\}.$$

For $-\infty < s \leqslant -1$, we only give the definition of $K^s(C_{\omega,+})$. For $-\infty < s \leqslant -1$, the spaces $K^s(C_{\omega,-})$ and $K^s(S_\omega)$ can be defined similarly. Assume

(i) $\underline{b} = \{b_n\}_{n=0}^{\infty} \in l^\infty$;

(ii) $\phi_{\underline{b}}(z) = \sum\limits_{n=0}^{\infty} b_n z^n$ is holomorphically defined in $C_{\omega,+}$;

(iii) The series $\phi_{\underline{b}}(1) = \sum\limits_{n=0}^{\infty} b_n$ is convergent.

Form the difference

$$\phi_{\underline{b}}(z) - \phi_{\underline{b}}(1) = b_1(z-1) + b_2(z^2-1) + \cdots + b_n(z^n-1) + \cdots + (z-1)\phi_{I(\underline{b})}(z),$$

where

$$I(\underline{b}) = \left(\sum_{k=n}^{\infty} b_k\right)_{n=1}^{\infty} \in l^\infty$$

and

$$\phi_{I(\underline{b})}(z) = \sum_{n=1}^{\infty}\left(\sum_{k=n}^{\infty} b_k\right) z^{n-1}.$$

Then by (ii), $\phi_{I(\underline{b})}$ is holomorphic in $C_{\omega,+}$.

The sequence $I(b)$ constructed above may or may not satisfy the condition (iii). If this sequence satisfies (iii), then it satisfies (i) automatically. Hence $(I(\underline{b}), \phi_{I(\underline{b})})$ satisfies (i), (ii) and (iii). Then we continue to consider if the sequence $I(I(\underline{b})) = I^2(\underline{b})$ satisfies (iii), and so on. Write

$$I(I^n(\underline{b})) = I^{n+1}(\underline{b}) \text{ and } I^0(\underline{b}) = \underline{b}.$$

If the above procedure can be applied at most k times, then the pairs

$$(I^j(\underline{b}),\ \phi_{I^j(\underline{b})}),\ 0 \leqslant j \leqslant k,$$

all satisfy (i), (ii) and (iii), but $I^{k+1}(\underline{b})$ does not satisfy (iii). In this case, we have

$$\phi_{\underline{b}}(z) = \phi_{\underline{b}}(1) + (z-1)\phi_{I(\underline{b})}(1) + \cdots + (z-1)^k \phi_{I^k(\underline{b})}(z). \tag{7.1}$$

Now we begin to define the function class $K^s(C_{\omega,+})$, $-\infty < s \leqslant -1$:

$$\begin{aligned} K^s(C_{\omega,+}) = \Big\{\phi_{\underline{b}} :\ & C_{\omega,+} \to \mathbb{C} \mid \underline{b} \in l^\infty,\ \text{the above procedure can be applied at most } k_s \text{ times},\\ & \text{where } k_s = [1-s] \text{ or } [-s] \text{ depending on whether } s \text{ is an integer or not},\\ & \text{and in any } C_{\mu,+}, 0 < \mu < \omega,\ |(z-1)^{k_s}\phi_{I^{k_s}(\underline{b})}(z)| \leqslant \frac{C_\mu}{|z-1|^{1+s}}\Big\}, \end{aligned}$$

where for $\alpha > 0$, $[\alpha]$ denotes the largest integer which does not exceed α, that is, $[\alpha] = \max\{n \in \mathbb{Z} \mid n \leqslant \alpha\}$.

For $s = -1, -2, \ldots$, we consider another class of functions

$$K^s_{\ln}(C_{\omega,+}) = \Big\{\phi_{\underline{b}} : C_{\omega,+} \to \mathbb{C} \mid \underline{b} \in l^\infty, \text{ the above procedure can be applied at most } -s-1 \text{ times, and in any } C_{\mu,+}, 0 < \mu < \omega,\ |(z-1)^{-s-1}\phi_{I^{-s-1}(\underline{b})}(z)| \leqslant C\frac{|\ln|z-1||}{|z-1|^{1+s}}\Big\}.$$

It is easy to see that the above spaces $\{H^s(S_{\omega,\pm})\}$ and $\{K^s(C_{\omega,\pm})\}$ are increasing classes along with $s \to \infty$. Now we state the main results of this section. In the rest of this section, the symbol "$\pm$" should be understood as either all $+$ or all $-$.

Theorem 7.1.1 *Let $-\infty < s < \infty$, $s \neq -1, -2, \ldots$, $b \in H^s(S_{\omega,\pm})$, and $\phi(z) = \sum_{n=\pm 1}^{\pm\infty} b(n)z^n$. Then $\phi \in K^s(C_{\omega,\pm})$.*

Proof We first consider the case $0 \leqslant s < \infty$. Define

$$\Psi(z) = \frac{1}{2\pi}\int_{\rho_\theta} \exp(iz\zeta)b(\zeta)d\zeta,\ z \in V_{\omega,+},$$

where

$$V_{\omega,+} = \Big\{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\Big\} \cup S_\omega$$

and ρ_θ denotes the ray: $r\exp(i\theta)$, $0 < r < \infty$. Here θ satisfies $\rho_\theta \in S_{\omega,+}$ and $\exp(iz\zeta)$ is exponentially decaying as $\zeta \to \infty$ along ρ_θ. It is easy to see that Ψ is well defined and holomorphic in $V_{\omega,+}$. In fact, the definition of Ψ is independent of the choice of θ. For any $\mu \in (0, \omega)$, we can see that

$$|\Psi(z)| \leqslant \frac{C_\mu}{|z|^{1+s}},\ z \in V_{\mu,+}.$$

We further define function

$$\Psi^1(z) = \int_{\delta(z)} \Psi(\zeta)d\zeta,\ z \in S_{\omega,+},$$

where $\delta(z)$ is any path from $-z$ to z in V_ω. It follows from Cauchy's formula that for any $\mu \in (0, \omega)$,

$$|\Psi^1(z)| \leqslant \frac{C_\mu}{|z|^s},\ z \in S_{\mu,+}.$$

By the Poisson summation formula, define

$$\psi(z) = 2\pi\sum_{n=-\infty}^{\infty} \Psi(z + 2n\pi),\ z \in \bigcup_{n=-\infty}^{\infty}(2n\pi + W_{\omega,+}),$$

where $\sum$ denotes the summation in the following sense:

(i) for $s > 0$, the series absolutely and locally uniformly converges to a 2π-periodic holomorphic function ψ, and the function $\phi = \psi \circ \ln /i \in K^s(C_{\omega,+})$;
(ii) for $s = 0$, there exists a subsequence $\{n_k\}_1^{\infty}$ such that the partial sum

$$s_{n_k}(z) = 2\pi \sum_{|n|\leqslant n_k} \Psi(z + 2n\pi)$$

locally uniformly converges to a 2π-periodic function ψ, and $\phi = \psi \circ \ln /i \in K^s(C_{\omega,+})$.

It can be proved that different functions Ψ defined via different subsequences $\{n_k\}$ differ by bounded constants. By use of the estimate of Ψ, it is easy to prove the case $s > 0$.

Now we consider the case $s = 0$. Consider the decomposition

$$\sum_{k=-n}^{n} \Psi(z + 2k\pi) = \Psi(z) + \sum_1 + \sum_2, \ z \in W_{\mu,+},$$

where

$$\sum_1 =: \sum_{k\neq 0}^{\pm n} \Big(\Psi(z + 2k\pi) - \Psi(2k\pi)\Big)$$

and

$$\sum_2 =: \sum_{k=1}^{n} (\Psi^1)'(2k\pi).$$

We will prove that $\sum_1$ is absolutely convergent and bounded, and $\sum_2$ is bounded and convergent in the sense mentioned above. Hence, as the principal part of the sum, $\Psi(z)$ is dominated by $C|z|^{-1}$ as $z \to 0$ and so is the function ψ. Therefore, the function $\phi = \psi \circ \ln /i$ satisfies the desired estimate. To deal with $\sum_1$, we need the following formula derived by Cauchy's formula:

$$|\Psi'(z)| \leqslant \frac{C_\mu}{|z|^{2+s}}, \ z \in W_{\mu,+}.$$

To deal with $\sum_2$, by the mean value theorem, we obtain

$$\begin{aligned}
&\sum_{k=1}^{n}(\Psi^1)'(2k\pi)\\
&=\left[\int_{2\pi}^{2(n+1)\pi}(\Psi^1)'(r)dr+\sum_{k=1}^{n}(\Psi^1)'(2k\pi)-\mathrm{Re}((\Psi^1)')(\xi_k)-i\mathrm{Im}((\Psi^1)')(\eta_k)\right]\\
&=\Psi^1(2(n+1)\pi)-\Psi^1(2\pi)+\sum_{k=1}\Big[(\Psi^1)'(2k\pi)-\mathrm{Re}((\Psi^1)')(\xi_k)-i\mathrm{Im}((\Psi^1)')(\eta_k)\Big],
\end{aligned}$$

where $\xi_k,\ \eta_k \in (2k\pi,\ 2(k+1)\pi)$. Then by the estimate of Ψ', the series part in the above expression is absolutely convergent. Because that part is bounded, by choosing a suitable subsequence $\{n_k\}$, we conclude that the part converges to a constant with the same bounds. This completes the proof of the case $s = 0$.

For the case $s < 0$, we apply induction to the interval $-k-1 \leqslant s < -k$, where $k \geqslant 0$ is an integer. We first consider $-1 < s < 0$. Let $b \in H^s(S_{\omega,+})$ and

$$\phi(z) = \sum_{n=1}^{\infty} b(n)z^n, \quad \phi_0(z) = \sum_{n=1}^{\infty} nb(n)z^n, z\phi'(z) = \phi_0(z).$$

Because $b \in H^s(S_{\omega,+})$, we have $(\cdot)b(\cdot) \in H^{s+1}(S_{\omega,+})$, where $0 < s+1 < 1$. As proved above, we get $\phi_0 \in K^{s+1}(C_{\omega,+})$, and the series ϕ_0 locally uniformly converges. This fact enables us to integrate the series $\phi_0(z)/z$ term by term. Notice that the region $C_{\omega,+}$ is starlike. Denote by $l(0,z)$ the segment from 0 to $1 \approx z = x+iy \in C_{\mu,+}$. By the estimate of the functions in $K^{s+1}(C_{\omega,+})$, we obtain

$$\begin{aligned}
|\phi(z)| &\leqslant \int_{l(0,z)} \left|\frac{\phi_0(\zeta)}{\zeta}\right||d\zeta|\\
&\leqslant C_\mu \int_{l(0,z)} \frac{|d\zeta|}{|1-\zeta|^{s+2}}\\
&\leqslant C_\mu \int_0^1 \frac{dt}{(|1-tx|+t|y|)^{s+2}}.
\end{aligned}$$

To complete the proof, we divide the rest of the proof into two cases: $x \leqslant 1$ and $x > 1$. For $x \leqslant 1$, the above estimate becomes

$$\begin{aligned}
\left|\int_0^1 \frac{dt}{(1-t(x-|y|))^{s+2}}\right| &= \frac{1}{s+1}\frac{1}{x-|y|}\left[\frac{1}{(|1-x|+|y|)^{s+1}}-1\right]\\
&\leqslant \frac{C_{\mu,s}}{|1-z|^{s+1}},
\end{aligned}$$

where we used the condition that $z \approx 1 \Longrightarrow x \approx 1, y \approx 0$.

For $x > 1$, because z belongs to the starlike region $C_{\mu,+}$, we can deduce that

$$x-1 = |1-x| \leqslant (\tan(\mu))|y|$$

and

$$|y| \geqslant C_\mu(|1-x|+|y|).$$

This fact together with $x \approx 1$ and $y \approx 0$ implies

$$\begin{aligned}
&\int_0^1 \frac{dt}{(|1-tx|+t|y|)^{s+2}} \\
&= \int_0^{1/x} \frac{dt}{(1-t(x-|y|))^{s+2}} + \int_{1/t}^1 \frac{dt}{(t(x+|y|)-1)^{s+2}} \\
&= \frac{1}{s+1}\left[\frac{2x}{x^2-y^2}\frac{x^{s+1}}{|y|^{s+1}} + \frac{1}{x+|y|}\frac{1}{(|1-x|+|y|)^{s+1}} - \frac{1}{x-|y|}\right] \\
&\leqslant \frac{C_\mu}{|1-z|^{s+1}}.
\end{aligned}$$

For $s=-1$, by using the result of the case $s=0$, we can apply a similar argument to obtain

$$|\phi(z)| \leqslant C_\mu \int_{l(0,z)} \frac{1}{|1-\zeta|}|d\zeta| \leqslant C_\mu |\ln|1-z||,$$

where $z \in C_{\mu,+}$.

This completes the proof for the case $-1 \leqslant s < 0$. Below we use induction to the index s :

Let $-k-1 \leqslant s < -k$, where $k \geqslant 0$ is an integer, and let $b \in H_\omega^s$. We define $\underline{b} = \{b(n)\}_{n=1}^\infty$ and get $\phi_{\underline{b}} \in K^s(C_{\omega,+})$.

Now we consider the case $-k-2 \leqslant s < -k-1$, where $k \geqslant 0$ is an integer and $b \in H^s(S_{\omega,+})$. Set

$$\begin{cases} \phi(z) = \sum\limits_{n=1}^\infty b(n)z^n, \\ \phi_0(z) = \sum\limits_{n=1}^\infty b_0(n)z^n, \end{cases}$$

where $b_0(z) = \sum\limits_{n=0}^\infty b(z+n)$. It is easy to see that $b_0 \in H^{s+1}(S_{\omega,+})$. Because $-k-1 \leqslant s+1 < -k$, by induction, we can obtain that $\phi_0 \in K_\omega^{s+1}$. Hence, if s is an integer, $\phi_{I^{[-s-2]}(\underline{b}_0)}$ can be extended to $C_{\omega,+}$ holomorphically. If s is not an integer, $\phi_{I^{[-s-1]}(\underline{b}_0)}$ can be extended to $C_{\omega,+}$ holomorphically. Here $\underline{b}_0 = \{b_0(n)\}_{n=1}^\infty$. In both cases, for $z \in C_{\mu,+}$, we have

$$|(z-1)^{[-s-2]}\phi_{I^{[-s-2]}(\underline{b}_0)}(z)| \leqslant C_\mu \frac{|\ln|z-1||}{|z-1|^{s+2}}$$

or

$$|(z-1)^{[-s-1]}\phi_{I^{[-s-1]}(\underline{b}_0)}(z)| \leqslant \frac{C_\mu}{|z-1|^{s+2}}.$$

Because $I^k \underline{b}_0 = I^{k+1}\underline{b}$ for any $k \to 0$, we have $\phi_{I^k(\underline{b}_0)} = \phi_{I^{k+1}(\underline{b})}$. When s is an integer,

$$|(z-1)^{[-s-1]}\phi_{I^{[-s-1]}(\underline{b})}(z)| \leqslant C_\mu \frac{|\ln|z-1||}{|z-1|^{s+1}}.$$

If s is not an integer,

$$|(z-1)^{[-s]}\phi_{I^{[-s]}(\underline{b})}(z)| \leqslant \frac{C_\mu}{|z-1|^{s+1}}.$$

This proves $\phi \in K_\omega^s$ for $b \in H_\omega^s$, $-k-2 \leqslant s < -k-1$. □

The cases "+" and "−" in Theorem 7.1.1 are associated with power series and negative power series, respectively. By these results, we obtain the result corresponding to the Laurent series.

Corollary 7.1.1 *Let* $-\infty < s < \infty$, $s \neq -1, -2, \ldots$, $b \in H^s(S_\omega)$ *and*

$$\phi(z) = \sum_{n=-\infty}^{\infty} b(n)z^n.$$

Then $\phi \in K^s(C_\omega)$.

The inverse of Theorem 7.1.1 is the following.

Theorem 7.1.2 *Let* $-\infty < s < \infty$ *and* $\phi \in K^s(C_{\omega,\pm})$. *Then for any* $\mu \in (0, \omega)$, *there exists a function* $b^\mu \in H^s(S_{\mu,\pm})$ *such that*

$$\phi(z) = \sum_{n=\pm 1}^{\pm\infty} b^\mu(n)z^n.$$

Moreover, for $s < 0$ *and* $z \in S_{\mu,\pm}^c$,

$$b^\mu(z) = \frac{1}{2\pi}\int_{\lambda_\pm(\mu)} \exp(-i\eta z)\phi(\exp(i\eta))d\eta, \tag{7.2}$$

where

$$\lambda_\pm(\mu) = \Big\{\eta \in H_{\omega,\pm}^c \mid \eta = r\exp(i(\pi \pm \mu)),\ r \textit{ is from } \pi\sec\mu \textit{ to } 0; \\ \textit{and } \eta = r\exp(\mp i\mu),\ r \textit{ is from } 0 \textit{ to } \pi\sec\mu\Big\}$$

and for $s \geqslant 0$, $z \in S_{\mu,\pm}^c$,

$$b^\mu(z) = \frac{1}{2\pi}\lim_{\varepsilon\to 0}\left(\int_{l(\varepsilon,|z|^{-1})\cup c_\pm(|z|^{-1},\mu)\cup\Lambda_\pm(|z^{-1}|,\mu)} \exp(-i\eta z)\phi(\exp(i\eta))d\eta + \phi_{\varepsilon,\pm}^{|s|}(z)\right),$$

where if $r \leqslant \pi$*,*

$$l(\varepsilon, r) = \left\{\eta = x + iy \mid y = 0, x \textit{ is from } -r \textit{ to } -\varepsilon, \textit{ and from } \varepsilon \textit{ to } r\right\},$$

$$c_{\pm}(r, \mu) = \left\{\eta = r\exp(i\alpha) \mid \alpha \textit{ from } \pi \pm \mu \textit{ to } \pi, \textit{ then from } 0 \textit{ to } \mp \mu\right\},$$

and

$$\Lambda_{\pm}(r, \mu) = \Big\{\eta \in W_{\omega,\pm} \mid \eta = \rho\exp(i(\pi \pm \mu)), \rho \textit{ is from } \pi \sec\mu \textit{ to } r; \textit{ and } \eta = \rho\exp(\mp i\mu), \rho \textit{ from } r \textit{ to } \pi \sec\mu\Big\};$$

If $r > \pi$*,*

$$l(\varepsilon, r) = l(\varepsilon, \pi), \quad c_{\pm}(r, \mu) = c_{\pm}(\pi, \mu)$$
$$\Lambda_{\pm}(r, \mu) = \Lambda_{\pm}(\pi, \mu).$$

In any case,

$$\phi^{[s]}_{\varepsilon,\pm}(z) = \int_{L_{\pm}(\varepsilon)} \phi(\exp(i\eta))\left(1 + (-i\eta z) + \cdots + \frac{(-i\eta z)^{[s]}}{[s]!}\right)d\eta,$$

where $L_{\pm}(\varepsilon)$ *is any contour from* $-\varepsilon$ *to* ε *in* $C_{\omega,\pm}$*.*

Proof Let $\phi \in K^s(C_{\omega,+})$, $-\infty < s < \infty$. We will apply (7.2) or (7.3) to prove that b^{μ} defined above belongs to $H^s(C_{\mu,+})$, and $\phi(z) = \sum_{n=1}^{\infty} b^{\mu}(n)z^n$.

We first consider the case $-\infty < s < 0$. By the expressions (7.2) and (7.1), using the estimate of the function ϕ and Cauchy's theorem, we can prove

$$\lim_{z\to 0} b^{\mu}(z) = \frac{1}{2\pi}\int_{\lambda(\mu)} \exp(i\eta z)\phi(\exp(i\eta))d\eta, \ z \in S_{\mu,+},$$

where

$$\lambda(\mu) = \Big\{\eta \in W_{\omega,+} \mid \eta = r\exp(i(\pi + \mu)), \ r \text{ is from } \pi\sec(\mu) \text{ to } 0, \text{ and } \eta = r\exp(-i\mu), \ r \text{ from } 0 \text{ to } \pi\sec(\mu)\Big\},$$

where $|\arg(z)| < \mu < \omega$. Let $|\arg(z)| < \theta < \mu$. By the estimate of ϕ and the property of the path $\lambda(\mu)$, the function b^{μ} satisfies the following estimate (Fig. 7.5):

$$|b^{\mu}(z)| \leqslant C_{\mu}\left(|z|^s + \int_0^{\infty} \exp(-\sin(\mu - \theta)|z|r)\frac{dr}{r^{1+s}}\right) \leqslant C_{\mu,\theta}|z|^s.$$

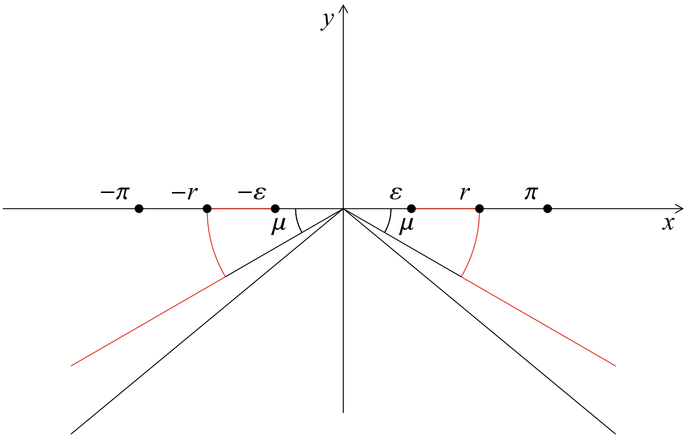


Fig. 7.5 $l_+(\varepsilon, r) \cup c_+(r, \mu) \cup \Lambda_+(r, \mu)$

Now we consider the case $0 \leqslant s < \infty$. By (7.2), without loss of generality, as $z \approx \infty$, assume that $|z|^{-1} \leqslant \pi$. We have

$$\begin{aligned} b^{\mu}(z) = & \frac{1}{2\pi} \lim_{\epsilon \to 0} \left\{ \left(\int_{\epsilon \leqslant |t| \leqslant |z|^{-1}} \exp(-itz)\phi(\exp(it))dt + \phi_{\epsilon}^{[s]}(z) \right) \right. \\ & + \int_{c_+(|z|^{-1}, \mu)} \exp(-i\eta z)\phi(\exp(i\eta))d\eta \\ & \left. + \int_{\Lambda_+(|z|^{-1}, \mu)} \exp(-i\eta z)\phi(\exp(i\eta))d\eta \right\} \\ = & \frac{1}{2\pi} \lim_{\epsilon \to 0} \left\{ I_1(\epsilon, z) + I_2(z, \mu) + I_3(z, \mu) \right\}, \end{aligned}$$

where $|\arg(z)| < \mu < \omega$,

$$c_+(r, \mu) = \left\{ \eta = r\exp(i\alpha) \mid \alpha \text{ is from } \pi + \mu \text{ to } \pi, \text{ and from } 0 \text{ to } -\mu \right\},$$

and

$$\begin{aligned} \Lambda_+(r, \mu) = \Big\{ & \eta \in W_{\omega,+} \mid \eta = \rho\exp(i(\pi + \mu)), \ \rho \text{ is from } \pi\sec(\mu) \text{ to } r, \\ & \text{and } \eta = \rho\exp(-i\mu), \ \rho \text{ is from } r \text{ to } \pi\sec(\mu) \Big\}. \end{aligned}$$

Now we prove that I_1, I_2, I_3 are uniformly dominated by the bounds indicated in the theorem, and the limit $\lim_{\epsilon \to 0} I_1$ exists.

By Cauchy's theorem, we have

$$\begin{aligned}
I_1(\epsilon, z) &= \int_{\epsilon\leqslant|t|\leqslant|z|^{-1}} \left(\exp(-itz) - 1 - \frac{(-itz)}{1!} - \cdots - \frac{(-itz)^{[s]}}{[s]!}\right)\phi(\exp(it))dt \\
&\quad + \int_{\epsilon\leqslant|t|\leqslant|z|^{-1}} \left(1 + \frac{(-itz)}{1!} + \cdots + \frac{(-itz)^{[s]}}{[s]!}\right)\phi(\exp(it))dt + \phi^{[s]}_{\epsilon,+}(z) \\
&= \int_{\epsilon\leqslant|t|\leqslant|z|^{-1}} \left(\exp(-itz) - 1 - \frac{(-itz)}{1!} - \cdots - \frac{(-itz)^{[s]}}{[s]!}\right)\phi(\exp(it))dt \\
&\quad + \phi^{[s]}_{|z|^{-1},+}(z).
\end{aligned}$$

Invoking the estimate of ϕ, we obtain

$$\begin{aligned}
&\left|\int_{\epsilon\leqslant|t|\leqslant|z|^{-1}} \left[\exp(-itz) - 1 - \frac{(-itz)}{1!} - \cdots - \frac{(-itz)^{[s]}}{[s]!}\right]\phi(\exp(it))dt\right| \\
&\leqslant C_\mu \int_{\epsilon\leqslant|t|\leqslant|z|^{-1}} |t|^{[s]+1}|z|^{[s]+1}\frac{1}{|t|^{1+s}}dt \\
&\leqslant C_\mu |z|^{[s]+1}\int_0^{|z|^{-1}} t^{[s]-s}dt \\
&= C_\mu |z|^s.
\end{aligned}$$

The above argument implies that $\lim_{\epsilon\to 0} I_1$ exists.

To estimate $\phi^{[s]}_{|z|^{-1},+}(z)$, we only need to estimate the integral

$$\int_{L_\pm(|z|^{-1})} \frac{(-i\eta z)^k}{k!}\phi(\exp(i\eta))d\eta,\ k = 0, 1, \ldots, [s]. \tag{7.3}$$

Taking the contour $L_+(|z|^{-1})$ as the upper half circle centered at 0 with radius $|z|^{-1}$, we get

$$\begin{aligned}
\left|\int_{L_+(|z|^{-1})} \frac{(-i\eta z)^k}{k!}\phi(\exp(i\eta))d\eta\right| &\leqslant C_\mu \int_{L_+(|z|^{-1})} |\eta z|^k |\eta|^{-1-s}|d\eta| \\
&\leqslant C_\mu |z|^s.
\end{aligned}$$

To estimate I_2, we have

$$|I_2(z, \mu)| \leqslant C_\mu \int_0^\mu \exp\Big(|\eta||z|\sin(\arg(z) + t)\Big)|\eta|\frac{dt}{|\eta|^{1+s}} \leqslant C_\mu |z|^s.$$

Now we consider I_3. Letting $|\arg(z)| < \theta < \mu$, we get

$$\begin{aligned}|I_3(z,\mu)| &\leqslant C_\mu \int_{\Lambda(|z|^{-1},\mu)} \exp(|\eta||z|\sin(\mu-\theta))\frac{|d\eta|}{|\eta|^{1+s}}\\ &\leqslant C_\mu \int_{|z|^{-1}}^{\infty} r^{-1-s}\exp(-r|z|\sin(\mu-\theta))dr\\ &\leqslant C_{\mu,\theta}|z|^s.\end{aligned}$$

For $z \approx 0$, assume that $|z|^{-1} > \pi$. We first prove that the integral on the contour $l(\epsilon,\pi)$ is uniformly bounded and has limit as $\epsilon \to 0$. Except that the contour in (7.3) should be replaced by $L_+(\pi)$, the argument dealing with $I_1(\epsilon,z)$ for $|z|^{-1} \leqslant \pi$ still applies to the integral on $l(\epsilon,\pi)$. Let the contour $L_+(\pi)$ be the upper half circle centered at 0 with radius π. We have

$$\begin{aligned}\left|\int_{L_+(\pi)} \frac{(-\eta z)^k}{k!}\phi(\exp(i\eta))d\eta\right| &\leqslant C_\mu \int_{L_+(\pi)} |\eta z|^k |\eta|^{-1-s}|d\eta|\\ &\leqslant C_\mu |z|^k\\ &\leqslant C_\mu,\end{aligned}$$

where $k = 1, 2, \ldots, [s]$.

To prove the integrals on $c_+(\pi,\mu)$ and $\Lambda_+(\pi,\mu)$ are bounded, we use Cauchy's theorem to change the contour to the following one:

$$\Big\{z = x+iy \mid x=-\pi,\ y \text{ is from } -\pi\tan(\mu) \text{ to } 0,\ \text{ and } x=-\pi,\ y \text{ is from } 0 \text{ to } -\pi\tan(\mu)\Big\}.$$

However, using the fact that $\mathrm{Re}(z) > 0$, we can conclude that the integrals on the above sets are bounded.

Now we are left to prove

$$\phi(z) = \sum_{n=1}^{\infty} b^\mu(n) z^n,\ -\infty < s < \infty,\ 0 < \mu < \omega.$$

This is equivalent to proving $b(n) = b^\mu(n)$, $n = 1, 2, \ldots$ in these cases.

Let $r \in (0,1)$. Since the series $\phi(rz) = \sum\limits_{n=1}^{\infty} b(n) r^n z^n$ is absolutely convergent in $|z| \leqslant 1$, we get

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \exp(-itn)\phi(r\exp(it))dt = r^n b_n. \tag{7.4}$$

We first deal with the case $s \geqslant 0$. Write $\delta = -\ln(r)$. Then $r \to 1-0$ if and only if $\delta \to 0+$. Taking the limits $\delta \to 0+$ and $r \to 1-0$ on both sides of (7.4), we conclude that the right hand side tends to b_n, while the limit of the left hand side is

$$\lim_{\delta\to 0+}\int_{-\pi}^{\pi} \exp(-itn)\phi(\exp(-\delta+it))dt.$$

For any fixed $\epsilon \in (0, \pi)$, we can get

$$\begin{aligned}
&\lim_{\delta\to 0+}\left(\int_{0\leqslant|t|\leqslant\epsilon}+\int_{\epsilon\leqslant|t|\leqslant\pi}\right)\exp(-itn)\phi(\exp(-\delta+it))dt \qquad (7.5)\\
&=\lim_{\delta\to 0+}\left\{\int_{0\leqslant|t|\leqslant\epsilon}\left(\exp(-itn)-1-\frac{(-itn)}{1!}-\frac{(-itn)^2}{2!}-\cdots-\frac{(-itn)^{[s]}}{[s]!}\right)\right.\\
&\times\phi(\exp(-\delta+it))dt\\
&+\int_{L_+(\epsilon)}\left(1+\frac{(-itn)}{1!}+\frac{(-itn)^2}{2!}+\cdots+\frac{(-itn)^{[s]}}{[s]!}\right)\phi(\exp(-\delta+it))dt\\
&\left.+\int_{\epsilon\leqslant|t|\leqslant\pi}\exp(-itn)\phi(\exp(-\delta+it))dt\right\}\\
&=\lim_{\delta\to 0+}\int_{0\leqslant|t|\leqslant\epsilon}\left(\exp(-itn)-1-\frac{(-itn)}{1!}-\frac{(-itn)^2}{2!}-\cdots-\frac{(-itn)^{[s]}}{[s]!}\right)\\
&\times\phi(\exp(-\delta+it))dt+\phi^{[s]}_{\epsilon,+}(n)+\int_{\epsilon\leqslant|t|\leqslant\pi}\exp(-itn)\phi(\exp(-\delta+it))dt,
\end{aligned}$$

where we used Cauchy's theorem and the fact that the last two integrals are absolutely integrable as $\delta \to 0+$. Invoking the estimate of ϕ, the last expression of (7.5) is dominated by

$$C_\mu\int_{0\leqslant|t|\leqslant\epsilon}|tn|^{[s]+1}\frac{1}{|t|^{s+1}}dt,$$

which is independent of $\delta > 0$. Taking the limits $\epsilon \to 0$ on (7.5), the integral tends to 0 and (7.5) reduces to

$$b_n=\lim_{\epsilon\to 0}\left(\int_{\epsilon\leqslant|t|\leqslant\pi}\exp(-itn)\phi(\exp(it))dt+\phi^{[s]}_{\epsilon,+}(n)\right),$$

which equals to (7.3). By the periodicity of the integrand function and Cauchy's theorem, this equals $b^\mu(n)$. The proof for the case $s \geqslant 0$ is complete.

For $s < 0$, by the estimate of the function ϕ and the Lebesgue dominated convergence theorem, we take the limit $r \to 1-0$ on both sides of (7.4) and therefore, obtain

$$b(n)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\exp(-itn)\phi(\exp(it))dt.$$

Then by the 2π-periodicity of the integral, Cauchy's theorem and (7.2), the above expression equals to $b^\mu(n)$. This completes the proof of the theorem. □

By Theorems 7.1.1 and 7.1.2, we obtain a result for the case $s \in \mathbb{Z}_-$.

Theorem 7.1.3 *Let s be a negative integer.*

(i) *If* $b \in H^s(S_{\omega,\pm})$ *and* $\phi(z) = \sum_{n=\pm 1}^{\pm\infty} b(n)z^n$, *then* $\phi \in K^s_{\ln}(C_{\omega,\pm})$.

(ii) *If $\phi \in K^s_{\ln}(C_{\omega,\pm})$, then for any $\nu \in (0,\omega)$, there exists a function b^μ such that $b^\mu \in H^s_{\ln}(S_{\mu,\pm})$, and*

$$\phi(z) = \sum_{n=\pm 1}^{\pm\infty} b^\mu(n) z^n.$$

Moreover, b^μ is given by (7.2).

Proof The conclusion (i) was obtained in Theorem 7.1.1. We only need to prove (ii). By (7.2), it is easy to prove that b^μ is bounded near the origin. For large z, invoking (7.1), we obtain that for $|\arg(z)| < \theta < \mu$,

$$\begin{aligned} |b^\mu(z)| &\leqslant C_\mu \left(|z|^s + \int_0^\infty \exp(-r|z|\sin(\mu-\theta)) |\ln r| r^{-s} \frac{dr}{r} \right) \\ &\leqslant C_\mu \left(|z|^s + |z|^s \int_0^\infty \exp(-r\sin(\mu-\theta)) |\ln r - \ln|z|| r^{-s} \frac{dr}{r} \right) \\ &\leqslant C_{\mu,\theta} |z|^s \ln|z|. \end{aligned}$$

This proves $b^\mu \in H^s_{\ln}(S_{\mu,+})$. The verification of $\phi(z) = \sum_{n=1}^{\infty} b^\mu(n) z^n$ is similar to the case $s < 0$ in Theorem 7.1.2. The proof is complete. □

Remark 7.1.1 For $\{b_n\}_{n=1}^\infty \in l^\infty$, the series

$$\phi(z) = \sum_{n=1}^{\infty} b_n z^n$$

is well-defined on the unit disc and holomorphic. Theorem 7.1.1 and (i) of Theorem 7.1.3 indicate that if there exists $b \in H^s(S_{\omega,+})$ such that $b_n = b(n)$, then ϕ can be extended to $C_{\omega,+}$ holomorphically. In any small $C_{\mu,+}$, when s is an integer, this function satisfies the conditions in the definition of $K^s_{\ln}(S_{\omega,+})$. When s is not an integer, this function satisfies the conditions in the definition of $K^s(S_{\omega,+})$. Theorem 7.1.2 and (ii) of Theorem 7.1.3 give the inverse result.

Remark 7.1.2 Under the assumption of Theorem 7.1.2, the mapping $\phi \to b$ satisfying $\phi(z) = \sum b(n) z^n$ is not single-valued. In fact, by Theorem 7.1.2, any b^μ, $0 < \mu < \omega$, gives a solution of b, and if $\mu_1 \neq \mu_2$, then generally, $b^{\mu_1} \neq b^{\mu_2}$, see also the example in Remark 7.1.3.

Remark 7.1.3 In the proof of Theorem 7.1.2, we need the following function space $\tilde{P}^+_\omega$ which consists of all finite linear combinations of the holomorphic functions with the following form

$$g_n(z) = \begin{cases} 1, & \text{if } z = n, \\ \dfrac{[\exp(i\pi(z-n)) - \exp(-i\pi(z-n))]\exp(-\pi(z-n)\tan\omega)}{2i\pi(z-n)}, & \text{if } z \neq n, \end{cases}$$

where n is a non-negative integer. It is easy to prove

$$|g_n(z)| \leqslant C_{\mu,n} \frac{\exp(-\pi(\mathrm{Re}(z)\tan\omega - |\mathrm{Im}(z)|))}{|z+1|}, \ z \in S_{\mu,+}, 0 < \mu < \omega.$$

Hence $g_n \in \bigcup_{s=-\infty}^{\infty} H^s(S_{\omega,+})$. It is remarkable that the functions in $\tilde{P}_{\omega}^{+}$ are the inverse Fourier transforms of the finite polynomials of z given by (7.2) in Theorem 7.1.2. Similarly, we can define the space $\tilde{P}^{-}$ with respect to the negative integer.

Remark 7.1.4 The holomorphic extension given in Theorem 7.1.1 is optimal in the following sense: if ω is the largest angle such that $b \in H^s(S_{\omega,+})$, then ϕ can not be holomorphically extended to any larger heart-shaped region $C_{\omega+\delta,+}$, $\delta > 0$, which satisfies the corresponding estimate. Or else, by Theorem 7.1.2, we can obtain contradiction.

Remark 7.1.5 (i) of Theorem 7.1.3 corresponds to the function $b(z) = z/(1+z^2)$. Take $s = -1$ for example, A. Baernstein studied that how to construct a holomorphic function in the unit disc such that when $z \to 1$,

$$\phi(z) = O(\ln|z-1|) \text{ and } \phi'(z) \neq O(1/|z-1|),$$

see [5]. At the same time he also proved that it is equivalent to considering the matter in the unit disc instead of in the heart-shaped region. The reason is that the estimates for $s = -1$ remain unchanged after applying a suitable conformal mapping. In Theorem 7.1.1, letting $s = 0$, we conclude that $b(z) \neq O(1/|z|)$ at ∞. However, it is still an open problem that the estimates given in (ii) of Theorem 7.1.3 are the best possible in those cases.

7.2 Fractional Fourier Multipliers on Starlike Lipschitz Surfaces

In this section, we consider a class of Fourier multiplier operators whose multipliers are dominated by a polynomial and give the estimates of the kernels of the integral operators associated with the Fourier multipliers. The main tool is still the generalized Fueter theorem obtained in [6] (see Sect. 3.5). The main idea is to establish a relation between the set O in the complex plane $\mathbb{C}$ and the set $\overrightarrow{O}$ in the $(n+1)$-dimensional space $\mathbb{R}_1^n$, and then transfer the estimate for the functions defined on $\overrightarrow{O}$ to the corresponding one defined on O.

As in Chap. 6, we still use the following intrinsic set. We recall

Definition 7.2.1

(i) A set O in the complex plane $\mathbb{C}$ is called an intrinsic set if the set is systemic about the real axis, that is, the set is unchanged under the complex conjugate.

(ii) If a function f^0 is defined on an intrinsic set in $\mathbb{C}$ and $\overline{f^0(z)} = f^0(\overline{z})$ in the domain, then the function f^0 is called an intrinsic function.

The functions of the form $\sum c_k(z-a_k)^k,\ k\in\mathbb{Z},\ a_k, c_k\in\mathbb{R}$, are all intrinsic functions. If $f = u + iv$, where u and v are real-valued, then f^0 is intrinsic if and only if in their domains, $u(x,-y) = u(x,y)$ and $v(x,-y) = -v(x,y)$.

We regard $\mathbb{R}_1^n$ as the $(n+1)$-dimensional Euclidean space and define the intrinsic set in $\mathbb{R}_1^n$ as follows.

Definition 7.2.2 We call a set in $\mathbb{R}_1^n$ an intrinsic set if it is invariant under all rotations in $\mathbb{R}_1^n$ that keep the e_0 axis fixed. If O is a subset in the complex plane, then in $\mathbb{R}_1^n$, we call the intrinsic set

$$\overrightarrow{O} = \left\{x\in\mathbb{R}_1^n : (x_0, |\underline{x}|)\in O\right\}$$

the set induced by O.

Definition 7.2.3 Let $f^0(z) = u(x,y) + iv(x,y)$ be the intrinsic function defined on the intrinsic set $U\subset\mathbb{C}$. Define the function $\overrightarrow{f^0}$ on the induced set $\overrightarrow{U}$ as follows:

$$\overrightarrow{f^0}(x_0+\underline{x}) = u(x_0, |\underline{x}|) + \frac{\underline{x}}{|\underline{x}|}v(x_0, |\underline{x}|).$$

We call $\overrightarrow{f^0}$ the function induced by f^0.

We denote by τ the mapping:

$$\tau(f^0) = k_n^{-1}\Delta^{(n-1)/2}\overrightarrow{f^0},$$

where $\Delta = D\overline{D}$ and $\overline{D} = D_0 - \underline{D}$, $k_n = (2i)^{n-1}\Gamma^2(\frac{n+1}{2})$ is the normalized constant such that $\tau((\cdot)^{-1}) = E$. The operator $\Delta^{(n-1)/2}$ is defined via the Fourier multiplier $m(\xi) = (2\pi i|\xi|)^{n-1}$ defined on the tempered distributions $\mathcal{M} : \mathcal{S}'\to\mathcal{S}'$. Precisely,

$$\mathcal{M}f = \mathcal{R}(m\mathcal{F}f),$$

where

$$\mathcal{F}f(\xi) = \int_{\mathbb{R}_1^n} e^{2\pi i\langle x,\xi\rangle}f(x)dx$$

and

$$\mathcal{R}h(x) = \int_{\mathbb{R}_1^n} e^{-2\pi i\langle x,\xi\rangle}h(\xi)d\xi.$$

The monogenic monomials in $\mathbb{R}_1^n$ are defined by

$$P^{(-k)} = \tau((\cdot)^{-k}) \text{ and } P^{(k-1)} = I(P^{(-k)}), k\in\mathbb{Z}^+,$$

where I denotes the Kelvin inversion $I(f)(x) = E(x)f(x^{-1})$.

We also need the following set in the complex plane. For $\omega \in (0, \frac{\pi}{2})$, let

$$\begin{aligned}
S^c_{\omega,\pm} &= \left\{ z \in \mathbb{C} : |arg(\pm z)| < \omega \right\}, \text{ the angle } \arg(z) \in (-\pi, \pi], \\
S^c_{\omega,\pm}(\pi) &= \left\{ z \in \mathbb{C} : |\mathrm{Re}\, z| \leqslant \pi,\ z \in S^c_{\omega,\pm} \right\}, \\
S^c_{\omega} &= S^c_{\omega,+} \cup S^c_{\omega,-} \text{ and } S^c_{\omega}(\pi) = S^c_{\omega,+}(\pi) \cup S^c_{\omega,-}(\pi), \\
W^c_{\omega,\pm}(\pi) &= \left\{ z \in \mathbb{C} : |\mathrm{Re}\, z| \leqslant \pi \text{ and } \pm\ \mathrm{Im} z > 0 \right\} \cup S^c_{\omega}(\pi), \\
H^c_{\omega,\pm} &= \left\{ z = \exp(i\eta) \in \mathbb{C}, \eta \in W^c_{\omega,\pm}(\pi) \right\} \\
H^c_{\omega} &= H^c_{\omega,+} \cap H^c_{\omega,-}.
\end{aligned}$$

We define the Fourier multipliers in the following function space

$$\begin{aligned}
K^s(H^c_{\omega,\pm}) = \Bigg\{ \phi^0 :\ & H^c_{\omega,\pm} \to \mathbb{C}, \phi^0 \text{ is holomorphic and} \\
& \text{in any } H^c_{\mu,\pm},\ 0 < \mu < \omega, |\phi^0(z)| \leqslant \frac{C_\mu}{|1-z|^{1+s}} \Bigg\},
\end{aligned}$$

and

$$K^s(H^c_{\omega}) = \left\{ \phi^0 :\ H^c_{\omega} \to \mathbb{C},\ \phi^0 = \phi^{0,+} + \phi^{0,-}, \phi^{0,\pm} \in K^s(H^c_{\omega,\pm}) \right\}.$$

The corresponding multiplier spaces are

$$\begin{aligned}
H^s(S^c_{\omega,\pm}) = \Big\{ b :\ & S^c_{\omega,\pm} \to \mathbb{C},\ b \text{ is holomorphic and in any } S^c_{\mu,\pm}, \\
& 0 < \mu < \omega, |b(z)| \leqslant C_\mu |z \pm 1|^s \Big\}.
\end{aligned}$$

and

$$H^s(S^c_{\omega}) = \left\{ b :\ S^c_{\omega} \to \mathbb{C},\ b_\pm = b\chi_{\{z\in\mathbb{C}:\pm Rez>0\}} \in H^s(S^c_{\omega,\pm}) \right\}.$$

Let

$$H_{\omega,\pm} = \left\{ x \in \mathbb{R}^n_1 : \frac{(\pm \ln |x|)}{\arg(e_0, x)} < \tan\omega \right\} = \overrightarrow{H^c_{\omega,\pm}},$$

and

$$H_{\omega} = H_{\omega,+} \cap H_{\omega,-} = \left\{ x \in \mathbb{R}^n_1 :\ \frac{|\ln |x||}{\arg(e_0, x)} < \tan\omega \right\} = \overrightarrow{H^c_{\omega}}.$$

Hence, the corresponding function spaces in $\mathbb{R}^n_1$ are

$$\begin{aligned}
K^s(H_{\omega,\pm}) = \Big\{ \phi :\ & H_{\omega,\pm} \to \mathbb{C}_{(n)}, \phi \text{ is monogenic and} \\
& |\phi(x)| \leqslant \frac{C_\mu}{|1-x|^{n+s}},\ x \in H_{\mu,\pm},\ 0 < \mu < \omega \Big\}
\end{aligned}$$

and

$$K^s(H_\omega) = \Big\{\phi : H_\omega \to \mathbb{C}_{(n)},\ \phi = \phi^+ + \phi^-,\ \phi^\pm \in K^s(H_{\omega,\pm})\Big\}.$$

Now we consider the multipliers $b \in H^s(S^c_{\omega,\pm})$. At first, in the following lemma, we estimate the jth derivative of the intrinsic function ϕ^0.

Lemma 7.2.1 *Assume that $b \in H^s(S^c_{\omega,-})$. For the multiplier defined by $\phi^0(z) = \sum_{k=1}^\infty b(-k)z^{-k}$, its jth derivative satisfies*

$$|(\phi^0)^{(j)}(z)| \leqslant \frac{C}{|1-z|^{s+j+1}},$$

where $z \in H^c_{\mu,-}$, $0 < \mu < \omega$ and j is a positive integer.

Proof Without loss of generality, for $b \in H^s(S^c_{\omega,-})$, we assume that $|b(-k)| \leqslant |k|^s$. By Theorem 7.1.1, for $\phi^0(z) = \sum\limits_{k=1}^{\infty} b(-k)z^{-k}$,

$$|\phi^0(z)| \leqslant \frac{C}{|1-z|^{s+1}}.$$

Take a circle $C(z,r)$ centered at z with radius r. By Cauchy's formula, we obtain

$$\left|(\phi^0)^{(j)}(z)\right| \leqslant \frac{C_j}{2\pi}\int_{C(z,r)} \frac{|\phi^0(\xi)|}{|z-\xi|^{j+1}}|d\xi|.$$

Let $r = \frac{1}{2}|1-z|$. Then $\xi \in C(z,r)$ implies that

$$|1-\xi| \geqslant |1-z| - |z-\xi| = |1-z| - \frac{1}{2}|1-z| = \frac{1}{2}|1-z|.$$

Therefore we obtain

$$\left|(\phi^0)^{(j)}(z)\right| \leqslant \frac{2j!C_\mu}{\delta^j(\mu)}\frac{1}{|1-z|^{j+s+2}}|1-z| \leqslant C_{\mu,j}\frac{1}{|1-z|^{j+s+1}}.$$

This proves Lemma 7.2.1. □

Lemma 7.2.2 enables us to estimate the kernels of the Fourier multipliers generated by the functions in $H^s(S^c_\omega)$ and the spherical monogenic functions.

Theorem 7.2.1 *For $s > 0$, if $b \in H^s(S^c_{\omega,\pm})$ and $\phi(x) = \sum\limits_{k=\pm 1}^{\pm\infty} b(k)P^{(k)}(x)$, then $\phi \in K^s(H_{\omega,\pm})$.*

Proof Similar to Theorem 6.1.1, we divide the proof into two cases according to the parity of n.

Case 1. n is odd: We assume that $n = 2m + 1$ and restrict the proof to $x \approx 1$. By Lemma 3.5.1, we only need to estimate the corresponding u_l and v_l. There are two subcases to be considered.

Subcase (1.1). $|\underline{x}| > (\delta(\mu)/2^{m+1/2})|1-x|$. For this case, we write $z = x_0 + i|\underline{x}|$. $x \approx 1$ implies that $z \approx 1$. We can write $z = s + it$, where $s = x_0$ and $t = |\underline{x}|$. We have $t = |\underline{x}| = |1-z|$.

For $l = 0$, $u_l = u_0 = u$ and $v_l = v_0 = v$. By the estimate of ϕ_0, we have

$$|u_0|,\ |v_0| \leqslant |\phi_0| \leqslant \frac{C}{\delta^0(\mu)} \frac{1}{|1-z|^{s+1}}.$$

For $l = 1$ and $t \approx |1-z|$, we get

$$|u_1| = \left| 2l \frac{1}{t} \frac{\partial u_0}{\partial t} \right| \leqslant \frac{1}{|1-z|} \frac{1}{|1-z|^{s+2}} = \frac{1}{|1-z|^{s+3}};$$

and

$$\begin{aligned} |v_1| &= \left| \frac{1}{t} \frac{\partial v_0}{\partial t} - \frac{v_0}{t^2} \right| \\ &\leqslant \frac{1}{|1-z|} \frac{1}{|1-z|^{s+2}} + \frac{1}{|1-z|^2} \frac{1}{|1-z|^{s+1}} \\ &= \frac{1}{|1-z|^{s+3}}. \end{aligned}$$

Because $\Delta^1 \phi^0(x) = u_1(x_0, |\underline{x}|) + \frac{\underline{x}}{|\underline{x}|} v_1(x_0, |\underline{x}|)$, we have

$$\left|\Delta^1 \phi^0(x)\right| \leqslant C \left|u_1(x_0, |\underline{x}|)\right| + \left|\frac{\underline{x}}{|\underline{x}|} v_1(x_0, |\underline{x}|)\right| \leqslant C \frac{1}{|1-z|^{s+3}}.$$

Repeating the above procedure m times, for u_m and v_m, we obtain

$$|u_m(x)|,\ |v_m(x)| \leqslant \frac{C}{|1-z|^{s+2m+1}} = \frac{1}{|1-z|^{n+s}}.$$

Subcase (1.2). $|\underline{x}| \leqslant (\delta(\mu)/2^{m+1/2})|1-x|$. The points x in $H_{\omega,-}$ satisfying $x \approx 1$, $x_0 \leqslant 1$ belong to *Subcase (1.1)*. Hence we assume that $x_0 > 1$. Now we prove the following conclusion: if $z = s + it \approx 1, s > 1, z \in H^c_{\mu,-}$ and $|t| \leqslant (\delta(\mu)/2^{m+1/2}|1-z|)$, then

(1) the function u_l is an even function with respect to the second variable t.

(2) the jth derivation satisfies

$$\left| \frac{\partial^j}{\partial t^j} u_l(s,t) \right| \leqslant \frac{C_\mu C_l 2^{lj} C_j}{\delta^{2l+j}} \frac{1}{|1-z|^{2l+j+s+1}},$$

where the constant C_j is

$$C_j = \begin{cases} (j+4l)!, & j \text{ is even}, \quad (7.7) \\ (j+5l)!, & j \text{ is odd}. \quad (7.7') \end{cases}$$

We apply the mathematical induction to l in order to to prove (1) and (2). Clearly, for $l = 0$, by Lemma 7.2.1, we have

$$\left|\frac{\partial^j}{\partial t^j} u_0(s,t)\right|, \ \left|\frac{\partial^j}{\partial t^j} v_0(s,t)\right| \leqslant \left|\frac{\partial^j}{\partial t^j} \phi^0(s,t)\right| \leqslant \frac{j!}{(\delta(\mu))^j} \frac{1}{|1-z|^{j+s+1}}.$$

Now we assume that (1) and (2) for $0 \leqslant l \leqslant m-1$. Because

$$u_{l+1} = 2(l+1)(1/t)(\partial u_l/\partial t)(s,t)$$

and u_l is even, u_{l+1} is also an even function. This proves (1).

For (2), we first consider the case that j is even. By the definition and (1), $\partial u_l/\partial t$ is an odd function with respect to the second variable t. We can obtain

$$\frac{\partial u_l}{\partial t}(s,0) = \frac{\partial^{2k+1} u_l}{\partial^{2k+1} t}(s,0) = 0.$$

By Taylor's expansion, we have

$$\begin{aligned} u_{l+1}(s,t) &= \frac{2(l+1)}{t}\left(\sum_{k=0}^{\infty} \frac{1}{(2k)!} \frac{\partial^{2k+1} u_l}{\partial t^{2k+1}}(s,0) t^{2k} + \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} \frac{\partial^{2k+2} u_l}{\partial t^{2k+2}}(s,0) t^{2k+1}\right) \\ &= \sum_{k=0}^{\infty} \frac{1}{(2k)!} \frac{\partial^{2k+1} u_l}{\partial t^{2k+1}}(s,0) t^{2k}. \end{aligned}$$

Letting $k = j/2 + k'$ and noticing that $\left(\frac{t}{\delta|1-z|}\right)^{2k'} \leqslant \left(\frac{1}{2^{m+1/2}}\right)^{2k'}$, we conclude that

$$\begin{aligned} &\left|\frac{\partial^j}{\partial t^j} u_{l+1}(s,t)\right| \\ &= \left|2(l+1) \sum_{k=j/2}^{\infty} \frac{(2k)(2k-1)\cdots(2k-j+1)}{(2k+1)!} \frac{\partial^{2k+2} u_l}{\partial t^{2k+2}}(s,0) t^{2k-j}\right| \\ &\leqslant 2(l+1) \sum_{k'=0}^{\infty} \frac{(2k'+j)(2k'+j-1)\cdots(2k'+1)}{(2k'+j+1)!} \frac{C_\mu C_l 2^{l(2k'+j+2)(2k'+j+2+4l)}}{\delta^{2l+2k'+j+2}} \\ &\times \frac{t^{2k'}}{|1-z|^{2l+2k'+j+2+s+1}} \end{aligned}$$

$$\leqslant 2(l+1)\frac{C_\mu C_l 2^{l(j+2)}}{\delta^{2(l+1)+j}|1-z|^{2(l+1)+j+1+s}}\sum_{k=0}^{\infty}\frac{(j+2k+2+4l)\cdots(2k+2)}{2^k}.$$

The rest of the proof is similar to that of Theorem 6.1.1. By use of (6.7), we obtain that the series in the last inequality converges and satisfies

$$\sum_{k=0}^{\infty}\frac{(j+2k+2+4l)\cdots(2k+2)}{2^k}\leqslant 2^{j+4l-1}(j+4l+4)!.$$

Finally, we have

$$\left|\frac{\partial^j}{\partial t^j}u_{l+1}(s,t)\right|\leqslant 2(l+1)\frac{C_\mu C_l 2^{l(j+2)}}{\delta^{2(l+1)+j}|1-z|^{2(l+1)+j+1+s}}2^{j+4l-1}(j+4l+4)!.$$

Now we verify that $\left|\frac{\partial^j}{\partial t^j}u_{l+1}(s,t)\right|$ satisfies the estimate for odd j. Similar to the proof for j even, by Taylor's expansion, we have

$$\frac{\partial^j}{\partial t^j}u_{l+1}(s,t)=2(l+1)t\sum_{k=\frac{j+1}{2}}^{\infty}\frac{2k(2k-1)\cdots(2k+1-j)}{(2k+1)!}\frac{\partial^{2k+2}u_l}{\partial t^{2k+2}}(s,0)t^{2k-1-j}.$$

Let $2k-1-j=2k'$. We can obtain

$$\begin{aligned}
&\left|\frac{\partial^j}{\partial t^j}u_{l+1}(s,t)\right|\\
&\leqslant 2(l+1)t\sum_{k=0}^{\infty}\frac{(2k+j+1)(2k+j)\cdots(2k+2)}{(2k+j+2)!}\frac{C_\mu C_l 2^{l(2k+3+j)}}{\delta^{2l(2k+3+j)}}\frac{(2k+3+j+5l)!}{|1-z|^{2l+2k+3+j+s+1}}t^{2k}\\
&\leqslant 2(l+1)\left(\frac{t}{\delta|1-z|}\right)\frac{1}{\delta^{2(l+1)+j}}\frac{C_\mu C_l 2^{l(j+3)}}{|1-z|^{2(l+1)+j+s+1}}\\
&\quad\sum_{k=0}^{\infty}\frac{(2k+j+1)(2k+j)\cdots(2k+2)}{(2k+j+2)!}2^{kl}\left(\frac{1}{2^{m+1/2}}\right)^{2k}(2k+3+j+5l)!\\
&\leqslant 2(l+1)\left(\frac{t}{\delta|1-z|}\right)\frac{1}{\delta^{2(l+1)+j}}\frac{C_\mu C_l 2^{l(j+3)}}{|1-z|^{2(l+1)+j+s+1}}2^{j+5l+4}((j+5l+3)/2)!
\end{aligned}$$

Letting $j=0$ and $l=m$, we have

$$|u_m(s,t)|\leqslant\frac{C_\mu C_0(4m)!}{\delta^{2m}}\frac{1}{|1-z|^{2m+s+1}}\leqslant\frac{C}{|1-z|^{n+s}}.$$

Now we estimate v_m. As before, we divide the discussion into two cases.
Subcase (1.3). $|\underline{x}|>(\delta(\mu)/2^{m+1/2})$. When $l=0$, noticing that $|t|\approx|1-z|$, we have

$$|v_0(s,t)| = |v(s,t)| \leqslant C\frac{2C_\mu}{|1-z|^{1+s}}.$$

For $l = 1$, because

$$|(\phi^0)^j(z)| \leqslant \frac{2j!C_\mu}{\delta^j(\mu)}\frac{1}{|1-z|^{1+j+s}},$$

we have

$$\begin{aligned}|v_1(s,t)| &\leqslant \frac{2C_\mu}{\delta(\mu)}\left(\frac{1}{|1-z|^{2+s}}\frac{1}{|1-z|} + \frac{1}{|1-z|^2}\frac{1}{|1-z|^{1+s}}\right)\\ &\leqslant \frac{C_\mu}{|1-z|^{s+3}}.\end{aligned}$$

Repeating this procedure m times, we know

$$|v_m(s,t)| \leqslant \frac{C_\mu}{|1-z|^{2m+1+s}} = \frac{C_\mu}{|1-z|^{n+s}}.$$

Subcase (1.4). $|\underline{x}| \leqslant (\delta(\mu)/2^{m+1/2})|1-x|$. For this case, we assume that $x_0 > 1$. For $0 \leqslant l \leqslant m$, we have the following conclusion:

Conclusion (1). $v_l(s,t)$ is odd with respect to the second variable t. In fact, for $l = 0$, $v_0(s,t) = \mathrm{Im}\phi^0(s,t)$. Because $\phi^0(z) = \sum_{k=1}^{\infty} b(-k)z^{-k}$, we have

$$\phi^0(\overline{z}) = \sum_{k=1}^{\infty} b(-k)\overline{z}^{-k} = \overline{\sum_{k=0}^{\infty} b(-k)z^{-k}} = \overline{\phi^0(z)}.$$

Let $\phi^0(z) = u(x,y) + iv(x,y)$, where u and v are real-valued functions. Then

$$u(x,-y) + iv(x,-y) = \overline{u(x,y)} - i\overline{v(x,y)} = u(x,y) - iv(x,y).$$

Hence $v(x,-y) = -v(x,y)$, that is, v_0 is an odd function for the second variable.

For $l = 1$, because (v_0/t) is an even function, $v_1 = 2\frac{\partial}{\partial t}(\frac{v_0}{t})$ is an odd function. We assume that for $0 \leqslant l \leqslant m-1$, v_l is odd. Hence

$$v_m = 2m\left(\frac{1}{t}\frac{\partial v_{m-1}}{\partial t} - \frac{v_{m-1}}{t^2}\right)$$

is also odd. This proves *Conclusion (1)*.

Conclusion (2). For $0 \leqslant l \leqslant m$,

$$\left|\frac{\partial^j}{\partial t^j}v_l(s,t)\right| \leqslant \frac{C_\mu C_l C_j j!}{\delta^j}\frac{1}{|1-z|^{2l+j+s+1}},$$

where the constant C_j is defined by

$$C_j = \begin{cases} (j+5l)!, & \text{if } j \text{ is even,} \\ (j+4l)!, & \text{if } j \text{ is odd.} \end{cases}$$

For simplicity, we only consider the case j is odd. When $l = 0$, it follows from the estimate of $|(\phi^0)^{(j)}|$ that

$$\left|\frac{\partial^j}{t^j} v_0(s,t)\right| \leqslant \frac{C_\mu C_j j!}{(\delta^j)} \frac{1}{|1-z|^{j+s+1}}.$$

Because $v_l(s,t)$ is odd with respect to the second variable, $(\partial^{2k} v_l/\partial t^{2k})(s,0) = 0$. By Taylor's expansion, we have

$$v_{l+1}(s,t) = 2(l+1)\frac{1}{t^2}\sum_{k=0}^{\infty}\left(\frac{1}{(2k)!} - \frac{1}{(2k+1)!}\right) t^{2k+1} \frac{\partial^{2k+1} v_l}{\partial t^{2k+1}}(s,0).$$

Let $k = k' + 1$ and write $k = k'$. We get

$$\frac{\partial^j v_{l+1}}{\partial t^j}(s,t) = 2(l+1)\sum_{k=0}^{\infty}\frac{2k+2}{(2k+3)!}\frac{\partial^{2k+3} v_l}{\partial t^{2k+3}}(s,0)(2k+1)\cdots(2k+2-j)t^{2k+1-j}.$$

We assume that *Conclusion (2)* holds for $1 \leqslant l \leqslant m-1$. Letting $2k - j = 2k'$, by $t/(\delta|1-z|) \leqslant 2^{-(m+1/2)}$, we have

$$\begin{aligned}
&\left|\frac{\partial^j v_{l+1}}{\partial t^j}(s,t)\right| \\
&\leqslant 2(l+1)\sum_{k=0}^{\infty}\frac{2k+2}{(2k+3)!}(2k+1)\cdots(2k+2-j)\left|\frac{\partial^{2k+3} v_l}{\partial t^{2k+3}}(s,0)\right| t^{2k+1-j} \\
&\leqslant 2(l+1)\frac{1}{2^{m+1/2}}\frac{2^{l(j+3)}}{\delta^{2(l+1)+j}}\frac{1}{|1-z|^{2(l+1)+j+s+1}} \\
&\sum_{k=0}^{\infty}\frac{(2k+j+3+5l)\cdots(2k+j+4)(2k+j+2)\cdots(2k+2)}{2^k}.
\end{aligned}$$

This proves Conclusion (2).

Similarly, we can prove the case that n are even. For $j = 0$ and $l = m$, we obtain

$$\left|v_m(s,t)\right| \leqslant \frac{C_\mu C_m (4m)!}{\delta^{2m}} \frac{1}{|1-z|^{2m+1+s}} \leqslant \frac{C_{\mu,\delta}}{|1-z|^{n+s}}.$$

Now we deal with the multipliers defined on the region $S^c_{\omega,+}$. By the Kelvin inversion, for $b \in H^{s,r}(S^c_{\omega,+})$, we estimate the function $\phi(x) = \sum_{i=1}^{\infty} b(i)P^{(i)}(x)$. We have

$$I(\phi)(x) = \sum_{i=-1}^{-\infty} \widetilde{b}(i)P^{(i-1)}(x),$$

where $\widetilde{b}(z) = b(-z) \in H^{s,r}(S^c_{\omega,-})$. Because $I(\phi) = \tau(\phi^0)$, where

$$\phi^0(z) = \sum_{i=-1}^{-\infty} \widetilde{b}(i)z^{i-1} = \frac{1}{z}\sum_{i=-1}^{-\infty} \widetilde{b}(i)z^{i} \in H^{s,c}_{\omega,-},$$

we have $\phi(x) = I^2(\phi) = E(x)I(\phi)(x^{-1})$ and

$$|\phi(x)| = \left|E(x)I(\phi)(x^{-1})\right| \leqslant \frac{1}{|x|^n}\frac{C_\mu}{|1-x^{-1}|^{n+s}} = \frac{C_\mu |x|^s}{|1-x|^{n+s}}.$$

Because $x \in H_{\nu,+} = \overrightarrow{H^c_{\nu,+}}$, we can see that $(x_0, |\underline{x}|) \in H^c_{\nu,+}$ and

$$|x| = (x_0^2 + |\underline{x}|^2)^{1/2} \leqslant 1 + e^{\tan\nu}.$$

Finally we obtain that $|\phi(x)| \leqslant C_\nu/|1-x|^{n+s}$. This completes the proof of Case 1.

Case 2. n is even. As above, we only need to estimate the kernel ϕ defined on $H_{\omega,-}$. Let $b \in H^{s,r}(S^c_{\omega,-})$. Consider $\phi(x) = \sum_{k=1}^{\infty} b(-k)P^{(-k)}_n(x)$. Because $n+1$ is odd, we have

$$\begin{aligned} c_{n+1}\phi(x) &= \sum_{k=1}^{\infty} b(-k)\int_{-\infty}^{\infty} P^{(-k)}_{n+1}(x + x_{n+1}e_{n+1})dx_{n+1} \\ &\leqslant c_\mu \int_{-\infty}^{\infty} \frac{1}{|1-(x+x_{n+1}e_{n+1})|^{n+1+s}}dx_{n+1} \\ &= \frac{1}{|1-x|^{n+s}}\int_0^{\infty} \frac{|1-x|}{\left[1+(x_{n+1}/|1-x|)^2\right]^{(n+1+s)/2}} d\left(\frac{x_{n+1}}{|1-x|}\right) \\ &\leqslant \frac{C}{|1-x|^{n+s}}. \end{aligned}$$

This completes the proof of Theorem 7.2.1. □

The following corollary can be deduced from Theorem 7.2.1.

Corollary 7.2.1 *Let $s > 0$, $b \in H^s(S^c_\omega)$ and*

$$\phi(x) = \left(\sum_{i=1}^{\infty} + \sum_{i=-1}^{-\infty}\right) b(i)P^{(i)}(x).$$

Then $\phi \in K^s(H_\omega)$.

For the case $s < 0$, the proof of the conclusion for the function ϕ is similar to that given in the above theorem. In the following theorem, we prove the conclusion of Theorem 7.2.1 holds for the spaces whose dimension n are odd.

Theorem 7.2.2 *For $s < 0$, $b \in H^s(S^c_{\omega,\pm})$ and $\phi(x) = \sum\limits_{k=\pm 1}^{\pm\infty} b(k)P^{(k)}(x)$, if the spatial dimension n is odd, we have $\phi \in K^s(H_{\omega,\pm})$.*

Proof Because the index s is negative, we can not use the method of Theorem 7.2.1 directly. Precisely, for $s < 0$, $|z|^s$ is unbounded as z approaches the origin. Hence, after getting the estimate of the function ϕ^0 on the region $S^c_{\omega,-}$, we will not use the Kelvin inversion to obtain the estimate on the region $S^c_{\omega,+}$.

To deal with this case, we estimate the function ϕ on the regions $H_{\omega,+}$ and $H_{\omega,-}$. On the region $H_{\omega,-}$, the estimate for the function ϕ is the same as that of Theorem 7.2.1. We omit the details.

For the region $H_{\omega,+}$, because the Kelvin inversion is invalid, we need to estimate the intrinsic function ϕ^0 in the region $H^c_{\omega,+}$. For this purpose, we use Theorem 3.5.1 to obtain that for the odd n, $P^{(k-1)} = \tau((\cdot)^{n+k-2})$, where the mapping τ denotes the operator $\tau(f^0) = k_n^{-1}\Delta^{(n-1)/2}\overrightarrow{f^0}$ and

$$\overrightarrow{f^0}(x) = u(x_0, |\underline{x}|) + \frac{\underline{x}}{|\underline{x}|}v(x_0, |\underline{x}|).$$

Now we complete the estimate for the kernel ϕ. We assume that $b \in H^{s,r}(S^c_{\omega,+})$ and consider $\phi(x) = \sum_{k=1}^{\infty} b(k)P^{(k)}(x)$. By Fueter's theorem, we have

$$\phi(x) = \Delta^m\phi^0(x_0, |\underline{x}|), \text{ where } \phi^0(z) = \sum_{k=1}^{\infty} b(k)z^{n+k-1}.$$

For simplicity, $\phi^0(z) = z^{n-1}\phi^0_1(z)$, where $\phi^0_1(z) = \sum\limits_{k=1}^{\infty} b(k)z^k$. By Theorem 7.1.1, for $b \in H^s(S^c_{\omega,+})$,

$$|\phi^0_1(z)| \leqslant \frac{C}{|1-z|^{1+s}},$$

where $z \in H^c_{\omega,+}$. Then we have

$$|\phi^0(z)| \leqslant \frac{|z|^{n-1}}{|1-z|^{1+s}} \leqslant \frac{C_\omega}{|1-z|^{1+s}},$$

where in the last inequality we have used the fact that the function $|z|^{n-1}$ is bounded on $H^c_{\omega,+}$. Then repeating the procedure used in Theorem 7.2.1, by the estimate of the intrinsic function ϕ^0, we can deduce the estimate of the induced function ϕ. This completes the proof. □

As a direct corollary of Theorem 7.2.2, we have

Corollary 7.2.2 *For the case that the spatial dimension n is odd, Corollary 7.2.1 holds for $s < 0$.*

On $\mathbb{R}^n$, The Fourier theory indicates that there exists a one-one correspondence between the kernels of singular integrals and the symbols of Fourier multipliers. By Theorem 7.2.1, for $b \in H^s(S^c_\omega)$, there exists a function $\phi \in K^s(H_\omega)$. Now we consider the converse of Theorem 7.2.1. For $\phi \in K^s(H_{\omega,\pm})$, we prove that there exists a function $b^\nu(z) \in H^s(S^c_{\nu,\pm})$ such that $b_k = b^\nu(k)$, $0 < \nu < \omega$.

Let $n = 3$. For the case $s = 0$, such function b^ν was obtained by T. Qian in [7]. The main tool is the following polynomial $P^{(k)}$. For any $z \in S^c_\omega$, let

$$\begin{cases} P^{(z)}_- = \tau^0((\cdot)^z), \ z \in S^c_{\omega,-}, \\ P^{(z)}_+ = \tau^0((\cdot)^{z+2}), \ z \in S^c_{\omega,+}, \end{cases}$$

where $(\cdot)^z = \exp(z\ln(\cdot))$. In the first case, the function ln is defined by cutting the positive half x-axis; while in the section case, the function is defined by cutting the negative half x-axis.

By the new functions $P^{(z)}_-$ and $P^{(z)}_+$, we can obtain the following result. For the sake of simplicity, we assume that $n = 3$.

Theorem 7.2.3 *Let $n = 3$ and $-\infty < s < -2$. If $\phi(x) = \sum_{k\in\mathbb{Z}\setminus\{0\}} b_k P^{(k)}(x) \in K^s(H_{\omega,\pm})$, then for any $\nu \in (0, \omega)$, there exists a function $b^\nu \in H^{s+2}(S^c_{\nu,\pm})$ such that $b_i = b^\nu(i)$, $i = \pm 1, \pm 2, \ldots$. In addition,*

$$b^\nu(z) = \lim_{r\to 1-} \frac{1}{2\pi^2} \int_{L^\pm(\nu)} P^{(z)}(y^{-1})E(y)n(y)\phi(r^{\pm 1}y)d\sigma(y),$$

where $L^\pm(\nu) = \overrightarrow{\exp(il^\pm(\nu))}$ and the path $l^\pm(\nu))$ is defined as

$$l^\pm(\nu) = \Big\{ z \in \mathbb{C} : z = r\exp(i(\pi \pm \nu)), r \text{ is from } \pi\sec(\nu) \text{ to } 0; \\ \text{and } z = r\exp(-(\pm i\nu)), r \text{ is from } 0 \text{ to } \pi\sec(\nu) \Big\}.$$

Proof Recall that $\tau^0 : f^0 \longrightarrow \frac{1}{4}\Delta\overrightarrow{f^0}$. Write $f^0 = \eta^z$, where $\eta = x + iy$. For $x = (x_0, |\underline{x}|) \in L^\pm(\nu)$, there exists $\eta \in \exp(il^\pm(\nu))$ such that $\eta = (x_0, |\underline{x}|)$. Write $\underline{e} = \underline{x}/|\underline{x}|$. We have

$$\Delta\overrightarrow{f^0} = \Delta\overrightarrow{((\cdot)^z)} = \frac{2}{|\underline{x}|}\frac{\partial u}{\partial y}(x_0, |\underline{x}|) + 2\mathbf{e}\left(\frac{1}{|\underline{x}|}\frac{\partial v}{\partial y}(x_0, |\underline{x}|) - \frac{1}{|\underline{x}|^2}v(x_0, |\underline{x}|)\right).$$

Now $f^0 = e^{i\eta z}$, where $\eta \in l^{\pm}(\nu)$. Then $f = u + iv$, where u and v are the real part and the imaginary part of f, respectively. We have $\frac{\partial}{\partial \eta}(e^{i\eta z}) = ize^{i\eta z}$. Let $\eta = re^{-i\mu}$ and $z = |z|e^{i\theta}$. We can get

$$e^{-i\eta z} = \exp(-ir|z|e^{i(\theta-\mu)}) = \exp(r|z|\sin(\theta-\mu))\exp(-ir|z|\cos(\theta-\mu)).$$

Because $\phi \in K^s(S_\omega)$, we have

$$|\phi(x)| \leqslant \frac{C}{|1-x|^{s+3}}, \text{ where } x = x_0 + \underline{x} \in L^{\pm}(\nu).$$

For such a x, there exists a $z = x + iy \in \exp(il^{\pm}(\nu))$ such that $z = e^{i\eta} = \exp(r\sin\mu + ir\cos\mu)$ and $|\underline{x}| = e^{r\sin\mu}\sin(r\cos\mu)$. Then we get

$$|b^\mu(z)| \leqslant C\int_0^{\pi\sec\mu} |z|e^{-r|z|\sin(\mu-\theta)}\frac{1}{|1-e^{i\eta}|^{s+3}}\frac{1}{|x|}\frac{1}{|\underline{x}|}r^2 dr.$$

For the factor $1/|1-e^{i\eta}|^{s+3}$, we have

$$|1-e^{i\eta}|^2 = 1 + e^{2r\sin\mu} - 2e^{r\sin\mu}\cos(r\cos\mu).$$

Let $f(r) = r^2$ and $g(r) = 1 + e^{2r\sin\mu} - 2e^{r\sin\mu}\cos(r\cos\mu)$. We obtain $\lim_{r\to 0}\frac{f(r)}{g(r)} = 1$. Hence we can find a constant C such that

$$\frac{r}{|1-e^{r\sin\mu}e^{ir\cos\mu}|} \leqslant C, \ r \in (0, \pi\sec\mu),$$

that is, $1/|1-e^{r\sin\mu}e^{ir\cos\mu}|^{s+3} \sim r^{s+3}$. Finally we have

$$\begin{aligned}|b^\mu(z)| &\leqslant C\int_0^{\pi\sin\mu} |z|e^{-r|z|\sin(\mu-\theta)}\frac{1}{r^{s+3}}\frac{1}{e^{3r\sin\mu}}\frac{r^2}{e^{r\sin\mu}\sin(r\cos\mu)}dr \\ &\leqslant C|z|\int_0^{\pi\sin\mu} e^{-r|z|\sin(\mu-\theta)}\frac{r^2}{r^{s+4}}e^{-4r\sin\mu}dr \\ &\leqslant C|z|^{s+2},\end{aligned}$$

where in the last inequality we used $s < -2$. □

Theorem 7.2.3 indicates that using the method in [7], for $s \neq 0$, we only get $b \in H^{s+2}(S^c_{\omega,\pm})$ rather than $b \in H^s(S^c_{\omega,\pm})$. To obtain a more precise result, we need apply a new method. It will be based on the following things. First, the desired function b is defined on $S^c_{\omega,\pm} \subset \mathbb{C}$. Secondly, by Proposition 6.1.1, we know that if the dimension n is odd, the polynomials $P^{(-k)}$ and $P^{(k-1)}$, $k \in \mathbb{Z}_+$, satisfy the following relation:

$$P^{(-k)} = \tau((\cdot)^{-k}), \quad P^{(k-1)} = \tau((\cdot)^{k+n-2}).$$

Our idea is to construct a function $\phi^0 \in K^s(H^c_{\omega,\pm})$ by use of $\phi \in K^s(H_{\omega,\pm})$. Then we can express the function b via ϕ^0 by using techniques in complex analysis. At first we give a lemma to show the relation between $H^c_{\omega,\pm}$ and $H_{\omega,\pm}$.

For any element $\mathbf{e}$ in the vector space Q, the linear span of 1 and $\mathbf{e}$ in $\mathbb{R}$ is called the complex plane induced by $\mathbf{e}$ in $\mathbb{R}^n_1$ denoted by $\mathbb{C}^{\mathbf{e}}$. Denote by $H^{\mathbf{e}}_{\omega,\pm}$ and $H^{\mathbf{e}}_{\omega}$ the images on $\mathbb{C}^{\mathbf{e}} \subset \mathbb{R}^{(n)}_1$ of the sets $H^c_{\omega,\pm}$ and H^c_{ω} in $\mathbb{C}$ under the mapping $i_{\mathbf{e}} : a + bi \longrightarrow a + b\mathbf{e}$, respectively. By the same method as that of [7, Lemma 4], we can prove the following lemma.

Lemma 7.2.2

$$H_{\omega,\pm} = \bigcup_{\mathbf{e}\in\mathbf{J}} H^{\mathbf{e}}_{\omega,\pm} \textit{ and } H_{\omega,\pm} = \bigcup_{\mathbf{e}\in\mathbf{J}} H^{\mathbf{e}}_{\omega,\pm},$$

where the index set is the set of all unit elements.

Lemma 7.2.2 establishes the relation between the class of monogenic functions and the corresponding holomorphic Fourier multipliers.

Theorem 7.2.4 *Let n be odd and $\phi(x) = \sum_{k\in\mathbb{Z}\setminus\{0\}} b_k P^{(k)}(x) \in K^s(H_{\omega,\pm})$. If the series $\sum_{k\in\mathbb{Z}\setminus\{0\}} b_k z^k$ converges in $H^c_{\omega,\pm}$, then for any $\nu \in (0, \omega)$, there exists a function $b^\nu \in H^s(S^c_{\nu,\pm})$ such that $b_k = b^\nu(k)$, $k \in \mathbb{Z} \setminus \{0\}$.*

Proof We already know that if n is odd, for $k \in \mathbb{Z}_+$,

$$P^{(-k)} = \tau^0((\cdot)^{-k}) \text{ and } P^{(k-1)} = \tau^0((\cdot)^{n+k-1}).$$

For $\phi(x) = \sum_{k\in\mathbb{Z}\setminus\{0\}} b_k P^{(k)}(x)$ on $H_{\omega,\pm}$, we define the following function ϕ^0 on $H^c_{\omega,\pm}$ as $\phi^0(z) = \sum_{k\in\mathbb{Z}\setminus\{0\}} b_k z^k$, where $z \in H^c_{\omega,\pm}$. For simplicity, we only estimate ϕ^0 in $H^c_{\omega,+}$. Let $\mathbf{e} = \frac{\mathbf{x}}{|\mathbf{x}|}$. For any $z = u + iv \in H^c_{\omega,+}$, by Lemma 7.2.2, we get $x = u + v\mathbf{e} = (x_0, \underline{x}) \in H^{\mathbf{e}}_{\omega,+} \subset H_{\omega,+}$. We have proved that for $z \in H^c_{\omega,+}$, there exists a constant $\delta(\nu) = \min\{1/2, \tan(\omega - \nu)\}$ such that the ball $S_r(z)$ is contained in $H^c_{\omega,\pm}$, where z is the center and the radius is $\delta(\nu)|1 - z|$. We denote by $B(x, r)$ the ball $\Big\{y \in \mathbb{R}^n_1, |x - y| < \delta(\nu)|1 - x|\Big\}$ and have $B(x, r) \subset H^{\mathbf{e}}_{\omega,+} \subset H_{\omega,+}$.

Assume that f and g are the real part and the imaginary part of $\phi^0(z)$, respectively. The induced function is defined by

$$\overrightarrow{\phi^0}(x) = f(x_0, |\underline{x}|) + \mathbf{e}g(x_0, |\underline{x}|)$$

and satisfies $\Delta^{(n-1)/2}\overrightarrow{\phi^0}(x) = \phi(x)$, where $x = (x_0, \underline{x}) = u + v\mathbf{e}$. We can see that

$$|\overrightarrow{\phi^0}(x)| \leqslant \int_{B(x,r)} \frac{c}{|x-y|^2} \frac{C_\nu}{|1-y|^{n+s}} dy.$$

For any $y \in B(x, \delta(\nu)|1-x|)$,

$$|1-y| \geqslant |1-x| - |x-y| > (1-\delta(\nu))|1-x|.$$

We get

$$\begin{aligned} |\overrightarrow{\phi^0}(x)| &\leqslant \frac{C_\nu}{|1-x|^{n+s}} \int_0^{\delta(\nu)|1-x|} \frac{1}{|x-y|^2} |x-y|^{n-1} d(|x-y|) \\ &\leqslant \frac{C_\nu}{|1-x|^{1+s}}. \end{aligned}$$

By the definition of $|\overrightarrow{\phi^0}|$, we have

$$|\phi^0(z)| = |\overrightarrow{\phi^0}(x)| \leqslant \frac{C_\nu}{|1-x|^{1+s}} = \frac{C_\nu}{|1-z|^{1+s}}.$$

By the above estimate, we can construct the function $b \in H^s(S^\omega_{\omega,\pm})$ as follows.

For $s < 0$ and $z \in S^c_{\mu,\pm}$,

$$b^\mu(z) = \frac{1}{2\pi} \int_{\lambda_\pm(\mu)} \exp(-i\eta z)\phi^0(\exp(i\eta))d\eta,$$

where

$$\begin{aligned} \lambda_\pm(\mu) = \Big\{ \eta \in H^c_{\omega,\pm} \mid \eta = r\exp(i(\pi \pm \mu)), r \text{ is from } \pi \sec\mu \text{ to } 0 \\ \text{and } \eta = r\exp(\mp i\mu), r \text{ is from } 0 \text{ to } \pi \sec\mu \Big\} \end{aligned}$$

and for $s \geqslant 0$, $z \in S^c_{\mu,\pm}$,

$$b^\mu(z) = \frac{1}{2\pi} \lim_{\varepsilon \to 0} \left(\int_{l(\varepsilon,|z|^{-1}) \cup c_\pm(|z|^{-1},\mu) \cup \Lambda_\pm(|z^{-1}|,\mu)} \exp(-i\eta z)\phi^0(\exp(i\eta))d\eta + \phi^{|s|}_{\varepsilon,\pm}(z) \right),$$

where if $r \leqslant \pi$,

$$l(\varepsilon, r) = \Big\{ \eta = x + iy \mid y = 0, x \text{ is from } -r \text{ to } -\varepsilon, \text{ then from } \varepsilon \text{ to } r \Big\},$$

$$c_\pm(r, \mu) = \Big\{ \eta = r\exp(i\alpha) \mid \alpha \text{ is from } \pi \pm \mu \text{ to } \pi, \text{ then from } 0 \text{ to } \mp\mu \Big\},$$

and

$$\Lambda_{\pm}(r,\mu) = \Big\{\eta \in W_{\omega,\pm} \mid \eta = \rho\exp(i(\pi \pm \mu)),\ \rho \text{ is from } \pi \sec\mu \text{ to } r;$$
$$\text{then } \eta = \rho\exp(\mp i\mu),\ \rho \text{ is from } r \text{ to } \pi\sec\mu\Big\},$$

and if $r > \pi$,

$$l(\varepsilon, r) = l(\varepsilon, \pi), \quad c_{\pm}(r,\mu) = c_{\pm}(\pi,\mu), \quad \Lambda_{\pm}(r,\mu) = \Lambda_{\pm}(\pi,\mu).$$

In any case,

$$\phi^{[s]}_{\varepsilon,\pm}(z) = \int_{L_{\pm}(\varepsilon)} \phi^0(\exp(i\eta))\left[1 + (-i\eta z) + \cdots + \frac{(-i\eta z)^{[s]}}{[s]!}\right] d\eta,$$

where $L_{\pm}(\varepsilon)$ is any contour from $-\varepsilon$ to ε in $C_{\omega,\pm}$.

By Cauchy's theorem and the Taylor series expansion, we can use the estimate for ϕ^0 to show $b^{\nu} \in H^s(S^c_{\omega})$ and $b_i = b^{\nu}(i)$, $i = \pm1, \pm2, \ldots$, see Sect. 7.1 for details. □

7.3 Integral Representation of Sobolev–Fourier Multipliers

In this section, we consider a class of Fourier multipliers defined on Sobolev spaces on starlike Lipschitz surfaces. If a Lipschitz surface Σ is n-dimensional and starlike about the origin and there exists a constant $M < \infty$ such that $x_1, x_2 \in \Sigma$,

$$\frac{\big|\ln|x_1^{-1}x_2|\big|}{\arg(x_1, x_2)} \leqslant M, \tag{7.8}$$

we call Σ a starlike Lipschitz surface. We denote by $N = Lip(\Sigma)$ the minimum of M such that (7.8) holds.

Let $s \in \mathbb{R}^n_1$. For $x \in \mathbb{R}^n_1$, we define the mapping $r_s :\ x \to sxs^{-1}$. By (i) and (iv) of Lemma 6.2.1, we can prove that if x' and x belong to a starlike Lipschitz surface with the Lipschitz constant N, then

$$\Big(\big|\ln|x^{-1}x'|\big| / \arg(x, x')\Big) = \big|\ln\big||x|^{-1}\tilde{x}\big|\big| / \arg(1, |x|^{-1}\tilde{x}) \leqslant N,$$

that is, $|x|^{-1}\widetilde{x} \in H_{\omega}$. This gives the relation between the set H_{ω} and the starlike Lipschitz surface.

We use $\mathcal{M}_k$ for the finite dimensional right module of k homogeneous monogenic functions in $\mathbb{R}^n_1$ and use $\mathcal{M}_{-(k+n)}$ for the right dimensional right module of $-(k+n)$-homogeneous monogenic functions in $\mathbb{R}^n_1 \setminus \{0\}$. The spaces $\mathcal{M}_k$ and $\mathcal{M}_{-(k+n)}$ are eigenspaces of the left Dirac operator Γ_{ξ}. We define

$$P_k : f \to P_k(f) \text{ and } Q_k : f \to Q_k(f)$$

as the projections on $\mathcal{M}_k$ and $\mathcal{M}_{-(k+n)}$, respectively.

The Fourier multipliers are defined on the following test function space:

$$\mathcal{A} = \Big\{ f : \text{ for some } s > 0, f(x) \text{ is left monogenic in } \rho - s < |x| < l + s \Big\}.$$

For $f \in \mathcal{A}$, in the annuals where f is defined, we have the Laurant series expansion

$$f(x) = \sum_{k=0}^{\infty} P_k(f)(x) + \sum_{k=0}^{\infty} Q_k(f)(x).$$

Here we have used the projection operators P_k and Q_k defined as follows:

$$P_k(f)(x) = \frac{1}{\Omega_n} \int_{\Sigma} |y^{-1}x|^k C^+_{n+1,k}(\xi, \eta) E(y) \mathfrak{n}(y) f(y) d\sigma(y)$$

and

$$Q_k(f)(x) = \frac{1}{\Omega_n} \int_{\Sigma} |y^{-1}x|^{-n-k} C^-_{n+1,k}(\xi, \eta) E(y) \mathfrak{n}(y) f(y) d\sigma(y),$$

where $x = |x|\xi, y = |y|\eta$ and $\mathfrak{n}(y)$ is the outer unit normal of Σ at y. Here $C^+_{n+1,k}(\xi, \eta)$ and $C^-_{n+1,k}(\xi, \eta)$ are the functions defined as

$$\begin{aligned} C^+_{n+1,k}(\xi, \eta) = \frac{1}{1-n} \Big[&-(n+k-1) C_k^{(n-1)/2}(\langle \xi, \eta \rangle) \\ &+ (1-n) C_{k-1}^{(n+1)/2}(\langle \xi, \eta \rangle)(\langle \xi, \eta \rangle - \overline{\xi}\eta) \Big] \end{aligned}$$

and

$$\begin{aligned} C^-_{n+1,k}(\xi, \eta) = \frac{1}{n-1} \Big[&(k+1) C_{k+1}^{(n-1)/2}(\langle \xi, \eta \rangle) \\ &+ (1-n) C_k^{(n+1)/2}(\langle \eta, \xi \rangle)(\langle \eta, \xi \rangle - \overline{\eta}\xi) \Big], \end{aligned}$$

where C_k^{ν} is the Gegenbaur polynomial of degree k associated with ν (see [8]).

Now, on the starlike Lipschitz surface Σ, we give the Fourier multiplier induced by the sequence $\{b_k\}$, where $b_k = b(k)$ for any function $b \in H^s(S^c_\omega)$. We can see from Theorem 7.2.1 that the corresponding kernel ϕ satisfies

$$|\phi(x)| \leqslant C_\mu / |1 - x|^{n+s} \text{ for } s > 0.$$

The regularity index s indicates that we can not define the Fourier multipliers for $f \in L^2(\Sigma)$ as the bounded Fourier multipliers in Sect. 6.2. To compensate the role of s, we need to restrict these multipliers on some subspace of $L^2(\Sigma)$. Hence we use the following Sobolev spaces on the starlike Lipschitz surface Σ.

Definition 7.3.1 Let $s \in \mathbb{Z}^+ \cup \{0\}$ and Σ be a starlike Lipschitz surface. For $1 \leqslant p < \infty$, define the norm of Sobolev space $\|\cdot\|_{W^{p,s}_{\Gamma_\xi}(\Sigma)}$ as

$$\|f\|_{W^{p,s}_{\Gamma_\xi}(\Sigma)} = \|f\|_{L^p(\Sigma)} + \sum_{j=0}^{s} \|\Gamma^j_\xi f\|_{L^p(\Sigma)}.$$

The Sobolev space associated with the spherical Dirac operator Γ_ξ is defines as the closure of the class $\mathcal{A}$ under the norm $\|\cdot\|_{W^{p,s}_{\Gamma_\xi}(\Sigma)}$, that is, $\overline{\mathcal{A}}^{\|\cdot\|_{W^{p,s}_{\Gamma_\xi}(\Sigma)}}$.

Now we give the definition of the Fourier multipliers. By Definition 7.3.1, $\mathcal{A}$ is dense in $W^{p,s}_{\Gamma_\xi}$. Hence when we define the Fourier multipliers, we assume that $f \in \mathcal{A}$.

Definition 7.3.2 For the sequence $\{b_k\}_{k\in\mathbb{Z}}$ satisfying $|b_k| \leqslant k^s$, we define the Fourier multiplier $M_{(b_k)}$ as follows:

$$M_{(b_k)} f(x) = \sum_{k=0}^{\infty} b_k P_k(f)(x) + \sum_{k=0}^{\infty} b_{-k-1} Q_k(f)(x).$$

Remark 7.3.1 When Σ is the unit sphere, if we take two sequences $\{b_k^{(1)}\}$and $\{b_k^{(2)}\}$, where $b_k^{(1)} = k^2$ and $b_k^{(2)} = k$, the Fourier multipliers in Definition 7.3.2 reduce to the boundary values of the Photogenic-Cauchy integrals on the hyperbolic unit sphere, see Example 7.0.1.

Now for $k \geqslant 0$, we define

$$\widetilde{P}^{(k)}(y^{-1}x) = |y^{-1}x|^k C^+_{n+1,k}(\xi, \eta)$$

and

$$\widetilde{P}^{(-k-1)}(y^{-1}x) = |y^{-1}x|^{-k-n} C^-_{n+1,k}(\xi, \eta).$$

The projections P_k and Q_k can be expressed by

$$P_k(f)(x) = \frac{1}{\Omega_n} \int_\Sigma \widetilde{P}^{(k)}(y^{-1}x) E(y) \mathfrak{n}(y) f(y) d\sigma(y)$$

and

$$Q_k(f)(x) = \frac{1}{\Omega_n} \int_{\Sigma} \widetilde{P}^{(-k-1)}(y^{-1}x)E(y)\mathfrak{n}(y)f(y)d\sigma(y).$$

If we use

$$\widetilde{\phi}(y^{-1}x) = \sum_{-\infty}^{\infty} b_k \widetilde{P}^{(-k)}(y^{-1}x)$$

to denote the kernel of the Fourier multiplier $M_{(b_k)}$ in Definition 7.3.2, we get the following estimate.

Theorem 7.3.1 *Let* $\omega \in (\arctan(N), \pi/2)$ *and* $b \in H^s(S^c_\omega)$. *The kernel* $\widetilde{\phi}(y^{-1}x) E(y)$ *associated with* $\{b_k\}$ *in the manner given above is monogeneically defined in a neighborhood of* $\Sigma \times \Sigma \setminus \{(x,y) : x = y\}$. *In addition, in this neighborhood,*

$$|\widetilde{\phi}(y^{-1}x)| \leqslant \frac{C}{|1 - y^{-1}x|^{n+s}}.$$

Proof The proof of this theorem is similar to Proposition 6.2.3. We omit the details. □

For $f \in \mathcal{A}$, the multiplier $M_{(b_k)}$ introduced above is well-defined. For $b \in H^s(S^c_\omega)$, we consider the following multiplier $M^r_{(b_k)}$:

$$M^r_{(b_k)}(f)(x) = \sum_{k=0}^{\infty} b_k P_k(f)(rx) + \sum_{k=0}^{\infty} b_{-k-1} Q_k(f)(r^{-1}x), \ \rho - s < |x| < l + s,$$

where $x \in \Sigma, \ r \approx 1$ and $r < 1$.

We use M_1 and M_2 to denote the two sums in the expression of $M^r_{(b_k)}$. Because $b \in H^s(S^c_\omega)$, b is bounded near the origin and $|b(z)| \leqslant |z|^s$ when $|z| > 1$. We deduce that for $|z| > 1$, $|b(z)| \leqslant |z|^s < |z|^{s_1}$. Hence for $s_1 = [s] + 1$, $b \in H^{s_1}(S^c_\omega)$. Write $b_1(z) = z^{-s_1}b(z)$. We see that $|b_1(z)| \leqslant |b(z)/z^{s_1}| \leqslant C$ implies $b_1(z) \in H^\infty(S^c_\omega)$, where

$$H^\infty(S^c_{\mu,\pm}) = \Big\{ b : \ S^c_{\mu,\pm} \to \mathbb{C} : \ b \text{ is holomorphic, and satisfies } |b(z)| \leqslant C_\nu \text{ in any } S^c_{\nu,\pm}, \ 0 < \nu < \mu \Big\}$$

and

$$H^\infty(S^c_\mu) = \Big\{ b : \ S^c_\mu \to \mathbb{C} : \ b_\pm = b\chi_{\{z\in\mathbb{C}:\ \pm \mathrm{Re} z > 0\}} \in H^\infty(S^c_{\mu,\pm}) \Big\},$$

where $S^c_{\mu,\pm}$ and S^c_μ are sectors.

For M_1, $|b_k| = |b(k)| \leqslant k^{s_1}$, we take $b_1(z) = z^{-s_1}b(z)$. It is easy to see that b_1 is also holomorphic in S^c_ω. Then we have

$$M_1 = \sum_{k=0}^{\infty} b_k P_k(f)(rx) = \sum_{k=0}^{\infty} b_{1,k} k^{s_1} P_k(f)(rx),$$

where $b_{1,k} = b_1(k) = \frac{b_k}{k^{s_1}}$. Because M_k is an eigenspace of the spherical Dirac operator Γ_ξ, we have

$$\Gamma_\xi P_k(f)(rx) = kP_k(f)(rx)$$

and

$$M_1 = \sum_{k=0}^{\infty} b_{1,k} \Gamma_\xi^{s_1} P_k(f)(rx) = \Gamma_\xi^{s_1} \left(\sum_{k=0}^{\infty} b_{1,k} P_k(f)(rx) \right).$$

By a result of [8], we obtain another expression of $P_k(f)$.

$$\begin{aligned} P_k(f)(x) &= \frac{1}{\Omega_n} \int_\Sigma \widetilde{P}^k(y^{-1}rx)E(y)\mathfrak{n}(y)f(y)d\sigma(y) \\ &= \frac{1}{\Omega_n} \int_\Sigma \sum_{|\underline{\alpha}|=k} V_{\underline{\alpha}}(rx) W_{\underline{\alpha}}(y)\mathfrak{n}(y)f(y)d\sigma(y), \end{aligned}$$

where we have used the Cauchy–Kovalevska expansion

$$\widetilde{P}^{(k)}(y^{-1}x)E(y) = \sum_{|\underline{\alpha}|=k} V_{\underline{\alpha}}(x) W_{\underline{\alpha}}(y),$$

where $V_{\underline{\alpha}}(x) \in \mathcal{M}_k$ and $W_{\underline{\alpha}}(y) \in \mathcal{M}_{-n-k}$ (see [8, Chap. 2, (1.15)]). By the above relation, we have

$$\begin{aligned} \Gamma_\xi P_k(f)(x) &= \frac{1}{\Omega_n} \int_\Sigma \sum_{|\underline{\alpha}|=k} (\Gamma_\xi V_{\underline{\alpha}})(x) W_{\underline{\alpha}}(y)\mathfrak{n}(y)f(y)d\sigma(y) \\ &= \frac{1}{\Omega_n} \int_\Sigma \sum_{|\underline{\alpha}|=k} k V_{\underline{\alpha}}(x) W_{\underline{\alpha}}(y)\mathfrak{n}(y)f(y)d\sigma(y) \\ &= \frac{1}{\Omega_n} \int_\Sigma \sum_{|\underline{\alpha}|=k} \frac{k}{n+k-2} V_{\underline{\alpha}}(x)(n+k-2) W_{\underline{\alpha}}(y)\mathfrak{n}(y)f(y)d\sigma(y) \\ &= \frac{k}{(n+k-2)\Omega_n} \int_\Sigma \sum_{|\underline{\alpha}|=k} V_{\underline{\alpha}}(x)(\Gamma_\eta W_{\underline{\alpha}})(y)\mathfrak{n}(y)f(y)d\sigma(y). \end{aligned}$$

Because the Fourier expansion of the functions in $\mathcal{A}$ is rapidly decaying, via integration by parts, we have

$$
\begin{aligned}
M_1 &= \sum_{k=1}^{\infty} b_{1,k} k^{s_1} P_k(f)(rx) \\
&= \sum_{k=1}^{\infty} b_{1,k} \left(\frac{k}{n+k-2}\right)^{s_1} \frac{r^k}{\Omega_n} \int_{\Sigma} \sum_{|\underline{\alpha}|=k} V_{\underline{\alpha}}(x)(\Gamma_{\eta}^{s_1} W_{\underline{\alpha}})(y)\mathfrak{n}(y) f(y) d\sigma(y) \\
&= \sum_{k=1}^{\infty} b_{1,k} \left(\frac{k}{n+k-2}\right)^{s_1} \frac{r^k}{\Omega_n} \int_{\Sigma} \sum_{|\underline{\alpha}|=k} V_{\underline{\alpha}}(x) W_{\underline{\alpha}}(y)\mathfrak{n}(y)(\Gamma_{\eta}^{s_1} f)(y) d\sigma(y).
\end{aligned}
$$

Since $|b_{1,k}(\frac{k}{n+k-2})^{s_1}| \leqslant C$, if we denote $b_{1,k}(\frac{k}{n+k-2})^{s_1}$ by $b_{1,k}$, we can obtain the following singular integral expression of M_1:

$$
\begin{aligned}
M_1 &= \sum_{k=1}^{\infty} b_{1,k} \frac{1}{\Omega_n} \int_{\Sigma} \widetilde{P}^k(y^{-1}rx) E(y)\mathfrak{n}(y)(\Gamma_{\eta}^{s_1} f(y)) d\sigma(y) \\
&= \frac{1}{\Omega_n} \int_{\Sigma} \left(\sum_{k=1}^{\infty} b_{1,k} \widetilde{P}^k(y^{-1}rx)\right) E(y)\mathfrak{n}(y)(\Gamma_{\eta}^{s_1} f(y)) d\sigma(y) \\
&= \frac{1}{\Omega_n} \int_{\Sigma} \widetilde{\phi}_1(y^{-1}rx) E(y)\mathfrak{n}(y)(\Gamma_{\eta}^{s_1} f(y)) d\sigma(y).
\end{aligned}
$$

Similarly, for M_2, applying the Cauchy–Kovalevska expansion again ([8, Chap. II, (1.16)]), we have

$$
\begin{aligned}
M_2 &= \sum_{k=0}^{\infty} b_{-k-1} Q_k(f)(r^{-1}x) \\
&= \sum_{k=0}^{\infty} \frac{b_{-k-1}}{(k+1)^{s_1}} \left(\frac{k+1}{k}\right)^{s_1} \frac{1}{\Omega_n} \int_{\Sigma} \sum_{|\underline{\alpha}|=k} W_{\underline{\alpha}}(r^{-1}x) k^{s_1} \overline{V}_{\underline{\alpha}}(y)\mathfrak{n}(y) f(y) d\sigma(y) \\
&= \sum_{k=0}^{\infty} \frac{b_{-k-1}}{(k+1)^{s_1}} \left(\frac{k+1}{k}\right)^{s_1} \frac{1}{\Omega_n} \int_{\Sigma} \sum_{|\underline{\alpha}|=k} W_{\underline{\alpha}}(r^{-1}x)(\Gamma_{\eta}^{s_1} \overline{V}_{\underline{\alpha}})(y)\mathfrak{n}(y) f(y) d\sigma(y) \\
&= \sum_{k=0}^{\infty} \frac{b_{-k-1}}{(k+1)^{s_1}} \left(\frac{k+1}{k}\right)^{s_1} \frac{1}{\Omega_n} \int_{\Sigma} \sum_{|\underline{\alpha}|=k} W_{\underline{\alpha}}(r^{-1}x) \overline{V}_{\underline{\alpha}}(y)\mathfrak{n}(y)(\Gamma_{\eta}^{s_1} f)(y) d\sigma(y).
\end{aligned}
$$

As above, we still denote $\frac{b_{-k-1}}{(k+1)^{s_1}} \left(\frac{k+1}{k}\right)^{s_1}$ by b_{-1-k}, and obtain the singular integral expression of M_2 as

$$M_2 = \sum_{k=0}^{\infty} b_{-k-1} \frac{1}{\Omega_n} \int_{\Sigma} \widetilde{P}^{-k-1}(y^{-1}r^{-1}x)E(y)\mathfrak{n}(y)(\Gamma_{\eta}^{s_1} f)(y)d\sigma(y)$$
$$= \frac{1}{\Omega_n} \int_{\Sigma} \left(\sum_{k=0}^{\infty} b_{-k-1}\widetilde{P}^{-k-1}(y^{-1}r^{-1}x) \right) E(y)\mathfrak{n}(y)(\Gamma_{\eta}^{s_1} f)(y)d\sigma(y)$$
$$= \frac{1}{\Omega_n} \int_{\Sigma} \widetilde{\phi}_2(y^{-1}r^{-1}x)E(y)\mathfrak{n}(y)(\Gamma_{\eta}^{s_1} f(y))d\sigma(y).$$

Finally we rewrite the multiplier $M^r_{(b_k)}(f)(x)$ as

$$M^r_{(b_k)}(f)(x) = \lim_{r \to 1-} \frac{1}{\Omega_n} \int_{\Sigma} (\widetilde{\phi}_1(y^{-1}rx) + \widetilde{\phi}_2(y^{-1}r^{-1}x))E(y)\mathfrak{n}(y)(\Gamma_{\xi}^{s_1} f)(y)d\sigma(y),$$

where we have used the fact that for $f \in \mathcal{A}$, the series which defines $M^r_{b_k}(f)$ is uniformly convergent as $r \to 1-$.

For $M_{(b_k)}(f)(x)$, we have the following boundary value result.

Theorem 7.3.2 *Let $s > 0$. If $b \in H^s(S^c_\omega)$, then for $f \in \mathcal{A}$ and $x \in \Sigma$, we have*

$$M_{(b_k)}(f)(x) = \lim_{r \to 1-} \frac{1}{\Omega_n} \int_{\Sigma} (\widetilde{\phi}_1(y^{-1}rx) + \widetilde{\phi}_2(y^{-1}r^{-1}x))E(y)\mathfrak{n}(y)(\Gamma_{\xi}^{s_1} f)(y)d\sigma(y)$$
$$= \lim_{\varepsilon \to 0} \frac{1}{\Omega_n} \left\{ \int_{|y-x|>\varepsilon, y \in \Sigma} [\widetilde{\phi}_1(y^{-1}x) + \widetilde{\phi}_2(y^{-1}x)]E(y)\mathfrak{n}(y)(\Gamma_{\xi}^{s_1} f)(y)d\sigma(y) \right.$$
$$\left. + (\widetilde{\phi}_1(\varepsilon, x) + \widetilde{\phi}_2(\varepsilon, x))f(x) \right\}.$$

Here

$$\widetilde{\phi}_1(\varepsilon, x) = \int_{S(\varepsilon,x,+)} \widetilde{\phi}_1(y^{-1}x)E(y)\mathfrak{n}(y)d\sigma(y)$$

and

$$\widetilde{\phi}_2(\varepsilon, x) = \int_{S(\varepsilon,x,-)} \widetilde{\phi}_2(y^{-1}x)E(y)\mathfrak{n}(y)d\sigma(y),$$

where $S(\varepsilon, x, \pm)$ is the part of the sphere $|y - x| = \varepsilon$ inside or outside Σ depending on the index of $\widetilde{\phi}_i$ taking $i = 1$ or $i = 2$.

Proof The proof of this theorem is similar to the classical Plemelj formula of the Cauchy integral. For simplicity, we only consider

$$\lim_{r \to 1-} I = \lim_{r \to 1-} \frac{1}{\Omega_n} \int_{\Sigma} \widetilde{\phi}_1(y^{-1}rx)E(y)\mathfrak{n}(y)(\Gamma_{\xi}^{s_1} f)(y)d\sigma(y).$$

The other integral can be dealt with similarly. For a fixed $\varepsilon > 0$, the above integral can be divided into three parts:

$$\begin{aligned}
I &= \frac{1}{\Omega_n}\int_{\Sigma}\widetilde{\phi}_1(y^{-1}rx)E(y)\mathfrak{n}(y)(\Gamma_{\xi}^{s_1}f)(y)d\sigma(y)\\
&= \frac{1}{\Omega_n}\int_{y\in\Sigma,|y-x|>\varepsilon}\widetilde{\phi}_1(y^{-1}rx)E(y)\mathfrak{n}(y)(\Gamma_{\xi}^{s_1}f)(y)d\sigma(y)\\
&+ \frac{1}{\Omega_n}\int_{y\in\Sigma,|y-x|\leqslant\varepsilon}\widetilde{\phi}_1(y^{-1}rx)E(y)\mathfrak{n}(y)[(\Gamma_{\xi}^{s_1}f)(y)-(\Gamma_{\xi}^{s_1}f)(x)]d\sigma(y)\\
&+ \frac{1}{\Omega_n}\int_{y\in\Sigma,|y-x|\leqslant\varepsilon}\widetilde{\phi}_1(y^{-1}rx)E(y)\mathfrak{n}(y)d\sigma(y)(\Gamma_{\xi}^{s_1}f)(x)\\
&=: I_1+I_2+I_3,
\end{aligned}$$

where the symbol $\Gamma_{\xi}f(y)$ denotes the spherical Dirac operator Γ_{ξ} acting on the variable η of f, where $y=|y|\eta$.

Let $r\to 1-$. The integral I_1 tends to

$$\frac{1}{\Omega_n}\int_{y\in\Sigma,|y-x|>\varepsilon}\widetilde{\phi}_1(y^{-1}x)E(y)\mathfrak{n}(y)(\Gamma_{\eta}^{s_1}f)(y)d\sigma(y).$$

For I_2, because $f\in\mathcal{A}$ implies $\Gamma_{\xi}^{s_1}f$ is a Lipschitz function, we have

$$\begin{aligned}
\lim_{\varepsilon\to 0}\lim_{r\to 1-} I_2 &= \lim_{r\to 1-}\lim_{\varepsilon\to 0}\int_{y\in\Sigma,\ |y-x|\leqslant\varepsilon}\widetilde{\phi}_1(y^{-1}rx)E(y)\mathfrak{n}(y)\\
&\quad\times\Big[(\Gamma_{\xi}^{s_1}f)(y)-(\Gamma_{\xi}^{s_1}f)(x)\Big]d\sigma(y)=0.
\end{aligned}$$

Finally we estimate I_3. By Cauchy's theorem, for any fixed $\varepsilon>0$, we have

$$\begin{aligned}
\lim_{r\to 1-} I_3 &= \lim_{r\to 1-}\int_{y\in\Sigma,\ |y-x|\leqslant\varepsilon}\widetilde{\phi}_1(y^{-1}rx)E(y)\mathfrak{n}(y)d\sigma(y)(\Gamma_{\xi}^{s_1}f)(x)\\
&= \widetilde{\phi}_1(\varepsilon,x)(\Gamma_{\xi}^{s_1}f)(x).
\end{aligned}$$

This completes the proof of the theorem. □

As a useful tool in the study of boundary value problems on the non-smooth domains, the theory of Hardy spaces on Lipschitz curves and surfaces has attracted attention of many mathematicians. In 1980s, Jerison and Kenig [9, 10] considered the complex variable case. In [11], Mitrea introduced the theory of Clifford-valued Hardy spaces on high-dimensional Lipschitz graphs.

Let Δ and Δ^c be the bounded and unbounded connected components of $\mathbb{R}_1^n\setminus\Sigma$, respectively. For $\alpha>0$, define the non-tangential approach regions $\Lambda_\alpha(x)$ and $\Lambda_\alpha^c(x)$ to a point $x\in\Sigma$ as

$$\Lambda_\alpha(x)=\Big\{x\in\Delta,\ |y-x|<(1+\alpha)dist(y,\Sigma)\Big\}$$

and

$$\Lambda_\alpha^c(x) = \Big\{ y \in \Delta^c, \ |y-x| < (1+\alpha)dist(y, \Sigma)\Big\}.$$

Let f be defined in Δ (Δ^c). The interior non-tangential maximal function $N_\alpha(f)$ is defined as

$$N_\alpha(f)(x) = \sup\Big\{ |f(y)| \ : \ y \in \Lambda_\alpha(x)(y \in \Lambda_\alpha^c(x))\Big\}.$$

For $0 < p < \infty$, Hardy spaces $\mathcal{H}^p(\Delta)$ and $\mathcal{H}^p(\Delta^c)$ are defined as

$$\mathcal{H}^p(\Delta) = \Big\{ f : f \text{ is left monogenic in } \Delta \text{ and } N_\alpha(f) \in L^p(\Sigma)\Big\},$$
$$\mathcal{H}^p(\Delta^c) = \Big\{ f : f \text{ is left monogenic in } \Delta^c \text{ and } N_\alpha(f) \in L^p(\Sigma)\Big\}.$$

The theory of monogenic Hardy spaces in [11] indicates that for $p > 1$, the $\mathcal{H}^p(\Delta)$-norm of a function is equivalent to the L^p-norm of its non-tangential maximal function on the boundary. For the spaces $\mathcal{H}^p(\Delta^c)$, similar conclusions hold. Precisely, if $f \in \mathcal{H}^p(\Delta)$ for $p > 1$, we have

$$C_1\|f\|_{\mathcal{H}^p(\Delta)} \leqslant \|f\|_{L^p(\Sigma)} \leqslant C_2\|f\|_{\mathcal{H}^p(\Delta)}.$$

If $f \in \mathcal{M}_k$ and $k \neq -1, -2, \ldots, -n+1$, because $\mathcal{M}_k$ is the subspace consisting of all k-homogeneous left monogenic functions, we have $\Gamma_\xi f(\xi) = kf(\xi)$. For $f \in \mathcal{A}$, we define $\Gamma(f|_\Gamma)$ as the restriction of the monogenic extension of $\Gamma_\xi(f|_{S_{\mathbb{R}_1^n}})$ to Γ. Then the definition of Γ_ξ can be extended to $\Gamma_\xi : \mathcal{A} \to \mathcal{A}$.

In [3], Eelbode studied the boundary value of the Photogenic-Cauchy transform C_P^α on the unit hyperbolic sphere. In Example 7.0.1, The occurrence of the factors $k^2P_k(f)$ and $k^2Q_k(f)$ implies that the boundary value $C_P^\alpha[f]\uparrow$ of C_P^α is not a bounded operator from $L^2(\mathbb{S}^{n-1})$ to itself. If we restrict this operator to some smaller subspaces of $L^2(\mathbb{S}^{n-1})$, we can obtain the corresponding boundedness.

Now we give the main result of this section.

Theorem 7.3.3 *Let* $\omega \in (\arctan(N), \pi/2)$. *If* $b \in H^s(S_\omega^c)$, $s > 0$, *then with the assumption* $b(0) = 0$, *the multipliers introduced in Definition 7.3.2 can be extended to a bounded operator from* $W^{2,s_1}_{\Gamma_\xi}(\Sigma)$ *to* $L^2(\Sigma)$, *where* $s_1 = \lceil s \rceil$. *In addition,*

$$\|M_{(b(k))}\|_{op} \leqslant C_\nu \left\| \frac{b}{|z+1|^s} \right\|_{L^\infty(S_\nu^c)}, \ \arctan N < \nu < \omega.$$

Proof For $f \in W^{2,s_1}_{\Gamma_\xi}(\Sigma) \subset L^2(\Sigma)$, by Proposition 6.2.7, we have $f = f^+ + f^-$, where $f^+ \in \mathcal{H}^2(\Delta)$ and $f^- \in \mathcal{H}^2(\Delta^c)$ such that

$$\|f^\pm\|_{L^2(\Sigma)} \leqslant C_N\|f\|_{W^{2,s_1}(\Sigma)}.$$

By the linearity and Theorem 7.3.2, we have $M_b(f) = M_{b^+}f^+ + M_{b^-}f^-$, where

$$M_{b^{\pm}}f^{\pm}(x) = \lim_{r\to -}\int_{\Sigma}\widetilde{\phi}_{\pm}(r^{\pm 1}y^{-1}x)E(y)\mathfrak{n}(y)f(y)d\sigma(y), x \in \Sigma.$$

Hence, we only need to prove

$$\|M_{b^{\pm}}f^{\pm}\|_{\mathcal{H}^2} \leqslant C_N\|\Gamma_{\xi}^{s_1}f^{\pm}\|_{\mathcal{H}^2}.$$

We only prove the above inequality for f^+. For the sake of simplicity, we omit the symbol "+". The f^- part can be similarly dealt.w

By Theorem 7.3.1, for $b \in H^s(S_{\omega}^c)$, we have

$$|\widetilde{\phi}(y^{-1}x)| \leqslant \frac{C}{|1-y^{-1}x|^{n+s}}.$$

Hence by Hölder's inequality, we obtain

$$\begin{aligned}
&|\Gamma_{\xi}^{1+s_1}M_bf(x)| \\
&\leqslant \left(\int_{\Sigma_{\sqrt{t}}}|\phi(y^{-1}x)|\frac{d\sigma(y)}{|y|^n}\right)^{1/2}\left(\int_{\Sigma_{\sqrt{t}}}|\phi(y^{-1}x)||\Gamma_{\xi}^{s_1+1}f(y)|^2\frac{d\sigma(y)}{|y|^n}\right)^{1/2} \\
&\leqslant C\left(\int_{\Sigma_{\sqrt{t}}}\frac{1}{|1-y^{-1}x|^{n+s}}\frac{d\sigma(y)}{|y|^n}\right)^{1/2}\left(\int_{\Sigma_{\sqrt{t}}}\frac{|\Gamma_{\xi}^{s_1+1}f(y)|^2}{|1-y^{-1}x|^{n+s}}\frac{d\sigma(y)}{|y|^n}\right)^{1/2}.
\end{aligned}$$

Through change of variable, we have

$$\begin{aligned}
|\Gamma_{\xi}^{1+s_1}M_bf(x)| \leqslant C&\left(\int_{\Sigma}\frac{1}{[(1-\sqrt{t})^2+\theta_0^2]^{\frac{n+s}{2}}}d\sigma(y)\right)^{1/2} \\
&\times\left(\int_{\Sigma}\frac{1}{[(1-\sqrt{t})^2+\theta_0^2]^{\frac{n+s}{2}}}|\Gamma_{\xi}^{1+s_1}f(y)|^2d\sigma(y)\right)^{1/2},
\end{aligned}$$

where the integral in the last inequality satisfies

$$\begin{aligned}
\int_{\Sigma}\frac{1}{[(1-\sqrt{t})^2+\theta_0^2]^{\frac{n+s}{2}}}d\sigma(y) &\leqslant \int_0^{\pi}\frac{\sin^{n-1}\theta_0}{[(1-\sqrt{t})^2+\theta_0^2]^{\frac{n+s}{2}}}d\theta_0 \\
&\leqslant C\frac{1}{(1-\sqrt{t})^s}.
\end{aligned}$$

Hence by the equivalent characterization given in Proposition 6.2.6, we have

$$
\begin{aligned}
&\|M_b f\|^2_{H^2(\Delta)} \\
&\leqslant \int_0^1 \int_\Sigma |\Gamma_\xi^{1+s_1} M_b f(tx)|^2 (1-t)^{2s_1+1} d\sigma(x) \frac{dt}{t} \\
&\leqslant C \int_0^1 \int_\Sigma \frac{(1-\sqrt{t})^{2s_1+1}}{(1-\sqrt{t})^s} \left(\int_\Sigma \frac{|\Gamma_\xi^{1+s_1} f(\sqrt{t}y)|^2}{[(1-\sqrt{t})^2+\theta_0^2]^{\frac{n+s}{2}}} d\sigma(y) \right) d\sigma(x) \frac{dt}{t} \\
&\leqslant C \int_0^1 \int_\Sigma |\Gamma_\xi^{1+s_1} f(\sqrt{t}y)|^2 \left(\int_\Sigma \frac{(1-\sqrt{t})^s}{[(1-\sqrt{t})^2+\theta_0^2]^{\frac{n+s}{2}}} d\sigma(x) \right) (1-\sqrt{t}) d\sigma(y) \frac{dt}{t} \\
&\leqslant C \int_0^1 \int_\Sigma \left| \Gamma_\xi (\Gamma_\xi^{s_1} f)(\sqrt{t}y) \right|^2 (1-\sqrt{t}) d\sigma(y) \frac{dt}{t} \\
&\leqslant C \|\Gamma_\xi^{s_1} f\|_{\mathcal{H}^2(\Delta)},
\end{aligned}
$$

where in the forth inequality we used the fact that for $t \in (0, 1)$,

$$
(1-\sqrt{t})^{2s_1+1-s} = (1-\sqrt{t})^{1+s+2s_1-s} \leqslant (1-\sqrt{t})^{1+s}
$$

and

$$
\int_\Sigma \frac{(1-\sqrt{t})^s}{[(1-\sqrt{t})^2+\theta_0^2]^{\frac{n+s}{2}}} d\sigma(x) \leqslant C(1-\sqrt{t})^s \frac{1}{(1-\sqrt{t})^s} \leqslant C.
$$

In the last inequality, we used Proposition 6.2.6. This completes the proof of Theorem 7.3.3. □

For the classical convolution singular integral operator T_ϕ on $\mathbb{R}^n$, one of the basic facts is the endpoint estimate, that is, the weak-(1, 1) boundedness. If for all $\lambda > 0$,

$$
|\{x \in \Sigma : |T(f)(x)| > \lambda\}| \leqslant \frac{C}{\lambda} \|f\|_1,
$$

we call an operator T is weak-(1, 1) bounded on Σ. In other words, we say that this operator is bounded from L^1 to the weak type space WL^1, see [12–14] and the reference therein. By this weak boundedness, we can use the interpolation theory and the duality of operators to deduce the L^p-boundedness of T_ϕ, $1 < p < \infty$. In the rest of this section, we study the endpoint estimate of the Fourier multipliers.

Theorem 7.3.4 *Let* $\omega \in (\arg(N), \frac{\pi}{2})$. *If* $b \in H^s(S_\omega^c)$, $s > 0$ *and* $b(0) = 0$. *Then the multiplier* $M_{(b_k)}$*:*

$$
M_{(b_k)}(f)(x) = \sum_{k=0}^{\infty} b_k P_k(f)(x) + \sum_{k=0}^{\infty} b_{-k-1} Q_k(f)(x)
$$

is bounded from $W^{1,s_1}_{\Gamma_\xi}(\Sigma)$ *to* $WL^1(\Sigma)$, *where* $s_1 = \lceil s \rceil$.

Proof For $b \in H^s(S^c_\omega)$ and $z \in S^c_\omega$, $|b(z)| \leqslant C|z|^s,\ s > 0$. Hence it is natural to get $|b(z)/z^s| \leqslant C$, where C is a constant. On the other hand, $b \in H^s(S^c_\omega)$ implies that b is holomorphic in S^c_ω. Then $z^{-s}b(z)$ is also holomorphic in S^c_ω. Now for the Fourier multiplier $M_{(b_k)}$, we have

$$\begin{aligned} M_{(b_k)}f(x) &= \sum_{k=0}^{\infty} b_k P_k(f)(x) + \sum_{k=0}^{\infty} b_{-k-1} Q_k(f)(x) \\ &= I + II. \end{aligned}$$

For simplicity, we only deal with the term I. As above, I can be represented as

$$I = \frac{1}{\Omega_n}\int_\Sigma \widetilde{\phi}(y^{-1}x)E(y)\mathfrak{n}(y)f(y)d\sigma(y).$$

If we write $b(z) = z^{s_1}b_1(z)$ and $b_1(z) \in H^\infty(S^c_\omega)$, then the corresponding sequence is $\{b_{1,k}\}$ whose the elements is $b_k = k^{s_1}b_{1,k}$. Therefore we can rewrite I as the following form

$$I = \sum_{k=0}^{\infty} b_{1,k}k^{s_1}P_k(f)(x).$$

The kernel associated to $M_{b_{1,k}}$ is denoted by $\widetilde{\phi}_1(y^{-1}x)E(y)$ that satisfies

$$\Gamma_\xi(\widetilde{\phi}_1(y^{-1}x))E(y) = \sum_{k=1}^{\infty} kb_1(k)\widetilde{P}^{(k)}(y^{-1}x)E(y).$$

By integration by parts, we get

$$\begin{aligned} I &= \frac{1}{\Omega_n}\int_\Sigma \Gamma_\xi^{s_1}(\widetilde{\phi}_1(y^{-1}x))E(y)\mathfrak{n}(y)f(y)d\sigma(y) \\ &= \frac{1}{\Omega_n}\int_\Sigma \widetilde{\phi}_1(y^{-1}x)E(y)\mathfrak{n}(y)\Gamma_\eta^{s_1}(f)(y)d\sigma(y). \end{aligned}$$

Similarly, if we take $s = 0$ in Theorem 7.3.1, $\widetilde{\phi}_1(y^{-1}x)$ satisfies

$$|\widetilde{\phi}_1(y^{-1}x)| \leqslant \frac{C}{|1 - y^{-1}x|^n}.$$

Hence the multiplier $M_{b_{1,k}}$ reduces to a H^∞-Fourier multiplier on starlike Lipschitz graph and is weak-(1, 1) bounded. Then we have

$$\begin{aligned} \left|\left\{x \in \Sigma :\ |M_{b_k}f(x)| > \lambda\right\}\right| &= \left|\left\{x \in \Sigma :\ |M_{b_{1,k}}(\Gamma_\xi^{s_1}f)(x)| > \lambda\right\}\right| \\ &\leqslant \frac{C}{\lambda}\left\|\Gamma_\xi^{s_1}f\right\|_{L^1}. \end{aligned}$$

This completes the proof of this theorem. □

At last, we consider the boundedness of the Fourier multipliers for $s < 0$. Let $-n < s < 0$ and $\{b_k\}$ be a sequence which satisfies $|b_k| \leqslant k^s$. We define the Fourier multiplier $M_{(b_k)}$ as follows.

$$M_{(b_k)}(f)(x) = \sum_{k=1}^{\infty} b_k P_k(f)(x) + \sum_{k=1}^{\infty} b_{-k-1} Q_k(f)(x).$$

Similar to the case $s > 0$, we can express the multiplier as

$$M_{(b_k)}(f)(x) = \frac{1}{\Omega_n} \int_{\Sigma} \widetilde{\phi}(y^{-1}x) E(y) \mathfrak{n}(y) f(y) d\sigma(y).$$

Here $x \in \Sigma$ and

$$\widetilde{\phi}(y^{-1}x) = \left(\sum_{k=1}^{\infty} + \sum_{-\infty}^{-1} \right) b_k \widetilde{P}^{(k)}(y^{-1}x),$$

where $\widetilde{P}^{(k)}$ is the polynomial defined as

$$\widetilde{P}^{(k)}(y^{-1}x) = |y^{-1}x|^k C^{+}_{n+1,k}(\xi, \eta)$$

or

$$\widetilde{P}^{(-k-1)}(y^{-1}x) = |y^{-1}x|^{-k-n} C^{-}_{n+1,k}(\xi, \eta).$$

To obtain the boundedness of the multiplier, we need to estimate the function $\widetilde{\phi}(x)$. By the method of Theorem 1.3.2, we can prove that the kernel $\phi(x) = \sum_{k=-\infty}^{\infty} b_k P^k(x)$ satisfies

$$|\phi(x)| \leqslant \frac{C|x|^s}{|1-x|^{n+s}}, \quad \text{where } x \in H_{\omega}.$$

For the kernel $\widetilde{\phi}(y^{-1}x)$ defined above, we can use the method of Proposition 6.2.3 to obtain

$$|\widetilde{\phi}(y^{-1}x)| \leqslant \frac{C|y^{-1}x|^s}{|1-y^{-1}x|^{n+s}}.$$

For any two points x_1, x_2 on the starlike Lipschitz surface, we have $x_2^{-1}x_1 \in H_{\omega}$, that is, there exist two constants C_1, C_2 such that $C_1 \leqslant |x_2^{-1}x_1| \leqslant C_2$. Hence for any points $x_1, x_2 \in \Sigma$, the equality

$$|x_1| = |x_2 x_2^{-1} x_1| = |x_2||x_2^{-1}x_1|$$

implies that $C_1|x_1| \leqslant |x_2| \leqslant C_2|x_1|$. In other words, the norms of the two points on the starlike Lipschitz surface are approximately a constant associated with Σ, denoted by C_Σ. Hence we can obtain the estimate

$$\begin{aligned}|\widetilde{\phi}(y^{-1}x)E(y)\mathfrak{n}(y)| &\leqslant \frac{C|y^{-1}x|^s}{|1-y^{-1}x|^{n+s}}\frac{1}{|y|^n}\\ &\leqslant \frac{C|x|^s}{|y-x|^{n+s}}\\ &\leqslant \frac{C_\Sigma}{|y-x|^{n+s}}.\end{aligned}$$

Because the Lipschitz surface is a homogeneous space, our Fourier multiplier $M_{(b_k)}f(x)$ can be regarded as the fractional integral operator on Σ. By the classical theory of the fractional integral operator on homogeneous spaces, we can obtain the $L^p - L^q$ boundedness of the Fourier multiplier as follows.

Theorem 7.3.5 *Let* $-n < s < 0$*,* $1 \leqslant p < q < \infty$ *and* $\frac{1}{q} = \frac{1}{p} + \frac{s}{n}$*. If* $b \in H^s(S^c_\omega)$*, the Fourier multipliers on starlike Lipschitz surface:*

$$M_{(b_k)}f(x) = \sum_{k=1}^{\infty} b_k P_k(f)(x) + \sum_{k=1}^{\infty} b_{-k-1}Q_k(f)(x)$$

with $b_k = b(k)$ *is bounded from* $L^p(\Sigma)$ *to* $L^q(\Sigma)$*.*

Proof For a starlike Lipschitz surface Σ, if $x_1, x_2 \in \Sigma$, then $x_2^{-1}x_1 \in H_\omega$, i.e., there exist two constants c_1, c_2 depending on ω and Σ such that $C_1 \leqslant |x_2^{-1}x_1| \leqslant C_2$. For any points $x_1, x_2 \in \Sigma$, the equality

$$|x_1| = |x_2x_2^{-1}x_1| = |x_2||x_2^{-1}x_1|$$

indicates that $C_1|x_1| \leqslant |x_2| \leqslant C_2|x_1|$. In other words, the norm of any point on Σ is about a constant C_Σ which is related to Σ. Then the kernel $\phi(y^{-1}x)E(y)$ satisfies

$$\begin{aligned}|\phi(y^{-1}x)E(y)| &= |\phi(y^{-1}x)||E(y)|\\ &\leqslant \frac{C}{|1-y^{-1}x|^{n+s}}\frac{1}{|y|^n}\\ &\leqslant \frac{C|y|^s}{|y-x|^{n+s}}\\ &\leqslant \frac{C_\Sigma}{|y-x|^{n+s}}.\end{aligned}$$

In addition, for any ball $B(x, r) = \left\{y \in \Sigma,\ |x-y| < r\right\}$, we have

$$\sigma(B(x, r)) = \int_{B(x,r)} d\sigma(y) \leqslant Cr^n,$$

that is, the surface measure of $B(x, r)$ is dominated by the area of a sphere in $\mathbb{R}^n$. Hence, we can use the classical method to prove the boundedness. Below we give the details. At first, we define the auxiliary function $\Omega(x)$ by

$$\Omega(x) = \sup_{r>0} \frac{\sigma(B(x, r))}{r^n}.$$

For the integral representation of M_b, we divide the integral into two parts.

$$|M_b(f)(x)| \leqslant \left(\int_{B(x,r)} + \int_{\Sigma \setminus B(x,r)} \right) |f(y)| \frac{1}{|y-x|^{n+s}} d\sigma(y) =: I_1 + I_2.$$

For I_1, we have

$$\begin{aligned} I_1 &\leqslant \int_{B(x,r)} |f(y)| \frac{1}{|y-x|^{n+s}} d\sigma(y) \\ &= \sum_{k=0}^{\infty} \int_{B(x,2^{-k}r)\setminus B(x,2^{-k-1}r)} |f(y)| \frac{1}{|y-x|^{n+s}} d\sigma(y). \end{aligned}$$

Because $|y - x| \leqslant 2^{-k} r$ for $y \in B(x, 2^{-k}r) \setminus B(x, 2^{-k-1}r)$, we can obtain

$$\begin{aligned} I_1 &\leqslant \sum_{k=0}^{\infty} (2^{-k-1}r)^{-n-s} \sigma(B(x, 2^{-k}r)) \frac{1}{\sigma(B(x, r))} \int_{B(x,2^{-k}r)} |f(y)| d\sigma(y) \\ &\leqslant \sum_{k=0}^{\infty} (2^{-k-1}r)^{-n-s} \sigma(B(x, 2^{-k}r)) M(f)(x). \end{aligned}$$

By the definition of $\Omega(x)$, we have

$$\sigma(B(x, 2^{-k}r)) = \frac{\sigma(B(x, 2^{-k}r))}{(2^{-k}r)^n} \leqslant \Omega(x)(2^{-k}r)^n.$$

Then by $-s > 0$, we get

$$I_1 \leqslant r^{-s} \Omega(x) M(f)(x) \sum_{k=0}^{\infty} (2^{-k-1})^{-s} \leqslant r^{-s} \Omega(x) M(f)(x).$$

For I_2, we have

$$
\begin{aligned}
I_2 &\leqslant \sum_{k=0}^{\infty} \int_{B(x,2^{k+1}r)\setminus B(x,2^k r)} \frac{|f(y)|}{|x-y|^{n+s}} d\sigma(y) \\
&\leqslant \sum_{k=0}^{\infty} (2^k r)^{-s-n} (\sigma(B(x,2^{k+1}r)))^{1-\lambda/p} (\sigma(B(x,2^{k+1}r)))^{\lambda/p-1} \int_{B(x,2^{k+1}r)} |f(y)| d\sigma(y) \\
&\leqslant \sum_{k=0}^{\infty} (2^k r)^{-s-n} (2^{k+1} r)^{n(1-\lambda/p)} (\Omega(x))^{1-\lambda/p} M_{\lambda/p}(f)(x) \\
&= r^{-s-n\lambda/p} \left(\sum_{k=0}^{\infty} 2^{k(-n-s)} 2^{nk(1-\lambda/p)} \right) (\Omega(x))^{1-\lambda/p} M_{\lambda/p}(f)(x).
\end{aligned}
$$

Because $s - n\lambda/p < 0$ for $1 \leqslant p < n\lambda/s$, then

$$
|M_b(f)(x)| \leqslant r^{-s}\Omega(x) M(f)(x) + r^{-s-n\lambda/p} (\Omega(x))^{1-\lambda/p} M_{\lambda/p}(f)(x).
$$

Letting

$$
r = \left(\frac{M_{\lambda/p}(f)(x)}{M(f)(x)} \right)^{p/n\lambda} \frac{1}{\Omega^{1/n}(x)},
$$

we obtain

$$
\begin{aligned}
|M_b(f)(x)| &\leqslant \left(M_{\lambda/p}(f)(x)\right)^{-sp/n\lambda} \left(\Omega(x)\right)^{1+s/n} \left(M(f)(x)\right)^{1+sp/n\lambda} \\
&\quad + \left(M_{\lambda/p}(f)(x)\right)^{-ps/n\lambda-1+1} \left(M(f)(x)\right)^{-sp/n\lambda+1} \left(\Omega(x)\right)^{1+s/n} \\
&\quad \leqslant \left(\Omega(x)\right)^{s/n+1} \left(M_{\lambda/p}(f)(x)\right)^{-sp/n\lambda} \left(M(f)(x)\right)^{1+sp/n\lambda}.
\end{aligned}
$$

Now we get

$$
\left\| (\Omega(x))^{-s/n-1} M_b(f)(x) \right\|_{L^q}^q \leqslant \int_\Sigma \left(M_{\lambda/p}(f)(x)\right)^{-spq/n\lambda} \left(M(f)(x)\right)^{(1+sp/n\lambda)q} d\sigma(x).
$$

Let $\lambda = 1$. Because $\sigma(B(x,r)) \leqslant cr^n$, then $\Omega^{-s/n-1}(x) \geqslant C^{-s/n-1}$ for $-n < s < 0$. By the fact that $M_{1/p} f(x) \leqslant C\|f\|_p$, we see that

$$
\begin{aligned}
\left\| (\Omega(x))^{-s/n-1} M_b(f)(x) \right\|_{L^q}^q &= \int_\Sigma |M_{1/p}(f)(x)|^{q-p} |M(f)(x)|^p d\sigma(x) \\
&\leqslant \|M_{1/p} f\|_\infty^{q-p} \|M(f)\|_p^p \\
&\leqslant C\|f\|_p^{q-p} \|f\|_p^p \\
&\leqslant C\|f\|_p^q.
\end{aligned}
$$

This completes the proof of Theorem 7.4.1. □

7.4 The Equivalence of Hardy–Sobolev Spaces

In this section, we give an application of Fourier multipliers on the starlike Lipschitz surface Σ. In the proof of Theorem 7.3.1, we used the Hardy decomposition of $L^2(\Sigma)$: for $f \in L^2(\Sigma)$, $f = f^+ + f^-$, where $f^+ \in \mathcal{H}^2(\Delta)$ and $f^- \in \mathcal{H}^2(\Delta^c)$. If $f \in W^{2,s}_{\Gamma_\xi}(\Sigma)$, f^+ and f^- belong to the so-called Hardy–Sobolev spaces. For these spaces, there exist two methods to given the definitions.

Method I. For $f \in L^2(\Sigma)$, $f = f^+ + f^-$, where $f^+ \in \mathcal{H}^{2,+}$ and $f^- \in \mathcal{H}^{2,-}$. That is f^+ belongs to the Hardy space, while f^- belongs to the conjugate Hardy space. We define the Hardy–Sobolev space on Σ as

$$\mathcal{H}^{2,s}_{+,1}(\Sigma) = \Big\{f : \text{ there exists a function } g \in L^2(\Sigma) \text{ such that } f = g^+ \in L^2(\Sigma) \text{ and } \Gamma^j_\xi(g^+) \in L^2(\Sigma),\ j = 1, 2, \ldots, s\Big\}$$

and

$$\mathcal{H}^{2,s}_{-,1}(\Sigma) = \Big\{f :\in L^2(\Sigma) \text{ there exists a function } g \in L^2(\Sigma) \text{ such that } f = g^- \in L^2(\Sigma) \text{ and } \Gamma^j_\xi(g^-) \in L^2(\Sigma),\ j = 1, 2, \ldots, s\Big\}.$$

Method II. At first for any $f \in W^{2,s}_{\Gamma_\xi}$, $\Gamma^j_\xi f \in L^2(\Sigma)$, $j = 1, 2, \ldots, s$. We obtain the decomposition $\Gamma^j_\xi f = (\Gamma^j_\xi f)^+ + (\Gamma^j_\xi f)^-$, where $(\Gamma^j_\xi f)^+ \in \mathcal{H}^{2,+}$ and $(\Gamma^j_\xi f)^- \in \mathcal{H}^{2,-}$. The Hardy–Sobolev spaces are defined as follows.

$$\mathcal{H}^{2,s}_{+,2}(\Sigma) = \Big\{f : \text{ there exists a function } g \in L^2(\Sigma) \text{ such that } f = g^+ \in L^2(\Sigma) \text{ and } (\Gamma^j_\xi g)^+ \in L^2(\Sigma),\ j = 1, 2, \ldots, s\Big\}$$

and

$$\mathcal{H}^{2,s}_{-,2}(\Sigma) = \Big\{f : \text{ there exists a function } g \in L^2(\Sigma) \text{ such that } f = g^- \in L^2(\Sigma) \text{ and } (\Gamma^j_\xi g)^- \in L^2(\Sigma),\ j = 1, 2, \ldots, s\Big\}.$$

On the unit sphere, because we can exchange the order of the Riesz transform and the Dirac operator, the above two Hardy–Sobolev spaces are the same one obviously. On a general starlike Lipschitz surface, we will use the theory of Fourier multipliers to show that the two kinds of Hardy–Sobolev spaces are equivalent on Σ.

Theorem 7.4.1 *For the starlike Lipschitz surface Σ, let s be a positive integer, the Hardy–Sobolev spaces $\mathcal{H}^{2,s}_{\pm,1}(\Sigma)$ and $\mathcal{H}^{2,s}_{\pm,1}(\Sigma)$ are equivalent.*

Proof Because $\mathscr{A}$ is dense in $L^2(\Sigma)$, without loss of generality, we assume that $f \in \mathscr{A}$. By the spherical harmonic expansion, we have

$$f = \sum_{k=1}^{\infty} P_k(f)(x) + \sum_{k=1}^{\infty} Q_k(f)(x).$$

Then letting $f^+ = \sum_{k=1}^{\infty} P_k(f)(x)$ and $f^- = \sum_{k=1}^{\infty} Q_k(f)(x)$, we get

$$\Gamma_\xi(f^+) = \Gamma_\xi\left(\sum_{k=1}^{\infty} P_k(f)(x)\right).$$

Because $P_k(f)(x)$ belongs to the k-homogeneous eigenspace $\mathcal{M}_k$, we can deduce that

$$\Gamma_\xi(f^+)(x) = \sum_{k=1}^{\infty} kP_k(f)(x) \text{ for } f \in \mathscr{A}.$$

On the other hand,

$$\begin{aligned} P_k(f)(x) &= \frac{1}{\Omega_n}\int_\Sigma \widetilde{P}^k(y^{-1}x)E(y)\mathfrak{n}(y)f(y)d\sigma(y) \\ &= \frac{1}{\Omega_n}\int_\Sigma \sum_{|\underline{\alpha}|=k} V_{\underline{\alpha}}(x)W_{\underline{\alpha}}(y)\mathfrak{n}(y)f(y)d\sigma(y), \end{aligned}$$

where we use the Cauchy–Kovalevska expansion

$$\widetilde{P}^k(y^{-1}x)E(y) = \sum_{|\underline{\alpha}|=k} V_{\underline{\alpha}}(x)W_{\underline{\alpha}}(y),$$

where $V_{\underline{\alpha}}(x) \in \mathcal{M}_k$ and $W_{\underline{\alpha}}(y) \in \mathcal{M}_{-3-k}$. Hence we can get

$$\begin{aligned} \Gamma_\xi(f^+)(x) &= \frac{1}{\Omega_n}\sum_{k=1}^{\infty}\int_\Sigma \sum_{|\underline{\alpha}|=k} V_{\underline{\alpha}}(x)\frac{k}{k+1}(k+1)W_{\underline{\alpha}}(y)\mathfrak{n}(y)f(y)d\sigma(y) \\ &= \frac{1}{\Omega_n}\sum_{k=1}^{\infty}\frac{k}{k+1}\int_\Sigma \sum_{|\underline{\alpha}|=k} V_{\underline{\alpha}}(x)\Gamma_\eta W_{\underline{\alpha}}(x)\mathfrak{n}(y)f(y)d\sigma(y). \end{aligned}$$

Because f decays fast for $f \in \mathscr{A}$, we can use integration by parts to obtain that

$$\Gamma_\xi(f^+)(x) = \frac{1}{\Omega_n}\sum_{k=1}^{\infty}\frac{k}{k+1}\int_\Sigma \widetilde{P}^{(k)}(y^{-1}x)E(y)\mathfrak{n}(y)(\Gamma_\eta f)(y)d\sigma(y)$$
$$= \frac{1}{\Omega_n}\sum_{k=1}^{\infty}\frac{k}{k+1}P_k(\Gamma_\xi f)(x).$$

Let $b_k = \frac{k}{k+1}$. We have $\Gamma_\xi(f^+)(x) = M_{(b_k)}((\Gamma_\xi f)^+)$. Since $|b_k| \leqslant C$, it follows from the theory of Fourier multipliers on Σ that $M_{(b_k)}$ is bounded on $L^2(\Sigma)$, that is, there exists a constant C_1 such that

$$\|(\Gamma_\xi f^+)\|_{L^2(\Sigma)} \leqslant C_1\|(\Gamma_\xi f)^+\|_{L^2(\Sigma)}.$$

Conversely, let $b_k' = \frac{k+1}{k}$. Similarly, we can get

$$(\Gamma_\xi f)^+(x) = \frac{1}{\Omega_n}\sum_{k=1}^{\infty}\frac{k+1}{k}(\Gamma_\xi P_k(f))(x) = M_{(b_k')}(\Gamma_\xi(f^+))(x),$$

and there exists a constant C_2 such that

$$\|(\Gamma_\xi f)^+\|_{L^2(\Sigma)} \leqslant C_1\|\Gamma_\xi(f^+)\|_{L^2(\Sigma)}.$$

This proves Theorem 7.4.1. □

7.5 Remarks

Remark 7.5.1 The definitions of $H^s_{\ln}$ & $K^s_{\ln}$ and Theorem 7.1.3 only concern the case of the first power of the log function. In fact, if k is a positive integer, by the same proof, we can extend (ii) of Theorem 7.1.3 to the kth power of the log function.

Remark 7.5.2 By the following method, we can obtain variations of Theorems 7.1.1–7.1.3. Denote by $\exp(-i\theta\cdot)$ the function $z \to \exp(i\theta z)$. Define the spaces

$$H^{s,\theta}(S_{\omega,\pm}) = \exp(i\theta\cdot)H^s(S_{\omega,\pm}), \quad H^{s,\theta}(S_\omega) = \exp(i\theta\cdot)H^s(S_\omega),$$

$$K^{s,\theta}(C_{\omega,\pm}) = \left\{\phi \mid \phi\circ\exp(-i\theta) \in K^s(C_{\omega,\pm})\right\}$$

and

$$K^{s,\theta}(S_\omega) = \left\{\phi \mid \phi\circ\exp(-i\theta) \in K^s(S_\omega)\right\}.$$

If we change the statements of the theorems by using these spaces with the parameter θ, then the singular point $z = 1$ of the functions ϕ_+ and ϕ will be shifted to the point $z = \exp(i\theta)$ on the unit circle.

Remark 7.5.3 For the case $s = 0$, the main results of Sect. 7.1 are corollaries of the Fourier theory of holomorphic functions on the sectors established in [15]. In [16], the authors proved that if the Lipschitz constant of the curve is smaller than $\tan(\omega)$, as the kernel, any element in $K^0(C_{\omega,\pm})$ and $K^0(S_\omega)$ induces a L^2-bounded convolution singular integral operator on this starlike Lipschitz curve. In fact, these operators can be represented as the H^∞-functional calculus of the Dirac operator $z(d/dz)$ on the closed curve. By the conformal mapping, we can deduce a corresponding singular integral operator on any simply-connected Lipschitz curve. The cases of $s \neq 0$ correspond to the fractional integrations and differentials on these curves. All those mentioned are closely related to boundary value problems associated with Lipschitz domains. We refer to [17–19] for further information.

Remark 7.5.4 In [20], D. Khavinson proved the following result. Let $f(z) = \sum_{n=1}^{\infty} b_n z^n$, where $b_n = g(n)$, g is a bounded holomorphic function in the sector $S_\phi = \left\{z : |\arg z| \leqslant \phi\right\}$, $0 < \phi \leqslant \frac{\pi}{2}$. Then f can be extended to a holomorphic function on the heart-shaped region $G_\phi = \left\{z = re^{i\theta},\ \ 2\pi - \cot\phi \cdot \log r > \theta > \cot\phi \cdot \log r\right\}$. Hence, in Sect. 7.1, the result of the fractional integrals on the closed Lipschitz curves can be deduced from the result of the unit circle.

Remark 7.5.5 If $b \in H^s(S^c_\omega)$, $s > 0$, there exists a holomorphic function b_1 such that $|b_1(z)| \leqslant C_\mu$ and $\phi(x) = \Gamma^{s_1}_\xi \phi_1(x)$, where $s_1 = [s] + 1$. Here ϕ_1 is the kernel associated with b_1 in Theorem 7.2.1. However, in this way, we only obtain the following estimate: $|\phi(x)| \leqslant C/|1 - x|^{n+s_1}$, which is not precise compared with the result of Theorem 7.2.1.

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Chapter 8
Fourier Multipliers and Singular Integrals on $\mathbb{C}^n$

In this chapter, we introduce a class of singular integral operators on the n-complex unit sphere. This class of singular integral operators corresponds to bounded Fourier multipliers. Similar to the results of Chaps. 6 and 7, we also develop the fractional Fourier multiplier theory on the unit complex sphere.

8.1 A Class of Singular Integral Operators on the *n*-Complex Unit Sphere

In this section, we study a class of singular integral operators defined on $n-$complex unit sphere. The Cauchy–Szegö kernel and the related theory of singular integrals of several variables have been studied extensively, see [1–4]. The singular integrals studied in this section can be represented as certain Fourier multiplier operators with bounded symbols defined on S_ω. This class of singular integrals constitute an operator algebra, that is, the bounded holomorphic functional calculus of the radial Dirac operator

$$D = \sum_{k=1}^{n} z_k \frac{\partial}{\partial z_k}.$$

A special example of these singular integrals is the Cauchy integral operator.

We will still use the following sector regions in the complex plane. For $0 \leqslant \omega < \pi/2$, let

$$S_\omega = \Big\{ z \in \mathbb{C} \mid z \neq 0, \text{ and } |\arg z| < \omega \Big\},$$

$$S_\omega(\pi) = \Big\{ z \in \mathbb{C} \mid z \neq 0, \ |\mathrm{Re} z| \leqslant \pi, \text{ and } |\arg(\pm z)| < \omega \Big\},$$

$$W_\omega(\pi) = \Big\{ z \in \mathbb{C} \mid z \neq 0, \ |\mathrm{Re} z| \leqslant \pi, \text{ and } \mathrm{Im}(z) > 0 \Big\} \cup S_\omega(\pi),$$

T. Qian and P. Li, *Singular Integrals and Fourier Theory on Lipschitz Boundaries*,
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$$H_\omega = \left\{ z \in \mathbb{C} \mid z = e^{iw},\ w \in W_\omega(\pi) \right\}.$$

The sets S_ω, $S_\omega(\pi)$, $W_\omega(\pi)$ and H_ω are cone-shaped, bowknot-shaped region, W-shaped region and heart-shaped region, respectively.

Let

$$\phi_b(z) = \sum_{k=1}^{\infty} b(k) z^k. \tag{8.1}$$

By Lemma 6.1.1, for $b \in H^\infty(S_\omega)$, ϕ_b can be extended to H_ω holomorphically, and

$$\left| \left(z \frac{d}{dz} \right)^l \phi_b(z) \right| \leqslant \frac{C_{\mu'} l!}{\delta^l(\mu,\ \mu')|1-z|^{1+l}},\ \ z \in H_\mu, 0 < \mu < \mu' < \omega,\ l = 0, 1, 2, \cdots,$$

where $\delta(\mu,\ \mu') = \min \left\{ 1/2,\ \tan(\mu' - \mu) \right\}$. $C_{\mu'}$ is the constant in the definition of $b \in H^\infty(S_\omega)$.

In the sequel, we use z to denote any element in $\mathbb{C}^n$, that is, $z = (z_1, \ldots, z_n)$, $z_i \in \mathbb{C}, i = 1, 2, \ldots, n, n \geqslant 2$. Write $\overline{z} = (\overline{z}_1, \cdots, \overline{z}_n)$. z can be seen as a row vector. Denote by B the open ball $\{z \in \mathbb{C}^n : |z| < 1\}$, where $|z| = \left(\sum_{i=1}^{n} |z_i|^2 \right)^{1/2}$, and ∂B is the boundary, i.e.,

$$\partial B = \left\{ z \in \mathbb{C}^n : |z| = 1 \right\}.$$

The open ball centered at z with radius r is denoted by $B(z, r)$. Any element on the unit sphere is usually denoted by ξ or ζ. Below the constant ω_{2n-1} occurring in the Cauchy–Szegö kernel is the surface area of $\partial B = S^{2n-1}$ and equals to $2\pi^n/\Gamma(n)$. For $z, w \in \mathbb{C}^n$, we use the notation $zw' = \sum_{k=1}^{n} z_k w_k$. The object of study in this section is the radial Dirac operator

$$D = \sum_{k=1}^{n} z_k \frac{\partial}{\partial z_k}.$$

We shall make some modifications on the basis of holomorphic function spaces in B and the corresponding function spaces on ∂B. We apply the form given in [1]. Let k be a non-negative integer. We consider the column vector $z^{[k]}$ with the components

$$\sqrt{\frac{k!}{k_1! \cdots k_n!}} z_1^{k_1} \cdots z_n^{k_n},\ k_1 + k_2 + \cdots k_n = k.$$

The dimension of $z^{[k]}$ is

$$N_k = \frac{1}{k!} n(n+1) \cdots (n+k-1) = C_{n+k-1}^k.$$

Set

$$\int_B \overline{z^{[k]'}} \cdot z^{[k]} dz = H_1^k$$

and

$$\int_{\partial B} \overline{\xi^{[k]'}} \xi^{[k]} d\sigma(\xi) = H_2^k,$$

where dz is the Lebesgue volume element in $\mathbb{R}^{2n} = \mathbb{C}^n$, and $d\sigma(\xi)$ is the Lebesgue area element of the unit sphere $S^{2n-1} = \partial B$. It is easy to prove that H_1^k and H_2^k is the positive definite Hermitian matrix of order N_k. Hence there exists a matrix Γ such that

$$\begin{cases} \overline{\Gamma'} \cdot H_1^k \cdot \Gamma = \Lambda, \\ \overline{\Gamma'} \cdot H_2^k \cdot \Gamma = I, \end{cases} \tag{8.2}$$

where $\Lambda = [\beta_1^k, \cdots, \beta_n^k]$ is the diagonal matrix and I is the identity matrix.

We set

$$\begin{cases} z_{[k]} = z^{[k]} \cdot \Gamma, \\ \xi_{[k]} = \xi^{[k]} \cdot \Gamma. \end{cases}$$

and use $\{p_\nu^k(z)\}$ to denote the components of the vector $z_{[k]}$. By (8.2), we have

$$\int_B p_\nu^k(z)\overline{p_\mu^l(z)} dz = \delta_{\nu\mu} \cdot \delta_{kl} \cdot \beta_\nu^k \tag{8.3}$$

and

$$\int_{\partial B} p_\nu^k(\xi)\overline{p_\mu^l(\xi)} d\sigma(\xi) = \delta_{\nu\mu} \cdot \delta_{kl}. \tag{8.4}$$

The following theorem is well-known.

Theorem 8.1.1 ([1]) *The function system*

$$(\beta_\nu^k)^{-1/2} p_\nu^k, \; k = 0, 1, 2, \ldots, \; \nu = 1, 2, \ldots, N_k,$$

is a complete orthogonal system of the holomorphic function space in B. In the space of continuous functions on ∂B, the function system $\{p_\nu^k(\xi)\}$ is orthogonal, but is not complete.

In [1], applying the function system $\{p_\nu^k\}$ and relation

$$H(z, \overline{\xi}) = \sum_{k=0}^{\infty} \sum_{\nu=1}^{N_k} p_\nu^k(z)\overline{p_\nu^k(\xi)}, \; z \in B, \; \xi \in \partial B,$$

L. Hua gave the explicit formula of the Cauchy–Szegö kernel on ∂B:

$$H(z, \overline{\xi}) = \frac{1}{\omega_n} \frac{1}{(1 - z\overline{\xi}')^n}. \tag{8.5}$$

In the following, we give a technical result.

Theorem 8.1.2 *Let $b \in H^\infty(S_\omega)$ and*

$$H_b(z, \overline{\xi}) = \sum_{k=1}^{\infty} b(k) \sum_{\nu=1}^{N_k} p_\nu^k(z)\overline{p_\nu^k(\xi)}, \ z \in B, \ \xi \in \partial B. \tag{8.6}$$

Then for any $z \in B$ and $\xi \in \partial B$ such that $z\overline{\xi}' \in H_\omega$,

$$H_b(z, \overline{\xi}) = \frac{1}{(n-1)!\omega_{2n-1}} (r^n \phi_b(r))^{(n-1)} \,|_{r=z\overline{\xi'}} \tag{8.7}$$

are all holomorphic, where ϕ_b is the function defined in (8.1). In addition, for $0 < \mu < \mu' < \omega$, $l = 0, 1, 2, \ldots,$

$$|D_z^l H_b(z, \overline{\xi})| \leqslant \frac{C_{\mu'} l!}{\delta^l(\mu, \mu')|1 - z\overline{\xi'}|^{n+l}}, z\overline{\xi'} \in H_\mu, \tag{8.8}$$

where $\delta(\mu, \mu') = \left\{1/2, \ \tan(\mu' - \mu)\right\}$; $C_{\mu'}$ is the constant in the definition of $H^\infty(S_\omega)$.

Proof In (8.5), letting $z = r\zeta$ and $|\zeta| = 1$, we obtain

$$H(r\zeta, \overline{\xi}) = \frac{1}{\omega_{2n-1}} \frac{1}{(1 - r\zeta\overline{\xi'})^n}. \tag{8.9}$$

Taking $H(r\zeta, \overline{\xi})$ as a function of r, we know that in the Taylor expansion of this function, the term with respect to r^k is

$$\begin{aligned} &\frac{1}{k!}\left(\frac{\partial}{\partial r}\right)^k \left(\frac{1}{\omega_{2n-1}} \frac{1}{(1 - r\zeta\overline{\xi'})^n}\right)\Big|_{r=0} r^k \\ &= \frac{1}{\omega_{2n-1}} \frac{n(n+1)\cdots(n+k-1)}{k!} (r\zeta\overline{\xi'})^k. \end{aligned} \tag{8.10}$$

Let $r\zeta = z$. We get the projection from $H(z, \overline{\xi})$ to the k-homogeneous function space of variable z is

$$\sum_{\nu=1}^{N_k} p_\nu^k(z)\overline{p_\nu^k(\xi)} = \frac{1}{\omega_{2n-1}} \frac{n(n+1)\cdots(n+k-1)}{k!} (z\overline{\xi'})^k.$$

By the definition of ϕ_b, a direct computation gives the formula of $H_b(z, \overline{\xi})$. The corresponding estimate can be deduced from Lemma 6.1.1. □

Remark 8.1.1 In the former chapters, the size of ω is very important and is related to the Lipschitz constant of Lipschitz curves or Lipschitz surfaces, see also [5–16]. Now, the Lipschitz constant of the unit sphere is 0, and ω can be chosen as any number in the interval $(0, \pi/2]$. In this section, we always assume that ω is any number in $(0, \pi/2]$ but should be determined via discussion. We also take $\mu = \omega/2$ and $\mu' = 3\omega'/4$ large enough to adapt to our theory.

For $z, w \in B \cup \partial B$, denote by $d(z, w)$ the anisotropic distance between z and w defined as

$$d(z,\ w) = |1 - z\overline{w'}|^{1/2}.$$

It is easy to prove d is a distance on $B \cup \partial B$. On ∂B, denote by $S(\zeta, \varepsilon)$ the ball centered at ζ with radius ε which is defined via d. The complementary set of $S(\zeta,\ \varepsilon)$ in ∂B is denoted by $S^c(\zeta,\ \varepsilon)$.

Let $f \in L^p(\partial B)$, $1 \leqslant p < \infty$. Then the Cauchy integral of f

$$C(f)(z) = \frac{1}{\omega_{2n-1}} \int_{\partial B} \frac{f(\xi)}{(1 - z\overline{\xi'})^n} d\sigma(\xi)$$

is well defined and is holomorphic in B.

It is fairly well known that the operator

$$P(f)(\zeta) = \lim_{r \to 1-0} C(f)(r\zeta)$$

is the projection from $L^p(\partial B)$ to the Hardy space $H^p(\partial B)$ and is bounded from $L^p(\partial B)$ to $H^p(\partial B)$, $1 < p < \infty$. Moreover, $P(f)$ has a singular integral expression [3, 4]

$$P(f)(\zeta) = \frac{1}{\omega_{2n-1}} \lim_{\varepsilon \to 0} \int_{S^c(\zeta,\ \varepsilon)} \frac{f(\xi)}{(1 - \zeta\overline{\xi'})^n} d\sigma(\xi) + \frac{1}{2} f(\zeta) \text{ a.e.} \zeta \in \partial B.$$

Let

$$\mathscr{A} = \Big\{ f :\ f \text{ is a holomorphic function in } B(0, 1+\delta) \text{ for some } \delta > 0 \Big\}.$$

It is easy to verify that $\mathscr{A}$ is dense in $L^p(\partial B)$, $1 \leqslant p < \infty$. If $f \in \mathscr{A}$, then

$$f(z) = \sum_{k=0}^{\infty} \sum_{\nu=0}^{N_k} c_{k\nu} p_\nu^k(z),$$

where $c_{k\nu}$ is the Fourier coefficient of f:

$$c_{k\nu} = \int_{\partial B} \overline{p_\nu^k(\xi)} f(\xi) d\sigma(\xi).$$

Also, for any positive integer l, the series

$$\sum_{k=0}^{\infty} k^l \sum_{\nu=0}^{N_k} c_{k\nu} p_\nu^k(z)$$

uniformly absolutely converges in any ball contained $B(0, 1+\delta)$ on which f is defined.

Let $\mathscr{U}$ be the unitary group consisting of all unitary operators in the sense of complex inner product $\langle z, w\rangle = z\overline{w'}$ on Hilbert spaces in $\mathbb{C}^n$. These operators are linear operators U which keep the inner product invariant:

$$\langle Uz, Uw\rangle = \langle z, w\rangle.$$

Obviously, $\mathscr{U}$ is a compact subset in $O(2n)$. It is easy to prove that $\mathscr{A}$ is invariant under the operation of $U \in \mathscr{U}$. If $f \in \mathscr{A}$, then f is determined by its value on ∂B. Below we shall regard $f|_{\partial B}$ as $f \in \mathscr{A}$. For a given function $b \in H^\infty(S_\omega)$, we define an operator $M_b : \mathscr{A} \to \mathscr{A}$ as

$$M_b(f)(\zeta) = \sum_{k=1}^{\infty} b(k) \sum_{\nu=0}^{N_k} c_{k\nu} p_\nu^k(\zeta), \ \zeta \in \partial B,$$

where $c_{k\nu}$ is the Fourier coefficient of the test function $f \in \mathscr{A}$.

The principal value of the Cauchy integral defined via the surface distance

$$d(\eta, \zeta) = |1 - \eta\overline{\zeta'}|^{1/2}$$

can be extended as in the following Theorem 8.1.3:

Theorem 8.1.3 *The operator M_b can be expressed as the form of the singular integral. Precisely, for $f \in \mathscr{A}$,*

$$M_b(f)(\zeta) = \lim_{\varepsilon\to 0}\Big[\int_{S^c(\zeta,\varepsilon)} H_b(\zeta, \overline{\xi}) f(\xi) d\sigma\xi + f(\zeta)\int_{S(\zeta,\varepsilon)} H_b(\zeta, \overline{\xi}) d\sigma(\xi)\Big], \tag{8.11}$$

where

$$\int_{S(\zeta,\varepsilon)} H_b(\zeta, \overline{\xi}) d\sigma(\xi)$$

are bounded functions for $\zeta \in \partial B$ and ε.

Proof Let $f \in \mathscr{A}$ and $\rho \in (0, 1)$. On the one hand,

$$M_b(f)(\rho\zeta) = \sum_{k=1}^{\infty} b(k) \sum_{\nu=1}^{N_k} c_{k\nu} p_\nu^k(\rho\zeta),$$

where $c_{k\nu}$ is the Fourier coefficient of f. Because $\{b(k)\}_{k=1}^{\infty} \in l^\infty$ and the Fourier expansion of $f \in \mathscr{A}$ is convergent, we obtain

$$\lim_{\rho \to 1-0} M_b(f)(\rho\zeta) = M_b(f)(\zeta). \tag{8.12}$$

On the other hand, applying the formula of the Fourier coefficients and the definition of $H_b(z, \overline{\xi})$ given in (8.5), we have

$$M_b(f)(\rho\zeta) = \int_{\partial B} H_b(\rho\zeta, \overline{\xi}) f(\xi) d\sigma(\xi).$$

For any $\varepsilon > 0$, we get

$$\begin{aligned} M_b(f)(\rho\zeta) = & \int_{S^c(\zeta, \xi)} H_b(\rho\zeta, \overline{\xi}) f(\xi) d\sigma(\xi) \\ & + \int_{S(\zeta, \xi)} H_b(\rho\zeta, \overline{\xi})(f(\xi) - f(\zeta)) d\sigma(\xi) \\ & + f(\zeta) \int_{S(\zeta, \xi)} H_b(\rho\zeta, \overline{\xi}) d\sigma(\xi) \\ = & I_1(\rho, \varepsilon) + I_2(\rho, \varepsilon) + f(\zeta) I_3(\rho, \varepsilon). \end{aligned}$$

For $\rho \to 1 - 0$, we have

$$I_1(\rho, \varepsilon) \to \int_{S^c(\zeta, \varepsilon)} H_b(\zeta, \overline{\xi}) f(\xi) d\sigma(\xi).$$

Now we consider $I_2(\rho, \varepsilon)$. Because the metric d, the Euclidean metric $|\cdot|$ and the function class $\mathscr{A}$ are all $\mathscr{U}-$invariant, without loss of generality, we can assume that $\zeta = (1, 0, \ldots, 0)$. For the variable $\xi \in \partial B$, we adopt the parameter system

$$\xi_1 = re^{1\theta}, \ \xi_2 = v_2, \ldots, \xi_n = v_n.$$

Write $v = (v_2, \ldots, v_n)$. The integral region $S(\zeta, \varepsilon)$ is defined by the following condition:

$$v\overline{v'} = 1 - r^2, \quad \cos\theta \geqslant \frac{1 + r^2 - \varepsilon^4}{2r}. \tag{8.13}$$

Now, because $\frac{1+r^2-\varepsilon^4}{2r} \leqslant \cos\theta \leqslant 1$, we have $(1 - r)^2 \leqslant \epsilon^4$. Then $1 - r \leqslant \varepsilon^2$, or $1 - \varepsilon^2 \leqslant r$. This implies that

$$v\overline{v'} = 1 - r^2 \leqslant 1 - (1 - \varepsilon^2)^2 = 2\varepsilon^2 - \varepsilon^4.$$

Write

$$a = a(r,\ \varepsilon) = \arccos\left(\frac{1 + r^2 - \varepsilon^4}{2r}\right).$$

Because $(1 - r)^2 \leqslant \varepsilon^4$ and $1 - y = O(\arccos^2(y))$, we obtain $a = O(\varepsilon^2)$.

It is not difficult to verify that

$$\begin{aligned} |\zeta - \xi|^2 &= |1 - re^{i\theta}|^2 + (|v_2|^2 + \cdots + |v_n|^2) \\ &= (1 + r^2 - 2r\cos\theta) + (1 - r^2) \\ &= 2 - 2r\cos\theta \end{aligned} \tag{8.14}$$

and

$$\begin{aligned} d^4(\zeta,\ \xi) &= |1 - \zeta\overline{\xi'}|^2 = 1 + r^2 - 2r\cos\theta \\ &= (2 - 2r\cos\theta) - (1 - r^2) \\ &= |\zeta - \xi|^2 - (1 + r)(1 - r). \end{aligned} \tag{8.15}$$

Now, it follows from (8.14) that $1 - r^2 \leqslant d^2(\zeta,\ \xi)$. This fact together with (8.15) implies that

$$d^4(\zeta,\ \xi) + (1 + r)d^2(\zeta,\ \xi) \geqslant |\zeta - \xi|^2.$$

Because $d^2(\zeta,\ \xi) < 2$, the last inequality indicates that

$$|\zeta - \xi| \leqslant 2d(\zeta,\ \xi). \tag{8.16}$$

Noticing that for $f \in \mathscr{A}$,

$$|f(\zeta) - f(\xi)| \leqslant C|\zeta - \xi|.$$

Hence

$$|f(\zeta) - f(\xi)| \leqslant Cd(\zeta,\ \xi).$$

For any $\rho \in (0, 1)$, because (8.13), we have

$$\begin{aligned} |I_2(\rho,\ \varepsilon)| &\leqslant \int_{S(\zeta,\ \varepsilon)} |H_b(\rho\zeta,\ \overline{\xi})||f(\zeta) - f(\xi)|d\sigma(\zeta) \\ &\leqslant C\int_{S(\zeta,\ \varepsilon)} \frac{1}{d^{2n-1}(\zeta,\ \xi)} d\sigma(\xi) \\ &\leqslant C\int_{v\overline{v'} \leqslant 2\varepsilon^2 - \varepsilon^4} \int_{-a}^{a} \frac{1}{|1 - re^{i\theta}|^{n-1/2}} d\theta dv. \end{aligned}$$

Now we estimate the inner integral. For $n = 2$, Hölder's inequality gives

$$\begin{aligned}\frac{1}{2a}\int_{-a}^{a}\frac{1}{|1-re^{i\theta}|^{2-1/2}}d\theta &\leqslant \left(\frac{1}{2a}\int_{-a}^{a}\frac{1}{|1-re^{i\theta}|^{2}}d\theta\right)^{3/4}\\ &\leqslant \left(\frac{1}{2a}\int_{-\pi}^{\pi}\frac{1}{|1-re^{i\theta}|^{2}}d\theta\right)^{3/4}\\ &\leqslant \left(\frac{1}{2a}\right)^{3/4}\frac{1}{(1-r^2)^{3/4}}.\end{aligned}$$

In this case, when $\varepsilon \to 0$,

$$\begin{aligned}|I_2(\rho,\ \varepsilon)| &\leqslant C\int_{v\overline{v'}\leqslant 2\varepsilon^2-\varepsilon^4} a^{1/4}\frac{1}{(1-r^2)^{3/4}}dv\\ &\leqslant C\varepsilon^{1/2}\int_{v\overline{v'}\leqslant 2\varepsilon^2-\varepsilon^4}\frac{1}{(v\overline{v'})^{3/4}}dv\\ &\leqslant C\varepsilon^{1/2}\int_{0}^{\sqrt{2\varepsilon^2-\varepsilon^4}}\frac{1}{t^{3/2}}dt\\ &\leqslant C\varepsilon \to 0.\end{aligned}$$

For $n > 2$, because r approaches 1, we have

$$\begin{aligned}\int_{-a}^{a}\frac{1}{|1-re^{i\theta}|^{n-(1/2)}}d\theta &\leqslant \frac{C}{(1-r^2)^{n-5/2}}\int_{-\pi}^{\pi}\frac{1}{|1-re^{i\theta}|^{2}}d\theta\\ &\leqslant \frac{C}{(1-r^2)^{n-3/2}}.\end{aligned}$$

Hence as $\varepsilon \to 0$,

$$|I_2(\rho,\ \varepsilon)| \leqslant C\int^{\sqrt{2\varepsilon^2-\varepsilon^4}} t^{2n-3}\frac{1}{t^{2n-3}}dt \leqslant C\varepsilon \to 0.$$

Now we prove that if $\rho \to 1-0$, then $I_3(\rho, \varepsilon)$ has a uniform bound for ε near 0. Similar to the above integral, we have

$$\begin{aligned}I_3(\rho,\ \varepsilon) &= \int_{S(\zeta,\ \varepsilon)} H_b(\rho\zeta,\ \overline{\xi})d\sigma(\xi)\\ &= \int_{v\overline{v'}\leqslant 2\varepsilon^2-\varepsilon^4}\int_{-a}^{a}(t^{n-1}\phi_b(t))^{(n-1)}\Big|_{t=\rho re^{i\theta}}d\theta dv\\ &= \frac{1}{i}\int_{v\overline{v'}\leqslant 2\varepsilon^2-\varepsilon^4}\int_{\rho re^{-ia}}^{\rho re^{ia}}\frac{(t^{n-1}\phi_b(t))^{(n-1)}}{t}dtdv.\end{aligned}$$

Using integration by parts, the inner product for the variable t reduces to

$$\Big[\sum_{k=1}^{n-1}(k-1)!\frac{(t^{n-1}\phi_b(t))^{(n-1-k)}}{t^k}\Big]_{\rho re^{-ia}}^{\rho re^{ia}} + (n-1)!\int_{\rho re^{-ia}}^{\rho re^{ia}}\frac{\phi_b(t)}{t}dt$$
$$= \sum_{k=1}^{n-1}\Big[J_k(t)\Big]_{\rho re^{-ia}}^{\rho re^{ia}} + L(r,\ a).$$

We first estimate the integral of J_k. We have

$$\int_{v\overline{v'}\leqslant 2\varepsilon^2-\varepsilon^4} J_k(\rho re^{\pm ia})dv \leqslant C\int_{v\overline{v'}\leqslant 2\varepsilon^2-\varepsilon^4}\frac{1}{|1-\rho re^{\pm ia}|^{n-k}}dv.$$

It can be directly verified that

$$|1-\rho re^{\pm ia}| \geqslant |1-re^{\pm ia}| = \varepsilon^2.$$

So the above integral is dominated by

$$\frac{1}{\varepsilon^{2n-2k}}\int_{v\overline{v'}\leqslant 2\varepsilon^2-\varepsilon^4} dv \leqslant \frac{1}{\varepsilon^{2n-2k}}\int_0^{\sqrt{2\varepsilon^2-\varepsilon^4}} t^{2n-3}dt$$
$$\leqslant C\varepsilon^{2k-2},$$

where the terms are bounded when $k=1$, tends to zero when $k\geqslant 2$. When $\rho\to 1-0$, the existence of the limit can be deduced from the Lebesgue dominated convergence theorem.

Now,

$$(n-1)!\int_{\rho re^{-ia}}^{\rho re^{ia}}\frac{\phi_b(t)}{t}dt = (n-1)!i\int_{-a}^{a}\phi_b(t)\Big|_{t=\rho re^{i\theta}}d\theta.$$

By Cauchy's theorem and the estimate of ϕ_b, we can prove that for any $\rho\to 1-0$, the above is a bounded function. This implies that

$$\lim_{\varepsilon\to 0}\int_{v\overline{v'}\leqslant 2\varepsilon^2-\varepsilon^4} L(\rho r,\ a)dv = 0.$$

At last we obtain $\lim\limits_{\rho\to 1-0} I_3(\rho,\ \varepsilon)$ exists and is bounded for small $\varepsilon>0$. This proves Theorem 8.1.3. □

Remark 8.1.2 A corollary of (8.14) is

$$d(\zeta,\ \xi)\leqslant |\zeta-\xi|^{1/2},$$

which is not used in the proof.

Theorem 8.1.4 *The operator M_b can be extended a bounded operator from $L^p(\partial B)$ to $L^p(\partial B)$, $1 < p < \infty$, and from $L^1(\partial B)$ to weak $L^1(\partial B)$.*

Proof The boundedness of $M_b = M_b P$ from $L^2(\partial B)$ to $H^2(\partial B)$ is a direct corollary of the orthogonality of the function system $\{p^k_\nu(\xi)\}$. We only prove the operator is bounded from $L^1(\partial B)$ to weak $-L^1(\partial B)$, that is, the operator is weak (1,1) type. For $1 < p < 2$, the $L^p(\partial B)-$boundedness can be deduced from Marcinkiewicz's interpolation. For $2 < p < \infty$, the L^p-boundedness can be obtained by the property of the kernel

$$\overline{H_b(\zeta,\ \overline{\xi})} = H_b(\xi,\ \overline{\zeta})$$

and the bilinear pair

$$\langle f,\ g\rangle = \int_{\partial B} f(\zeta)\overline{g(\zeta)}d\sigma(\zeta),$$

in the standard duality method.

The weak (1, 1) type boundedness of M_b is based on a Hömander type inequality. The proof given below is different from that of the Cauchy integral in [3]. We will use the non-tangential approach regions

$$D_\alpha(\zeta) = \left\{z \in \mathbb{C}^n :\ |1 - z\overline{\zeta'}| < \frac{a}{2}(1 - |z|^2)\right\},\ \zeta \in \partial B,\ a > 1.$$

□

We shall prove

Lemma 8.1.1 *Assume that $\xi,\ \zeta,\ \eta \in \partial B$, $d(\xi,\ \zeta) < \delta$, $d(\xi,\ \eta) > 2\delta$, and $z \in D_\alpha(\eta)$. Then*

$$\left|H_b(z,\ \overline{\xi}) - H_b(z,\ \overline{\zeta})\right| \leqslant \delta C_\alpha |1 - \xi\overline{\eta'}|^{-n-1/2}.$$

Proof By the estimate

$$\left|\left(r^{n-1}\phi_b(r)\right)^{(n)}\right| \leqslant \frac{C_\omega}{|1 - r|^{n+1}},$$

and the mean value theorem, for some $t \in (0, 1)$, the real part

$$\begin{aligned} &\left|\mathrm{Re}(r^{n-1}\phi_b(r))^{(n-1)}\ |_{r=z\overline{\xi'}} - \mathrm{Re}(r^{n-1}\phi_b(r))^{(n-1)}\ |_{r=z\overline{\zeta'}}\right| \\ &\leqslant \left|\mathrm{Re}(r^{n-1}\phi_b(r))^{(n)}\ |_{r=z\overline{w'}}\right| \cdot |z\overline{\xi'} - z\overline{\zeta'}| \\ &\leqslant \frac{C_\omega |z\overline{\xi'} - z\overline{\zeta'}|}{|1 - z\overline{w'}_t|^{n+1}}, \end{aligned} \tag{8.17}$$

where $w_t = t\overline{\xi'} + (1 - t)\overline{\zeta'} \in B$.

The imaginary part satisfies a similar inequality.

Denote by ξ_t the projection onto ∂B of ω_t. We can easily prove

(i) as $\delta \to 0$, $|\xi_t - w_t| = 1 - |z_t| = A(t) \to 0$;
(ii) $\xi_t \in S(\xi, \delta) \cap S(\zeta, \delta)$.

It follows from (i) that $\xi_t = \frac{1}{1-A(t)} w_t$. Because $D_\alpha(\eta)$ is an open set, for small $\delta > 0$, i.e., $0 < \delta \leqslant \delta_0$, we have $z_t = (1 - A(t))z \in D_\alpha(\eta)$. We write

$$|1 - z\overline{w'}_t| = |1 - z_t\overline{\xi'}_t|. \tag{8.18}$$

On the other hand, by (4) on page 92 of [3], we have

$$\begin{aligned} |z\overline{\xi'} - z\overline{\zeta'}| &= \frac{1}{1 - A(t)} |z_t\overline{\xi'} - z_t\overline{\zeta'}| \\ &\leqslant \frac{1}{1 - A(t)} \Big(|z_t\overline{\xi'} - z_t\overline{\xi'}_t| + |z_t\overline{\zeta'} - z_t\overline{\xi'}_t| \Big) \\ &\leqslant \frac{6}{1 - A(t)} \delta\alpha^{1/2} |1 - z_t\overline{\xi'}_t|^{1/2} \\ &\leqslant \delta C_\alpha |1 - z_t\overline{\xi'}_t|^{1/2}. \end{aligned} \tag{8.19}$$

By (3) on page 92 of [3], we have

$$|1 - z_t\overline{\xi'}_t|^{-1} \leqslant 16\alpha |1 - \xi\overline{\eta'}|^{-1}. \tag{8.20}$$

The relations (8.18)–(8.20) imply that for $\delta \leqslant \delta_0$, the last part of the inequality (8.17) is dominated by $\delta C_\alpha |1 - \xi\overline{\eta'}|^{-n-1/2}$.

For $\delta \geqslant \delta_0$, on the right hand side of the desired inequality,

$$\delta |1 - \xi\overline{\eta'}|^{-n-1/2}$$

has a positive lower bound which depends on δ_0. Hence it is easy to choose $C = C_{\alpha,\ \delta_0}$ such that the inequality holds. This proves Lemma 8.1.1. □

The weak (1, 1) type boundedness is a special case of Theorem 8.1.5.

Theorem 8.1.5 *For any* $\alpha > 1$, *there exists a constant* $C_\alpha < \infty$ *such that for any* $f \in \mathscr{A}$ *and* $t > 0$,

$$\sigma\Big(\{M_\alpha M_b(f) > t\Big\} \leqslant C_\alpha t^{-1} \|f\|_{L^1(\partial B)},$$

where

$$M_\alpha M_b(f)(\zeta) = \sup \Big\{ |M_b(f)(z)| : \ z \in D_\alpha(\zeta) \Big\}$$

is defined as the non-tangential maximal function of $M_b(f)$ *in the region* $D_\alpha(\zeta)$.

The proof of Theorem 8.1.5 is based on Lemma 8.1.1 and a covering lemma [3]. To adapt to this case, we can make some modifications on the proof for the corresponding result of the Cauchy integral operator in [3].

It should be pointed out that the class of bounded operators M_b generates an operator algebra. In fact, this operator class is equivalent to the Cauchy–Dunford bounded holomorphic functional calculus of DP, where D is the radial Dirac operator and P is the projection operator from L^p to H^p.

The operator M_b has the following properties, and hence the operator class $\{M_b, b \in H^\infty(S_\omega)\}$ is called the bounded holomorphic functional calculus.

Let $b, b_1, b_2 \in H^\infty(S_\omega)$, and $\alpha_1, \alpha_2 \in \mathbb{C}$, $1 < p < \infty$, $0 < \mu < \omega$. Then

$$\|M_b\|_{L^p(\partial B)\to L^p(\partial B)} \leqslant C_{p,\ \mu}\|b\|_{L^\infty(S_\mu)},$$

$$M_{b_1 b_2} = M_{b_1} \circ M_{b_2},$$

$$M_{\alpha_1 b_1 + \alpha_2 b_2} = \alpha_1 M_{b_1} + \alpha_2 M_{b_2}.$$

The first property follows from Theorem 8.1.4. The second and the third properties can be obtained by the Taylor series expansion of test functions.

Denote by

$$R(\lambda,\ DP) = (\lambda I - DP)^{-1}$$

the resolvent operator of DP at $\lambda \in \mathbb{C}$. For $\lambda \notin [0,\ \infty)$, we prove

$$R(\lambda,\ DP) = M_{\frac{1}{\lambda - (\cdot)}}.$$

In fact, by the relation

$$DP(f)(\zeta) = \sum_{k=1}^{\infty} k \sum_{\nu=1}^{N_k} c_{k\nu} p_\nu^k(\zeta),\ \ f \in \mathscr{A},$$

where $c_{k\nu}$ are the Fourier coefficients of f, the Fourier multiplier $(\lambda - k)$ is associated with the operator $\lambda I - DP$. Hence the Fourier multiplier $(\lambda - k)^{-1}$ is associated with $R(\lambda,\ DP)$. The properties of the functional calculus in relation to the boundedness indicate that for $1 < p < \infty$,

$$\|R(\lambda,\ DP)\|_{L^p(\partial B)\to L^p(\partial B)} \leqslant \frac{C_\mu}{|\lambda|},\ \ \lambda \notin S_\mu.$$

By this estimate, for a function $b \in H^\infty(S_\omega)$ with good decay properties at both the origin and the infinity, the Cauchy–Dunford integral

$$b(DP)f = \frac{1}{2\pi i}\int_{II} b(\lambda)R(\lambda,\ DP)d\lambda f$$

is well defined and is a bounded operator, where II denotes the path containing two rays in

$$S_\omega = \left\{ s\exp(i\theta):\ s \text{ is from } \infty \text{ to } 0 \right\} \bigcup \left\{ s\exp(-i\theta):\ s \text{ is from } 0 \text{ to } \infty \right\},\ 0 < \theta < \omega.$$

Such functions b generate a dense subclass of $H^\infty(S_\omega)$ in the sense of the covering lemma of [17]. By this lemma, we can generalize the definition given by the Cauchy–Dunford integral and define a functional calculus for $b \in H^\infty(S_\omega)$.

Now we prove $b(DP) = M_b$. Assume that b has good decay properties at both the origin and at the infinity, and $f \in \mathscr{A}$. In the following deductions, the order of the integral and the summation can be exchanged. Then we have

$$\begin{aligned} b(DP)(f)(\zeta) &= \frac{1}{2\pi i}\int_{II} b(\lambda)R(\lambda,\ DP)d\lambda f(\zeta) \\ &= \frac{1}{2\pi i}\int_{II} b(\lambda)\sum_{k=1}^{\infty}(\lambda-k)^{-1}\sum_{\nu=1}^{N_p} c_{k\nu}p_\nu^k(\zeta)d\lambda \\ &= \sum_{k=1}^{\infty}\left(\frac{1}{2\pi i}\int_{II} b(\lambda)(\lambda-k)^{-1}d\lambda\right)\sum_{\nu=1}^{N_p} c_{k\nu}p_\nu^k(\zeta) \\ &= \sum_{k=1}^{\infty} b(k)\sum_{\nu=1}^{N_p} c_{k\nu}p_\nu^k(\zeta) \\ &= M_b(f)(\zeta). \end{aligned}$$

It follows from the estimate of the norm of the resolvent operator $R(\lambda,\ DP)$ that DP is a type ω operator (see [17]). For the bilinear pair and the dual pair $(L^2(\partial B),\ L^2(\partial B))$ used in the proof of Theorem 8.1.4, the operator DP equals to the dual operator on $L^2(\partial B)$, that is,

$$\left\langle DP(f),\ g\right\rangle = \left\langle f,\ DP(g)\right\rangle,\ f, g \in \mathscr{A},$$

which can be deduced from the Parseval identity

$$\sum_{k=0}^{\infty}\sum_{\nu=1}^{N_k} c_{k\nu}\overline{c'_{k\nu}} = \int_{\partial B} f(\zeta)\overline{g(\zeta)}d\sigma(\zeta).$$

The Parseval identity follows from the orthogonality of $\{p_\nu^k\}$, where $c_{k\nu}$ and $c'_{k\nu}$ are the Fourier coefficients of f and g, respectively.

Under the same bilinear pair, a counterpart result holds for the Banach space dual pair $(L^p(\partial B),\ L^{p'}(\partial B))$, $1 < p < \infty$, $1/p + 1/p' = 1$. In [17, 18], the authors studied the properties on Hilbert spaces and Banach spaces for the generalized type ω operator. It can be verified, without difficulty, that the results of [17, 18] hold for the operator DP.

8.2 Fractional Multipliers on the Unit Complex Sphere

The contents of this section is an extension of the results in Sect. 8.1. We state some new developments of the study on unbounded Fourier multipliers on the unit complex ball, see Li–Qian–Lv [19]. Let

$$\begin{aligned} S_\omega &= \Big\{z \in \mathbb{C} \mid z \neq 0 \ \text{ and } \ |\arg z| < \omega\Big\}, \\ S_\omega(\pi) &= \Big\{z \in \mathbb{C} \mid z \neq 0, |\mathrm{Re}(z)| \leqslant \pi \ \text{ and } \ |\arg(\pm z)| < \omega\Big\}, \\ W_\omega(\pi) &= \Big\{z \in \mathbb{C} \mid z \neq 0, |\mathrm{Re}(z)| \leqslant \pi \ \text{ and } \ \mathrm{Im}(z) > 0\Big\} \bigcup S_\omega(\pi), \\ H_\omega &= \Big\{z \in \mathbb{C} \mid z = e^{i\omega}, \omega \in W_\omega(\pi)\Big\}. \end{aligned}$$

We also need the following function space:

Definition 8.2.1 Let $-1 < s < \infty$. $H^s(S_\omega)$ is defined as the set of all functions in S_ω which satisfy the following conditions:

(1) for $|z| < 1$, b is bounded;
(2) $|b(z)| \leqslant C_\mu |z|^s$, $z \in S_\mu$, $0 < \mu < \omega$.

Remark 8.2.1 The spaces $H^s(S_\omega)$ are extensions of $H^\infty(S_\omega)$ introduced by A. McIntosh et al. For further information on $H^\infty(S_\omega)$, see [10, 17, 20, 21] and the reference therein.

Letting

$$\varphi_b(z) = \sum_{k=1}^{\infty} b(k) z^k.$$

we have the following result.

Lemma 8.2.1 *Let $b \in H^s(S_\omega)$, $-1 < s < \infty$. Then φ_b can be extended holomorphically to H_ω. In addition, for $0 < \mu < \mu' < \omega$ and $l = 0, 1, 2, \ldots$,*

$$\left|\left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)^l \varphi_b(z)\right| \leqslant \frac{C_{\mu'} l!}{\delta^l(\mu, \mu') \, |1 - z|^{l+1+s}}, \ z \in H_\mu,$$

where $\delta(\mu, \mu') = \min\{1/2, \tan(\mu, \mu'\}$ and $C_{\mu'}$ is the constant in Definition 8.2.1.

Proof Let

$$V_\omega = \Big\{z \in \mathbb{C} : \mathrm{Im}(z) > 0\Big\} \bigcup S_\omega \bigcup (-S_\omega),$$

$$W_\omega = V_\omega \cap \Big\{z \in \mathbb{C} : \ -\pi \leqslant \mathrm{Re} z \leqslant \pi\Big\}$$

and ρ_θ is the ray $r\exp(i\theta)$, $0 < r < \infty$, where θ is chosen such that $\rho_\theta \subsetneq S_\omega$. Define

$$\Psi_b(z) = \frac{1}{2\pi}\int_{\rho(\theta)} \exp(i\xi z)b(\xi)\mathrm{d}\xi,\ z \in V_\omega,$$

where as $\xi \to \infty$, $\exp(iz\xi)$ is decreasing exponentially along ρ_θ. Then we obtain

$$\begin{aligned}\left||z|^{1+s}\,\Psi_b(z)\right| &= \left|\frac{1}{2\pi}\int_{\rho(\theta)} \exp(i\xi z)\,|z|^{1+s}\,b(\xi)\mathrm{d}z\right| \\ &\leqslant \frac{C_{\mu'}}{2\pi}\int_0^\infty \exp(-r|z|\sin(\theta+\arg z))(r|z|)^s\mathrm{d}(r|z|)^s \\ &\leqslant C_{\mu'}.\end{aligned} \tag{8.21}$$

Hence we get $|\Psi_b(z)| \leqslant 1/|z|^{1+s}$. Define

$$\psi_b(z) = 2\pi\sum_{n=-\infty}^{\infty}\Psi_b(z+2n\pi),\quad z \in \bigcup_{n=-\infty}^{\infty}(2n\pi + W_\omega).$$

It is easy to see that ψ_b is holomorphic, 2π-periodic and satisfies $|\psi_b(z)| \leqslant C/|z|^{1+s}$. Let

$$\varphi_b(z) = \psi_b\left(\frac{\log z}{i}\right).$$

For $z \in \exp(iS_\omega)$, we write $z = e^{iu}$, where $u \in S_\omega$. Then $\sin(|u|/2) \leqslant c|u|/2$. This implies that $2 - 2\cos|u| \leqslant c|u|^2$ and $|1 - e^{i|u|}| \leqslant c|u|$. Therefore, (8.21) yields

$$\begin{aligned}|\varphi_b(z)| &\leqslant \frac{C_{\mu'}}{|\log z|^{1+s}} \leqslant \frac{C_{\mu'}}{|\log|z||^{1+s}} \\ &\leqslant \frac{C_{\mu'}}{|1-z|^{1+s}}.\end{aligned}$$

Take the ball

$$B(z,r) = \left\{\xi : |z-\xi| < \delta(\mu,\mu')|1-z|\right\}.$$

By Cauchy's formula, we have

$$\varphi_b^{(l)}(z) = \frac{l!}{2\pi i}\int_{\partial B(z,r)} \frac{\varphi(\eta)}{(\eta-z)^{1+l}}d\eta.$$

For any $\eta \in \partial B(z,r)$, we have $|\eta - z| \geqslant (1-\delta(\mu,\mu'))|1-z|$. Then we obtain

$$\left|\varphi_b^{(l)}(z)\right| \leqslant \frac{Cl!\|b\|_{H^s(S_\omega^c)}}{\delta^l(\mu,\mu')|1-z|^l}\left|\int_{\partial B(z,r)}\frac{1}{|1-\eta|^{1+s}}d\eta\right|$$
$$\leqslant \frac{Cl!}{\delta^l(\mu,\mu')|1-z|^{l+1+s}}.$$

□

Theorem 8.2.1 *Let $b \in H^s(S_\omega)$ and*

$$H_b(z,\bar{\xi}) = \sum_{k=1}^{\infty} b(k)\sum_{v=1}^{N_k} p_v^k(z)\overline{p_v^k(\xi)}, \quad z \in \mathbb{B}_n,\ \xi \in \partial\mathbb{B}_n.$$

Then for $z \in \mathbb{B}_n$, $\xi \in \partial\mathbb{B}_n$ such that $z\bar{\xi}' \in H_\omega$,

$$H_b(z,\overline{\xi}) = \frac{1}{(n-1)!\omega_{2n-1}}(r^{n-1}\varphi_b(r))^{(n-1)}\Big|_{r=z\bar{\xi}'}$$

is holomorphic, where φ_b is the function defined in Lemma 8.2.1. In addition, for $0 < \mu < \mu' < \omega$ and $l = 0, 1, 2, \ldots,$

$$\left|D_z^l H_b(z,\bar{\xi})\right| \lesssim \frac{C_{\mu'} l!}{\delta^l(\mu,\mu')\left|1-z\bar{\xi}'\right|^{n+l+s}}, \quad z\bar{\xi}' \in H_\mu,$$

where $\delta(\mu,\mu') = \min\{1/2, \tan(\mu'-\mu)\}$ and $C_{\mu'}$ is the constant in the definition of the function space $H^s(S_\omega)$.

Proof We know that

$$\begin{cases} \varphi_b(z) = \sum_{k=1}^{\infty} b(k)z^k, \\ r^{n-1}\varphi_b(r) = \sum_{k=1}^{\infty} b(k)r^{n+k-1}. \end{cases}$$

Then we have

$$\frac{1}{(n-1)!}\left(r^{n-1}\varphi_b(r)\right)^{(n-1)} = \frac{1}{(n-1)!}\sum_{k=1}^{\infty} b(k)(n+k-1)(n+k-2)\ldots(k+1)r^k$$
$$= \sum_{k=1}^{\infty} b(k)r^k\frac{(n+k-1)!}{(n-1)!k!}$$
$$= \sum_{k=1}^{\infty}\frac{(n+k-1)(n+k-2)(n+1)n}{k!}b(k)r^k.$$

Therefore,

$$\begin{aligned}\frac{1}{(n-1)!}\left(r^{n-1}\varphi_b(r)\right)^{(n-1)}\Big|_{r=z\bar{\xi}'} &= \sum_{k=1}^{\infty} b(k)\frac{(n+k-1)(n+k-2)(n+1)n}{k!}(z\bar{\xi}')^k \\ &= \omega_{2n-1}\sum_{k=1}^{\infty} b(k)\sum_{v=1}^{N_k} p_v^k(z)\overline{p_v^k(\xi)} \\ &= \omega_{2n-1}H_b(z,\bar{\xi}).\end{aligned}$$

□

By [12, Theorem 3], we can get the following result.

Theorem 8.2.2 *Let s be a negative integer. If $b \in H^s(S_{\omega,\pm})$,*

$$H_b(z,\xi) = \sum_{k=1}^{\infty} b(k)\sum_{v=1}^{N_k} p_v^k(z)p_\mu^l(\xi),\ z \in \mathbb{B},\ \xi \in \partial\mathbb{B}_n,$$

then

$$\left|D_z^l H_b(z,\bar{\xi})\right| \lesssim \frac{C_\mu l!\left[|\ln|1-z\bar{\xi}'||+1\right]}{\delta^l(\mu,\mu')|1-z\bar{\xi}'|^{n+l+s}}.$$

Proof The proof is similar to that of Theorem 8.2.1. We omit the details. □

Given $b \in H^s(S_\omega)$. We define the Fourier multiplier operator $M_b : \mathcal{A} \to \mathcal{A}$ as

$$M_b(f)(\xi) = \sum_{k=1}^{\infty} b(k)\sum_{v=0}^{N_k} c_{kv}p_v^k(\xi),\ \xi \in \partial\mathbb{B}_n,$$

where $\{c_{kv}\}$ is the Fourier coefficient of the test function $f \in \mathcal{A}$.

For the above operator M_b, there holds a Plemelj type formula.

Theorem 8.2.3 *Let $b \in H^s(S_\omega)$, $s > 0$. Take $b_1(z) = z^{-s_1}b(z)$, where $s_1 = [s]+1$. The operator M_b has a singular integral expression. Precisely, for $f \in \mathcal{A}$,*

$$M_b(f)(\xi) = \lim_{\varepsilon\to 0}\left[\int_{S^c(\xi,\varepsilon)} H_{b_1}(\xi,\overline{\eta})D_\eta^{s_1}f(\eta)d\sigma(\eta) + (D_z^{s_1}f)(\xi)\int_{S^c(\xi,\varepsilon)} H_{b_1}(\xi,\overline{\eta})d\sigma(\eta)\right],$$

where $\int_{S(\xi,\varepsilon)} H_{b_1}(\xi,\overline{\eta})d\sigma(\eta)$ is a bounded function of $\xi \in \partial\mathbb{B}_n$ and ε.

Proof Let

$$M_b(f)(\rho\xi) = \sum_{k=1}^{\infty} b(k)\sum_{v=1}^{N_k} c_{kv}p_v^k(\rho\xi), \quad \xi \in \partial\mathbb{B}_n,$$

where

$$c_{kv} = \int_{\partial B}\overline{p_v^k(\eta)}f(\eta)\mathrm{d}\sigma(\eta).$$

We can see that

$$\begin{aligned}
D_z z^{[l]} &= \sqrt{\frac{l!}{l_1!l_2!\cdots l_n!}}\sum_{k=1}^{n} z_k \frac{\partial}{\partial z_k}\left(z_1^{l_1} z_2^{l_2}\cdots z_n^{l_n}\right)\\
&= \sqrt{\frac{l!}{l_1!l_2!\cdots l_n!}}\sum_{k=1}^{n} z_k l_k z_1^{l_1} z_2^{l_2}\cdots z_{k-1}^{l_{k-1}} z_k^{l_k-1} z_{k+1}^{l_{k+1}}\cdots z_n^{l_n}\\
&= \sqrt{\frac{l!}{l_1!l_2!\cdots l_n!}}\Big(\sum_{k=1}^{n} l_k\Big) z_1^{l_1} z_2^{l_2}\cdots z_n^{l_n}\\
&= l z^{[l]},
\end{aligned}$$

which yields $D_z p_v^k = k p_v^k$. Then we have

$$\begin{aligned}
M_b(f)(\rho\xi) &= \sum_{k=1}^{\infty} b(k) \sum_{v=1}^{N_k}\int_{\partial B} p_v^k(\rho\xi)\overline{p_v^k(\eta)} f(\eta)\mathrm{d}\sigma(\eta)\\
&= \sum_{k=1}^{\infty} b(k)\frac{1}{k^{s_1}} \sum_{v=1}^{N_k}\int_{\partial B} p_v^k(\rho\xi)k^{s_1}\overline{p_v^k(\eta)} f(\eta)\mathrm{d}\sigma(\eta)\\
&= \sum_{k=1}^{\infty} b(k)\frac{1}{k^{s_1}} \sum_{v=1}^{N_k}\int_{\partial B} p_v^k(\rho\xi)D_\eta^{s_1}\overline{p_v^k(\eta)} f(\eta)\mathrm{d}\sigma(\eta).
\end{aligned}$$

By integration by parts,

$$\begin{aligned}
M_b(f)(\rho\xi) &= \sum_{k=1}^{\infty} b(k)\frac{1}{k^{s_1}} \sum_{v=1}^{N_k}\int_{\partial B} p_v^k(\rho\xi)\overline{p_v^k(\eta)}(D_\eta^{s_1} f)(\eta)\mathrm{d}\sigma(\eta)\\
&= \sum_{k=1}^{\infty} b_1(k) \sum_{v=1}^{N_k}\int_{\partial B} p_v^k(\rho\xi)\overline{p_v^k(\eta)}(D_\eta^{s_1} f)(\eta)\mathrm{d}\sigma(\eta).
\end{aligned}$$

For any $\varepsilon > 0$, we have

$$\begin{aligned}
M_b(f)(\rho\xi) \;&=\; \int_{S^c(\xi,\varepsilon)} H_{b_1}(\rho\xi,\bar{\eta})D_\eta^{s_1} f(\eta)\mathrm{d}\sigma(\eta)\\
&\;+\; \int_{S(\xi,\varepsilon)} H_{b_1}(\rho\xi,\bar{\eta})(-D_\xi^{s_1} f(\xi) + D_\eta^{s_1} f(\eta))\mathrm{d}\sigma(\eta)\\
&\;+\; D_\xi^{s_1} f(\xi)\int_{S(\xi,\varepsilon)} H_{b_1}(\rho\xi,\bar{\eta})\mathrm{d}\sigma(\eta)\\
&=: I_1(\rho,\varepsilon) + I_2(\rho,\varepsilon) + D_\xi^{s_1} f(\xi) I_3(\rho,\varepsilon),
\end{aligned}$$

where

$$I_1(\rho,\varepsilon)=\int_{S^c(\xi,\varepsilon)} H_{b_1}(\rho\xi,\bar\eta)D_\eta^{s_1}f(\eta)\mathrm{d}\sigma(\eta),$$
$$I_2(\rho,\varepsilon)=\int_{S(\xi,\varepsilon)} H_{b_1}(\rho\xi,\bar\eta)(-D_\xi^{s_1}f(\xi)+D_\eta^{s_1}f(\eta))\mathrm{d}\sigma(\eta),$$
$$I_3(\rho,\varepsilon)=\int_{S(\xi,\varepsilon)} H_{b_1}(\rho\xi,\bar\eta)\mathrm{d}\sigma(\eta).$$

For $\rho\to 1-0$, we have

$$\begin{aligned}\lim_{\rho\to 1-0} I_1(\rho,\varepsilon)&=\lim_{\rho\to 1-0}\int_{S^c(\xi,\varepsilon)} H_{b_1}(\rho\xi,\bar\eta)D_\eta^{s_1}f(\eta)\mathrm{d}\sigma(\eta)\\&=\int_{S^c(\xi,\varepsilon)} H_{b_1}(\xi,\bar\eta)D_\eta^{s_1}f(\eta)\mathrm{d}\sigma(\eta).\end{aligned}$$

Now we consider $I_2(\rho,\varepsilon)$. Let $\xi=(1,0,\ldots,0)$. For $\eta\in\partial\mathbb{B}_n$, write

$$\begin{cases}\eta_1=re^{i\theta},\eta_2=v_2,\eta_3=v_3,\ldots,\eta_n=v_n,\\ v=[v_2,v_3,\ldots,v_n].\end{cases}$$

For such $\eta\in\partial\mathbb{B}_n$, $v\bar v'=1-r^2$. Without loss of generality, assume that $\xi=1$. We get

$$\left|1-\xi\bar\eta'\right|^{1/2}=\left|1-re^{i\theta}\right|^{1/2}=[(1-r\cos\theta)^2+(r\sin\theta)^2]^{1/4}\leqslant\varepsilon.$$

This implies

$$\cos\theta\geqslant\frac{1+r^2-\varepsilon^4}{2r}.$$

The above estimate indicates

$$S(\xi,\varepsilon)=\left\{\eta\mid v\bar v'=1-r^2,\cos\theta\geqslant\frac{1+r^2-\varepsilon^4}{2r}\right\}.$$

Because

$$\frac{1+r^2-\varepsilon^4}{2r}\leqslant\cos\theta\leqslant 1,$$

we obtain $1-r\leqslant\varepsilon^2$ and

$$v\bar v'=1-r^2\leqslant 1-(1-\varepsilon^2)^2=2\varepsilon^2-\varepsilon^4.$$

Set

$$a=a(r,\varepsilon)=\arccos\left(\frac{1+r^2-\varepsilon^4}{2r}\right).$$

Because $(1-r)^2 \leqslant \varepsilon^4$ and $1-y = O(\arccos^2 y)$, we get $a = O(\varepsilon^2)$. It is easy to see

$$\begin{aligned}|\xi-\eta|^2 &= |1-re^{i\theta}|^2 + \sum_{k=2}^{n} |v_k|^2 \\ &= (1+r^2-2r\cos\theta) + (1-r^2) \\ &= 2-2r\cos\theta\end{aligned}$$

and

$$\begin{aligned}d^4(\xi,\eta) &= 1+r^2-2r\cos\theta \\ &= (2-2r\cos\theta)-(1-r^2) \\ &= |\xi-\eta|^2-(1+r)(1-r),\end{aligned}$$

that is, $d^2(\xi,\eta) \leqslant |\xi-\eta|$. Since

$$d^2(\xi,\eta) = [1+r^2-2r\cos\theta]^{1/2} \geqslant 1-r,$$

we have $1-r \leqslant d^2(\xi,\eta)$, and thus

$$|\xi-\eta|^2 \leqslant d^4(\xi,\eta) + (1+r)d^2(\xi,\eta).$$

The fact that $d^2(\xi,\eta) \leqslant 2$ implies

$$|\xi-\eta|^2 \leqslant 2d^2(\xi,\eta) + 2d^2(\xi,\eta) = 4d^2(\xi,\eta),$$

that is, $|\xi-\eta| \leqslant 2d(\xi,\eta)$. Since $f \in \mathcal{A}$, we have

$$|f(\xi)-f(\eta)| \leqslant C|\xi-\eta| \leqslant Cd(\xi,\eta).$$

For $\rho \in (0,1)$

$$\begin{aligned}\Big|I_2(\rho,\varepsilon)\Big| &\leqslant C\int_{S(\xi,\varepsilon)} \Big|H_{b_1}(\rho\xi,\bar{\eta})\Big|\Big|f(\xi)-f(\eta)\Big|\mathrm{d}\sigma(\eta) \\ &\leqslant C\int_{S(\xi,\varepsilon)} \frac{d(\xi,\eta)}{|1-\xi\bar{\eta}'|^n}\mathrm{d}\sigma(\eta) \\ &\leqslant C\int_{v\bar{v}'\leqslant 2\varepsilon^2-\varepsilon^4}\int_{-a}^{a} \frac{1}{|1-re^{i\theta}|^{n-1/2}}\mathrm{d}\theta\mathrm{d}v.\end{aligned}$$

For $n=2$,

$$\begin{aligned}\frac{1}{2a}\int_{-a}^{a}\frac{1}{\left|1-re^{i\theta}\right|^{2-1/2}}\mathrm{d}\theta &\leqslant \Big(\frac{1}{2a}\int_{-a}^{a}\frac{1}{\left|1-re^{i\theta}\right|^{2}}\mathrm{d}\theta\Big)^{3/4}\\ &\leqslant \Big(\frac{1}{2a}\int_{-\pi}^{\pi}\frac{1}{\left|1-re^{i\theta}\right|^{2}}\mathrm{d}\theta\Big)^{3/4}\\ &\leqslant \Big(\frac{1}{2a}\Big)^{3/4}\frac{1}{(1-r^2)^{3/4}}.\end{aligned}$$

Then we obtain

$$\begin{aligned}|I_2(\rho,\varepsilon)| &\lesssim \int_{v\bar{v}'\leqslant 2\varepsilon^2-\varepsilon^4} a^{1/4}\frac{1}{\left(1-r^2\right)^{3/4}}\mathrm{d}v\\ &\lesssim \varepsilon^{1/2}\int_{v\bar{v}'\leqslant 2\varepsilon^2-\varepsilon^4}\frac{1}{(v\bar{v}')^{3/4}}\mathrm{d}v\\ &= \varepsilon^{1/2}\int_0^{\sqrt{2\varepsilon^2-\varepsilon^4}}\frac{t}{t^{3/2}}\mathrm{d}t\\ &\lesssim \varepsilon \to 0.\end{aligned}$$

For $n > 2$, we have

$$\begin{aligned}\int_{-a}^{a}\frac{1}{\left|1-re^{i\theta}\right|^{n-1/2}}\mathrm{d}\theta &\leqslant C\int_{-a}^{a}\frac{\left|1-r^2\right|^{n-1/2-2}}{\left|1-re^{i\theta}\right|^{n-1/2}}\frac{1}{\left|1-r^2\right|^{n-1/2-2}}\mathrm{d}\theta\\ &\leqslant C\frac{1}{\left|1-r^2\right|^{n-1/2-1}}\int_{-\pi}^{\pi}\frac{1}{\left|1-re^{i\theta}\right|^{2}}\mathrm{d}\theta\\ &\leqslant C\frac{1}{\left|1-r^2\right|^{n-1/2-1}}.\end{aligned}$$

Then we obtain

$$|I_2(\rho,\varepsilon)| \lesssim \int_0^{\sqrt{2\varepsilon^2-\varepsilon^4}} t^{2n-3}\frac{1}{t^{2n-3}}\mathrm{d}t \lesssim \sqrt{2\varepsilon^2} \to 0.$$

Now we prove that if $\rho \to 1-0$, $I_3(\rho,\varepsilon)$ has a uniformly bounded limit for ε near 0. Integrating as above, we can deduce that

$$\begin{aligned}I_3(\rho,\varepsilon) &= \int_{S(\xi,\varepsilon)} H_{b_1}(\rho\xi,\bar{\eta})\mathrm{d}\sigma(\eta)\\ &= \int_{v\bar{v}'\leqslant 2\varepsilon^2-\varepsilon^4}\int_{-a}^{a}\left(t^{n-1}\varphi_{b_1}(t)\right)^{(n-1)}\Big|_{t=\rho re^{i\theta}}\mathrm{d}\theta\mathrm{d}v.\end{aligned}$$

Let $s=\rho re^{i\theta}$. Then $\mathrm{d}s=is\mathrm{d}\theta$. We can obtain

$$I_3(\rho,\varepsilon)=-i\int_{v\bar{v}'\leqslant 2\varepsilon^2-\varepsilon^4}\int_{\rho re^{-ia}}^{\rho re^{ia}}\left(s^{n-1}\varphi_{b_1}(s)\right)^{(n-1)}\mathrm{d}s\mathrm{d}v.$$

Using integration by parts, we can see that the inner integral for the variable t reduces to

$$\begin{aligned}&\int_{-a}^{a}\left(t^{n-1}\varphi_{b_1}(t)\right)^{(n-1)}\Big|_{t=\rho re^{i\theta}}\mathrm{d}\theta\\&=\left[\sum_{k=1}^{n-1}(k-1)!\frac{\left(t^{n-1}\varphi_{b_1}(t)\right)^{(n-k-1)}}{t^k}\right]\Bigg|_{\rho re^{-ia}}^{\rho re^{ia}}+(n-1)!\int_{\rho re^{-ia}}^{\rho re^{ia}}\frac{\varphi_{b_1}(t)}{t}\mathrm{d}t\\&=\sum_{k=1}^{n-1}[J_k(t)]_{\rho re^{-ia}}^{\rho re^{ia}}+L(r,a).\end{aligned}$$

We first estimate J_k as

$$\begin{aligned}&\int_{v\bar{v}'\leqslant 2\varepsilon^2-\varepsilon^4}J_k\left(\rho re^{\pm ia}\right)\mathrm{d}v\\&\leqslant C\int_{v\bar{v}'\leqslant 2\varepsilon^2-\varepsilon^4}(k-1)!\frac{\left(\rho re^{\pm ia}\right)^k}{\left(\rho re^{\pm ia}\right)^k}\frac{1}{\left|1-\rho re^{\pm ia}\right|^{n-k}}\mathrm{d}v\\&\leqslant C\int_{v\bar{v}'\leqslant 2\varepsilon^2-\varepsilon^4}\frac{1}{\left|1-\rho re^{\pm ia}\right|^{n-k}}\mathrm{d}v.\end{aligned}$$

Since $\left|1-\rho re^{\pm ia}\right|^2=1+\rho^2r^2-2\rho r\cos a$, we have

$$\begin{aligned}\left|1-\rho re^{\pm ia}\right|^2-\left|1-re^{\pm ia}\right|^2&=\rho^2r^2-2\rho r\cos a-(r^2-2r\cos a)\\&=r^2(\rho^2-1)+2r\cos a(1-\rho).\end{aligned}$$

It follows from the relation $\cos a=(1+r^2-\varepsilon^4)/2r$ that we have

$$\begin{aligned}\left|1-\rho re^{\pm ia}\right|^2-\left|1-re^{\pm ia}\right|^2&=r^2(\rho^2-1)+(1+r^2-\varepsilon^4)(1-\rho)\\&=(1-\rho)[1+r^2-\varepsilon^4-(1+\rho)r^2]\\&=(1-\rho)(1-\rho r^2-\varepsilon^4)>0.\end{aligned}$$

Therefore,

$$\left|1-\rho re^{\pm ia}\right|\geqslant\left|1-re^{\pm ia}\right|=\varepsilon^2.$$

For any fixed k, as $\varepsilon\to 0$, we obtain

$$\begin{aligned}\int_{v\bar{v}'\leqslant 2\varepsilon^2-\varepsilon^4} J_k\left(\rho r e^{\pm ia}\right)\mathrm{d}v &\leqslant C\frac{1}{\varepsilon^{2n-2k}}\int_{v\bar{v}'\leqslant 2\varepsilon^2-\varepsilon^4}\mathrm{d}v\\ &\leqslant C\frac{1}{\varepsilon^{2n-2k}}\int_0^{\sqrt{2\varepsilon^2-\varepsilon^4}} t^{2n-3}\mathrm{d}t\\ &\leqslant C\frac{\varepsilon^{2n-2}}{\varepsilon^{2n-2k}} \lesssim 1.\end{aligned}$$

On the other hand, as $\rho \to 0$,

$$\begin{aligned}(n-1)!\int_{\rho r e^{-ia}}^{\rho r e^{ia}} \frac{\varphi_{b_1}(t)}{t}\mathrm{d}t &= i(n-1)!\int_{-a}^{a} \varphi_{b_1}(t)\big|_{t=\rho r e^{i\theta}}\,\mathrm{d}\theta\\ &\leqslant C,\end{aligned}$$

which implies

$$\int_{v\bar{v}'\leqslant 2\varepsilon^2-\varepsilon^4} L(\rho r, a)\mathrm{d}v.$$

□

8.3 Fourier Multipliers and Sobolev Spaces on Unit Complex Sphere

We define Sobolev spaces on the n-complex unit sphere $\partial\mathbb{B}_n$ through defining as follows. We define the fractional integrals $\mathcal{I}^s$ on $\partial\mathbb{B}_n$. Let

$$f(z) = \sum_{k=0}^{\infty}\sum_{v=0}^{N_k} c_{kv} p_v^k(z).$$

For $-\infty < s < \infty$, the operator $\mathcal{I}^s$ is defined as

$$\mathcal{I}^s f(z) = \sum_{k=0}^{\infty}\sum_{v=0}^{N_k} k^s c_{kv} p_v^k(z).$$

For $s \in \mathbb{Z}_+$, we see that the operator $\mathcal{I}^s$ reduces to the high-order ordinary differential operator.

Theorem 8.3.1 *Let $s \in \mathbb{Z}_+$. $D_z^s = \mathcal{I}^s$ on $L^2(\partial\mathbb{B}_n)$.*

Proof Without loss of generalization, we assume that $f \in \mathcal{A}$. Then

$$f(z) = \sum_{k=0}^{\infty}\sum_{v=0}^{N_k} c_{kv} p_v^k(z),$$

where c_{kv} is the Fourier coefficient of f:

$$c_{kv} = \int_{\partial \mathbb{B}_n} \overline{p_v^k(\xi)} f(\xi) d\sigma(\xi).$$

So

$$\begin{aligned} D_z^s f(z) &= \sum_{k=0}^{\infty} \sum_{v=0}^{N_k} \int_{\partial \mathbb{B}_n} \overline{p_v^k(\xi)} f(\xi) d\sigma(\xi) D_z^s(p_v^k)(z) \\ &= \sum_{k=0}^{\infty} k^s \sum_{v=0}^{N_k} \int_{\partial \mathbb{B}_n} \overline{p_v^k(\xi)} f(\xi) d\sigma(\xi) p_v^k(z). \end{aligned}$$

□

Definition 8.3.1 Let $s \in [0, +\infty)$. The Sobolev norm $\| \cdot \|_{W^{2,s}(\partial \mathbb{B}_n)}$ on $\partial \mathbb{B}_n$ is defined as

$$\| f \|_{W^{2,s}(\partial \mathbb{B}_n)} =: \| \mathcal{I}^s f \|_2 < \infty.$$

The Sobolev space on $\partial \mathbb{B}_n$ is defined as the closure of $\mathcal{A}$ under the norm $\| \cdot \|_{W^{2,s}(\partial \mathbb{B}_n)}$, that is,

$$W^{2,s}(\partial \mathbb{B}_n) = \overline{\mathcal{A}}^{\| \cdot \|_{W^{2,s}(\partial \mathbb{B}_n)}}.$$

Remark 8.3.1 According to Plancherel's theorem, $f \in W^{2,s}(\partial \mathbb{B}_n)$ if and only if

$$\left(\sum_{k=1}^{\infty} k^{2s} \sum_{v=0}^{N_k} |c_{kv}|^2 \right)^{1/2} < \infty.$$

Now we study the boundedness properties of M_b on Sobolev spaces.

Theorem 8.3.2 *Given $r, s \in [0, +\infty)$ and $b \in H^s(S_\omega)$. The Fourier multiplier operator M_b is bounded from $W^{2,r+s}(\partial \mathbb{B}_n)$ to $W^{2,r}(\partial \mathbb{B}_n)$.*

Proof Set

$$\mathcal{I}^s f(z) = \sum_{k=0}^{\infty} \sum_{v=0}^{N_k} c_{kv}^s p_v^k(z).$$

By the orthogonality of $\{p_v^k\}$, we see that $c_{kv}^s = k^s c_{kv}$. Let $b(z) = z^{-s} b(z)$. Because $b \in H^s(S_\omega)$, we have $b_1 \in H^\infty(S_\omega)$. This implies that

$$
\begin{aligned}
\mathcal{I}^r(M_b(f))(\xi) &= \sum_{k=1}^{\infty} b(k)k^r \sum_{v=0}^{N_k} c_{kv} p_v^k(\xi) \\
&= \sum_{k=1}^{\infty} b_1(k)k^{r+s} \sum_{v=0}^{N_k} c_{kv} p_v^k(\xi) \\
&= M_{b_1}(\mathcal{I}^{r+s} f)(\xi).
\end{aligned}
$$

Finally, by Theorem 8.1.4, we get

$$
\begin{aligned}
\|M_b(f)\|_{W^{2,r}} &= \|\mathcal{I}^r(M_b(f))\|_2 \\
&= \|M_{b_1}(\mathcal{I}^{r+s} f)\|_2 \\
&\leqslant C\|\mathcal{I}^{r+s} f\|_2.
\end{aligned}
$$

This completes the proof of Theorem 8.3.2.

□

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T. Qian and P. Li, *Singular Integrals and Fourier Theory on Lipschitz Boundaries*,
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Index

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