Daniel Breaz
Michael Th. Rassias *Editors*

Advancements in Complex Analysis

From Theory to Practice

Springer

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Daniel Breaz • Michael Th. Rassias
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Preface

This volume presents papers devoted to a broad spectrum of areas of Complex Analysis, ranging from pure to applied and interdisciplinary mathematical research. Topics treated within this book include holomorphic approximation, hypercomplex analysis, special functions of complex variables, automorphic groups, zeros of the Riemann zeta function, Gaussian multiplicative chaos, non-constant frequency decompositions, minimal kernels, one-component inner functions, power moment problems, complex dynamics, biholomorphic cryptosystems, fermionic and bosonic operators.

The papers have been contributed by experts from the international community, who have presented the state-of-the-art research in the corresponding problems treated. Effort has been made for the present volume to be a valuable source for both graduate students and research mathematicians as well as physicists, engineers and scientists conducting research in related interdisciplinary subjects.

We would like to express our warmest thanks to all the authors of papers in this volume who contributed in this collective effort. Last but not least, we would like to extend our appreciation to the Springer staff for their valuable help throughout the publication process of this work.

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A Theory on Non-Constant Frequency Decompositions and Applications

Qiuhui Chen, Tao Qian, and Lihui Tan

Abstract Positive time-varying frequency representation of transient signals has been a hearty desire of signal analysts due to its theoretical and practical importance. During approximately the last two decades there has been formulated a signal decomposition and reconstruction method rooting in harmonic and complex analysis and giving rise to the desired signal representation. The method decomposes a signal into a few basic signals that possess positive-instantaneous frequencies. The theory has profound relations with classical mathematics and can be generalized to signals defined in higher dimensions with vector or matrix values. Such representations, in particular, promote rational approximations in higher dimensions. This article mainly serves as a survey. It also gives a new proof of a general convergence result, as well as a proof of a result concerning multiple selections of the parameters.

Expositorily, for a given real-valued signal f one can associate it with a Hardy space function F whose real part coincides with f. Such function F has the form $F = f + iHf$, where H stands for the Hilbert transformation of the context. We develop fast converging expansions of F in orthogonal terms of the form

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$$F = \sum_{k=1}^{\infty} c_k B_k,$$

where B_k's are also Hardy space functions but with the additional properties

$$B_k(t) = \rho_k(t)e^{i\theta_k(t)}, \quad \rho_k \geq 0, \quad \theta_k'(t) \geq 0, \quad \text{a.e.}$$

The original real-valued function f is accordingly expanded

$$f = \sum_{k=1}^{\infty} \rho_k(t) \cos \theta_k(t)$$

which, besides the properties of ρ_k and θ_k given above, also satisfies the relation

$$H(\rho_k \cos \theta_k)(t) = \rho_k(t) \sin \theta_k(t).$$

Real-valued functions $f(t) = \rho(t) \cos \theta(t)$ that satisfy the condition

$$\rho \geq 0, \quad \theta'(t) \geq 0, \quad H(\rho \cos \theta)(t) = \rho(t) \sin \theta(t)$$

are called mono-components. Phase derivative in the above definition will be interpreted in a wider sense. If f is a mono-component, then the phase derivative $\theta'(t)$ is defined to be instantaneous frequency of f. The above defined positive-instantaneous frequency expansion is a generalization of the Fourier series expansion. Mono-components are crucial to understand the concept of instantaneous frequency. We will present several most important mono-component function classes. Decompositions of signals into their principal or intrinsic mono-components are called adaptive Fourier decompositions (AFDs). We note that some scopes of the study of the 1D mono-components and AFDs can be extended to vector-valued or even matrix-valued signals defined on higher dimensional manifolds. We provide an account of the related studies in pure and applied mathematics, and in signal analysis, as well as applications of the developed theory.

1 Introduction

It is a common sense among analysts that "The study on the unit circle is harmonic analysis; and inside the unit circle is complex analysis", and the same is true for a manifold and its neighborhood regions. In general, the following mechanism may be regarded as complex analysis method of harmonic analysis. When studying analysis on the boundary of a region, say, for instance, in an Euclidean space, one can at the cost of one more (or p-more) dimension (dimensions), imbed the region together with its boundary into a larger space, where the latter is equipped with a Cauchy complex structure, including essentially the Cauchy theorem, the Cauchy kernel, and the Cauchy formula. That is, one treats the boundary of the region as a co-dimension 1 (or co-dimension $p + 1$) manifold in the larger space with a Cauchy structure. With the complex structure one can define complex Hardy spaces consisting of suitable complex holomorphic functions in the regions divided by the manifold. Here by "suitable" we mean, in particular, the complex Hardy functions defined in the regions having non-tangential boundary limits as projections into the corresponding function spaces on the manifold. Conversely, functions in suitably defined function classes on the manifold can be made to be associated with those non-tangential boundary limits, the latter being called analytic signals. Those ideas appeared in the lectures of M.-T. Cheng and D.-G. Deng given in Beijing University [12], in the book of Gorusin translated by J.-G. Chen [38], in works of A. McIntosh and, separately, of C. Kenig and other authors, on complex Hardy spaces, singular integrals, boundary value problems and related topics on Lipschitz curves and surfaces. This article serves as a survey on the study that the author and his collaborators have been undertaking by implementing the complex analysis method to harmonic and signal analysis.

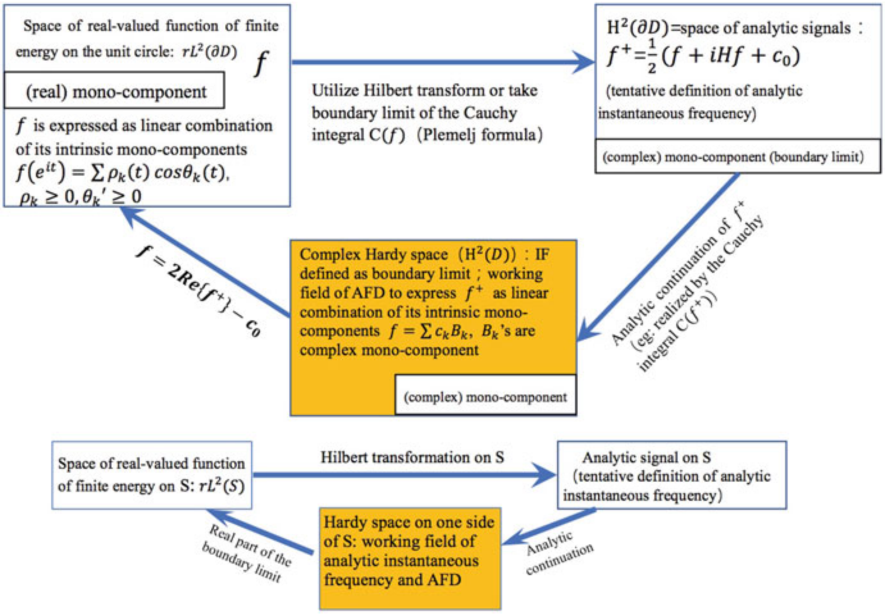
The study can be divided into two parts of which one is mono-component function theory, studying signals possessing a non-negative instantaneous frequency function; and the other is approximation to analytic functions by using mono-components. Note that the monomials $z^n, n = 0, 1, \cdots$, are particular cases of mono-components, and the Fourier series expansion is a mono-component approximation. The study that we are going to explore is generalization of the Fourier theory in relation to the scope of the Beurling–Lax theorem involving forward shift and backward shift invariant subspaces.

The study at beginning was motivated by the tentative definition of the concept *instantaneous frequency* (IF), or, in brief, the *frequency* function, by Gabor [33]. The concept instantaneous frequency is, so far, still one to be accepted by signal analysts. People tend to believe that for a general signal there is a certain "frequency" at each moment of time. This belief is supported by sinusoidal functions that possess constant frequencies. Justification of existence of a frequency function crucially depends on how to define IF. Unfortunately, the IF concept itself appears to be paradoxical: "frequency" is the oscillation number (or in the averaging sense) per unit time duration, hence a time interval is required in order to determine it; while "instantaneous" involves only a time moment. A great variety of engineering definitions of IF have been proposed those, in the author's opinion, are mostly vague or self-contradictory. None of the existing theories, nor the applications, are satisfied [8, 13]. It is believed that there does not exist an anticipated IF concept for a general signal. One can, however, propose a mathematical definition of instantaneous frequency based on which signals can be effectively analyzed. The proposed definition of IF is based on the Möbius transform. A coherent theory that has close and profound relations with the classical analysis and great potential in applications has been initialized. As a new trend of Fourier analysis it emphasizes on non-linear phase phenomenon, consisting of two parts: defining the IF concept, and decomposing a general signal into those possessing IF. We call the signals possessing an IF function as *mono-components* (or MCs). Signals that do not possess an IF are called *multi-components*. There are several classical function classes belonging to the mono-component class. There are also several newly constructed function classes belonging to the mono-component function class. The mono-component function theory is a combined effort of world harmonic and signal analysts (see Section 2 and the related literature in the references).

G. Weiss and M. Weiss published a paper in 1962 re-proving the Nevanlinna factorization theorem in the complex Hardy spaces of one complex variable that sheds lights on Blaschke expansions of functions [117]. The factorization result is a crucial tool in the complex Hardy spaces theory. Directly related to the Nevanlinna factorization, M. Nahon, in 2000, in his Ph.D. thesis at Yale University, under supervision of R. Coifman, developed the non-linear phase unwinding algorithm (UWA) to expand any analytic signal into a series of Blaschke products [64]. In the 2016 paper [15] R. Coifman and S. Steinerberger published the UWA theory and algorithm, and further developed some aspects initialized in [64]. A later paper by the same authors together with Wu developed certain practical aspects of the unwinding method with computations of the IFs [16]. More recently a new paper by R. Coifman and J. Peyriére studies invariant subspace decompositions including the Schauder basis property of the unwinding series [14]. Being unaware of Nahon's thesis, Qian independently studied the UWA method and proved its H^2-convergence in [74] (2010), and coincidently uses the same terminology "unwinding" in [92] (2013). It is noted that UWA is a special case of UWAFD, the latter being incorporated with a maximal sifting process involving a generalized backward shifting operator together [74, 81, 95].

As already mentioned the unwinding method is only one of the two main strategies in the adaptive approximation methodology. The other one is incorporated with a maximal selection principle (MSP). The terminology adaptive Fourier decomposition (AFD) that we use at its very beginning started from the MSP type [81], and further extended to the UWA type, as indeed the latter being also adaptive, and of the Fourier type. The 1D maximal selection type AFD heavily depends on the factorization properties of one complex variable. For multi-variables cases, either with the several complex variables or the Clifford algebra settings, the AFD methods are not directly applicable. In the latest studies, the AFD methodology, in fact, was extended to the reproducing kernel Hilbert spaces context with certain boundary vanishing property, called pre-orthogonal AFD, or POAFD in brief, that includes the multi-variables cases with scalar- or vector-, or even matrix-valued functions [1, 2, 5, 77, 91, 94]. The related Hilbert transform and phase derivative theory may be found in [83, 122]. The mono-component function and the related AFD approximation theory have found significant applications, including those in system identification, signal and image processing, etc. [20, 45, 51, 60, 61, 90, 116, 126]. We will include some literature with short descriptions on engineering applications.

The writing of the paper is organized as follows. In Section 2 we present the main results of mono-component function theory, including the definition of mono-component function, the inner function type, the Bedrosian type, and the starlike type mono-components. In Section 3 we give an account on various kinds of AFD algorithms in the classical setting, as well as in the reproducing kernel Hilbert space setting. In Section 4 we provide information on the related studies and applications.



Complex Harmonic Analysis Method in Analyzing Signals

2 Mono-component Function Theory

2.1 Mono-component and IF

In 1946 Gabor proposed his *analytic signal* approach [35]. Throughout this article we restrict ourselves to only signals with finite energy, or L^2-functions. The theory on the unit circle is parallel with that on the real line. To explain the idea we most time restrict ourselves to the unit circle case. We occasionally jump into the upper half space context, including, for instance, when we describe the ideas in relation to the Bedrosian type results in terms of Fourier transform. Let $s(t)$ be a real-valued signal of finite energy on the unit circle $\partial\mathbf{D}$, where $\mathbf{D}$ denotes the unit disc. The associated analytic signal, denoted by $s^+(t)$, is defined

$$s^+(e^{it}) = \frac{1}{2}\Big(s(e^{it}) + i\tilde{H}s(e^{it}) + c_0\Big), \tag{1}$$

where $\tilde{H}$ is the circular Hilbert transformation, and c_0 is the 0-th Fourier coefficient, i.e., the average of s on the circle. That is,

$$\tilde{H}s(e^{it}) = \frac{1}{2\pi}\text{p.v.}\int_0^{2\pi} f(e^{iu})\cot\left(\frac{t-u}{2}\right)du, \quad c_0 = \frac{1}{2\pi}\int_0^{2\pi} f(e^{iu})du.$$

We note s^+ is the non-tangential boundary limit of the Cauchy integral of s (the Plemelj formula):

$$s^+(e^{it}) = \lim_{z\to e^{it}} \frac{1}{2\pi}\int_0^{2\pi} \frac{f(e^{iu})}{z-e^{iu}} e^{iu}du, \qquad a.e.$$

The fact that s is real-valued makes the Hilbert transform $\frac{1}{2}\tilde{H}f$ in (1) the purely imaginary part of s^+, and $s = 2\text{Re}s^+ - c_0$. There also holds the following relation that in the real line context corresponds to the Laplace transform

$$s^+(e^{it}) = \sum_{k=0}^{\infty} c_k e^{ikt}.$$

What is important is that $s^+(e^{it})$ has a holomorphic continuation into the interior of the disc

$$s^+(z) = \sum_{k=0}^{\infty} c_k z^k, \quad |z| < 1,$$

as a Hardy $H^2(\mathbf{D})$ function in the sense that whose non-tangential boundary limit coincides with $s^+(e^{it})$. The Fourier multiplier of the circular Hilbert transformation is $-i\text{sgn}$, that is, if

$$s(e^{it}) = \sum_{k=-\infty}^{\infty} c_k e^{ikt}, \quad c_k = \frac{1}{2\pi}\int_0^{2\pi} s(e^{iu})e^{-iku}du,$$

where $\text{sgn}(k) = 1$, if $k > 0$; and $\text{sgn}(k) = -1$, if $k < 0$; and $\text{sgn}(0) = 0$. Then

$$\tilde{H}s(e^{it}) = \sum_{k=-\infty}^{\infty} (-i)\text{sgn}(k)c_k e^{ikt}.$$

In the sequel we drop the tilde sign above $\tilde{H}$ and write it simply as H.

This Fourier multiplier form of the Hilbert transform gives rise to the Hilbert transform characterization of the Hardy spaces. If restricted to the L^2 cases, it is: A function s of finite energy belongs to the Hardy H^2 space if and only if $Hs = -is$

[70]. This result holds in general contexts including the upper half space cases in one and higher dimensions [25, 27, 28].

In writing $s^+(e^{it}) = \rho(t)e^{i\theta(t)}$, Gabor defined that the derivative of the phase function, $\theta'(t)$, to be the instantaneous frequency of $s(e^{it})$. In commenting on this definition we would say that the definition is "good," because if we take the example that for a positive integer n, $s(e^{it}) = \cos(nt)$, then in such way, $s^+(e^{it}) = e^{int}$, and the phase derivative is n, being complementary with the common sense. Gabor's definition, however, is not valid for general signals $s \in L^2(\partial\mathbf{D})$, but only a tentative one, due to the following reasons. Firstly s, and thus s^+ as well, is an equivalent class of Lebesgue square-integrable functions that cannot be expected to be smooth and hence has pointwise phase derivatives; and secondly, the derivative, if exists, cannot be expected to be non-negative, as required in physics, and thus cannot stand as a qualified instantaneous frequency function. It is, in fact, the signal analysts who decide that the IFs should be non-negative and thus can be effectively analyzed in engineering applications. The primary importance is that the instantaneous frequency concept is generated from physics practice: it is an extension of the vibrating frequency. In the average sense the phase derivative of an analytic signal is non-negative as read out from the relation

$$\frac{1}{2\pi}\int_0^{2\pi} \theta'(t)|s^+(e^{it})|^2 dt = \sum_{k=0}^{\infty} k|c_k|^2,$$

[13, 21, 66]. Pointwisely, however, the phase derivative of an analytic signal can be negative. For instance, for any non-trivial outer function in the complex Hardy space we have a set of positive Lebesgue measure on which the phase derivatives are strictly less than zero [73].

The idea is to define a function set consisting of the signals having well-defined non-negative analytic phase derivatives. The functions defined in the following definition are called mono-components. The terminology first appeared in [9]. The rigorous definition was given by [72]

Definition Let s be a real- or complex-valued signal of finite energy on the unit circle. We call s a mono-component, or real-mono-component, if its analytic signal, or equivalently its projection into the Hardy space H^2, viz., $s^+(t) = \frac{1}{2}(s(t) + iHs(t) + c_0)$, in its phase-amplitude representation $s^+(t) = \rho(t)e^{i\theta(t)}$ satisfies $\theta'(t) \geq 0$, a.e., where the phase derivative $\theta'(t)$ is defined through the non-tangential limit of the same quantity from inside of the region. Precisely, in the unit circle case,

$$\theta'(t) = \lim_{r\to 1-} \theta_r'(t), \quad \text{a.e.,}$$

where $s^+(re^{it}) = \rho_r(t)e^{i\theta_r(t)}$ is the holomorphic continuation inside the unit disc. When s is a mono-component we call s^+ a complex-mono-component, or simply mono-component as well. When and only when s is a mono-component it has an instantaneous frequency function defined as its non-negative analytic phase derivative $\theta'(t)$.

Since s^+ is the non-tangential boundary limit of a Hardy space function inside the unit disc, $\theta_r'(t)$ everywhere exists, and

$$\theta_r'(t) = \mathrm{Re}\{\frac{re^{it}s^{+'}(re^{it})}{s^+(re^{it})}\}.$$

We note that the class of mono-component functions is closed under the multiplication operation but not the addition.

2.2 *The Inner Function Type Mono-Components*

There exist a number of interesting mono-component subclasses. First we mention the class of inner functions. Through simple computation one asserts that the boundary function of the canonical Möbius transform mapping $a \in \mathbf{D}$ to zero

$$e_a(e^{it}) = \frac{e^{it} - a}{1 - \overline{a}z} = e^{i\theta_a(t)}$$

is an analytic signal whose phase derivative θ_a' is the Poisson kernel of the disc [35, 71, 85]. The early study along this direction was joined by Qiu-Hui Chen and Luo-Qing Li. This implies that finite Blaschke products (Blaschke products with finitely many zeros) are all mono-components. The question is whether infinite Blaschke products are mono-components. As an application of the Julia–Wolff–Carathéodory theorem the following result for general inner functions (containing finite and infinite Blaschke products and singular inner functions) is proved [73].

Theorem 2.1 (Tao Qian 2009 [73]) *Let θ be a real-valued Lebesgue measurable function on the unit circle. Then the phase function $e^{i\theta}$ is a complex mono-component if and only if $e^{i\theta}$ is the non-tangential boundary limit of an inner function, or, equivalently, if and only if $H(e^{i\theta}) = -ie^{i\theta}$.*

The earlier study in [111] gives good observations and partial results. It is noted that in the earlier digital signal processing (DSP) literature, as far as being aware by the author, the fact that Blaschke products possess positive phase derivative functions were stated, but without a valid proof [11]. DSP researchers and engineers had been using the concept physically realizable signals with minimum phase that are outer functions, which is a related result, but without a rigorous proof either. The minimum phase property is a direct consequence of the result that boundary limit functions of inner functions possess non-negative phase derivative. The reference [73] also proves the opposite property for outer functions: Under mild conditions that guarantee absolute continuity of the phase function $\theta(t)$ of an outer function then there holds

$$\int_0^{2\pi} \theta'(t)dt = 0.$$

The inner and outer functions are thus characterized by the sign properties of their phase derivatives.

2.3 The Bedrosian Type Mono-Components

The second interesting class of mono-components is regarded as the Bedrosian type. The classical Bedrosian theorem declares the relation

$$H(fg) = fHg$$

under two lots of sufficient conditions, both being based on certain Fourier spectrum properties of the involved functions. The first lot of sufficient conditions is that there exists $\sigma > 0$ such that supp$\hat{f} \subset [-\sigma, \sigma]$, and supp $\hat{g} \subset (-\infty, \sigma] \cap [\sigma, \infty)$. The second lot sufficient conditions is that both functions f and g are in the Hardy H^2 space. The latter, through invoking the Paley–Wiener theorem for the Hardy space functions, is equivalent with $f, g \in L^2$, supp$\hat{f} \subset [0, \infty)$ and supp $\hat{g} \subset [0, \infty)$. The idea of using the Bedrosian type results is as follows: Suppose that $e^{i\theta}$ is an analytic signal with the property $\theta'(t) \geq 0$, a.e. This type of mono-components now have all been characterized by Theorem 2.1. One wishes to find a non-negative function $\rho(t)$ that makes the Bedrosian type relation $H(\rho(t)e^{i\theta} = \rho H(e^{i\theta})$ hold. For such a function ρ there holds

$$H(\rho e^{i\theta}) = \rho H(e^{i\theta}) = (-i)\rho e^{i\theta}. \tag{2}$$

By recalling the Hilbert transform characterization of the Hardy space functions the last equality implies that $\rho e^{i\theta}$ is an analytic signal, and, due to the positivity of the phase derivative $\theta'(t)$, it is a mono-component.

To find the above desired functions ρ the classical Bedrosian theorem cannot be directly implemented. The first lot sufficient conditions refers to bandlimiting properties of the functions f and g. That, unfortunately, are not our case: The inner functions $g = e^{i\theta}$ do have the full spectrum range. The second lot of sufficient conditions requires that the real-valued amplitude function ρ itself is the boundary limit of some complex Hardy space function. This is impossible either (see the example given in (3)). To implement the idea in (2) one has to find new sufficient conditions for the Bedrosian relation to hold in our specific circumstance $f = \rho, g = e^{i\theta}$.

In order to enrich the mono-component class new sufficient and necessary conditions for the Bedrosian relation (2) to hold were seeking by mainly a group of Chinese harmonic and signal analysts using Fourier analysis methods and complex analysis methods [79, 87, 105, 113, 121, 123]. One of the most comprehensive results along this line is based on the following observation.

The essential structure of Bedrosian type mono-components is as follows:

$$s(e^{it}) = \left(\frac{1}{1-\overline{a}_1 e^{it}} + \frac{1}{1-a_1 e^{-it}}\right) \frac{e^{it}-a_1}{1-\overline{a}_1 e^{it}} \frac{e^{it}-a_2}{1-\overline{a}_2 e^{it}}. \tag{3}$$

On the circle it is a real-valued function multiplied with an order-2 Blaschke product. In verifying that $s(e^{it})$ is a Bedrosian type mono-component, the key point is that

$$\frac{1}{1-\overline{a}_1 z} \frac{z-a_1}{1-\overline{a}_1 z} \frac{z-a_2}{1-\overline{a}_2 z}$$

is an analytic function in the disc; and for $|z| = 1$, the product

$$\frac{1}{1-a_1 \overline{z}} \frac{z-a_1}{1-\overline{a}_1 z} \frac{z-a_2}{1-\overline{a}_2 z}$$

has an analytic continuation to the interior part of the disc. As a result, s(z) is a bounded analytic function. Since $\frac{1}{1-\overline{a}_1 e^{it}} + \frac{1}{1-a_1 e^{-it}}$ is real-valued and has finitely many sign-change points on $|z| = 1$, it is, therefore, a so-called *generalized amplitude* on the circle. We have the following general result ([79], the finite order Blaschke products case is proved in [106]).

Theorem 2.2 *Let $\phi(e^{it})$ be an infinite Blaschke product, where $a_1, \cdots, a_n, \cdots$ are the totality of its zeros, the multiples being all counted. Then (1) $\rho(t)$ is a real-valued function such that $\rho(t)\phi(e^{it}) \in H^p(\partial\mathbf{D}), 1 \le p \le \infty$, if and only if ρ is the real part of some function in the backward shift invariant subspace induced by the Blaschke product $\phi(e^{it})$, that is, $\rho \in \mathrm{Re}\{H^p(\partial\mathbf{D}) \cap \phi(e^{it})\overline{H^p(\partial\mathbf{D})}\}$; and (2) For $1 < p < \infty$, $\rho \in \mathrm{Re}\{H^p(\partial\mathbf{D}) \cap \phi(e^{it})\overline{H^p(\partial\mathbf{D})}\}$ if and only if, in the L^p norm sense,*

$$\rho(t) = \mathrm{Re}\{\sum_{k=1}^{\infty} c_k B_k(e^{it})\},$$

where $c_k = \langle \rho(t), B_k(e^{it})\rangle = \frac{1}{2\pi}\int_0^{2\pi} \rho(t)\overline{B_k(e^{it})}dt, k = 1, 2, \cdots$, and $\{B_k\}_{k=1}^{\infty}$ is the rational orthonormal system (or TM system) generated by $a_1, \cdots, a_k, \cdots$, that is,

$$B_k(z) = \frac{\sqrt{1-|a_k|^2}}{1-\overline{a}_k z} \prod_{l=1}^{k-1} \frac{z-a_l}{1-\overline{a}_l z}. \tag{4}$$

2.4 The Non-Bedrosian Type Mono-Components: The Starlike and Boundary Starlike Type

The third type mono-components are non-Bedrosian type which contains all p-starlike, as well as boundary starlike functions in one complex variable. This kind of mono-components exhibits a different type of connections between mono-components and conformal mappings. Let f denote a univalent conformal mapping that, with $f(0) = 0$, maps the unit disc together with its boundary to a region of a rectifiable boundary. Obviously, if $f(e^{it})$ is starlike, then f is a complex mono-component as its phase function is increasing along with increasing of the angular variable t. Below we will denote by $\mathcal{S}^*$ the set of such starlike functions. Next we define several other function classes including p-starlike functions as follows.

Definition Let p be any positive integer. Denote by $\mathcal{S}(p)$ the set of p-valent holomorphic functions satisfying the following conditions:

(i) There exists $r : 0 < r < 1$, such that for all $z : r < |z| < 1$, there holds $\mathrm{Re}\{\frac{zf'(z)}{f(z)}\} > 0$; and
(ii) $\int_0^{2\pi} \mathrm{Re}\{\frac{zf'(z)}{f(z)}\}dt = 2p\pi$ for all $z : r < |z| < 1$. Functions belonging to $\mathcal{S}(p)$ are called p-starlike functions.

Definition A function f is said to be a weak p-valent starlike function, and denoted $f \in \mathcal{S}_w(p)$, if and only if it is holomorphic in $\mathbf{D}$ with precisely p zeros in $\mathbf{D}$ (including multiples) and with the expression

$$f(z) = [h(z)]^p \prod_{k=1}^{p} \frac{(z - a_k)(1 - \overline{a}_k z)}{z},$$

where $h \in \mathcal{S}^*$.

With $p = 1$ and $a_1 = 0$ we obtain $\mathcal{S}_w(1) = \mathcal{S}(1)$. The article [40] shows that $\mathcal{S}(p)$ is a proper subset of $\mathcal{S}_w(p)$. The advantage of the latter is that functions in $\mathcal{S}_w(p)$ have an explicit representation formula. In order to reveal the essential structure we assume the convenient property that functions under study have a holomorphic continuation to an open neighborhood of the closed unit disc. Denote by $\mathcal{A}$ the set of such holomorphic functions, one can show $\mathcal{A} \cap \mathcal{S}(p) = \mathcal{A} \cap \mathcal{S}_w(p)$ [79]. To describe the relation between mono-components and various types of starlike functions we need two more definitions.

Definition [53] A univalent function is said to be a boundary starlike function with respect to the origin if f is holomorphic in $\mathbf{D}$, $\lim_{r\to 1-} f(r) = 0$, $f(\mathbf{D})$ is starlike with respect to the origin, and $\mathrm{Re}\{e^{i\alpha} f(z)\} > 0$ for some real number α and all $z \in \mathbf{D}$. Denote by $\mathcal{G}^*$ the set of all boundary starlike functions with respect to the origin.

The following definition specifies a class of mono-components.

Definition Let $f(e^{it}) = \rho(t)e^{i\theta(t)} \in L^p(\partial\mathbf{D})$, $p \geq 1$. Then f is called a Hilbert-n, or H-n atom, if it satisfies the following conditions:

(1) $H(\rho\cos\theta) = \rho\sin\theta$;
(2) $\rho \geq 0, \theta' \geq 0$ a.e.; and
(3) $\int_0^{2\pi} \theta'(t)dt = n\pi$.

Note that due to (1) f has a holomorphic continuation into the unit disc as a Hardy space function. In (2) the phase derivative θ' takes the sense given in 2.1. The condition (3) refers to the multivalent degree of f. The concept H-p atom was first proposed in [72] for $p = 2$ with the result that a function f is a H-2 atom if and only if f is a starlike function about the origin. Some further studies along this line for $p = 2n$ are given in [106]. The following result ultimately reveals the relation between the H-n atoms and the starlike-boundary starlike functions.

Theorem 2.3 *Assume that f is holomorphic in $\overline{\mathbf{D}}$ having p zeros in the open disc $\mathbf{D}$. Then $f(e^{it})$ is a H-n atom, $n \geq 1$, if and only if*

$$f^2(z)=\left[\prod_{i=1}^{p} h_i(z)\right]^2 \prod_{j=1}^{n-2p} g_j^2(z)=\left[\prod_{k=1}^{p}(z-a_k)\left(\frac{1}{z}-\overline{a}_k\right)\right]^2\left[\prod_{k=1}^{n-2p}(z-b_k)\left(\frac{1}{z}-\overline{b}_k\right)\right][h(z)]^n,$$

where $\{a_k\}_{k=1}^{p}$ are the zeros of $f(z)$ inside the unit disc, $\{b_k\}_{k=1}^{n-2p}$ are the zeros of $f(z)$ on the unit circle (both can be with multiples), $h(z) \in \mathcal{S}^*$, $h_i \in \mathcal{S}_w(1)$, and $g_j(b_j z) \in \mathcal{G}^*$ are all holomorphic in $\overline{\mathbf{D}}$, $i = 1, \cdots, p$, $j = 1, \cdots, n-2p$.

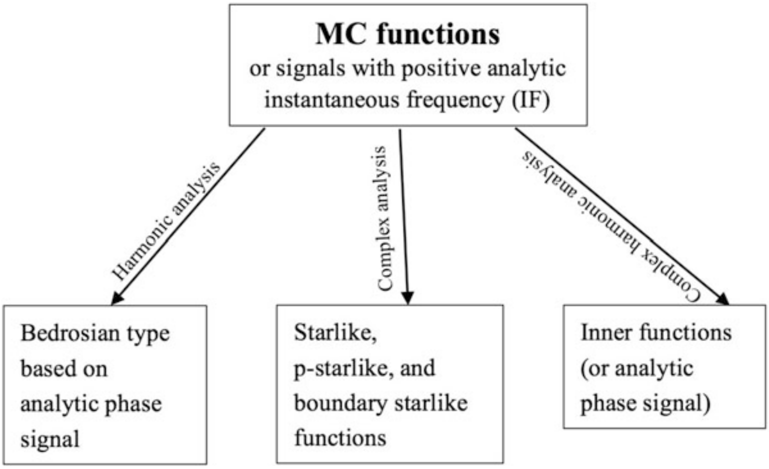
The results on mono-component functions in particular with the three categories, viz., the inner function type, the Bedrosian type, and the starlike type, are not only important in themselves in the theory, but also bring new understanding to related topics in the classical harmonic and complex analysis.

3 Adaptive Fourier Approximations

In this part we will give descriptions of adaptive Fourier approximations (AFD). In the one complex variable cases AFD gives rise to positive-frequency expansions of signals into rational holomorphic functions, while in higher dimensions AFD offers at this stage mainly fast converging rational or reproducing kernel approximations.

3.1 Mono-component Decomposition of Signals in General

The idea of positive-frequency decompositions of signals is not new, it goes back to more than 200 years ago in relation to the name Jean Baptiste Joseph Fourier and



Basic Types of Mono-Component Functions

other names. Fourier series will be a particular case of the general theory that we are now to present.

Let s be a real-valued function defined on the unit circle $\partial\mathbf{D}$ with finite energy. We recall that its Hardy H^2 space projection is $s^+ = \frac{1}{2}(s + iHs + c_0)$. The simple relation $s = 2\mathrm{Re}\{s^+\} - c_0$ implies that a complex-mono-component decomposition $s^+(e^{it}) = \sum_{k=1}^{\infty} \rho_k(t)e^{i\theta_k(t)}$ gives rise to a real-mono-component decomposition, or, in other words, positive-frequency decomposition $s : s(e^{it}) = -c_0 + \sum_{k=1}^{\infty} \rho_k(t)\cos\theta_k(t)$. We are hence reduced to decomposing the complex Hardy space function s^+.

The ultimate purpose is to find the intrinsic constructing blocks with positive-time varying-instantaneous frequency. The word "intrinsic" has a profound meaning, but here we understand it simply as fast convergence. The words "fast convergence", however, would need to be further justified. It may be shown that for any Hardy space function s^+ and any $\epsilon > 0$, there exist a constant c and two 1-starlike functions m_1 and m_2 such that [88]

$$\|s^+ - (c + m_1 + m_2)\| \le \epsilon.$$

The two starlike functions m_1 and m_2 are not unique, and, according to their construction, are very irregular. To get meaningful intrinsic positive-frequency decompositions one would then use function systems of a certain type. The one we use consists of rational functions, essentially built up from parameterized reproducing kernels of the underlying space.

3.2 One Dimensional Core-Adaptive Fourier Decomposition (Core-AFD) and Its Variations

Due to the above-mentioned reason we decide to use the rational orthonormal system, or by another name the Takenaka–Malmquist, or TM system in brief, introduced in Theorem 2.2. We note that TM systems in general cannot be avoided for they are Gram–Schmidt (G-S) orthogonalization of the partial fractions with poles outside the closed unit disc, the latter being fundamental constructive building blocks of rational functions in the Hardy spaces. TM systems consist of functions of positive frequency due to their construction in (finite) Blaschke products. The difference between our use and the traditional use of TM systems is that we make the parameters defining the system to be adaptive: For every individual function or signal we expand it by using a suitable TM system while the determining parameters are deliberately selected according to the data of the given function. The TM system itself may not be a basis. Whether or not the system in use is a basis, is, in fact, not interested or required. On the other hand, the adaptive expansion in the selected TM system converges very fast. And, additionally, each expanding term has positive non-constant and non-linear instantaneous frequencies. In contrast, the traditional use of a TM system is based on a fixed collection of parameters making the corresponding TM system a basis of the underlying space. The reason of use of a particular and fixed collection of parameters, however, is, as usual, not be well justified. Laguerre and two-parameter Kautz systems are examples of such fixed-parameter TM bases.

In the sequel we change our function notation s^+ in the Hardy $H^2(\mathbf{D})$ to f. In the unit circle context we have $f(z) = \sum_{l=1}^{\infty} c_l z^l, \sum_{l=1}^{\infty} |c_l|^2 < \infty$. Now we seek a decomposition of f into a TM system with adaptively selected parameters. The collection of the functions

$$e_a(z) = \frac{\sqrt{1-|a|^2}}{1-\overline{a}z}, \qquad a \in \mathbf{D},$$

consists of normalized Szegö kernels of the disc. Below we present AFD, or more specifically, Core-AFD algorithm. Set $f = f_1$. First write

$$f(z) = \langle f_1, e_{a_1}\rangle e_{a_1}(z) + \frac{f_1(z) - \langle f_1, e_{a_1}\rangle e_{a_1}(z)}{\frac{z-a_1}{1-\overline{a}_1 z}} \frac{z-a_1}{1-\overline{a}_1 z}.$$

We note that in this stage a_1 can be any complex number in the unit disc and the above is an identity. Denoting

$$f_2(z) = \frac{f_1(z) - \langle f_1, e_{a_1}\rangle e_{a_1}(z)}{\frac{z-a_1}{1-\overline{a}_1 z}}, \tag{5}$$

calling it the *reduced remainder,* the identity is re-written as

$$f(z) = \langle f_1, e_{a_1}\rangle e_{a_1}(z) + f_2(z)\frac{z - a_1}{1 - \overline{a}_1 z}, \tag{6}$$

We call the operator defined by (5) mapping f_1 to f_2 the *generalized* a_1-*backward shift operator* and f_2 *the generalized* a_1*-backward shift of* f_1. The terminology is a generalization of the classical backward shift operator

$$S(f)(z) = a_1 + a_2 z + \cdots + c_{k+1}z^k + \cdots = \frac{f(z) - f(0)}{z}.$$

Recognizing that $f(0) = \langle f, e_0\rangle e_0(z)$, the operator S is the generalized 0-backward shift operator.

Notice that the Szegö kernel is the reformulation of the Cauchy kernel in the arc-length measure and hence it has the reproducing kernel property. As a consequence of the orthogonality property and the modular one property of Möbius transform we have the energy relations

$$\|f\|^2 = \|\langle f_1, e_{a_1}\rangle e_{a_1}\|^2 + \|f_2\|^2 = (1 - |a_1|^2)|f_1(a_1)|^2 + \|f_2\|^2.$$

The purpose now is to extract the maximal energy portion of the form of $\|\langle f_1, e_{a_1}\rangle e_{a_1}\|^2$ from the totality $\|f\|^2$ and thus the remainder has the smallest energy. This is reduced to maximize $(1 - |a_1|^2)|f_1(a_1)|^2$ among all $a_1 \in \mathbf{D}$. Although $\mathbf{D}$ is an open set we can show that there exists a_1 in $\mathbf{D}$ such that

$$a_1 = \arg\max\{(1 - |a|^2)|f_1(a)|^2 \ : \ a \in \mathbf{D}\}$$

[81]. The existence of such maximal selection is called *Maximal Selection Principle*. Selecting such a_1 and repeating the process for f_2, and so on. We call f_2 as *maximal sifting from* f_1 *through* a_1. After n maximal siftings we have

$$f(z) = \sum_{k=1}^{n}\langle f_k, e_{a_k}\rangle B_k(z) + f_{n+1}\prod_{k=1}^{n}\frac{z - a_k}{1 - \overline{a}_k z},$$

where for $k = 1, \cdots, n$,

$$a_k = \arg\max\{(1 - |a|^2)|f_k(a)|^2 \ : \ a \in \mathbf{D}\},$$

$$B_k(z) = B_{\{a_1,\cdots,a_k\}}(z) = \frac{\sqrt{1 - |a_k|^2}}{1 - \overline{a}_k z}\prod_{l=1}^{k-1}\frac{z - a_l}{1 - \overline{a}_l z},$$

and, for $k = 2, \ldots, n+1$, f_k is the maximal sifting of f_{k-1} through a_{k-1}, that is,

$$f_k(z) = \frac{f_{k-1}(z) - \langle f_{k-1}, e_{a_{k-1}} \rangle e_{a_{k-1}}(z)}{\frac{z - a_{k-1}}{1 - \overline{a}_{k-1} z}}.$$

We have the following convergent theorem.

Theorem 3.1 For any given function f in the Hardy H^2 space, by applying the maximum sifting process at each step we have

$$f(z) = \sum_{k=1}^{\infty} \langle f_k, e_{a_k} \rangle B_k(z).$$

This result was first proved in [81] based on the complex modular 1 property of the Möbius transform. Below we provide a new proof releasing the modular 1 requirement for the system functions but only based on maximal selections of the parameters. The essence of the proof is contained in several proofs of the reference [76]. There is a similar expansion developed in [89] but the computation is less efficient.

Proof We prove the convergence by contradiction. Assume that through a sequence of maximally selected parameters $\mathbf{a} = \{a_1, \cdots, a_n, \cdots\}$ we arrive

$$f = \sum_{k=1}^{\infty} \langle f_k, e_{a_k} \rangle B_k + h, \qquad h \neq 0. \tag{7}$$

The routine argument by using the Riesz–Fisher theorem shows that both the functions $\sum_{k=1}^{\infty} \langle f_k, e_{a_k} \rangle B_k$ and h are in H^2. We note that h is orthogonal with all B_k, and thus also with $\sum_{k=1}^{\infty} \langle f_k, e_{a_k} \rangle B_k$.

The relation (7) can be re-written as

$$f = \left(\sum_{k=1}^{M} + \sum_{k=M+1}^{\infty} \right) \langle f_k, e_{a_k} \rangle B_k + h,$$

where by our notation,

$$g_{M+1} = \sum_{k=M+1}^{\infty} \langle f_k, e_{a_k} \rangle B_k + h = G_{M+1} + h.$$

To proceed we note that

$$\langle f_k, e_{a_k} \rangle = \langle f, B_k \rangle = \langle g_k, B_k \rangle, \tag{8}$$

where $g_k = f - \sum_{l=1}^{k-1} \langle f_l, e_{a_l} \rangle B_l$ is the k-th *standard remainder*.

Therefore, we have

$$g_{M+1} = \sum_{k=M+1}^{\infty} \langle g_k, B_k \rangle B_k + h. \tag{9}$$

Due to the density of the span of $\{e_a\}_{a\in\mathbf{D}}$ in H^2, there exists $a \in \mathbf{D}$ such that $\delta \triangleq |\langle h, e_a \rangle| > 0$. We can in particular choose a to be different from all the selected a_k's. We are now to explore a contradiction in relation to the selections of a_{M+1} for large M. Now, on the one hand, by the Bessel inequality applied to the infinite series part in (7), taking into account (9), we have

$$|\langle g_{M+1}, B_{M+1} \rangle| \to 0, \qquad \text{as} \quad M \to 0. \tag{10}$$

On the other hand, we will show, for large M,

$$|\langle g_{M+1}, B_{M+1}^a \rangle| > \frac{\delta}{2}. \tag{11}$$

This is clearly a contradiction.

The rest part of the proof is devoted to showing (11). Due to the relations

$$|\langle g_{M+1}, B_{M+1}^a \rangle| \geq |\langle h, B_{M+1}^a \rangle| - |\langle G_{M+1}, B_{M+1}^a \rangle| \tag{12}$$

and

$$|\langle G_{M+1}, B_{M+1}^a \rangle| \leq \|G_{M+1}\| \to 0, \qquad \text{as} \quad M \to \infty, \tag{13}$$

for large M the lower bounds of $|\langle g_{M+1}, B_{M+1}^a \rangle|$ depend on the quantity of $|\langle h, B_{M+1}^a \rangle|$. Now for any positive integer M denote by X_{M+1}^a the $(M+1)$-dimensional subspace spanned by $\{e_a, B_1, \cdots, B_M\}$. We have two ways to compute the energy of the projection of h into X_{M+1}^a, being denoted as $\|h/X_{M+1}^a\|^2$. One way is based on the orthonormality of $\{B_1, \cdots, B_M, B_{M+1}^a\}$ obtained from G-S orthonormalization of the system $\{B_1, \cdots, B_M, e_a\}$. Due to the orthogonality of h with $B_1, \cdots, B_M$, we have

$$\|h/X_{M+1}^a\|^2 = |\langle h, B_{M+1}^a \rangle|^2.$$

The other way is based on G-S orthonormalization of the same $(M+1)$-tuple of functions but in the order $\{e_a, B_1, \cdots, B_M\}$. Then we have

$$\|h/X_{M+1}^a\|^2 \geq |\langle h, e_a \rangle|^2 = \delta^2.$$

Hence we have, for any M, $|\langle h, B^a_{M+1}\rangle| \geq \delta$. In view of this last estimation and (13) and (12), for large M we have the contradiction given by (11), (10). The proof is complete.

Remark It is noted that the selected parameters $a_1, \cdots, a_n, \cdots$ according the maximal principle may not satisfy the hyperbolic non-separable condition

$$\sum_{k=1}^{\infty}(1-|a_k|)=\infty$$

and thus the generated TM system $\{B_k\}$ may not be a basis. By doing such decomposition one is not interested in whether the resulted TM system is a basis, but only in whether it can effectively expand the given signal f. The AFD algorithm, in fact, achieves very fast convergence.

Remark For arbitrary selections of $a_1, \ldots, a_n, \ldots,$ we arrive at a pre-mono-component decomposition in the following sense: Each of the B_k's in the infinite sum after being multiplied by e^{it} becomes a mono-component. If we choose $a_1 = 0$, then all B_k's are mono-components, and AFD offers a mono-component decomposition.

Remark AFD is different from all the other existing greedy type algorithms [56, 109] due to the following features: (i) The sifting process to get a reduced remainder makes the system automatically orthogonal. (ii) Owing to the relations (8) the optimization in AFD is more optimal than that in orthogonal greedy algorithm. (iii) In AFD the supreme is attainable and therefore the algorithm attains the maximum energy portion in each of the iterations.

Remark Restricted to a practical subclass the convergence rate for AFD is $M/\sqrt{n}$, where n is the order of the AFD partial sum. This is considered as a good convergence rate for the boundary limits of Hardy space functions may be non-smooth.

3.3 Unwinding AFD (UWAFD)

Let $f = hg$, where f, g are Hardy $H^2(\mathbf{D})$ functions, and h is an inner function. Let f and g be expanded into their respective Fourier series, viz.,

$$f(z)=\sum_{k=0}^{\infty}c_k z^k, \qquad g(z)=\sum_{k=0}^{\infty}d_k z^k.$$

The Plancherel theorem and the modular 1 property of inner functions assert that

$$\sum_{k=0}^{\infty} |c_k|^2 = \|f\|^2 = \|g\|^2 = \sum_{k=0}^{\infty} |d_k|^2.$$

In digital signal processing (DSP) there is the following result: For any n,

$$\sum_{k=n}^{\infty} |c_k|^2 \geq \sum_{k=n}^{\infty} |d_k|^2$$

(see, for instance, [11, 19]).

In DSP this is referred as energy-front-loading property of minimum phase signals. This amounts to saying that through factorizing out the inner function factor the convergence rate of the Fourier series of the remaining outer function becomes higher. This fact suggests that the AFD process would be better to incorporate with the factorization process for speeding up the convergence. This instructs that when a signal by its nature is of high frequency, one should first perform "unwinding" before extracting out from it a maximal portion of lower frequency. We proceed as follows [74, 92]. First we do factorization $f = f_1 = I_1 O_1$, where I_1 and O_1 are, respectively, the inner and outer factors of f. The factorization is based on Nevanlinna's factorization theorem, also see [117]. The outer function has the explicit integral representation

$$O_1(z) = e^{\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} \log|f_1(e^{it})| dt}.$$

The outer function is computed by using the boundary value of f_1. On the boundary the above integral is taken to be of the principal integral sense. The imaginary part of the integral reduces to the circular Hilbert transform of $\log|f_1(e^{it})|$. Next, we do a maximal sifting to O_1. That gives

$$f(z) = I_1(z)[\langle O_1, e_{a_1}\rangle e_{a_1}(z) + f_2(z)\frac{z - a_1}{1 - \overline{a}_1 z}],$$

where f_2 is the maximal shifting of O_1 through a_1 :

$$f_2(z) = \frac{O_1(z) - \langle O_1, e_{a_1}\rangle e_{a_1}(z)}{\frac{z-a_1}{1-\overline{a}_1 z}}.$$

By factorizing f_2 into its inner and outer factors, $f_2 = I_1 O_2$, we have

$$f(z) = I_1(z)[\langle O_1, e_{a_1}\rangle e_{a_1}(z) + I_2(z)O_2(z)\frac{z - a_1}{1 - \overline{a}_1 z}].$$

We next proceed a maximal sifting to O_2, and so on. In such way we arrive at the unwinding AFD decomposition [74]:

Theorem 3.2 *The above procedure gives rise to the unwinding AFD (UWAFD) decomposition*

$$f(z) = \sum_{k=1}^{n} \prod_{l=1}^{k} I_l(z) \langle O_k, e_{a_k} \rangle B_k(z) + f_{n+1}(z) \prod_{k=1}^{n} \frac{z - a_k}{1 - \overline{a}_k z} \prod_{l=1}^{n} I_l(z),$$

where $f_{k+1} = I_{k+1} O_{k+1}$ is the maximal shifting of O_k through $a_k, k = 1, \ldots, n$, and I_{k+1} and O_{k+1} are, respectively, the inner and outer functions of f_{k+1}. Furthermore,

$$f(z) = \sum_{k=1}^{\infty} \prod_{l=1}^{k} I_l(z) \langle O_k, e_{a_k} \rangle B_k(z).$$

Remark Like AFD, unwinding AFD (UWAFD) is a mono-component or pre-mono-component decomposition. Experiments, in particular on singular inner functions, show that among various AFD type algorithms UWAFD converges most rapidly [92].

Remark If we do not incorporate with a maximal sifting process as what we have done in UWAFD, the algorithm falls into UWA, which was first developed in [64] 2000. Below we denote Blaschke products as ϕ_k, which can have finite or infinite zeros, denote products of a singular inner function and an outer function by ψ_k, and denote $f = f_1, f_k(z) = \psi_{k-1}(z) - \psi_{k-1}(0), k = 2, \cdots, c_k = \psi_k(0), k = 1, 2, \cdots$. UWA proceeds as

$$\begin{aligned} f(z) = f_1(z) &= \phi_1(z)\psi_1(z) \\ &= \phi_1(z) \left(\psi_1(z) - \psi_1(0) + \psi_1(0)\right) \\ &= c_1\phi_1(z) + \phi_1(z) f_2(z) \\ &= c_1\phi_1(z) + \phi_1(z)\phi_2(z) \left(\psi_2(z) - \psi_2(0) + \psi_1(0)\right) \\ &= c_1\phi_1(z) + c_2\phi_1(z)\phi_2(z) + \phi_1(z)\phi_2(z) f_3(z) \\ &= \cdots \\ &= \sum_{k=1}^{\infty} c_k \phi_1(z) \cdots \phi_k(z). \end{aligned}$$

The convergence in H^2 was first proved in Remark 4.4, [74]. Several generalized convergence results were proved in [15]. In the recent papers [108] and [104] the computation aspect of UWA is studied.

3.4 Cyclic AFD for n-Best Rational Approximation

In core-AFD the parameters $a_1, \ldots, a_n, \ldots$ are selected in the one by one manner to obtain an optimal sequence of Blaschke forms to approximate the given function

$$\sum_{k=1}^{n} \langle f, B_{\{a_1,\cdots,a_k\}}\rangle B_{\{a_1,\cdots,a_k\}}(z). \tag{14}$$

Now we change the question to the following: Given $f \in H^2(\mathbf{D})$ and a fixed positive integer n, find n parameters $\tilde{a}_1, \ldots, \tilde{a}_n$ such that the associated n-Blaschke form best approximates f, that is,

$$\|f - \sum_{k=1}^{n} \langle f, B_{\{\tilde{a}_1,\cdots,\tilde{a}_k\}}\rangle B_{\{\tilde{a}_1,\cdots,\tilde{a}_k\}}(z)\| \tag{15}$$

$$= \min\{\|f - \sum_{k=1}^{n} \langle f, B_{\{b_1,\cdots,b_k\}}\rangle B_{\{b_1,\cdots,b_k\}}(z)\| \ : \ \{b_1, \cdots, b_n\} \in \mathbf{D}^n\}. \tag{16}$$

This amounts an optimization with simultaneous selected n parameters that is obviously better than one on selections of n parameters in the one by one manner. Simultaneous selection of the parameters in an approximating n-Blaschke form is equivalent with the so-called *optimal approximation by rational functions of degrees not larger than n*. The latter problem was phrased as *n-best rational approximation*. It has been a long standing open problem, presented as follows.

Let p and q denote polynomials of one complex variable. We say that (p, q) is an *n-pair* if p and q are co-prime, both of degrees less than or equal to n, and q does not have zero in the unit disc. Denote by $\mathcal{R}_n$ the set of all such n-pairs. If $(p, q) \in \mathcal{R}_n$, then the rational function p/q is said to be a rational function of degree less or equal n. Let f be a function in the Hardy H^2 space in the unit disc. To find an n-best rational approximation to f is to find an n-pair (p_1, q_1) such that

$$\|f - p_1/q_1\| = \min\{\|f - p/q\| \ : \ (p, q) \in \mathcal{R}_n\}.$$

Existence of such a minimum solution was proved many decades ago [4, 112], a practical algorithm to get a solution, however, has been an open problem till now. The best n-Blaschke form approximation is essentially equivalent with the n-best rational approximation. There are separate proofs for existence of the solution of optimization problem (15) [75, 84]. By taking advantages of the explicit form and the orthogonality of Blaschke forms we get a practical algorithm for the classical n-best rational approximation problem.

By using *cyclic* AFD *algorithm* we can easily get a solution of the mentioned problem (15) if there is only one critical point for the objective function [75]. In general, cyclic AFD offers a *conditional solution* depending on the initial values to star with. Besides cyclic AFD there was a previously existing algorithm, RARL2, by the French institute INRIA, that again can only get a conditional solution [4, 32]. The theory and algorithm of cyclic AFD seem more explicit than RARL2. It uses the poles of the approximating rational functions as parameters. The other known rational approximation models all use the coefficients of p and q as parameters. Using coefficients of polynomials, which is double amount of the parameter number of the n-Blaschke form setting, involves tedious analysis and computation. The ultimate solution of the optimization problem lays on suitable selections of initial values to start with. Finding suitable optimal initial values, however, is an NP hard problem. Below we describe cyclic AFD algorithm.

For a given positive integer number n the objective function for the n-Blaschke optimization problem is

$$A(f; a_1, \ldots, a_n) = \|f\|^2 - \sum_{k=1}^{n} |\langle f, B_k\rangle|^2. \tag{17}$$

Definition 2 An n-tuple $(a_1, \ldots, a_n)$ is said to be a coordinate-minimum point (CMP) of the objective function $A(f; z_1, \ldots, z_n)$ if for any chosen k among 1,. . . ,n, whenever we fix the rest $n-1$ variables, being $z_1 = a_1, \ldots, z_{k-1} = a_{k-1}, z_{k+1} = a_{k+1}, \ldots, z_n = a_n$, and select the kth variable z_k to minimize the objective function, we have

$$a_k \in \arg\min\{A(f; a_1, \ldots, a_{k-1}, z_k, a_{k+1}, \ldots, a_n) \ : \ z_k \in \mathbf{D}\}.$$

In the core-AFD algorithm we proceed the following procedure: For a $(k-1)$-tuple $\{a_1, \ldots, a_{k-1}\}$ in $\mathbf{D}$ we produce the reduced remainders $f_2, \ldots, f_k$, and for f_k we apply the maximal selection principle to find an a_k giving rise to $\max\{|\langle f_n, e_a\rangle| : a \in \mathbf{D}\}$. The proposed cyclic AFD algorithm repeats such procedure always for $k = n$: For any permutation P of $1, \ldots, n$, for the first $(n-1)$ parameters in the order $a_{P(1)}, \ldots, a_{P(n-1)}$ we produce the corresponding reduced remainders $f_2, \ldots, f_n$, and then use the maximal selection principle to select an optimal $a_{P(n)}$.

The proposed cyclic AFD algorithm is valid by the following theorem.

Theorem 3.3 *Suppose that f is not an m-Blaschke form for any $m < n$. Let $s_0 = \{b_1^{(0)}, \ldots, b_n^{(0)}\}$ be any n-tuple of parameters inside $\mathbf{D}$. Fix some $n-1$ parameters of s_0 and make an optimal selection of the single remaining parameter under the "maximal selection principle" in accordance with the objective function (17). Denote the obtained new n-tuple of parameters by s_1. We repeat this procedure and make cyclic optimal selections over the n parameters. In the process we obtain a sequence of n-tuples $s_0, s_1, \ldots, s_l, \ldots,$ with decreasing objective function values $d_1, \ldots, d_l, \ldots$ that tend to a limit $d \geq 0$, where, in the notation and formulation of (17),*

$$d_l = A(f; b_1^{(l)}, \ldots, b_n^{(l)}) = \|f\|^2 - \sum_{k=1}^{n} (1 - |b_k^{(l)}|^2)|f_k^{(l)}(b_k^{(l)})|^2. \tag{18}$$

Then, (i) If $\overline{s}$, as an n-tuple, is a limit of a subsequence of $\{s_l\}_{l=0}^{\infty}$, then $\overline{s}$ is in $\mathbf{D}$; *(ii) $\overline{s}$ is a CMP of $A(f;\cdots)$; (iii) If the correspondence between a CMP and the corresponding value of $A(f;\cdots)$ is one to one, then the sequence $\{s_l\}_{l=0}^{\infty}$ itself converges to the CMP, being dependent of the initial n-tuple s_0; (iv) If $A(f;\ldots)$ has only one CMP, then $\{s_l\}_{l=0}^{\infty}$ converges to a limit $\overline{s}$ in* $\mathbf{D}$ *at which $A(f;\cdots)$ attains its global minimum value.*

We refer the reader to [75] for details and examples of cyclic AFD. In a recent paper the algorithm is improved by incorporating a complex gradient decent method [82].

3.5 Pre-Orthogonal Adaptive Fourier Decomposition (POAFD) for Reproducing Kernel Hilbert Spaces

The approximation theory and algorithm that were developed in the previous sections can be extended to more general contexts. To explain just the idea we restrict ourselves to the simplest cases, including the weighted Bergman spaces and weighted Hardy spaces, etc. Assume that Hilbert space $\mathcal{H}$ consists of functions defined in an open connected region $\mathcal{E}$ (can be unbounded) in the complex plane, and the reproducing kernel k_a is an analytic function of the variable a in $\mathcal{E}$ satisfying the relation

$$f^{(l)}(a) = \langle f, \left(\frac{\partial}{\partial \overline{a}}\right)^l k_a\rangle, \quad l = 1, 2, \cdots \tag{19}$$

Let $\{a_1, \cdots, a_n, \cdots\}$ be a finite or infinite sequence. For a fixed n we define the multiple of a_n, denoted by $l(a_n)$, to be the repeating times of a_n in the n-tuple $\{a_1, \cdots, a_n\}$. With this definition, for instance, the multiple of a_1 is just 1, and the multiple of a_2 will depend on whether $a_2 = a_1$. If yes, then $l(a_2) = 2$, and, if not, $l(a_2) = 1$, and so on. Note that it is a little abuse of notation for it is not dependent on the value of a_n but on the repeating times of a_n in the corresponding n-tuple. We accordingly define

$$\tilde{k}_{a_n} \triangleq \left[\left(\frac{\partial}{\partial \overline{a}}\right)^{l(a_n)-1} k_a\right]_{a=a_n} \triangleq \left(\frac{\partial}{\partial \overline{a}}\right)^{l(a_n)-1} k_{a_n}. \tag{20}$$

We further assume the following boundary vanishing condition, implying the maximal selection principle in every individual context, as follows: Let $a_1, \cdots, a_{n-1}$ be previously given, and $\{B_1, \cdots, B_{n-1}\}$ be the Gram–Schmidt orthonormalization of $\{\tilde{k}_{a_1}, \cdots, \tilde{k}_{a_{n-1}}\}$, then for every $f \in \mathcal{H}$, the pre-orthogonal system has the property

$$\lim_{a\to\partial\mathcal{E}} \langle f, B_n^a\rangle = 0, \tag{21}$$

where $\{B_1, \cdots, B_{n-1}, B_n^a\}$ is the Gram–Schmidt orthonormalization of $\{\tilde{k}_{a_1}, \cdots, \tilde{k}_{a_{n-1}}, k_a\}$, with $a \neq a_k, k = 1, \cdots, n-1$. We note (1) if $a \to \partial\mathcal{E}$, then a is different from any already selected $a_k, k = 1, \cdots, n-1$; and (2) in any case the limit $a \to \partial\mathcal{E}$ is in the sense of the topology of the one-point-compactification of the complex plane while the "one point" takes to be ∞. With boundary vanishing assumption we conclude the maximal selection principle of POAFD: Under the assumption (21), through a compact argument, there exists a sequence $\{b_j\}_{j=1}^{\infty}$ such that none of the b_j's take any values $a_1, \cdots, a_{n-1}$, and $\lim_{j\to\infty} b_j \triangleq a_n \in \mathcal{E}$, and

$$\lim_{j\to\infty} |\langle f, B_n^{b_j}\rangle| = \max\{|\langle f, B_n^a\rangle| \ : \ a \in \mathcal{E}\}. \tag{22}$$

To understand what would happen if a_n coincides with a previous $a_k, k < n$, we prove the following lemma.

Lemma 3.4

$$\lim_{l\to\infty} B_n^{b_j} = B_n^{a_n},$$

where $\{B_1, \cdots, B_{n-1}, B_n^{a_n}\}$ *is the Gram–Schmidt orthonormalization of* $\{\tilde{k}_{a_1}, \cdots, \tilde{k}_{a_{n-1}}, \tilde{k}_{a_n}\}$.

Proof If a_n does not coincide with any $a_k, k = 1, \cdots, a_{n-1}$, then $\lim_{j\to\infty} B_n^{b_j} = B_n^{a_n}$, where $\{B_1, \cdots, B_{n-1}, B_n^{a_n}\}$ is the Gram–Schmidt orthonormalization of $\{\tilde{k}_{a_1}, \cdots, \tilde{k}_{a_{n-1}}, k_{a_n}\} = \{\tilde{k}_{a_1}, \cdots, \tilde{k}_{a_{n-1}}, \tilde{k}_{a_n}\}$. Now consider the case that a_n coincides with some of the earlier $a_1, \cdots, a_{n-1}$, or in other words, $l(a_n) > 1$. That means that, in the notation (20), the $(l-1)$ functions $k_{a_n}, \frac{\partial}{\partial \overline{a}} k_{a_n}, \cdots, \left(\frac{\partial}{\partial \overline{a}}\right)^{(l-2)} k_{a_n}$ have already appeared in the sequence $\{\tilde{k}_{a_1}, \cdots, \tilde{k}_{a_{n-1}}\}$. As a consequence, the function

$$T_{l-2}(b_j, a_n) = k_{a_n} + \frac{\frac{\partial}{\partial \overline{a}} k_{a_n}}{1!}(\overline{b_j} - \overline{a}_n) + \cdots + \frac{\left(\frac{\partial}{\partial \overline{a}}\right)^{(l-2)} k_{a_n}}{(l-2)!}(\overline{b_j} - \overline{a}_n)^{l-2},$$

as the order-$(l-2)$ Taylor expansion of the function $k_a(z)$ in $\overline{b}_j$ about $\overline{a}_n$, is in the linear span of $B_1, \cdots, B_{n-1}$. This last statement amounts to the identical relation

$$T_{l-2}(b_j, a_n) - \sum_{k=1}^{n} \langle T_{l-2}(b_j, a_n), B_k\rangle B_k = 0. \tag{23}$$

Since b_j's are taken differently from all $a_k, k = 1, \cdots, n-1$, we have, with the G-S orthonormalization,

$$B_n^{b_j}(z) = \frac{k_{b_j}(z) - \sum_{k=1}^{n-1}\langle k_{b_j}, B_k\rangle B_k(z)}{\|k_{b_j} - \sum_{k=1}^{n-1}\langle k_{b_j}, B_k\rangle B_k\|}, \tag{24}$$

Inserting (23) into (24), and dividing by $(\overline{b_j} - \overline{a}_n)^{l-1}$ and $|\overline{b_j} - \overline{a}_n|^{l-1}$ to the numerator and the denominator parts, respectively, we have

$$B_n^{b_j}(z) = e^{-i(l-1)\theta} \frac{\frac{k_{b_j}(z) - T_{l-2}(k_{b_j}, a_n)(z)}{(\overline{b_j} - \overline{a}_n)^{l-1}} - \sum_{k=1}^{n-1}\langle \frac{k_{b_j} - T_{l-2}(b_j, a_n)(z)}{\overline{w} - \overline{a}_n)^{l-1}}, B_k\rangle B_k(z)}{\|\frac{k_{b_j}(z) - T_{l-2}(b_j, a_n)(z)}{(\overline{b_j} - \overline{a}_n)^{l-1}} - \sum_{k=1}^{n-1}\langle \frac{k_{b_j}(z) - T_{l-2}(b_j, a_n)(z)}{(\overline{b_j} - \overline{a}_n)^{l-1}}, B_k\rangle B_k\|}, \tag{25}$$

where $e^{i\theta}$ is the tangential direction of the limiting $b_j \to a_n$. We can, in fact, take any direction, including $\theta = 0$. Letting $b_j \to a_n$ with $\theta = 0$, and using the Lagrange type remainder of the Taylor expansion, we obtain

$$\lim_{j\to\infty} B_n^{b_j}(z) = \frac{\tilde{k}_{a_n}(z) - \sum_{k=1}^{n-1}\langle \tilde{k}_{a_n}, B_k\rangle B_k(z)}{\|\tilde{k}_{a_n} - \sum_{k=1}^{n-1}\langle \tilde{k}_{a_n}, B_k\rangle B_k\|}.$$

Therefore, $\{B_1, \cdots, B_{n-1}, B_n^{a_n}\}$ is a Gram–Schmidt orthonormalization of $\{\tilde{k}_{a_1}, \cdots, \tilde{k}_{a_{n-1}}, \tilde{k}_{a_n}\}$. The proof is complete.

Remark The essence of the proof is contained in [76, 77, 96].

We have the pre-orthogonal adaptive Fourier decomposition (POAFD) convergence theorem as follows.

Theorem 3.5 *Selecting* $\{a_1, \cdots, a_n, \cdots\}$ *according to the maximal selection principle set by (22), we have*

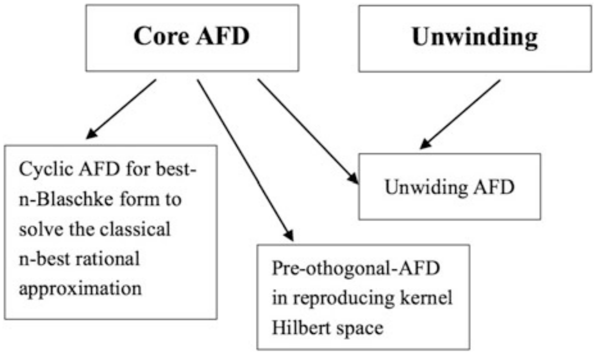
$$f = \sum_{k=1}^{\infty}\langle f, B_n\rangle B_n,$$

where for any positive integer n, $\{B_1, \cdots, B_{n-1}, B_n\}$ *is the Gram–Schmidt orthonormalization of* $\{\tilde{k}_{a_1}, \cdots, \tilde{k}_{a_{n-1}}, \tilde{k}_{a_n}\}$.

One can adopt the same proof for the AFD convergence (Theorem 3.1) in which only the standard remainders g_k's are involved. As a matter of fact, the sifting process and the role of the induced remainders are taken place by the pre-orthogonal process.

Remark Repeating selection of a parameter is a natural thing under maximal selection principle. For POAFD we refer the reader to the references [54, 55, 76, 77, 84, 91], where POAFD is called by other names including POGA or PreOGA, etc. Since the method inherits the AFD idea, that is especially seen from the relation (8), we decide to justify the name and call it by POAFD.

Various Adaptive Fourier Decomposition Methods



4 Related Studies and Applications

4.1 Aspects in Relation to Beurling–Lax Shift-Invariant Subspaces

The AFD type expansions is in a great extent related to the Beurling–Lax shift-invariant subspaces of the Hardy H^2 spaces. In the unit disc case,

$$H^2(\mathbf{D}) = \overline{\text{span}}\{B_k\}_{k=1}^{\infty} \oplus \phi H^2(\mathbf{D}), \tag{26}$$

where $\{B_k\}_{k=1}^{\infty}$ is the TM system generated by a sequence $\{a_1, \cdots, a_n, \cdots\}$, where multiples are counted, and ϕ is the Blaschke product with the zeros $\{a_1, \cdots, a_n, \cdots\}$ including the multiples. Note that when a Blaschke product ϕ having a_k's as all its zeros does not exist, corresponding to the condition

$$\sum_{k=1}^{\infty}(1 - |a_k|) < \infty,$$

then the associated TM system is a basis. Although this has been well known over a long time, its relations with adaptive expansions, as far as what are aware by the author, have not been brought up. The fact that TM systems being Schauder systems was proved in [93]. The space decomposition relation (26) was extended to H^p spaces, where $p \neq 2$ [80]. Relations between backward shift invariant subspaces and bandlimited functions and Bedrosian identity [80, 107] were studied. There are open questions on whether there exist adaptive and fast converging expansions by using TM systems for the cases $p \neq 2$, and for $p = 2$ how far one can extend AFD (26) to higher dimensions. The study has a great room to be further developed.

4.2 Extra-Strong Uncertainty Principle

The phase and frequency studies in mono-component function theory lay certain foundations in digital signal processing. In related studies what is called extra-strong uncertainty principle

$$\sigma_t^2\sigma_\omega^2 \geq \frac{1}{4} + \left(\int_{-\infty}^{\infty} |t - \langle t\rangle||\phi(t) - \langle\omega\rangle||f(t)|^2 dt\right)^2 \tag{27}$$

was recently established [22], where f is a real-valued signal, σ_t^2 and σ_ω^2 are the standard deviations with respect to the time and the Fourier frequency, and $\langle t\rangle$ and $\langle\omega\rangle$ are the corresponding means. A weaker uncertainty principle of the same type was previously given by L. Cohen

$$\sigma_t^2\sigma_\omega^2 \geq \frac{1}{4} + |\int_{-\infty}^{\infty} (t - \langle t\rangle)(\phi(t) - \langle\omega\rangle)|f(t)|^2 dt|^2$$

[13]. We further extended the above result to multi-dimensional contexts [21–24, 26].

4.3 The Dirac-Type Time-Frequency Distributions Based on Mono-component Decompositions

The Dirac-type time-frequency distribution (DTFD) of the form

$$P(t,\omega) = \rho^2(t)\delta(\omega - \theta'(t)) \tag{28}$$

is the ultimate desire of signal analysts. Several time-frequency distributions, including windowed Fourier transform and Wigner–Ville transform, etc., have been used by signal analysts, of which none are entirely satisfied. The existing time-frequency distributions do not give explicit and clear frequency components, and, they often depend on parameter selections. Positive-frequency decompositions of signals offered by the AFD decompositions naturally give rise to Dirac-type time-frequency distributions. For a single mono-component $m_1(t) = \rho_1(t)\cos\theta_1(t)$ the corresponding DTFD according to (28) is the graph $(t, \theta_1'(t))$ of the function $\omega = \theta_1'(t)$ in the ω-t plane, while the weight $\rho_1^2(t)$ may be represented by colors continuously changing along with changing of the values $\rho_1^2(t)$. If a signal f is expanded into a series of "intrinsic composing" mono-components, then its DTFD is the bunch of color-weighted graphs of which each is made from a composing mono-component [20, 126]. This definition has been interested and being paid attention by signal analysts including Leon Cohen and Lorenzo Galleani, etc., and has been used in practice (see below the application section).

4.4 Higher Dimensional AFDs

To develop an AFD like approximation theory in higher dimensions a Cauchy type structure is necessary that is mainly for use of the reproducing kernel property in conjunction with maximal selection principle. By using the Cauchy structure in the Clifford algebra or the several complex variables setting we achieved the AFD type theories and the associated algorithms for function spaces with those settings, including several real variables on the plane (Clifford Hardy spaces and Hardy spaces on tubes), on the real spheres, on the n-torus, and on the n-complex spheres [1, 2, 91, 94, 114]. With D. Alpay, F. Colombo, I. Sabadini we achieved analogous theory involving matrix-valued Blaschke products [1, 2]. This study has impacts to rational approximation in a number of spaces [5].

4.5 Fourier Spectrum Characterization of Hardy Spaces: Analytic Signals Revised

The Paley–Wiener theorem for the classical Hardy H^2 space over the upper half complex space addresses the fact that if $f \in L^2(\mathbf{R})$, then further $f \in H^2(\mathbf{C}^+)$ if and only if $\mathrm{supp}\hat{f} \subset [0, \infty)$. This result is systematically extended to $H^p(\mathbf{C}^+)$ for all $p \in [1, \infty]$, where the Fourier transform is in some occasions defined in the distribution sense [70, 86]. The results are summarized as: Letting $f \in L^p(\mathbf{R})$, then f is further the non-tangential boundary limit of some function in the complex Hardy space $H^p(\mathbf{C}^+)$ if and only if $\hat{f} = \chi_+ \hat{f}$, where χ_+ is the indicator (characteristic) function of the right-half-real line, and the Fourier transform may take the distribution sense. The generalization to the Hardy spaces on tubes (extending the $p = 2$ case in [103] to $1 \leq p \leq \infty$) was given in [49]. The results of the same type but with the Clifford algebra setting is given in [28] proving that a Clifford-valued function $f \in L^p(\mathbf{R}^n)$ is the non-tangential boundary limit of some Clifford-valued Hardy space function in the upper half space if and only if $\hat{f} = \chi_+ \hat{f}$, where $\chi_+(\underline{\xi}) = \frac{1}{2}\left(1 + i\frac{\underline{\xi}}{|\underline{\xi}|}\right)$, $\underline{\xi} = \xi_1 e_1 + \cdots + \xi_n e_n$ (the Hardy space projection function). This extends some partial cases proved in [103] for conjugate harmonic systems. In various contexts Fourier spectrum characterizations give rise to Hardy spaces decompositions for $L^p,\ 1 < p \leq \infty$, that further induce Hardy space decompositions of $L^p,\ 0 < p < 1$, [30, 48]. Hardy space decomposition is the strategy of our study of functions of various integrability in the Lebesgue sense. The strategy is extensively implemented along with the mono-component and AFD approximation theories. In particular, for any signal f, by multiplying $\hat{f}$ with χ_+ and then taking the inverse Fourier transform, we obtain the associated analytic signal. This is philosophically valid in any context. We finally note that the Hardy space decomposition issue has been extended to the L^p-vector fields and one obtains the Hardy–Hodge decomposition [6].

4.6 Hilbert Transforms as Singular Integral Operators: Analytic Signals Revised

As in the 1-D case [7], on higher dimensional manifolds one defines the non-scalar part of the non-tangential boundary limit of a hypercomplex holomorphic function in a certain sense to be the Hilbert transform of the scalar part [83]. Hilbert transform, therefore, is a particular singular integral. One is to study singular integrals to understand Hilbert transforms. On one dimensional manifolds, including Lipschitz perturbations of the real line and the circle, certain singular integrals of holomorphic kernels form an operator algebra as studied in a series of work of A. McIntosh, C. Li, S. Semmes, T. Qian, R.-L. Long, and S.-L. Wang [37, 58, 59, 67]. The theory on the plane was earlier established in the work, or under influence, of A. McIntosh [36, 46, 47]. Through using and generalizing the results of Fueter and Sce to arbitrary Euclidean spaces (as technical necessity) the second author acquired the necessary techniques to establish the theory of the operator algebra of singular integrals of quaternionic or Clifford monogenic kernels on Lipschitz perturbations of the unit sphere for all dimensions [3, 68, 69]. Based on the established singular integral theory Hilbert transformations of the plane and of the sphere became well understood. Analytic signals on the sphere, for instance, are constructed as follows: Let f be a real-valued signal of finite energy on a manifold $\mathcal{S}$. Denote by $H_{\mathcal{S}}$ the Hilbert transform of f on the manifold. Then the analytic signal on $\mathcal{S}$ is defined to be $f^+ = f + H_{\mathcal{S}}f$, where $H_{\mathcal{S}}f$ is the non-scalar part (on sphere it is a 2-form valued function). f^+ has Clifford monogenic extension to one of the two regions divided by $\mathcal{S}$. One can derive, if ζ is on the plane or on the sphere,

$$f^+(\zeta) = \rho_f(\zeta)\left(\frac{f(\zeta)}{\rho_f(\zeta)} + \frac{H_{\mathcal{S}}(f)(\zeta)}{|H_{\mathcal{S}}(f)(\zeta)|}\frac{|H_{\mathcal{S}}(f)(\zeta)|}{\rho_f(\zeta)}\right) \tag{29}$$

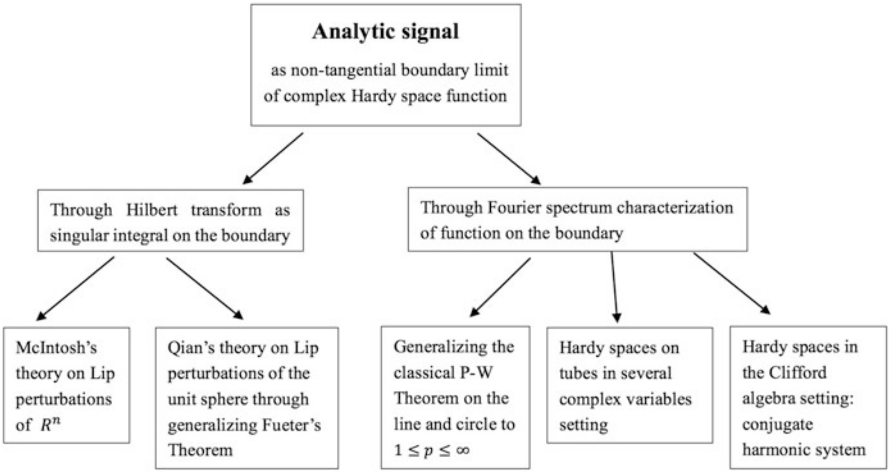
$$= \rho_f(\zeta)\left(\cos\theta(\zeta) + \frac{H_{\mathcal{S}}(f)(\zeta)}{|H_{\mathcal{S}}(f)(\zeta)|}\sin\theta(\zeta)\right) \tag{30}$$

$$= \rho_f(\zeta)e^{\frac{H_{\mathcal{S}}(f)(\zeta)}{|H_{\mathcal{S}}(f)(\zeta)|}\theta(\zeta)}, \tag{31}$$

where $\rho_f(\zeta) = \sqrt{|f(\zeta)|^2 + |H_{\mathcal{S}}f(\zeta)|^2}$, and $\left(\frac{H_{\mathcal{S}}(f)(\zeta)}{|H_{\mathcal{S}}(f)(\zeta)|}\right)^2 = -1$, the latter being a varying imaginary element just like the complex imaginary element with the property $i^2 = -1$. The instantaneous frequency is defined, as in the classical case through the mentioned monogenic extension, which can be formally read

$$\theta'(\zeta) = \mathrm{Re}\{\left[(\Gamma_\zeta - I)f^+(\zeta)\right]\left[(f^+(\zeta))^{-1}\right]\},$$

the latter can be expressed in terms of the angle $\theta(\zeta)$ [83, 122], where Γ_ζ is the surface Dirac operator on the manifold. Such formulation gives IF a certain sense in higher dimensions. The related studies published, in, respectively, 2015 and 2017 with the Chinese Science Press two monographs books [76] and [78]. With the Szegö and Bergman kernel of the related regions, following what is done in the standard domains, approximation theory for certain hypercomplex analytic function spaces can be established [41]. We finally note that Hilbert transformation may be characterized by commutativity with the affine groups in the underlying symmetric manifold which shows that the three objects the Hilbert transformation, the Dirac differential operator, and the group representation theory have intimate relations [25, 27].



Analytic Signals in Various Contexts

4.7 Applications

Mono-component function and AFD theories have demonstrative applications in signal and image analysis as well as in system identification. A number of signal and image analysts promoted the AFD method. Below we summarize part of the applications found in the literature.

It is commented in [118] that, as a new method, AFD was proposed in the recent years that could be used to decompose and reconstruct signals. It contains the classical Fourier method as a particular case. Experiments show that the 1D AFDs achieve excellent signal decomposition and reconstruction results. The article [119] compares 2D AFD with the traditional frequency digital watermark methods, including discrete cosine transform DCT, discrete wavelet transform DWT, discrete Fourier transform DFT, etc., and concludes that 2D AFD has better transparency and robustness under attacking. A recent watermarking method based on 2D AFD is developed in [51]. The article [120] revises the 2D AFD algorithm and, as a result, increases its speed, and uses it in denoising.

In [52] Y. Liang et al. propose a new fault diagnosis method of rolling bearing faults based on AFD. They show that AFD can avoid using band-pass filters, the latter often suffering from the difficulty of algorithm parameter selection, they show that AFD adaptively, efficiently, and accurately diagnose all kinds of rolling bearing problems in their study.

In [115] the authors study interference and separation between the lung sound (LS) and the heart sound (HS) signals. Due to the overlap in their frequency spectra, it is difficult to separate them. The article proposes a novel separation method based on AFD. This AFD-based separation method is validated on real HS signals from the University of Michigan Heart Sound and Murmur Library, as well as on real LS signals from the 3M repository. Simulation results indicate that the proposed method is more effective than the extraction methods based on the recursive least square (RLS), than the standard empirical mode decomposition (EMD) and its various extensions, including the ensemble EMD (EEMD), the multivariate EMD (M-EMD), and the noise assisted M-EMD (NAM-EMD) [39, 102].

Over the years people have made unremitting studies in predicting the stock price movements. In [125] a novel automatic stock movement forecasting system is proposed, which is based on the newly developed signal decomposition approach—adaptive Fourier decomposition (AFD). AFD can effectively extract the signal primary trend, which is specifically suitable in the Dow theory based automatic technique analysis. Effectiveness of the proposed approach is assessed through the comparison with the direct BP approach and manual observation. The result is proved to be promising.

In [124] an AFD based time-frequency speech analysis approach is proposed. Given the fact that the fundamental frequency of speech signals often undergoes fluctuation, the classical short-time Fourier transform (STFT) based spectrogram analysis suffers from the difficulty of window size selection. AFD is a newly developed signal decomposition theory. The outstanding characteristic of AFD is to provide instantaneous frequency for each decomposed component, so the time-frequency analysis becomes accessible. Experiments are conducted based on the sample sentence in TIMIT Acoustic-Phonetic Continuous Speech Corpus. The results show that the AFD based time-frequency distribution outperforms the STFT.

AFD has already been employed to the productions of CASA Environmental Technology Co., Ltd, including the second generation of BEWS (biological early warning system) and ETBES (ecological toxicity biological exposed system). The two systems take advantages of AFD and unwinding to analyze the biological behavioral signal. Compared with the traditional Fourier, wavelet, and EMD algorithms, the AFD approach can efficiently solve early warning judgment for low concentration pollutants and disturbance of fish biological clock and other problems.

In control theory the authors of the article [44] introduce an AFD algorithm to eliminate the channel noise superimposed on the output signal in the wireless transmission process. In the frequency domain, based on AFD, an ILC method for discrete linear system with wireless transmission is proposed. Simulation results show that the AFD algorithm is able to achieve signal denoising well in the case of small decomposition threshold compared with Fourier decomposition. Thus the goal that the output signal of ILC system can track the desired signal is better achieved.

Indian researchers in their article [34] assert that to analyze biomedical signals in relation to e-health devices the frequency domain method outperforms the time domain method, and among numerate frequency domain methods (Hermit, Fourier, Karhunen–Loeve, Wavelet) AFD appears to have features of a greater variety, and more stable for the data compression. Based on compression using AFD they started to manufacture economic, accurate, and stable domestic e-health devices.

Apart from China and Asia, AFD has also achieved international influence. Interests, studies, and applications of AFD are found in relevant literature, by Ph.D. thesis of F. D. Fulle at Michigan University on oxygenic photosynthesis; by A. Kirkbas et al. on optimal basis pursuit based on Jaya optimization for adaptive Fourier decomposition [42]; by V. Vatchev, on a class of intrinsic trigonometric mode polynomials [110]; by J. Mashreghi et al. on Blaschke Products and Applications [57]; by R.S. Krausshar et al. on Clifford and harmonic analysis on cylinders and tori [43]; by F. Colombo et al. on the Fueter mapping theorem in integral form and the $\mathcal{F}$-functional calculus [18]; by M.I. Falc*ão* et al. on remarks on the generation of monogenic functions; by F. Colombo et al. on the Fueter primitive of bi-axially monogenic functions [17]; by L. Salomon on analysis of the anisotropy in image textures [101]; by F. Sakaguchi on the related integral-type method in higher order differential equations [97–100]; by P. León on instantaneous frequency estimation and representation of the audio signal through complex wavelet additive synthesis [29]; by F.E. Mozes on computing the instantaneous frequency for an ECG signal [63]; by N.R. Gomes, as Doctoral dissertation, on compressive sensing in Clifford analysis; by T. Eisner et al. on discrete orthogonality of the Malmquist–Takenaka system on the upper half plane and rational approximation [31]; and by A. Perotti on his article in directional quaternionic Hilbert operators [65].

The AFD methods have found promising applications, especially in model reduction, in system identification [10, 50, 60–62, 116].

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One-Component Inner Functions II

Joseph Cima and Raymond Mortini

Abstract We continue our study of the set $\mathfrak{I}_c$ of inner functions u in $H^\infty(\mathbb{D})$ with the property that there is $\eta \in]0,1[$ such that the level set $\Omega_u(\eta) := \{z \in \mathbb{D} : |u(z)| < \eta\}$ is connected. These functions are called one-component inner functions. Here we show that the composition of two one-component inner functions is again in $\mathfrak{I}_c$. We also give conditions under which a factor of one-component inner function belongs to $\mathfrak{I}_c$.

1 Introduction

Let $H^\infty = H^\infty(\mathbb{D})$ be the space of all bounded holomorphic functions in the open unit disk $\mathbb{D}$. In this paper we study an important class of inner functions, the so-called *one-component inner functions*. Recall that a function $u \in H^\infty$ is said to be *inner* if the boundary values of u have modulus one almost everywhere. Such an inner function u now is said to be a *one-component inner function* if there is $\eta \in]0,1[$ such that the level set (also called sublevel set or filled level set)

$$\Omega_u(\eta) := \{z \in \mathbb{D} : |u(z)| < \eta\}$$

is connected. We denote the collection of all one-component inner functions by $\mathfrak{I}_c$. Unimodular constants are considered to belong to $\mathfrak{I}_c$. These functions were first studied by B. Cohn [7] in connection with embedding theorems and Carleson measures. It was shown in [7, p. 355], for instance, that arc length on $\{z \in \mathbb{D} : |u(z)| = \varepsilon\}$ is a Carleson measure whenever $\Omega_u(\eta)$ is connected for some

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$\eta < \varepsilon < 1$. Many operator-theoretic applications appear in [1–3, 5]. A detailed study of the elements in $\mathfrak{I}_c$ was undertaken by A.B. Aleksandrov [1]. Classes of explicit examples of one-component inner functions were given by the present authors in [6]. The most fundamental ones are finite Blaschke products and singular inner functions S_μ with finite singularity set (or spectrum) $\text{Sing}(S_\mu)$. Infinite interpolating Blaschke products with real zeros (x_n) satisfying $0 < \eta_1 \leq \rho(x_n, x_{n+1}) \leq \eta_2 < 1$ (where ρ is the pseudohyperbolic distance in $\mathbb{D}$) were also shown to belong to $\mathfrak{I}_c$. On the other hand, no finite product of thin interpolating Blaschke products (these are (infinite) Blaschke products B whose zeros (z_n) satisfy $\lim_n \prod_{k:k\neq n} \rho(z_n, z_k) = 1$) can be in $\mathfrak{I}_c$. It also turned out that the class of one-component inner functions is invariant under taking finite products. In the present note, we are considering when a factor of a one-component inner function is in $\mathfrak{I}_c$ again. A sufficient criterion is provided. On the other hand, as it is shown, there exist two non-one-component inner functions u and v such that $uv \in \mathfrak{I}_c$. Our main result will show that the class of one-component inner functions is also invariant under taking compositions, generalizing special cases dealt with in [6]. The results of this note stem from December 2016. A paper by A. Reijonen [13] provides other classes of one-component inner functions.

2 Main Tools

Our results will mainly be based on the following known results which we recall for citational reasons.

Lemma 1 *Given a non-constant inner function u and $\eta \in]0, 1[$, let $\Omega := \Omega_u(\eta) = \{z \in \mathbb{D} : |u(z)| < \eta\}$ be a level set. Suppose that Ω_0 is a component (=maximal connected subset) of Ω. Then*

(1) Ω_0 is a simply connected domain; that is, $\mathbb{C} \setminus \Omega_0$ has no bounded components.
(2) $\inf_{\Omega_0} |u| = 0$.
(3) Either $\overline{\Omega_0} \subseteq \mathbb{D}$ or $\overline{\Omega_0} \cap \mathbb{T}$ has measure zero.

A detailed proof of parts (1) and (2) is given in [6]; part (3) is in [4, p. 733].

Recall that the spectrum $\text{Sing}(u)$ of an inner function u is the set of all boundary points ζ for which u does not admit a holomorphic extension; or equivalently, for which $Cl(u, \zeta) = \overline{\mathbb{D}}$, where

$$Cl(u, \zeta) = \{w \in \mathbb{C} : \exists (z_n) \in \mathbb{D}^{\mathbb{N}}, \ \lim z_n = \zeta \text{ and } \lim u(z_n) = w\}$$

is the cluster set of u at ζ (see [9, p. 80]). The pseudohyperbolic disk of center $z_0 \in \mathbb{D}$ and radius r is denoted by $D_\rho(z_0, r)$.

Theorem 2 (Aleksandrov) *Let u be an inner function. The following assertions are equivalent:*

(1) $u \in \mathfrak{I}_c$.

(2) There is a constant $C > 0$ such that for every $\zeta \in \mathbb{T} \setminus \operatorname{Sing}(u)$ we have

i) $|u''(\zeta)| \leq C\, |u'(\zeta)|^2$,

ii) $\liminf_{r \to 1} |u(r\zeta)| < 1$ *for all* $\zeta \in \operatorname{Sing}(u)$.

Note that as a consequence of this result, which is due to A. B. Aleksandrov [1, Theorem 1.11 and Remark 2, p. 2915], $u \in \mathfrak{I}_c$ necessarily implies that $\operatorname{Sing}(u)$ has measure zero.

3 Splitting Off Factors

In this section we give a condition under which a factor of a one-component inner function is in $\mathfrak{I}_c$ again. Recall from [6] that for the atomic inner function $S(z) = \exp(-\frac{1+z}{1-z})$ and a thin Blaschke product with positive zeros, $SB \in \mathfrak{I}_c$, but not B. For $a \neq 0$, let

$$\phi_a(z) = \frac{|a|}{a} \frac{a - z}{1 - \overline{a}z}$$

and $\phi_0(z) = z$. A Blaschke product B is written as $B = e^{i\theta} \prod_{j=1}^{\infty} \phi_{a_j}$, where we have $\sum_{j=1}^{\infty}(1 - |a_j|) < \infty$, and each a_j appearing as often as its multiplicity needs. The following result tells us that one can split off finitely many zeros without leaving the class of one-component inner functions. Any inner function u has the form $u = BS_\mu$, where B is a Blaschke product and S_μ a singular inner function

$$S_\mu(z) := \exp\left(-\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z}\, d\mu(\zeta)\right)$$

associated with a positive Borel measure μ which is singular with respect to Lebesgue measure on $\mathbb{T}$.

Proposition 3 *Let $\Theta \in \mathfrak{I}_c$ and $a \in \mathbb{D}$. If $\Theta(a) = 0$, then $v := \Theta/\varphi_a \in \mathfrak{I}_c$.*

Proof Note that $\Theta = \varphi_a v$. We may assume that v is not constant, otherwise we are done. Choose $\eta \in \,]0, 1[$ so that $\Omega_\Theta(\eta)$ is connected. Let

$$\delta := \inf\{|\varphi_a(z)| : |\Theta(z)| = \eta\}.$$

We claim that $\eta < \delta < 1$. In fact, since the set $L := \{z \in \mathbb{D} : |\Theta(z)| = \eta\}$ is not empty, and $|\varphi_a| < 1$ in $\mathbb{D}$, we see that $\delta < 1$. Moreover, if $z_0 \in L$, then

$$L' := \{|\varphi_a(z)| : |\Theta(z)| = \eta, |\varphi_a(z)| \leq |\varphi_a(z_0)|\}$$

is a compact set in [0, 1], and so

$$\inf\{|\varphi_a(z)| : |\Theta(z)| = \eta\} = \inf L' = \min L'.$$

Hence $\delta = |\varphi_a(z_1)|$ for some $z_1 \in L$. Since v is not a unimodular constant, we deduce from $|\Theta(z_1)| = |\varphi_a(z_1)|\, |v(z_1)|$ that $\eta < \delta$. Consequently, if $|\Theta(z)| = \eta$,

$$|v(z)| = \frac{|\Theta(z)|}{|\varphi_a(z)|} \leq \frac{\eta}{\delta} := \eta' < 1. \tag{1}$$

We claim that

$$\Omega_v(\eta) \subseteq \Omega_\Theta(\eta) \subseteq \Omega_v(\eta').$$

Notice that the first inclusion is obvious. To verify the second inclusion, let $z_0 \in \Omega_\Theta(\eta)$. We discuss three cases: $\rho(z, a) < \delta$, $\rho(z, a) = \delta$, and $\rho(z, a) > \delta$.

To this end, we first note that $D_\rho(a, \delta) \subseteq \Omega_\Theta(\eta)$. In fact, if $\rho(a, z) = |\varphi_a(z)| < \delta$, then $|\Theta(z)| < \eta$, since otherwise $\Theta(a) = 0$ implies the existence of $z_0 \in D_\rho(a, \delta)$ with $|\Theta(z_0)| = \eta$ and so, by the definition of δ, $|\varphi_a(z)| \geq \delta$. This is an obvious contradiction.

Hence $|\Theta(z)| \leq \eta$ for $\rho(z, a) = \delta$. Thus (1) holds true for $z \in \partial D_\rho(a, \delta)$. By the maximum principle, $|v(z)| < \eta'$ on $D_\rho(a, \delta)$. If $\rho(z, a) \geq \delta$ and $|\Theta(z)| < \eta$, then, as in (1), $|v(z)| < \eta'$, too. We deduce that $\Omega_\Theta(\eta) \subseteq \Omega_v(\eta')$.

Now we are able to prove that $\Omega_v(\eta')$ is connected. Assuming the contrary, there would exist a component Ω_1 of $\Omega_v(\eta')$ distinct (and so disjoint) from that containing the connected set $\Omega_\Theta(\eta)$. In particular, $|v| \geq |\Theta| \geq \eta$ on Ω_1. By Lemma 1, $\inf_{\Omega_1} |v| = 0$, which is an obvious contradiction. □

The preceding result admits the following generalization.

Proposition 4 *Let u, v be two non-constant inner functions and put $\Theta = uv$. Suppose that*

(i) $\Theta \in \mathfrak{I}_c$ and that $\eta \in]0, 1[$ is chosen so that $\Omega_\Theta(\eta)$ is connected.
(ii) $\sigma := \sup_{|\Theta|=\eta} |v| \in]\eta, 1[$ (or equivalently, $\delta := \inf_{|\Theta|=\eta} |u| \in]\eta, 1[$).

Then $v \in \mathfrak{I}_c$. The assertion does not necessarily hold if $\sigma = 1$ (or, equivalently, if $\delta = \eta$).

Proof Due to hypothesis (ii), we have the following estimate on $|\Theta| = \eta$:

$$|u| = \frac{|\Theta|}{|v|} \geq \frac{\eta}{\sigma} = \delta. \tag{2}$$

Note that $\delta \in]\eta, 1[$. We claim that

$$\Omega_u(\delta) \subseteq \Omega_\Theta(\eta) \cap \Omega_v(\sigma). \tag{3}$$

To this end, we first show that $|\Theta| < \eta$ on $\Omega_u(\delta)$. In fact, assuming the contrary, there exists $z_0 \in \Omega_u(\delta)$ such that $|\Theta(z_0)| \geq \eta$. Let Ω_0 be that component of $\Omega_u(\delta)$ containing z_0. By Lemma 1(2), $\inf_{\Omega_0} |u| = 0$. Since u is a factor of Θ, we conclude that there exists $z_1 \in \Omega_0 \subseteq \Omega_u(\delta)$ such that $|\Theta(z_1)| < \eta$. Thus, the connected set Ω_0 meets $\{|\Theta| < \eta\}$ as well as its complement. Hence Ω_0 meets the topological boundary of $\Omega_\Theta(\eta)$. Because $\Omega_0 \subseteq \mathbb{D}$, we obtain $z_2 \in \Omega_0$ such that $|\Theta(z_2)| = \eta$. Hence, by (ii), $|v(z_2)| \leq \sigma$ and so $|u(z_2)| \geq \delta$ by (2). Both assertions $|u(z_2)| \geq \delta$ and $z_2 \in \Omega_0 \subseteq \Omega_u(\delta)$ cannot hold. Thus our assumption right at the beginning of this paragraph was wrong. We deduce that

$$\Omega_u(\delta) \subseteq \Omega_\Theta(\eta). \tag{4}$$

By continuity, this inclusion implies that $|\Theta| \leq \eta$ on $\{|u| = \delta\}$. Hence, for $|u(z)| = \delta$,

$$|v(z)| = \frac{|\Theta(z)|}{|u(z)|} \leq \frac{\eta}{\delta} \overset{(2)}{=} \sigma. \tag{5}$$

Now $\partial\Omega_u(\delta) \cap \mathbb{D} = \{|u| = \delta\}$. If Ω is a component of $\Omega_u(\delta)$ whose closure belongs to $\mathbb{D}$, then by the maximum principle and (5), $|v| < \sigma$ on Ω. If $E := \overline{\Omega} \cap \mathbb{T} \neq \emptyset$, then E has measure zero by Lemma 1 (3). The maximum principle with exceptional points (see [4, p. 729] or [8]) now implies that $|v| < \sigma$ on Ω. Consequently,

$$\Omega_u(\delta) \subseteq \Omega_v(\sigma). \tag{6}$$

Thus (3) holds. Next we will deduce that

$$\Omega_v(\eta) \subseteq \Omega_\Theta(\eta) \subseteq \Omega_v(\sigma). \tag{7}$$

To see this, observe that the first inclusion is obvious because v is a factor of Θ. To prove the second inclusion, we write the η-level set of Θ as

$$\Omega_\Theta(\eta) = \Big(\Omega_\Theta(\eta) \cap \Omega_u(\delta)\Big) \cup \Big(\Omega_\Theta(\eta) \setminus \Omega_u(\delta)\Big).$$

By (6), the first set in this union is contained in $\Omega_v(\sigma)$. The second set is also contained in $\Omega_v(\sigma)$, because if $|u(z)| \geq \delta$ and $z \in \Omega_\Theta(\eta)$, then

$$|v(z)| = \frac{|\Theta(z)|}{|u(z)|} < \frac{\eta}{\delta} \overset{(2)}{=} \sigma. \tag{8}$$

To sum up, we have shown that for every $z \in \Omega_\Theta(\eta)$ we have $|v(z)| < \sigma$ both in the case where $|u(z)| < \delta$ and $|u(z)| \geq \delta$. Thus

$$\Omega_\Theta(\eta) \subseteq \Omega_v(\sigma),$$

and so, (7) holds. Using these inclusions (7), we are now able to prove that $\Omega_v(\sigma)$ is connected. Assuming the contrary, there would exist a component Ω_1 of $\Omega_v(\sigma)$, distinct (and so disjoint) from that containing the connected set $\Omega_\Theta(\eta)$. In particular, $|v| \geq |\Theta| \geq \eta$ on Ω_1. By Lemma 1 (2), $\inf_{\Omega_1} |v| = 0$, which is an obvious contradiction.

Finally we construct an example showing that in (ii) the parameter σ cannot be taken to be 1. In fact, let v be a thin interpolating Blaschke product with positive zeros clustering at 1, for example

$$v(z) = \prod_{n=1}^{\infty} \frac{1 - 1/n! - z}{1 - (1 - 1/n!)z},$$

and let $u(z) = S(z) := \exp[-(1+z)/(1-z)]$ be the atomic inner function. Then, by [6, Proposition 11], $\Theta = uv \in \mathfrak{I}_c$. However, $v \notin \mathfrak{I}_c$, see [6, Corollary 21]. Thus, by the main assertion of this proposition, $\sigma = \sup_{|\Theta|=\eta} |v| = 1$. A direct proof of the assertion $\sigma = 1$ can also be given using [10, p. 55], by noticing that the boundary of the component $\Omega_\Theta(\eta)$ is a closed curve in $\mathbb{D} \cup \{1\}$. □

Observation We know from [6, Proposition 12] that $u, v \in \mathfrak{I}_c$ implies $uv \in \mathfrak{I}_c$. Here is an example showing that neither u nor v must belong to $\mathfrak{I}_c$ for uv to be in $\mathfrak{I}_c$. In fact, let b be a thin Blaschke product with real zeros clustering at 1 and -1 (just consider $b(z) = v(z)v(-z)$, v as above). Let $\tilde{u} := Sb$ and $\tilde{v}(z) := S(-z)b(z)$. Then $\Theta := \tilde{u}\tilde{v} \in \mathfrak{I}_c$, because $\Theta(z) = \big(S(z)v^2(z)\big)\big(S(-z)v^2(-z)\big)$ is the product of two functions in $\mathfrak{I}_c$ (same proof as in [6, Proposition 11]), but neither $\tilde{u}$ nor $\tilde{v}$ belongs to $\mathfrak{I}_c$. This can be seen as follows: since $S(-1) = 1$, $\tilde{u} = Sb$ behaves as b close to -1. Thus, for η arbitrarily close to 1, the level set $\Omega_{\tilde{u}}(\eta)$ is contained in a union of pairwise disjoint pseudohyperbolic disks $D_\rho(x_n, \eta^*)$, $n = 0, 1, 2, \cdots$, together with some tangential disk D at 1, where $x_0 = 0$ and x_n is the n-th negative zero of b (this works similarly as in [6, Corollary 21] and [6, Proposition 11]).

4 Composition of One-Component Inner Functions

In [6] we showed that for every finite Blaschke product B, the atomic singular inner function S and $\Theta \in \mathfrak{I}_c$, the compositions $S \circ B \in \mathfrak{I}_c$ and $B \circ \Theta \in \mathfrak{I}_c$. Using the following standard lemma, we will extend this to arbitrary one-component inner functions.

Lemma 5

1) *Let B be a Blaschke product with zero sequence $(a_n)_{n\in\mathbb{N}}$. Then the following inequalities hold for every $\xi \in \mathbb{T} \setminus \mathrm{Sing}(B)$ and $n_0 \in \mathbb{N}$:*

$$|B'(\xi)| = \sum_{n\in\mathbb{N}} \frac{1-|a_n|^2}{|a_n-\xi|^2} \geq \frac{1-|a_{n_0}|}{1+|a_{n_0}|} > 0.$$

2) *If u is an inner function for which $\mathrm{Sing}(u) \neq \mathbb{T}$, then*

$$\delta_u := \inf\{|u'(\xi)| : \xi \in \mathbb{T} \setminus \mathrm{Sing}(u)\} > 0.$$

Proof

1) Just compute the logarithmic derivative B'/B and note that on $\mathbb{T} \setminus \mathrm{Sing}(B)$ the Blaschke product B converges.
2) Let $\varphi_a(z) = (a-z)/(1-\overline{a}z)$. By Frostman's theorem (see [9, p. 79]) there is $a \in \mathbb{D}$ such that $B := \varphi_a \circ u$ is a Blaschke product. Of course, $\mathrm{Sing}(u) = \mathrm{Sing}(B)$, $u = \varphi_a \circ B$, and $\varphi_a'(z) = -(1-|a|^2)/(1-\overline{a}z)^2$. Hence, for $\xi \in \mathbb{T} \setminus \mathrm{Sing}(u)$,

$$\begin{aligned} |u'(\xi)| = |\varphi_a'(B(\xi))|\,|B'(\xi)| &\geq \frac{1-|a|^2}{|1-\overline{a}B(\xi)|^2}\,\delta_B \\ &\geq \frac{1-|a|}{1+|a|}\,\delta_B > 0. \end{aligned}$$

This concludes the proof. □

Theorem 6 *If u and v are two non-constant inner functions in $\mathfrak{I}_c$, then $u \circ v \in \mathfrak{I}_c$.*

Proof As in [6], we shall use Aleksandrov's theorem (Theorem 2).

(1) Let $\Theta := u \circ v$. It is well known that Θ is an inner function again (see e.g. [12, p. 83]). Now

$$\mathrm{Sing}(\Theta) = \mathrm{Sing}(v) \cup \{\xi \in \mathbb{T} \setminus \mathrm{Sing}(v) : v(\xi) \in \mathrm{Sing}(u)\}.$$

Since $v \in \mathfrak{I}_c$, $\liminf_{r\to 1} |v(r\zeta)| < 1$ for every $\zeta \in \mathrm{Sing}(v)$ (Theorem 2). Hence there exists a sequence (r_n) in $]0,1[$, $r_n \to 1$, such that $v(r_n\zeta) \to w_0 \in \mathbb{D}$. Then

$$\Theta(r_n\zeta) = u(v(r_n\zeta)) \to u(w_0) \in \mathbb{D}. \tag{9}$$

If $\xi \in \mathrm{Sing}(\Theta) \setminus \mathrm{Sing}(v)$, then $v(r\xi) \to v(\xi) = e^{i\theta} \in \mathrm{Sing}(u)$ for some $\theta \in \mathbb{R}$. By Lemma 5, $v'(\xi) \neq 0$; hence v is a conformal map in small neighborhoods of ξ; in particular, due to the angle conservation law, the curve $\gamma : r \mapsto v(r\xi)$ stays in a cone

$$C = C(\theta) := \{z \in \mathbb{D} : |z| \geq r_0, |\arg z - \theta| < \sigma\}$$

with curved base, aperture $0 < 2\sigma < \pi$ and cusp at $e^{i\theta} \in \text{Sing}(u)$ (see the figure, where we sketched the situation for $\theta = 0$).

Since $u \in \mathfrak{I}_c$, $\liminf |u(re^{i\theta})| < 1$. We claim that $\liminf |u(v(r\xi))| < 1$, too. To see this, choose a pseudohyperbolic radius ρ_0 so big that for some $r_0 \in]0, 1[$ the cone C is entirely contained in the domain

$$V := \bigcup_{-1<x<1} D_\rho(xe^{i\theta}, \rho_0).$$

Note that by [11], the boundary of V is the union of two arcs of circles cutting the line $\{se^{i\theta} : s \in \mathbb{R}\}$ at $e^{i\theta}$ under an angle α with $\sigma < \alpha < \pi/2$ (see Figure 1).

Choose r_n so that $\lim u(r_n e^{i\theta}) = a \in \mathbb{D}$. Then the curve γ cuts the boundary of infinitely many disks $D_\rho(r_n e^{i\theta}, \rho_0)$ twice. But for $z \in D_\rho(r_n e^{i\theta}, \rho_0)$ we have

$$\frac{|u(z)| - |u(r_n e^{i\theta})|}{1 - |u(r_n e^{i\theta})|\, |u(z)|} \le \rho(u(z), u(r_n e^{i\theta})) \le \rho(z, r_n e^{i\theta}) \le \rho_0,$$

and so

$$|u(z)| \le \frac{\rho_0 + |u(r_n e^{i\theta})|}{1 + |u(r_n e^{i\theta})|\, \rho_0}.$$

This clearly implies that

$$\liminf |u(v(r\xi))| < 1.$$

Consequently, $\liminf |\Theta(r\xi)| < 1$ for every $\xi \in \text{Sing}(\Theta)$. Next we verify the first condition in Aleksandrov's theorem (Theorem 2). By an elementary calculation, we obtain

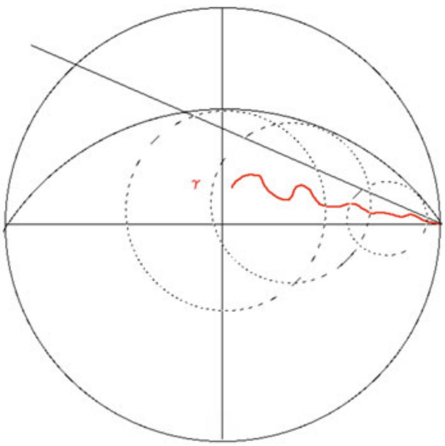

Fig. 1 The curve γ, the cone $C(0)$ with cusp at 1, and the envelope of a series of pseudohyperbolic disks of fixed radius ρ_0.

$$A := \frac{(u \circ v)''}{[(u \circ v)']^2} = \frac{u'' \circ v}{(u' \circ v)^2} + \frac{(u' \circ v)}{(u' \circ v)^2} \frac{v''}{(v')^2} \tag{10}$$
$$= \frac{u'' \circ v}{(u' \circ v)^2} + \frac{1}{u' \circ v} \frac{v''}{(v')^2}.$$

If $\zeta \in \mathbb{T} \setminus \mathrm{Sing}(u \circ v)$, then $|v(\zeta)| = 1$ and $\xi := v(\zeta) \notin \mathrm{Sing}(u)$. Since $u, v \in \mathfrak{I}_c$, we deduce from Lemma 5 and Aleksandrov's theorem 2 that

$$|A(\zeta)| \leq \sup_{\beta \notin \mathrm{Sing}(u)} \frac{|u''(\beta)|}{|u'(\beta)|^2} + \frac{1}{\delta_u} \sup_{\alpha \notin \mathrm{Sing}(v)} \frac{|v''(\alpha)|}{|v'(\alpha)|^2} =: C < \infty,$$

where

$$\delta_u := \inf\{|u'(\xi)| : \xi \in \mathbb{T} \setminus \mathrm{Sing}(u)\}.$$

Hence $\Theta \in \mathfrak{I}_c$. □

Theorem 7

1) Let $E \subseteq \mathbb{T}$ be a closed finite set. Then there exists a one-component inner function u such that for some $\eta_0 \in \,]0, 1[$ (and hence for all $\eta \in [\eta_0, 1[$) the associated level set $\Omega_u(\eta)$ is connected and has the property that

$$\overline{\Omega_u(\eta)} \cap \mathbb{T} = \mathrm{Sing}(u) = E. \tag{11}$$

2) There exists $u \in \mathfrak{I}_c$ such that $\overline{\Omega_u(\eta)} \cap \mathbb{T} = \mathrm{Sing}(u)$ is an infinite set.

Proof

1) Let $E = \{\lambda_1, \ldots, \lambda_N\}$ be finite. Then the function S_μ given by

$$S_\mu(z) = \prod_{j=1}^{N} \exp\left(-\frac{\lambda_j + z}{\lambda_j - z}\right)$$

belongs to $\mathfrak{I}_c$ (by [6, Corollary 17]) and satisfies (11).

2) Let $E = S^{-1}(1)$ be the countably infinite set of points where the atomic inner function $S(z) = \exp(-(1+z)/(1-z))$ takes the value 1, and let b be the interpolating Blaschke product with zeros $1 - 2^{-n}$. Then b and S belong to $\mathfrak{I}_c$ (see [6, Theorem 6]). By Theorem 6, $u := b \circ S \in \mathfrak{I}_c$. It is easy to see that $\overline{\Omega_u(\eta)} \cap \mathbb{T} = \mathrm{Sing}(u) = E$. The same holds true for $S \circ b$ as well; just note that the argument function of b on $\mathbb{T} \setminus \{1\}$ is unbounded when approaching 1 from both sides on the circle (see [9, p. 92]), so that $b^{-1}(\{1\})$ is infinite. Thus we have a singular inner function in $\mathfrak{I}_c$ with infinitely many singularities. □

To conclude, we want to give some useful (surely known) information on the relations between $\mathrm{Sing}(u)$ and $\Omega_u(\eta)$. To this end, let $\overline{X}$ and ∂X denote the closure, respectively, boundary, in $\mathbb{C}$ of a set X, $X \subseteq \mathbb{D}$.

Observation 8 *Let u be an inner function. Then the following assertions hold:*

(1) For all $\eta \in]0, 1[$, we have

$$E := \overline{\Omega_u(\eta)} \cap \mathbb{T} = \partial\Omega_u(\eta) \cap \mathbb{T} = \mathrm{Sing}(u),$$

(2) $X := \bigcup_n \partial\Omega_n$ may be a strict subset of $\partial\Omega_v(\eta)$ (but not always) whenever Ω_n are the components of $\Omega_v(\eta)$ for a non-one-component inner function v.

Proof

(1) Note that $E = \emptyset$ whenever u is a finite Blaschke product. Hence (1) obviously holds in that case. Now suppose that $E \neq \emptyset$. We first show that $\overline{\Omega_u(\eta)} \cap \mathbb{T} \subseteq \mathrm{Sing}(u)$. To see this, let $\xi \in E$. Then there is $z_n \in \Omega_u(\eta)$ such that $z_n \to \xi$. In particular, by taking a subsequence, $u(z_{n_k}) \to w$ for some $w \in \mathbb{D}$ with $|w| \leq \eta$. Hence $w \in Cl(u, \xi)$, and so $\xi \in \mathrm{Sing}(u)$.

One may also see this in the following way: if $\xi \in \mathbb{T} \setminus \mathrm{Sing}(u)$, then u has an analytic extension u^* around ξ with $|u^*(\xi)| = 1$. Hence, given $0 < \eta < \eta' < 1$, we see that $|u^*(z)| \geq \eta'$ for every $z \in U$, where U is a neighborhood of ξ in $\mathbb{C}$. In particular, $U \cap \overline{\Omega_u(\eta)} = \emptyset$. Thus $\xi \notin \overline{\Omega_u(\eta)}$.

To prove the other inclusion, let $\xi \in \mathrm{Sing}(u)$. Then there is a sequence (z_n) in $\mathbb{D}$ with $z_n \to \xi$ and $u(z_n) \to 0$. Hence $z_n \in \Omega_u(\eta)$ for almost all n. Thus $\xi \in \overline{\Omega_u(\eta)}$. Consequently, $\overline{\Omega_u(\eta)} \cap \mathbb{T} = \mathrm{Sing}(u)$.

Since $\Omega_u(\eta)$ is an open set with $\Omega_u(\eta) \cap \mathbb{T} = \emptyset$, we also obtain that

$$\partial\Omega_u(\eta) \cap \mathbb{T} = \left(\overline{\Omega_u(\eta)} \setminus \Omega_u(\eta)\right) \cap \mathbb{T} = \mathrm{Sing}(u).$$

(2) Let B be a thin Blaschke product and S the atomic inner function with $\mathrm{Sing}(S) = \{1\}$. Then for any $\eta \in]0, 1[$, $\Omega_B(\eta)$ has infinitely many components all of them are relatively compact in $\mathbb{D}$. So $\mathrm{Sing}(B) \cap X = \emptyset$. On the other hand, if B is a thin Blaschke product with negative zeros clustering at -1, then $u = SB$ is a non-one-component inner function whose level sets $\Omega_u(\eta)$ consist of infinitely many components which are relatively compact in $\mathbb{D}$ (and clustering at -1) and a component whose closure contains 1. □

Questions

i) Given any countable closed subset E of $\mathbb{T}$, does there exist $u \in \mathfrak{I}_c$ such that $\overline{\Omega_u(\eta)} \cap \mathbb{T} = E$?

ii) Is the set $\overline{\Omega_u(\eta)} \cap \mathbb{T}$ necessarily countable whenever $u \in \mathfrak{I}_c$? We do not think so. As indicated by Carl Sundberg [14], the usual Cantor ternary set may be the support of some singular measure μ whose associated singular inner function S_μ belongs to $\mathfrak{I}_c$.

iii) Give a description of those closed subsets E of $\mathbb{T}$ such that for some singular inner function S_μ with $\mathrm{Sing}(S_\mu) = E$ every inner factor of S_μ belongs to $\mathfrak{I}_c$.
For example, finite subsets of $\mathbb{T}$ have this property [6, Corollary 17].

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Biholomorphic Cryptosystems

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Abstract We present a physical adaptation of classical cryptological discrete structures within the environment of complex variables.

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1 Introduction

Historically, the first overstepping of the discrete nature of classical cryptology structures was made by applying the methods of Chaos theory in cryptography. Already by the 1950s, Claude Shannon pointed out that the mechanisms of contraction and expansion of chaos could be exploited in cryptology. After a thirty-year recession, during the 1990s, about 30 published papers gave various encryption algorithms focusing on analog circuits. After 2000, the Chaos theory became recognized in several applications and inaugurated the Crypstic from Lexicon Data Limited (http://eleceng.dit.ie/arg/downloads/Crypstic.zip) .

Generalizing to this direction, it is reasonable to look for an efficient adaptation of the discrete structures of classical cryptology within a suitable constant environment, in such a way that the arranging to be computationally functional and free of any possibility of revocation due to the narrow technological tracking. The purpose of this paper is to present a physical adaptation of classical cryptological discrete structures within the environment of complex variables.

Thus, in the next section we will determine biholomorphic rules of encoding and decoding into a simple connected domain in the complex plane. According to these rules, the alphabet source is embedded into an initial simple connected domain of the complex plane $\mathbb{C}$, which is then transformed successively to other

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connected domains of $\mathbb{C}$. The sequential selection of these simply connected domains, as well as the number and the representation formulas of the biholomorphic transformations between these domains, constitute the exclusive information elements characterizing the result of each coding. In the third section, we will give basic convergence properties of biholomorphic rules for encoding and decoding, while in the fourth section, we will investigate the dynamic behaviour of these biholomorphic codes. Next, in the fifth section, we introduce and study the *biholomorphic cryptosystems*, as well as the *evolutionary biholomorphic cryptosystems*. In biholomorphic cryptosystems, all plain texts are embedded into an initial domain of the complex space $\mathbb{C}^n$ which, by means of successive biholomorphic mappings, is transformed several times in other domains of $\mathbb{C}^n$, resulting in a final domain of $\mathbb{C}^n$. The initial and final domain of $\mathbb{C}^n$, as well as the form, number and type of the successive transformations are elements that characterize exclusively the encryption. Especially, in evolutionary biholomorphic cryptosystems, the successive domains are all *parametrized* to provide an additional option of determining continuously variable cryptosystems, depending on any (real or complex) parameter.

From several examples developed through the sections, it will be clear that the definitions of the biholomorphic rules of encoding and decoding, as well as the (evolutionary) biholomorphic cryptosystems may be computationally functional and free of any possibility of revocation due to the technological tracking. Finally, in the last section, we will study some basic dynamic properties of biholomorphic cryptosystems and evolutionary biholomorphic cryptosystems.

2 Biholomorphic Codes

In communications and information processing, code is a system of rules to convert information – such as a letter, word, sound, image, or gesture – into another, sometimes shortened or secret, form or representation for communication through a channel or storage in a medium. The process of encoding converts information from a source into symbols for communication or storage. Decoding is the reverse process, converting code symbols back into a form that the recipient understands.

A code is usually considered as an algorithm which uniquely represents symbols from some source alphabet, by encoded strings, which may be in some other target alphabet. An extension of the code for representing sequences of symbols over the source alphabet is obtained by concatenating the encoded strings. Using terms from formal language theory, the precise mathematical definition of this concept is as follows: Let Σ and T be two finite sets, called the ***source*** and ***target alphabets***, respectively. A ***code*** is a mapping

$$\sigma : \Sigma \longrightarrow T^*$$

which assigns each symbol from Σ to a sequence of symbols over T . Any extension of σ to a homomorphism of Σ^* into T^*, mapping each sequence of source symbols of Σ to a sequence of target symbols of T, is referred to be an ***extension*** of σ.

Definition 1 Let Ω be a domain in the complex plan $\mathbb{C}$.

i. A **biholomorphic codification chain** on Ω is an infinite forward composition

$$F = \ldots \circ f_M \circ \ldots \circ f_1 \circ f_0$$

of biholomorphic mappings $f_\alpha : U_\alpha \longrightarrow \mathbb{C}$, with

$$f_\alpha\left(U_\alpha\right) \bigcap U_{\alpha+1} \neq \emptyset \text{ and } U_\alpha = \text{open subset of } \mathbb{C} \text{ such that } U_0 = \Omega.$$

ii. A **biholomorphic encoding rule** with length $M + 1$ ($M \in \mathbb{N}_0$) on Ω subordinate to F is a section

$$F^{(M)} = f_M \circ \ldots \circ f_1 \circ f_0$$

of F starting from the beginning of the biholomorphic codification chain. The biholomorphic mappings f_α are the **codification links** of the rule $F^{(M)}$.

iii. The set of all biholomorphic encoding rules on Ω will be denoted by

$$\mathcal{E}_\Omega.$$

iv. Given a biholomorphic codification chain F, a **biholomorphic decodification chain** of F on Ω is an infinite backward composition

$$G = g_0 \circ \ldots g_{N-1} \circ g_N \ldots$$

of biholomorphic mappings $g_\beta : V_\beta \longrightarrow \mathbb{C}$, with

$$g_\beta\left(V_\beta\right) \bigcap V_{\beta-1} \neq \emptyset \text{ and } V_\beta = \text{open subset of } \mathbb{C},$$

such that for every biholomorphic encoding rule $F^{(M)}$ on Ω subordinate to F, there is a section $G^{(N)} = g_0 \circ \ldots g_{N-1} \circ g_N$ ($N \in \mathbb{N}_0$), with $V_0 \subset \Omega$, satisfying

$$G^{(N)} \circ F^{(M)} = z \text{ whenever } z \in \Omega.$$

v. The backward composition

$$G^{(N)} = g_0 \circ \ldots \circ g_{N-1} \circ g_N$$

is called a **biholomorphic decoding rule** of $F^{(M)}$ with length $N+1$ and range in Ω. The biholomorphic mappings g_β are the **decodification links** of $G^{(N)}$.

vi. The set of all biholomorphic decoding rules on Ω will be denoted by

$$\mathcal{D}_\Omega.$$

vii. A **biholomorphic code** on Ω is a pair (E, D) satisfying the following two properties:

- $E \subset \mathcal{E}_\Omega$ and
- for each $F^{(M)} \in \mathcal{E}_\Omega$, there is a $G^{(N)} \in \mathcal{D}_\Omega$ with $N = N(M)$ and such that $G^{(N)} \circ F^{(M)} = z$ whenever $z \in \Omega$.

viii. A **biholomorphic cipher** on Ω is such any pair $\left(F^{(M)}, G^{(N)}\right) \in (\mathcal{E}_\Omega, \mathcal{D}_\Omega)$.
ix. The collection of all the criteria determining the selection of biholomorphic ciphers is the **guideline** of the biholomorphic code.

Remark 1 If $f_\alpha = f$, for any α, then the study of the biholomorphic codification chain is limited just on the study of the iterations and dynamics of f. And, if $g_\beta = g$, for any β, then the study of the biholomorphic decodification chain is limited just on the study of the iterations and dynamics of g.

Below, we give the general algorithmic framework for the construction of encoding rules, as well as of respective decoding rules.

CONSTRUCTING BIHOLOMORPHIC CIPHERS AND CODES

Let Ω be a domain in the complex plan $\mathbb{C}$. Let also Σ be a (finite) source alphabet and let σ be an arbitrary mapping $\sigma : \Sigma \longrightarrow \Omega$.

1. Consider a biholomorphic code on Ω, that is a pair (E, D) such that
   - $E \subset \mathcal{E}_\Omega$ and $D \subset \mathcal{D}_\Omega$,
   - if $F^{(M)} = f_M \circ \ldots \circ f_1 \circ f_0 \in E$, the mappings $f_M, \ldots, f_1$ and f_0 are biholomorphic,
   - if $G^{(N)} = g_0 \circ \ldots \circ g_{N-1} \circ g_N \in D$, the mappings $g_0, \ldots, g_{N-1}$ and g_N are biholomorphic,
   - for each biholomorphic encoding rule $F^{(M)} \in E$ with length $M + 1$ on Ω there exists a biholomorphic decoding rule $G^{(N)} \in D$ with length $N + 1$ and range in Ω such that

$$G^{(N)} \left(F^{(M)}\right) = z \text{ whenever } z \in \Omega.$$

2. Choose a biholomorphic cipher (F, G) in (E, D):
   - the composition $F \circ \sigma$ defines a new code with target alphabet $(F \circ \sigma)(\Sigma)$ and
   - the composition $G \circ F \circ \sigma$ decodes the target alphabet and returns the source alphabet:

$$G(F(\sigma))(\Sigma) = \Sigma.$$

Let us give some indicative examples.

Example 1 Let Σ be any source alphabet. Obviously, each character of Σ can be illustrated at one point of the right half-plane $\mathcal{H}_\delta = \{z \in \mathbb{C} : Rez > 0\}$, according to a map $\sigma : \Sigma \longrightarrow \mathcal{H}_\delta$. Set

- $U_0 = \Omega = \mathcal{H}_\delta = \{z \in \mathbb{C} : Rez > 0\}$ (: the right half-plane),
- $U_1 = U_2 = \Delta(0; 1) = \{z \in \mathbb{C} : |z| < 1\}$ (: the open unit disk) and
- $U_3 = \mathcal{H}_\pi = \{z \in \mathbb{C} : Imz > 0\}$ (: the upper half-plane).

We may consider the following chain of biholomorphic mappings:

$$\mathcal{H}_\delta \overset{f_0}{\to} \Delta(0; 1) \overset{f_1}{\to} \Delta(0; 1) \overset{f_2}{\to} \mathcal{H}_\pi$$

$$z \overset{f_0}{\mapsto} \underbrace{e^{i\theta}\frac{z - A}{z + \bar{A}}}_{w} \overset{f_1}{\mapsto} \underbrace{e^{i\varphi}\frac{w + B}{1 + \bar{B}w}}_{u} \overset{f_2}{\mapsto} \underbrace{\frac{\bar{C}u - CD}{u - D}}_{v}$$

with $0 \le \theta, \varphi \le 2\pi$, $ImA > 0$, $|B| < 1$, $ImC > 0$ and $|D| = 1$. The composition

$$F(z) = (f_2 \circ f_1 \circ f_0)(z) = \frac{\left(ab\bar{C} - aCD\bar{B}\right)(z - A) + \left(bB\bar{C} - CD\right)\left(z + \bar{A}\right)}{\left(ab - aD\bar{B}\right)(z - A) + (bB - D)\left(z + \bar{A}\right)}, z \in \Omega,$$

with $a = e^{i\theta}$ and $b = e^{i\varphi}$ is a biholomorphic encoding rule with length 3 of Ω. Defining the chain of the inverse mappings

$$\mathcal{H}_\pi \overset{g_2}{\to} \Delta(0; 1) \overset{g_1}{\to} \Delta(0; 1) \overset{g_0}{\to} \mathcal{H}_\delta$$

$$v \overset{g_2}{\mapsto} \underbrace{D\frac{v - C}{v - \bar{C}}}_{u} \overset{g_1}{\mapsto} \underbrace{\frac{e^{i\varphi}B - u}{u\bar{B} - e^{i\varphi}}}_{w} \overset{g_0}{\mapsto} \underbrace{\frac{\bar{A}w + e^{i\theta}A}{e^{i\theta} - w}}_{z}$$

it is easily verified that its composition

$$G(v) = (g_0 \circ g_1 \circ g_2)(v) = \frac{D\left(aA\bar{B} - A\right)(v - C) + b\left(\bar{A}B - aA\right)\left(v + \bar{C}\right)}{D\left(a\bar{B} + 1\right)(v - C) - b(B + a)\left(v - \bar{C}\right)}, v \in \mathcal{H}_\pi,$$

satisfies $G(F(z)) = z$ for every $z \in \mathcal{H}_\delta$ and therefore it constitutes a biholomorphic decoding rule with length 3 in Ω over the (time) interval $[0, 2\pi]$. Here, the biholomorphic cipher $(F, G) = (f_0, f_1, f_2; g_2, g_1, g_0)$ defines the biholomorphic code $(\mathcal{E}, \mathcal{D}) = (f_0, f_1, f_2; g_2, g_1, g_0)$ on Ω with guideline

$$\{\Omega = U_0 = \mathcal{H}_\delta, U_1 = U_2 = \Delta(0; 1), U_3 = \mathcal{H}_\pi; 0 \le \theta, \varphi \le 2\pi, ImA > 0,$$
$$|B| < 1, ImC > 0 \text{ and } |D| = 1\}.$$

The composition $F \circ \sigma$ defines the target alphabet $(F \circ \sigma)(\Sigma)$ associated with the source Σ.

Example 2 Let Σ be the set of the 255 characters considered by the extended ASCII code (see, for instance, http://www.ascii-code.com/). Each such character is illustrated at one point of the right half-plane $\mathcal{H}_\delta = \{z \in \mathbb{C} : Rez > 0\}$, according to the identity map $\sigma : \Sigma \to \mathcal{H}_\delta : z \mapsto \sigma(z) := z + 1$ induced by the well-known ASCII table. Set

- $U_0 = \Omega = \Delta(0; 260) \cap \mathcal{H}_\delta = \{z \in \mathbb{C} : Rez > 0\}$ (:the intersection of the disc of radius 260 with the right half-plane),
- $U_1 = \Delta_\pi(0; 1) = \{z \in \mathbb{C} : |z| < 1, Imz > 0\}$ (: the upper half-disk),
- $U_2 = \mathcal{Q}_2 = \{z \in \mathbb{C} : Rez < 0, Imz > 0\}$ (: the second quadrant),
- $U_3 = \mathcal{H}_\kappa = \{z \in \mathbb{C} : Imz < 0\}$ (: the lower half-plane) and
- $U_4 = \Delta(0; 1) = \{z \in \mathbb{C} : |z| < 1\}$ (: the open unit disk).

We consider the following chain of biholomorphic mappings:

$$\Delta(0; 260) \cap \mathcal{H}_\delta \overset{f_0}{\to} \Delta_\pi(0; 1) \overset{f_1}{\to} \mathcal{Q}_2 \overset{f_2}{\to} \mathcal{H}_\kappa \overset{f_3}{\to} \Delta(0; 1)$$

$$z \overset{f_0}{\mapsto} \left(\underbrace{\frac{i}{260}z}_{w}\right) \overset{f_1}{\mapsto} \left(\underbrace{\frac{w-1}{w+1}}_{u}\right) \overset{f_2}{\mapsto} \left(\underbrace{u^2}_{v}\right) \overset{f_3}{\mapsto} \left(\underbrace{\frac{v+i}{v-i}}_{h}\right).$$

Then the composition

$$F(z) = (f_3 \circ f_2 \circ f_1 \circ f_0)(z) = \frac{(1+i)(260-z)(260+z)}{(1-i)(260-z)(260+z) - 1040i}, \; z \in \Omega,$$

is a biholomorphic encoding rule with length 4 of Ω. If we define the chain of the inverse biholomorphic mappings

$$\Delta(0; 1) \overset{g_3}{\to} \mathcal{H}_\kappa \overset{g_2}{\to} \mathcal{Q}_2 \overset{g_1}{\to} \Delta_\pi(0; 1) \overset{g_0}{\to} \Delta(0; 260) \cap \mathcal{H}_\delta$$

$$h \overset{g_3}{\mapsto} \left(\underbrace{i\frac{h+1}{h-1}}_{v}\right) \overset{g_2}{\mapsto} \left(\underbrace{e^{-\frac{1}{2}logv}}_{u}\right) \overset{g_1}{\mapsto} \left(\underbrace{\frac{1+u}{1-u}}_{w}\right) \overset{g_0}{\mapsto} (\underbrace{-260wi}_{z}),$$

then its composition

$$G(h) = (g_0 \circ g_1 \circ g_2 \circ g_3)(h) = -260i\frac{1 + exp\left(\frac{1}{2}log\left(i\frac{h+1}{h-1}\right)\right)}{1 - exp\left(-\frac{1}{2}log\left(i\frac{h+1}{h-1}\right)\right)}, \; h \in \Delta(0; 1),$$

satisfies

$$G\,(F(z)) = z \text{ for every } z \in \mathcal{H}_\delta,$$

and therefore it is a biholomorphic decoding rule with length 4 in Ω. Here, the biholomorphic cipher $(F, G) = (f_0, f_1, f_2, f_3; g_3, g_2, g_1, g_0)$ defines the biholomorphic code $(\mathcal{E}, \mathcal{D}) = (f_0, f_1, f_2, f_3; g_3, g_2, g_1, g_0)$ on Ω with guideline

$$\{\Omega = U_0 = \Delta(0; 260) \cap \mathcal{H}_\delta, U_1 = \Delta_\pi(0; 1), U_2 = \mathcal{Q}_2, U_3 = \mathcal{H}_\kappa, U_4 = \Delta(0; 1)\}.$$

The composition $F \circ \sigma$ defines the target alphabet $(F \circ \sigma)\,(\Sigma)$ associated with the source Σ.

Example 3 Let Σ be any collection of source symbols. Each such symbol can be associated with a point of the open unit disk $\Delta(0; 1)$, according to an arbitrary map $\sigma : \Sigma \to \Delta(0; 1)$. Let

$$z_0, z_1, \ldots, z_M$$

be M arbitrarily chosen points in the open unit and let

$$t_0, t_1, \ldots, t_M$$

be M arbitrarily chosen positive numbers in the closed interval $[0, 2\pi]$. For $j = 0, 1, \ldots, M$, we define the function

$$f_j(z) = e^{it_j} \frac{z + z_j}{1 + \bar{z}_j z}.$$

Let us consider the following chain of biholomorphic mappings:

$$\Delta(0; 1) \overset{f_0}{\to} \Delta(0; 1) \overset{f_1}{\to} \ldots \overset{f_{M-1}}{\to} \Delta(0; 1) \overset{f_M}{\to} \Delta(0; 1)$$

$$z \overset{f_0}{\mapsto} \underbrace{f_0(z)}_{w_0} \overset{f_1}{\mapsto} \ldots \overset{f_{M-1}}{\mapsto} \underbrace{f_{M-1}(z)}_{w_{M-1}} \overset{f_M}{\mapsto} \underbrace{f_M(z)}_{w_M}.$$

Then the composition $F(z) = (f_M \circ f_{M-1} \circ \ldots \circ f_1 \circ f_0)\,(z)$, $z \in \Delta(0; 1)$, is a biholomorphic encoding rule with length $M + 1$ of $\Delta(0; 1)$. If we define the chain of the inverse biholomorphic mappings

$$\Delta(0; 1) \overset{g_M = f_M^{-1}}{\to} \Delta(0; 1) \overset{g_{M-1} = f_{M-1}^{-1}}{\to} \ldots \overset{g_1 = f_1^{-1}}{\to} \Delta(0; 1) \overset{g_0 = f_0^{-1}}{\to} \Delta(0; 1),$$

then its composition $G(h) = (g_0 \circ g_1 \circ \ldots \circ g_{M-1} \circ g_M)\,(h)$, $h \in \Delta(0; 1)$, satisfies

$$G\,(F(z)) = z \text{ for every } z \in \Delta(0; 1),$$

and therefore it is a biholomorphic decoding rule with length $M+1$ in $\Delta(0; 1)$. Here, the biholomorphic cipher $(F, G) = (f_0, f_1, \ldots, f_{M-1}, f_M; g_M, g_{M-1}, \ldots, g_1, g_0)$ defines the biholomorphic code $(\mathcal{E}, \mathcal{D}) = (F, G)$ on $\Delta(0; 1)$ with guideline

$$\{\Omega = U_0 = \ldots = U_M = \Delta(0; 1), z_0, \ldots, z_M \in \Delta(0; 1); t_0, \ldots, t_M \in [0, 2\pi]\}.$$

The composition $F \circ \sigma$ defines the target alphabet $(F \circ \sigma)(\Sigma)$ associated with the source Σ.

Example 4 Let Ω be any simply connected domain in the complex plan $\mathbb{C}$. Let also Σ be any collection of source symbols. Each such symbol can be associated with a point of the open unit disk $\Delta(0; 1)$, according to an arbitrary map $\sigma : \Sigma \to \Delta(0; 1)$. Let $f(z)$ be an automorphism of Ω. If, for example, $\Omega = \Delta(0; 1)$, then $f(z)$ can be expressed as follows:

$$f(z) = e^{it_0} \frac{z + z_0}{1 + \bar{z}_0 z}$$

for a point z_0 in the open unit disk and a $t_0 \in [0, 2\pi]$; and if $\Omega = \mathbb{C}$, then

$$f(z) = az + b$$

for $a \in \mathbb{C} \setminus 0$ and $b \in \mathbb{C}$. For any arbitrarily chosen positive integer M, we consider the iterate of order M of f:

$$F = f^M = \underbrace{f \circ f \circ \ldots \circ f \circ f}_{Mtimes}.$$

Obviously, F is a biholomorphic encoding rule of length M on Ω. If we define the iterate of order M of the inverse mapping $g = f^{-1}$

$$G(h) = g^M = \underbrace{g \circ g \circ \ldots \circ g \circ g}_{Mtimes},$$

it is immediately seen that

$$G\left(F(z)\right) = z \text{ for every } z \in \Omega$$

and therefore it is a biholomorphic decoding rule with length M in Ω. Here, the biholomorphic cipher $(F, G) = \left(f^M; g^M\right)$ defines the biholomorphic code $(\mathcal{E}, \mathcal{D}) = F, G)$ on Ω. If $\Omega = \Delta(0; 1)$,the guideline is

$$\{M; \Omega = \Delta(0; 1), z_0 \in \Delta(0; 1), t_0 \in [0, 2\pi]\},$$

and if $\Omega = \mathbb{C}$, the guideline of the code is the set

$$\{M, \Omega = \mathbb{C}, a \in \mathbb{C}, b \in \mathbb{C}\}.$$

The composition $F \circ \sigma$ defines the target alphabet $(F \circ \sigma)(\Sigma)$ associated with the source Σ.

3 Convergence Properties of Biholomorphic Codes

Possibly, one could think that an endless elongation of the biholomorphic encoding rule, by using increasingly longer lengths in the backward compositions of biholomorphic mappings, will shield the encoding process, making it more difficult to decode. And conversely, one would think that increasingly longer forward compositions of biholomorphic mappings would make the decoding process more feasible.

In this section, we will give a series of examples, for several ordinary cases, through which it will be clearly seen that *the arbitrary choice of finite length for the biholomorphic encoding rules, as well as the arbitrary choice of the domains of definition and the different intrinsic forms of the biholomorphic mappings participating in the biholomorphic codification chains*, create an "insurmountable" computational protect of the codification, in the sense that they constitute absolutely essential conditions that cannot substituted or even be approached by large and simple encoding rules. And similarly, the attempts for biholomorphic decoding should be specific, of finite length, with different domains of definition and with different intrinsic forms of biholomorphic mappings in the decoding rules.

To this end, suppose a biholomorphic code $(\mathcal{E}, \mathcal{D})$ on a non-empty domain $\Omega \subset \mathbb{C}$ consists in a given sequence

$$\left(\left(F^{(M)}{=}f_M \circ f_{M-1} \circ \ldots \circ f_1 \circ f_0\right), \left(G^{(M)}{=}g_0 \circ g_1 \circ \ldots \circ g_{M-1} \circ g_M\right)\right)_{M,N \in \mathbb{N}_0}$$

of biholomorphic ciphers on Ω. An application of the *Contraction Theorem for holomorphic functions* (see, for instance, [9]) shows that

Theorem 1 (Contraction of Biholomorphic Encoding Rules) *If for any $M \in \mathbb{N}_0$, the codification links $f_0, f_1, \ldots, f_M$ are all equal to the same function $f \in \bar{A}(\Omega)$ having bounded range $f(\Omega)$ in Ω, then the biholomorphic encoding rule*

$$F^{(M)}(z) = f^M(z) = (\underbrace{f \circ f \circ \ldots \circ f \circ f}_{M+1\ times})(z)$$

trivializes in Ω as its length $M+1$ grows illimitably, in the sense that

$$F^{(M)}(z) \underset{M \to \infty}{\to} A,$$

where A is the attractive fixed point of f in Ω.

Using Lorentzen–Gill's results on infinite compositions of contractive functions [6, 14], we can also easily show the following.

Theorem 2

i. *Assume that, for any M, the codification links f_0, f_1,..., f_M are all defined on the same simply connected domain Ω (i.e., with the notation of Definition 2, $U_\alpha = \Omega$, $\forall \alpha = 1, 2, \ldots, M$) and there exists a compact set $E \subset\subset \Omega$ such that for each M, $f_\alpha(\Omega) \subset E$ whenever α. Then the sequence of biholomorphic encoding rules $\left(F^{(M)} = f_M \circ \ldots \circ f_1 \circ f_0\right)_{M=0,1,2,\ldots}$ converges uniformly on compact subsets of Ω to a constant function $\mathcal{F}(z) \equiv c$ if and only if the sequence of fixed points $\left(\gamma_\alpha\right)$ of the codification links f_α converge to the point $c \in E$.*

ii. *If, for any N, the decodification links g_N, g_{N-1},..., g_0 are all defined on the same simply connected domain Ω (i.e., with the notation of Definition 1, $V_\beta = \Omega$, $\forall \beta = 1, 2, \ldots, N$) and there exists a compact set $K \subset\subset \Omega$ such that for each N, $g_\beta(\Omega) \subset K$ whenever β, then the sequence of biholomorphic decoding rules $\left(G^{(N)} = g_0 \circ \ldots \circ g_{N-1} \circ g_N\right)_{N=0,1,2,\ldots}$ converges uniformly on compact subsets of Ω to a constant function $\mathcal{G}(z) \equiv \tilde{c} \in K$.*

Remark 2 Recall that the sequence of iterates of a biholomorphic self-map of the unit disc, which fix one point of the disk, converges uniformly on the compact subsets of the disk to a constant, the interior fixed point (see [16]).

Theorem 3

i. *Assume that all the codification links f_α in a biholomorphic codification chain F are all defined in the same disk $D_R = \{z \in \mathbb{C} : |z| < R\}$. Suppose there exists a sequence (p_α) of non-negative numbers such that $C \sum_{\alpha=1}^{\infty} p_\alpha < R$ and $|f_\alpha - z| < C p_\alpha$, $\forall z \in D_R$ and $\alpha = 0, 1, 2, \ldots$, for a $C > 0$. Set*

$$r := R - C \sum\nolimits_{\alpha=1}^{\infty} p_\alpha \quad (> 0).$$

Then the sequence of biholomorphic encoding rules $\left(F^{(M)} = f_M \circ \ldots \circ f_0\right)_{M=0,1,2,\ldots}$ converges to a function $\mathcal{F}(z)$ uniformly on compact subsets of the open disk $D_r = \{z \in \mathbb{C} : |z| < r\}$.

ii. *Assume that all the links g_β in a biholomorphic decodification chain G are defined on the open disk $D_R = \{z \in \mathbb{C} : |z| < R\}$ and satisfy $g_\beta(z) = z\left(1 + u_\beta(z)\right)$, with $|u_\beta(z)| \le C q_\beta(, z \in D_R)$ and $\sum_{\beta=1}^{\infty} q_\beta < \infty$, for a $C > 0$. Choose a $0 < r < R$ and define*

$$\varrho := \varrho(r) := \frac{R - r}{\prod_{\beta=1}^{\infty}\left(1 + C q_\beta\right)}.$$

Then the sequence of biholomorphic decoding rules $\left(G^{(N)} = g_0 \circ \ldots \circ g_N\right)_{N=0,1,2,\ldots}$ converges uniformly on the compact subsets of the disk $D_\varrho = \{z \in \mathbb{C} : |z| < \varrho\}$ to a biholomorphic function $\mathcal{G}(z)$ satisfying

$$|\mathcal{G}'(z)| \leq \prod_{\beta=1}^{\infty} \left(1 + C\frac{R}{r}q_\beta\right).$$

As it has been seen in the examples of Section 1, an important case of biholomorphic encodings and decodings is derived by the use of *non-singular Möbius transformations*:

$$\frac{ax+b}{cx+d}, \; (ad - bc \neq 0).$$

Below, we will deal particularly with the case of biholomorphic decodings whose links are described by these transformations.

Firstly, according to a slight adaptation of the wording of earlier known results due to Gill [8] and Piranian–Thron [21], we have the following two general propositions.

Proposition 1 *Suppose, for any N, the biholomorphic decoding rule $G^{(N)} = g_0 \circ \ldots \circ g_N$ is a non-singular Möbius transformation. Then, within the set of convergence of the sequence $\left(G^{(N)}\right)_{N=0,1,2,\ldots}$, the limit function is either*

(i). *a non-singular Möbius transformation,*
(ii). *a function taking on two distinct values or*
(iii). *a constant.*

In (i)*, the sequence converges everywhere in the extended plane. In* (ii)*, the sequence converges either everywhere, and to the same value everywhere except at one point, or it converges at only two points. Case* (iii) *can occur with every possible set of convergence.*

Proposition 2 *If the sequence $\left(G^{(N)}\right)_{N=0,1,2,\ldots}$ of the biholomorphic decoding rules converges to a Möbius transformation, then the sequence of the decodification links g_N converges to the identity function.*

Further, by exploring the conditions which shape the form of links of the chains, we infer the next theorem, based on known results due to DePree and Thron [2], Magnus and Mandell [17] and Gill [5].

To state the theorem, we need some preparatory material. For this purpose, we recall that a Möbius transformation with three or more fixed points coincides with the identity (and so fixes every point). Otherwise, it has either:

(i). *one* fixed point (: in that case the transformation is conjugate to the mapping $z \mapsto z + 1$ and we say that the Möbius transformation is *parabolic*;
(ii). *two* distinct fixed points (: in that case, if the transformation is conjugate to a transformation of the form$z \mapsto \lambda z$, where $|\lambda| = 1$, we say that the Möbius transformation is *elliptic*; if the transformation is conjugate to a transformation of the form $z \mapsto \lambda z$, where $|\lambda| \notin \{0, 1\}$, we say that the Möbius transformation is *loxodromic*; in the particular case of λ real positive and not equal to 0 or 1, the Möbius transformation is called *hyperbolic*.

Note that the location of the fixed points is a delicate question: if, for instance, a parabolic Möbius transformation maps the upper half-plane $\mathcal{H}_\pi$ onto itself, then it has one fixed point in $\partial\mathcal{H}_\pi$ and none in $\mathcal{H}_\pi$; an elliptic Möbius transformation mapping $\mathcal{H}_\pi$ onto itself has one fixed point in $\mathcal{H}_\pi$ and none in $\partial\mathcal{H}_\pi$; and an hyperbolic Möbius transformation mapping $\mathcal{H}_\pi$ onto itself has two fixed points in $\partial\mathcal{H}_\pi$ and none in $\mathcal{H}_\pi$. Moreover, we need the following.

Lemma 1

- **i**. [3] *For every loxodromic, hyperbolic, or elliptic Möbius transformation h, we have $(h(z) - A)\,/\,(h(z) - B) = (z - A)\,/\,(z - B)$, where $K := (a - cA)\,/\,(a - cB)$ and A, B are the two distinct fixed points of h. The constant K is called the **multiplier** of h.*
- **ii**. *If $|K| < 1$, the Möbius transformation h is loxodromic or hyperbolic and satisfies $h^k(z) \overset{k\to\infty}{\to} A$ for all $z \neq A$ and $h^k(B) \equiv B$. A is called the **attractive fixed point** of h; B is called the **repelling** or **repulsive fixed point** of h.*

Based on the above background, we are in position to state the announced theorem.

Theorem 4 *Assume that all the links g_β in a biholomorphic decodification chain G are Möbius transformations that converge to a Möbius transformation g.*

- **i**. *If the links g_β and their limit g are all hyperbolic or loxodromic, then the sequence of biholomorphic decoding rules $\left(G^{(N)} = g_0 \circ \ldots \circ g_{N-1} \circ g_N\right)_{N=0,1,2,\ldots}$ converges to a constant function $G(z) \equiv c$ for all values $z \neq B$: the repelling fixed point of g.*
- **ii**. *Suppose any decodification link g_β has two fixed points A_β and B_β and the limit g of the decodification links is a parabolic Möbius transformation. If $\sum_{\beta=1}^{\infty} |A_\beta - B_\beta| < \infty$ and $\sum_{\beta=1}^{\infty} \beta|B_{\beta+1} - B_\beta| < \infty$, then the sequence of the biholomorphic decoding rules $\left(G^{(N)}\right)_{N=0,1,2,\ldots}$ converges to a constant everywhere in the extended complex plane $\bar{\mathbb{C}}$.*

Finally, for the sake of completeness, it should also be given the following result which refers to the case of the biholomorphic encodings whose links are Möbius transformations.

Theorem 5 ([7]) *Assume that all the links f_α in a biholomorphic codification chain F are Möbius transformations that converge to a Möbius transformation f. If the links f_α and their limit f are all hyperbolic or loxodromic, then the sequence of biholomorphic encoding rules $\left(F^{(M)} = f_M \circ f_{M-1} \circ \ldots \circ f_1 \circ f_0\right)_{M=0,1,2,\ldots}$ converges to the attractive fixed point A of f for all values of z except possibly one z_0. In this exceptional case $F^{(M)}(z_0) \overset{M\to\infty}{\to} B$, the repelling fixed point of f.*

4 Dynamic Properties of Biholomorphic Codes

Certainly, it is of interest, for our intentions, to study the recurring points into biholomorphic codification chains $F = \ldots \circ f_M \circ \ldots \circ f_1 \circ f_0$ which can entirely be covered by segments from successive sub-chains of the F, each of which maps a set into itself. Observe that if $f_0 = f_1 = \ldots = f_M = \ldots = f$, then the study is limited just on the study of the iterations and dynamics of the biholomorphic mapping f.

To this end, we must look for characteristic properties of points that are kept fixed through the parts $f_{n_{j+1}-1} \circ \ldots \circ f_{n_j+1} \circ f_{n_j}$ of a biholomorphic codification chain F on an open domain $\Omega \subset \mathbb{C}$.

Notation 1 *We will use the symbol*

$$F_n^{(M)}$$

to denote the segment of the biholomorphic codification chain F that involves the composition of all functions which are at the part of F that starts from the function in the $(n+1)^{th}$ place of F and ends with the function in the $(M+1)^{th}$ place of F. It is clear that

$$F_n^{(M)} = f_M \circ \ldots \circ f_{n+1} \circ f_n.$$

Thus, for instance, we may write

$$F_3^{(8)} = f_8 \circ \ldots \circ f_4 \circ f_3 \text{ and } F_{n_j}^{(n_{j+1}-1)} = f_{n_{j+1}-1} \circ \ldots \circ f_{n_j+1} \circ f_{n_j}.$$

In particular, we have

$$F_M^{(M)}(p) = f_M(p),\ F_M^{(M+1)}(p) = (f_{M+1} \circ f_M)(p), \ldots, F_M^{M+n}(p) = (f_{M+n} \circ \ldots \circ f_{M+1} \circ f_M)(p).$$

If $n > 0$ and $M = \infty$, then we set

$$F_n^{(\infty)} = \ldots f_M \circ \ldots, f_{n+1} \circ f_n$$

and we say that $F^{(\infty)}$ is a ***truncated codification chain*** *with late start in position n. For example, the segment*

$$F_6^{(\infty)} = \ldots f_8 \circ f_7 \circ f_6$$

of F is a truncated codification chain with late start in position 6.

If the segment starts from the beginning of the codification chain (: $n = 0$) and ends with the function in the $(M+1)^{th}$ place, then we adopt often the simpler notation

$$F^{(M)} = f_M \circ \ldots \circ f_1 \circ f_0.$$

For example,

$$F^{(1000)} = f_{1000} \circ \ldots \circ f_1 \circ f_0 \text{ and } F^{(\infty)} = \ldots \circ f_M \circ \ldots \circ f_1 \circ f_0.$$

In particular, we have

$$F^{(0)}(p) = f_0(p), F^{(1)}(p) = (f_1 \circ f_0)(p), \ldots, F^{(M)}(p) = (f_M \circ \ldots \circ f_1 \circ f_0)(p).$$

Finally, we point out that with the notation of Definition 2, we have $F^{(0)}(\Omega) = U_0 = \Omega$, $F^{(1)}(\Omega) \subset U_1, \ldots$ *and in general*

$$F^{(M)}(\Omega) \subset U_M,$$

for any $M = 0, 1, 2, \ldots$

We will need some preparatory material.

Definition 2 Let $F = \ldots \circ f_M \circ \ldots \circ f_1 \circ f_0$ be a biholomorphic codification chain on an open domain $\Omega \subset \mathbb{C}$.

- **i**. Let $p \in F^{(M)}(\Omega)$. The **orbit** of the point p through the codification F is the sequence of points is the sequence of points

$$\bigcirc^{+}(p) := \left\{p, F_M^{(M)}(p), F_M^{(M+1)}(p), \ldots, F_M^{(M+n)}(p), \ldots\right\}.$$

  If $M > 0$, we also say that $\bigcirc^{+}(p)$ is a **truncated orbit** with late start in position M. If $M = 0$, then the orbit has the simple form

$$\bigcirc^{+}(p) := \left\{p, F^{(0)}(p), F^{(1)}(p), \ldots, F^{(n)}(p), \ldots\right\}.$$

- **ii**. A point $z \in \bigcirc^{+}(p)$ is **periodic** in the chain F, if the value of z occurs more than once in its orbit $\bigcirc^{+}(p)$.
- **iii**. A periodic point z is said to be **of restricted periodicity**, if the number of times of its occurrence in the chain is finite; it is said to be of extensive periodicity, if the number of times of its occurrence is infinite.

It is straightforward to show the following two propositions.

Proposition 3 *A point* $p \in F^{(M)}(\Omega)$ *is periodic in the codification chain* F*, if there is an* $n \in \mathbb{N}$ *such that* p *is a fixed point of* $F_M^{(M+n)}$*, that is,* p *is a solution of the equation* $F_M^{(M+n)}(z) = z \Leftrightarrow (f_{M+n} \circ \ldots \circ f_{M+1} \circ f_M)(z) = z$.

Proposition 4 *Let* $M \in \mathbb{N}$. *The codification chain* F *has a restricted (respectively, an extensive) periodicity at* $z \in F^{(M)}(\Omega)$*, if there is a finite (respectively, an infinite) sequence* $(n_j) \subset M, M+1, \ldots$ *so that* $n_1 = M$ *and the value* $F_M^{(\infty)}(z)$ *of the codification chain* $F_M^{(\infty)} = (\ldots f_{M+n} \circ \ldots \circ f_{M+1} \circ f_M)$ *at the point* p *can be* *expressed as infinite succession of finite segmentation sub-chains* $F_{n_j}^{(n_{j+1}-1)}(p) = \left(\ldots f_{n_{j+1}-1} \circ \ldots \circ f_{n_j+1} \circ f_{n_j}\right)(p)$ *satisfying* $F_{n_j}^{(n_{j+1}-1)}(p) = F_{n_{j+1}}^{(n_{j+2}-1)}(p)$*. The sequence* $\left(n_j\right)$ *is called the* ***sequence of periods*** *of the point* p *in the codification chain* F*, while the sequence* $\left(F_{n_j}^{(n_{j+1}-1)}\right)$ *is the* ***segmentation sequence*** *of* F*.*

Remark 3 With this notation, it is clear that if $M > 0$, then we say that the codification chain F has a **period with late start** in position M and F can be represented as follows:

$$F = \ldots \circ \underbrace{f_{n_{j+1}-1} \circ \ldots \circ f_{n_j+1} \circ f_{n_j}}_{F_{n_j}^{(n_{j+1}-1)}} \circ \ldots \circ \underbrace{f_{n_2-1} \circ \ldots \circ f_{n_1+1} \circ f_{n_1}}_{F_{n_1}^{(n_2-1)}} \circ \left(f_M \circ \ldots \circ f_0\right).$$

If $M = 0$, the codification chain F takes the form

$$F = \ldots \circ \underbrace{f_{n_{j+1}-1} \circ \ldots \circ f_{n_j+1} \circ f_{n_j}}_{F_{n_j}^{(n_{j+1}-1)}} \circ \ldots \circ \underbrace{f_{n_3-1} \circ \ldots \circ f_{n_2+1} \circ f_{n_2}}_{F_{n_2}^{(n_3-1)}} \circ \underbrace{f_{n_2-1} \circ \ldots \circ f_1 \circ f_0}_{F_0^{(n_2-1)}}.$$

Definition 3 Let $p \in F^{(M)}(\Omega) \subset U_M$ be a periodic point with sequence of periods $\left(n_j\right)$ in the chain F. We say that p is

i. **repelling** if $\mid \left(F_{n_j}^{(n_{j+1}-1)}\right)'(p) \mid > \mu$ for some positive constant $\mu > 1$, whenever $j = 1, 2, \ldots$;

ii. **attracting** if $\mid \left(F_{n_j}^{(n_{j+1}-1)}\right)'(p) \mid < \lambda$ for some positive constant $\lambda < 1$, whenever $j = 1, 2, \ldots$;

iii. **super-attracting** if $\left(F_{n_j}^{(n_{j+1}-1)}\right)'(p) = 0$, whenever $j = 1, 2, \ldots$;

iv. **rationally indifferent** if $\left(F_{n_j}^{(n_{j+1}-1)}\right)'(p)$ is a root of unity, whenever $j = 1, 2, \ldots$;

v. **irrationally indifferent** if $\mid \left(F_{n_j}^{(n_{j+1}-1)}\right)'(p) \mid = 1$ and $\left(F_{n_j}^{(n_{j+1}-1)}\right)(p)$ is a root of unity, whenever $j = 1, 2, \ldots$.

Respectively, the orbit $\bigcirc^{+}(p)$ of p is called a **repelling cycle**, an **attracting cycle**, a **super-attracting cycle**, a **rationally indifferent cycle** or an **irrationally indifferent cycle**.

Definition 4

i. The **Fatou set** for F is the set $\mathbb{F} = \mathbb{F}_{\Omega}(F)$ of all points $p \in \Omega$ such that the family $\mathcal{F} = (F^{(M)})$ is normal in a neighbourhood of p, i.e., for every sequence $(F^{(M_k)}) \subset \mathbb{F}$ one can extract a subsequence $\left(F^{\left(M_{k_j}\right)}\right)$ which converges uniformly on a compact neighbourhood of a either to ∞ or to a holomorphic function.

ii. The **Julia set** $\mathbb{J} = \mathbb{J}_{\Omega}(F)$ for F is the complement of the Fatou set $\mathbb{F}_{\Omega}(F)$.

By definition, *the Fatou set is open*, while *its Julia set is closed.*

Proposition 5

i. *Every attracting fixed point $p \in \Omega$ in the codification chain F is in the Fatou set $\mathbb{F} = \mathbb{F}_{\Omega}(F)$ of F.*

ii. *Furthermore, the set $\mathcal{A}$ of all $z \in \Omega$ whose orbits converge to a $p \in \Omega$ is an open subset of the Fatou set. This set is called the* ***basin of attraction*** *of p in the codification chain F. The connected component of $\mathcal{A}$ containing p is called the* ***immediate basin of attraction*** *of p in the codification chain F.*

Proof

i. Choosing local coordinates, we can assume that $p = 0$. By Taylor's theorem, there is some ball (in the Euclidean metric) $B \subset \Omega$ around 0 such that $\mid F_{n_j}^{(n_{j+1}-1)}(z) \mid < \lambda \mid z \mid$ for all $z \in B$ whenever $j = 1, 2, \ldots$ It then follows that $\lim_{j \to \infty} F_{n_j}^{(n_{j+1}-1)}(z) = 0$ uniformly on B. Hence B is a neighbourhood of normality.

ii. Now suppose $z_0 \in \mathcal{A}$. Then $F_{n_j}^{(n_{j+1}-1)}(z_0) \in B$ for some n_j. It follows that

$$\left(F_{n_j}^{(n_{j+1}-1)}\right)^{-1}(z_0)$$

is a neighbourhood of z_0 contained in $\mathcal{A}$. Thus $\mathcal{A}$ is open. The segmentation sequence $\left(F_{n_j}^{(n_{j+1}-1)}\right)$ converges uniformly on B to 0, so $\lim_{j \to \infty}\left(F_{n_j}^{(n_{j+1}-1)}\right)(z_0) = 0$, which shows that $\mathcal{A}$ is contained in the Fatou set.

Proposition 6 *Every repelling fixed point $p \in \Omega$ in the codification chain F is in the Julia set $\mathbb{J} = \mathbb{J}_{\Omega}(F)$ of F.*

Proof Choosing local coordinates, we can assume that $p = 0$. In the domain of these coordinates, the derivative of any $F_{n_k}^{(n_{k+1}-1)} \circ \ldots \circ F_{n_2}^{(n_3-1)} \circ F_{n_1}^{(n_2-1)}$ at 0 is equal to

$$\left(F_{n_k}^{(n_{k+1}-1)}\right)'(p) \ldots \left(F_{n_2}^{(n_3-1)}\right)'(p)\left(F_{n_1}^{(n_2-1)}\right)'(p)(k = 1, 2, \ldots).$$

Since $\mid \left(F_{n_j}^{(n_{j+1}-1)}\right)'(p) \mid > \mu > 1$ whenever $j = 1, 2, \ldots$, no subsequence of these derivatives will converges to a finite value. On the other hand, the derivative of a sequence of biholomorphic mappings converges to the derivative of the locally uniform limit of the mappings, assuming such a limit exists. Hence, the sequence $\left(F_{n_k}^{(n_{k+1}-1)} \circ \ldots \circ F_{n_2}^{(n_3-1)} \circ F_{n_1}^{(n_2-1)}\right)$ cannot form a normal family on any neighbourhood of $p = 0$, and so $\{p\}$ is in the Julia set.

5 Biholomorphic Cryptosystems

In classical cryptography, a ***cryptosystem*** is a five-tuple $(\mathcal{P}, \mathcal{C}, \mathcal{K}, \mathcal{E}, \mathcal{D})$, where the following conditions are satisfied. $\mathcal{P}$ is a finite set of possible ***plaintexts*** $\mathcal{C}$ is a finite set of possible ***ciphertexts*** $\mathcal{K}$, the keyspace, is a finite set of possible ***keys***. For each $K \in \mathcal{K}$, there is an ***encryption rule*** $e_K \in \mathcal{E}$ and a corresponding ***decryption rule*** $d_K \in \mathcal{D}$. Each $e_K : \mathcal{P} \to \mathcal{C}$ and $d_K : \mathcal{C} \to \mathcal{P}$ are functions such that $d_K(e_K)(x) = x$ for every plaintext element $x \in \mathcal{P}$.

The purpose of the present section is to give an extension of this definition which is based on theory of complex variables and to show the differences of this extension with the classical case. In this direction, we point out that translating such a classical definition to the context of complex variables induces a natural change in the keyspace, from the discrete state to an uncountable infinite structure. It follows that an encryption method which based on the theory of complex analysis is beyond the capacities of modern computers, even of future quantum computers. As a consequence, the rules in the context of theory of complex analysis can become so complicated, from the point of view of constructive approximations, that it becomes impossible to achieve decryption by using electronic machines or advanced computer technology.

After these brief and very general introductory remarks, we are able to go to the foundation of complex cryptosystems. To this end, we first give the following general definition. To do so, we may remark that, in practice, the capacity of each message may not exceed a certain number of characters, so we can assume that *the length of each plaintext in $\mathcal{P}$ equals a given number, say n*. Otherwise, you may add at the end of the plain text, the symbol of blank space, so many times so that the length of the resulting final plaintext which will result equals to n. Under this assumption, we are now in position to define biholomorphic cryptosystems.

Definition 5 Let us consider two domains $\Omega \subset \mathbb{C}^n$ and $D \subset \mathbb{C}^n$ which together constitute the **encryption environment**. A finite **biholomorphic cryptosystem** is a four-tuple $(\mathcal{P}, \mathcal{K}, \mathcal{E}, \mathcal{D})$, where the following conditions are satisfied.

1. $\mathcal{P}$ is a fixed finite set of possible **plaintexts** embedded into Ω.
2. $\mathcal{E}$ is the set of **biholomorphic encryption rules** on Ω with range in D, that is, a subset of the set:

$$\{F : \Omega \to D : F \textit{ is holomorphic mapping}\}.$$

3. $\mathcal{D}$ is the set of **biholomorphic decryption rules** on D with range in Ω, that is a subset of the set:

$$\{G : D \to \Omega : G \; is \; holomorphic \; mapping\}.$$

4. $\mathcal{K}$ is a subset (not necessarily finite) of the set $u = u_N \circ \ldots \circ u_1 : u_\alpha$ is holomorphic mapping of U_α, $U_1 = \Omega$, $u_\alpha (U_\alpha) \subset U_{\alpha+1} \subset \mathbb{C}^n$, $u_N (U_N) \subset D$, $N \in \mathbb{N}$. The elements of $\mathcal{K}$ are the **keymappings**, while $\mathcal{K}$ is the **keyspace**.
5. For each $u \in \mathcal{K}$, there exist a mapping $F_u \in \mathcal{E}$ and a mapping $G_u \in \mathcal{D}$ such that $G_u (F_u)(z) = z$ for every plaintext element $z \in \mathcal{P}$. F_u is a finite composition of biholomorphic mappings $f_1^{(u)}, \ldots, f_N^{(u)}$ (satisfying $f_\alpha^{(u)} (U_\alpha) \subset U_{\alpha+1}$, $U_1 = \Omega$ and $f_N^{(u)} (U_N) \subset \Omega$, for some $N \in \mathbb{N}$) and is called a **biholomorphic encryption chain**, while G_u is a finite composition of biholomorphic mappings $g_M^{(u)}, \ldots, g_1^{(u)}$ (with $g_{\beta+1} \left(V_{\beta+1}\right) \subset V_\beta$, $V_1 = D$ and $g_M (V_M) \subset \Omega$, for some $M \in \mathbb{N}$) and is said to be a corresponding **biholomorphic decryption finite chain**.

Remark 4

- **i**. If the length of the plaintext is very long, it is appropriate to take a partition of the long plain text into others with shorter length, embedded into open subsets of Euclidean spaces of several complex variables, so that it will be possible to apply the process of the above definition into these subsets. Thus, without loss of generality, in what follow, **we will always assume that each plain text has enough small length**.
- **ii**. Compared with the classical case, Definition 5 is purely functional, and does not claim that the set of possible ciphertexts is given. Instead, it completely bypasses this set and the role of the ciphertext set is played from the range $\left(f_N^{(u)} \circ \ldots \circ f_1^{(u)}\right)(\mathcal{P})$ of a composite function. Further, Definition 5 does not require as the keyspace is finite. Instead, **the space $\mathcal{K}$ may be infinite dimensional**.
- **iii**. It is well known that a self-map of a domain in $\mathbb{C}$ that fixes 3 points is necessarily the identity [19]. However, if $n > 1$, then an automorphism $G_u \circ F_u$ of a domain Ω in $\mathbb{C}^n$ fixing the finite set $\mathcal{P}$ is not necessarily the identity map id_Ω (see, for instance, [4]).

We can generalize the definition 5, so that the resulting parametrized encryption form depends on the particular time moments.

Definition 6 Let $\Omega \subset \mathbb{C}^n$ and $D \subset \mathbb{C}^n$ be two domains . An **evolutionary biholomorphic cryptosystem** is a four-tuple $(\mathcal{P}, \mathcal{K}, \mathcal{E}, \mathcal{D})$, where the following conditions are satisfied.

1. $\mathcal{P}$ is a fixed finite set of possible **plaintexts** embedded into Ω.
2. $\mathcal{E}$ is the set of **evolutionary biholomorphic encryption rules** on Ω with range in D, that is, a subset of the set:

$$\{F : \Omega \times \mathbb{R} \to D : F \; is \; holomorphic \; in \; \Omega \; and \; continuous \; in \; \mathbb{R}\}.$$

3. $\mathcal{D}$ is the set of **evolutionary biholomorphic decryption rules** on D with range in Ω, that is a subset of the set:

$$\{G : D \times \mathbb{R} \to \Omega : G\ is\ holomorphic in\ D\ and\ continuous\ in\ \mathbb{R}\}.$$

4. $\mathcal{K}$ is a subset (not necessarily finite) of the set
$\{u = u_N \circ \ldots \circ u_1 : u_\alpha : U_\alpha \times \mathbb{R} \to \mathbb{C}^{n+1}$ is holomorphic in the open set $U_\alpha \subset \mathbb{C}^n$ and continuous in $\mathbb{R}$, $U_1 = \Omega \times \mathbb{R}$, $u_\alpha (U_\alpha \times \mathbb{R}) \cap U_{\alpha+1} \times \mathbb{R} \neq \emptyset, u_N (U_N \times \mathbb{R}) \subset D \times \mathbb{R}, N \in \mathbb{N}\}$.
The elements of $\mathcal{K}$ are the **evolutionary keymappings**, while $\mathcal{K}$ is the **evolutionary keyspace**.

5. For each $u \in \mathcal{K}$, there exist a mapping $F_u \in \mathcal{E}$ and a mapping $G_u \in \mathcal{D}$ such that $G_u (F_u) (z) = z$ for every plaintext element $z \in \mathcal{P}$. F_u is a finite composition of biholomorphic mappings $f_1^{(u)}, \ldots, f_N^{(u)}$ (satisfying $f_\alpha^{(u)} (U_\alpha \times \mathbb{R}) \subset U_{\alpha+1} \times \mathbb{R}$, $U_1 \times \mathbb{R} = \Omega \times \mathbb{R}$ and $f_N^{(u)} (U_N \times \mathbb{R}) \subset \Omega \times \mathbb{R}$, for some $N \in \mathbb{N}$) and is called a **evolutionary biholomorphic encryption chain**, while G_u is a finite composition of biholomorphic mappings $g_M^{(u)}, \ldots, g_1^{(u)}$ (with $g_{\beta+1} \left(V_{\beta+1} \times \mathbb{R}\right) \subset V_\beta \times \mathbb{R}$, $V_1 \times \mathbb{R} = D \times \mathbb{R}$ and $g_M (V_M \times \mathbb{R}) \subset \Omega \times \mathbb{R}$, for some $M \in \mathbb{N}$) and is said to be a corresponding **evolutionary biholomorphic decryption finite chain**.

Remark 5 In contrast to the classical case, the (non-finite) keyspace $\mathcal{K}$ contains keymappings depending continuously on the time parameter $t \in \mathbb{R}$. This allows consideration of a dynamic evolution of cryptosystems, eventually by means of a differential equation. Practically, this means introducing of *continuously rolling codes*, which guarantees increased degree of security (safety).

Let us see how Definition 5 describes the complex-valued cryptographic procedures, having adapted to the context of the classical case. To this end, we will give two indicative cases. The first case refers to the description of the encryption environment in which the domains $\Omega \subset \mathbb{C}^n$ and $D \subset \mathbb{C}^n$ coincide and are bounded sets. The second case describes the encryption environment in which both the two domains coincide with whole $\mathbb{C}^n$.

Recall that the finite set $\mathcal{P}$ of possible plaintexts is formed by characters of the source alphabet $\mathcal{S}$. According to the method developed in the first section, the source alphabet $\mathcal{S}$ can be embedded into an open subset U of the complex plane $\mathbb{C}$, via a biholomorphic code $(\mathcal{E}, \mathcal{D})$. Thus, each character χ_j of any plaintext element $\chi = \chi_1\chi_2 \cdots \chi_n \in \mathcal{P}$ is coded to a corresponding element $z_j \in U : \chi_j \mapsto z_j$, $j = 1, 2, \ldots, n$. This implies that every plaintext χ in the set $\mathcal{P}$ is represented as a point $z = z^{(\chi)}$ of U^n: $\chi = \chi_1\chi_2 \cdots \chi_n \in \mathcal{P} \mapsto z = (z_1, \ldots, z_n) \in U^n$.

1st Case: The Bounded Encryption Environment

In order to refer to the case where the encryption environment consists of two bounded sets, we let $\lambda_B \geq \max_{\chi \in \mathcal{P}} \sum_{j=1}^n \mid z_j \mid^2$ and $\lambda_\Delta \geq \max_{\chi \in \mathcal{P}} \max_j \mid z_j \mid$. Dividing the coordinates of the point $z = z^{(\chi)} \in U^n$ by λ_B or λ_Δ, we may suppose that U^n is the open unit ball $B^n \equiv B^n(0;1)$ of $\mathbb{C}^n$ or the open unit polydisk $\Delta^n \equiv \Delta^n(0;1)$ of $\mathbb{C}^n$. As it is explained in [25],

Proposition 7

i. *The group of automorphisms of the ball B^n (: i.e., the group of biholomorphisms of the ball B^n onto the ball B^n consists of all fractional-linear transformations*

$$\Phi : B^n \to B^n : \left(z_1, \dots, z_j, \dots, z_n\right) \mapsto$$

$$\left(\underbrace{\frac{\alpha_{1,0} + \sum_{\nu=1}^{n} \alpha_{1,\nu} z_\nu}{\alpha_{0,0} + \sum_{\nu=1}^{n} \alpha_{0,\nu} z_\nu}}_{w_1}, \dots, \underbrace{\frac{\alpha_{j,0} + \sum_{\nu=1}^{n} \alpha_{j,\nu} z_\nu}{\alpha_{0,0} + \sum_{\nu=1}^{n} \alpha_{0,\nu} z_\nu}}_{w_j}, \dots, \underbrace{\frac{\alpha_{n,0} + \sum_{\nu=1}^{n} \alpha_{n,\nu} z_\nu}{\alpha_{0,0} + \sum_{\nu=1}^{n} \alpha_{0,\nu} z_\nu}}_{w_n}\right)$$

which satisfy the relationships
$\sum_{j=1}^{n} \alpha_{j,\nu}\bar{\alpha}_{j,\kappa} = \alpha_{0,\nu}\bar{\alpha}_{0,\kappa}$ *for any* $\nu \neq \kappa$ *and*
$\sum_{j=1}^{n} \mid \alpha_{j,\nu} \mid^2 - \mid \alpha_{0,\nu} \mid^2 = -\sum_{j=1}^{n} \mid \alpha_{j,0} \mid^2 + \mid \alpha_{0,0} \mid^2 \neq 0.$

ii. *The group of automorphisms of the polydisk Δ^n (: i.e., the group of biholomorphisms of Δ^n onto Δ^n consists of all fractional-linear transformations*

$$\Psi : \Delta^n \to \Delta^n : \left(z_1, \dots, z_j, \dots, z_n\right) \mapsto$$

$$\left(\underbrace{e^{i\theta_{\tau(1)}} \frac{z_{\sigma(1)} - \alpha_{\sigma(1)}}{1 - \bar{\alpha}_{\sigma(1)} z_{\sigma(1)}}}_{u_1}, \dots, \underbrace{e^{i\theta_{\tau(j)}} \frac{z_{\sigma(j)} - \alpha_{\sigma(j)}}{1 - \bar{\alpha}_{\sigma(j)} z_{\sigma(j)}}}_{u_j}, \dots, \underbrace{e^{i\theta_{\tau(n)}} \frac{z_{\sigma(n)} - \alpha_{\sigma(n)}}{1 - \bar{\alpha}_{\sigma(n)} z_{\sigma(n)}}}_{u_n}\right)$$

where $\theta = (\theta_1, \dots, \theta_n) \in \mathcal{S}^n$, $\alpha = (\alpha_1, \dots, \alpha_n) \in \Delta^n$ and σ, τ are two permutations of 1,…,n (See http://mathoverflow.net/questions/154612/automorphism-groups-of-unit-disk-mathbfdn-and-unit-ball-bn; also recall that all automorphisms of the complex unit disk $\Delta = z \in \mathbb{C} : |z| < 1$ to itself can be written in the form $f_a(z) = e^{ij}\frac{(z-a)}{(1-\bar{a}z)}$ where $a \in \Delta$ and $\theta \in S_1$. This map sends a to 0, $\frac{1}{\bar{a}}$ to ∞ and the unit circle to the unit circle; see http://planetmath.org/automorphismsofunitdisk).

According to the above proposition, we can give the following two general frameworks for the construction of a finite biholomorphic cryptosystem into a bounded encryption environment.

A General Framework for Constructing Biholomorphic Cryptosystems with Bounded Encryption Environment

1. Let

   (a) $\mathcal{P}$ be the fixed finite set of possible plaintexts;
   (b) Ω be a domain in the complex plan $\mathbb{C}$;

(c) Σ be a (finite) source alphabet;
(d) σ be an arbitrary mapping $\sigma : \Sigma \to \Omega$.

2. Consider a biholomorphic code on Ω.
3. Applying the biholomorphic codification, every plaintext χ in the set $\mathcal{P}$ is represented as a point

$$\zeta = \zeta^{(\chi)} \in \mathbb{C}^n.$$

4. Choose $U^n \in \{B^n, \Delta^n\}$.
5. If $U^n = B^n$, then do the following steps

(a) Let us choose an *arbitrary set* **N** of random numbers λ_B, such that

$$\lambda_B \geq \max_{\chi \in \mathcal{P}} \sum_{n=1}^{n} \mid \zeta_j \mid^2 .$$

(b) For any $\lambda_B \in \mathbb{N}$, dividing the coordinates of the point $\zeta = \zeta^{(\chi)} \in \Omega^n$ by λ_B, every plaintext χ in the set $\mathcal{P}$ is represented as a point z of the unit open ball:

$$z = z^{(\chi)} = (z_1, \ldots, z_n) \in B^n.$$

(c) Take $\mathcal{E} = \mathcal{D} = \{\Phi : automorphism\ of\ B^n\}$.
(d) Denote by **M** the set of all $(n+1) \times (n+1)$ matrices

$$\mathbf{I}= \begin{pmatrix} \alpha_{0,0} & \alpha_{0,1} & \ldots & \alpha_{0,n} \\ \alpha_{1,0} & \alpha_{1,1} & \ldots & \alpha_{1,n} \\ \ldots & \ldots & \ldots & \ldots \\ \alpha_{n,0} & \alpha_{n,1} & \ldots & \alpha_{n,n} \end{pmatrix}$$

such that

– the row $\alpha_{0,0}, \alpha_{0,1}, \ldots, \alpha_{0,n}$ is arbitrarily chosen into $\mathbb{C}^n \setminus \{0\}$ and
– the entries of the other n rows are $n(n+1)$ complex numbers satisfying the following system of $n(n+1)$ equations:

(E1) $\sum_{j=1}^{n} \alpha_{j,\nu}\bar{\alpha}_{j,\kappa} = \alpha_{0,\nu}\bar{\alpha}_{0,\kappa}$ $(\nu \neq \kappa)$ and
(E2) $\sum_{j=1}^{n} \mid \alpha_{j,\nu} \mid^2 - \mid \alpha_{0,\nu} \mid^2 = -\sum_{j=1}^{n} \mid \alpha_{j,0} \mid^2 + \mid \alpha_{0,0} \mid^2 \neq 0$ $(\nu = 1, 2, \ldots, n)$.

Note that the solutions of the these equations are not uniquely defined;

(e) According to Proposition 7.i, to each *attributive* matrix $[\Phi] \in \mathbf{M}$, there corresponds a unique automorphism $\Phi \in Aut\ (B^n)$ defined by

$$\Phi : B^n \to B^n : \left(z_1, \ldots, z_j, \ldots, z_n\right) \mapsto$$
$$\left(\underbrace{\frac{\alpha_{1,0} + \sum_{\nu=1}^{n} \alpha_{1,\nu} z_\nu}{\alpha_{0,0} + \sum_{\nu=1}^{n} \alpha_{0,\nu} z_\nu}}_{w_1}, \ldots, \underbrace{\frac{\alpha_{j,0} + \sum_{\nu=1}^{n} \alpha_{j,\nu} z_\nu}{\alpha_{0,0} + \sum_{\nu=1}^{n} \alpha_{0,\nu} z_\nu}}_{w_j}, \ldots, \underbrace{\frac{\alpha_{n,0} + \sum_{\nu=1}^{n} \alpha_{n,\nu} z_\nu}{\alpha_{0,0} + \sum_{\nu=1}^{n} \alpha_{0,\nu} z_\nu}}_{w_n}\right)$$

(f) Choose $\mathcal{M} \subset \mathbf{M}$.

(g) To define the keyspace $\mathcal{K}$, do the following steps.

i. For any finite family with M elements of $\mathcal{M}$:

$$m_M = \{[\Phi]_s : s = 1, 2, \ldots, M\} \subset \mathcal{M},$$

where

$$[\Phi]_s = \begin{pmatrix} \alpha_{0,0}^{(s)} & \alpha_{0,1}^{(s)} & \cdots & \alpha_{0,n}^{(s)} \\ \alpha_{1,0}^{(s)} & \alpha_{1,1}^{(s)} & \cdots & \alpha_{1,n}^{(s)} \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_{n,0}^{(s)} & \alpha_{n,1}^{(s)} & \cdots & \alpha_{n,n}^{(s)} \end{pmatrix}$$

A. construct the sums

$$\underbrace{\frac{\alpha_{1,0}^{(s)} + \sum_{\nu=1}^{n} \alpha_{1,\nu}^{(s)} z_\nu}{\alpha_{0,0}^{(s)} + \sum_{\nu=1}^{n} \alpha_{0,\nu}^{(s)} z_\nu}}_{w_1^{(s)}}, \ldots, \underbrace{\frac{\alpha_{n,0}^{(s)} + \sum_{\nu=1}^{n} \alpha_{n,\nu}^{(s)} z_\nu}{\alpha_{0,0}^{(s)} + \sum_{\nu=1}^{n} \alpha_{0,\nu}^{(s)} z_\nu}}_{w_{n^{(s)}}}, s{=}1, 2, \ldots, M;$$

B. for $z = (z_1, \ldots, z_n) \in B^n$ define the automorphism

$$\Phi_s\left(z_1, \ldots, z_n\right) = \left(w_1^{(s)}, \ldots, w_n^{(s)}\right), s = 1, 2, \ldots, M$$

C. choose a permutation τ of the indices $1, 2, \ldots, M$, $M \in \mathbb{N}$;

D. consider the composition $\Phi_{\tau(M)} \circ \ldots \circ \Phi_{\tau(2)} \circ \Phi_{\tau(1)}$;

E. define $\mathcal{K}_{m_M} = \left\{\Phi_{\tau(M)} \circ \ldots \circ \Phi_{\tau(1)} : \Phi_{\tau(s)} \in m_M \ (s \leq M)\right\}$.

ii. Define

$$\mathcal{K} := \left[\bigcup_{m_M: finite\ family\ with\ M\ elements\ of\, \mathcal{M}} \mathcal{K}_{m_M}\right] \times \mathbb{N}.$$

It is clear that for each keymapping $(\Phi, \lambda_B) \in \mathcal{K}$, there exist a mapping $F_\Phi \in \mathcal{E}$ and a mapping $G_\Phi \in \mathcal{D}$ such that $G_\Phi\left(F_\Phi(z)\right) = z$ for every $z \in B^n$. In particular,

$$G_\Phi\left(F_\Phi\left(\frac{\zeta^{(\chi)}}{\lambda_B}\right)\right)=\frac{\zeta^{(\chi)}}{\lambda_B}$$

for every plaintext element $\chi \in \mathcal{P}$ and $\lambda_B \in \mathbb{N}$. Indeed, any finite composition $F_\Phi = \Phi_{i_M} \circ \ldots \circ \Phi_{i_1} \in \mathcal{E}$ is a ***biholomorphic encryption chain***, and the finite composition $G_\Phi = \Phi_{i_1}^{-1} \circ \ldots \circ \Phi_{i_M}^{-1} \in \mathcal{D}$ is the corresponding ***biholomorphic decryption chain***.

6. If $U^n = \Delta^n$, then do the following steps
   (a) Let us choose an *arbitrary set* Λ of random numbers λ_Δ, such that

$$\lambda_\Delta \geq \max_{\chi\in\mathcal{P}} \sum\nolimits_{n=1}^{n} \mid \zeta_j \mid^2 .$$

   (b) For any $\lambda_\Delta \in \Lambda$, dividing the coordinates of the point $\zeta = \zeta^{(\chi)} \in \Omega^n$ by λ_Δ, every plaintext χ in the set $\mathcal{P}$ is represented as a point z of the unit open ball:

$$z = z^{(\chi)} = (z_1, \ldots, z_n) \in \Delta^n.$$

   (c) Take $\mathcal{E} = \mathcal{D} = \{\Phi : automorphism\ of\ \Delta^n\}$.
   (d) Denote by $\mathcal{N}$ an arbitrary collection of complex vectors in $\mathbb{C}^n$:

$$\mathcal{N} = \left\{a = (a_1, \ldots, a_n) \in \mathbb{C}^n\right\}.$$

   (e) Denote by $\mathcal{L}$ an arbitrary collection of real vectors in S^n:

$$\mathcal{L} = \theta = (\theta_1, \ldots, \theta_n) \in S^n.$$

   (f) Let $\mathcal{T}$ be a set of arbitrary permutations of $1, 2, \ldots, n$:

$$\mathcal{T} = \{\tau = (\tau_1, \ldots, \tau_n)\}.$$

   (g) According to Proposition 7.ii, to each $(a_1, \ldots, a_n) \in \mathcal{N}$ and any two permutations $\tau, \varsigma \in T$, there corresponds a unique automorphism defined by

$$\Psi^{(a,\tau,\varsigma)} : \Delta^n \to \Delta^n : \left(z_1, \ldots, z_j, \ldots, z_n\right) \mapsto \Psi^{(a,\tau,\varsigma)}\left(z_1, \ldots, z_j, \ldots, z_n\right) =$$

$$\left(\underbrace{e^{i\theta_{\tau}(1)}\frac{z_{\varsigma(1)}-\alpha_{\varsigma(1)}}{1-\bar{\alpha}_{\varsigma(1)}z_{\varsigma(1)}}}_{u_1}, \ldots, \underbrace{e^{i\theta_{\tau}(j)}\frac{z_{\varsigma(j)}-\alpha_{\varsigma(j)}}{1-\bar{\alpha}_{\varsigma(j)}z_{\varsigma(j)}}}_{u_j}, \ldots, \underbrace{e^{i\theta_{\tau}(n)}\frac{z_{\varsigma(n)}-\alpha_{\varsigma(n)}}{1-\bar{\alpha}_{\varsigma(n)}z_{\varsigma(n)}}}_{u_n}\right)$$

(h) Define the keyspace

$$\mathcal{K} = \left\{ \Psi^{(a,\tau,\varsigma)} : (a, \tau, \varsigma) \in (\mathcal{N}, \mathcal{T}, \mathcal{T}) \right\} \times \Lambda.$$

It is clear that for each keymapping $(\Psi, \lambda_\Delta) \in \mathcal{K}$, there exist a mapping $F_\Psi \in \mathcal{E}$ and a mapping $G_\Psi \in \mathcal{D}$ such that $G_\Psi (F_\Psi(z)) = z$ for every $z \in \Delta^n$. In particular,

$$G_\Psi \left(F_\Psi \left(\frac{\zeta^{(\chi)}}{\lambda_\Delta} \right) \right) = \frac{\zeta^{(\chi)}}{\lambda_\Delta}$$

for every plaintext element $\chi \in \mathcal{P}$ and $\lambda_\Delta \in \Lambda$. Indeed, any finite composition $F_\Psi = \Psi_{i_M} \circ \ldots \circ \Psi_{i_1} \in \mathcal{E}$ is a ***biholomorphic encryption chain***, and the finite composition $G_\Psi = \Psi_{i_1}^{-1} \circ \ldots \circ \Psi_{i_M}^{-1} \in \mathcal{D}$ is the corresponding ***biholomorphic decryption chain***.

Let us give a concrete example.

Example 5 By considering the extended ASCII code, each character of the word `Evelpis` corresponds to the sequence of seven numbers `69 118 101 108 112 105 115`. Using the biholomorphic encoding rule with length 4 of Example 2, we see that

a). the integer `69` is coded to the complex number

$$\xi_{\mathtt{E}} = \frac{-4509326640 + i12406806482}{22065630722} \approx -(0.2043597437486) + i(0.5622683981953);$$

b). the integer `118` is coded to the complex number

$$\xi_{\mathtt{v}} = \frac{-6587118720 + i12349344672}{33996661792} \approx -(0.1937578095256) + i(0, 3632516847553);$$

c). the integer `101` is coded to the complex number

$$\xi_{\mathtt{e}} = \frac{-6029190960 + i12618481362}{29681073922} \approx -(0.2031325071271) + i(0.4251356064528);$$

d). the integer `108` is coded to the complex number

$$\xi_{\mathtt{l}} = \frac{-6282731520 + i12540403712}{31438917632} \approx -(0.1998393072415) + i(0.398881534625);$$

e). the integer `112` is coded to the complex number

$$\xi_{\texttt{p}} = \frac{-6412922880 + i\,12475249152}{32455762432} \approx -(0.1975896543314) + i(0.384377017121);$$

f). the integer `105` is coded to the complex number

$$\xi_{\texttt{i}} = \frac{-6347250000 + i\,13104281250}{31376171250} \approx -(0.2022952370264)+i(0.4176507434762);$$

g). the integer `115` is coded to the complex number

$$\xi_{\texttt{s}} = \frac{-6503250000 + i\,12416531250}{33223941250} \approx -(0.1957398717709)+i(0.3737224056764).$$

First Choice $U^n = B^n$.

We take

$$\mathcal{E} = \mathcal{D} = \left\{\Phi : \textit{automorphism of } B^7 \subset \mathbb{C}^7\right\} \text{ and } M = 2.$$

To define the keymapping, we may consider the biholomorphic encryption chain $\Phi = \Phi_2 \circ \Phi_1 \in \mathcal{E}$, where

$$\Phi_j : B^7 \to B^7 : (z_1, \dots, z_7) \mapsto \Phi_j (z_1, \dots, z_7) =$$

$$\left(\underbrace{\frac{\alpha_{1,0}^{(j)} + \sum_{\nu=1}^{7} \alpha_{1,\nu}^{(j)} z_\nu}{\alpha_{0,0}^{(j)} + \sum_{\nu=1}^{7} \alpha_{0,\nu}^{(j)} z_\nu}}_{w_1^{(j)}}, \dots, \underbrace{\frac{\alpha_{n,0}^{(j)} + \sum_{\nu=1}^{7} \alpha_{n,\nu}^{(j)} z_\nu}{\alpha_{0,0}^{(j)} + \sum_{\nu=1}^{7} \alpha_{0,\nu}^{(j)} z_\nu}}_{w_7^{(j)}} \right).$$

The mapping $\Phi^{-1} = \Phi_1^{-1} \circ \Phi_2^{-1} \in \mathcal{D}$ is the corresponding ***biholomorphic decryption finite chain***. According to Proposition 7.i, for any $s = 1, 2$, the coefficients $a_{j,\nu}^{(s)}$ satisfy the system of equations `(E1)` and `(E2)`. It is easily seen that the entries of the matrix

$$[\Phi]^{(1)} =$$

$$\begin{pmatrix}
\alpha_{0,0}^{(1)} = \frac{1\mp i\sqrt{3}}{2} & \alpha_{0,1}^{(1)} = 0 & \alpha_{0,2}^{(1)} = 0 & \alpha_{0,3}^{(1)} = 0 & \alpha_{0,4}^{(1)} = 0 & \alpha_{0,5}^{(1)} = 0 & \alpha_{0,6}^{(1)} = 0 & \alpha_{0,7}^{(1)} = \frac{1\pm i\sqrt{3}}{2} \\
\alpha_{1,0}^{(1)} = \frac{1\mp i\sqrt{3}}{2} & \alpha_{1,1}^{(1)} = 0 & \alpha_{1,2}^{(1)} = 0 & \alpha_{1,3}^{(1)} = 0 & \alpha_{1,4}^{(1)} = 0 & \alpha_{1,5}^{(1)} = 0 & \alpha_{1,6}^{(1)} = 0 & \alpha_{1,7}^{(1)} = \frac{1\pm i\sqrt{3}}{2} \\
\alpha_{2,0}^{(1)} = \frac{1\mp i\sqrt{3}}{2} & \alpha_{2,1}^{(1)} = 0 & \alpha_{2,2}^{(1)} = 0 & \alpha_{2,3}^{(1)} = 0 & \alpha_{2,4}^{(1)} = 0 & \alpha_{2,5}^{(1)} = 0 & \alpha_{2,6}^{(1)} = 0 & \alpha_{2,7}^{(1)} = \frac{1\pm i\sqrt{3}}{2} \\
\alpha_{3,0}^{(1)} = \frac{1\mp i\sqrt{3}}{2} & \alpha_{3,1}^{(1)} = 0 & \alpha_{3,2}^{(1)} = 0 & \alpha_{3,3}^{(1)} = 0 & \alpha_{3,4}^{(1)} = 0 & \alpha_{3,5}^{(1)} = 0 & \alpha_{3,6}^{(1)} = 0 & \alpha_{3,7}^{(1)} = \frac{1\pm i\sqrt{3}}{2} \\
\alpha_{4,0}^{(1)} = \frac{1\mp i\sqrt{3}}{2} & \alpha_{4,1}^{(1)} = 0 & \alpha_{4,2}^{(1)} = 0 & \alpha_{4,3}^{(1)} = 0 & \alpha_{4,4}^{(1)} = 0 & \alpha_{4,5}^{(1)} = 0 & \alpha_{4,6}^{(1)} = 0 & \alpha_{4,7}^{(1)} = \frac{1\pm i\sqrt{3}}{2} \\
\alpha_{5,0}^{(1)} = \frac{1\mp i\sqrt{3}}{2} & \alpha_{5,1}^{(1)} = 0 & \alpha_{5,2}^{(1)} = 0 & \alpha_{5,3}^{(1)} = 0 & \alpha_{5,4}^{(1)} = 0 & \alpha_{5,5}^{(1)} = 0 & \alpha_{5,6}^{(1)} = 0 & \alpha_{5,7}^{(1)} = \frac{1\pm i\sqrt{3}}{2} \\
\alpha_{6,0}^{(1)} = \frac{1\mp i\sqrt{3}}{2} & \alpha_{6,1}^{(1)} = 0 & \alpha_{6,2}^{(1)} = 0 & \alpha_{6,3}^{(1)} = 0 & \alpha_{6,4}^{(1)} = 0 & \alpha_{6,5}^{(1)} = 0 & \alpha_{6,6}^{(1)} = 0 & \alpha_{6,7}^{(1)} = \frac{1\pm i\sqrt{3}}{2} \\
\alpha_{7,0}^{(1)} = \frac{1\mp i\sqrt{3}}{2} & \alpha_{7,1}^{(1)} = 0 & \alpha_{7,2}^{(1)} = 0 & \alpha_{7,3}^{(1)} = 0 & \alpha_{7,4}^{(1)} = 0 & \alpha_{7,5}^{(1)} = 0 & \alpha_{7,6}^{(1)} = 0 & \alpha_{7,7}^{(1)} = \frac{1\pm i\sqrt{3}}{2}
\end{pmatrix}$$

satisfy the system of equations `(E1)` and `(E2)` and, thus, one can choose

$$\Phi\,(z_1, \dots, z_7) =$$

$$\left(\underbrace{\frac{\frac{1\pm i\sqrt{3}}{2} + \frac{1\mp i\sqrt{3}}{2}z_7}{\frac{1\mp i\sqrt{3}}{2} + \frac{1\pm i\sqrt{3}}{2}z_7}}_{w_1}, \underbrace{\frac{(-3)^{\frac{1}{4}} z_1}{\frac{1\mp i\sqrt{3}}{2} + \frac{1\pm i\sqrt{3}}{2}z_7}}_{w_2}, \underbrace{\frac{(-3)^{\frac{1}{4}} z_2}{\frac{1\mp i\sqrt{3}}{2} + \frac{1\pm i\sqrt{3}}{2}z_7}}_{w_3}, \right.$$

$$\left. \underbrace{\frac{(-3)^{\frac{1}{4}} z_3}{\frac{1\mp i\sqrt{3}}{2} + \frac{1\pm i\sqrt{3}}{2}z_7}}_{w_4}, \underbrace{\frac{(-3)^{\frac{1}{4}} z_4}{\frac{1\mp i\sqrt{3}}{2} + \frac{1\pm i\sqrt{3}}{2}z_7}}_{w_5}, \underbrace{\frac{(-3)^{\frac{1}{4}} z_5}{\frac{1\mp i\sqrt{3}}{2} + \frac{1\pm i\sqrt{3}}{2}z_7}}_{w_6}, \underbrace{\frac{(-3)^{\frac{1}{4}} z_6}{\frac{1\mp i\sqrt{3}}{2} + \frac{1\pm i\sqrt{3}}{2}z_7}}_{w_7} \right).$$

Further, taking

$$[\Phi]^{(2)} = [\Phi]^{(2)}(\alpha)$$

$$\begin{pmatrix}
\alpha_{0,0}^{(2)} = 1 & \alpha_{0,1}^{(2)} = -\bar{\alpha} & \alpha_{0,2}^{(2)} = 0 & \alpha_{0,3}^{(2)} = 0 & \alpha_{0,4}^{(2)} = 0 & \alpha_{0,5}^{(2)} = 0 & \alpha_{0,6}^{(2)} = 0 & \alpha_{0,7}^{(2)} = 0 \\
\alpha_{1,0}^{(2)} = -\alpha & \alpha_{1,1}^{(2)} = 1 & \alpha_{1,2}^{(2)} = 0 & \alpha_{1,3}^{(2)} = 0 & \alpha_{1,4}^{(2)} = 0 & \alpha_{1,5}^{(1)} = 0 & \alpha_{1,6}^{(2)} = 0 & \alpha_{1,7}^{(2)} = 0 \\
\alpha_{2,0}^{(2)} = 0 & \alpha_{2,1}^{(2)} = 0 & \alpha_{2,2}^{(2)} = \mathfrak{B} & \alpha_{2,3}^{(2)} = 0 & \alpha_{2,4}^{(2)} = 0 & \alpha_{2,5}^{(2)} = 0 & \alpha_{2,6}^{(2)} = 0 & \alpha_{2,7}^{(2)} = 0 \\
\alpha_{3,0}^{(2)} = 0 & \alpha_{3,1}^{(1)} = 0 & \alpha_{3,2}^{(2)} = 0 & \alpha_{3,3}^{(2)} = \mathfrak{B} & \alpha_{3,4}^{(2)} = 0 & \alpha_{3,5}^{(2)} = 0 & \alpha_{3,6}^{(2)} = 0 & \alpha_{3,7}^{(2)} = 0 \\
\alpha_{4,0}^{(2)} = 0 & \alpha_{4,1}^{(2)} = 0 & \alpha_{4,2}^{(2)} = 0 & \alpha_{4,3}^{(2)} = 0 & \alpha_{4,4}^{(2)} = \mathfrak{B} & \alpha_{4,5}^{(2)} = 0 & \alpha_{4,6}^{(2)} = 0 & \alpha_{4,7}^{(2)} = 0 \\
\alpha_{5,0}^{(2)} = 0 & \alpha_{5,1}^{(2)} = 0 & \alpha_{5,2}^{(2)} = 0 & \alpha_{5,3}^{(2)} = 0 & \alpha_{5,4}^{(2)} = 0 & \alpha_{5,5}^{(2)} = \mathfrak{B} & \alpha_{5,6}^{(2)} = 0 & \alpha_{5,7}^{(2)} = 0 \\
\alpha_{6,0}^{(2)} = 0 & \alpha_{6,1}^{(2)} = 0 & \alpha_{6,2}^{(2)} = 0 & \alpha_{6,3}^{(2)} = 0 & \alpha_{6,4}^{(2)} = 0 & \alpha_{6,5}^{(2)} = 0 & \alpha_{6,6}^{(2)} = \mathfrak{B} & \alpha_{6,7}^{(2)} = 0 \\
\alpha_{7,0}^{(2)} = 0 & \alpha_{7,1}^{(2)} = 0 & \alpha_{7,2}^{(2)} = 0 & \alpha_{7,3}^{(2)} = 0 & \alpha_{7,4}^{(2)} = 0 & \alpha_{7,5}^{(2)} = 0 & \alpha_{7,6}^{(2)} = 0 & \alpha_{7,7}^{(2)} = \mathfrak{B}
\end{pmatrix}$$

with $\mathfrak{B} := \sqrt{1- \mid \alpha \mid^2}$ and $\mid \alpha \mid < 1$, it is easily seen that the entries $\alpha_{j,\nu}^{(2)}$ satisfy the system of equations `(E1)` and `(E2)` and, thus, one can choose

$$\Phi_2^{(\alpha)}(w_1, \ldots, w_7) =$$

$$\left(\frac{w_1 - \alpha}{1 - \bar{\alpha} w_1}, \frac{\sqrt{1- \mid \alpha \mid^2} w_2}{1 - \bar{\alpha} w_1}, \frac{\sqrt{1- \mid \alpha \mid^2} w_3}{1 - \bar{\alpha} w_1}, \frac{\sqrt{1- \mid \alpha \mid^2} w_4}{1 - \bar{\alpha} w_1}, \frac{\sqrt{1- \mid \alpha \mid^2} w_5}{1 - \bar{\alpha} w_1}, \frac{\sqrt{1- \mid \alpha \mid^2} w_6}{1 - \bar{\alpha} w_1}, \frac{\sqrt{1- \mid \alpha \mid^2} w_7}{1 - \bar{\alpha} w_1} \right).$$

According to these choices, the corresponding **biholomorphic encryption chain** is given by

$$\Phi_\alpha(z_1, \ldots, z_7) = \left(\Phi_2^{(\alpha)} \circ \Phi_1 \right)(z_1, \ldots, z_7)$$

$$= \left(\frac{\dfrac{\dfrac{1 \pm i\sqrt{3}}{2} + \dfrac{1 \mp i\sqrt{3}}{2} z_7}{\dfrac{1 \mp i\sqrt{3}}{2} + \dfrac{1 \pm i\sqrt{3}}{2} z_7} - \alpha}{1 - \bar{\alpha} \dfrac{\dfrac{1 \pm i\sqrt{3}}{2} + \dfrac{1 \mp i\sqrt{3}}{2} z_7}{\dfrac{1 \mp i\sqrt{3}}{2} + \dfrac{1 \pm i\sqrt{3}}{2} z_7}}, \frac{\sqrt{1- \mid \alpha \mid^2} \dfrac{(-3)^{\frac{1}{4}} z_1}{\dfrac{1 \mp i\sqrt{3}}{2} + \dfrac{1 \pm i\sqrt{3}}{2} z_7}}{1 - \bar{\alpha} \dfrac{\dfrac{1 \pm i\sqrt{3}}{2} + \dfrac{1 \mp i\sqrt{3}}{2} z_7}{\dfrac{1 \mp i\sqrt{3}}{2} + \dfrac{1 \pm i\sqrt{3}}{2} z_7}}, \right.$$

$$\frac{\sqrt{1- \mid \alpha \mid^2} \dfrac{(-3)^{\frac{1}{4}} z_2}{\dfrac{1 \mp i\sqrt{3}}{2} + \dfrac{1 \pm i\sqrt{3}}{2} z_7}}{1 - \bar{\alpha} \dfrac{\dfrac{1 \pm i\sqrt{3}}{2} + \dfrac{1 \mp i\sqrt{3}}{2} z_7}{\dfrac{1 \mp i\sqrt{3}}{2} + \dfrac{1 \pm i\sqrt{3}}{2} z_7}}, \frac{\sqrt{1- \mid \alpha \mid^2} \dfrac{(-3)^{\frac{1}{4}} z_3}{\dfrac{1 \mp i\sqrt{3}}{2} + \dfrac{1 \pm i\sqrt{3}}{2} z_7}}{1 - \bar{\alpha} \dfrac{\dfrac{1 \pm i\sqrt{3}}{2} + \dfrac{1 \mp i\sqrt{3}}{2} z_7}{\dfrac{1 \mp i\sqrt{3}}{2} + \dfrac{1 \pm i\sqrt{3}}{2} z_7}},$$

$$\frac{\dfrac{(-3)^{\frac{1}{4}} z_4}{\dfrac{1 \mp i\sqrt{3}}{2} + \dfrac{1 \pm i\sqrt{3}}{2} z_7}}{1 - \bar{\alpha} \dfrac{\dfrac{1 \pm i\sqrt{3}}{2} + \dfrac{1 \mp i\sqrt{3}}{2} z_7}{\dfrac{1 \mp i\sqrt{3}}{2} + \dfrac{1 \pm i\sqrt{3}}{2} z_7}}, \frac{\sqrt{1- \mid \alpha \mid^2} \dfrac{(-3)^{\frac{1}{4}} z_5}{\dfrac{1 \mp i\sqrt{3}}{2} + \dfrac{1 \pm i\sqrt{3}}{2} z_7}}{1 - \bar{\alpha} \dfrac{\dfrac{1 \pm i\sqrt{3}}{2} + \dfrac{1 \mp i\sqrt{3}}{2} z_7}{\dfrac{1 \mp i\sqrt{3}}{2} + \dfrac{1 \pm i\sqrt{3}}{2} z_7}},$$

$$\left. \frac{\sqrt{1- \mid \alpha \mid^2} \dfrac{(-3)^{\frac{1}{4}} z_6}{\dfrac{1 \mp i\sqrt{3}}{2} + \dfrac{1 \pm i\sqrt{3}}{2} z_7}}{1 - \bar{\alpha} \dfrac{\dfrac{1 \pm i\sqrt{3}}{2} + \dfrac{1 \mp i\sqrt{3}}{2} z_7}{\dfrac{1 \mp i\sqrt{3}}{2} + \dfrac{1 \pm i\sqrt{3}}{2} z_7}} \right), (\mid \alpha \mid < 1).$$

In particular, we have

$$\Phi_\alpha\left(\xi_{\mathtt{E}},\xi_{\mathtt{v}},\xi_{\mathtt{e}},\xi_{\mathtt{l}},\xi_{\mathtt{p}},\xi_{\mathtt{i}},\xi_{\mathtt{s}}\right)=\left(\Phi_2^{(\alpha)}\circ\Phi_1\right)\left(\xi_{\mathtt{E}},\xi_{\mathtt{v}},\xi_{\mathtt{e}},\xi_{\mathtt{l}},\xi_{\mathtt{p}},\xi_{\mathtt{i}},\xi_{\mathtt{s}}\right)$$

$$=\left(\frac{\dfrac{\frac{1\pm i\sqrt{3}}{2}+\frac{1\mp i\sqrt{3}}{2}\xi_{\mathtt{s}}}{\frac{1\mp i\sqrt{3}}{2}+\frac{1\pm i\sqrt{3}}{2}\xi_{\mathtt{s}}}-\alpha}{1-\bar{\alpha}\dfrac{\frac{1\pm i\sqrt{3}}{2}+\frac{1\mp i\sqrt{3}}{2}\xi_{\mathtt{s}}}{\frac{1\mp i\sqrt{3}}{2}+\frac{1\pm i\sqrt{3}}{2}\xi_{\mathtt{s}}}},\frac{\sqrt{1-\mid\alpha\mid^2}\dfrac{(-3)^{\frac{1}{4}}\xi_{\mathtt{E}}}{\frac{1\mp i\sqrt{3}}{2}+\frac{1\pm i\sqrt{3}}{2}\xi_{\mathtt{s}}}}{1-\bar{\alpha}\dfrac{\frac{1\pm i\sqrt{3}}{2}+\frac{1\mp i\sqrt{3}}{2}\xi_{\mathtt{s}}}{\frac{1\mp i\sqrt{3}}{2}+\frac{1\pm i\sqrt{3}}{2}\xi_{\mathtt{s}}}},\right.$$

$$\frac{\sqrt{1-\mid\alpha\mid^2}\dfrac{(-3)^{\frac{1}{4}}\xi_{\mathtt{v}}}{\frac{1\mp i\sqrt{3}}{2}+\frac{1\pm i\sqrt{3}}{2}\xi_{\mathtt{s}}}}{1-\bar{\alpha}\dfrac{\frac{1\pm i\sqrt{3}}{2}+\frac{1\mp i\sqrt{3}}{2}\xi_{\mathtt{s}}}{\frac{1\mp i\sqrt{3}}{2}+\frac{1\pm i\sqrt{3}}{2}\xi_{\mathtt{s}}}},\frac{\sqrt{1-\mid\alpha\mid^2}\dfrac{(-3)^{\frac{1}{4}}\xi_{\mathtt{e}}}{\frac{1\mp i\sqrt{3}}{2}+\frac{1\pm i\sqrt{3}}{2}\xi_{\mathtt{s}}}}{1-\bar{\alpha}\dfrac{\frac{1\pm i\sqrt{3}}{2}+\frac{1\mp i\sqrt{3}}{2}\xi_{\mathtt{s}}}{\frac{1\mp i\sqrt{3}}{2}+\frac{1\pm i\sqrt{3}}{2}\xi_{\mathtt{s}}}},$$

$$\frac{\dfrac{(-3)^{\frac{1}{4}}\xi_{\mathtt{l}}}{\frac{1\mp i\sqrt{3}}{2}+\frac{1\pm i\sqrt{3}}{2}\xi_{\mathtt{s}}}}{1-\bar{\alpha}\dfrac{\frac{1\pm i\sqrt{3}}{2}+\frac{1\mp i\sqrt{3}}{2}\xi_{\mathtt{s}}}{\frac{1\mp i\sqrt{3}}{2}+\frac{1\pm i\sqrt{3}}{2}\xi_{\mathtt{s}}}},\frac{\sqrt{1-\mid\alpha\mid^2}\dfrac{(-3)^{\frac{1}{4}}\xi_{\mathtt{p}}}{\frac{1\mp i\sqrt{3}}{2}+\frac{1\pm i\sqrt{3}}{2}\xi_{\mathtt{s}}}}{1-\bar{\alpha}\dfrac{\frac{1\pm i\sqrt{3}}{2}+\frac{1\mp i\sqrt{3}}{2}\xi_{\mathtt{s}}}{\frac{1\mp i\sqrt{3}}{2}+\frac{1\pm i\sqrt{3}}{2}\xi_{\mathtt{s}}}},$$

$$\left.\frac{\sqrt{1-\mid\alpha\mid^2}\dfrac{(-3)^{\frac{1}{4}}\xi_{\mathtt{i}}}{\frac{1\mp i\sqrt{3}}{2}+\frac{1\pm i\sqrt{3}}{2}\xi_{\mathtt{s}}}}{1-\bar{\alpha}\dfrac{\frac{1\pm i\sqrt{3}}{2}+\frac{1\mp i\sqrt{3}}{2}\xi_{\mathtt{s}}}{\frac{1\mp i\sqrt{3}}{2}+\frac{1\pm i\sqrt{3}}{2}\xi_{\mathtt{s}}}}\right),\ (\mid\alpha\mid<1).$$

To find the corresponding biholomorphic decryption finite chain, observe that, putting

$$A:=\frac{1\pm i\sqrt{3}}{2},\ B:=\frac{1\mp i\sqrt{3}}{2},\ C:=(-3)^{\frac{1}{4}}\text{ and }D_\alpha:=\sqrt{1-\mid\alpha\mid^2},$$

we have

$$\Phi_1^{-1}(w_1,\ldots,w_7)=$$

$$\Bigg(\underbrace{w_2\left[\frac{B}{C}+\frac{A}{C}\frac{Bw_1-A}{B-Aw_1}\right]}_{z_1},\underbrace{w_3\left[\frac{B}{C}+\frac{A}{C}\frac{Bw_1-A}{B-Aw_1}\right]}_{z_2},\underbrace{w_4\left[\frac{B}{C}+\frac{A}{C}\frac{Bw_1-A}{B-Aw_1}\right]}_{z_3},$$

$$\underbrace{w_5\left[\frac{B}{C}+\frac{A}{C}\frac{Bw_1-A}{B-Aw_1}\right]}_{z_4},\underbrace{w_6\left[\frac{B}{C}+\frac{A}{C}\frac{Bw_1-A}{B-Aw_1}\right]}_{z_5},\underbrace{w_7\left[\frac{B}{C}+\frac{A}{C}\frac{Bw_1-A}{B-Aw_1}\right]}_{z_6},\underbrace{\left[\frac{Bw_1-A}{B-Aw_1}\right]}_{z_7}\Bigg),$$

and

$$\Phi_2^{(\alpha)^{-1}}(u_1,\ldots,u_7)=$$

$$\Bigg(\underbrace{\left[\frac{u_1+\alpha}{1+\bar{\alpha}u_1}\right]}_{w_1},\underbrace{\frac{u_2}{D}\left[1-\bar{\alpha}\frac{u_1+\alpha}{1+\bar{\alpha}u_1}\right]}_{w_2},\underbrace{\frac{u_3}{D}\left[1-\bar{\alpha}\frac{u_1+\alpha}{1+\bar{\alpha}u_1}\right]}_{w_3},\underbrace{\frac{u_4}{D}\left[1-\bar{\alpha}\frac{u_1+\alpha}{1+\bar{\alpha}u_1}\right]}_{w_4},$$

$$\underbrace{\frac{u_5}{D}\left[1-\bar{\alpha}\frac{u_1+\alpha}{1+\bar{\alpha}u_1}\right]}_{w_5},\underbrace{\frac{u_6}{D}\left[1-\bar{\alpha}\frac{u_1+\alpha}{1+\bar{\alpha}u_1}\right]}_{w_6},\underbrace{\frac{u_7}{D}\left[1-\bar{\alpha}\frac{u_1+\alpha}{1+\bar{\alpha}u_1}\right]}_{w_7}\Bigg).$$

Thus, the corresponding **biholomorphic decryption finite chain** is given by

$$\Phi^{-1}(u_1,\ldots,u_7)=\left(\Phi_1^{-1}\circ\Phi_2^{(\alpha)^{-1}}\right)(u_1,\ldots,u_7)=$$

$$\Bigg(\underbrace{u_2\left[\frac{A+B}{CD}\left(1-\bar{\alpha}\frac{u_1+\alpha}{1+\bar{\alpha}u_1}\right)\frac{B\frac{u_1+\alpha}{1+\bar{\alpha}u_1}-A}{B-A\frac{u_1+\alpha}{1+\bar{\alpha}u_1}}\right]}_{z_1},\underbrace{u_3\left[\frac{A+B}{CD}\left(1-\bar{\alpha}\frac{u_1+\alpha}{1+\bar{\alpha}u_1}\right)\frac{B\frac{u_1+\alpha}{1+\bar{\alpha}u_1}-A}{B-A\frac{u_1+\alpha}{1+\bar{\alpha}u_1}}\right]}_{z_2},$$

$$\underbrace{u_4\left[\frac{A+B}{CD}\left(1-\bar{\alpha}\frac{u_1+\alpha}{1+\bar{\alpha}u_1}\right)\frac{B\frac{u_1+\alpha}{1+\bar{\alpha}u_1}-A}{B-A\frac{u_1+\alpha}{1+\bar{\alpha}u_1}}\right]}_{z_3},\underbrace{u_5\left[\frac{A+B}{CD}\left(1-\bar{\alpha}\frac{u_1+\alpha}{1+\bar{\alpha}u_1}\right)\frac{B\frac{u_1+\alpha}{1+\bar{\alpha}u_1}-A}{B-A\frac{u_1+\alpha}{1+\bar{\alpha}u_1}}\right]}_{z_4},$$

$$\underbrace{u_6\left[\frac{A+B}{CD}\left(1-\bar{\alpha}\frac{u_1+\alpha}{1+\bar{\alpha}u_1}\right)\frac{B\frac{u_1+\alpha}{1+\bar{\alpha}u_1}-A}{B-A\frac{u_1+\alpha}{1+\bar{\alpha}u_1}}\right]}_{z_5},\underbrace{u_7\left[\frac{A+B}{CD}\left(1-\bar{\alpha}\frac{u_1+\alpha}{1+\bar{\alpha}u_1}\right)\frac{B\frac{u_1+\alpha}{1+\bar{\alpha}u_1}-A}{B-A\frac{u_1+\alpha}{1+\bar{\alpha}u_1}}\right]}_{z_6},$$

$$\underbrace{\frac{B\frac{u_1+\alpha}{1+\bar{\alpha}u_1}-A}{B-A\frac{u_1+\alpha}{1+\bar{\alpha}u_1}}}_{z_7}\Bigg),(\mid\alpha\mid<1).$$

Second Choice $U^n = \Delta^n$. Take

$$E = D = \left\{\Psi : automorphism\ of\ \Delta^7 \subset \mathbb{C}^7\right\} \text{ and } M = 3.$$

Let us choose

a). three **arbitrary** complex vectors in $\mathbb{C}^7$:

$\alpha^{(1)} = \left(\alpha_1^{(1)} = \frac{i}{2}, \alpha_2^{(1)} = \frac{i}{3}, \alpha_3^{(1)} = \frac{i}{4}, \alpha_4^{(1)} = \frac{i}{5}, \alpha_5^{(1)} = \frac{i}{6}, \alpha_6^{(1)} = \frac{i}{7}, \alpha_7^{(1)} = \frac{i}{8}\right)^T$,

$\alpha^{(2)} = \left(\alpha_1^{(2)} = \frac{1-i}{2}, \alpha_2^{(2)} = \frac{1+2i}{3}, \alpha_3^{(2)} = \frac{1-3i}{4}, \alpha_4^{(2)} = \frac{1+4i}{5}, \alpha_5^{(2)} = \frac{1-5i}{6}, \alpha_6^{(2)} = \frac{1+6i}{7}, \alpha_7^{(2)} = \frac{1-7i}{8}\right)^T$,

$\alpha^{(3)} = \left(\alpha_1^{(3)} = \frac{1+i}{2}, \alpha_2^{(3)} = \frac{1-2i}{3}, \alpha_3^{(3)} = \frac{1+3i}{4}, \alpha_4^{(3)} = \frac{1-4i}{5}, \alpha_5^{(3)} = \frac{1+5i}{6}, \alpha_6^{(3)} = \frac{1-6i}{7}, \alpha_7^{(3)} = \frac{1+7i}{8}\right)^T$;

for each $\alpha^{(j)}$, we get an arbitrary permutation of $1, 2, \ldots, 7$:

$\sigma^{(1)} = \left(\sigma^{(1)}(1) = 2, \sigma^{(1)}(2) = 6, \sigma^{(1)}(3) = 4, \sigma^{(1)}(4) = 1, \sigma^{(1)}(5) = 7, \sigma^{(1)}(6) = 5, \sigma^{(1)}(7) = 3\right)$,

$\sigma^{(2)} = \left(\sigma^{(2)}(1) = 5, \sigma^{(2)}(2) = 4, \sigma^{(2)}(3) = 3, \sigma^{(2)}(4) = 7, \sigma^{(2)}(5) = 6, \sigma^{(2)}(6) = 1, \sigma^{(2)}(7) = 2\right)$

and

$\sigma^{(3)} = \left(\sigma^{(3)}(1) = 7, \sigma^{(3)}(2) = 3, \sigma^{(3)}(3) = 1, \sigma^{(3)}(4) = 5, \sigma^{(3)}(5) = 2, \sigma^{(3)}(6) = 4, \sigma^{(3)}(7) = 6\right)$.

b). three **arbitrary** real vectors in $\mathbb{R}^7$:

$$\theta^{(1)} = \left(\theta_1^{(1)} = \theta_2^{(1)} = \theta_3^{(1)} = \theta_4^{(1)} = \theta_5^{(1)} = \theta_6^{(1)} = \theta_7^{(1)} = \frac{\pi}{3}\right)^T,$$

$$\theta^{(2)} = \left(\theta_1^{(2)} = \theta_2^{(2)} = \theta_3^{(2)} = \theta_4^{(2)} = \theta_5^{(2)} = \theta_6^{(2)} = \theta_7^{(2)} = 0\right)^T,$$

$$\theta^{(3)} = \left(\theta_1^{(3)} = \theta_2^{(3)} = \frac{\pi}{3}, \theta_3^{(3)} = \theta_4^{(3)} = \theta_5^{(3)} = \frac{\pi}{4}, \theta_6^{(3)} = \theta_7^{(3)} = \pi\right)^T;$$

for each $\theta^{(j)}$, we get another permutation of $1, 2, \ldots, 7$:

$\tau^{(1)} = \left(\tau^{(1)}(1) = 3, \tau^{(1)}(2) = 7, \tau^{(1)}(3) = 1, \tau^{(1)}(4) = 5, \tau^{(1)}(5) = 4, \tau^{(1)}(6) = 2, \tau^{(1)}(7) = 6\right)$,

$\tau^{(2)} = \left(\tau^{(2)}(1) = 4, \tau^{(2)}(2) = 1, \tau^{(2)}(3) = 3, \tau^{(2)}(4) = 2, \tau^{(2)}(5) = 6, \tau^{(2)}(6) = 7, \tau^{(2)}(7) = 5\right)$

and

$\tau^{(3)} = \left(\tau^{(3)}(1) = 6, \tau^{(3)}(2) = 7, \tau^{(3)}(3) = 5, \tau^{(3)}(4) = 1, \tau^{(3)}(5) = 3, \tau^{(3)}(6) = 4, \tau^{(3)}(7) = 2\right)$.

According to Proposition 7.ii,

c). to $\left(\alpha^{(1)}, \theta^{(1)}, \sigma^{(1)}, \tau^{(1)}\right)$, there corresponds a unique automorphism defined by

$$\Psi_1 = \Psi^{\left(\alpha^{(1)}, \theta^{(1)}, \sigma^{(1)}, \tau^{(1)}\right)} : \Delta^7 \to \Delta^7 : (z_1, \ldots, z_7) \mapsto \Psi_1(z_1, \ldots, z_7) =$$

$$\Bigg(\underbrace{e^{i\pi} \frac{z_2 - \frac{i}{2}}{1 + \frac{i}{3} z_2}}_{u_1}, \underbrace{e^{i\frac{\pi}{3}} \frac{z_6 - \frac{i}{7}}{1 + \frac{i}{7} z_6}}_{u_2}, \underbrace{e^{i\pi} \frac{z_4 - \frac{i}{5}}{1 + \frac{i}{5}} z_4}_{u_3},$$

$$\underbrace{e^{i\frac{\pi}{3}} \frac{z_1 - \frac{i}{2}}{1 + \frac{i}{2} z_1}}_{u_4}, \underbrace{e^{i\frac{\pi}{3}} \frac{z_7 - \frac{i}{8}}{1 + \frac{i}{8} z_7}}_{u_5}, \underbrace{e^{i\pi} \frac{z_5 - \frac{i}{6}}{1 + \frac{i}{6} z_5}}_{u_6}, \underbrace{e^{i\frac{\pi}{3}} \frac{z_3 - \frac{i}{4}}{1 + \frac{i}{4} z_3}}_{u_7} \Bigg) \in E;$$

d). to $\left(\alpha^{(2)}, \theta^{(2)}, \sigma^{(2)}, \tau^{(2)}\right)$, there corresponds a unique automorphism defined by

$$\Psi_2 = \Psi^{\left(\alpha^{(2)},\theta^{(2)},\sigma^{(2)},\tau^{(2)}\right)} : \Delta^7 \to \Delta^7 : (z_1, \ldots, z_7) \mapsto \Psi_2\,(z_1, \ldots, z_7) =$$

$$\Bigg(\underbrace{\frac{u_5 - \frac{1-5i}{6}}{1 - \frac{1+5i}{6} u_5}}_{w_1}, \underbrace{\frac{u_4 - \frac{1+4i}{5}}{1 - \frac{1-4i}{5} u_4}}_{w_2}, \underbrace{\frac{u_3 - \frac{1+3i}{4}}{1 - \frac{1-3i}{4} u_3}}_{w_3},$$

$$\underbrace{\frac{u_7 - \frac{1-7i}{8}}{1 - \frac{1+7i}{8} u_7}}_{w_4}, \underbrace{\frac{u_6 - \frac{1+6i}{7}}{1 - \frac{1-6i}{7} u_6}}_{w_5}, \underbrace{\frac{u_1 - \frac{1-i}{2}}{1 - \frac{1+i}{2} u_1}}_{w_6}, \underbrace{\frac{u_2 - \frac{1+2i}{3}}{1 - \frac{1-2i}{3} u_2}}_{w_7} \Bigg) \in E;$$

e). to $\left(\alpha^{(3)}, \theta^{(3)}, \sigma^{(3)}, \tau^{(3)}\right)$, there corresponds a unique automorphism defined by

$$\Psi_3 = \Psi^{\left(\alpha^{(3)},\theta^{(3)},\sigma^{(3)},\tau^{(3)}\right)} : \Delta^7 \to \Delta^7 : (z_1, \ldots, z_7) \mapsto \Psi_3\,(z_1, \ldots, z_7) =$$

$$\Bigg(\underbrace{e^{i\pi} \frac{w_7 - \frac{1+7i}{8}}{1 - \frac{1-7i}{8} w_7}}_{v_1}, \underbrace{e^{i\pi} \frac{w_3 - \frac{1+3i}{4}}{1 - \frac{1-3i}{4} w_3}}_{v_2}, \underbrace{e^{i\frac{\pi}{4}} \frac{w_1 - \frac{1+i}{2}}{1 - \frac{1-i}{2} w_1}}_{v_3},$$

$$\underbrace{e^{i\frac{\pi}{3}} \frac{w_5 - \frac{1+5i}{6}}{1 - \frac{1+5i}{6} w_5}}_{v_4}, \underbrace{e^{i\frac{\pi}{4}} \frac{w_2 - \frac{1-2i}{3}}{1 - \frac{1+2i}{3} w_2}}_{v_5}, \underbrace{e^{i\frac{\pi}{4}} \frac{w_4 - \frac{1-4i}{5}}{1 - \frac{1+4i}{5} w_4}}_{v_6}, \underbrace{e^{i\frac{\pi}{3}} \frac{w_6 - \frac{1-6i}{7}}{1 - \frac{1+6i}{7} w_6}}_{v_7} \Bigg) \in E.$$

So, according to these choices, the **biholomorphic encryption chain** is given by

$$\Psi\,(z_1, z_2, z_3, z_4, z_5, z_6, z_7) = (\Psi_3 \circ \Psi_2 \circ \Psi_1)\,(z_1, z_2, z_3, z_4, z_5, z_6, z_7)$$

$$= \Bigg(\underbrace{e^{i\pi} \frac{\dfrac{e^{i\frac{\pi}{3}} \frac{z_6 - \frac{i}{7}}{1+\frac{i}{7}z_6} - \frac{1+2i}{3}}{1-\frac{1-2i}{3} e^{i\frac{\pi}{3}} \frac{z_6-\frac{i}{7}}{1+\frac{i}{7}z_6}} - \dfrac{1+7i}{8}}{1 - \dfrac{1-7i}{8} \dfrac{e^{\frac{\pi}{3}} \frac{z_6 - \frac{i}{7}}{1+\frac{i}{7}z_6} - \frac{1+2i}{3}}{1-\frac{1-2i}{3} e^{i\frac{\pi}{3}} \frac{z_6-\frac{i}{7}}{1+\frac{i}{7}z_6}}}}_{v_1}, \underbrace{e^{i\pi} \frac{\dfrac{e^{i\pi} \frac{z_4 - \frac{i}{5}}{1+\frac{i}{5}z_4} - \frac{1+3i}{4}}{1-\frac{1-3i}{4} e^{i\pi} \frac{z_4-\frac{i}{5}}{1+\frac{i}{5}z_4}} - \dfrac{1+3i}{4}}{1 - \dfrac{1-3i}{4} \dfrac{e^{i\pi} \frac{z_4 - \frac{i}{5}}{1+\frac{i}{5}z_4} - \frac{1+3i}{4}}{1-\frac{1-3i}{4} e^{i\pi} \frac{z_4-\frac{i}{5}}{1+\frac{i}{5}z_4}}}}_{v_2},$$

$$\underbrace{e^{i\frac{\pi}{4}} \frac{\dfrac{e^{i\frac{\pi}{3}} \frac{z_7 - \frac{i}{8}}{1+\frac{i}{8}z_7} - \frac{1-5i}{6}}{1-\frac{1+5i}{6} e^{i\frac{\pi}{3}} \frac{z_7-\frac{i}{8}}{1+\frac{i}{8}z_7}} - \dfrac{1+i}{2}}{1 - \dfrac{1-i}{2} \dfrac{e^{\frac{\pi}{3}} \frac{z_7 - \frac{i}{8}}{1+\frac{i}{8}z_7} - \frac{1-5i}{6}}{1-\frac{1+5i}{6} e^{i\frac{\pi}{3}} \frac{z_7-\frac{i}{8}}{1+\frac{i}{8}z_7}}}}_{v_3}, \underbrace{e^{i\frac{\pi}{3}} \frac{\dfrac{e^{i\pi} \frac{z_5 - \frac{i}{6}}{1+\frac{i}{6}z_5} - \frac{1+6i}{7}}{1-\frac{1-6i}{7} e^{i\pi} \frac{z_5-\frac{i}{6}}{1+\frac{i}{6}z_5}} - \dfrac{1+5i}{6}}{1 - \dfrac{1+5i}{6} \dfrac{e^{i\pi} \frac{z_5 - \frac{i}{6}}{1+\frac{i}{6}z_5} - \frac{1+6i}{7}}{1-\frac{1-6i}{7} e^{i\pi} \frac{z_5-\frac{i}{6}}{1+\frac{i}{6}z_5}}}}_{v_4}, \underbrace{e^{i\frac{\pi}{4}} \frac{\dfrac{e^{i\frac{\pi}{3}} \frac{z_1 - \frac{i}{2}}{1+\frac{i}{2}z_1} - \frac{1+4i}{5}}{1-\frac{1-4i}{5} e^{i\frac{\pi}{3}} \frac{z_1-\frac{i}{2}}{1+\frac{i}{2}z_1}} - \dfrac{1-2i}{3}}{1 - \dfrac{1+2i}{3} \dfrac{e^{i\frac{\pi}{3}} \frac{z_1 - \frac{i}{2}}{1+\frac{i}{2}z_1} - \frac{1+4i}{5}}{1-\frac{1-4i}{5} e^{i\frac{\pi}{3}} \frac{z_1-\frac{i}{2}}{1+\frac{i}{2}z_1}}}}_{v_5},$$

$$\underbrace{e^{i\frac{\pi}{4}}\frac{\dfrac{e^{i\frac{\pi}{3}}\frac{z_3-\frac{i}{4}}{1+\frac{i}{4}z_3}-\frac{1-7i}{8}}{1-\frac{1+7i}{8}e^{i\frac{\pi}{3}}\frac{z_3-\frac{i}{4}}{1+\frac{i}{4}z_3}}-\dfrac{1-4i}{5}}{1-\dfrac{1+4i}{5}\dfrac{e^{\frac{\pi}{3}}\frac{z_3-\frac{i}{4}}{1+\frac{i}{4}z_3}-\frac{1-7i}{8}}{1-\frac{1+7i}{8}e^{i\frac{\pi}{3}}\frac{z_3-\frac{i}{4}}{1+\frac{i}{4}z_3}}}}_{v_6},\ \underbrace{e^{i\frac{\pi}{3}}\frac{\dfrac{e^{i\pi}\frac{z_2-\frac{i}{3}}{1+\frac{i}{3}z_2}-\frac{1-i}{2}}{1-\frac{1+i}{2}e^{i\pi}\frac{z_2-\frac{i}{3}}{1+\frac{i}{3}z_2}}-\dfrac{1-6i}{7}}{1-\dfrac{1+6i}{7}\dfrac{e^{i\pi}\frac{z_2-\frac{i}{3}}{1+\frac{i}{3}z_2}-\frac{1-i}{2}}{1-\frac{1+i}{2}e^{i\pi}\frac{z_2-\frac{i}{3}}{1+\frac{i}{3}z_2}}}}_{v_7}\Bigg).$$

In particular, we have

$$\Psi\left(\xi_{\mathtt{E}},\xi_{\mathtt{v}},\xi_{\mathtt{e}},\xi_{\mathtt{l}},\xi_{\mathtt{p}},\xi_{\mathtt{i}},\xi_{\mathtt{s}}\right)=\left(\Psi_3\circ\Psi_2\circ\Psi_1\right)\left(\xi_{\mathtt{E}},\xi_{\mathtt{v}},\xi_{\mathtt{e}},\xi_{\mathtt{l}},\xi_{\mathtt{p}},\xi_{\mathtt{i}},\xi_{\mathtt{s}}\right)$$

$$=\Bigg(\underbrace{e^{i\pi}\frac{\dfrac{e^{i\frac{\pi}{3}}\frac{\xi_{\mathtt{i}}-\frac{i}{7}}{1+\frac{i}{7}\xi_{\mathtt{i}}}-\frac{1+2i}{3}}{1-\frac{1-2i}{3}e^{i\frac{\pi}{3}}\frac{\xi_{\mathtt{i}}-\frac{i}{7}}{1+\frac{i}{7}\xi_{\mathtt{i}}}}-\dfrac{1+7i}{8}}{1-\dfrac{1-7i}{8}\dfrac{e^{\frac{\pi}{3}}\frac{\xi_{\mathtt{i}}-\frac{i}{7}}{1+\frac{i}{7}\xi_{\mathtt{i}}}-\frac{1+2i}{3}}{1-\frac{1-2i}{3}e^{i\frac{\pi}{3}}\frac{\xi_{\mathtt{i}}-\frac{i}{7}}{1+\frac{i}{7}\xi_{\mathtt{i}}}}}}_{v_1},\ \underbrace{e^{i\pi}\frac{\dfrac{e^{i\pi}\frac{\xi_{\mathtt{l}}-\frac{i}{5}}{1+\frac{i}{5}\xi_{\mathtt{l}}}-\frac{1+3i}{4}}{1-\frac{1-3i}{4}e^{i\pi}\frac{\xi_{\mathtt{l}}-\frac{i}{5}}{1+\frac{i}{5}\xi_{\mathtt{l}}}}-\dfrac{1+3i}{4}}{1-\dfrac{1-3i}{4}\dfrac{e^{i\pi}\frac{\xi_{\mathtt{l}}-\frac{i}{5}}{1+\frac{i}{5}\xi_{\mathtt{l}}}-\frac{1+3i}{4}}{1-\frac{1-3i}{4}e^{i\pi}\frac{\xi_{\mathtt{l}}-\frac{i}{5}}{1+\frac{i}{5}\xi_{\mathtt{l}}}}}}_{v_2},$$

$$\underbrace{e^{i\frac{\pi}{4}}\frac{\dfrac{e^{i\frac{\pi}{3}}\frac{\xi_{\mathtt{s}}-\frac{i}{8}}{1+\frac{i}{8}\xi_{\mathtt{s}}}-\frac{1-5i}{6}}{1-\frac{1+5i}{6}e^{i\frac{\pi}{3}}\frac{\xi_{\mathtt{s}}-\frac{i}{8}}{1+\frac{i}{8}\xi_{\mathtt{s}}}}-\dfrac{1+i}{2}}{1-\dfrac{1-i}{2}\dfrac{e^{\frac{\pi}{3}}\frac{\xi_{\mathtt{s}}-\frac{i}{8}}{1+\frac{i}{8}\xi_{\mathtt{s}}}-\frac{1-5i}{6}}{1-\frac{1+5i}{6}e^{i\frac{\pi}{3}}\frac{\xi_{\mathtt{s}}-\frac{i}{8}}{1+\frac{i}{8}\xi_{\mathtt{s}}}}}}_{v_3},\ \underbrace{e^{i\frac{\pi}{3}}\frac{\dfrac{e^{i\pi}\frac{\xi_{\mathtt{p}}-\frac{i}{6}}{1+\frac{i}{6}\xi_{\mathtt{p}}}-\frac{1+6i}{7}}{1-\frac{1-6i}{7}e^{i\pi}\frac{\xi_{\mathtt{p}}-\frac{i}{6}}{1+\frac{i}{6}\xi_{\mathtt{p}}}}-\dfrac{1+5i}{6}}{1-\dfrac{1+5i}{6}\dfrac{e^{i\pi}\frac{\xi_{\mathtt{p}}-\frac{i}{6}}{1+\frac{i}{6}\xi_{\mathtt{p}}}-\frac{1+6i}{7}}{1-\frac{1-6i}{7}e^{i\pi}\frac{\xi_{\mathtt{p}}-\frac{i}{6}}{1+\frac{i}{6}\xi_{\mathtt{p}}}}}}_{v_4},\ \underbrace{e^{i\frac{\pi}{4}}\frac{\dfrac{e^{i\frac{\pi}{3}}\frac{\xi_{\mathtt{E}}-\frac{i}{2}}{1+\frac{i}{2}\xi_{\mathtt{E}}}-\frac{1+4i}{5}}{1-\frac{1-4i}{5}e^{i\frac{\pi}{3}}\frac{\xi_{\mathtt{E}}-\frac{i}{2}}{1+\frac{i}{2}\xi_{\mathtt{E}}}}-\dfrac{1-2i}{3}}{1-\dfrac{1+2i}{3}\dfrac{e^{i\frac{\pi}{3}}\frac{\xi_{\mathtt{E}}-\frac{i}{2}}{1+\frac{i}{2}\xi_{\mathtt{E}}}-\frac{1+4i}{5}}{1-\frac{1-4i}{5}e^{i\frac{\pi}{3}}\frac{\xi_{\mathtt{E}}-\frac{i}{2}}{1+\frac{i}{2}\xi_{\mathtt{E}}}}}}_{v_5},$$

$$\underbrace{e^{i\frac{\pi}{4}}\frac{\dfrac{e^{i\frac{\pi}{3}}\frac{\xi_{\mathtt{e}}-\frac{i}{4}}{1+\frac{i}{4}\xi_{\mathtt{e}}}-\frac{1-7i}{8}}{1-\frac{1+7i}{8}e^{i\frac{\pi}{3}}\frac{\xi_{\mathtt{e}}-\frac{i}{4}}{1+\frac{i}{4}\xi_{\mathtt{e}}}}-\dfrac{1-4i}{5}}{1-\dfrac{1+4i}{5}\dfrac{e^{\frac{\pi}{3}}\frac{\xi_{\mathtt{e}}-\frac{i}{4}}{1+\frac{i}{4}\xi_{\mathtt{e}}}-\frac{1-7i}{8}}{1-\frac{1+7i}{8}e^{i\frac{\pi}{3}}\frac{\xi_{\mathtt{e}}-\frac{i}{4}}{1+\frac{i}{4}\xi_{\mathtt{e}}}}}}_{v_6},\ \underbrace{e^{i\frac{\pi}{3}}\frac{\dfrac{e^{i\pi}\frac{z_2-\frac{i}{3}}{1+\frac{i}{3}z_2}-\frac{1-i}{2}}{1-\frac{1+i}{2}e^{i\pi}\frac{\xi_{\mathtt{v}}-\frac{i}{3}}{1+\frac{i}{3}\xi_{\mathtt{v}}}}-\dfrac{1-6i}{7}}{1-\dfrac{1+6i}{7}\dfrac{e^{i\pi}\frac{\xi_{\mathtt{v}}-\frac{i}{3}}{1+\frac{i}{3}\xi_{\mathtt{v}}}-\frac{1-i}{2}}{1-\frac{1+i}{2}e^{i\pi}\frac{\xi_{\mathtt{v}}-\frac{i}{3}}{1+\frac{i}{3}\xi_{\mathtt{v}}}}}}_{v_7}\Bigg).$$

To find the corresponding **biholomorphic decryption finite chain**, observe that

$$\Psi_1^{-1}\left(u_1,u_2,u_3,u_4,u_5,u_6,u_7\right)=$$

$$\Bigg(\underbrace{\frac{u_1+e^{i\pi}\frac{i}{3}}{e^{i\pi}-\frac{i}{3}u_1}}_{z_2},\underbrace{\frac{u_2+e^{i\frac{\pi}{3}}\frac{i}{7}}{e^{i\frac{\pi}{3}}-\frac{i}{7}u_2}}_{z_6},\underbrace{\frac{u_3+e^{i\pi}\frac{i}{5}}{e^{i\pi}-\frac{i}{5}u_3}}_{z_4},\underbrace{\frac{u_4+e^{i\frac{\pi}{3}}\frac{i}{2}}{e^{i\frac{\pi}{3}}-\frac{i}{2}u_4}}_{z_1},\underbrace{\frac{u_5+e^{i\frac{\pi}{3}}\frac{i}{3}}{e^{i\frac{\pi}{3}}-\frac{i}{3}u_5}}_{z_7},\underbrace{\frac{u_6+e^{i\pi}\frac{i}{6}}{e^{i\pi}-\frac{i}{6}u_6}}_{z_5},\underbrace{\frac{u_7+e^{i\frac{\pi}{3}}\frac{i}{4}}{e^{i\frac{\pi}{3}}-\frac{i}{4}u_7}}_{z_3}\Bigg),$$

$$\Psi_2^{-1}\left(w_1,w_2,w_3,w_4,w_5,w_6,w_7\right)=$$

$$\Bigg(\underbrace{\frac{w_1+\frac{1-5i}{6}}{1+\frac{1+5i}{6}w_1}}_{u_5},\underbrace{\frac{w_2+\frac{1+4i}{5}}{1+\frac{1-4i}{5}w_2}}_{u_4},\underbrace{\frac{w_3+\frac{1+3i}{4}}{1+\frac{1-3i}{4}w_3}}_{u_3},\underbrace{\frac{w_4+\frac{1-7i}{8}}{1+\frac{1+7i}{8}w_4}}_{u_7},\underbrace{\frac{w_5+\frac{1+6i}{7}}{1+\frac{1-6i}{7}w_5}}_{u_6},\underbrace{\frac{w_6+\frac{1-i}{2}}{1+\frac{1+i}{2}w_6}}_{u_1},\underbrace{\frac{w_7+\frac{1+2i}{3}}{1+\frac{1-2i}{3}w_7}}_{u_2}\Bigg),$$

$$\Psi_3^{-1}\left(v_1,v_2,v_3,v_4,v_5,v_6,v_7\right)=$$

$$\Bigg(\underbrace{\frac{v_1+e^{i\pi}\frac{1+7i}{8}}{e^{i\pi}+\frac{1-7i}{8}v_1}}_{w_7},\underbrace{\frac{v_2+e^{i\pi}\frac{1+3i}{4}}{e^{i\pi}+\frac{1-3i}{4}v_2}}_{w_3},\underbrace{\frac{v_3+e^{i\frac{\pi}{4}}\frac{1+i}{2}}{e^{i\frac{\pi}{4}}+\frac{1-i}{2}v_3}}_{w_1},\underbrace{\frac{v_4+e^{i\frac{\pi}{3}}\frac{1+5i}{6}}{e^{i\frac{\pi}{3}}+\frac{1-5i}{6}v_4}}_{w_5},$$

$$\underbrace{\frac{v_5+e^{i\frac{\pi}{4}}\frac{1-2i}{3}}{e^{i\frac{\pi}{4}}+\frac{1+2i}{3}v_5}}_{w_2},\underbrace{\frac{v_6+e^{i\frac{\pi}{4}}\frac{1-4i}{5}}{e^{i\frac{\pi}{4}}+\frac{1+4i}{5}v_6}}_{w_4},\underbrace{\frac{v_7+e^{i\frac{\pi}{3}}\frac{1-6i}{7}}{e^{i\frac{\pi}{3}}+\frac{1+6i}{7}v_7}}_{w_6}\Bigg).$$

Thus, the corresponding **biholomorphic decryption finite chain** is

$$\Psi^{-1}\left(v_1,v_2,v_3,v_4,v_5,v_6,v_7\right)=\left(\Psi_1^{-1}\circ\Psi_2^{-1}\circ\Psi_3^{-1}\right)\left(v_1,v_2,v_3,v_4,v_5,v_6,v_7\right)$$

$$=\Bigg(\underbrace{\frac{\frac{\frac{v_7+e^{i\frac{\pi}{3}}\frac{1-6i}{7}}{e^{i\frac{\pi}{3}}+\frac{1+6i}{7}v_7}+\frac{1-i}{2}}{1+\frac{1+i}{2}\frac{v_7+e^{i\frac{\pi}{3}}\frac{1-6i}{7}}{e^{i\frac{\pi}{3}}+\frac{1+6i}{7}v_7}}+e^{i\pi}\frac{i}{3}}{e^{i\pi}-\frac{i}{3}\frac{\frac{v_7+e^{i\frac{\pi}{3}}\frac{1-6i}{7}}{e^{i\frac{\pi}{3}}+\frac{1+6i}{7}v_7}+\frac{1-i}{2}}{1+\frac{1+i}{2}\frac{v_7+e^{i\frac{\pi}{3}}\frac{1-6i}{7}}{e^{i\frac{\pi}{3}}+\frac{1+6i}{7}v_7}}}}_{z_2},\underbrace{\frac{\frac{\frac{v_1+e^{i\pi}\frac{1+7i}{8}}{e^{i\pi}+\frac{1-7i}{8}v_1}+\frac{1+2i}{3}}{1+\frac{1-2i}{3}\frac{v_1+e^{i\pi}\frac{1+7i}{8}}{e^{i\pi}+\frac{1-7i}{8}v_1}}+e^{i\frac{\pi}{3}}\frac{i}{7}}{e^{i\frac{\pi}{3}}-\frac{i}{7}\frac{\frac{v_1+e^{i\pi}\frac{1+7i}{8}}{e^{i\pi}+\frac{1-7i}{8}v_1}+\frac{1+2i}{3}}{1+\frac{1-2i}{3}\frac{v_1+e^{i\pi}\frac{1+7i}{8}}{e^{i\pi}+\frac{1-7i}{8}v_1}}}}_{z_6},$$

$$\underbrace{\frac{\dfrac{\dfrac{v_2+e^{i\pi}\frac{1+3i}{4}}{e^{i\pi}+\frac{1-3i}{4}v_2}+\frac{1+3i}{4}}{1+\frac{1-3i}{4}\,\dfrac{v_2+e^{i\pi}\frac{1+3i}{4}}{e^{i\pi}+\frac{1-3i}{4}v_2}}+e^{i\pi}\,\frac{i}{5}}{e^{i\pi}-\frac{i}{5}\,\dfrac{\dfrac{v_2+e^{i\pi}\frac{1+3i}{4}}{e^{i\pi}+\frac{1-3i}{4}v_2}+\frac{1+3i}{4}}{1+\frac{1-3i}{4}\,\dfrac{v_2+e^{i\pi}\frac{1+3i}{4}}{e^{i\pi}+\frac{1-3i}{4}v_2}}}}_{z_4},\ \underbrace{\frac{\dfrac{\dfrac{v_5+e^{i\frac{\pi}{4}}\frac{1-2i}{3}}{e^{i\frac{\pi}{4}}+\frac{1+2i}{3}v_5}+\frac{1+4i}{5}}{1+\frac{1-4i}{5}\,\dfrac{v_5+e^{i\frac{\pi}{4}}\frac{1-2i}{3}}{e^{i\frac{\pi}{4}}+\frac{1+2i}{3}v_5}}+e^{i\frac{\pi}{3}}\,\frac{i}{2}}{e^{i\pi}-\frac{i}{2}\,\dfrac{\dfrac{v_5+e^{i\frac{\pi}{4}}\frac{1-2i}{3}}{e^{i\frac{\pi}{4}}+\frac{1+2i}{3}v_5}+\frac{1+4i}{5}}{1+\frac{1-4i}{5}\,\dfrac{v_5+e^{i\frac{\pi}{4}}\frac{1-2i}{3}}{e^{i\frac{\pi}{4}}+\frac{1+2i}{3}v_5}}}}_{z_1},$$

$$\underbrace{\frac{\dfrac{\dfrac{v_3+e^{i\frac{\pi}{4}}\frac{1+i}{2}}{e^{i\frac{\pi}{4}}+\frac{1-i}{2}v_3}+\frac{1-5i}{6}}{1+\frac{1+5i}{6}\,\dfrac{v_3+e^{i\frac{\pi}{4}}\frac{1+i}{2}}{e^{i\frac{\pi}{4}}+\frac{1-i}{2}v_3}}+e^{i\frac{\pi}{3}}\,\frac{i}{3}}{e^{i\frac{\pi}{3}}-\frac{i}{3}\,\dfrac{\dfrac{v_3+e^{i\frac{\pi}{4}}\frac{1+i}{2}}{e^{i\frac{\pi}{4}}+\frac{1-i}{2}v_3}+\frac{1-5i}{6}}{1+\frac{1+5i}{6}\,\dfrac{v_3+e^{i\frac{\pi}{4}}\frac{1+i}{2}}{e^{i\frac{\pi}{4}}+\frac{1-i}{2}v_3}}}}_{z_7},\ \underbrace{\frac{\dfrac{\dfrac{v_4+e^{i\frac{\pi}{3}}\frac{1+5i}{6}}{e^{i\frac{\pi}{3}}+\frac{1-5i}{6}v_4}+\frac{1+6i}{7}}{1+\frac{1-6i}{7}\,\dfrac{v_4+e^{i\frac{\pi}{3}}\frac{1+5i}{6}}{e^{i\frac{\pi}{3}}+\frac{1-5i}{6}v_4}}+e^{i\pi}\,\frac{i}{6}}{e^{i\pi}-\frac{i}{6}\,\dfrac{\dfrac{v_4+e^{i\frac{\pi}{3}}\frac{1+5i}{6}}{e^{i\frac{\pi}{3}}+\frac{1-5i}{6}v_4}+\frac{1+6i}{7}}{1+\frac{1-6i}{7}\,\dfrac{v_4+e^{i\frac{\pi}{3}}\frac{1+5i}{6}}{e^{i\frac{\pi}{3}}+\frac{1-5i}{6}v_4}}}}_{z_5},$$

$$\left.\underbrace{\frac{\dfrac{\dfrac{v_6+e^{i\frac{\pi}{4}}\frac{1-4i}{5}}{e^{i\frac{\pi}{4}}+\frac{1+4i}{5}v_6}+\frac{1-7i}{8}}{1+\frac{1+7i}{8}\,\dfrac{v_6+e^{i\frac{\pi}{4}}\frac{1-4i}{5}}{e^{i\frac{\pi}{4}}+\frac{1+4i}{5}v_6}}+e^{i\frac{\pi}{3}}\,\frac{i}{4}}{e^{i\frac{\pi}{3}}-\frac{i}{4}\,\dfrac{\dfrac{v_6+e^{i\frac{\pi}{4}}\frac{1-4i}{5}}{e^{i\frac{\pi}{4}}+\frac{1+4i}{5}v_6}+\frac{1-7i}{8}}{1+\frac{1+7i}{8}\,\dfrac{v_6+e^{i\frac{\pi}{4}}\frac{1-4i}{5}}{e^{i\frac{\pi}{4}}+\frac{1+4i}{5}v_6}}}}_{z_3}\right).$$

The arrangement in coordinates of Ψ and Ψ^{-1} is given in the following table:

$$\begin{pmatrix} \Psi = \Psi_3 \circ \Psi_2 \circ \Psi_1 & \Psi^{-1} = \Psi_1^{-1} \circ \Psi_2^{-1} \circ \Psi_3^{-1} \\ \left(z_1 \overset{\Psi_1}{\to} u_4 \overset{\Psi_2}{\to} w_2 \overset{\Psi_3}{\to} v_5\right) & \left(v_1 \overset{\Psi_3^{-1}}{\to} w_7 \overset{\Psi_2^{-1}}{\to} u_2 \overset{\Psi_1^{-1}}{\to} z_6\right) \\ \left(z_2 \overset{\Psi_1}{\to} u_1 \overset{\Psi_2}{\to} w_6 \overset{\Psi_3}{\to} v_7\right) & \left(v_2 \overset{\Psi_3^{-1}}{\to} w_3 \overset{\Psi_2^{-1}}{\to} u_3 \overset{\Psi_1^{-1}}{\to} z_4\right) \\ \left(z_3 \overset{\Psi_1}{\to} u_7 \overset{\Psi_2}{\to} w_4 \overset{\Psi_3}{\to} v_6\right) & \left(v_3 \overset{\Psi_3^{-1}}{\to} w_1 \overset{\Psi_2^{-1}}{\to} u_5 \overset{\Psi_1^{-1}}{\to} z_7\right) \\ \left(z_4 \overset{\Psi_1}{\to} u_3 \overset{\Psi_2}{\to} w_3 \overset{\Psi_3}{\to} v_2\right) & \left(v_4 \overset{\Psi_3^{-1}}{\to} w_5 \overset{\Psi_2^{-1}}{\to} u_6 \overset{\Psi_1^{-1}}{\to} z_5\right) \\ \left(z_5 \overset{\Psi_1}{\to} u_6 \overset{\Psi_2}{\to} w_5 \overset{\Psi_3}{\to} v_4\right) & \left(v_5 \overset{\Psi_3^{-1}}{\to} w_2 \overset{\Psi_2^{-1}}{\to} u_4 \overset{\Psi_1^{-1}}{\to} z_1\right) \\ \left(z_6 \overset{\Psi_1}{\to} u_2 \overset{\Psi_2}{\to} w_7 \overset{\Psi_3}{\to} v_1\right) & \left(v_6 \overset{\Psi_3^{-1}}{\to} w_4 \overset{\Psi_2^{-1}}{\to} u_7 \overset{\Psi_1^{-1}}{\to} z_3\right) \\ \left(z_7 \overset{\Psi_1}{\to} u_5 \overset{\Psi_2}{\to} w_1 \overset{\Psi_3}{\to} v_3\right) & \left(v_7 \overset{\Psi_3^{-1}}{\to} w_6 \overset{\Psi_2^{-1}}{\to} u_1 \overset{\Psi_1^{-1}}{\to} z_2\right) \end{pmatrix}.$$

Remark 6

i. It is clear that due to the absence of biholomorphic equivalence between the ball and the polydisk, the Step 5 of the proposed framework is crucial, because the choice of the complex domain (ball or polydisk) for conducting an encryption process is absolutely substantial. Indeed, if he who makes the cryptanalysis decides to conduct the cryptanalysis into the ball, while the cryptographer has conducted his encryption method into the polydisk, then he will certainly come to erroneous conclusions. And conversely, if he who makes the cryptanalysis decides to work into the polydisk while the cryptographer has conducted the encryption method into the ball, then he will also infer incorrectly.

ii. However, the limitation of the reference domains exclusively in the two sets of the proposed framework (: ball and polydisk) is not binding. One could also, for instance, choose another open set that is not biholomorphic to the ball, such as an ellipsoid (see, for instance, [1, 10, 12] and [18]). However, as it has been pointed above, even with the option of the ball or the polydisk, the possibilities of cryptanalysis are severely limited, since the ball and the polydisk are not biholomorphically equivalent and the person who conducts the cryptanalysis cannot never able to know with certainty the domain of cryptanalysis.

iii. Another crucial choice that shields the result of encryption is the choice of the parameter M. The choice of this parameter determines the length of the chain of automorphisms, and hence the number of selected consecutive automorphisms. If he who makes the cryptanalysis chooses a different number of consecutive automorphisms than the number chosen by the cryptographer, then he will certainly be driven to erroneous conclusions.

iv. Determining automorphisms of the ball through effective management of Equations (`E1`) and (`E2`) is often quite difficult. For this reason, it may be preferred to use automorphisms belonging to a more manageable subset. Towards this direction, fix $a \in B^n$. According to [24], we consider the orthogonal projection P_a of $\mathbb{C}^n$ onto the subspace $[a]$ generated by a:

$$P_a(z) = \begin{cases} 0, & if\ a = 0 \\ \frac{<z,a>}{<a,a>}a, & if\ a \neq 0. \end{cases}$$

Let also $Q_a = I - P_a$ be the projection onto the orthogonal complement of $[a]$:

$$Q_a(z) = z - P_a \begin{cases} z, & if\ a = 0 \\ z - \frac{<z,a>}{<a,a>}a, & if\ a \neq 0. \end{cases}$$

Put $s_a = \left(1 - |a|^2\right)^{\frac{1}{2}}$ and define

$$\phi_a(z) := \frac{a - P_a(z) - s_a Q_a(z)}{1- < z, a >}.$$

It is easily seen that if $a \in B^n$ and $b \in B^n$, then $\phi_b \circ \phi_a$ is an automorphism of B^n that takes a to b. Put

1. $\Omega = B^n \subset \mathbb{C}^n$,
2. $\mathcal{E} = \mathcal{D} = \{\phi_a(z) := \frac{a - P_a(z) - s_a Q_a(z)}{1-<z,a>}, a \in B^n\}$ and
3. $\mathcal{K} = \{\phi = \phi_{a^{(M)}} \circ \ldots \circ \phi_{a^{(1)}} : M \in \mathbb{N}\ and\ a^{(1)}, \ldots, a^{(M)} \in B^n\ with\ a^{(i)} \neq a^{(j)}\ whenever\ i \neq j\}$. (Remind that if $c^{(i)} \neq c^{(j)}$, then $\phi_{c^{(i)}}\left(\phi_{c^{(j)}}(z)\right) = z$, for any $z \in B^n$.)

It is clear that for each keymapping $\phi \in \mathcal{K}$, there exist a mapping $F_\phi \in \mathcal{E}$ and a mapping $G_\phi \in \mathcal{D}$ such that $G_\phi(F_\phi(z)) = z$ for every $z \in B^n$. In particular, $G_\phi(F_\phi(z)) = z$ for every plaintext element $z \in \mathcal{P}$. Indeed, the finite composite function $F_\phi = \phi_{a^{(M)}} \circ \ldots \circ \phi_{a^{(2)}} \circ \phi_{a^{(1)}} \in \mathcal{E}$ is the **biholomorphic encryption chain**, while the finite composite function $G_\phi = \phi^{-1}_{a^{(1)}} \circ \phi^{-1}_{a^{(2)}} \circ \ldots \circ \phi^{-1}_{a^{(M)}} \in \mathcal{D}$ is the corresponding **biholomorphic decryption finite chain**.

2nd Case: The Unbounded Encryption Environment Let us now turn to the case where the encryption environment consists of two domains that coincide with $\mathbb{C}^n$. We may consider the set $Aut\ (\mathbb{C}^n)$ of (holomorphic) automorphisms of $\mathbb{C}^n$.

The automorphisms of the complex plan $\mathbb{C}$ are simply the affine maps $z \mapsto az + b$, $a, b \in \mathbb{C}$, $a \neq 0$. For $n > 1$, the group of automorphisms of n-dimensional complex affine space $\mathbb{C}^n$ is very large and complicated.

Automorphisms of $\mathbb{C}^n$, $n > 1$, have been studied starting with Rosay and Rudin's seminal paper [22]). In the early 1990s, Andersén and Lempert answered a question by Rosay and Rudin and showed that the subgroup generated by shears and overshears is a proper dense subgroup of $Aut\ (\mathbb{C}^n)$ [13]. A *shear* on $\mathbb{C}^n$ is an element of $Aut\ (\mathbb{C}^n)$ that is obtained by choosing a j ($1 \leq j \leq n$) and adding a holomorphic function of the other $n - 1$ variables to z_j. For instance, any map

$$F\big(z_1, \ldots, z_n\big) = \Big(\underbrace{z_1 + f\big(z_2, \ldots, z_n\big)}_{w_1}, \underbrace{z_2}_{w_2}, \ldots, \underbrace{z_n}_{w_n}\Big)$$

is a shear (in the direction of e_1) (see p. 49 in [22]). An *overshear* on $\mathbb{C}^n$ is a mapping $F : \mathbb{C}^n \to \mathbb{C}^n$ of the form

$$F\big(z_1, \ldots, z_n\big) =$$

$$\Big(\underbrace{z_1}_{w_1}, \ldots, \underbrace{z_{j-1}}_{w_{j-1}}, \underbrace{z_j e^{g(z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n)} + h\big(z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n\big)}_{w_j}, \underbrace{z_{j+1}}_{w_{j+1}}, \ldots, \underbrace{z_n}_{w_n}\Big),$$

where g and h are entire functions of $\mathbb{C}^{n-1}$. For instance, an overshear on $\mathbb{C}^2$ is an automorphism of the form $F\big(z_1, z_2\big) = \Big(e^{g(z_2)} z_1 + h(z_2, z_2\Big)$ or $F\big(z_1, z_2\big) = \Big(z_1, e^{g(z_1)} z_2 + h(z_1\Big)$ where g and h are entire functions of one variable. If $g \equiv 0$, the overshear becomes a shear.

A First Framework for Constructing Biholomorphic Cryptosystems into Unbounded Encryption Environment

1. Let
   (a) $\mathcal{P}$ be the fixed finite set of possible plaintexts;
   (b) Σ be a (finite) source alphabet;
   (c) σ be an arbitrary mapping $\sigma : \Sigma \to \mathbb{C}$.
2. Consider a biholomorphic code on $\mathbb{C}$.
3. Applying the biholomorphic codification, every plaintext χ in the set $\mathcal{P}$ is represented as a point

$$\zeta = \zeta^{(\chi)} \in \mathbb{C}^n.$$

4. Choose $\mathcal{E} = \mathcal{D} = \{F : composition\ of\ overshears\ of\ \mathbb{C}^n\}$.
5. To define the keyspace $\mathcal{K}$, do the following steps.
   (a) Choose a finite subset $\mathcal{N}$ of the set $\mathbb{N}$ of natural numbers;
   (b) For any $N \in \mathcal{N}$
      i. choose two sets $\mathcal{G}_N$ and $\mathcal{H}_N$ of entire functions;
      ii. for $m = 1, 2, \ldots, N$,
         A. choose $j_m \in \{1, 2, \ldots, n\}$, $g_m \in \mathcal{G}_N$ and $h_m \in \mathcal{H}_N$;
         B. define the overshear

$$F_{j_m}^{(g_m,h_m)}\big(z_1,\ldots,z_n\big) = \Bigg(\underbrace{z_1}_{w_1},\ldots,\underbrace{z_{j_m-1}}_{w_{j_m-1}},$$

$$\underbrace{z_{j_m}e^{g_m\left(z_1,\ldots,z_{j_m-1},z_{j_m+1},\ldots,z_n\right)}+h_m\big(z_1,\ldots,z_{j_m-1},z_{j_m+1},\ldots,z_n\big)}_{w_{j_m}},$$

$$\underbrace{z_{j_m+1}}_{w_{j_m+1}},\ldots,\underbrace{z_n}_{w_n}\Bigg);$$

C. consider the composition $F_{j_N}^{(g_N,h_N)}\circ\ldots\circ F_{j_2}^{(g_2,h_2)}\circ F_{j_1}^{(g_1,h_1)}$.

(c) Define

$$\mathcal{K} := \{\Big(F_{j_N}^{(g_N,h_N)}\circ\ldots\circ F_{j_2}^{(g_2,h_2)}\circ F_{j_1}^{(g_1,h_1)};\mathcal{G}_N;\mathcal{H}_N\Big): g_1,\ldots,g_N\in\mathcal{G}_N, h_1,\ldots,h_N\in\mathcal{H}_N, N\in\mathcal{N}\}.$$

6. It is clear that for each keymapping

$$F = F_{j_N}^{(g_N,h_N)}\circ\ldots\circ F_{j_2}^{(g_2,h_2)}\circ F_{j_1}^{(g_1,h_1)},$$

we have $F\in\mathcal{E}$ and there exists a mapping $G = F_F\in\mathcal{D}$ such that $G(F(z)) = z$ for every $z\in\mathbb{C}^n$. In particular, $G(F(\zeta^{(\chi)})) = \zeta^{(\chi)}$ for every plaintext $\chi\in\mathcal{P}$. Indeed, any finite composition $F = F_{j_N}^{(g_N,h_N)}\circ\ldots\circ F_{j_2}^{(g_2,h_2)}\circ F_{j_1}^{(g_1,h_1)}\in\mathcal{E}$ is a **biholomorphic encryption chain**, while the finite composition $G = F_{j_1}^{(g_1,h_1)^{-1}}\circ\ldots\circ F_{j_{N-1}}^{(g_{N-1},h_{N-1})^{-1}}\circ F_{j_N}^{(g_N,h_N)^{-1}}\in\mathcal{D}$ is the corresponding **biholomorphic decryption finite chain**.

Let us give a concrete example.

Example 6 By considering the extended ASCII code, each character of the word `Ichor` corresponds to the sequence of seven numbers `73 99 104 111 114`. Using the biholomorphic encoding rule with length 4 of Example 2, we see that

a). the integer `73` is coded to the complex number

$$\xi_{\mathtt{I}} = \frac{-4727614320+i12482969202}{22974429922}\approx -(0.2057772199811)+i(0.5433418476272);$$

b). the integer `99` is coded to the complex number

$$\xi_{\mathtt{c}} = \frac{-5950985040+i12632433842}{29184180482}\approx -(0.203911329416)+i(0.4328521011509);$$

c). the integer `104` is coded to the complex number

$$\xi_{\mathtt{h}} = \frac{-6141757440 + i9366236880}{29184180482} \approx -(0.203911329416) + i(0.4328521011509);$$

d). the integer `111` is coded to the complex number

$$\xi_{\mathtt{o}} = \frac{-6387543729 + i12499079411}{32238656741} \approx -(0.1981330605774)+i(0.3877047208082);$$

e). the integer `114` is coded to the complex number

$$\xi_{\mathtt{r}} = \frac{-6473850240 + i12437043872}{32967367712} \approx -(0.1963714633378)+i(0.3772531668482).$$

We take

$$\mathcal{E} = \mathcal{D} = \left\{F : compositions of overshears of \mathbb{C}^5\right\} \text{ and } N = 2.$$

Choose

$\mathcal{G}_N = \mathbb{P}(\mathbb{C}) =$ the space of holomorphic polynomials in $\mathbb{C}$,

$\mathcal{H}_N = \mathcal{O}(\mathbb{C}) =$ the space of entire functions in $\mathbb{C}$ and

$j_1 = 3$, $j_2 = 1$.

To define the keymapping, we take

$$g_1(z_1, z_2, z_4, z_5),\ \ g_2(w_2, w_3, w_4, w_5) \in \mathcal{G}_N = \mathbb{P}(\mathbb{C}) and$$
$$h_1(z_1, z_2, z_4, z_5),\ \ h_2(w_2, w_3, w_4, w_5) \in \mathcal{H}_N = \mathcal{O}(\mathbb{C})$$

such that

- $$g_1(z_1, z_2, z_4, z_5) = \underbrace{5\sum_{\nu=0}^{5}(-2)^{\nu}\frac{(5+\nu-1)!}{(5-\nu)!(2\nu)!}(1-z_1)^{\nu}}_{the\ Chebyshev\ Polynomial\ of\ degree\ 5\ in\ z_1} \times \underbrace{\frac{5z_2^3 - 3z_2}{2}}_{the\ Legendre\ Polynomial\ of\ degree\ 3\ in\ z_2} \times \underbrace{\frac{z_4^2 - 4z_4 + 2}{2}}_{the\ Laguerre\ Polynomial\ of\ degree\ 2\ in\ z_4}$$

$$\times \qquad \underbrace{16z_5^4 - 48z_5^2 + 12}_{\textit{the Hermite Polynomial of degree } 4 \textit{ in } z_5}$$

- $h_1(z_1, z_2, z_4, z_5) = \sin(z_1 z_5) + \cos\left(z_2^{\frac{1}{4}}\right) + \cosh\left(z_4^{\frac{1}{4}}\right)$
- $g_2(w_2, w_3, w_4, w_5) = \frac{\left(1+i\frac{w_2}{3}\right)^3 + \left(1-i\frac{w_2}{3}\right)^3}{2} w_3 \left(1 - \frac{(w_4-6)^2}{4}\right)^4 (w_5 - 18)$ and
- $h_2(w_2, w_3, w_4, w_5) = \exp\left(e^{-w_2}\right) \cos\left(w_3^{\frac{1}{8}}\right) \left(\sum_{\nu=0}^{20} (2)^{-\nu^2} w_4^{\nu}\right) \left(\sum_{\nu=2}^{20} (\nu \ln \nu)^{-\nu} w_5^{\nu}\right).$

Notice that the Chebyshev polynomial $g_1(z_1, z_2, z_4, z_5)$ is also equal to

$$\frac{\left(z_1 - (z_1^2 - 1)^{\frac{1}{2}}\right)^5 + \left(z_1 + (z_1^2 - 1)^{\frac{1}{2}}\right)^5}{2}.$$

Let $F_3^{(g_1,h_1)}$ and $F_1^{(g_2,h_2)}$ be the overshears

$$F_3^{(g_1,h_1)}(z_1, z_2, z_3, z_4, z_5)$$
$$= \Bigg(\underbrace{z_1}_{w_1}, \underbrace{z_2}_{w_2}, \underbrace{z_3 e^{g_1(z_1,z_2,z_4,z_5)} + h_1(z_1, z_2, z_4, z_5)}_{w_3}, \underbrace{z_4}_{w_4}, \underbrace{z_5}_{w_5} \Bigg),$$
$$F_1^{(g_2,h_2)}(w_1, w_2, w_3, w_4, w_5)$$
$$= \Bigg(\underbrace{w_1 e^{g_2(w_2,w_3,w_4,w_5)} + h_2(w_2, w_3, z_4, z_5)}_{v_1}, \underbrace{w_2}_{v_2}, \underbrace{w_3}_{v_3}, \underbrace{w_4}_{v_4}, \underbrace{w_5}_{v_5} \Bigg).$$

According to this option, our biholomorphic encryption chain is given by the composition

$$\left(F_1^{(g_2,h_2)} \circ F_3^{(g_1,h_1)}\right)(z_1, z_2, z_3, z_4, z_5) = (v_1, v_2, v_3, v_4, v_5)$$

with

- $v_1 = z_1 \exp\left(g_2\left(z_2, z_3 e^{g_1(z_1,z_2,z_4,z_5)} + h_1(z_1, z_2, z_4, z_5), z_4, z_5\right)\right)$
  $+h_2\left(z_2, z_3 e^{g_1(z_1,z_2,z_4,z_5)} + h_1(z_1, z_2, z_4, z_5), z_4, z_5\right),$
- $v_2 = z_2,$
- $v_3 = z_3 e^{g_1(z_1,z_2,z_4,z_5)} + h_1(z_1, z_2, z_4, z_5),$
- $v_4 = z_4$ and
- $v_5 = z_5.$

Especially,

- $v_1 = z_1 \exp\Bigg(\frac{\left(1+i\frac{w_2}{3}\right)^3 + \left(1-i\frac{w_2}{3}\right)^3}{2}$
  $\times \Bigg(z_3 \exp\left\{ \frac{1}{8} \left[\left(z_1 - (z_1^2 - 1)^{\frac{1}{2}}\right)^5 + \left(z_1 + (z_1^2 - 1)^{\frac{1}{2}}\right)^5 \right] \right.$

$\left(5z_2^3 - 3z_2\right)\left(z_4^2 - 4z_4 + 2\right)\left(16z_5^4 - 48z_5^2 + 12\right)\Big\} + \sin(z_1 z_5)\Big) + \cos\left(z_2^{\frac{1}{4}}\right) +$
$\cosh\left(z_4^{\frac{1}{4}}\right)$

$$\left(\left(1 - \frac{(z_4-6)^2}{4}\right)^4 (w_5 - 18) + \sin(z_1 z_5) + \cos\left(z_2^{\frac{1}{4}}\right) + \cosh\left(z_4^{\frac{1}{4}}\right)\right)$$

$+\exp\left(-z_2\right)\left(\sum_{\nu=0}^{20} 2^{-\nu^2} z_4^{\nu}\right)\left(\sum_{\nu=2}^{20} (\nu ln\nu)^{-\nu} z_5^{\nu}\right)$

$\times\cos\left(\left\{z_3 \exp\left(\frac{\left[z_1-\left(z_1^2-1\right)^{\frac{1}{2}}\right]^5+\left[z_1+\left(z_1^2-1\right)^{\frac{1}{2}}\right]^5}{8}\left(5z_2^3 - 3z_2\right)\left(z_4^2 - 4z_4 + \right.\right.\right.\right.$
$\left.\left.\left.\left.2\right)\left(16z_5^4 - 48z_5^2 + 12\right)\right) + \sin(z_1 z_5) + \cos\left(z_2^{\frac{1}{4}}\right) + \cosh\left(z_4^{\frac{1}{4}}\right)\right\}^{\frac{1}{8}}\right),$

- $v_3 = z_3 \exp\left\{\frac{1}{8}\left[\left(z_1 - \left(z_1^2 - 1\right)^{\frac{1}{2}}\right)^5 + \left(z_1 + \left(z_1^2 - 1\right)^{\frac{1}{2}}\right)^5\right] \times \left(5z_2^3 - 3z_2\right)\left(z_4^2 - \right.\right.$
$\left.\left.4z_4 + 2\right)\left(16z_5^4 - 48z_5^2 + 12\right)\right\} + \sin(z_1 z_5) + \cos\left(z_2^{\frac{1}{4}}\right) + \cosh\left(z_4^{\frac{1}{4}}\right).$

In particular, we have

$$\left(F_1^{(g_2,h_2)} \circ F_3^{(g_1,h_1)}\right)(\xi_{\mathtt{I}}, \xi_{\mathtt{c}}, \xi_{\mathtt{h}}, \xi_{\mathtt{o}}, \xi_{\mathtt{r}}) = (-0.116019521095477$$
$$-0.149956792269831i, \xi_{\mathtt{c}}, 1.202171374296308e + 0.09i, \xi_{\mathtt{o}}, \xi_{\mathtt{r}}).$$

Remark 7 In this example, they are encrypted only two (:$\xi_{\mathtt{I}}$ and $\xi_{\mathtt{h}}$) of the coded values of 5 characters of the word **Ichor**. Of course, a complete form of encryption should also include the encryption of the other three coded values (: $\xi_{\mathtt{c}}$, $\xi_{\mathtt{o}}$ and $\xi_{\mathtt{r}}$). Such a process would require the composition of at least five over shears, to cover all the characters of the word. However, for efficient practical description of the proposed framework, it was preferred an **indicative** application showing the typical target structure of the frame work.

The use of overshears seems that it significantly complicates the construction of mathematical formulas. For this reason, it arises the substantial need of finding an alternative method for easier identification of many automorphisms of $\mathbb{C}^n$. To this end, a good idea is to use polynomial automorphisms. One of the simplest forms of such automorphisms is described by the lower triangular polynomial mappings (see p.14 in [20]).

Definition 7 A mapping $F = \left(f_1, \ldots, f_n\right) : \mathbb{C}^n \to \mathbb{C}^n$ is called a **lower triangular polynomial mapping** if there are constants $s_j \in \mathbb{C}$ and polynomial mappings $p_2, p_3, \ldots, p_n$ with $p_j = (0) = 0$ such that

$$f_1(z) = s_1 z_1 \text{ and } f_j(z) = s_j z_j + p_j\left(z_1, \ldots, z_{j-1}\right)$$

for every $j = 2, \ldots, n$.

It is easy to demonstrate the following result.

Proposition 8 *If* $0 < |s_j| < 1$ *for every* j*, then* F *is a polynomial automorphism of* $\mathbb{C}^n$ *with an attracting fixed point at the origin.*

Using Proposition 8, we can give the following general framework for the construction of a finite biholomorphic cryptosystem into the encryption environment in which both the two domains coincide with whole of $\mathbb{C}^n$.

A Second Framework for Constructing Biholomorphic Cryptosystems into Unbounded Encryption Environment

1. Let
   (a) $\mathcal{P}$ be the fixed finite set of possible plaintexts;
   (b) Σ be a (finite) source alphabet;
   (c) σ be an arbitrary mapping $\sigma : \Sigma \to \mathbb{C}$.
2. Consider a biholomorphic code on $\mathbb{C}$.
3. Applying the biholomorphic codification, every plaintext χ in the set $\mathcal{P}$ is represented as a point

$$\zeta = \zeta^{(\chi)} \in \mathbb{C}^n.$$

4. Choose $\mathcal{E} = \mathcal{D} = \{F : \mathit{composition\ of\ lower\ triangular\ polynomial\ mappings\ of\ } \mathbb{C}^n\}$.
5. To define the keyspace $\mathcal{K}$, do the following steps.
   (a) For every $j = 1, 2, \ldots, n$, choose a set $S_j \subset \{s_j \in \mathbb{C} : 0 < |s_j| < 1\}$
   (b) For any $j = 23, , \ldots, n$, choose a set $\mathbb{P}_j(\mathbb{C}^{j-1})$ of polynomial functions p_j in $\mathbb{C}^{j-1}$ with $p_j(0) = 0$
   (c) Define

$$f_1(z) = s_1 z_1, \text{ with } s_1 \in S_1$$

   (d) For every $j = 2, 3, , \ldots, n$, define

$$f_j(z) = s_j z_j + p_j\left(z_1, z_2, \ldots, z_{j-1}\right), \text{ with } s_j \in S_j \text{ and } p_j \in \mathbb{P}_j(\mathbb{C}^{j-1})$$

   (e) Define

$$\mathcal{K} := \left\{F^{(N)} \circ \ldots \circ F^{(1)} : F^{(\nu)} = \left(f_1^{(\nu)}, \ldots, f_n^{(\nu)}\right) : \mathbb{C}^n \to \mathbb{C}^n : f_1^{(\nu)}(z) = s_1^{\nu} z_1 \text{ with } s_1 \in S_1 \text{ and } f_j^{(\nu)}(z) = s_j^{\nu} z_j + p_j^{\nu} \in \mathbb{P}_j(\mathbb{C}^{j-1})\right\}.$$

6. It is clear that for each keymapping $F^{(N)} \circ \ldots \circ F^{(1)}$, there is a $F \in \mathcal{E}$ and a $G \in \mathcal{D}$ such that $F = F^{(N)} \circ \ldots \circ F^{(1)}$ and $G(F(z)) = z$ for every $z \in \mathbb{C}^n$. In particular, $G(F(\zeta^{(\chi)})) = \zeta^{(\chi)}$ for every plaintext $\chi \in \mathcal{P}$. Indeed, any mapping $F = F^{(N)} \circ \ldots \circ F^{(1)} \in \mathcal{E}$ is a **biholomorphic encryption chain**, while its inverse $G = \left(f_1^{-1}, \ldots, f_n^{-1}\right) \in \mathcal{D}$ is the corresponding **biholomorphic decryption chain**.

Let us give a simple example.

Example 7 By considering the extended ASCII code, each character of the word `Information` corresponds to the sequence of seven numbers `73 110 102 111 114 109 97 116 105 111 110`. Using the biholomorphic encoding rule with length 4 of Example 2, we see that

a). the integer `73` is coded to the complex number

$$\xi_{\mathtt{I}} = \frac{-4727614320 + i\,12482969202}{22974429922} \approx -(0.2057772199811) + i(0.5433418476272);$$

b). the integer `110` is coded to the complex number

$$\xi_{\mathtt{n}} = \frac{-6349200000 + i\,12509700000}{31946260000} \approx -(0.1987462695164) + i(0.3915857443093);$$

c). the integer `102` is coded to the complex number

$$\xi_{\mathtt{f}} = \frac{-6067351680 + i\,12610116512}{29930434592} \approx -(0.2027151213375) + i(0.4213141801613);$$

d). the integer `111` is coded to the complex number

$$\xi_{\mathtt{o}} = \frac{-6387543729 + i\,12499079411}{32238656741} \approx -(0.1981330605774) + i(0.3877047208082);$$

e). the integer `114` is coded to the complex number

$$\xi_{\mathtt{r}} = \frac{-6473850240 + i\,12437043872}{32967367712} \approx -(0.1963714633378) + i(0.3772531668482);$$

f). the integer `109` is coded to the complex number

$$\xi_{\mathtt{m}} = \frac{-6316305840 + i\,12525519762}{31692315202} \approx -(0.1993008652016) + i(0.395226172864);$$

g). the integer `97` is coded to the complex number

$$\xi_{\mathtt{a}} = \frac{-5870308080 + i12642693042}{28689775522} \approx -(0.2046132454225)+i(0.4406689425752);$$

h). the integer `116` is coded to the complex number

$$\xi_{\mathtt{t}} = \frac{-6531932160 + i12395077632}{33481019392} \approx -(0.1950935867132)+i(0.3702120740972);$$

i). the integer `105` is coded to the complex number

$$\xi_{\mathtt{i}} = \frac{-6347250000 + i13104281250}{31376171250} \approx -(0.2022952370264)+i(0.4176507434762);$$

j). the integer `111` is coded to the complex number

$$\xi_{\mathtt{o}} = \frac{-6387543729 + i12499079411}{32238656741} \approx -(0.1981330605774)+i(0.3877047208082);$$

k). the integer `110` is coded to the complex number

$$\xi_{\mathtt{n}} = \frac{-6349200000 + i12509700000}{31946260000} \approx -(0.1987462695164)+i(0.3915857443093).$$

We take

$$\mathcal{E} = \mathcal{D} =$$

$$\left\{ F : \textit{composition of lower triangular polynomial mappings of } \mathbb{C}^{11} \right\}$$

and we define the components $(f_1, \ldots, f_{11})$ of the biholomorphic encryption chain F as follows:

- $f_1(z) = 0.577350269189626z_1$,
- $f_2(z) = 0.774596669241483z_2 + 12155z_1^9 - 25740z_1^7 + 18018z_1^5 - 4620z_1^3 + 315z_1$,
- $f_3(z) = 0.861136311594053z_3 + 13z_1^3 - 9z_1^2z_2 + z_2^2 + 2z_2^3$,
- $f_4(z) = 0.906179845938664z_4 + z_1^2z_2^2 + z_1^2z_3^2 + z_2^2z_3^2 + z_1^2z_2z_3 + z_1z_2^2z_3 + z_1z_2z_3^2$,
- $f_5(z) = 0.932469514203152z_5 + 6z_1^5z_3 + z_1^2z_2 - 2z_1z_2^2z_4^2 + 2z_2^3 - z_3^4$,
- $f_6(z) = 0.949107912342759z_6 + 12z_1^2z_3^6z_4^3z_5^8 + 10z_1^3z_2^3z_3^3z_4^3z_5^2 + z_1^5z_2^2z_3^4z_4^6 + z_1^7z_2^2z_3^2z_4^2z_5^3 - 6z_1^6z_2^2z_3^3z_4^6z_5^6$,
- $f_7(z) = 0.960289856497536z_7 + 8z_2^3z_3^2z_5^6z_6^5 + 27z_1^4z_2^9z_4^2 - 18z_1^5z_2z_3^9z_4^2z_6^6 + z_6^7$,

- $f_8(z) = 0.968160239507626 z_8 + 6z_1^6 z_2 z_3^9 z_4^6 z_5 z_6^{12} z_7^{10} - 999 z_1 z_5^3 z_7^7 + 16 z_2^2 z_4^4 z_6^6 - z_1^{11} z_2^3 z_3^5 z_4^{19} z_5^7 z_6^{13} z_7^{17}$,
- $f_9(z) = 0.973906528517172 z_9 + z_1^{10} z_2^5 z_3^6 z_6^8 z_7^{20} z_8 - z_1 z_2^{22} z_3 z_4^{31} z_6^{12} z_8 + z_1^{25} z_3^{18} z_4 + z_4^{30} z_5^{21} z_6^{16} z_8^2 - z_1 z_2 z_4^3 z_5 z_7^4 z_8 - z_2^{14} z_6^{19}$,
- $f_{10}(z) = 0.978228658146057 z_{10} + z_2^{23} z_9^{24} + z_4^{26} z_6^{28} z_7^{29} + z_1^{10} z_3^6 z_5^8 - z_4^7 z_7^9 z_8^5 z_9^4 + z_1^3 z_3^{15} z_6^2 z_8^{16} z_9^{18}$ and
- $f_{11}(z) = 0.981560634246719 z_{11} - 90090 z_1^6 z_5^9 z_8^5 z_{10}^4 + 128 z_1^3 z_4^{10} z_7^{12} z_9^{14} z_{10}^{18}$.

Here, we have taken s_j to be the greater zero of the Legendre polynomial of degree $j+1$ (see, for instance, [15]):

- $s_1 = 0.577350269189626$,
- $s_2 = 0.774596669241483$,
- $s_3 = 0.861136311594053$,
- $s_4 = 0.906179845938664$,
- $s_5 = 0.932469514203152$,
- $s_6 = 0.949107912342759$,
- $s_7 = 0.960289856497536$,
- $s_8 = 0.968160239507626$,
- $s_9 = 0.973906528517172$,
- $s_{10} = 0.978228658146057$ and
- $s_{11} = 0.981560634246719$.

Further, we got

- $p_2(z_1) = 12155 z_1^9 - 25740 z_1^7 + 18018 z_1^5 - 4620 z_1^3 + 315 z_1$,
- $p_3(z_1, z_2) = 13 z_1^3 - 9 z_1^2 z_2 + z_2^2 + 2 z_2^3$,
- $p_4(z_1, z_2, z_3) = z_1^2 z_2^2 + z_1^2 z_3^2 + z_2^2 z_3^2 + z_1^2 z_2 z_3 + z_1 z_2^2 z_3 + z_1 z_2 z_3^2$ (:the Schur polynomial) $S_{(2,2,0)}(z_1, z_2, z_3)$),
- $p_5(z_1, z_2, z_3, z_4) = 6 z_1^5 z_3 + z_1^2 z_2 - 2 z_1 z_2^2 z_4^2 + 2 z_2^3 - z_3^4$,
- $p_6(z_1, z_2, z_3, z_4, z_5) = 12 z_1^2 z_3^6 z_4^3 z_5^8 + 10 z_1^3 z_2^3 z_3^3 z_4^3 z_5^2 + z_1^5 z_2^2 z_3^4 z_4^6 + z_1^7 z_2^2 z_3^2 z_4^2 z_5^3 - 6 z_1^6 z_2^2 z_3^3 z_4^6 z_5^6$,
- $p_7(z_1, z_2, z_3, z_4, z_5, z_6) = 8 z_2^3 z_3^2 z_5^6 z_6^5 + 27 z_1^4 z_2^9 z_4^2 - 18 z_1^5 z_2 z_3^9 z_4^2 z_6^6 + z_6^7$,
- $p_8(z_1, z_2, z_3, z_4, z_5, z_6, z_7) = 6 z_1^6 z_2 z_3^9 z_4^6 z_5 z_6^{12} z_7^{10} - 999 z_1 z_5^3 z_7^7 + 16 z_2^2 z_4^4 z_6^6 - z_1^{11} z_2^3 z_3^5 z_4^{19} z_5^7 z_6^{13} z_7^{17}$,
- $p_9(z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8) = z_1^{10} z_2^5 z_3^6 z_6^8 z_7^{20} z_8 - z_1 z_2^{22} z_3 z_4^{31} z_6^{12} z_8 + z_1^{25} z_3^{18} z_4 + z_4^{30} z_5^{21} z_6^{16} z_8^2 - z_1 z_2 z_4^3 z_5 z_7^4 z_8 - z_2^{14} z_6^{19}$,
- $p_{10}(z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8, z_9) = z_2^{23} z_9^{24} + z_4^{26} z_6^{28} z_7^{29} + z_1^{10} z_3^6 z_5^8 - z_4^7 z_7^9 z_8^5 z_9^4 + z_1^3 z_3^{15} z_6^2 z_8^{16} z_9^{18}$ and
- $p_{11}(z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8, z_9, z_{10}) = -90090 z_1^6 z_5^9 z_8^5 z_{10}^4 + 128 z_1^3 z_4^{10} z_7^{12} z_9^{14} z_{10}^{18}$.

Making substitutions and performing calculations, we obtain

$$F\big(\xi_{\mathtt{I}}, \xi_{\mathtt{n}}, \xi_{\mathtt{n}}, \xi_{\mathtt{f}}, \xi_{\mathtt{o}}, \xi_{\mathtt{r}}, \xi_{\mathtt{m}}, \xi_{\mathtt{a}}, \xi_{\mathtt{t}}, \xi_{\mathtt{i}}, \xi_{\mathtt{o}}, \xi_{\mathtt{n}},\big)$$

$$= \Big(f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9, f_{10}, f_{11}\Big)$$

$$\big(\xi_{\mathtt{I}}, \xi_{\mathtt{n}}, \xi_{\mathtt{n}}, \xi_{\mathtt{f}}, \xi_{\mathtt{o}}, \xi_{\mathtt{r}}, \xi_{\mathtt{m}}, \xi_{\mathtt{a}}, \xi_{\mathtt{t}}, \xi_{\mathtt{i}}, \xi_{\mathtt{o}}, \xi_{\mathtt{n}},\big)$$

$$\begin{aligned}
= \big(&-0.118805533349187 + 0.313698561989525i,\\
&-2.343204437536524e + 003 - 2.526159691928734e + 002i,\\
&0.894305471781336 - 0.516891292632848i,\\
&-0.223358439219717 + 0.706281472441953i,\\
&0.105276770297329 + 0.188338893707562i,\\
&-0.189399688549816 + 0.375136254919529i,\\
&-0.196366106091943 + 0.422849701303160i,\\
&0.094640729838655 + 0.395579708885873i,\\
&-0.196813809147348 + 0.406802504494881i,\\
&-0.193819477038854 + 0.379263835097444i,\\
&-0.194944665196051 + 0.385079778780525i\big).
\end{aligned}$$

6 Dynamics of Biholomorphic Cryptosystems

The idea of considering biholomorphic cryptosystems is not new. As a concept, the biholomorphic cryptosystem is nested in the meaning of the chain of a sequence of biholomorphic mappings. Already, in 2005, Han Peters in his doctoral thesis examined the dynamic behaviour of the composition of a sequence of automorphisms, in the special case in which each mapping which participates in the composition has a single attracting fixed point. In this section, we discuss the generalization of the results of Han Peters.

Peters having as a springboard earlier work of Rudin and Rosay raised the following question. Let $f_0, f_1, \ldots$ be a sequence of automorphisms of a complex manifold all having a single attracting fixed point. Under what conditions is the basin of attraction biholomorphically equivalent to a complex Euclidean space? Here, by a **basin of attraction**, it is meant the set of p points whose (non-autonomous) orbits converge to fixed point. This question was motivated by a question about stable manifolds. A stable manifold is a generalization of a basin of attraction to the case where there is not a fixed point. Peters proved a more general proposition.

Theorem 6 *A stable manifold is always biholomorphic to complex Euclidean space if the following conjecture holds:Let $f_0, f_1, \ldots$ be a sequence of automorphisms of $\mathbb{C}^n$ that fix the origin. Assume that there exist $a, b \in \mathbb{R}$ with $0 < a < b < 1$ so that for any z in the unit ball and any $k \in \mathbb{N}$ the following holds:*

$$\text{(C)} \qquad a|z| < |f_k(z)| < b|z|.$$

Then the basin of attraction of the sequence $f_0, f_1, \ldots$ *is biholomorphic to* $\mathbb{C}^n$.

Several examples show that a basin of attraction of a sequence of biholomorphic mappings is **not** biholomorphic to $\mathbb{C}^n$ unless some assumptions are made on the rate at which different orbits converge to the attracting fixed point. However, it is showed that given any sequence of automorphisms with a common attracting fixed point, the basin of attraction is biholomorphic to $\mathbb{C}^n$ if the mappings are repeated often enough.

In what follow we will discuss the extension of the results of Peters in the case of biholomorphic cryptosystems. To this end, without loss of generality and by expanding the interpretation of Definition 2, we can extend the concept of a (finite) holomorphic cryptosystem in the case of an *infinite* encryption chain.

Definition 8 Let Ω be a domain in the $\mathbb{C}^n$.

An **infinite biholomorphic cryptosystem** is a four-tuple $(\mathcal{P}, \mathcal{K}, \mathcal{E}, \mathcal{D})$, where the following conditions are satisfied.

1. $\mathcal{P}$ is a fixed finite set of possible **plaintexts** embedded into Ω.
2. $\mathcal{E}$ is the set of **infinite biholomorphic encryption rules** on Ω with indeterminate range, that is a subset of the set of compositions $F = \ldots \circ f_M \circ \ldots \circ f_1 \circ f_0$ of an infinite number of biholomorphic mappings $f_\alpha : U_\alpha \to \mathbb{C}^n$ (U_α=open subset of $\mathbb{C}^n$ and $f_\alpha(U_\alpha) \cap U_{(\alpha+1)} \neq \varnothing$), such that $U_0 = \Omega$. Given an $F \in E$ and a $M \in \mathbb{N}$, the finite sequence $F^{(M)} = f_M \circ \ldots \circ f_1 \circ f_0$ defines a **biholomorphic encryption rule on Ω of length M subordinate to the chain F**.
3. $\mathcal{D}$ is the set of **infinite biholomorphic decryption chains with range in** Ω, that is a subset of compositions $G = \ldots \circ g_N \circ \ldots \circ g_1 \circ g_0$ of an infinite number of biholomorphic mappings $g_\beta : V_\beta \to \mathbb{C}^n$ (V_β=open subset of $\mathbb{C}^n$ and $g_\beta\left(V_\beta\right) \cap V_{(\beta+1)} \neq \varnothing$),with the following property:

   For every biholomorphic encryption rule $F^{(M)}$ subordinate to an $F \in E$, there is a $G \in D$ and a section $G^{(N)} = g_N \circ \ldots \circ g_1 \circ g_0$ of G such that

   $$\left(G^{(N)} \circ F^{(M)}\right)(z) = z \text{ whenever } z \in \Omega.$$

   The section $G^{(N)}$ is said to be a biholomorphic decryption rule on Ω of length N subordinate to the chain G.

4. $\mathcal{K}$ is a subset (not necessarily finite) of the set $u = u_N \circ \ldots \circ u_1 \colon u_\alpha$ is holomorphic mapping of U_α, $U_0 = \Omega$, $u_\alpha(U_\alpha) \subset U_{\alpha+1} \subset \mathbb{C}^n$, $N \in \mathbb{N}$. The elements of $\mathcal{K}$ are the **keymappings**, while $\mathcal{K}$ is the **keyspace**.
5. For each $u \in \mathcal{K}$, there exist a biholomorphic encryption rule $F^{(M),u}$ of length M subordinate to some encryption chain $F \in E$ and a biholomorphic decryption rule $G^{(N),u}$ of length N subordinate to some decryption chain $G \in D$ such that

   $$G^{(M),u}\left(F^{(M),u}(z)\right) = z \text{ for every plaintext element } z \in \mathcal{P}.$$

Definition 9 Let $F = \ldots \circ f_M \circ \ldots \circ f_1 \circ f_0$ be an infinite biholomorphic encryption chain on an open domain $\Omega \subset \mathbb{C}^n$. If $z \in \Omega$, the **orbit of the point** z is the sequence of points

$$\bigcirc^+(z) := \left\{z, F_1(z), F_2(z), \ldots, F_M(z), \ldots\right\},$$

where we have used the notation $F^{(M)} := f_M \circ \ldots \circ f_1 \circ f_0$.

Recall that if $|\cdot|$ is any norm in $\mathbb{C}^n$, the numbers

$$||A|| := sup_{z \in \mathbb{C}^n \backslash 0} \frac{|A \cdot z|}{|z|}$$

define a norm in the set of $n \times n$ matrices $A = \left(a_{(i,j)}\right) \in \mathbb{C}$. If, for instance, $|z| = \sum_{j=1}^n |z_j|$, then $||A|| = max_{1 \leq j \leq n} \sum_{i=1}^n |a_{i,j}|$; if $|z| = \left(|z_j|^2\right)^{\frac{1}{2}}$, then $||A|| = \varrho\left(A^T \cdot A\right)^{\frac{1}{2}}$, where $\varrho(\cdot)$ is a notation for the spectral radius; and, if $|z| = max_{1 \leq j \leq n} |z_j|$, then $||A|| = max_{1 \leq i \leq n} \sum_{j=1}^n |a_{i,j}|$.

Definition 10 Let $F = \ldots \circ f_M \circ \ldots \circ f_1 \circ f_0$ be an infinite biholomorphic encryption chain on an open domain $\Omega \subset \mathbb{C}$ and let $z_0 \in \Omega$. Let $||\cdot||$ be any norm in the set of $n \times n$ matrices. Suppose the set

$$\Upsilon_{z_0} := \left\{\left[f_M\right]' |_{(f_{M-1} \circ \ldots \circ f_1 \circ f_0)(z_0)}: M = 0, 1, 2, \ldots\right\}$$

is uniformly bounded.

i. We say that z_0 is **attracting**, if there is a positive constant $\lambda < 1$ such that the modulus of all eigenvalues of $\left[f_M\right]' |_{(f_{M-1} \circ \ldots \circ f_1 \circ f_0)(z_0)}$ is strictly smaller than λ, whenever $M = 1, 2, \ldots$; the orbit $\bigcirc^+(z_0)$ of z_0 is called an **attracting cycle**.

ii. We say that z_0 is **repelling**, if there is a positive constant $\mu > 1$ such that the modulus of all eigenvalues of $\left[f_M\right]' |_{(f_{M-1} \circ \ldots \circ f_1 \circ f_0)(z_0)}$ is strictly larger than μ, whenever $M = 1, 2, \ldots$; the orbit $\bigcirc^+(z_0)$ of z_0 is called a **repelling cycle**.

Proposition 9 *Under the assumptions of Definition 10, the point $z_0 \in \Omega$ is attracting if and only if there exists a neighbourhood $\mathcal{N}_{z_0}$ of z_0 such that the orbit of any $z \in \mathcal{N}_{z_0}$ converges to z_0. The set of all points $z \in \Omega$ whose orbits converge to z_0 is called the* ***basin of attraction*** *of F at z_0 in Ω:*

$$\mathfrak{B}_{z_0}(F) := \left\{z \in \Omega : F^{(M)}(z) \overset{M \to \infty}{\to} z_0\right\}.$$

Proof Choosing local coordinates, we can assume that $z_0 = 0$. Since the set $\Upsilon_0 := \Upsilon_{z_0}$ is bounded, any left-infinite product

$$\sqcap := \ldots \left[f_{M_k}\right]' |_{(f_{M_k-1} \circ \ldots \circ f_0)(0)} \cdots \left[f_{M_2}\right]' |_{(f_{M_2-1} \circ \ldots \circ f_0)(0)} \left[f_{M_1}\right]' |_{(f_{M_1-1} \circ \ldots \circ f_0)(0)}$$

converges to zero if and only if the joint spectral radius $\hat{\varrho}(\Upsilon_0)$ of the set Υ_0 is less than one: $\hat{\varrho}(\Upsilon_0) < 1$ (see [11]). Recall that, by definition, the **joint spectral radius** of the bounded set Υ_0 is defined by $\hat{\varrho}(\Upsilon_0) := lim_{t\to\infty}\hat{\varrho}_t(\Upsilon_0, ||\cdot||)$, where $\hat{\varrho}_t(\Upsilon_0, ||\cdot||) := sup\{||A||^{\frac{1}{t}} : A \in \Upsilon_0^t\}$ (see [23]). This definition is independent of the norm used by the equivalence of the norms in $\mathbb{C}^n$, so Daubechies and Lagarias showed that $\hat{\varrho}(\Upsilon_0) = limsup_{t\to\infty}\{\hat{\varrho}(A)^{\frac{1}{t}} : A \in \Upsilon_0^t\}$. An application of Gelfand's formula gives $\hat{\varrho}(A)^{\frac{1}{t}} = \hat{\varrho}(A_1A_2\ldots A_t)^{\frac{1}{t}} \leq \hat{\varrho}(A_1)^{\frac{1}{t}}\hat{\varrho}(A_2)^{\frac{1}{t}}\ldots\hat{\varrho}(A_t)^{\frac{1}{t}}$, whenever $A = A_1A_2\ldots A_t \in \Upsilon_0^t$. Since z_0 is attracting, $\hat{\varrho}(A_1)^{\frac{1}{t}}\hat{\varrho}(A_2)^{\frac{1}{t}}\ldots\hat{\varrho}(A_t)^{\frac{1}{t}} \leq \lambda^{\frac{1}{t}}\lambda^{\frac{1}{t}}\ldots\lambda^{\frac{1}{t}} = \lambda < 1$, whenever $A \in \Upsilon_0^t$. Hence $\hat{\varrho}(\Upsilon_0) < 1$. This implies that any left-infinite product $\sqcap$ converges to zero. In particular, the infinite product

$$\sqcap^M := \ldots\left[f_M\right]' |_{(f_{M-1}\circ\ldots\circ f_0)(0)} \cdots \left[f_2\right]' |_{(f_1\circ f_0)(0)} \left[f_1\right]' |_{(f_0)(0)} \left[f_0\right]' |_0$$

converges to zero. Since, by the chain rule $\sqcap^M = \left[f_M\right]'(0)$, we obtain

$$lim_{M\to\infty}\left[f_M\right]'(0) = 0.$$

By Taylor's theorem, there is an open ball (in the Euclidean metric) $B \subset \Omega$ around 0 and $0 < c < 1$ such that $|F^{(M)}(z)| < c|z|$ for all $z \in B$ whenever $M = 1, 2, \ldots$ It then follows that, for the neighbourhood $\mathcal{N}_0 = B$ of $0 \in \Omega$, we have

$$lim_{M\to\infty}F^{(M)} = 0, \text{ uniformly on } B.$$

This means that $z \in \mathfrak{B}_0(F)$ and the proof is complete.

Let us give an interpretation of the significance of this proposition in the framework of biholomorphic cryptosystems. According to the result of Proposition 5, ***if some encryption data are within the basin of attraction of a attracting point of the biholomorphic cryptosystem, then, over time, all these encryption data will tend to coincide on this point***.

It is interesting for our purposes to consider the particular case of biholomorphic cryptosystems. In such a case, all the rules encryption with finite length M can be reversed and therefore be decrypted. It is therefore important to know the conditions under which all mappings f_M can be inverted in a neighbourhood of a point $z_0 \in \Omega$ and there exists the inverse of the entire cryptosystem $F = \ldots\circ f_M \circ \ldots \circ f_1 \circ f_0$ around z_0. To examine whether a mapping f_M can be inverted in a neighbourhood of z_0, we recall that, according to the inverse function theorem, the matrix inverse of the Jacobian matrix of an invertible function is the Jacobian matrix of the inverse function. That is, *if the Jacobian* $\left[f_M\right]'$ *is continuous and non-singular at* z_0, *then* f_M *is invertible when restricted to some neighbourhood of* z_0 *and* $\left(\left[f_M\right]^{-1}\right)' |_{(f_M\circ f_{M-1}\circ\ldots\circ f_1\circ f_0)(z_0)} = \left(\left[f_M\right]' |_{(f_M\circ f_{M-1}\circ\ldots\circ f_1\circ f_0)(z_0)}\right)^{-1}$. Conversely, *if the Jacobian determinant* $det\left[f_M\right]'$ *is not zero at* z_0, *then the function is locally invertible near this point*, i.e., *there is neighbourhood of this point, in which the* *function is invertible.* Following the above discussion, it is clear that *we should consider sufficient conditions for inversion throughout the length of all of the chain.*

Definition 11 Let $F = \ldots \circ f_M \circ \ldots \circ f_1 \circ f_0$ be an infinite biholomorphic encryption chain on an open domain $\Omega \subset \mathbb{C}$ and let $z_0 \in \Omega$. For any $s \geq 1$, we define

$$\mathfrak{F}_s := \prod_{M=1}^{s} \left[f_M \right]' |_{(f_{M-1} \circ \ldots \circ f_1 \circ f_0)(z_0)} .$$

We say that the infinite product $\mathfrak{F} := \prod_{M=1}^{\infty} \mathfrak{F}_s$ converges invertibly if

- the matrix $\left[f_M \right]' |_{(f_{M-1} \circ \ldots \circ f_1 \circ f_0)(z_0)}$ is invertible for any M and
- the matrix $lim_{s \to \infty} \prod_{M=1}^{\infty} \left[f_M \right]' |_{(f_{M-1} \circ \ldots \circ f_1 \circ f_0)(z_0)}$ exists and is invertible.

The following result is easily verified.

Proposition 10 *The product $\mathfrak{F}$ converges invertibly, if and only if*

1. $lim_{M \to \infty} \left[f_M \right]' |_{(f_{M-1} \circ \ldots \circ f_1 \circ f_0)(z_0)} = \mathbb{I}$ *[26]*
2. $\sum_{M=1}^{\infty} || \left[f_M \right]' |_{(f_{M-1} \circ \ldots \circ f_1 \circ f_0)(z_0)} - \mathbb{I} \; || < \infty$*, where* $|| \cdot ||$ *is any* $p-$*norm in the set of* $n \times n$ *matrices (see [27]).*

Notation 2 *As in Section 2, we use the symbol* $F_n^{(M)}$ *to denote the segment of the infinite biholomorphic encryption chain* $F = \ldots \circ f_M \circ \ldots \circ f_1 \circ f_0$ *that involves the composition of all mappings which are at the part of* F *that starts from the function in the* $(n+1)-$*th place of* F *and ends with the function in the* $(M+1)-$*th place of* F*. It is clear that*

$$F_n^{(M)} = f_M \circ \ldots \circ f_{n+1} \circ f_n .$$

If $n > 0$ *and* $M = \infty$*, then we set*

$$F_n^{(\infty)} = \ldots \circ f_M \circ \ldots . \circ f_{(n+1)} \circ f_n$$

and we say that $F_n^{(\infty)}$ *is a* ***truncated infinite encryption chain****. If the segment starts from the beginning of the codification chain (:*$n = 0$*) and ends with the function in the* $(M+1)-$*th place, then we adopt often the simpler notation*

$$F^{(M)} = f_M \circ \ldots \circ f_1 \circ f_0 .$$

We point out that with the notation of Definition 8, we have $F^{(0)}(\Omega) = U_0 = \Omega$*,* $F^{(1)}(\Omega) \subset U_1, \ldots$ *and in general* $F^{(M)}(\Omega) \subset U_M$*, for any* $k = 0, 1, 2, \ldots$

Definition 12 Let $F = \ldots \circ f_M \circ \ldots \circ f_1 \circ f_0$ be an infinite biholomorphic encryption chain on an open domain $\Omega \subset \mathbb{C}$.

- A point $z_0 \in F^{(M)}(\Omega)$ is **periodic** in the chain F, if the value of z_0 occurs more than once in its orbit $\bigcirc^{+}(z_0)$.
- A periodic point p is said to be of **restricted periodicity**, if the number of times of its occurrence in the chain is finite; it is said to be of **extensive periodicity**, if the number of times of its occurrence is infinite.

Based on this Definition, it is straightforward to show the following two results.

Proposition 11 *A point $z_0 \in F^{(M)}(\Omega)$ is periodic in the encryption chain F, if there is a $n \in \mathbb{N}$ such that z_0 is a fixed point of $F_M^{(M+n)}$, that is, z_0 is a solution of the equation $F_M^{(M+n)}(z) = z \Longleftrightarrow \big(f_{M+n} \circ \ldots \circ f_{M+1} \circ f_M\big)(z) = z$.*

Proposition 12 *Let $M \in \mathbb{N}$. The infinite biholomorphic encryption chain F has a restricted (respectively, an extensive) periodicity at the point $z_0 \in F^{(M)}(\Omega)$, if there is a finite (respectively, an infinite) sequence $\big(n_j\big) \subset M, M+1, \ldots$ so that $n_1 = M$ and the value $F_M^{(\infty)}(z_0)$ of the truncated infinite encryption chain $F_M^{(\infty)} = \big(\ldots \circ f_{M+n} \circ \ldots . \circ f_{(M+1)} \circ f_M\big)$ at the point z_0 can be expressed as infinite succession of finite segmentation sub-chains $F_{n_j}^{(n_{j+1}-1)}(z_0) = \big(f_{n_{j+1}-1} \circ \ldots \circ f_{n_j+1} \circ f_{n_j}\big)(z_0)$ satisfying $F_{n_j}^{(n_{j+1}-1)}(z_0) = F_{n_{j+1}}^{(n_{j+2}-1)}(z_0)$. The sequence $\big(n_j\big)$ is called the* ***sequence of periods*** *of the point z_0 in the encryption chain F, while the sequence $F_{n_j}^{(n_{j+1}-1)}(z_0)\big)$ is called the* ***segmentation sequence*** *of F.*

Remark 8 With these notations, it is clear that if $M > 0$, then we say that the infinite biholomorphic encryption chain F has a period with **late start in position** M and F can be represented as follows:

$$F = \ldots \circ \underbrace{\big(f_{n_{j+1}-1} \circ \ldots \circ f_{n_j+1} \circ f_{n_j}\big)}_{F_{n_j}^{n_{j+1}-1}} \circ \ldots \circ$$

$$\underbrace{\big(f_{n_3-1} \circ \ldots \circ f_{n_2+1} \circ f_{n_2}\big)}_{F_{n_2}^{n_3-1}} \circ \underbrace{\big(f_{n_2-1} \circ \ldots \circ f_{n_1+1} \circ f_{n_1}\big)}_{F_{n_1}^{n_2-1}} \circ \big(f_M \circ \ldots \circ f_0\big).$$

If $M = 1$, the encryption chain F takes the simpler form

$$F = \ldots \circ \underbrace{\big(f_{n_{j+1}-1} \circ \ldots \circ f_{n_j+1} \circ f_{n_j}\big)}_{F_{n_j}^{n_{j+1}-1}} \circ \ldots \circ$$

$$\underbrace{\big(f_{n_3-1} \circ \ldots \circ f_{n_2+1} \circ f_{n_2}\big)}_{F_{n_2}^{n_3-1}} \circ \underbrace{\big(f_{n_2-1} \circ \ldots \circ f_1 \circ f_0\big)}_{F_{n_1}^{n_2-1}}.$$

Definition 13 Let $z_0 \in F^{(M)}(\Omega)$ be a periodic point with late start in position M. Suppose $\big(n_j\big)$ is the sequence of periods in the infinite biholomorphic encryption chain F, such that the set

$$\Upsilon_{z_0} := \left(\left[F_{n_j}^{(n_{j+1}-1)}\right]'(z_0)\right)_j$$

is bounded.

i. We say that z_0 is **attracting**, if there is a positive constant $\lambda < 1$ such that the modulus of all eigenvalues of $\left[F_{n_j}^{(n_{j+1}-1)}\right]'(z_0)$ is strictly smaller than λ, whenever $j = 1, 2, \ldots$; the orbit $\bigcirc^+(z_0)$ of z_0 is called an **attracting cycle**.

ii. We say that z_0 is **repelling**, if there is a positive constant $\mu > 1$ such that the modulus of all eigenvalues of $\left[F_{n_j}^{(n_{j+1}-1)}\right]'(z_0)$ is strictly larger than μ, whenever $j = 1, 2, \ldots$; the orbit $\bigcirc^+(z_0)$ of z_0 is called a **repelling cycle**.

Proposition 13 *Under the assumptions of the above Definition, the point $z_0 \in F^{(M)}(\Omega)$ is attracting if and only if there exists a neighbourhood $\mathcal{N}_{z_0}$ of z_0 such that the orbit of any $z \in mathcal N_{z_0}$ converges to z_0. The set of all points $z \in F^{(M)}(\Omega)$ whose orbits converge to p is called the* ***basin of attraction*** *of F at z_0 in $F^M(W)$:*

$$\mathfrak{B}_{z_0}(F^M) := \left\{z \in F^{(M)}(\Omega) : F_{n_i}^{(n_{j+1}-1)}(z) \overset{j\to\infty}{\to} z_0\right\}.$$

Proof Choosing local coordinates, we can assume that $p = 0$. Since the set $\Upsilon_0 := \Upsilon_{z_0}$ is bounded, any left-infinite product

$$\left[F_{n_{i_k}}^{(n_{i_k}-1)}\right]'(0)\left[F_{n_{i_{k-1}}}^{(n_{i_{k-1}}-1)}\right]'(0)\ldots\left[F_{n_{i_2}}^{(n_{i_2}-1)}\right]'(0)\left[F_{n_{i_1}}^{(n_{i_1}-1)}\right]'(0)$$

converges to zero if and only if the joint spectral radius $\hat{\varrho}(\Upsilon)_0 < 1$ (see [11]). Recall that, by definition, the **joint spectral radius** of the bounded set Υ_0 is defined by $\hat{\varrho}\left(\Upsilon_0\right) := lim_{t\to\infty}\hat{\varrho}_t\left(\Upsilon_0, ||\cdot||\right)$, where $\hat{\varrho}_t\left(\Upsilon_0, ||\cdot||\right) := sup\left\{||A||^{\frac{1}{t}} : A \in \Upsilon_0^t\right\}$ (see [23]). This definition is independent of the norm used by the equivalence of the norms in $\mathbb{C}^n$, so Daubechies and Lagarias showed that $\hat{\varrho}\left(\Upsilon_0\right) = limsup_{t\to\infty}\left\{\hat{\varrho}(A)^{\frac{1}{t}} : A \in \Upsilon_0^t\right\}$. An application of Gelfand's formula gives $\hat{\varrho}(A)^{\frac{1}{t}} = \hat{\varrho}(A_1A_2\ldots A_t)^{\frac{1}{t}} \leq \hat{\varrho}(A_1)^{\frac{1}{t}}\hat{\varrho}(A_2)^{\frac{1}{t}}\ldots\hat{\varrho}(A_t)^{\frac{1}{t}}$, whenever $A = A_1A_2\ldots A_t \in \Upsilon_0^t$. Since z_0 is attracting, $\hat{\varrho}(A_1)^{\frac{1}{t}}\hat{\varrho}(A_2)^{\frac{1}{t}}\ldots\hat{\varrho}(A_t)^{\frac{1}{t}} \leq \lambda^{\frac{1}{t}}\lambda^{\frac{1}{t}}\ldots\lambda^{\frac{1}{t}} = \lambda < 1$, whenever $A \in \Upsilon_0^t$. Hence $\hat{\varrho}\left(\Upsilon_0\right) < 1$. This implies that any left-infinite product $\sqcap$ converges to zero. In particular, the infinite product

$$\sqcap^M := \ldots\left[F_{n_{i_k}}^{(n_{i_k}-1)}\right]'(0)\left[F_{n_{i_{k-1}}}^{(n_{i_{k-1}}-1)}\right]'(0)\ldots\left[F_{n_{i_2}}^{(n_{i_2}-1)}\right]'(0)\left[F_{n_1}^{(n_{i_1}-1)}\right]'(0)$$

converges to zero. In particular, the infinite product

$$\sqcap^M := \ldots\left[F_{n_{i_k}}^{(n_{i_k}-1)}\right]'(0)\left[F_{n_{i_{k-1}}}^{(n_{i_{k-1}}-1)}\right]'(0)\ldots\left[F_{n_{i_2}}^{(n_{i_2}-1)}\right]'(0)\left[F_{n_1}^{(n_{i_1}-1)}\right]'(0)$$

converges to zero. From the periodicity of 0, it follows that also the product

$$\big[F_{n_j}^{(n_{j+1}-1)}\big]'\Big(\big(F_{n_{j-1}}^{(n_j-1)}\circ\ldots\circ F_{n_2}^{(n_3-1)}\circ F_{n_1}^{(n_2-1)}\big)(0)\Big)$$

$$\ldots\big[F_{n_2}^{(n_3-1)}\big]'\big(F_{n_1}^{(n_2-1)}(0)\big)big[F_{n_1}^{(n_2-1)}\big]'(0)$$

converges to zero. By the chain rule, this product equals $big[F_{n_1}^{(n_{j+1}-1)}\big]'(0)$. Hence

$$lim_{j\to\infty}big[F_{n_1}^{(n_{j+1}-1)}\big]'(0)=0.$$

By Taylor's theorem, there is some ball (in the Euclidean metric) $B\subset F^M(W)$ around 0 and $0<c<1$ such that $|F_{n_1}^{n_{j+1}-1}(z)|<c|z|$ for all $z\in B$ whenever $j=1,2,\ldots$ It then follows that, for the neighbourhood $\mathcal{N}_0=B$ of $0\in F^M(W)$, we have

$$lim_{j\to\infty}F_{n_1}^{(n_{j+1}-1)}(z)=0 \text{ uniformly on } B.$$

This means that $z\in\mathfrak{B}_{z_0}(F^{(M)})$ and the proof is complete.

Let us give an interpretation of the significance of this proposition in the framework of biholomorphic cryptosystems. According to the result of Proposition 5, ***if at some step of the segmentation sequence some encryption data are within the basin of attraction of a periodic attracting point of this step, then, over time, all these encryption data will tend to coincide on this point***.

It is interesting for our purposes to consider the particular case of biholomorphic cryptosystems. In such a case, all the rules encryption with finite length M can be reversed and therefore be decrypted. It is therefore important to know the conditions under which all mappings $F_{n_j}^{(n_{j+1}-1)}$ can be inverted in a neighbourhood of a periodic point z_0 and there exists the inverse of the entire segmented chain $F=\ldots\circ F_{n_j}^{(n_{j+1}-1)}\circ\ldots\circ F_{n_2}^{(n_3-1)}\circ F_{n_1}^{(n_2-1)}$ around z_0. To examine whether a mapping $F_{n_j}^{(n_{j+1}-1)}$ can be inverted in a neighbourhood of z_0, we recall that, according to the inverse function theorem, the matrix inverse of the Jacobian matrix of an invertible function is the Jacobian matrix of the inverse function. That is, *if the Jacobian $\big[F_{n_j}^{(n_{j+1}-1)}\big]'$ of the mapping $F_{n_j}^{(n_{j+1}-1)}$ is continuous and non-singular at z_0, then $F_{n_j}^{(n_{j+1}-1)}$ is invertible when restricted to some neighbourhood of z_0.* Conversely, if the Jacobian determinant $det\big[F_{n_j}^{(n_{j+1}-1)}\big]'$ is not zero at z_0, then the function is locally invertible near this point, i.e., there is neighbourhood of this point, in which the function is invertible. Following the above discussion, it is clear that we should consider sufficient conditions for inversion throughout the length of the chain.

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Third-Order Fermionic and Fourth-Order Bosonic Operators

Chao Ding, Raymond Walter, and John Ryan

Abstract This paper continues the work of our previous paper (Ding et al., *Higher Order Fermionic and Bosonic Operators*, Springer Series), where we generalize kth-powers of the Euclidean Dirac operator D_x to higher spin spaces in the case the target space is a degree one homogeneous polynomial space. To generalize the results in (Ding et al., *Higher Order Fermionic and Bosonic Operators*, Springer Series) to more general cases, i.e., the target space is a degree k homogeneous polynomial space, we reconsider the generalizations of D_x^3 and D_x^4 to higher spin spaces in the case the target space is a degree k homogeneous polynomial space in this paper. Constructions of $3rd$- and $4th$-order conformally invariant operators in higher spin spaces are given; these are the $3rd$-order fermionic and $4th$-order bosonic operators. They are consistent with the 3rd- and 4th-order conformally invariant differential operators obtained in our paper (Ding et al., J. Geometric Anal. **27**(3), 2418–2452 (2017)) with a different technique. Further, we point out that the generalized symmetry technique used in (De Bie et al., Potential Analysis **47**(2), 123–149 (2017); Eelbode and Roels, Compl. Anal. Oper. Theory, 1–27 (2014)) is not applicable for higher order cases because the computations are

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infeasible. Fundamental solutions and intertwining operators of both operators are also presented here. These results can be easily generalized to cylinders and Hopf manifolds as in Ding et al. (J. Indian Math. Soc. **83**(3-4), 231–240 (2016)). To conclude this paper, we investigate ellipticity property for our $3rd$- and $4th$-order conformally invariant operators.

1 Introduction

The *higher spin theory* in Clifford analysis began with the Rarita–Schwinger operator [5], which is named analogously to the Dirac operator and reproduces the wave equations for a massless particle of arbitrary half-integer spin in four dimensions with appropriate signature [25]. The former operator takes its name from the 1941 work of Rarita and Schwinger [24] that simply formulated the theory of particles of arbitrary half-integer spin $k + \frac{1}{2}$ and in particular considered its implications for particles of spin $\frac{3}{2}$. The higher spin theory considers generalizations of classical Clifford analysis techniques to higher spin spaces [4–6, 12, 15, 20], focusing on operators acting on functions on $\mathbb{R}^m$ that take values in arbitrary irreducible representations of $Spin(m)$. Generally these are polynomial representations, such as k-homogeneous monogenic (harmonic) polynomials corresponding to particles of half-integer spin (integer spin). The highest weight vector of the spin representation as a whole may even be taken as a parameter [7], but we consider a narrower scope.

Slovák [27] provided a non-constructive classification of all conformally invariant differential operators on locally conformally flat manifolds in higher spin theory, but this shows only between which vector bundles these operators exist and what is their order; explicit expressions of these operators are still being found. Eelbode and Roels [15] noted the Laplace operator Δ_x is no longer conformally invariant when acting on $C^\infty(\mathbb{R}^m, \mathcal{H}_1)$, where $\mathcal{H}_1$ is the degree one homogeneous harmonic polynomial space (correspondingly $\mathcal{M}_1$ for monogenic polynomials). They construct a second-order conformally invariant operator on $C^\infty(\mathbb{R}^m, \mathcal{H}_1)$, the (generalized) Maxwell operator, reproducing the Maxwell equation for appropriate dimension and signature [15]. De Bie and his co-authors [6] generalize this Maxwell operator from $C^\infty(\mathbb{R}^m, \mathcal{H}_1)$ to $C^\infty(\mathbb{R}^m, \mathcal{H}_k)$ to provide the higher spin Laplace operators, which are the second-order conformally invariant operators generalizing the Laplace operator to arbitrary integer spins. Our earlier work [10] generalizes D_x^k in higher spin spaces in the case the target space is a degree one homogeneous polynomial space, encompassing the spin-1 and spin-$\frac{3}{2}$ cases, which simplifies the constructions of our conformally invariant differential operators, since all higher order derivative terms ($\geq$2) with respect to target spaces disappear. In this paper, we consider $3rd$-order fermionic and $4th$-order bosonic operators corresponding to the appropriate degree-k homogeneous polynomial space ($\mathcal{M}_k$ or $\mathcal{H}_k$).

The paper is organized as follows: We briefly introduce Clifford algebras, Clifford analysis, and representation theory of the Spin group in Section 2. In Section 3, we introduce the $3rd$-order higher spin operators $\mathcal{D}_3$ as the generalization of D_x^3 when acting on $C^\infty(\mathbb{R}^m, \mathcal{M}_k)$ and $4th$-order higher spin operators $\mathcal{D}_4$ as the generalization of D_x^4 when acting on $C^\infty(\mathbb{R}^m, \mathcal{H}_k)$. Nomenclature for general higher order higher spin operators is given: bosonic and fermionic operators. The construction and conformal invariance of both operators are given with the help of the concept of *generalized symmetry* as in [6, 10, 15]. Then we provide the intertwining operators for $\mathcal{D}_3$ and $\mathcal{D}_4$ with similar techniques as in [10], which also reveal that these operators are conformally invariant. However, from the calculation of the constructions in this section, we realize that the generalized symmetries approach used in [6, 10, 15] and here does not apply for other higher order conformally invariant differential operators, because of the infeasible computation. A different approach for the higher order cases will be demonstrated elsewhere.

Section 4 presents the fundamental solutions and intertwining operators of $\mathcal{D}_3$ and $\mathcal{D}_4$ using similar techniques as in [10]. The expressions of the fundamental solutions also suggest that $\mathcal{D}_3$ and $\mathcal{D}_4$ are generalizations of D_x^3 and D_x^4 in higher spin spaces and these can be generalized to conformally flat manifolds, for instance, cylinders and Hopf manifolds, as in [9]. In Section 5, we show the connections between our third (fourth)-order conformally invariant differential operator and the first (second)-order conformally invariant differential operator. Section 6 is devoted to the investigation of ellipticity property for our $3rd$-order and $4th$-order higher spin operators with the technique developed in [6].

2 Preliminaries

2.1 Clifford Algebra

A real Clifford algebra, $\mathcal{C}l_m$, can be generated from $\mathbb{R}^m$ by considering the relationship

$$\underline{x}^2 = -\|\underline{x}\|^2$$

for each $\underline{x} \in \mathbb{R}^m$. We have $\mathbb{R}^m \subseteq \mathcal{C}l_m$. If $\{e_1, \ldots, e_m\}$ is an orthonormal basis for $\mathbb{R}^m$, then $\underline{x}^2 = -\|\underline{x}\|^2$ tells us that

$$e_i e_j + e_j e_i = -2\delta_{ij},$$

where δ_{ij} is the Kronecker delta function. An arbitrary element of the basis of the Clifford algebra can be written as $e_A = e_{j_1} \cdots e_{j_r}$, where $A = \{j_1, \cdots, j_r\} \subset \{1, 2, \cdots, m\}$ and $1 \le j_1 < j_2 < \cdots < j_r \le m$. Hence for any element $a \in \mathcal{C}l_m$, we have $a = \sum_A a_A e_A$, where $a_A \in \mathbb{R}$. Similarly, the complex Clifford algebra $\mathcal{C}l_m(\mathbb{C})$ is defined as the complexification of the real Clifford algebra

$$\mathcal{C}l_m(\mathbb{C}) = \mathcal{C}l_m \otimes \mathbb{C}.$$

We consider real Clifford algebra $\mathcal{C}l_m$ throughout this subsection, but in the rest of the paper we consider the complex Clifford algebra $\mathcal{C}l_m(\mathbb{C})$ unless otherwise specified.

The Pin and Spin groups play an important role in Clifford analysis. The Pin group can be defined as

$$Pin(m) = \{a \in \mathcal{C}l_m : a = y_1 y_2 \ldots y_p,\ y_1, \ldots, y_p \in \mathbb{S}^{m-1}, p \in \mathbb{N}\},$$

where $\mathbb{S}^{m-1}$ is the unit sphere in $\mathbb{R}^m$. $Pin(m)$ is clearly a group under multiplication in $\mathcal{C}l_m$.

Now suppose that $a \in \mathbb{S}^{m-1} \subseteq \mathbb{R}^m$, if we consider axa, we may decompose $x = x_{a\parallel} + x_{a\perp}$, where $x_{a\parallel}$ is the projection of x onto a and $x_{a\perp}$ is the rest, perpendicular to a. Hence $x_{a\parallel}$ is a scalar multiple of a and we have $axa = ax_{a\parallel}a + ax_{a\perp}a = -x_{a\parallel} + x_{a\perp}$. So the action axa describes a reflection of x in the direction of a. By the Cartan–Dieudonné theorem each $O \in O(m)$ is the composition of a finite number of reflections. If $a = y_1 \cdots y_p \in Pin(m)$, we define $\tilde{a} := y_p \cdots y_1$ and observe that $ax\tilde{a} = O_a(x)$ for some $O_a \in O(m)$. Choosing $y_1, \ \ldots, \ y_p$ arbitrarily in $\mathbb{S}^{m-1}$, we see that the group homomorphism

$$\theta : \ Pin(m) \longrightarrow O(m) \ : \ a \mapsto O_a, \tag{1}$$

with $a = y_1 \cdots y_p$ and $O_a x = ax\tilde{a}$ is surjective. Further $-ax(-\tilde{a}) = ax\tilde{a}$, so $1, \ -1 \in Ker(\theta)$. In fact $Ker(\theta) = \{1, \ -1\}$. See [23]. The Spin group is defined as

$$Spin(m) = \{a \in \mathcal{C}l_m : a = y_1 y_2 \ldots y_{2p},\ y_1, \ldots, y_{2p} \in \mathbb{S}^{m-1}, p \in \mathbb{N}\}$$

and it is a subgroup of $Pin(m)$. There is a group homomorphism

$$\theta : \ Spin(m) \longrightarrow SO(m) \ ,$$

which is surjective with kernel $\{1, \ -1\}$. It is defined by (1). Thus $Spin(m)$ is the double cover of $SO(m)$. See [23] for more details.

For a domain U in $\mathbb{R}^m$, a diffeomorphism $\phi : U \longrightarrow \mathbb{R}^m$ is said to be conformal if, for each $x \in U$ and each $\mathbf{u}, \mathbf{v} \in TU_x$, the angle between $\mathbf{u}$ and $\mathbf{v}$ is preserved under the corresponding differential at x, $d\phi_x$. For $m \geq 3$, a theorem of Liouville tells us the only conformal transformations are Möbius transformations. Ahlfors and Vahlen show that given a Möbius transformation on $\mathbb{R}^m \cup \{\infty\}$ it can be expressed as $y = (ax + b)(cx + d)^{-1}$ where $a, \ b, \ c, \ d \in \mathcal{C}l_m$ and satisfy the following conditions [1]:

1. $a, \ b, \ c, \ d$ *are all products of vectors in* $\mathbb{R}^m$;
2. $a\tilde{b}, \ c\tilde{d}, \ \tilde{b}c, \ \tilde{d}a \in \mathbb{R}^m$;
3. $a\tilde{d} - b\tilde{c} = \pm 1$.

Since $y = (ax+b)(cx+d)^{-1} = ac^{-1}+(b-ac^{-1}d)(cx+d)^{-1}$, a conformal transformation can be decomposed as compositions of translation, dilation, reflection, and inversion. This gives an *Iwasawa decomposition* for Möbius transformations. See [20] for more details.

The Dirac operator in $\mathbb{R}^m$ is defined to be $D_x := \sum_{i=1}^m e_i \partial_{x_i}$. Note $D_x^2 = -\Delta_x$, where Δ_x is the Laplacian in $\mathbb{R}^m$. A $\mathcal{C}l_m$-valued function $f(x)$ defined on a domain U in $\mathbb{R}^m$ is left monogenic if $D_x f(x) = 0$. Since multiplication of Clifford numbers is not commutative in general, there is a similar definition for right monogenic functions. Sometimes we will consider the Dirac operator D_u in vector u rather than x.

Let $\mathcal{M}_k$ denote the space of $\mathcal{C}l_m$-valued monogenic polynomials, homogeneous of degree k. Note that if $h_k \in \mathcal{H}_k$, the space of $\mathcal{C}l_m$-valued harmonic polynomials homogeneous of degree k, then $D_u h_k \in \mathcal{M}_{k-1}$, but $D_u u p_{k-1}(u) = (-m-2k+2)p_{k-1}(u)$, where $p_{k-1} \in \mathcal{M}_{k-1}$. Hence,

$$\mathcal{H}_k = \mathcal{M}_k \oplus u\mathcal{M}_{k-1},\ h_k = p_k + up_{k-1}.$$

This is an *Almansi–Fischer decomposition* of $\mathcal{H}_k$. See [5] for more details. In this Almansi–Fischer decomposition, we define P_k as the projection map

$$P_k : \mathcal{H}_k \longrightarrow \mathcal{M}_k.$$

Suppose U is a domain in $\mathbb{R}^m$. Consider a differentiable function $f : U \times \mathbb{R}^m \longrightarrow \mathcal{C}l_m$, such that for each $x \in U$, $f(x,u)$ is a left monogenic polynomial homogeneous of degree k in u, then the Rarita–Schwinger operator [5, 12] is defined by

$$R_k f(x,u) := P_k D_x f(x,u) = \left(\frac{uD_u}{m+2k-2}+1\right)D_x f(x,u).$$

2.2 Irreducible Representations of the Spin Group

The following three representation spaces of the Spin group are frequently used as the target spaces in Clifford analysis. The spinor representation is the most commonly used spin representation in classical Clifford analysis and the other two polynomial representations are often used in higher spin theory.

Spinor Representation of $Spin(m)$

Consider the complex Clifford algebra $\mathcal{C}l_m(\mathbb{C})$ with even dimension $m = 2n$. Then $\mathbb{C}^m$ or the space of vectors is embedded in $\mathcal{C}l_m(\mathbb{C})$ as

$$(x_1, x_2, \cdots, x_m) \mapsto \sum_{j=1}^{m} x_j e_j : \mathbb{C}^m \hookrightarrow \mathcal{C}l_m(\mathbb{C}).$$

Define the *Witt basis* elements of $\mathbb{C}^{2n}$ as

$$f_j := \frac{e_j - ie_{j+n}}{2}, \quad f_j^\dagger := -\frac{e_j + ie_{j+n}}{2}.$$

Let $I := f_1 f_1^\dagger \ldots f_n f_n^\dagger$. The space of *Dirac spinors* is defined as

$$\mathcal{S} := \mathcal{C}l_m(\mathbb{C})I.$$

This is a representation of $Spin(m)$ under the following action:

$$\rho(s)I := sI, \ \ for \ s \in Spin(m).$$

Note that $\mathcal{S}$ is a left ideal of $\mathcal{C}l_m(\mathbb{C})$. For more details, we refer the reader to [8]. An alternative construction of spinor spaces is given in the classical paper of Atiyah, Bott, and Shapiro [2].

Homogeneous Harmonic Polynomials on $\mathcal{H}_k(\mathbb{R}^m, \mathbb{C})$

The space of harmonic polynomials is invariant under the action of $Spin(m)$ because the Laplacian Δ_m is an $SO(m)$-invariant operator, but this space is not irreducible for $Spin(m)$, decomposing into the infinite sum of spaces of k-homogeneous harmonic polynomials, $0 \leq k < \infty$, each of which is irreducible for $Spin(m)$. This brings us to a familiar representation of $Spin(m)$, that is $\mathcal{H}_k$. The following action has been shown to be an irreducible representation of $Spin(m)$ [29]:

$$\rho : Spin(m) \longrightarrow Aut(\mathcal{H}_k), \ s \longmapsto \big(f(x) \mapsto \tilde{s} f(sy\tilde{s})s\big),$$

with $x = sy\tilde{s}$. This can also be realized as follows:

$$Spin(m) \xrightarrow{\theta} SO(m) \xrightarrow{\rho} Aut(\mathcal{H}_k);$$
$$a \longmapsto O_a \longmapsto \big(f(x) \mapsto f(O_a x)\big),$$

where θ is the double covering map and ρ is the standard action of $SO(m)$ on a function $f(x) \in \mathcal{H}_k$ with $x \in \mathbb{R}^m$. The function $\phi(z) = (z_1 + iz_m)^k$ is the highest weight vector for $\mathcal{H}_k(\mathbb{R}^m, \mathbb{C})$ having highest weight $(k, 0, \cdots, 0)$ (for more details, see [18]). Accordingly, spin representations given by $\mathcal{H}_k(\mathbb{R}^m, \mathbb{C})$ are said to have integer spin k; we can either specify an integer spin k or degree of homogeneity k of harmonic polynomials.

Homogeneous Monogenic Polynomials on $\mathcal{C}l_m$

In $\mathcal{C}l_m$-valued function theory, the previously mentioned Almansi–Fischer decomposition shows that we can also decompose the space of k-homogeneous harmonic polynomials as follows:

$$\mathcal{H}_k = \mathcal{M}_k \oplus u\mathcal{M}_{k-1}.$$

If we restrict $\mathcal{M}_k$ to the spinor-valued subspace, we have another important representation of $Spin(m)$: the space of k-homogeneous spinor-valued monogenic polynomials on $\mathbb{R}^m$, henceforth denoted by $\mathcal{M}_k := \mathcal{M}_k(\mathbb{R}^m, \mathcal{S})$. More specifically, the following action has been shown to be an irreducible representation of $Spin(m)$:

$$\pi \ : \ Spin(m) \longrightarrow Aut(\mathcal{M}_k), \ s \longmapsto (f(x) \mapsto \tilde{s} f(s x \tilde{s})).$$

When m is odd, in terms of complex variables $z_s = x_{2s-1} + i x_{2s}$ for all $1 \leq s \leq \frac{m-1}{2}$, the highest weight vector is $\omega_k(x) = (\overline{z_1})^k I$ for $\mathcal{M}_k(\mathbb{R}^m, \mathcal{S})$ having highest weight $(k + \frac{1}{2}, \frac{1}{2}, \cdots, \frac{1}{2})$, where $\overline{z_1}$ is the conjugate of z_1, $\mathcal{S}$ is the Dirac spinor space, and I is defined as in Section 2.2.1; for details, see [29]. Accordingly, the spin representations given by $\mathcal{M}_k(\mathbb{R}^m, \mathcal{S})$ are said to have half-integer spin $k + \frac{1}{2}$; we can either specify a half-integer spin $k + \frac{1}{2}$ or the degree of homogeneity k of monogenic spinor-valued polynomials.

3 Construction and Conformal Invariance

Slovák [27] established the existence of conformally invariant differential operators of arbitrary order and spin, provided that operators of odd order (respectively, even order) have half-integer spin $k + \frac{1}{2}$ (integer spin k) and are between spaces of k-homogeneous monogenic polynomials $\mathcal{M}_k$ (harmonic polynomials $\mathcal{H}_k$), more details can be found in [10]. The spin-$\frac{1}{2}$ and spin-0 cases are well established to arbitrary order: these are the powers of the Dirac and Laplace operators. We recently established the cases of spin-$\frac{3}{2}$ and spin-1 to arbitrary order [10]. In the first-order case for arbitrary (half-integer) spin, the explicit form of the operator is well known: the Rarita–Schwinger operators [5]. Preceding our work, Eelbode and Roels followed by De Bie et al. worked out the second-order case for arbitrary (integer) spin in the generalized Maxwell operator and higher spin Laplace operators [6, 15]. We push further here, working out the third- and fourth-order cases for arbitrary spin: in our terminology, these are the $3rd$-order fermionic operators and $4th$-order bosonic operators. Our nomenclature emphasizes the motivation by mathematical physics: particles of half-integer spin are known as fermions and particles of integer spin are known as bosons, so the operators of half-integer spin take the name fermionic operators and those of integer spin take the name bosonic operators.

3.1 3*rd*-Order Higher Spin Operator $\mathcal{D}_3$

The technique used here closely follows the treatment in our paper [10]. Our main result in the $3rd$-order higher spin case is the following theorem.

Theorem 1 *Up to a multiplicative constant, the unique* 3*rd-order conformally invariant differential operator is* $\mathcal{D}_3 : C^\infty(\mathbb{R}^m, \mathcal{M}_k) \longrightarrow C^\infty(\mathbb{R}^m, \mathcal{M}_k)$*, where*

$$\mathcal{D}_3 = D_x^3 + \frac{4}{m+2k}\langle u, D_x\rangle\langle D_u, D_x\rangle D_x - \frac{4||u||^2\langle D_u, D_x\rangle^2 D_x}{(m+2k)(m+2k-2)} - \frac{2u\langle D_u, D_x\rangle D_x^2}{m+2k}$$
$$-\frac{8u\langle u, D_x\rangle\langle D_u, D_x\rangle^2}{(m+2k)(m+2k-2)} - \frac{8u^3\langle D_u, D_x\rangle^3}{(m+2k)(m+2k-2)(m+6k-10)}$$

and $\langle\ ,\ \rangle$ *is the standard inner product in Euclidean space.*

Hereafter we may suppress the k index for the operator since there is little risk of confusion. Note the target space $\mathcal{M}_k$ is a function space, so any element in $C^\infty(\mathbb{R}^m, \mathcal{M}_k)$ has the form $f(x, u) \in \mathcal{M}_k$ for each fixed $x \in \mathbb{R}^m$ and x is the variable on which $\mathcal{D}_3$ acts.

Our proof of conformal invariance of this operator follows closely the method of [10, 15]. In order to explain what conformal invariance means, we begin with the concept of a generalized symmetry (see for instance [13]):

Definition 1 An operator η_1 is a generalized symmetry for a differential operator $\mathcal{D}$ if and only if there exists another operator η_2 such that $\mathcal{D}\eta_1 = \eta_2\mathcal{D}$. Note that for $\eta_1 = \eta_2$, this reduces to a definition of a (proper) symmetry: $\mathcal{D}\eta_1 = \eta_1\mathcal{D}$.

One determines the first-order generalized symmetries of an operator, which span a Lie algebra [15, 21]. In this case, the first-order symmetries will span a Lie algebra isomorphic to the conformal Lie algebra $\mathfrak{so}(1, m+1)$; in this sense, the operators we consider are conformally invariant. The operator $\mathcal{D}_3$ is $\mathfrak{so}(m)$-invariant (rotation-invariant) because it is the composition of $\mathfrak{so}(m)$-invariant (rotation-invariant) operators, which means the angular momentum operators $L_{ij}^x + L_{ij}^u$ that generate these rotations are proper symmetries of $\mathcal{D}_3$. The infinitesimal translations ∂_{x_j}, $j = 1, \cdots, n$, corresponding to linear momentum operators are proper symmetries of $\mathcal{D}_3$; this is an alternative way to say that $\mathcal{D}_3$ is invariant under translations that are generated by these infinitesimal translations. Readers familiar with quantum mechanics will recognize the connection to isotropy and homogeneity of space, the rotational and translational invariance of Hamiltonian, and the conservation of angular and linear momentum [26]; see also [3] concerning Rarita–Schwinger operators.

The remaining two of the first-order generalized symmetries of $\mathcal{D}_3$ are the Euler operator and special conformal transformations. The Euler operator $\mathbb{E}_x$ that measures degree of homogeneity in x is a generalized symmetry because $\mathcal{D}_3\mathbb{E}_x = (\mathbb{E}_x + 3)\mathcal{D}_3$; this is an alternative way to say that $\mathcal{D}_3$ is invariant under dilations, which are generated by the Euler operator. The special conformal transformations are defined in Lemma 1 in terms of harmonic inversion for $\mathcal{H}_k$-valued functions; harmonic inversion is defined in Definition 2 and is an involution mapping solutions of $\mathcal{D}_3$ to $\mathcal{D}_3$. Readers familiar with conformal field theory will recognize that invariance under dilation corresponds to scale-invariance and that special conformal transformations are another class of conformal transformations arising on spacetime [17]. An alternative method of proving conformal invariance of $\mathcal{D}_3$ is to prove the invariance of $\mathcal{D}_3$ under those finite transformations generated by these first-order generalized symmetries (rotations, dilations, translations, and special conformal transformations) to show invariance of $\mathcal{D}_3$ under actions of the conformal group; this may be phrased in terms of Möbius transformations and the Iwasawa decomposition. However, the first-order generalized symmetry method emphasizes the connection to mathematical physics and is more amenable to our proof of a certain property of harmonic inversion. It is also that used by earlier authors [6, 15].

Definition 2 The monogenic inversion is a conformal transformation defined as

$$\mathcal{J}_3 : C^\infty(\mathbb{R}^m, \mathcal{M}_k) \longrightarrow C^\infty(\mathbb{R}^m, \mathcal{M}_k) : f(x, u) \mapsto \mathcal{J}_3[f](x, u) := \frac{x}{||x||^{m-2}} f\left(\frac{x}{||x||^2}, \frac{xux}{||x||^2}\right).$$

Note that this inversion consists of Kelvin inversion $\mathcal{J}$ on $\mathbb{R}^m$ in the variable x composed with a reflection $u \mapsto \omega u \omega$ acting on the dummy variable u (where $x = ||x||\omega$) and a multiplication by a conformal weight term $\frac{x}{||x||^{m-2}}$; it satisfies $\mathcal{J}_3^2 = -1$.

Then we have the special conformal transformation defined in the following lemma. The definition is an infinitesimal version of the fact that finite special conformal transformations consist of a translation preceded and followed by an inversion [17]: an infinitesimal translation preceded and followed by monogenic inversion. The second equality in the lemma shares some terms in common with the generators of special conformal transformations in conformal field theory [17] and is a particular case of a result in [14].

Lemma 1 *The special conformal transformation defined as* $\mathcal{C}_3 := \mathcal{J}_3 \partial_{x_j} \mathcal{J}_3$ *satisfies*

$$\mathcal{J}_3 \partial_{x_j} \mathcal{J}_3 = x e_j - 2\langle u, x\rangle \partial_{u_j} + 2u_j \langle x, D_u\rangle - ||x||^2 \partial_{x_j} + x_j(2\mathbb{E}_x + m - 2).$$

Proof A similar calculation as in *Proposition A.1* in [6] will show the conclusion. □

Then, we have the main proposition as follows.

Proposition 1 *The special conformal transformations* $\mathcal{C}_3$*, with* $j \in \{1, 2, \ldots, m\}$ *are generalized symmetries of* $\mathcal{D}_3$*. More specifically,*

$$[\mathcal{D}_3, \mathcal{C}_3] = 6x_j \mathcal{D}_3,$$

where [,] *is the commutator. In particular, this shows that*

$$\mathcal{J}_3\mathcal{D}_3\mathcal{J}_3 = ||x||^6\mathcal{D}_3, \tag{2}$$

which is the generalization of D_x^3 in classical Clifford analysis [22]. This also implies $\mathcal{D}_3$ is invariant under inversion.

If the main proposition holds, then the conformal invariance can be summarized in the following theorem:

Theorem 2 *The first-order generalized symmetries of $\mathcal{D}_3$ are given by:*

1. *The infinitesimal rotations $L_{i,j}^x + L_{i,j}^u - \frac{1}{2}e_ie_j$, with $1 \le i < j \le m$.*
2. *The shifted Euler operator $m + 2\mathbb{E}_x - 3$.*
3. *The infinitesimal translations ∂_{x_j}, with $1 \le j \le m$.*
4. *The special conformal transformations $\mathcal{J}_3\partial_{x_j}\mathcal{J}_3$, with $1 \le j \le m$.*

These operators span a Lie algebra which is isomorphic to the conformal Lie algebra $\mathfrak{so}(1, m+1)$, whereby the Lie bracket is the ordinary commutator.

Proof The proof is similar as in [14] via transvector algebras. □

Detailed Proof of Proposition 1

To prove this proposition, we first introduce the following technical lemmas:

Lemma 2 *For all $1 \le j \le m$, we have*

$$[D_x^3, \mathcal{C}_3] = 4\langle u, D_x\rangle D_x\partial_{u_j} - 2u\partial_{u_j}D_x^2 - 4u_jD_x\langle D_u, D_x\rangle + 6x_jD_x^3.$$

Proof Recall that the special conformal transformation $\mathcal{C}_2$ is defined (see [6] equation (1)) as follows:

$$\mathcal{C}_2 = 2\langle u, x\rangle\partial_{u_j} - 2u_j\langle x, \partial_u\rangle + ||x||^2\partial_{x_j} - x_j(2\mathbb{E}_x + m - 2).$$

It is easy to see that $\mathcal{C}_3 = xe_j - \mathcal{C}_2$. Since

$$[AB, C] = A[B, C] + [A, C]B, \tag{3}$$

where $A,\ B,\ C$ are operators, then a straightforward calculation shows that

$$\begin{aligned}
[D_x^3, xe_j] &= -2x\partial_{x_j}\Delta_x + e_j(m + 2\mathbb{E}_x + 2)\Delta_x,\\
[D_x^3, \mathcal{C}_2] &= -4\langle u, D_x\rangle D_x\partial_{u_j} + 2u\partial_{u_j}D_x^2 + 4u_jD_x\langle D_u, D_x\rangle + 2x\partial_{x_j}D_x^2\\
&\quad -(2\mathbb{E}_x + m + 2)e_jD_x^2 - 6x_jD_x^3.
\end{aligned}$$

Combining above three questions completes the proof. □

Lemma 3 *For all* $1 \leq j \leq m$*, we have*

$$[\langle u, D_x\rangle\langle D_u, D_x\rangle D_x, \mathcal{C}_3] = -(m+2k)\langle u, D_x\rangle D_x\partial_{u_j} - e_j u\langle D_u, D_x\rangle D_x$$
$$+(m+2k-2)u_j\langle D_u, D_x\rangle D_x - 2u\langle u, D_x\rangle\langle D_u, D_x\rangle\partial_{u_j} - 2|u|^2\langle D_u, D_x\rangle D_x\partial_{u_j}$$
$$+6x_j\langle u, D_x\rangle\langle D_u, D_x\rangle D_x.$$

Proof Similar as in the proof of the previous lemma, we also use the identity (3) and

$$[AB, CD] = A[B, C]D + AC[B, D] + [A, C]DB + C[A, D]B, \tag{4}$$

where $A,\ B,\ C,\ D$ are operators.

$$[\langle u, D_x\rangle\langle D_u, D_x\rangle D_x, xe_j]$$
$$= \langle u, D_x\rangle\langle D_u, D_x\rangle[D_x, x]e_j + \langle u, D_x\rangle\langle D_u, D_x\rangle x[D_x, e_j] + [\langle u, D_x\rangle\langle D_u, D_x\rangle, x]e_j D_x$$
$$+x[\langle u, D_x\rangle\langle D_u, D_x\rangle, e_j]D_x$$
$$= -(m+2\mathbb{E}_x+2)\langle u, D_x\rangle\langle D_u, D_x\rangle e_j + 2u\langle D_u, D_x\rangle\partial_{x_j} + 2x\langle u, D_x\rangle\langle D_u, D_x\rangle\partial_{x_j}$$
$$-2\langle u, D_x\rangle D_x\partial_{u_j} - e_j u\langle D_u, D_x\rangle D_x - 2u_j\langle D_u, D_x\rangle D_x.$$

Since we already have (see [6] Lemma $A.2.$)

$$[\langle u, D_x\rangle\langle D_u, D_x\rangle, \mathcal{C}_2] = 2\|u\|^2\partial_{u_j}\langle D_u, D_x\rangle - 4x_j\langle u, D_x\rangle\langle D_u, D_x\rangle + (\langle u, D_x\rangle\partial_{u_j}$$
$$-u_j\langle D_u, D_x\rangle)(2\mathbb{E}_u + m - 2),$$

we can obtain that

$$[\langle u, D_x\rangle\langle D_u, D_x\rangle D_x, \mathcal{C}_2] = \langle u, D_x\rangle\langle D_u, D_x\rangle[D_x, \mathcal{C}_2] + [\langle u, D_x\rangle\langle D_u, D_x\rangle, \mathcal{C}_2]D_x$$
$$= (m+2k-2)\langle u, D_x\rangle D_x\partial_{u_j} - (m+2k)u_j\langle D_u, D_x\rangle D_x + 2u\langle u, D_x\rangle\langle D_u, D_x\rangle\partial_{u_j}$$
$$+2||u||^2\langle D_u, D_x\rangle D_x\partial_{u_j} + 2u\langle D_u, D_x\rangle\partial_{x_j} + 2x\langle u, D_x\rangle\langle D_u, D_x\rangle\partial_{x_j}$$
$$-(m+2\mathbb{E}_x+2)\langle u, D_x\rangle\langle D_u, D_x\rangle e_j - 6x_j\langle u, D_x\rangle\langle D_u, D_x\rangle D_x.$$

Since $\mathcal{C}_3 = xe_j - \mathcal{C}_2$, combining above two equations completes the proof. □

Lemma 4 *For all* $1 \leq j \leq m$*, we have*

$$[\|u\|^2\langle D_u, D_x\rangle^2 D_x, \mathcal{C}_3] = 2\|u\|^2\langle D_u, D_x\rangle^2 e_j - (2m+4k-4)\|u\|^2\langle D_u, D_x\rangle D_x\partial_{u_j}$$
$$-2u\|u\|^2\langle D_u, D_x\rangle^2\partial_{u_j} + 6x_j\|u\|^2\langle D_u, D_x\rangle^2 D_x.$$

Proof Similarly, with the help of (4) and a straightforward calculation, we have

$$\begin{aligned}
&[\|u\|^2\langle D_u, D_x\rangle^2 D_x, xe_j]\\
=&\ \|u\|^2\langle D_u, D_x\rangle^2[D_x, x]e_j+\|u\|^2\langle D_u, D_x\rangle^2 x[D_x, e_j]+[\|u\|^2\langle D_u, D_x\rangle^2, x]e_j D_x\\
&+x[\|u\|^2\langle D_u, D_x\rangle^2, e_j]D_x\\
=&\ -(m+2\mathbb{E}_x)\|u\|^2\langle D_u, D_x\rangle^2 e_j+2x\|u\|^2\langle D_u, D_x\rangle^2\partial_{x_j}-4\|u\|^2\langle D_u, D_x\rangle D_x\partial_{u_j}.
\end{aligned}$$

Since (see [6] Lemma $A.3$.)

$$[\|u\|^2\langle D_u, D_x\rangle^2, \mathcal{C}_2] = -4x_j\|u\|^2\langle D_u, D_x\rangle^2 + 2\|u\|^2\partial_{u_j}\langle D_u, D_x\rangle(2\mathbb{E}_u + m - 4),$$

we use (3) to get

$$\begin{aligned}
&[\|u\|^2\langle D_u, D_x\rangle^2 D_x, \mathcal{C}_2] = \|u\|^2\langle D_u, D_x\rangle^2[D_x, \mathcal{C}_2] + [\|u\|^2\langle D_u, D_x\rangle^2, \mathcal{C}_2]\\
=&\ (2m + 4k - 8)\|u\|^2\langle D_u, D_x\rangle D_x\partial_{u_j} + 2x\|u\|^2\langle D_u, D_x\rangle^2\partial_{x_j}\\
&-(m + 2\mathbb{E}_x + 2)\|u\|^2\langle D_u, D_x\rangle^2 e_j - 6x_j\|u\|^2\langle D_u, D_x\rangle^2 D_x.
\end{aligned}$$

Combining above two equations completes the proof. □

Lemma 5 *For all $1 \leq j \leq m$, we have*

$$\begin{aligned}
&[u\langle D_u, D_x\rangle D_x^2, \mathcal{C}_3] = -2e_j u\langle D_u, D_x\rangle D_x - 4u_j\langle D_u, D_x\rangle D_x - (m + 2k)uD_x^2\partial_{u_j}\\
&+4u\langle u, D_x\rangle\langle D_u, D_x\rangle\partial_{u_j} - 4u_j u\langle D_u, D_x\rangle^2 + 6x_j u\langle D_u, D_x\rangle D_x^2.
\end{aligned}$$

Proof Similarly, we have

$$\begin{aligned}
&[u\langle D_u, D_x\rangle D_x^2, xe_j]\\
=&\ u\langle D_u, D_x\rangle[D_x^2, x]e_j+u\langle D_u, D_x\rangle x[D_x^2, e_j]+[u\langle D_u, D_x\rangle, x]e_j D_x^2+x[u\langle D_u, D_x\rangle, e_j]D_x^2\\
=&\ -2e_j u\langle D_u, D_x\rangle D_x - 4u_j\langle D_u, D_x\rangle D_x + 4u\langle D_u, D_x\rangle\partial_{x_j} - 2u\partial_{u_j}D_x^2\\
&+2u_j x\langle D_u, D_x\rangle D_x^2 - 2\langle u, x\rangle\langle D_u, D_x\rangle e_j D_x^2,
\end{aligned}$$

and

$$\begin{aligned}
&[u\langle D_u, D_x\rangle D_x^2, \mathcal{C}_2] = u\langle D_u, D_x\rangle D_x[D_x, \mathcal{C}_2] + [u\langle D_u, D_x\rangle D_x, \mathcal{C}_2]D_x\\
=&\ (m+2k-2)uD_x^2\partial_{u_j}+4u\langle D_u, D_x\rangle\partial_{x_j}-4u\langle u, D_x\rangle\langle D_u, D_x\rangle\partial_{u_j}+4u_j u\langle D_u, D_x\rangle^2\\
&-6x_j u\langle D_u, D_x\rangle D_x^2 - 2e_j\langle u, x\rangle\langle D_u, D_x\rangle D_x^2 + 2u_j x\langle D_u, D_x\rangle D_x^2.
\end{aligned}$$

Combining above two completes the proof. □

Lemma 6 *For all* $1 \leq j \leq m$, *we have*

$$[u\langle u, D_x\rangle\langle D_u, D_x\rangle^2, \mathcal{C}_3] = -e_j\|u\|^2\langle D_u, D_x\rangle^2 - (2m+4k-4)u\langle u, D_x\rangle\langle D_u, D_x\rangle\partial_{u_j}$$
$$-2u|u|^2\langle D_u, D_x\rangle^2\partial_{u_j} + (m+2k-2)u_j u\langle D_u, D_x\rangle^2 + 6x_j u\langle u, D_x\rangle\langle D_u, D_x\rangle^2.$$

Proof Similarly, we have

$$\begin{aligned}
&[u\langle u, D_x\rangle\langle D_u, D_x\rangle^2, xe_j]\\
=\ & u\langle u, D_x\rangle[\langle D_u, D_x\rangle^2, x]e_j + u\langle u, D_x\rangle x[\langle D_u, D_x\rangle^2, e_j] + [u\langle u, D_x\rangle, x]e_j\langle D_u, D_x\rangle^2\\
&+x[u\langle u, D_x\rangle, e_j]\langle D_u, D_x\rangle^2\\
=\ & -4u\langle u, D_x\rangle\langle D_u, D_x\rangle\partial_{u_j} - e_j\|u\|^2\langle D_u, D_x\rangle^2 + 2u_j x\langle u, D_x\rangle\langle D_u, D_x\rangle^2\\
&-2e_j\langle u, x\rangle\langle u, D_x\rangle\langle D_u, D_x\rangle^2,
\end{aligned}$$

and

$$\begin{aligned}
&[u\langle u, D_x\rangle\langle D_u, D_x\rangle^2, \mathcal{C}_2] = u[\langle u, D_x\rangle\langle D_u, D_x\rangle^2, \mathcal{C}_2] + [u, \mathcal{C}_2]\langle u, D_x\rangle\langle D_u, D_x\rangle^2\\
=\ & (2m+4k-8)u\langle u, D_x\rangle\langle D_u, D_x\rangle\partial_{u_j} + 2u\|u\|^2\langle D_u, D_x\rangle^2\partial_{u_j}\\
&-2e_j\langle u, x\rangle\langle u, D_x\rangle\langle D_u, D_x\rangle^2 - 2u\langle u, x\rangle\langle D_u, D_x\rangle^2\partial_{x_j} - 2u_j u(\mathbb{E}_u - \mathbb{E}_x)\langle D_u, D_x\rangle^2\\
&+2u_j x\langle u, D_x\rangle\langle D_u, D_x\rangle^2 + 2u\langle u, x\rangle\langle D_u, D_x\rangle^2\partial_{x_j} - (m+2\mathbb{E}_x+2)u_j u\langle D_u, D_x\rangle^2\\
&-6x_j u\langle u, D_x\rangle\langle D_u, D_x\rangle^2.
\end{aligned}$$

Combining above two equations completes the proof. □

Lemma 7 *For all* $1 \leq j \leq m$, *we have*

$$[u^3\langle D_u, D_x\rangle^3, \mathcal{C}_3] = -(m+6k-10)u^3\langle D_u, D_x\rangle^2\partial_{u_j} + 6x_j u^3\langle D_u, D_x\rangle^3.$$

Proof Similarly, we have

$$\begin{aligned}
&[u^3\langle D_u, D_x\rangle^3, xe_j]\\
=\ & u^3\langle D_u, D_x\rangle[\langle D_u, D_x\rangle^2, x]e_j + u^3\langle D_u, D_x\rangle x[\langle D_u, D_x\rangle^2, e_j]\\
&+[u^3\langle D_u, D_x\rangle, x]e_j\langle D_u, D_x\rangle^2 + x[u^3\langle D_u, D_x\rangle, e_j]\langle D_u, D_x\rangle^2\\
=\ & -4u^3\langle D_u, D_x\rangle^2\partial_{u_j} - 2u^3\langle D_u, D_x\rangle^2\partial_{u_j} + 2xu_j u^2\langle D_u, D_x\rangle^3 - 2e_j u^2\langle u, x\rangle\langle D_u, D_x\rangle^3,
\end{aligned}$$

and

$$[u^3\langle D_u, D_x\rangle^3, \mathcal{C}_2] = u^3\langle D_u, D_x\rangle[\langle D_u, D_x\rangle^2, \mathcal{C}_2] + [u^3\langle D_u, D_x\rangle, \mathcal{C}_2]\langle D_u, D_x\rangle^2$$

$$= (m+6k-16)u^3\langle D_u, D_x\rangle^2\partial_{u_j}+2u^3\langle x, D_u\rangle\langle D_u, D_x\rangle^2\partial_{x_j}-2e_j u^2\langle u, x\rangle\langle D_u, D_x\rangle^3$$
$$-6x_j u^3\langle D_u, D_x\rangle^3.$$

Combining above two questions completes the proof. □

Combining Lemma 2 to 7 gives the results. We use these lemmas to obtain

$$[\mathcal{D}_3, \mathcal{C}_3] = 6x_j\mathcal{D}_3.$$

Similar arguments as in [10] give that $\mathcal{J}_3\mathcal{D}_3\mathcal{J}_3 = ||x||^6\mathcal{D}_3$, which can be rewritten as

$$\mathcal{D}_{3,y,w}\frac{x}{||x||^{m-2}}f(y, w) = \frac{x}{||x||^{m+2}}\mathcal{D}_{3,x,u}f(x, u),\ \forall f(x, u) \in C^\infty(\mathbb{R}^m, \mathcal{M}_k),$$

where $y = x^{-1}$ and $w = \dfrac{xux}{||x||^2}$. Therefore, we have proved $\mathcal{D}_3$ is invariant under inversion. The uniqueness is determined by the results given in [27, 28]. More details on this can also be found in [10].

3.2 4th-Order Higher Spin Operator $\mathcal{D}_4$

Now for the main result in the $4th$-order higher spin case.

Theorem 3 *Up to a multiplicative constant, the unique 4th-order conformally invariant differential operator is $\mathcal{D}_4 : C^\infty(\mathbb{R}^m, \mathcal{H}_k) \longrightarrow C^\infty(\mathbb{R}^m, \mathcal{H}_k)$, where*

$$\mathcal{D}_4 = \mathcal{D}_2^2 - \frac{8}{(m+2k-2)(m+2k-4)}\mathcal{D}_2\Delta_x.$$

Hereafter we may suppress the k index for the operator since there is little risk of confusion. The strategy is similar to that used above. It is sufficient to show only invariance under inversion. We have the definition for harmonic inversion as follows.

Definition 3 Harmonic inversion is a (conformal) transformation defined as

$$\mathcal{J}_4 : C^\infty(\mathbb{R}^m, \mathcal{H}_k) \longrightarrow C^\infty(\mathbb{R}^m, \mathcal{H}_k) : f(x, u) \mapsto \mathcal{J}_4[f](x, u) := ||x||^{4-m}f(\frac{x}{||x||^2}, \frac{xux}{||x||^2}).$$

Note this inversion consists of the classical Kelvin inversion $\mathcal{J}$ on $\mathbb{R}^m$ in the variable x composed with a reflection $u \mapsto \omega u\omega$ acting on the dummy variable u (where $x = ||x||\omega$) and a multiplication by a conformal weight term $||x||^{4-m}$. It satisfies $\mathcal{J}_4^2 = 1$. Then a similar calculation as in *Proposition A.1* in [6] provides the following lemma.

Lemma 8 *The special conformal transformation is defined as*

$$\mathcal{C}_4 := \mathcal{J}_4 \partial_{x_j} \mathcal{J}_4 = 2\langle u, x\rangle \partial_{u_j} - 2u_j \langle x, D_u\rangle + ||x||^2 \partial_{x_j} - x_j(2\mathbb{E}_x + m - 4).$$

Proposition 2 *The special conformal transformations $\mathcal{C}_4$, with $j \in \{1, 2, \ldots, m\}$ are generalized symmetries of $\mathcal{D}_4$. More specifically,*

$$[\mathcal{D}_4, \mathcal{C}_4] = -8x_j \mathcal{D}_4.$$

In particular, this shows $\mathcal{J}_4\mathcal{D}_4\mathcal{J}_4 = ||x||^8\mathcal{D}_4$, which generalizes the case of the classical higher order Dirac operator D_x^4. This also implies $\mathcal{D}_4$ is invariant under inversion and hence conformally invariant.

If the previous proposition holds, then the conformal invariance of $\mathcal{D}_4$ can be summarized in the following theorem:

Theorem 4 *The first-order generalized symmetries of $\mathcal{D}_4$ are given by:*

1. *The infinitesimal rotations $L^x_{i,j} + L^u_{i,j}$, with $1 \leq i < j \leq m$.*
2. *The shifted Euler operator $m + 2\mathbb{E}_x - 4$.*
3. *The infinitesimal translations ∂_{x_j}, with $1 \leq j \leq m$.*
4. *The special conformal transformations $\mathcal{J}_4 \partial_{x_j} \mathcal{J}_4$, with $1 \leq j \leq m$.*

These operators span a Lie algebra which is isomorphic to the conformal Lie algebra $\mathfrak{so}(1, m + 1)$, whereby the Lie bracket is the ordinary commutator.

Proof The proof is similar as in [14] via transvector algebras. □

Detailed Proof of Proposition 2

The previous proposition follows immediately with the help of the following two lemmas.

Lemma 9

$$\begin{aligned}
\left[\mathcal{D}_2^2, \mathcal{C}_4\right] = &-8x_j\mathcal{D}_2^2 + \frac{32\langle u, D_x\rangle \Delta_x \partial_{u_j}}{(m+2k-2)^2} - \frac{32u_j\langle D_u, D_x\rangle \Delta_x}{(m+2k-2)^2} \\
&- \frac{128\langle u, D_x\rangle^2 \langle D_u, D_x\rangle \partial_{u_j}}{(m+2k-2)^2(m+2k-4)} + \frac{128||u||^2\langle D_u, D_x\rangle \Delta_x \partial_{u_j}}{(m+2k-2)^2(m+2k-4)^2} \\
&- \frac{128||u||^2\langle D_u, D_x\rangle^2 \partial_{x_j}}{(m+2k-2)^2(m+2k-4)^2} + \frac{128u_j\langle u, D_x\rangle \langle D_u, D_x\rangle^2}{(m+2k-2)^2(m+2k-4)} \\
&+ \frac{128||u||^2\langle u, D_x\rangle \langle D_u, D_x\rangle^2 \partial_{u_j}}{(m+2k-2)^2(m+2k-4)^2} - \frac{128u_j||u||^2\langle D_u, D_x\rangle^3}{(m+2k-2)^2(m+2k-4)^2}.
\end{aligned}$$

Proof Similar as in the odd order case, we use the identity

$$\mathcal{C}_4 = \mathcal{C}_2 + 2x_j$$

and (see [6] page 25)

$$[\mathcal{D}_2, \mathcal{C}_2] = -4x_j\mathcal{D}_2, \tag{5}$$

then, we have

$$\begin{aligned} &[\mathcal{D}_2^2, \mathcal{C}_4]{=}[\mathcal{D}_2^2, \mathcal{C}_2]{+}[\mathcal{D}_2^2, 2x_j]{=}\mathcal{D}_2[\mathcal{D}_2, \mathcal{C}_2]{+}[\mathcal{D}_2, \mathcal{C}_2]\mathcal{D}_2{+}2\mathcal{D}_2^2x_j - 2x_j\mathcal{D}_2^2\\ &= -4\mathcal{D}_2x_j\mathcal{D}_2 - 4x_j\mathcal{D}_2^2 + 2\mathcal{D}_2^2x_j - 2x_j\mathcal{D}_2^2\\ &= 2\mathcal{D}_2^2x_j - 2\mathcal{D}_2x_j\mathcal{D}_2 - 2\mathcal{D}_2x_j\mathcal{D}_2 + 2x_j\mathcal{D}_2^2 - 8x_j\mathcal{D}_2^2\\ &= 2[\mathcal{D}_2, [\mathcal{D}_2, x_j]] - 8x_j\mathcal{D}_2^2. \end{aligned} \tag{6}$$

It is easy to check that

$$[\mathcal{D}_2, x_j]{=}2\partial_{x_j}{-}\frac{4(\langle u, D_x\rangle\partial_{u_j}{+}u_j\langle D_u, D_x\rangle)}{m+2k-2}{+}\frac{8\|u\|^2\langle D_u, D_x\rangle\partial_{u_j}}{(m+2k-2)(m+2k-4)}. \tag{7}$$

Then we can obtain that

$$\begin{aligned} [\mathcal{D}_2, [\mathcal{D}_2, x_j]] = {}& \frac{16\langle u, D_x\rangle\Delta_x\partial_{u_j}}{(m+2k-2)^2} - \frac{16u_j\langle D_u, D_x\rangle\Delta_x}{(m+2k-2)^2} - \frac{64\langle u, D_x\rangle^2\langle D_u, D_x\rangle\partial_{u_j}}{(m{+}k2{-}2)^2(m{+}2k{-}4)}\\ &+\frac{64\|u\|^2\langle D_u, D_x\rangle\Delta_x\partial_{u_j}}{(m+2k-2)^2(m+2k-4)^2} - \frac{64\|u\|^2\langle D_u, D_x\rangle^2\partial_{x_j}}{(m+2k-2)^2(m+2k-4)^2}\\ &+\frac{64u_j\langle u, D_x\rangle\langle D_u, D_x\rangle^2}{(m+2k-2)^2(m+2k-4)} + \frac{64\|u\|^2\langle u, D_x\rangle\langle D_u, D_x\rangle^2\partial_{u_j}}{(m+2k-2)^2(m+2k-4)^2}\\ &-\frac{64\|u\|^2u_j\langle D_u, D_x\rangle^3}{(m+2k-2)^2(m+2k-4)^2}. \end{aligned}$$

We leave the details of the calculation above as an exercise. Plugging the previous equation into (6) completes the proof. □

Lemma 10

$$\begin{aligned} [\mathcal{D}_2\Delta_x, \mathcal{C}_4] = {}& -8x_j\mathcal{D}_2\Delta_x{+}\frac{4m{+}8k{-}16}{m{+}2k{-}2}\langle u, D_x\rangle\Delta_x\partial_{u_j}{-}\frac{16\langle u, D_x\rangle^2\langle D_u, D_x\rangle\partial_{u_j}}{m{+}2k{-}2}\\ &+\frac{16||u||^2\langle D_u, D_x\rangle\Delta_x\partial_{u_j}}{(m+2k-2)(m+2k-4)} + \frac{16||u||^2\langle u, D_x\rangle\langle D_u, D_x\rangle^2\partial_{u_j}}{(m+2k-2)(m+2k-4)} \end{aligned}$$

$$-\frac{4m+8k-16}{m+2k-2}u_j\langle D_u, D_x\rangle\Delta_x + \frac{16u_j\langle u, D_x\rangle\langle D_u, D_x\rangle^2}{m+2k-2}$$
$$-\frac{16||u||^2\langle D_u, D_x\rangle^2\partial_{x_j}}{(m+2k-2)(m+2k-4)} - \frac{16u_j||u||^2\langle D_u, D_x\rangle^3}{(m+2k-2)(m+2k-4)}.$$

Proof With the help of (3), we have

$$[\mathcal{D}_2\Delta_x, \mathcal{C}_4] = [\mathcal{D}_2\Delta_x, \mathcal{C}_2] + 2[\mathcal{D}_2\Delta_x, x_j]$$
$$= \mathcal{D}_2[\Delta_x, \mathcal{C}_2] + [\mathcal{D}_2, \mathcal{C}_2]\Delta_x + 2\mathcal{D}_2[\Delta_x, x_j] + 2[\mathcal{D}_2, x_j]\Delta_x. \quad (8)$$

Since we already have (see [6] Lemma A.1.)

$$[\Delta_x, \mathcal{C}_2] = -4x_j\Delta_x + 4\langle u, D_x\rangle\partial_{u_j} - 4u_j\langle D_u, D_x\rangle,$$

and with the help of (5), (7) and $[\Delta_x, x_j] = 2\partial_{x_j}$, plugging them into (8), a straightforward calculation completes the proof. □

With Lemma 9 and 10, Proposition 2 is followed immediately. The uniqueness is also determined by results in [27, 28]. For more details, we refer the readers to [10].

4 Fundamental Solutions and Intertwining Operators

Using similar arguments as in [10], we obtain the fundamental solutions (up to a multiplicative constant) and intertwining operators of $\mathcal{D}_3$ and $\mathcal{D}_4$ as follows.

Theorem 5 (Fundamental Solutions of $\mathcal{D}_3$) *Let $Z_k(u, v)$ be the reproducing kernel of $\mathcal{M}_k$, then the fundamental solutions of $\mathcal{D}_3$ are*

$$c_1\frac{x}{||x||^{m-2}}Z_k\Big(\frac{xux}{||x||^2}, v\Big),$$

where the constant c_1 is determined from [11]:

$$\frac{(m+2k-4)}{2(m-2)(m-4)\omega_{m-1}},$$

where ω_{m-1} is the surface area of $(m-1)$-dimensional unit sphere.

Theorem 6 (Fundamental Solutions of $\mathcal{D}_4$) *Let $Z_k(u, v)$ be the reproducing kernel of $\mathcal{H}_k$, then the fundamental solutions of $\mathcal{D}_4$ are*

$$c_2||x||^{4-m}Z_k\Big(\frac{xux}{||x||^2}, v\Big),$$

where the constant c_2 is also determined from [11]:

$$\frac{(m+2k-2)(m+2k-4)\Gamma(\frac{m}{2}-1)}{32(m-2)(m-4)\pi^{\frac{m}{2}}}.$$

Theorem 7 (Intertwining Operators) *Let $y=\phi(x)=(ax+b)(cx+d)^{-1}$ be a Möbius transformation. Then*

$$\frac{\widetilde{cx+d}}{||cx+d||^{m+4}}\mathcal{D}_{3,y,\omega}f(y,\omega)=\mathcal{D}_{3,x,u}\frac{\widetilde{cx+d}}{||cx+d||^{m-2}}f(\phi(x),\frac{(cx+d)u(\widetilde{cx+d})}{||cx+d||^2}),$$

where $\omega=\dfrac{(cx+d)u(\widetilde{cx+d})}{||cx+d||^2}$ *and* $f(y,\omega)\in C^\infty(\mathbb{R}^m,\mathcal{M}_k)$*;*

$$||cx+d||^{-m-4}\mathcal{D}_{4,y,\omega}f(y,\omega)=\mathcal{D}_{4,x,u}||cx+d||^{4-m}f(\phi(x),\frac{(cx+d)u(\widetilde{cx+d})}{||cx+d||^2}),$$

where $\omega=\dfrac{(cx+d)u(\widetilde{cx+d})}{||cx+d||^2}$ *and* $f(y,\omega)\in C^\infty(\mathbb{R}^m,\mathcal{H}_k)$.

It is worth pointing out that our above results generalize to conformally flat manifolds according to the method in our paper on cylinders and Hopf manifolds [9].

5 Connection with Lower Order Conformally Invariant Operators

To construct higher order conformally invariant operators, one possible method is by composing and combining lower order conformally invariant operators. In this section, we will rewrite our operators $\mathcal{D}_3$ and $\mathcal{D}_4$ in terms of first-order and second-order conformally invariant operators. This might help us to construct higher order conformally invariant differential operators by induction from the lower order ones.

Recall $\mathcal{D}_3$ maps $C^\infty(\mathbb{R}^m,\mathcal{M}_k)$ to $C^\infty(\mathbb{R}^m,\mathcal{M}_k)$. If we fix $x\in\mathbb{R}^m$, then for any $f(x,u)\in\mathcal{M}_k$, we have $\mathcal{D}_3f(x,u)\in\mathcal{M}_k$. In other words, $\mathcal{D}_3$ should be equal to the sum of contributions to $\mathcal{M}_k$ of all terms in $\mathcal{D}_3$. Notice that if we apply each term of $\mathcal{D}_3$ to $f(x,u)\in C^\infty(\mathbb{R}^m,\mathcal{M}_k)$, we will get a k-homogeneous polynomial in u that is in the kernel of Δ_u^2. Hence, we can decompose it by harmonic decomposition as follows:

$$\mathcal{P}_k=\mathcal{H}_k\oplus u^2\mathcal{H}_{k-2},$$

where $\mathcal{P}_k$ is the k-homogeneous polynomial space and $\mathcal{H}_k$ is the k-homogeneous harmonic polynomial space. The Almansi–Fischer decomposition provides further

$$\mathcal{H}_k = \mathcal{M}_k \oplus u\mathcal{M}_{k-1},$$

where $\mathcal{M}_k$ is the k-homogeneous monogenic polynomial space; therefore, the contribution of each term to $\mathcal{M}_k$ can be written with two projections. For instance, the contribution of $u^3\langle D_u, D_x\rangle^3 f(x,u)$ to $\mathcal{M}_k$ is $P_k P_1 u^3\langle D_u, D_x\rangle^3 f(x,u)$, where

$$\mathcal{P}_k \xrightarrow{P_1} \mathcal{H}_k \xrightarrow{P_k} \mathcal{M}_k,$$

and

$$P_1 = 1 + \frac{u^2\Delta_u}{2(m+2k-4)},\ \ P_k = 1 + \frac{uD_u}{m+2k-2}.$$

We also notice that for fixed $x \in \mathbb{R}^m$ and $f(x,u) \in \mathcal{M}_k$,

$$u^3\langle D_u, D_x\rangle^3 f(x,u),\ \ ||u||^2\langle D_u, D_x\rangle^2 D_x f(x,u) \in u^2\mathcal{H}_{k-2},$$

and $u\langle D_u, D_x\rangle D_x^2 \in u\mathcal{M}_{k-1}$. Hence, their contributions to $\mathcal{M}_k$ are all zero. Therefore,

$$\mathcal{D}_3 = P_k P_1\bigg(D_x^3 + \frac{4}{m+2k}\langle u, D_x\rangle\langle D_u, D_x\rangle D_x - \frac{8u\langle u, D_x\rangle\langle D_u, D_x\rangle^2}{(m+2k)(m+2k-2)}\bigg).$$

It is useful to recall some first- and second-order conformally invariant operators in higher spin spaces [5, 6]:

$$R_k : C^\infty(\mathbb{R}^m, \mathcal{M}_k) \longrightarrow C^\infty(\mathbb{R}^m, \mathcal{M}_k),\ R_k = P_k D_x = (1 + \frac{uD_u}{m+2k-2})D_x;$$

$$T_k : C^\infty(\mathbb{R}^m, u\mathcal{M}_{k-1}) \longrightarrow C^\infty(\mathbb{R}^m, \mathcal{M}_k),\ T_k{=}P_k D_x{=}(1{+}\frac{uD_u}{m{+}2k{-}2})D_x;$$

$$T_k^* : C^\infty(\mathbb{R}^m, \mathcal{M}_k) \longrightarrow C^\infty(\mathbb{R}^m, u\mathcal{M}_{k-1}),\ T_k^*{=}(I{-}P_k)D_x{=}\frac{-uD_u}{m{+}2k{-}2}D_x;$$

$$\mathcal{D}_2 : C^\infty(\mathbb{R}^m, \mathcal{H}_k) \longrightarrow C^\infty(\mathbb{R}^m, \mathcal{H}_k),\ \mathcal{D}_2{=}P_1(\Delta_x{-}\frac{4}{m{+}2k{-}2}\langle u, D_x\rangle\langle D_u, D_x\rangle D_x).$$

Hence,

$$\mathcal{D}_3{=}P_k P_1\bigg(D_x^3{+}\frac{4\langle u, D_x\rangle\langle D_u, D_x\rangle D_x}{m{+}2k{-}2}{-}\frac{8\langle u, D_x\rangle\langle D_u, D_x\rangle D_x}{(m{+}2k)(m{+}2k{-}2)}{-}\frac{8u\langle u, D_x\rangle\langle D_u, D_x\rangle^2}{(m{+}2k)(m{+}2k{-}2)}\bigg)$$

$$={-}P_k P_1\mathcal{D}_2 D_x - \frac{8}{(m+2k)(m+2k-2)}P_k P_1\big(\langle u, D_x\rangle\langle D_u, D_x\rangle D_x{+}u\langle u, D_x\rangle\langle D_u, D_x\rangle^2\big).$$

Since for $f(x,u) \in C^\infty(\mathbb{R}^m, \mathcal{M}_k)$, we have [6]

$$\mathcal{D}_2 = -R_k^2 + \frac{4u\langle D_u, D_x\rangle}{(m+2k-2)(m+2k-4)} R_k.$$

A straightforward calculation leads to

$$\mathcal{D}_3 = R_k^3 - \frac{4}{(m+2k)(m+2k-4)} T_k T_k^* R_k.$$

Recall these conformally invariant second-order twistor and dual-twistor operators [6]:

$$T_{k,2} = \langle u, D_x\rangle - \frac{||u||^2\langle D_u, D_x\rangle}{m+2k-4} : C^\infty(\mathbb{R}^m, \mathcal{H}_{k-1}) \longrightarrow C^\infty(\mathbb{R}^m, \mathcal{H}_k),$$

$$T_{k,2}^* = \langle D_u, D_x\rangle : C^\infty(\mathbb{R}^m, \mathcal{H}_k) \longrightarrow C^\infty(\mathbb{R}^m, \mathcal{H}_{k-1}), \text{ and}$$

$$\mathcal{D}_2 = \Delta_x - \frac{4T_{k,2}T_{k,2}^*}{m+2k-2}.$$

Hence

$$\begin{aligned}
\mathcal{D}_4 &= \mathcal{D}_2^2 - \frac{8\mathcal{D}_2\Delta_x}{(m+2k-2)(m+2k-4)} \\
&= \mathcal{D}_2^2 - \frac{8\mathcal{D}_2}{(m+2k-2)(m+2k-4)}\Big(\mathcal{D}_2 + \frac{4T_{k,2}T_{k,2}^*}{m+2k-2}\Big) \\
&= \frac{(m+2k)(m+2k-6)}{(m+2k-2)(m+2k-4)}\mathcal{D}_2^2 - \frac{32\mathcal{D}_2 T_{k,2}T_{k,2}^*}{(m+2k-2)^2(m+2k-4)}.
\end{aligned}$$

Remark The 3rd-order fermionic and 4th-order bosonic operators constructed here are consistent with the 3rd- and 4th-order conformally invariant differential operators obtained from our paper [11]. Further, from the expressions of $\mathcal{D}_3$ and $\mathcal{D}_4$ obtained previously, we notice that it is hardly to find general expressions for higher order conformally invariant differential operators by induction. This also reveals the limit of the generalized symmetries approach used in [6, 15]. Therefore, a different approach is introduced for other higher order cases in our paper [11].

6 Ellipticity

We start with the definition of ellipticity as follows.

Definition 4 A linear homogeneous differential operator of k-th order $\mathcal{D}_k$: $C^\infty(\mathbb{R}^m, V_\lambda) \longrightarrow C^\infty(\mathbb{R}^m, V_\mu)$ is elliptic if for every non-zero vector $x \in \mathbb{R}^m$

its principal symbol, the linear map $\sigma_x(\mathcal{D}_k) : V_\lambda \longrightarrow V_\mu$ obtained by replacing its partial derivatives ∂_{x_j} with the corresponding variables x_j, is a linear isomorphism.

Note V_λ stands for a representation space of $Spin(m)$ with a dominant weight λ. The proofs below will be extended to their full generality in a subsequent paper.

6.1 Ellipticity for 3rd-Order Higher Spin Operator $\mathcal{D}_3$

Theorem 8 *The* 3rd*-order higher spin operator, which is explicitly given by*

$$\mathcal{D}_3 = R_k^3 - \frac{4}{(m+2k)(m+2k-4)} T_k T_k^* R_k,$$

is an elliptic operator if $m > 4$.

Proof The technique used here is motivated by [6]. The critical point in this technique is the following: when proving the principal symbol is a linear map from $\mathcal{M}_k$ to $\mathcal{M}_k$, we choose a basis obtained from the classical CK extension for monogenic polynomials [19]. This helps us to see that the symbol is an injective map. On the other hand, the symbol is obviously linear, which completes our proof.

Let $R_k(x)$, $T_k(x)$, and $T_k^*(x)$ be the symbols of R_k, T_k, and T_k^*, respectively. We will show that for fixed $x \in \mathbb{R}^m$, the symbol of $\mathcal{D}_3$, which is given by

$$\sigma_x(\mathcal{D}_3) = R_k(x)^3 - \frac{4}{(m+2k)(m+2k-4)} T_k(x) T_k^*(x) R_k(x) : \ \mathcal{M}_k \longrightarrow \mathcal{M}_k,$$

is a linear isomorphism. Since this symbol is obviously a linear map, it remains to be showed that this map is injective. Notice that

$$\sigma_x(\mathcal{D}_3) = \big(R_k(x)^2 - \frac{4}{(m+2k)(m+2k-4)} T_k(x) T_k^*(x)\big) R_k(x) : \ \mathcal{M}_k \longrightarrow \mathcal{M}_k,$$

and R_k is an elliptic operator [16]. Therefore, we only need to show what the term in the parenthesis above

$$\begin{aligned}
&\sigma_x(\mathcal{D}_3)' := R_k(x)^2 - \frac{4}{(m+2k)(m+2k-4)} T_k(x) T_k^*(x) \\
&= -||x||^2 + \frac{4ux\langle x, D_u\rangle}{(m+2k)(m+2k-4)} + \frac{4(m+2k-2)\langle u, x\rangle\langle x, D_u\rangle}{(m+2k)(m+2k-4)} \\
&\quad - \frac{4||u||^2\langle x, D_u\rangle^2}{(m+2k)(m+2k-4)} : \ \mathcal{M}_k \longrightarrow \mathcal{M}_k,
\end{aligned}$$

is injective. To do so, we choose a basis for $\mathcal{M}_k(\mathbb{R}^m, \mathcal{S})$ as in [6, 19]. First, we need the monogenic inversion

$$\mathcal{J} : \mathcal{M}_k(\mathbb{R}^m, \mathcal{S}) \longrightarrow \mathcal{M}_k(\mathbb{R}^m, \mathcal{S}) : \ f(u) \mapsto \mathcal{J}[f](u) := \frac{u}{||u||^m} f(\frac{u}{||u||^2}).$$

In this case, we also have that

$$\mathfrak{sl}(2) \cong Span(\mathcal{J}\partial_{u_j}\mathcal{J}, \partial_{u_j}, m + 2\mathbb{E}_u - 1),$$

where

$$\mathcal{J}\partial_{u_j}\mathcal{J} = ue_j + u_j(m + 2\mathbb{E}_u) - ||u||^2\partial_{u_j}.$$

For fixed $x \in \mathbb{R}^m\backslash\{0\}$, we have

$$\mathcal{J}\langle x, D_u\rangle\mathcal{J} = ux + \langle u, x\rangle(m + 2\mathbb{E}_u) - ||u||^2\langle x, D_u\rangle. \tag{9}$$

This means we can rewrite $\sigma_x(\mathcal{D}_3)'$ as

$$||x||^2\big(- 1 + \frac{4}{(m + 2k)(m + 2k - 4)}\mathcal{J}\langle\omega, D_u\rangle\mathcal{J}\langle\omega, D_u\rangle\big),$$

where $\omega = \dfrac{x}{||x||}$. The branching rules for $\mathfrak{so}(m)$ state that when we restrict the action on the irreducible representation with highest weight $(k + \frac{1}{2}, \frac{1}{2}, \cdots, \frac{1}{2})$ to $\mathfrak{so}(m - 1)$, we get the following decomposition:

$$(k + \frac{1}{2}, \frac{1}{2}, \cdots, \frac{1}{2})\Big|^{\mathfrak{so}(m)}_{\mathfrak{so}(m-1)} = \bigoplus_{j=0}^{k}(k - j + \frac{1}{2}, \frac{1}{2}, \cdots, \frac{1}{2}).$$

This implies that an arbitrary monogenic polynomial $f_k(u) \in \mathcal{M}_k(\mathbb{R}^m, \mathcal{S})$ can be written as

$$f_k(u) = \sum_{j=0}^{k}(\mathcal{J}\langle\omega, D_u\rangle\mathcal{J})^j f^*_{k-j}(u),$$

where $f^*_{k-j}(u) \in \mathcal{M}_{k-j}(\mathbb{R}^m, \mathcal{S})$ and $\langle\omega, D_u\rangle f^*_{k-j}(u) = 0$. It is obvious that the right-hand side of the equation above is invariant under the action of $\mathfrak{so}(m - 1)$, where $\mathfrak{so}(m - 1)$ is understood as the Lie algebra corresponding to the subgroup of $SO(m - 1)$, which contains rotations in the hyperplane perpendicular to $\omega \in \mathbb{R}^m$. Now, we claim that

$$[\langle\omega, D_u\rangle, (\mathcal{J}\langle\omega, D_u\rangle\mathcal{J})^j] = j(\mathcal{J}\langle\omega, D_u\rangle\mathcal{J})^{j-1}(m + 2\mathbb{E}_u + j - 2).$$

Indeed, with the expression (9), it is easy to obtain that

$$[\langle\omega, D_u\rangle, \mathcal{J}\langle\omega, D_u\rangle\mathcal{J}] = m + 2\mathbb{E}_u - 1.$$

Then, suppose it is true for $j - 1$ and with the identity that $[A, BC] = [A, B]C + B[A, C]$, where A, B, and C are operators. We have

$$\begin{aligned}
&[\langle\omega, D_u\rangle, (\mathcal{J}\langle\omega, D_u\rangle\mathcal{J})^j]\\
&= [\langle\omega, D_u\rangle, (\mathcal{J}\langle\omega, D_u\rangle\mathcal{J})^{j-1}]\mathcal{J}\langle\omega, D_u\rangle\mathcal{J} + (\mathcal{J}\langle\omega, D_u\rangle\mathcal{J})^{j-1}[\langle\omega, D_u\rangle, \mathcal{J}\langle\omega, D_u\rangle\mathcal{J}]\\
&= (j-1)(\mathcal{J}\langle\omega, D_u\rangle\mathcal{J})^{j-2}(m+2\mathbb{E}_u+j-3)\mathcal{J}\langle\omega, D_u\rangle\mathcal{J} + (\mathcal{J}\langle\omega, D_u\rangle\mathcal{J})^{j-1}(m+2\mathbb{E}_u-1)\\
&= (j-1)(\mathcal{J}\langle\omega, D_u\rangle\mathcal{J})^{j-1}(m + 2\mathbb{E}_u+j-1) + (\mathcal{J}\langle\omega, D_u\rangle\mathcal{J})^{j-1}(m+2\mathbb{E}_u-1)\\
&= j(\mathcal{J}\langle\omega, D_u\rangle\mathcal{J})^{j-1}(m + 2\mathbb{E}_u + j - 2).
\end{aligned}$$

This completes the proof for our claim above. Therefore, the equation

$$||x||^2\Big(- 1 + \frac{4}{(m + 2k)(m + 2k - 4)}\mathcal{J}\langle\omega, D_u\rangle\mathcal{J}\langle\omega, D_u\rangle\Big) f_k(u) = 0$$

leads to the following equation:

$$\sum_{j=1}^{k}\left(- 1 + \frac{4j(m + 2k - j - 2)}{(m + 2k)(m + 2k - 4)}\right)(\mathcal{J}\langle\omega, D_u\rangle\mathcal{J})^j f^*_{k-j}(u) = 0.$$

Since the polynomials $f^*_{k-j}(u) \in \mathcal{M}_k(\mathbb{R}^m, \mathcal{S})$ are linearly independent for $1 \leq j \leq k$, we have that either $f^*_{k-j}(u) = 0$ for all $1 \leq j \leq k$, which means that $ker\sigma_x(\mathcal{D}_3)' = 0$ or that

$$\begin{aligned}
&-1 + \frac{4j(m + 2k - j - 2)}{(m + 2k)(m + 2k - 4)} = 0,\\
\iff\ &(m + 2k)(m + 2k - 4) - 4j(m + 2k - j - 2) = 0.
\end{aligned}$$

It is easy to find the roots are

$$m_1 = -2(k - j) \text{ and } m_2 = -2(k - j) + 4.$$

Notice that $k \geq j$, it is easy to see that for $k \in \mathbb{N}$ fixed, only $m \leq 4$ causes trouble. This means that $ker\sigma_x(\mathcal{D}_3)' = 0$, whenever, $m > 4$. This also means that $ker\sigma_x(\mathcal{D}_3) = 0$ whenever, $m > 4$. This completes the proof. □

6.2 Ellipticity for $4th$-Order Higher Spin Operator $\mathcal{D}_4$

Theorem 9 *The $4th$-order higher spin operator, which is explicitly given by*

$$\mathcal{D}_4 = \frac{(m+2k)(m+2k-6)}{(m+2k-2)(m+2k-4)}\mathcal{D}_2^2 - \frac{32\mathcal{D}_2 T_{k,2} T_{k,2}^*}{(m+2k-2)^2(m+2k-4)},$$

is an elliptic operator if $m > 6$.

Proof Here we use similar argument as in the $3rd$-order case and [6]. Let $\mathcal{D}_2(x)$, $T_{k,2}(x)$, and $T_{k,2}^*(x)$ be the symbols of $\mathcal{D}_2$, $T_{k,2}$, and $T_{k,2}^*$, respectively. To prove this theorem, we will show that for fixed $x \in \mathbb{R}^m$, the symbol of $\mathcal{D}_4$, which is given by

$$\sigma_x(\mathcal{D}_4)=\frac{(m+2k)(m+2k-6)}{(m+2k-2)(m+2k-4)}\mathcal{D}_2(x)^2-\frac{32\mathcal{D}_2(x)T_{k,2}(x)T_{k,2}^*(x)}{(m+2k-2)^2(m+2k-4)} : \mathcal{H}_k \longrightarrow \mathcal{H}_k,$$

is a linear isomorphism. Since this symbol is obviously a linear map, it remains to be showed that this map is injective. Notice that

$$\sigma_x(\mathcal{D}_4)=\mathcal{D}_2(x)\Big(\frac{(m+2k)(m+2k-6)}{(m+2k-2)(m+2k-4)}\mathcal{D}_2(x)-\frac{32T_{k,2}(x)T_{k,2}^*(x)}{(m+2k-2)^2(m+2k-4)}\Big),$$

and $\mathcal{D}_2$ is an elliptic operator when $m > 4$ [6]. Therefore, we only need to show what the term in the parenthesis above

$$\begin{aligned}
\sigma_x(\mathcal{D}_4)' &:= \frac{(m+2k)(m+2k-6)}{(m+2k-2)(m+2k-4)}\mathcal{D}_2(x) - \frac{32T_{k,2}(x)T_{k,2}^*(x)}{(m+2k-2)^2(m+2k-4)}\\
&= \frac{(m+2k)(m+2k-6)}{(m+2k-2)(m+2k-4)}||x||^2-\frac{4\langle u,x\rangle\langle x,D_u\rangle}{m+2k-2}+\frac{4||u||^2\langle x,D_u\rangle^2}{(m+2k-2)(m+2k-4)}\\
\sigma_x(\mathcal{D}_4)' &: \mathcal{H}_k \longrightarrow \mathcal{H}_k,
\end{aligned}$$

is injective. To do so, we choose a basis for $\mathcal{H}_k(\mathbb{R}^m,\mathbb{C})$ as in [6, 19]. First, we need the harmonic inversion [6]

$$\mathcal{J}' : \mathcal{H}_k(\mathbb{R}^m,\mathbb{C}) \longrightarrow \mathcal{H}_k(\mathbb{R}^m,\mathbb{C}) : \ f(u) \mapsto \mathcal{J}'[f](u) := ||u||^{2-m} f(\frac{u}{||u||^2}).$$

In this case, we also have that

$$\mathfrak{sl}(2) \cong Span(\mathcal{J}'\partial_{u_j}\mathcal{J}', \partial_{u_j}, m+2\mathbb{E}_u-2),$$

where

$$\mathcal{J}'\partial_{u_j}\mathcal{J}' = ||u||^2\partial_{u_j} - u_j(m + 2\mathbb{E}_u - 2),$$

For fixed $x \in \mathbb{R}^m\backslash\{0\}$, we have

$$\mathcal{J}'\langle x, D_u\rangle\mathcal{J}' = ||u||^2\langle x, D_u\rangle - \langle u, x\rangle(m + 2\mathbb{E}_u - 2). \tag{10}$$

This means we can rewrite $\sigma_x(\mathcal{D}_4)'$ as

$$||x||^2\Big(\frac{(m+2k)(m+2k-6)}{(m+2k-2)(m+2k-4)}+\frac{4}{(m+2k-2)(m+2k-4)}\mathcal{J}'\langle\omega, D_u\rangle\mathcal{J}'\langle\omega, D_u\rangle\Big),$$

where $\omega = \dfrac{x}{||x||}$. As described in [6], an arbitrary harmonic polynomial $g_k(u) \in \mathcal{H}_k(\mathbb{R}^m, \mathbb{C})$ can be written as

$$g_k(u) = \sum_{j=0}^{k}(\mathcal{J}'\langle\omega, D_u\rangle\mathcal{J}')^j g^*_{k-j}(u),$$

where $g^*_{k-j}(u) \in \mathcal{H}_{k-j}(\mathbb{R}^m, \mathbb{C})$ and $\langle\omega, D_u\rangle f^*_{k-j}(u) = 0$. Now, we can prove that

$$[\langle\omega, D_u\rangle, (\mathcal{J}'\langle\omega, D_u\rangle\mathcal{J}')^j] = -j(\mathcal{J}'\langle\omega, D_u\rangle\mathcal{J}')^{j-1}(m + 2\mathbb{E}_u + j - 3),$$

by induction as we did in the $3rd$-order case. Therefore, the equation

$$\Big(||x||^2\Big(\frac{(m + 2k)(m + 2k - 6)}{(m + 2k - 2)(m + 2k - 4)} + \frac{4\mathcal{J}'\langle\omega, D_u\rangle\mathcal{J}'\langle\omega, D_u\rangle}{(m + 2k - 2)(m + 2k - 4)}\Big)g_k(u) = 0$$

leads to the following equation:

$$\sum_{j=1}^{k}\left(\frac{(m+2k)(m+2k-6)}{(m+2k-2)(m+2k-4)}-\frac{4j(m+2k-j-3)}{(m+2k-2)(m+2k-4)}\right)(\mathcal{J}'\langle\omega, D_u\rangle\mathcal{J}')^j g^*_{k-j}(u)=0.$$

Since the polynomials $g^*_{k-j}(u) \in \mathcal{H}_k(\mathbb{R}^m, \mathbb{C})$ are linearly independent for $1 \leq j \leq k$, we have that either $g^*_{k-j}(u) = 0$ for all $1 \leq j \leq k$, which means that $ker\sigma_x(\mathcal{D}_4)' = 0$ or that

$$\begin{aligned}&\frac{(m + 2k)(m + 2k - 6)}{(m + 2k - 2)(m + 2k - 4)} - \frac{4j(m + 2k - j - 3)}{(m + 2k - 2)(m + 2k - 4)} = 0,\\ \iff\ &(m + 2k)(m + 2k - 6) - 4j(m + 2k - j - 3) = 0.\end{aligned}$$

It is easy to find the roots are

$$m_1 = -2(k-j) \text{ and } m_2 = -2(k-j)+6.$$

Notice that $k \geq j$, it is easy to see that for $k \in \mathbb{N}$ fixed, only $m \leq 6$ causes trouble. This means that $ker\sigma_x(\mathcal{D}_4)' = 0$, whenever, $m > 6$. This also means that $ker\sigma_x(\mathcal{D}_4) = 0$ whenever, $m > 6$. This completes the proof. □

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Holomorphic Approximation: The Legacy of Weierstrass, Runge, Oka–Weil, and Mergelyan

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Abstract In this paper we survey the theory of holomorphic approximation, from the classical nineteenth century results of Runge and Weierstrass, continuing with the twentieth century work of Oka and Weil, Mergelyan, Vitushkin, and others, to the most recent ones on higher dimensional manifolds. The paper includes some new results and applications of this theory, especially to manifold-valued maps.

1 Introduction

The aim of this paper is to provide a review and synthesis of holomorphic approximation theory from classical to modern. The emphasis is on recent results and applications to manifold-valued maps.

Approximation theory plays a fundamental role in complex analysis, holomorphic dynamics, the theory of minimal surfaces in Euclidean spaces, and in many other related fields of Mathematics and its applications. It provides an indispensable tool in constructions of holomorphic maps with desired properties between complex manifolds. Applications of this theory are too numerous to be presented properly in a short space, but we mention several of them at appropriate places and provide references that the reader might pursue. We are hoping that the paper will bring a new stimulus for future developments in this important area of analysis.

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Although this is largely a survey, it includes some new results, especially those concerning Mergelyan approximation in higher dimension (see Section 6), and applications of these techniques to manifold-valued maps (see Section 7). We also mention open problems and indicate promising directions. Proofs are outlined where possible, especially of those result which introduce major new ideas. More advanced results are only mentioned with references to the original sources. Of course we included proofs of the new results.

There exist a number of surveys on holomorphic approximation theory; see, e.g., [25, 69–72, 75, 77–79, 109, 180], among others. However, ours seems the first attempt at a unified picture, from the highlights of the classical theory to results in several variables and for manifold-valued maps. On the other hand, several of the surveys mentioned above include discussions of certain finer topics of approximation theory that we do not cover here, also for solutions of more general elliptic partial differential equations. It is needless to say that the higher dimensional approximation theory is much less developed and the problems tend to be considerably more complex. It is also clear that further progress in many areas of complex analysis and its applications hinges upon developing new and more powerful approximation techniques for holomorphic mappings.

Organization of the paper. In Sections 2–4 we review the main achievements of the classical approximation theory for functions on the complex plane $\mathbb{C}$ and on Riemann surfaces. Our main goal is to identify those key ideas and principles which may serve as guidelines when considering approximation problems in several variables and for manifold-valued maps. We begin in Section 2 with theorems of K. Weierstrass, C. Runge, S. N. Mergelyan, and A. G. Vitushkin. In Section 3 we discuss approximation on closed unbounded subsets of $\mathbb{C}$ and of Riemann surfaces. There are two main lines in the literature, one following the work of T. Carleman on approximation in the fine topology, and another the work of N. U. Arakelian on uniform approximation. In Section 4 we survey results on $\mathscr{C}^k$ Mergelyan approximation of smooth functions on Riemann surfaces. The remainder of the paper is devoted to the higher dimensional theory. In Section 5 we recall the Oka–Weil approximation theorem on Stein manifolds and some generalizations; these are higher dimensional analogues of Runge's theorem. In Section 6 we discuss Mergelyan and Carleman approximation of functions and closed forms on $\mathbb{C}^n$ and on Stein manifolds. In Section 7 we look at applications of these and other techniques to local and global approximation problems of Runge, Mergelyan, Carleman, and Arakelian type for maps from Stein manifolds to more general complex manifolds; these are especially interesting when the target is an *Oka manifold*. Section 7.2 contains very recent results on Mergelyan approximation of manifold-valued maps. In Section 8 we mention some recent progress on weighted approximation in L^2 spaces.

Notation and terminology. We denote by $\mathbb{N} = \{1, 2, 3, \ldots\}$ the natural numbers, by $\mathbb{Z}$ the ring of integers, $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$, and by $\mathbb{R}$ and $\mathbb{C}$ the fields of real and complex numbers, respectively. For any $n \in \mathbb{N}$ we denote by $\mathbb{R}^n$ the n-dimensional real Euclidean space, and by $\mathbb{C}^n$ the n-dimensional complex Euclidean space with complex coordinates $z = (z_1, \ldots, z_n)$, where $z_i = x_i + \mathrm{i} y_i$ with $x_i, y_i \in \mathbb{R}$ and

$\mathrm{i} = \sqrt{-1}$. We denote the Euclidean norm by $|z|^2 = \sum_{i=1}^n |z_i|^2$. Given $a \in \mathbb{C}$ and $r > 0$, we set $\mathbb{D}(a, r) = \{z \in \mathbb{C} : |z - a| < r\}$ and $\mathbb{D} = \mathbb{D}(0, 1)$. Similarly, $\mathbb{B}^n$ denotes the unit ball in $\mathbb{C}^n$ and $\mathbb{B}^n(a, r)$ the ball centered at $a \in \mathbb{C}^n$ of radius r. The corresponding balls in $\mathbb{R}^n$ are denoted $\mathbb{B}^n_{\mathbb{R}}$ and $\mathbb{B}^n_{\mathbb{R}}(a, r)$.

Let X be a complex manifold. We denote by $\mathscr{C}(X)$ and $\mathscr{O}(X)$ the Fréchet algebras of all continuous and holomorphic functions on X, respectively, endowed with the compact-open topology. Given a compact set K in X, we denote by $\mathscr{C}(K)$ the Banach algebra of all continuous complex valued functions on K with the supremum norm, by $\mathscr{O}(K)$ the set of all functions that are holomorphic in a neighborhood of K (depending on the function), and by $\overline{\mathscr{O}}(K)$ the uniform closure of $\{f|_K : f \in \mathscr{O}(K)\}$ in $\mathscr{C}(K)$. By $\mathscr{A}(K) = \mathscr{C}(K) \cap \mathscr{O}(\mathring{K})$ we denote the set of all continuous functions $K \to \mathbb{C}$ which are holomorphic in the interior $\mathring{K}$ of K. If $r \in \mathbb{Z}_+ \cup \{\infty\}$, we let $\mathscr{C}^r(K)$ denote the space of all functions on K which extend to r-times continuously differentiable functions on X, and $\mathscr{A}^r(K) = \mathscr{C}^r(K) \cap \mathscr{O}(\mathring{K})$. Given a complex manifold Y, we use the analogous notation $\mathscr{O}(X, Y)$, $\mathscr{O}(K, Y)$, $\mathscr{A}^r(K, Y)$, etc., for the corresponding classes of maps into Y. We have the inclusions

$$\mathscr{O}(K, Y) \subset \overline{\mathscr{O}}(K, Y) \subset \mathscr{A}(K, Y) \subset \mathscr{C}(K, Y). \tag{1}$$

A compact set K in a complex manifold X is said to be $\mathscr{O}(X)$-*convex* if

$$K = \widehat{K}_{\mathscr{O}(X)} := \{p \in X : |f(p)| \le \max_{x \in K} |f(x)| \ \ \forall f \in \mathscr{O}(X)\}. \tag{2}$$

A compact $\mathscr{O}(\mathbb{C}^n)$-convex set K in $\mathbb{C}^n$ is said to be *polynomially convex*. A compact set K in a complex manifold X is said to be a *Stein compact* if it admits a basis of open Stein neighborhoods in X.

2 From Weierstrass and Runge to Mergelyan

In this and the following two sections we survey the main achievements of the classical holomorphic approximation theory. More comprehensive surveys of this subject are available in [25, 69–72, 75, 77, 78, 180], among other sources.

The approximation theory for holomorphic functions has its origin in two classical theorems from 1885. The first one, due to K. Weierstrass [170], concerns the approximation of continuous functions on compact intervals in $\mathbb{R}$ by polynomials.

Theorem 1 (Weierstrass (1885), [170]) *Suppose f is a continuous function on a closed bounded interval $[a, b] \subset \mathbb{R}$. For every $\epsilon > 0$ there exists a polynomial p such that for all $x \in [a, b]$ we have $|f(x) - p(x)| < \epsilon$.*

Proof We use convolution with the Gaussian kernel. After extending f to a continuous function on $\mathbb{R}$ with compact support, we consider the family of entire functions

$$f_\epsilon(z) = \frac{1}{\epsilon\sqrt{\pi}} \int_{\mathbb{R}} f(x) e^{-(x-z)^2/\epsilon^2} dx, \qquad z \in \mathbb{C},\ \epsilon > 0. \tag{3}$$

As $\epsilon \to 0$, we have that $f_\epsilon \to f$ uniformly on $\mathbb{R}$. Hence, the Taylor polynomials of f_ϵ approximate f uniformly on compact intervals in $\mathbb{R}$. If furthermore f is of class $\mathscr{C}^k$, then by a change of variable $u = x - z$ and placing the derivatives on f it follows that we get convergence also in the $\mathscr{C}^k$ norm. □

The paper by A. Pinkus [136] (2000) contains a more complete survey of Weierstrass's results and of his impact on the theory of holomorphic approximation. As we shall see in Section 6.1, the idea of using convolutions with the Gaussian kernel gives major approximation results also on certain classes of real submanifolds in complex Euclidean space $\mathbb{C}^n$ and, more generally, in Stein manifolds.

One line of generalizations of Weierstrass's theorem was discovered by M. Stone in 1937, [154, 155]. The *Stone–Weierstrass theorem* says that, if X is a compact Hausdorff space and A is a subalgebra of the Banach algebra $\mathscr{C}(X, \mathbb{R})$ which contains a nonzero constant function, then A is dense in $\mathscr{C}(X, \mathbb{R})$ if and only if it separates points. It follows in particular that any complex valued continuous function on a compact set $K \subset \mathbb{C}$ can be uniformly approximated by polynomials in z and $\bar{z}$. Stone's theorem opened a major direction of research in Banach algebras.

Another line of generalizations concerns approximation of continuous functions on curves in the complex plane by holomorphic polynomials and rational functions. This led to Mergelyan and Carleman theorems discussed in the sequel.

However, we must first return to the year 1885. The second of the two classical approximation theorems proved that year is due to C. Runge [144].

Theorem 2 (Runge (1885), [144]) *Every holomorphic function on an open neighborhood of a compact set K in $\mathbb{C}$ can be approximated uniformly on K by rational functions without poles in K, and by holomorphic polynomials if $\mathbb{C} \setminus K$ is connected.*

The maximum principle shows that the condition that K does not separate the plane is necessary for polynomial approximation on K.

Proof The simplest proof of Runge's theorem, and the one given in most textbooks on the subject (see, e.g., [143, p. 270]), goes as follows. Assume that f is a holomorphic function on an open set $U \subset \mathbb{C}$ containing K. Choose a smoothly bounded domain D with $K \subset D$ and $\overline{D} \subset U$. By the Cauchy integral formula we have that

$$f(z) = \frac{1}{2\pi \mathrm{i}} \int_{bD} \frac{f(\zeta)}{\zeta - z} d\zeta, \qquad z \in D.$$

Approximating the integral by Riemann sums provides uniform of f on K by linear combinations of functions $\frac{1}{a-z}$ with poles $a \in \mathbb{C} \setminus K$. Assuming that $\mathbb{C} \setminus K$ is connected, we can push the poles to infinity as follows. Pick a disc $\Delta \subset \mathbb{C}$ containing K. Since $\mathbb{C} \setminus K$ is connected, there is a path $\lambda : [0, 1] \to \mathbb{C} \setminus K$ connecting $a = \lambda(0)$ to a point $b = \lambda(1) \in \mathbb{C} \setminus \overline{\Delta}$. Let $\delta = \inf\{\mathrm{dist}(\lambda(t), K) : t \in [0, 1]\} > 0$.

Choose points $a = a_0, a_1, \ldots, a_N = b \in \lambda([0,1])$ such that $|a_j - a_{j+1}| < \delta$ for $j = 0, \ldots, N-1$. For $z \in K$ and $j = 0, 1, \ldots, N-1$ we then have that

$$\frac{1}{a_j - z} = \frac{1}{(a_{j+1} - z) - (a_{j+1} - a_j)} = \sum_{k=0}^{\infty} \frac{(a_{j+1} - a_j)^k}{(a_{j+1} - z)^{k+1}},$$

where the geometric series converges uniformly on K. It follows by a finite induction that $\frac{1}{a-z}$ is a uniform limit on K of polynomials in $\frac{1}{b-z}$. Since $b \in \mathbb{C} \setminus \overline{\Delta}$, the function $\frac{1}{b-z}$ is a uniform limit on Δ of holomorphic polynomials in z and the proof is complete. If $\mathbb{C} \setminus K$ is not connected, a modification of this argument gives uniform approximations of f by rational functions with poles in a given set $\Lambda \subset \mathbb{C} \setminus K$ containing a point in every bounded connected component of $\mathbb{C} \setminus K$.

Another proof uses the *Cauchy–Green formula*, also called the *Pompeiu formula* for compactly supported function $f \in \mathscr{C}_0^1(\mathbb{C})$:

$$f(z) = \frac{1}{\pi} \int_{\mathbb{C}} \frac{\overline{\partial} f(\zeta)}{z - \zeta} \, du \, dv, \qquad z \in \mathbb{C}, \ \zeta = u + \mathrm{i}v. \tag{4}$$

Here, $\overline{\partial} f(\zeta) = (\partial f / \partial \bar{\zeta})(\zeta)$. If f is holomorphic in an open set $U \subset \mathbb{C}$ containing a compact set K, we choose a smooth function $\chi : \mathbb{C} \to [0,1]$ which equals 1 on a smaller neighborhood V of K and satisfies $\mathrm{supp}(\chi) \subset U$. Then,

$$f(z) = \frac{1}{\pi} \int_{\mathbb{C}} \frac{\overline{\partial} \chi(\zeta) \, f(\zeta)}{z - \zeta} \, du \, dv, \qquad z \in V.$$

Since the integrand is supported on $\mathrm{supp}(\overline{\partial}\chi)$ which is disjoint from K, approximating the integral by Riemann sums shows that f can be approximated uniformly on K by rational functions with poles in $\mathbb{C} \setminus K$, and the proof is concluded as before. □

We now digress for a moment to recall the main properties of the Cauchy–Green operator in (4) which is used in many approximation results discussed in the sequel.

Given a compact set $K \subset \mathbb{C}$ and an integrable function g on K, we set

$$T_K(g)(z) = \frac{1}{\pi} \int_K \frac{g(\zeta)}{z - \zeta} \, du dv, \qquad \zeta = u + \mathrm{i}v. \tag{5}$$

It is well known (see, e.g., L. Ahlfors [1, Lemma 1, p. 51] or A. Boivin and P. Gauthier [25, Lemma 1.5]) that for any $g \in L^p(K)$, $p > 2$, $T_K(g)$ is a bounded continuous function on $\mathbb{C}$ that vanishes at infinity and satisfies the uniform Hölder condition with exponent $\alpha = 1 - 2/p$; moreover, $T_K : L^p(K) \to \mathscr{C}^\alpha(\mathbb{C})$ is a continuous linear operator. (A closely related operator is actually bounded from $L^p(\mathbb{C})$ to $\mathscr{C}^{1-2/p}(\mathbb{C})$ without any support condition.) The key property of T_K is that it solves the nonhomogeneous Cauchy–Riemann equation, that is,

$$\overline{\partial}\, T_K(g) = g$$

holds in the sense of distributions, and in the classical sense on any open subset on which g is of class $\mathscr{C}^1$. In particular, $T_K(g)$ is holomorphic on $\mathbb{C} \setminus K$. The optimal sup-norm estimate of $T_K(g)$ for $g \in L^\infty(K)$ is obtained from Mergelyan's estimate

$$\int_{\zeta \in K} \frac{du\,dv}{|z-\zeta|} \;\le\; \sqrt{4\pi \,\mathrm{Area}(K)}, \qquad z \in \mathbb{C}, \tag{6}$$

which is sharp when K is the union of a closed disc centered at z and a compact set of measure zero. (See S. N. Mergelyan [124, 125] or A. Browder [29, Lemma 3.1.1].) The related Ahlfors–Beurling estimate which is also sharp is that

$$|T_K(1)(z)| = \left| \frac{1}{\pi} \int_{\zeta \in K} \frac{du\,dv}{z-\zeta} \right| \;\le\; \sqrt{\frac{\mathrm{Area}(K)}{\pi}}, \qquad z \in \mathbb{C}.$$

Another excellent source for this topic is the book of K. Astala, T. Iwaniec, and G. Martin [10]; see in particular Sect. 4.3 therein.

Coming back to the topic of approximation, the situation becomes considerably more delicate when the function f to be approximated is only continuous on K and holomorphic in the interior $\mathring{K}$; that is, $f \in \mathscr{A}(K)$. The corresponding approximation problem for compact sets in $\mathbb{C}$ with connected complement was solved by S. N. Mergelyan in 1951.

Theorem 3 (Mergelyan (1951), [123–125]) *If K is a compact set in $\mathbb{C}$ with connected complement, then every function in $\mathscr{A}(K)$ can be approximated uniformly on K by holomorphic polynomials.*

Mergelyan's theorem generalizes both Runge's and Weierstrass's theorem. It also contains as special cases the theorems of J. L. Walsh [168] (1926) in which K is the closure of a Jordan domain, F. Hartogs and A. Rosenthal [90] (1931) in which K has Lebesgue measure zero, M. Lavrentieff [107] (1936) in which K is nowhere dense, and M. V. Keldysh [99] (1945) in which K is the closure of its interior.

In light of Runge's theorem, the main new point in Mergelyan's theorem is to approximate functions in $\mathscr{A}(K)$ by functions holomorphic in open neighborhoods of K, that is, to show that

$$\mathscr{A}(K) = \overline{\mathscr{O}}(K).$$

If this holds, we say that K (or $\mathscr{A}(K)$) enjoys the *Mergelyan property*. Hence, Mergelyan's theorem is essentially of local nature, where *local* now pertains to *neighborhoods of K*. This aspect is emphasized further by Bishop's localization theorem, Theorem 6, and its converse, Theorem 14.

Some generalizations of Mergelyan's theorem can be found in his papers [124, 125]. Subsequently to Mergelyan, another proof was given by E. Bishop in 1960,

[20], and yet another by L. Carleson in 1964, [32]. Expositions are available in many sources; see, for instance, D. Gaier [70, p. 97], T. W. Gamelin [72], and W. Rudin [143]. We outline the proof and refer to the cited sources for the details.

Sketch of Proof of Theorem 3. By Tietze's extension theorem, every $f \in \mathscr{A}(K)$ extends to a continuous function with compact support on $\mathbb{C}$. Fix a number $\delta > 0$. Let $\omega(\delta)$ denote the modulus of continuity of f. By convolving f with the function $A_\delta \colon \mathbb{C} \to \mathbb{R}_+$ defined by $A_\delta(z) = 0$ for $|z| > \delta$ and

$$A_\delta(z) = \frac{3}{\pi\delta^2}\left(1 - \frac{|z|^2}{\delta^2}\right)^2, \qquad 0 \le |z| \le \delta,$$

we obtain a function $f_\delta \in \mathscr{C}^1_0(\mathbb{C})$ with compact support such that

$$|f(z) - f_\delta(z)| < \omega(\delta) \quad \text{and} \quad \left|\frac{\partial f_\delta}{\partial \bar z}(z)\right| < \frac{2\omega(\delta)}{\delta}, \quad z \in \mathbb{C},$$

and $f_\delta = f$ on $K_\delta = \{z \in K : \operatorname{dist}(z, \mathbb{C} \setminus K) > \delta\}$. By the Cauchy–Green formula (4),

$$f_\delta(z) = \frac{1}{\pi}\int_{\mathbb{C}} \frac{\overline{\partial} f_\delta(\zeta)}{z - \zeta}\, du\, dv, \qquad z \in \mathbb{C}.$$

Next, we cover the compact set $X = \operatorname{supp}(\overline{\partial} f_\delta)$ by finitely many open discs $D_j = \mathbb{D}(z_j, 2\delta)$ $(j = 1, \ldots, n)$ with centers $z_j \in \mathbb{C} \setminus K$ such that each D_j contains a compact Jordan arc $E_j \subset D_j \setminus K$ of diameter at least 2δ. (Such discs D_j and arcs E_j exist because $\mathbb{C} \setminus K$ is connected.) The main point now is to approximate the Cauchy kernel $\frac{1}{z-\zeta}$ for $z \in \mathbb{C} \setminus E_j$ and $\zeta \in D_j$ sufficiently well by a function of the form

$$P_j(z, \zeta) = g_j(z) + (\zeta - b_j) g_j(z)^2,$$

where $g_j \in \mathscr{O}(\mathbb{C} \setminus E_j)$ and $b_j \in \mathbb{C}$. This is accomplished by Mergelyan's lemma which says that g_j and b_j can be chosen such that the inequalities

$$|P_j(z, \zeta)| < \frac{50}{\delta} \quad \text{and} \quad \left| P_j(z, \zeta) - \frac{1}{z - \zeta} \right| < \frac{4000\,\delta^2}{|z - \zeta|^3} \tag{7}$$

hold for all $z \in \mathbb{C} \setminus E_j$ and $\zeta \in D_j$. (See also [70, p. 104] or [143, Lemma 20.2].) Set

$$X_1 = X \cap \overline{D}_1, \qquad X_j = X \cap \overline{D}_j \setminus (X_1 \ldots \cup X_{j-1}) \;\text{ for }\; j = 2, \ldots, n.$$

The open set $\Omega = \mathbb{C} \setminus \bigcup_{j=1}^n E_j$ clearly contains K. The function

$$F_\delta(z) = \sum_{j=1}^{n} \frac{1}{\pi} \int_{X_j} \frac{\partial f_\delta}{\partial \bar\zeta}(\zeta) P_j(z,\zeta)\, du\, dv$$

is holomorphic in Ω (since every function $P_j(z,\zeta)$ is holomorphic for $z \in \Omega$), and it follows from (7) that $|F_\delta(z) - f_\delta(z)| < 6000\omega(\delta)$ for all $z \in \Omega$. As $\delta \to 0$, we have that $\omega(\delta) \to 0$ and hence $F_\delta \to f$ uniformly on K. □

We now consider approximation problems on Riemann surface. Fundamental discoveries concerning function theory on *open Riemann surfaces* were made by H. Behnke and K. Stein [17] in 1949. They proved the following extension of Runge's theorem to open Riemann surfaces (see [17, Theorem 6]); the case for X compact was pointed out by H. L. Royden in 1967, [142, Theorem 10], and again by H. Köditz and S. Timmann in 1975 [102, Satz 1].

Theorem 4 (Runge's Theorem on Riemann Surfaces; [17, 102, 142]) *If K is a compact set in a Riemann surface X, then every holomorphic function f on a neighborhood of K can be approximated uniformly on K by meromorphic functions F on X without poles in K, and by holomorphic functions on X if $X \setminus K$ has no relatively compact connected components.*

In the papers of Royden [142] and Köditz and Timmann [102] the function f is assumed to be meromorphic on a neighborhood of K (with at most finitely many poles on K), the approximating meromorphic function F on X has no poles on K except those of f, and its poles in $X \setminus K$ are located in a set E having one point in each connected component of $X \setminus K$. Furthermore, Royden showed that F can be chosen to agree with f to a given finite order at a given finite set of points in K.

A relatively compact connected component of $X \setminus K$ is called a *hole* of K. A compact set without holes in an open Riemann surface X is also called a *Runge compact* in X. The following is a corollary to Theorem 4 and the maximum principle.

Corollary 1 *Let X be an open Riemann surface.*

(a) *Holomorphic functions on X separate points, that is, for any pair of distinct points $p, q \in X$ there exists $f \in \mathscr{O}(X)$ such that $f(p) \neq f(q)$.*
(b) *For every compact set K in X, its $\mathscr{O}(X)$-convex hull $\widehat{K}_{\mathscr{O}(X)}$ (see (2)) is the union of K and all holes of K in X; in particular, $\widehat{K}_{\mathscr{O}(X)}$ is compact.*

Conditions (a) and (b) in Corollary 1 were used in 1951 by K. Stein [152] to introduce the class of *Stein manifolds* of any dimension. (The third of Stein's axioms is a consequence of these two.) Thus, open Riemann surfaces are the same thing as 1-dimensional Stein manifolds. Theorem 4 is a special case of the *Oka–Weil theorem* on Stein manifolds; see Section 5.

The proof of Runge's theorem in the plane is based on Cauchy's integral formula. To prove Runge's theorem on open Riemann surfaces, Behnke and Stein constructed Cauchy type kernels, the so-called *elementary differentials*; see [17, Theorem 3] and Remark 1 below where additional references are given. More precisely, on any open Riemann surface X there is a meromorphic 1-form ω on $X_z \times X_\zeta$ which is holomorphic off the diagonal and which in any pair of local coordinates has an expression

$$\omega(z,\zeta) = \left(\frac{1}{\zeta - z} + h(z,\zeta)\right) d\zeta, \tag{8}$$

with h a holomorphic function. (Note that ω is a form only in the second variable ζ, but its coefficient is a meromorphic function of both variables (z,ζ).) In particular, ω has simple poles with residues one along the diagonal of $X \times X$. For any $\mathscr{C}^1$-smooth domain $\Omega \Subset X$ and $f \in \mathscr{C}^1(\overline{\Omega})$ one then obtains the Cauchy–Green formula

$$f(z) = \frac{1}{2\pi \mathrm{i}} \int_{\partial\Omega} f(\zeta)\omega(z,\zeta) - \frac{1}{2\pi \mathrm{i}} \int_{\Omega} \overline{\partial} f(\zeta) \wedge \omega(z,\zeta). \tag{9}$$

By using this formula when f is holomorphic on an open neighborhood of the set K in Theorem 4, one can approximate f by meromorphic functions with poles on $X \setminus K$, and the rest of the argument (pushing the poles) is similar to the one in Theorem 2.

Note that, just as in the complex plane, if we consider $(0,1)$-forms α with compact support in Ω, we get that the mapping $\alpha \mapsto T(\alpha)$, given by

$$T(\alpha)(z) = -\frac{1}{2\pi \mathrm{i}} \int_{\Omega} \alpha(\zeta) \wedge \omega(z,\zeta), \tag{10}$$

is a bounded linear operator satisfying $\overline{\partial}(T(\alpha)) = \alpha$. This will be used below where we give a simple proof of Bishop's localization theorem.

A functional analytic proof of Theorem 4 using Weyl's lemma was given by B. Malgrange [115] in 1955; see also O. Forster's monograph [51, Sect. 25].

Remark 1 H. Behnke and K. Stein constructed Cauchy type kernels on relatively compact domains in any open Riemann surface [17, Theorem 3]; see also H. Behnke and F. Sommer [16, p. 584]. The existence of globally defined Cauchy kernels (8) was shown by S. Scheinberg [148] and P. M. Gauthier [74] in 1978–79. Their proof uses the theorem of R. C. Gunning and R. Narasimhan [89] (1967) which says that every open Riemann surface X admits a holomorphic immersion $g : X \to \mathbb{C}$. The pull-back by g of the Cauchy kernel on $\mathbb{C}$ is a Cauchy kernel on X with the correct behavior along the diagonal $D = \{(z,z) : z \in X\}$ (see (8)), but with additional poles if g is not injective. Since the diagonal D has a basis of Stein neighborhoods in $X \times X$ and its complement $X \times X \setminus D$ is also Stein, one can remove the extra poles by solving a Cousin problem. Furthermore, Gauthier and Scheinberg found Cauchy kernels satisfying the symmetry condition $F(p,q) = -F(q,p)$. □

Theorem 4 implies the analogous approximation result for meromorphic functions. Indeed, we may write a meromorphic function f on an open neighborhood $U \subsetneq X$ of the compact set K as the quotient $f = g/h$ of two holomorphic functions (this follows from the Weierstrass interpolation theorem on open Riemann surfaces; see [47, 169]) and apply the same result separately to g and h. Since meromorphic functions are precisely holomorphic maps to the Riemann sphere $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$, this extension of Theorem 4 has the following corollary.

Corollary 2 *Let K be a compact set in an arbitrary Riemann surface X. Then, every holomorphic map from a neighborhood of K to $\mathbb{CP}^1$ may be approximated uniformly on K by holomorphic maps $X \to \mathbb{CP}^1$.*

In 1958, E. Bishop [19] proved the following extension of Mergelyan's theorem.

Theorem 5 (Bishop–Mergelyan Theorem; Bishop (1958), [19]) *If K is a compact set without holes in an open Riemann surface X, then every function in $\mathscr{A}(K)$ can be approximated uniformly on K by functions in $\mathscr{O}(X)$.*

More generally, if X is an arbitrary Riemann surface, ρ is a metric on X, and there is a $c > 0$ such that every hole of a compact subset $K \subset X$ has ρ-diameter at least c, then every function in $\mathscr{A}(K)$ is a uniform limit of meromorphic functions on X with poles off K. This holds in particular if K has at most finitely many holes.

Bishop's proof depends on investigation of measures on K annihilating the algebra $\mathscr{A}(K)$. This approach was further developed by L. K. Kodama [101] in 1965. In 1968, J. Garnett observed [73, p. 463] that Theorem 5 can be reduced to Mergelyan's theorem on polynomial approximation (see Theorem 3) by means of the following localization theorem due to Bishop [19] (see also [101, Theorem 5]).

Theorem 6 (Bishop's Localization Theorem; (1958), [19]) *Let K be a compact set in a Riemann surface X and $f \in \mathscr{C}(K)$. If every point $x \in K$ has a compact neighborhood $D_x \subset X$ such that $f|_{K\cap D_x} \in \overline{\mathscr{O}}(K \cap D_x)$, then $f \in \overline{\mathscr{O}}(K)$.*

Let us first indicate how Theorems 3, 4, and 6 imply Theorem 5. We cover K by open coordinate discs $U_1, \ldots, U_N$ of diameter at most c (the number in the second part of Theorem 5; no condition is needed for the first part). Choose closed discs $D_j \subset U_j$ for $j = 1, \ldots, N$ whose interiors still cover K. Then, $U_j \setminus (K \cap D_j)$ is connected. (Indeed, every relatively compact connected component of $U_j \setminus (K \cap D_j)$ is also a connected component of $X \setminus D_j$ of diameter $< c$, contradicting the assumption.) Since U_j is a planar set, Theorem 3 implies $\mathscr{A}(K \cap D_j) = \overline{\mathscr{O}}(K \cap D_j)$. Thus, the hypothesis of Theorem 6 is satisfied, and hence $\mathscr{A}(K) = \overline{\mathscr{O}}(K)$. Theorem 5 then follows from Runge's theorem (see Theorem 4).

Proof of Theorem 6 The following simple proof, based on solving the $\overline{\partial}$-equation, was given by A. Sakai [146] in 1972. We may assume that f is continuous in a neighborhood of K. Cover K by finitely many neighborhoods D_j as in the theorem, such that the family of open sets $\mathring{D}_j$ is an open cover of K. Let χ_j be a partition of unity with respect to this cover. Now by the assumption we obtain for any $\epsilon > 0$ functions $f_j \in \mathscr{C}(D_j) \cap \mathscr{O}(K \cap D_j)$ such that $\|f_j - f\|_{\mathscr{C}(K\cap D_j)} < \epsilon$. Set $g := \sum_{j=1}^m \chi_j f_j$. Then on some open neighborhood U of K we have that $\|g - f\|_{\mathscr{C}(U)} = O(\epsilon)$ and

$$\overline{\partial} g = \sum_{j=1}^{m} \overline{\partial}\chi_j \cdot f_j = \sum_{j=1}^{m} \overline{\partial}\chi_j \cdot (f_j - f) = O(\epsilon).$$

(We have used that $\sum_{j=1}^m \overline{\partial}\chi_j = 0$ in a neighborhood of K.) Let $\chi \in \mathscr{C}_0^\infty(U)$ be a cut-off function with $0 \le \chi \le 1$ and $\chi \equiv 1$ near K. Then we have $\|\chi \cdot \overline{\partial} g\|_{\mathscr{C}(\overline{U})} = O(\epsilon)$, and so $T(\chi \cdot \overline{\partial} g) = O(\epsilon)$, where T is the Cauchy–Green operator (10). Hence, the function $g - T(\chi \cdot \overline{\partial} g)$ is holomorphic on some open neighborhood of K and it approximates f to a precision of order ϵ on K. □

Remark 2 Sakai's proof also applies to a compact set K in a higher dimensional complex manifold, provided K admits a basis of Stein neighborhoods on which one can solve the $\overline{\partial}$-equation with uniform estimates with a constant independent of the neighborhood. This holds, for instance, when K is the closure of a strongly pseudoconvex domain; see Theorem 24 on p. 165. □

Remark 3 It was observed by K. Hoffman and explained by J. Garnett [73] in 1968 that Bishop's localization theorem in the plane is a simple consequence of the properties of the Cauchy transform. Given a function $\phi \in \mathscr{C}_0^\infty(\mathbb{C})$ with compact support and a bounded continuous function f on $\mathbb{C}$, we consider the *Vitushkin localization operator*:

$$\begin{aligned} T_\phi(f)(z) &= \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{C}} \frac{f(\zeta) - f(z)}{\zeta - z}\, \overline{\partial}\phi(\zeta) \wedge d\zeta \\ &= f(z)\phi(z) + \frac{1}{\pi} \int_{\mathbb{C}} \frac{f(\zeta)}{\zeta - z} \frac{\partial \phi}{\partial \bar\zeta}(\zeta)\, du dv. \end{aligned} \tag{11}$$

(We used the Cauchy–Green formula (4).) From properties of the operator T_K (5) we see that $T_\phi(f)$ is a bounded continuous function on $\mathbb{C}$ vanishing at ∞, it is holomorphic where f is holomorphic and in $\mathbb{C} \setminus \mathrm{supp}(\phi)$, and $f - T_\phi(f)$ is holomorphic in the interior of the level set $\{\phi = 1\}$. If f has compact support and $\{\phi_j\}_{j=1}^N$ is a partition of unity on $\mathrm{supp}(f)$, then $f = \sum_{j=1}^N T_{\phi_j}(f)$. Finally, it follows from (6) that

$$\|T_\phi(f)\|_\infty \le c_0 \delta \omega_f(\delta) \|\partial\phi/\partial\bar\zeta\|_\infty, \tag{12}$$

where $\delta > 0$ is the radius of a disc containing the support of ϕ, $\omega_f(\delta)$ is the δ-modulus of continuity of f, and $c_0 > 0$ is a universal constant. (See T. Gamelin [72, Lemma II.1.7] or D. Gaier [70, p. 114] for the details.)

Suppose now that $f \colon K \to \mathbb{C}$ satisfies the hypothesis of Theorem 6. By Tietze's theorem we may extend f to a continuous function with compact support on $\mathbb{C}$. Let $U_1, \ldots, U_N \subset \mathbb{C}$ be a finite covering of $\mathrm{supp}(f)$ by bounded open sets such that, setting $K_j = K \cap \overline{U}_j$, we have $f|_{K_j} \in \overline{\mathscr{O}}(K_j)$ for each j. Let $\phi_j \in \mathscr{C}_0^\infty(\mathbb{C})$ be a smooth partition of unity on $\mathrm{supp}(f)$ with $\mathrm{supp}(\phi_j) \subset U_j$. By the hypothesis, given $\epsilon > 0$ there is a holomorphic function $h_j \in \mathscr{O}(W_j)$ on an open neighborhood of K_j which is uniformly ϵ-close to f on K_j. Shrinking W_j around K_j, we may assume that h_j is 2ϵ-close to f on W_j. Choose a smooth function $\chi_j : \mathbb{C} \to [0, 1]$ which equals one on a neighborhood $V_j \subset W_j$ of K_j and has $\mathrm{supp}(\chi_j) \subset W_j$. The function $\tilde{h}_j = \chi_j h_j + (1 - \chi_j) f$ then equals h_j on V_j (hence is holomorphic there), it equals f on $\mathbb{C} \setminus W_j$, and is uniformly 2ϵ-close to f on $\mathbb{C}$. The function $g_j = T_{\phi_j}(\tilde{h}_j) \in \mathscr{C}(\mathbb{C})$ is holomorphic on V_j (since g_j is holomorphic there) and on $\mathbb{C} \setminus \mathrm{supp}(\phi_j)$. Since the union of the latter two sets contains K, g_j is holomorphic in a neighborhood of K. Furthermore, g_j approximates $f_j = T_{\phi_j}(f)$ in view of (12). The sum $\sum_{j=1}^N g_j$ is then holomorphic in a neighborhood of K and uniformly close to $\sum_{j=1}^N f_j = f$ on $\mathbb{C}$. (Further details can be found in Gaier [70, pp. 114–118].)

By using the Cauchy type kernels in Remark 1, P. Gauthier [74] and S. Scheinberg [148] adapted this approach to extend Bishop's localization theorem to closed (not necessarily compact) sets of essentially finite genus in any Riemann surface. See also Section 3 and in particular Theorem 15.

Another proof of Mergelyan's theorem on Riemann surfaces (Theorem 5) can be found in [98, Chapter 1.11]. It is based on a proof of Bishop's localization theorem (Theorem 6) which avoids the use of Cauchy type kernels on Riemann surfaces, such as those given by Behnke and Stein in [17]. □

After Mergelyan proved his theorem on polynomial approximation and Bishop extended it to open Riemann surfaces (Theorem 5), a major challenging problem was to characterize the class of compact sets K in $\mathbb{C}$, or in a Riemann surface X, which enjoy the Mergelyan property $\mathscr{A}(K) = \overline{\mathscr{O}}(K)$. In view of Runge's theorem (Theorem 4), this is equivalent to approximation of functions in $\mathscr{A}(K)$ by meromorphic functions on X with poles off K, and by rational functions if $X = \mathbb{C}$:

$$\mathscr{A}(K) \stackrel{?}{=} \mathscr{R}(K). \tag{13}$$

The study of this question led to powerful new methods in approximation theory. There are examples of compact sets of *Swiss cheese* type (with a sequence of holes of K clustering on K) for which $\mathscr{R}(K) \subsetneq \mathscr{A}(K)$; see D. Gaier [70, p. 110]. An early positive result is the theorem of F. Hartogs and A. Rosenthal [90] from 1931 which states that if K is a compact set in $\mathbb{C}$ with Lebesgue measure zero, then $\mathscr{C}(K) = \mathscr{R}(K)$. After partial results by S. N. Mergelyan [124, 125], E. Bishop [19, 20], and others, the problem was completely solved by A. G. Vitushkin in 1966, [166, 167]. To state his theorem, we recall the notion of continuous capacity. Let M be a subset of $\mathbb{C}$. Denote by $\mathfrak{R}(M)$ the set of all continuous functions f on $\mathbb{C}$ with $\|f\|_\infty \leq 1$ which are holomorphic outside some compact subset K of M and whose Laurent expansion at infinity is $f(z) = \frac{c_1(f)}{z} + O\left(\frac{1}{z^2}\right)$. The *continuous capacity* of M is defined by

$$\alpha(M) = \sup\left\{|c_1(f)| : f \in \mathfrak{R}(M)\right\}.$$

Theorem 7 (Vitushkin (1966/1967), [166, 167]) *Let K be a compact set in $\mathbb{C}$. Then, $\mathscr{R}(K) = \mathscr{A}(K)$ if and only if $\alpha(D \setminus K) = \alpha(D \setminus \mathring{K})$ for every open disc D in $\mathbb{C}$.*

Vitushkin's proof relies on the localization operators (11) which he introduced (see [167, Ch. 2, §3]). Theorem 7 is a corollary of Vitushkin's main result in [167] which provides a criterium for rational approximation of individual functions in $\mathscr{A}(K)$. The most advanced form of Vitushkin-type results is due to Paramonov [134]. Major results on the behavior of the (continuous) capacity and estimates of Cauchy integrals over curves were obtained by M. Mel'nikov [120, 121], X. Tolsa [162, 163], and Mel'nikov and Tolsa [122].

3 Approximation on Unbounded sets in Riemann Surfaces

It seems that the first result concerning the approximation of functions on unbounded closed subsets of $\mathbb{C}$ by entire functions is the following generalization of Weierstrass's Theorem 1, due to T. Carleman [31].

Theorem 8 (Carleman (1927), [31]) *Given continuous functions $f : \mathbb{R} \to \mathbb{C}$ and $\epsilon : \mathbb{R} \to (0, +\infty)$, there exists an entire function $F \in \mathscr{O}(\mathbb{C})$ such that*

$$|F(x) - f(x)| < \epsilon(x) \quad \textit{for all} \;\; x \in \mathbb{R}. \tag{14}$$

This says that continuous functions on $\mathbb{R}$ can be approximated in the fine $\mathscr{C}^0$ topology by restriction to $\mathbb{R}$ of entire functions on $\mathbb{C}$. The proof amounts to inductively applying Mergelyan's theorem on polynomial approximation (Theorem 3).

Proof Recall that $\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \le 1\}$. For $j \in \mathbb{Z}_+ = \{0, 1, \ldots\}$ set

$$K_j = j\overline{\mathbb{D}} \cup [-j-2, j+2], \qquad \epsilon_j = \min\{\epsilon(x) : |x| \le j+2\}.$$

Note that $\epsilon_j \ge \epsilon_{j+1} > 0$ for all $j \in \mathbb{Z}_+$. We construct a sequence of continuous functions $f_j : (j+1/3)\mathbb{D} \cup \mathbb{R} \to \mathbb{C}$ satisfying the following conditions for all $j \in \mathbb{N}$:

(a_j) f_j is holomorphic on $(j+1/3)\mathbb{D}$,
(b_j) $f_j(x) = f(x)$ for $x \in \mathbb{R}$ with $|x| \ge j + 2/3$, and
(c_j) $|f_j - f_{j-1}| < 2^{-j-1}\epsilon_{j-1}$ on K_{j-1}.

To construct f_0, we pick a smooth function $\chi : \mathbb{R} \to [0, 1]$ such that $\chi(x) = 1$ for $|x| \le 1/3$ and $\chi(x) = 0$ for $|x| \ge 2/3$. Mergelyan's theorem (see Theorem 3) gives a holomorphic polynomial h such that, if we define f_0 to equal h on $(1/3)\mathbb{D}$ and set $f_0(x) = \chi h(x) + (1-\chi)f(x)$ for $|x| \ge 1/3$, then f_0 satisfies conditions (a_0) and (b_0), while condition (c_0) is vacuous.

The inductive step $(j-1) \to j$ is as follows. Mergelyan's theorem (see Theorem 3) gives a holomorphic polynomial h satisfying $|h - f_{j-1}| < 2^{-j-1}\epsilon_{j-1}$ on K_{j-1}. Pick a smooth function $\chi \colon \mathbb{R} \to [0,1]$ such that $\chi(x) = 1$ for $|x| \le j + 1/3$ and $\chi(x) = 0$ for $|x| \ge j + 2/3$. Set $f_j = h$ on $(j+1/3)\mathbb{D}$ and $f_j = \chi h + (1-\chi) f_{j-1}$ on $\mathbb{R}$. It is easily verified that the sequence f_j satisfies conditions (a$_j$), (b$_j$), and (c$_j$). In view of (b$_j$) we have $f_0 = f_1 = \ldots = f_{k-1}$ on $\{|x| \ge k\}$ for any $k \in \mathbb{N}$. From this and (c$_j$) it follows that the sequence f_j converges to an entire function $F \in \mathscr{O}(\mathbb{C})$ such that for every $k \in \mathbb{Z}_+$ the following inequality holds on $\{x \in \mathbb{R} : k \le |x| \le k+1\}$:

$$|F(x) - f(x)| \le \sum_{j=0}^{\infty} |f_{j+1}(x) - f_j(x)| < \sum_{j=k-1}^{\infty} 2^{-j-2}\epsilon_j \le \epsilon_{k-1} \le \epsilon(x).$$

This proves Theorem 8. □

The above proof is easily adapted to show that every function $f \in \mathscr{C}^r(\mathbb{R})$ for $r \in \mathbb{N}$ can be approximated in the fine $\mathscr{C}^r(\mathbb{R})$ topology by restrictions to $\mathbb{R}$ of entire functions, i.e., (14) is replaced by the stronger condition on the derivatives:

$$|F^{(k)}(x) - f^{(k)}(x)| < \epsilon(x) \quad \text{for all } x \in \mathbb{R} \text{ and } k = 0, 1, \ldots, r.$$

In 1973, L. Hoischen [93] proved a similar result on $\mathscr{C}^r$-Carleman approximation on more general curves in the complex plane.

When trying to adapt the proof of Carleman's theorem to more general closed sets $E \subset \mathbb{C}$ without holes, a complication appears in the induction step since the union of E with a closed disc may contain holes. Consider the following notion.

Definition 1 Let D be a domain in $\mathbb{C}$. A closed subset E of D is a *Carleman set* if each function in $\mathscr{A}(E)$ can be approximated in the fine $\mathscr{C}^0$ topology on E by functions in $\mathscr{O}(D)$. (More precisely, given $f \in \mathscr{A}(E)$ and a continuous function $\epsilon \colon E \to (0, +\infty)$, there exists $F \in \mathscr{O}(D)$ such that $|F(z) - f(z)| < \epsilon(z)$ for all $z \in E$.)

The following characterization of Carleman sets was given by A. A. Nersesyan in 1971, [127, 128]. Given a domain $D \subsetneq \mathbb{CP}^1$, let $V_\epsilon(bD)$ denote the set of all points having chordal (spherical) distance less than ϵ from the boundary bD.

Theorem 9 (Nersesyan (1971/1972), [127, 128]) *A closed set E in a domain $D \subsetneq \mathbb{CP}^1$ is a Carleman set if and only if it satisfies the following two conditions.*

(a) *For each $\epsilon > 0$ there exists a δ, with $0 < \delta < \epsilon$, such that none of the components of $\mathring{E}$ intersects both $V_\delta(bD)$ and $D \setminus V_\epsilon(bD)$.*
(b) *For each $\epsilon > 0$ there is a $\delta > 0$ such that each point of the set $(D \setminus E) \cup V_\delta(bD)$ can be connected to bD by an arc lying in $(D \setminus E) \cup V_\epsilon(bD)$.*

We now look at the related problem of *uniform approximation* of functions in the space $\mathscr{A}(E)$ by holomorphic functions on D. This type of approximation was considered by N. U. Arakelian [6–8] who proved the following result characterizing *Arakelian sets*.

Theorem 10 (Arakelian (1964), [6–8]) *Let E be a closed set in a domain $D \subset \mathbb{C}$. The following two conditions are equivalent.*

(a) *Every function in $\mathscr{A}(E)$ is a uniform limit of functions in $\mathscr{O}(D)$.*
(b) *The complement $D^* \setminus E$ of E in the one point compactification $D^* = D \cup \{*\}$ of D is connected and locally connected.*

When E is compact, condition (b) simply says that $D \setminus E$ is connected, and in this case, (a) is Mergelyan's theorem. Note that local connectivity of $D^* \setminus E$ is a nontrivial condition only at the point $\{*\} = D^* \setminus D$. This condition has a more convenient interpretation. For simplicity, we consider the case $D = \mathbb{C}$. Given a closed set F in $\mathbb{C}$, we denote by H_F the union of all holes of F, an open set in $\mathbb{C}$. (Recall that a hole of F is a bounded connected components of $\mathbb{C} \setminus F$.)

Definition 2 (Bounded Exhaustion Hulls Property) A closed set E in $\mathbb{C}$ with connected complement has the *bounded exhaustion hulls property* (BEH) if the set $H_{E \cup \Delta}$ is bounded (relatively compact) for every closed disc Δ in $\mathbb{C}$.

It is well known and easily seen that the BEH property of a closed subset $E \subset \mathbb{C}$ is equivalent to $\mathbb{CP}^1 \setminus E$ being connected and locally connected at $\{\infty\} = \mathbb{CP}^1 \setminus \mathbb{C}$. Furthermore, this property may be tested on any sequence of closed discs (or more general compact simply connected domains) exhausting $\mathbb{C}$. For the corresponding condition in higher dimensions, see Definition 6 on p. 169.

We now present a simple proof of sufficiency of condition (b) for the case $D = \mathbb{C}$ in Arakelian's theorem, due to J.-P. Rosay and W. Rudin (1989), [140].

Proof of (b)⇒(a) in Theorem 10 Since the set $E \subset \mathbb{C}$ has the BEH property (see Definition 2), we can find a sequence of closed discs $\Delta_1 \subset \Delta_2 \subset \cdots \subset \bigcup_{i=1}^{\infty} \Delta_i = \mathbb{C}$ such that, setting $H_i = H_{E \cup \Delta_i}$ (the union of holes of $E \cup \Delta_i$), we have that

$$\Delta_i \cup \overline{H}_i \subset \mathring{\Delta}_{i+1}, \quad i = 1, 2, \ldots .$$

Set $E_0 = E$ and $E_i = E \cup \Delta_i \cup H_i$ for $i \in \mathbb{N}$. Note that E_i is a closed set with connected complement in $\mathbb{C}$, $E_i \subset E_{i+1}$, $\bigcup_{i=0}^{\infty} E_i = \mathbb{C}$, and $E \setminus \Delta_{i+1} = E_i \setminus \Delta_{i+1}$.

Choose a function $f = f_0 \in \mathscr{A}(E)$ and a number $\epsilon > 0$. We shall inductively construct a sequence $f_i \in \mathscr{A}(E_i)$ for $i = 1, 2, \ldots$ such that $|f_i - f_{i-1}| < 2^{-i}\epsilon$ on E_{i-1}; since the sets E_i exhaust $\mathbb{C}$, it follows that $F = \lim_{i\to\infty} f_i$ is an entire function satisfying $|F - f| < \epsilon$ on $E = E_0$. Let us explain the induction step $(i-1) \to i$. Assume that $f_{i-1} \in \mathscr{A}(E_{i-1})$. Pick a closed disc Δ such that $\Delta_i \cup \overline{H}_i \subset \Delta \subset \mathring{\Delta}_{i+1}$, and a smooth function $\chi \colon \mathbb{C} \to [0, 1]$ satisfying $\chi = 1$ on Δ and $\mathrm{supp}(\chi) \subset \Delta_{i+1}$. Note that $E_i \cup \Delta = E_{i-1} \cup \Delta$. Since the compact set $E_{i-1} \cap \Delta_{i+1}$ has no holes, Mergelyan's Theorem 3 furnishes a holomorphic polynomial h on $\mathbb{C}$ satisfying

$$|f_{i-1} - h| < 2^{-i-1}\epsilon \quad \text{on } E_{i-1} \cap \Delta_{i+1},$$

and

$$\frac{1}{\pi}\int_{\zeta\in E_{i-1}}|f_{i-1}(\zeta)-h(\zeta)|\cdot|\overline{\partial}\chi(\zeta)|\frac{dudv}{|z-\zeta|}<2^{-i-1}\epsilon,\quad z\in\mathbb{C}. \tag{15}$$

Note that the integrand is supported on $E_{i-1}\cap(\Delta_{i+1}\setminus\Delta)$, and hence the integral is bounded uniformly on $\mathbb{C}$ by the supremum of the integrand (which may be as small as desired by the choice of h) and the diameter of Δ_{i+1}. Let

$$g(z)=\frac{1}{\pi}\int_{\zeta\in E_{i-1}}(f_{i-1}(\zeta)-h(\zeta))\cdot\overline{\partial}\chi(\zeta)\frac{dudv}{z-\zeta},\quad z\in\mathbb{C},$$

and define the next function $f_i\colon E_i\cup\Delta\to\mathbb{C}$ by setting

$$f_i=\chi h+(1-\chi)f_{i-1}+g. \tag{16}$$

Note that g is continuous on $E_i\cup\Delta$, smooth on $\mathring{E}_i\cup\Delta$, it satisfies $\overline{\partial}g=(f_{i-1}-h)\overline{\partial}\chi$ on $\mathring{E}_{i-1}\cup\mathring{\Delta}$, and $|g|<2^{-i-1}\epsilon$ in view of (15). Since $E_i\cup\Delta=E_{i-1}\cup\Delta$, it follows that f_i is continuous on E_i and $\overline{\partial}f_i=(h-f_{i-1})\overline{\partial}\chi+\overline{\partial}g=0$ on $\mathring{E}_i$. Furthermore, on E_{i-1} we have $f_i=f_{i-1}+\chi(h-f_{i-1})+g$ and hence

$$|f_i-f_{i-1}|\le|\chi|\cdot|h-f_{i-1}|+|g|<2^{-i}\epsilon.$$

This completes the induction step and hence proves (b)⇒(a) in Theorem 10. □

Comparing with the proof of Theorem 8, we see that it was now necessary to solve a $\overline{\partial}$-equation since the set $E_{i-1}\cap(\Delta_{i+1}\setminus\Delta)$, on which we glued the approximating polynomial h with f_{i-1}, might have nonempty interior. This prevents us from obtaining Carleman approximation in the setting of Theorem 10 without additional hypotheses on E (compare with Nersesyan's theorem 9). On the other hand, the same proof yields the following special case of Nersesyan's theorem on Carleman approximation which is of interest in many applications.

Corollary 3 (On Carleman Approximation) *Assume that $E\subset\mathbb{C}$ is a closed set with connected complement satisfying the BEH property (see Definition 2). If there is a disc $\Delta\subset\mathbb{C}$ such that $E\setminus\Delta$ has empty interior, then every function in $\mathscr{A}(E)$ can be approximated in the fine $\mathscr{C}^0$ topology by entire functions.*

To prove Corollary 3 one follows the proof of Theorem 10, choosing the first disc Δ_1 big enough such that $E\setminus\Delta_1$ has empty interior. This allows us to define each function f_i (16) in the sequence without the correction term g (i.e., $g=0$).

The definition of the BEH property (see Definition 2) extends naturally to closed sets E in an arbitrary domain $\Omega\subset\mathbb{C}$. For such sets, an obvious modification of proof of Theorem 10 and Corollary 3 provide approximation of functions in $\mathscr{A}(E)$ by functions in $\mathscr{O}(\Omega)$ in the uniform and fine topology on E, respectively.

In 1976, A. Roth [141] proved several results on uniform and Carleman approximation of functions in $\mathscr{A}(E)$, where E is a closed set in a domain $\Omega \subset \mathbb{C}$, by meromorphic functions on Ω without poles on E. Her results are based on the technique of *fusing rational functions*, given by the following lemma.

Lemma 1 (Roth (1976), [141]) *Let K_1, K_2, and K be compact sets in $\mathbb{CP}^1$ with $K_1 \cap K_2 = \varnothing$. Then there is a constant $a = a(K_1, K_2) > 0$ such that for any pair of rational functions r_1, r_2 with $|r_1(z) - r_2(z)| < \epsilon$ $(z \in K)$ there is a rational function r such that $|r(z) - r_j(z)| < a\epsilon$ for $z \in K \cup K_j$ for $j = 1, 2$.*

The proof of this lemma is fairly elementary. In the special case of holomorphic functions, this amounts to the solution of a Cousin-I problem with bounds. As an application, A. Roth proved the following result [141, Theorem 1] on approximation of functions in $\mathscr{A}(E)$ by meromorphic functions without poles on E.

Theorem 11 (Roth (1976), [141]) *Let Ω be open in $\mathbb{C}$, and let $E \subsetneq \Omega$ be a closed subset of Ω. A function $f \in \mathscr{A}(E)$ may be uniformly approximated on E by functions in $\mathscr{M}(\Omega)$ without poles on E if and only if $f|_K \in \mathscr{R}(K)$ for every compact $K \subset E$.*

The paper [141] of A. Roth also contains results on tangential and Carleman approximation by meromorphic functions on closed subsets of planar domains.

The following result [141, Theorem 2] was proved by A. A. Nersesyan [128] for $\Omega = \mathbb{C}$; this extends Vitushkin's theorem (Theorem 7) to closed subsets of $\mathbb{C}$.

Theorem 12 (Nersesyan (1972), [128]; Roth (1976), [141]) *Let $E \subset \Omega$ be as in Theorem 11. A necessary and sufficient condition that every function in $\mathscr{A}(E)$ can be approximated uniformly on E by meromorphic functions on Ω with poles off E is that $\mathscr{R}(E \cap K) = \mathscr{A}(E \cap K)$ holds for every closed disc $K \subset \Omega$.*

The results presented above have been generalized to open Riemann surfaces to a certain extent, although the theory does not seem complete. In 1975, P. M. Gauthier and W. Hengartner [76] gave the following necessary condition for uniform approximation. (As before, X^* denotes the one point compactification of X.)

Theorem 13 *Let E be a closed subset of a Riemann surface X. If every function in $\overline{\mathscr{O}}(E)$ is a uniform limit of functions in $\mathscr{O}(X)$, then $X^* \setminus E$ is connected and locally connected, i.e., E is an Arakelian set.*

However, an example in [76] shows that the converse does not hold in general. In particular, *Arakelian's Theorem 10 cannot be fully generalized to Riemann surfaces.* Further examples to this effect can be found in [25, p. 120].

The situation is rather different for *harmonic* functions: if E is a closed Arakelian set in an open Riemann surface X, then every continuous function on E which is harmonic in the interior $\mathring{E}$ can be approximated uniformly on E by entire harmonic functions on X (see T. Bagby and P. M. Gauthier [11, Corollary 2.5.2]).

In 1986, A. Boivin [23] extended Nersesyan's Theorem 9 to a characterization of sets of holomorphic Carleman approximation in open Riemann surfaces, and he provided a sufficient condition on sets of meromorphic Carleman approximation.

For Carleman approximation of harmonic functions, we refer to T. Bagby and P. M. Gauthier [11, Theorem 3.2.3]. Furthermore, in [27], A. Boivin, P. Gauthier, and P. Paramonov established new Roth, Arakelian, and Carleman type theorems for solutions of a large class of elliptic partial differential operators L with constant complex coefficients.

We return once more to Bishop's localization theorem (see Theorem 6). We have already mentioned (cf. Remark 1) that in the late 1970s, P. M. Gauthier [74] and S. Scheinberg [148] constructed on any open Riemann surface X a meromorphic kernel $F(p,q)$ such that $F(p,q) = -F(q,p)$ and the only singularities of F are simple poles with residues $+1$ on the diagonal. With this kernel in hand, they extended Bishop's localization theorem to closed sets of essentially finite genus in any Riemann surface. (See also [24, 146].) The most precise results in this direction were obtained by S. Scheinberg [149] in 1979. Under certain restrictions on the Riemann surface X and the closed set $E \subset X$, he completely described those sets $P \subset X \setminus E$ such that every function in $\mathscr{A}(E)$ may be approximated uniformly on E by functions meromorphic on X whose poles lie in P. His theorems provide an elegant synthesis of all previously known results of this type and a summary of localization results.

The following converse to Bishop's localization theorem on an arbitrary Riemann surface was proved by A. Boivin and B. Jiang [26] in 2004. Recall that a *closed parametric disc* in a Riemann surface X is the inverse image $D = \phi^{-1}(\Delta)$ of a closed disc $\Delta \subset \phi(U) \subset \mathbb{C}$, where (U, ϕ) is a holomorphic chart on X.

Theorem 14 (Boivin and Jiang (2004), Theorem 1 in [26]) *Let E be a closed subset of a Riemann surface X. If $\mathscr{A}(E) = \overline{\mathscr{O}}(E)$, then $\mathscr{A}(E \cap D) = \overline{\mathscr{O}}(E \cap D)$ holds for every closed parametric disc $D \subset X$.*

Their proof relies on Vitushkin localization operators (11), adapted to Riemann surfaces by P. Gauthier [74] and S. Scheinberg [149] by using the Cauchy kernels mentioned above. (See also Remark 1.)

Note that Theorem 14 generalizes one of the implications in Theorem 12 to Riemann surfaces. A result of this kind does not seem available for compact sets in higher dimensional complex manifolds. We shall discuss this question again in connection with the Mergelyan approximation problem for manifold-valued maps (see Section 7.2, in particular Definition 8 and Remark 9).

The following is an immediate corollary to Theorem 14 and Bishop's localization theorem for closed sets in Riemann surfaces [74, 149]. It provides an optimal version of Vitushkin's approximation theorem (see Theorem 7) on Riemann surfaces.

Theorem 15 (Boivin and Jiang (2004), Theorem 2 in [26]) *Let E be a closed subset of a Riemann surface X, and assume either that E is weakly of infinite genus (this holds in particular if E is compact) or $\mathring{E} = \varnothing$. Then, the following are equivalent:*

1. *Every function in $\mathscr{A}(E)$ is a uniform limit of meromorphic functions on X with poles off E.*
2. *For every closed parametric disc $D \subset X$ we have $\mathscr{A}(E \cap D) = \overline{\mathscr{O}}(E \cap D)$.*

3. *For every point $x \in X$ there exists a closed parametric disc D_x centered at x such that $\mathscr{A}(E \cap D_x) = \overline{\mathscr{O}}(E \cap D_x)$.*

4 Mergelyan's Theorem for $\mathscr{C}^r$ Functions on Riemann Surfaces

In applications, one is often faced with the approximation problem for functions of class $\mathscr{C}^r$ ($r \in \mathbb{N}$) on compact or closed sets in a Riemann surface. Such problems arise not only in complex analysis (for instance, in constructions of closed complex curves in complex manifolds, see [44], or in constructions of proper holomorphic embeddings of open Riemann surfaces into $\mathbb{C}^2$, see [67, 68] and [62, Chap. 9]), but also in related areas such as the theory of minimal surfaces in Euclidean spaces $\mathbb{R}^n$ (see the recent survey [3]), the theory of holomorphic Legendrian curves in complex contact manifolds (see [2, 4]), and others. In most geometric constructions it suffices to consider compact sets of the following type.

Definition 3 (Admissible Sets in Riemann Surfaces) A compact set S in a Riemann surface X is *admissible* if it is of the form $S = K \cup M$, where K is a finite union of pairwise disjoint compact domains with piecewise $\mathscr{C}^1$ boundaries in X and $M = S \setminus \mathring{K}$ is a union of finitely many pairwise disjoint smooth Jordan arcs and closed Jordan curves meeting K only in their endpoints (or not at all) and such that their intersections with the boundary bK of K are transverse.

Clearly, the complement $X \setminus S$ of an admissible set has at most finitely many connected components, and hence Theorem 5 applies.

A function $f : S = K \cup M \to \mathbb{C}$ on an admissible set is said to be of class $\mathscr{C}^r(S)$ if $f|_K \in \mathscr{C}^r(K)$ (this means that it is of class $\mathscr{C}^r(\mathring{K})$ and all its partial derivatives of order $\le r$ extend continuously to K) and $f|_M \in \mathscr{C}^r(M)$. Whitney's jet-extension theorem (see Theorem 46) shows that any $f \in \mathscr{A}^r(S)$ extends to a function $f \in \mathscr{C}^r(X)$ which is $\overline{\partial}$*-flat* to order r on S, meaning that

$$\lim_{x \to S} D^{r-1}(\overline{\partial} f)(x) = 0. \tag{17}$$

Here, D^k denotes the total derivative of order k (the collection of all partial derivatives of order $\le k$). We define the $\mathscr{C}^r(S)$ norm of f as the maximum of derivatives of f up to order r at points $z \in S$, where for points $z \in M \setminus K$ we consider only the tangential derivatives. (This equals the r-jet norm on S of a $\overline{\partial}$-flat extension of f.)

We have the following approximation result for functions of class $\mathscr{A}^r$ on admissible sets in Riemann surfaces. Corollary 9 in Section 7.2 gives an analogous result for manifold-valued maps.

Theorem 16 ($\mathscr{C}^r$ Approximation on Admissible Sets in Riemann Surfaces) *If S is an admissible set in a Riemann surface X, then every function $f \in \mathscr{A}^r(S)$ ($r \in \mathbb{N}$) can be approximated in the $\mathscr{C}^r(S)$-norm by meromorphic functions on X, and by holomorphic functions if S has no holes.*

Proof We give a proof by induction on r, reducing it to $\mathscr{C}^0$ approximation. The result can also be proved by the method in the proof of Theorems 24 and 25 below.

Pick an open neighborhood $\Omega \subsetneq X$ of S such that there is a deformation retraction of Ω onto S. (It follows in particular that S has no holes in Ω.) It suffices to show that any function $f \in \mathscr{A}^r(S)$ can be approximated in $\mathscr{C}^r(S)$ by functions holomorphic on Ω; the conclusion then follows from Runge's theorem (Theorem 4) and the Cauchy estimates. We may assume that S (and hence Ω) is connected. There are smooth closed oriented Jordan curves $C_1, \ldots, C_l \subset S$ generating the first homology group $H_1(S, \mathbb{Z}) = H_1(\Omega, \mathbb{Z}) \cong \mathbb{Z}^l$ such that $C = \bigcup_{i=1}^l C_i$ is a compact Runge set in Ω. Let θ be a nowhere vanishing holomorphic 1-form on Ω. (Such θ exists by the Oka–Grauert principle, see [62, Theorem 5.3.1]. Furthermore, by the Gunning–Narasimhan theorem [89] there exists a holomorphic function $\xi \colon \Omega \to \mathbb{C}$ without critical points, and we may take $\theta = d\xi$.) Consider the period map $P = (P_1, \ldots, P_l) \colon \mathscr{C}(C) \to \mathbb{C}^l$ given by

$$P_i(h) = \int_{C_i} h\theta, \qquad h \in \mathscr{C}(C), \ \ i = 1, \ldots, l.$$

It is elementary to find continuous functions $h_1, \ldots, h_l \colon C \to \mathbb{C}$ such that $P_i(h_j) = \delta_{i,j}$ (Kronecker's delta). By Mergelyan's theorem (Theorem 5) we can approximate each h_i uniformly on C by a holomorphic function $g_i \in \mathscr{O}(\Omega)$. Assuming that the approximations are close enough, the $l \times l$ matrix A with the entries $P_i(g_j)$ is invertible. Replacing the vector $g = (g_1, \ldots, g_l)^t$ by $A^{-1}g$ we obtain $P_i(g_j) = \delta_{i,j}$. Fix an integer $r \in \mathbb{Z}_+$. Consider the function $\Phi \colon \mathscr{A}^r(S) \times S \times \mathbb{C}^l \to \mathbb{C}$ defined by

$$\Phi(h, x, t) = h(x) + \sum_{j=1}^{l} t_j g_j(x),$$

where $h \in \mathscr{A}^r(S)$, $x \in S$, and $t = (t_1, \ldots, t_l) \in \mathbb{C}^l$. Then, $P(\Phi(h, \cdot, t)) = P(h) + \sum_{j=1}^l t_j P(g_j)$, and hence

$$\left. \frac{\partial P_i(\Phi(h, \cdot, t)}{\partial t_j} \right|_{t=0} = P_i(g_j) = \delta_{i,j}, \qquad i, j = 1, \ldots, l.$$

This period domination condition implies, in view of the implicit function theorem, that for every $h_0 \in \mathscr{A}^r(S)$ the equation $P(\Phi(h, \cdot, t)) = P(h_0)$ can be solved on $t = t(h)$ for all $h \in \mathscr{A}^r(S)$ near h_0, with $t(h_0) = P(h_0)$.

We can now prove the theorem by induction on $r \in \mathbb{Z}_+$. By Theorem 5, the result holds for $r = 0$. Assume that $r \in \mathbb{N}$ and the theorem holds for $r - 1$.

Pick $f \in \mathscr{A}^r(S)$. The function $f'(x) := df(x)/\theta(x)$ ($x \in S$) then belongs to $\mathscr{A}^{r-1}(S)$. (At a point $x \in S \setminus K$ we understand $df(x)$ as the $\mathbb{C}$-linear extension to T_xX of the differential of $f|_M$.) Note that $P(f') = \left(\int_{C_j} df\right)_j = 0 \in \mathbb{C}^l$. By the induction hypothesis, we can approximate f' in $\mathscr{C}^{r-1}(S)$ by holomorphic functions $h \in \mathscr{O}(\Omega)$. If the approximation is close enough, there is a $t = t(h) \in \mathbb{C}^l$ near $P(f') = 0$ such that the holomorphic function $\tilde{h} := \Phi(h, \cdot, t)$ on Ω satisfies $P(\tilde{h}) = 0$. Fix a point $p_0 \in S$ and define $\tilde{f}(p) = \int_{p_0}^{p} \tilde{h}\theta$ for $p \in \Omega$. Since the holomorphic 1-form $\tilde{h}\theta$ has vanishing period, the integral is independent of the choice of a path of integration. If $p \in S$, then the path may be chosen to lie in S, and hence $\tilde{f}$ approximates f in $\mathscr{C}^r(S)$. This completes the induction step and therefore proves the theorem. □

The following optimal approximation result for smooth functions on compact sets in $\mathbb{C}$ was proved by J. Verdera in 1986, [165].

Theorem 17 (Verdera (1986), [165]) *Let K be a compact set in $\mathbb{C}$, and let f be a compactly supported function in $\mathscr{C}^r(\mathbb{C})$, $r \in \mathbb{N}$, such that $\partial f/\partial \bar{z}$ vanishes on K to order $r-1$ (see (17)). Then, f can be approximated in $\mathscr{C}^r(\mathbb{C})$ by functions which are holomorphic in neighborhoods of K.*

Theorem 17 shows that the obstacles to rational approximation of functions in $\mathscr{A}(K)$ in Vitushkin's theorem (see Theorem 7) are no longer present when considering rational approximation of $\mathscr{C}^r$ functions which are $\bar{\partial}$-flat of order r for $r > 0$. Results in the same direction, concerning rational approximation on compact sets in $\mathbb{C}$ in Lipschitz and Hölder norms, were obtained by A. G. O'Farrell during 1977–79, [129–131].

Verdera's proof of Theorem 17 is somewhat simpler for $r \geq 2$ than for $r = 1$. In the case $r \geq 2$, he follows Vitushkin's scheme for rational approximation, using in particular the localization operators (11); here is the outline. Fix a number $\delta > 0$. Choose a covering of $\mathbb{C}$ by a countable family of discs Δ_j of radius δ such that every point $z \in \mathbb{C}$ is contained in at most 21 discs. Also, let $\phi_j \in \mathscr{C}_0^\infty(\mathbb{C})$ be a smooth function with values in $[0, 1]$, with compact support contained in Δ_j, such that $\sum_j \phi_j = 1$ and $|D^k \phi_j| \leq C\delta^{-k}$ for some absolute constant $C > 0$. Set

$$f_j(z) = T_{\phi_j}(f)(z) = \frac{1}{\pi} \int_{\mathbb{C}} \frac{f(\zeta) - f(z)}{\zeta - z} \frac{\partial \phi_j(\zeta)}{\partial \bar{\zeta}} \, du dv, \quad z \in \mathbb{C}.$$

Then, f_j is holomorphic on $\mathbb{C} \setminus \mathrm{supp}(\phi_j)$, $f_j = 0$ if $\mathrm{supp}(f) \cap \Delta_j = \varnothing$, and $f = \sum_j f_j$ (a finite sum). Let $g = \sum_j' f_j$ where the sum is over those indices j for which $\Delta_j \cap K = \varnothing$ and $h = f - g = \sum_j'' f_j$ is the sum over the remaining j's. Thus, g is holomorphic in a neighborhood of K, and Verdera shows that the $\mathscr{C}^r(\mathbb{C})$ norm of h goes to zero as $\delta \to 0$. The analytic details are considerable, especially for $r = 1$.

In conclusion, we mention that many of the results on holomorphic approximation, presented in this and the previous two sections, have been generalized to solutions of more general elliptic differential equations in various Banach space norms; see in particular J. Verdera [164], P. Paramonov and J. Verdera [135], A. Boivin, P. Gauthier, and P. Paramonov [27], and P. Gauthier and P. Paramonov [78].

5 The Oka–Weil Theorem and Its Generalizations

The analogue of Runge's theorem (see Theorems 2 and 4) on Stein manifolds and Stein spaces is the following theorem due to K. Oka [132] and A. Weil [171]. All complex spaces are assumed to be reduced.

Theorem 18 (The Oka–Weil Theorem) *If X is a Stein space and K is a compact $\mathscr{O}(X)$-convex subset of X, then every holomorphic function in an open neighborhood of K can be approximated uniformly on K by functions in $\mathscr{O}(X)$.*

Proof Two proofs of this result are available in the literature. The original one, due to K. Oka and A. Weil, proceeds as follows. A compact $\mathscr{O}(X)$-convex subset K in a Stein space X admits a basis of open Stein neighborhoods of the form

$$P = \{x \in X : |h_1(x)| < 1, \ldots, |h_N(x)| < 1\}$$

with $h_1, \ldots, h_N \in \mathscr{O}(X)$. We may assume that the function $f \in \mathscr{O}(K)$ to be approximated is holomorphic on P. By adding more functions if necessary, we can ensure that the map $h = (h_1, \ldots, h_N) \colon X \to \mathbb{C}^N$ embeds P onto a closed complex subvariety $A = h(P)$ of the unit polydisc $\mathbb{D}^N \subset \mathbb{C}^N$. Hence, there is a function $g \in \mathscr{O}(A)$ such that $g \circ h = f$ on P. By the Oka–Cartan extension theorem [62, Corollary 2.6.3], g extends to a holomorphic function G on $\mathbb{D}^N$. Expanding G into a power series and precomposing its Taylor polynomials by h gives a sequence of holomorphic functions on X converging to f uniformly on K.

Another approach uses the method of L. Hörmander for solving the $\overline{\partial}$-equation with L^2-estimates (see [94, 96]). We consider the case $X = \mathbb{C}^n$; the general case reduces to this one by standard methods of Oka–Cartan theory. Assume that f is a holomorphic function in a neighborhood $U \subset \mathbb{C}^n$ of K. Choose a pair of neighborhoods $W \Subset V \Subset U$ of K and a smooth function $\chi \colon \mathbb{C}^n \to [0, 1]$ such that $\chi = 1$ on $\overline{V}$ and $\mathrm{supp}(\chi) \subset U$. By choosing $W \supset K$ small enough, there is a nonnegative plurisubharmonic function $\rho \geq 0$ on $\mathbb{C}^n$ that vanishes on W and satisfies $\rho \geq c > 0$ on $U \setminus V$. Note that the smooth $(0, 1)$-form

$$\alpha = \overline{\partial}(\chi f) = f\,\overline{\partial}\chi = \sum_{i=1}^{n} \alpha_i \, d\bar{z}_i$$

is supported in $U \setminus V$. Hörmander's theory for the $\overline{\partial}$-complex (see [96, Theorem 4.4.2]) furnishes for any $t > 0$ a smooth function h_t on $\mathbb{C}^n$ satisfying

$$\overline{\partial} h_t = \alpha \qquad \text{and} \quad \int_{\mathbb{C}^n} \frac{|h_t|^2}{(1+|z|^2)^2}\, e^{-t\rho} d\lambda \le \int_{\mathbb{C}^n} \sum_{i=1}^{n} |\alpha_i|^2 e^{-t\rho} d\lambda. \tag{18}$$

(Here, $d\lambda$ denotes the Lebesgue measure on $\mathbb{C}^n$.) As $t \to +\infty$, the right-hand side approaches zero since $\rho \ge c > 0$ on $\mathrm{supp}(\alpha) \subset U \setminus V$. Since $\rho|_W = 0$, it follows that $\lim_{t\to 0} \|h_t\|_{L^2(W)} = 0$. The interior elliptic estimates (see [66, Lemma 3.2]) imply that $h_t|_K \to 0$ in $\mathscr{C}^r(K)$ for every fixed $r \in \mathbb{Z}_+$. The functions

$$f_t = \chi f - h_t : \mathbb{C}^n \longrightarrow \mathbb{C}$$

are then entire and converge to f uniformly on K as $t \to +\infty$. □

We also have the following parametric version of the Cartan–Oka–Weil theorem which is useful in applications (see [62, Theorem 2.8.4]).

Theorem 19 (Cartan–Oka–Weil Theorem with Parameters) *Let X be a Stein space. Assume that K is an $\mathscr{O}(X)$-convex subset of X, X' is a closed complex subvariety of X, and $P_0 \subset P$ are compact Hausdorff spaces. Let $f : P \times X \to \mathbb{C}$ be a continuous function such that*

(a) *for every $p \in P$, $f(p,\cdot) : X \to \mathbb{C}$ is holomorphic in a neighborhood of K (independent of p) and $f(p,\cdot)|_{X'}$ is holomorphic, and*
(b) *$f(p,\cdot)$ is holomorphic on X for every $p \in P_0$.*

Then there exists for every $\epsilon > 0$ a continuous function $F : P \times X \to \mathbb{C}$ satisfying the following conditions:

(i) *$F_p = F(p,\cdot)$ is holomorphic on X for all $p \in P$,*
(ii) *$|F - f| < \epsilon$ on $P \times K$, and*
(iii) *$F = f$ on $(P_0 \times X) \cup (P \times X')$.*

The same result holds for sections of any holomorphic vector bundle over X.

The proof can be obtained by any of the two schemes outlined above. For the second one, note that there is a linear solution operator to the $\overline{\partial}$-problem (18), and hence continuous dependence on the parameter comes for free. One needs to include the interpolation condition into the scheme to take care of the interpolation condition (iii). We refer to [62, Theorem 2.8.4] for the details.

A similar approximation theorem holds for sections of coherent analytic sheaves over Stein spaces (see, e.g., H. Grauert and R. Remmert [85, p. 170]).

The extension of the Oka–Weil theorem to maps $X \to Y$ from a Stein space X to more general complex manifolds Y is the subject of *Oka theory*. A complex manifold Y for which the analogue of Theorem 19 holds in the absence of topological obstructions is called an *Oka manifold*. We discuss this topic in Section 7.1.

6 Mergelyan's Theorem in Higher Dimensions

As we have seen in Sections 2–4, the Mergelyan approximation theory in the complex plane and on Riemann surfaces was a highly developed subject around mid twentieth century. Around the same time, it became clear that the situation is much more complicated in higher dimensions. For example, in 1955 J. Wermer [173] constructed an arc in $\mathbb{C}^3$ which fails to have the Mergelyan property. This suggests that, in several variables, one has to be much more restrictive about the sets on which one considers Mergelyan type approximation problems.

There are two lines of investigations in the literature: approximation on submanifolds of $\mathbb{C}^n$ of various degrees of smoothness and approximation on closures of bounded pseudoconvex domains. In neither category the problem is completely understood, and even with these restrictions, the situation is substantially more complicated than in dimension one. For example, R. Basener (1973), [14] (generalizing a result of B. Cole (1968), [39]) showed that Bishop's peak point criterium does not suffice even for smooth polynomially convex submanifolds of $\mathbb{C}^n$. Even more surprisingly, it was shown by K. Diederich and J. E. Fornæss in 1976 [42] that there exist bounded pseudoconvex domains with smooth boundaries in $\mathbb{C}^2$ on which the Mergelyan property fails. The picture for curves is more complete; see G. Stolzenberg [153], H. Alexander [5], and P. Gauthier and E. Zeron [80].

In this section we outline the developments starting around the 1960s, give proofs in some detail in the cases of totally real manifolds and strongly pseudoconvex domains, and provide some new results on combinations of such sets.

Definition 4 Let (X, J) be a complex manifold, and let $M \subset X$ be a $\mathscr{C}^1$ submanifold.

(a) M is *totally real* at a point $p \in M$ if $T_pM \cap JT_pM = \{0\}$. If M is totally real at all points, we say that M is a totally real submanifold of X.
(b) M is a *stratified totally real submanifold* of X if $M = \bigcup_{i=1}^{l} M_i$, with $M_i \subset M_{i+1}$ locally closed sets, such that M_1 and $M_{i+1} \setminus M_i$ are totally real submanifolds.

We now introduce suitable types of sets for Mergelyan approximation. The following notion is a generalization of the one for Riemann surfaces in Definition 3. Recall that a compact set S in a complex manifold X is a *Stein compact* if S admits a basis of open Stein neighborhoods in X.

Definition 5 (Admissible Sets) Let S be a compact set in a complex manifold X.

(a) S is *admissible* if it is of the form $S = K \cup M$, where S and K are Stein compacts and $M = S \setminus K$ is a totally real submanifold of X (possibly with boundary).
(b) S is *stratified admissible* if instead $M = \bigcup_{i=1}^{l} M_i$ is a stratified totally real submanifold such that $S_i = K \cup M_i$ is compact for every $i = 1, \dots, l$.
(c) An admissible set $S = K \cup M$ is *strongly admissible* if, in addition to the conditions in (a), K is the closure of a strongly pseudoconvex Stein domain, not necessarily connected.

Remark 4 We emphasize that, in the definition of an admissible set, it is the decomposition of S into the union $K \cup M$ that matters, so one might think of them as pairs (K, M) with the indicated properties. We will show (see Lemma 2) that if in (a) only the set S is assumed to be a Stein compact (and M is totally real), then K is nevertheless automatically a Stein compact. It follows that if $S = K \cup M$ is a stratified admissible set and $M = \bigcup_{i=1}^{m} M_i$ is a totally real stratification, then the set $S_i = K \cup M_i$ is a stratified admissible set for every i (see Corollary 4). □

Remark 5 We wish to compare the class of admissible sets with those considered by L. Hörmander and J. Wermer [97] and F. Forstnerič [54, Sect. 3], [62, Sects. 3.7–3.8]. A compact set S in a complex manifold X is said to be *holomorphically convex* if it admits a Stein neighborhood $\Omega \subset X$ such that S is $\mathscr{O}(\Omega)$-convex. Clearly, such S is a Stein compact, but the converse is false in general. Let us call a compact set $S = K \cup M$ an *HW set* (for Hörmander and Wermer) if S is holomorphically convex and $M = S \setminus K$ is a totally real submanifold of X. In the cited works, approximation results similar to those presented here are proved on HW sets. Clearly, every HW set is admissible. By combining the techniques in the proof of Proposition 2 and Theorem 20 one can prove the following partial converse.

Proposition 1 *If $S = K \cup M$ is an admissible set in complex manifold X and $U \subset X$ is a neighborhood of K, there exists a Stein neighborhood Ω of S such that*

$$h(S) := \overline{\widehat{S}_{\mathscr{O}(\Omega)} \setminus S} \subset U.$$

Thus, taking $S' = \widehat{S}_{\mathscr{O}(\Omega)}$, $K' = K \cup h(S)$, and $M' = M \setminus h(S)$, we see that $S' = K' \cup M'$ is an HW set with $K' \subset U$. Thus, every admissible set can be approximated from the outside by HW sets, enlarging K only slightly.

It was shown by L. Hörmander and J. Wermer [97] (see also [62, Theorem 3.7.1]) that if $S = K \cup M$ is an HW set and $S' = K \cup M'$ is another compact set, with M' a totally real submanifold, such that $S \cap U = S' \cap U$ holds for some open neighborhood of K, then S' is also admissible (i.e., any such S' is a Stein compact). In view of the above proposition, the same holds true for admissible sets, i.e., this class is stable under changes of the totally real part which are fixed near K. □

We will consider two types of approximations in higher dimensions. On admissible sets $S = K \cup M$ we will consider Runge–Mergelyan approximation, *i.e.*, we assume that the object we want to approximate (function, form, map, etc.) is holomorphic on a neighborhood of K and continuous or smooth on M. On strongly admissible sets we will consider true Mergelyan approximation, assuming that the object to be approximated is of class $\mathscr{A}^r(S)$ for some $r \in \mathbb{Z}_+$.

6.1 Approximation on Totally Real Submanifolds and Admissible Sets

In this section we present an optimal $\mathscr{C}^k$-approximation result on totally real submanifolds. With essentially no extra effort we get approximation results on stratified totally real manifolds and on admissible sets (see Theorems 20 and 21).

There is a long history on approximation on totally real submanifolds, starting with J. Wermer [173] on curves and R. O. Wells [172] on real analytic manifolds. The first general result on approximation on totally real manifolds with various degrees of smoothness is due to L. Hörmander and J. Wermer [97]. Their work is based on L^2-methods for solving the $\overline{\partial}$-equation, and the passage from L^2 to $\mathscr{C}^k$-estimates led to a gap between the order m of smoothness of the manifold M on which the approximation takes place, and the order k of the norm of the Banach space $\mathscr{C}^k(M)$ in which the approximation takes place. Subsequently, several authors worked on decreasing the gap between m and k, introducing more precise integral kernel methods for solving $\overline{\partial}$. The optimal result with $m = k$ was eventually obtained by M. Range and Y.-T. Siu [139]. Subsequent improvements were made by F. Forstnerič, E. Løw, and N. Øvrelid [66] in 2001. They developed Henkin-type kernels adapted to this situation and obtained optimal results on approximation of $\overline{\partial}$-flat functions in tubes around totally real manifolds by holomorphic functions. In 2009, B. Berndtsson [18] used L^2-theory to give a new approach to uniform approximation by holomorphic functions on compact zero sets of strongly plurisubharmonic functions. A novel byproduct of his method is that, in the case of polynomial approximation, one gets a bound on the degree of the approximating polynomial in terms of the closeness of the approximation.

We will not go into the details of the L^2 or the integral kernel approaches, but will instead present a method based on convolution with the Gaussian kernel which originates in the proof of Weierstrass's Theorem 1 on approximating continuous functions on $\mathbb{R}$ by holomorphic polynomials. This approach is perhaps the most elementary one, and is particularly well suited for proving Runge–Mergelyan type approximation results with optimal regularity on (strongly) admissible sets. It seems that the first modern application of this method was made in 1981 by S. Baouendi and F. Treves [12] to obtain local approximation of Cauchy–Riemann (CR) functions on CR submanifolds. The use of this method on totally real manifolds was developed further by P. Manne [118] in 1993 to obtain Carleman approximation on totally real submanifolds (see also [119]).

We define the bilinear form $\langle \cdot\,, \cdot \rangle$ on $\mathbb{C}^n$ by

$$\langle z, w \rangle = \sum_{i=1}^{n} z_i w_i, \qquad z^2 = \langle z, z \rangle = \sum_{i=1}^{n} z_i^2. \tag{19}$$

We consider first the real subspace $\mathbb{R}^n$ of $\mathbb{C}^n$. Recall that

$$\int_{\mathbb{R}^n} e^{-x^2} dx = \int_{\mathbb{R}^n} e^{-\sum_{i=1}^n x_i^2} dx_1 \cdots dx_n = \left(\int_{\mathbb{R}} e^{-t^2} dt \right)^n = \pi^{n/2}.$$

It follows that $\int_{\mathbb{R}^n} e^{-x^2/\epsilon^2} dx = \epsilon^n \pi^{n/2}$, so the family of functions $\pi^{-n/2}\epsilon^{-n}e^{-x^2/\epsilon^2}$ is an approximate identity on $\mathbb{R}^n$. Given $f \in \mathscr{C}_0^k(\mathbb{R}^n)$, consider the entire functions

$$f_\epsilon(z) = \pi^{-n/2} \int_{\mathbb{R}^n} \frac{1}{\epsilon^n} f(x) e^{-(x-z)^2/\epsilon^2} dx, \qquad z \in \mathbb{C}^n, \ \epsilon > 0.$$

(See (3) for $n = 1$.) It is straightforward to verify that $f_\epsilon \to f$ uniformly on $\mathbb{R}^n$, and by a change of variables $u = z - x$ we get convergence also in the $\mathscr{C}^k$ norm.

It is remarkable that the same procedure gives local approximation in the $\mathscr{C}^k$ norm on any totally real submanifold of class $\mathscr{C}^k$. Recall that $\mathbb{B}^n_{\mathbb{R}} \subset \mathbb{R}^n$ is the unit ball and $\mathbb{B}^n_{\mathbb{R}}(\epsilon) = \epsilon \mathbb{B}^n_{\mathbb{R}}$ for any $\epsilon > 0$.

Proposition 2 *Let $\psi : \mathbb{B}^n_{\mathbb{R}} \to \mathbb{R}^n$ be a map of class $\mathscr{C}^k$ ($k \in \mathbb{N}$) with $\psi(0) = (d\psi)_0 = 0$, and set $\phi(x) = x + i\psi(x) \in \mathbb{C}^n$. Then there exists a number $0 < \delta < 1$ such that the following holds. Let $N \subset \mathbb{B}^n_{\mathbb{R}}$ be a closed set, and let $M = \phi(\mathbb{B}^n_{\mathbb{R}}(\delta) \cap N) \subset \mathbb{C}^n$ and $bM = \phi(b\mathbb{B}^n_{\mathbb{R}}(\delta) \cap N) \subset \mathbb{C}^n$. Given $f \in \mathscr{C}_0(M)$, there exists a family of entire functions $f_\epsilon \in \mathscr{O}(\mathbb{C}^n)$, $\epsilon > 0$, such that the following hold as $\epsilon \to 0$:*

(a) *$f_\epsilon \to f$ uniformly on M, and*
(b) *$f_\epsilon \to 0$ uniformly on $U = \{z \in \mathbb{C}^n : \operatorname{dist}(z, bM) < \eta\}$ for some $\eta > 0$.*

Moreover, if N is a $\mathscr{C}^k$-smooth submanifold of $\mathbb{B}^n_{\mathbb{R}}$ and $f \in \mathscr{C}_0^k(M)$, then the approximation in (a) may be achieved in the $\mathscr{C}^k$-norm on M.

Remark 6 This proposition is local. However, Condition (b) and Cartan's theorem B make it very simple to globalize the approximation in the case that M is a totally real piece of an admissible set (see Theorem 20 below). □

Proof of Proposition 2 Since functions on N extend to $\mathbb{B}^n_{\mathbb{R}}$ in the appropriate classes, it is enough to prove the proposition in the case $N = \mathbb{B}^n_{\mathbb{R}}$.

Note that $\phi'(x) = I + i\psi'(x)$. We will need (see Hörmander [95, p. 85]) that if A is a symmetric $n \times n$ matrix with positive definite real part, then

$$\int_{\mathbb{R}^n} e^{-\langle Au, u \rangle} du = \pi^{n/2} (\det\ A)^{-1/2}. \tag{20}$$

We shall use this with the matrix $A(x) = \phi'(x)^T \phi'(x)$ whose real part equals $\Re A(x) = I - \psi'(x)^T \psi'(x)$. Since $\psi'(0) = 0$, there is a number $0 < \delta_0 < 1$ such that $\Re A(x) > 0$ is positive definite for all $x \in \mathbb{B}^n_{\mathbb{R}}(\delta_0)$, and ψ is Lipschitz-α with $\alpha < 1$ on $\mathbb{B}^n_{\mathbb{R}}(\delta_0)$. By using a smooth cut-off function, we extend ψ to $\mathbb{R}^n$ such that $\operatorname{supp}(\psi) \subset \mathbb{B}^n_{\mathbb{R}}$, without changing its values on $\mathbb{B}^n_{\mathbb{R}}(\delta_0)$. (This does not affect the lemma.) We will show that the lemma holds for any number δ with $0 < \delta < \delta_0$.

Set $M = \phi(\mathbb{B}^n_{\mathbb{R}}(\delta))$ and $M_0 = \phi(\mathbb{B}^n_{\mathbb{R}}(\delta_0))$, so $\overline{M} \subset M_0$. Given $f \in \mathscr{C}_0^k(M)$, set

$$f_\epsilon(z) = \frac{1}{\pi^{n/2}\epsilon^n} \int_M f(w) e^{-(w-z)^2/\epsilon^2} dw, \quad z \in \mathbb{C}^n, \tag{21}$$

where $dw = dw_1 \ldots dw_n$.

We begin by showing that condition (b) holds by inspecting the integral kernel. Writing $z = x + \mathrm{i}y \in \mathbb{C}^n$ and $w = u + \mathrm{i}v = \mathbb{C}^n$, we have that

$$\left| e^{-(w-z)^2} \right| = e^{-\Re(w-z)^2} = e^{(y-v)^2-(x-u)^2}.$$

For a fixed $w = u + \mathrm{i}v$, let $\Gamma_w = \{z = x + \mathrm{i}y \in \mathbb{C}^n : (y - v)^2 < (x - u)^2\}$. On Γ_w, the function $e^{-(w-z)^2/\epsilon^2}$ clearly converges to zero as $\epsilon \to 0$. Since the map ψ is Lipschitz-α with $\alpha < 1$ on $\mathbb{B}^n_{\mathbb{R}}(\delta_0)$, we see that for every $w \in M_0$ we have that $M_0 \setminus \{w\} \subset \Gamma_w$. Hence, there exists and open neighborhood $U \subset \mathbb{C}^n$ of bM with $U \subset \Gamma_w$ for all $w \in \operatorname{supp}(f)$. This establishes (b).

Let us now prove (a). Since the function $x \mapsto f(\phi(x))$ is supported in $\mathbb{B}^n_{\mathbb{R}}(\delta)$, we can extend the product of it with any other function on $\mathbb{B}^n_{\mathbb{R}}(\delta)$ to all of $\mathbb{R}^n$ by letting it vanish outside $\mathbb{B}^n_{\mathbb{R}}(\delta)$. Fix a point $z_0 = \phi(x_0) \in M$ with $x_0 \in \mathbb{B}^n_{\mathbb{R}}(\delta)$. Using the notation (19), we have that

$$\begin{aligned} f_\epsilon(z_0) &= \pi^{-n/2} \int_M \frac{1}{\epsilon^n} f(w) e^{-(w-z_0)^2/\epsilon^2}\, dw \\ &= \pi^{-n/2} \int_{\mathbb{R}^n} \frac{1}{\epsilon^n} f(\phi(x))\, e^{-(\phi(x)-\phi(x_0))^2/\epsilon^2} \det \phi'(x)\, dx \\ &= \pi^{-n/2} \int_{\mathbb{R}^n} f(\phi(x_0 + \epsilon u))\, e^{-(u+\mathrm{i}(\psi(x_0+\epsilon u)-\psi(x_0))/\epsilon)^2} \det \phi'(x_0 + \epsilon u) du. \end{aligned}$$

(We applied the change of variable $x = x_0 + \epsilon u$.) The Lipschitz condition on ψ gives

$$\left| e^{-(u+\mathrm{i}(\psi(x_0+\epsilon u)-\psi(x_0))/\epsilon)^2} \right| \le e^{-(1-\alpha)|u|^2}$$

for all $x_0 \in \mathbb{B}^n_{\mathbb{R}}(\delta)$ and $0 < \epsilon < \delta_0 - \delta$. The dominated convergence theorem implies

$$\begin{aligned} \lim_{\epsilon \to 0} f_\epsilon(z_0) &= \pi^{-n/2} \int_{\mathbb{R}^n} f(\phi(x_0)) e^{-\langle \phi'(x_0)u, \phi'(x_0)u \rangle} \det \phi'(x_0)\, du \\ &= \pi^{-n/2} \int_{\mathbb{R}^n} f(z_0) e^{-\langle \phi'(x_0)^T \phi'(x_0)u, u \rangle} \det \phi'(x_0)\, du \\ &= f(z_0). \end{aligned}$$

The last line follows from (20) applied with the matrix $A = \phi'(x_0)^T \phi'(x_0)$, noting also that $\det A = \det \phi'(x_0)^2$. The estimates are clearly independent of $x_0 \in \mathbb{B}^n_{\mathbb{R}}(\delta)$, and hence of $z_0 \in M$, so the convergence $f_\epsilon \to f$ is uniform on M.

To get convergence in the $\mathscr{C}^k$ norm, one replaces partial differentiation of the kernel in (21) with respect to z by partial differentiation of f with respect to w (see P. Manne [118, p. 524] for the details). $\square$

We now globalize Proposition 2 to obtain the approximation of $\mathscr{C}^k$ functions on totally real manifolds of class $\mathscr{C}^k$ and on (stratified) admissible sets.

Theorem 20 *Let $S = K \cup M$ be an admissible set in a complex manifold X (see Definition 5), with M a totally real submanifold (possibly with boundary) of class $\mathscr{C}^k$. Then for any $f \in \mathscr{C}^k(S) \cap \mathscr{O}(K)$ there exists a sequence $f_j \in \mathscr{O}(S)$ such that*

$$\lim_{j\to\infty} \|f_j - f\|_{\mathscr{C}^k(S)} = 0.$$

Proof We begin by considering the case when $\mathrm{supp}(f)$ is contained in the totally real manifold $M = S \setminus K$, that is, $\mathrm{supp}(f) \cap K = \varnothing$. We cover the compact set $\mathrm{supp}(f) \subset M \setminus K$ by finitely many open domains (coordinate balls) $M_1, \dots, M_m \subset M$ such that Proposition 2 holds for each M_j and $\bigcup_{j=1}^m \overline{M}_j \subset M \setminus K$. Let $\chi_j \in \mathscr{C}_0^k(M_j)$ be a partition of unity on a neighborhood of $\mathrm{supp}(f)$, so $f = \sum_j \chi_j f$. Clearly, it suffices to prove the theorem separately for each $\chi_j f$, so we assume without loss of generality that f is compactly supported in M_1.

Let $U \subset X$ be a neighborhood of bM_1 satisfying condition (b) in Proposition 2. Let $B \subset X$ be an open set with $M_1 \subset B$ and let $A \subset X$ be an open set containing $S \setminus M_1$, such that $A \cap B \subset U$. Let $\Omega \subset X$ be a Stein neighborhood of S with $\Omega \subset A \cup B$, and set $\Omega_A := \Omega \cap A$ and $\Omega_B = \Omega \cap B$. Consider the map $\Gamma : \mathscr{O}(\Omega_A) \oplus \mathscr{O}(\Omega_B) \to \mathscr{O}(\Omega_A \cap \Omega_B)$ defined by $(f_A, f_B) \mapsto f_A - f_B$. Then Γ is surjective by Cartan's theorem B, and so by the open mapping theorem, Γ is an open mapping with respect to the Fréchet topologies on the respective spaces. Let now f_ϵ be a family as in Proposition 2. Then $f_\epsilon \to 0$ on $\Omega_A \cap \Omega_B$, so there is a sequence $F_\epsilon = (f_{A,\epsilon}, f_{B,\epsilon}) \in \mathscr{O}(\Omega_A) \oplus \mathscr{O}(\Omega_B)$ converging to zero in the Fréchet topology, with $\Gamma(F_\epsilon) = f_\epsilon$. Pick a sequence $\epsilon_j \to 0$, and set $f_j := f_{\epsilon_j} + f_{B,\epsilon_j}$ on Ω_B and $f_j := f_{A,\epsilon_j}$ on Ω_A. The conclusion now follows by restricting f_j to any domain Ω' with $S \subset \Omega' \Subset \Omega$.

It remains to consider the general case when the support of f intersects K. To this end, we note that what we have proved so far gives the following useful lemma.

Lemma 2 *If $S = K \cup M$ is a Stein compact in a complex manifold X, if $S \setminus K$ is totally real, and $U \subset X$ is an open set containing K, then there exists a Stein neighborhood $\Omega \subset X$ of S such that $\widehat{K}_{\mathscr{O}(\Omega)} \subset U$. In particular, K is a Stein compact.*

Proof For each point $p \in M \setminus K$ there is a disc $M_p \subset M \setminus K$ around p on which Proposition 2 holds. As we have just shown, we may use Theorem 20 to approximate a continuous function which is zero near K and one at the point p, and so there exists a holomorphic function $f_p \in \mathscr{O}(S)$ such that $|f_p|$ is as small as desired on K and $|f_p| > 1/2$ on a neighborhood of p. By taking the sum $\rho = \sum_j |f_{p_j}|^2$ over finitely many such functions, we get a plurisubharmonic function $\rho \geq 0$ on a neighborhood

V of S which is > 1 on a neighborhood W of the compact set $\overline{M \setminus U}$ and is close to 0 on a neighborhood of K. Note that $S \subset U \cup W$. By choosing a Stein neighborhood Ω of S such that $\Omega \subset (U \cup W) \cap V$, it follows that $\widehat{K}_{\mathscr{O}(\Omega)} \subset U$. □

We now conclude the proof of Theorem 20. Assume that the function $f \in \mathscr{C}^k(S)$ to be approximated is holomorphic in an open set $U \supset K$. Choose a Stein neighborhood Ω of S as in Lemma 2 such that $K_0 := \widehat{K}_{\mathscr{O}(\Omega)} \subset U$. Pick an $\mathscr{O}(\Omega)$-convex compact set $K_1 \subset U$ containing K_0 in its interior. Choose a smooth cut-off function χ supported on K_1 such that $\chi = 1$ on a neighborhood K_0. By the Oka–Weil theorem (see Theorem 18) there is a sequence $g_j \in \mathscr{O}(\Omega)$ such that $g_j \to f$ uniformly on K_1 as $j \to \infty$. Then, we clearly have that

$$\tilde{f}_j := \chi g_j + (1-\chi) f = g_j + (1-\chi)(f - g_j) \to f \quad \text{as } j \to \infty$$

in $\mathscr{C}^k(S)$. As $g_j \in \mathscr{O}(\Omega)$, it remains to approximate the functions $(1-\chi)(f-g_j) \in \mathscr{C}^k(S)$ whose support does not intersect $K_0 \supset K$, so we have our reduction. □

For approximation on stratified admissible sets, we need the following.

Corollary 4 *Let $S = K \cup M$ be a stratified admissible set, with a totally real stratification $M = \bigcup_{i=1}^{l} M_i$. Then $S_i := K \cup M_i$ is a Stein compact (and hence a stratified admissible set) for each $i = 1, \dots, l-1$.*

Proof Note that the top stratum $\widetilde{M} := M \setminus M_{l-1}$ is a totally real submanifold and $S = S_{l-1} \cup \widetilde{M}$ is an admissible set. Lemma 2 implies that S_{l-1} is a Stein compact. The result now follows by a finite downward induction on l. □

Theorem 21 *If $S = K \cup M$ is a stratified admissible set in a complex manifold X, then for any $f \in \mathscr{C}(S) \cap \mathscr{O}(K)$ there exists a sequence $f_j \in \mathscr{O}(S)$ such that*

$$\lim_{j \to \infty} \|f_j - f\|_{\mathscr{C}(S)} = 0.$$

Proof By assumption there is a stratification $M = \bigcup_{i=1}^{l} M_i$ with M_1 and $M_{i+1} \setminus M_i$ totally real manifolds for $i = 1, \dots, l-1$. Let $M_0 = \varnothing$. It suffices to apply Theorem 20 successively with $K_i = K \cup M_i$ and $S_i = K_i \cup M_{i+1}$ $(i = 0, \dots, l-1)$. □

Remark 7 It is not possible in general to obtain $\mathscr{C}^k$-approximation on stratified totally real manifolds M, even if M is itself $\mathscr{C}^k$-smooth. Suppose, for instance, that $M \subset \mathbb{C}^n$ is a $\mathscr{C}^1$-smooth submanifold which is a Stein compact, and which is totally real except at a point $p \in M$. Then, M has an obvious stratification by totally real manifolds, but it is clearly impossible to achieve $\mathscr{C}^1$-approximation at the point p due to the Cauchy–Riemann equations. However, one sees immediately that one may achieve $\mathscr{C}^k$-approximation on compact subsets of each $M_{i+1} \setminus M_i$.

Theorem 21 holds in the more general case when $S = K \cup M$ is a Stein compact with $M = \bigcup_{i=1}^{l} M_i$ a *stratified totally real set*, meaning that M_1 and each difference

$M_i \setminus M_{i-1}$ $(i = 2, \dots, l)$ is a locally closed totally real set. We refer to P. Manne [117, 118] and to E. Løw and E. F. Wold [113] for these extensions. □

A further generalization of Theorem 20 is provided by Theorem 34 in Section 7; we state it there as it concerns manifold-valued maps.

Although holomorphically convex smooth submanifolds of $\mathbb{C}^n$ do not in general admit Mergelyan approximation, E. L. Stout [156] gave the following general result in the real analytic setting, also allowing for varieties.

Theorem 22 (E. L. Stout (2006), [156]) *Let X be a Stein space. If $M \subset X$ is a compact real analytic subvariety such that $M = \operatorname{spec} \mathscr{O}(M)$, then $\mathscr{C}(M) = \overline{\mathscr{O}}(M)$.*

Recall that $M = \operatorname{spec} \mathscr{O}(M)$ means that any continuous character on the algebra $\overline{\mathscr{O}}(M)$ may be represented by a unique point evaluation on M. An example is if M is a countable intersection of Stein domains. We will not give a proof of the full theorem here, but we will use Theorem 21, together with some fundamental results due to K. Diederich and J. E. Fornæss [43] and E. Bishop [21], to give a relatively short proof under the stronger assumption that M is a Stein compact.

Proof of Theorem 22 under the assumption that M is a Stein compact Without loss of generality we may assume that $M \subset \mathbb{C}^n$. It was proved by K. Diederich and J.-E. Fornæss [43] that M does not contain a nontrivial analytic disc. Now, M has a stratification $M = \bigcup_{i=1}^m M_i$ such that each difference $V_i = M_{i+1} \setminus M_i$ is a real analytic submanifold. We claim that every V_i is totally real outside a real analytic submanifold $\widetilde{V}_i$ of positive codimension. If not, there is an open subset $U \subset V_i$ such that U is a CR-manifold, and according to Bishop [21] one may attach families of holomorphic discs to U shrinking towards any given point $p \in U$. By the assumption about holomorphic convexity, the discs will eventually be contained in U, but this contradicts the result of Diederich and Fornæss [43]. This argument may be used repeatedly to refine the initial stratification of M into a stratification by totally real submanifolds, and hence the result follows from Theorem 21. □

6.2 Approximation on Strongly Pseudoconvex Domains and on Strongly Admissible Sets

As we have seen, proofs of the Mergelyan theorem in one complex variable depend heavily on integral representations of holomorphic or $\overline{\partial}$-flat functions. The single most important reason why the one-dimensional proofs work so well is that the Cauchy–Green kernel (4) provides a solution to the inhomogeneous $\overline{\partial}$-equation which is uniformly bounded on all of $\mathbb{C}$ in terms of sup-norm of the data and the area of its support (see (6)). This allows uniform approximation of functions in $\mathscr{A}(K)$ on any compact set $K \subset \mathbb{C}$ with not too rough boundary by functions in $\mathscr{O}(K)$ (see Vitushkin's Theorem 7). Nothing like that holds in several variables, and the question of uniform approximability is highly sensitive to the shape of the boundary even for smoothly bounded domains.

The idea of developing holomorphic integral kernels for domains in $\mathbb{C}^n$ with comparable properties to those of the Cauchy kernel in one variable was promoted by H. Grauert already around 1960; however, it took almost a decade to be realized. The first such constructions were given in 1969 by G. M. Henkin [92] and E. Ramírez de Arellano [138] for the class of strongly pseudoconvex domains. These kernels provide solution operators for the $\overline{\partial}$-equation which are bounded in the $\mathscr{C}^k$ norms and improve the regularity by $1/2$. We state here a special case of their results for (0, 1)-forms, but in a more precise form which can be found in the works by I. Lieb and M. Range [112, Theorem 1], I. Lieb and J. Michel [111], and [62, Theorem 2.7.3]. A brief historical review of the kernel method is given in [66, pp. 392–393].

Given a domain $\Omega \subset \mathbb{C}^n$, we denote by $\mathscr{C}^k_{(0,1)}(\overline{\Omega})$ the space of all differential (0, 1)-forms of class $\mathscr{C}^k$ on $\overline{\Omega}$.

Theorem 23 *If Ω is a bounded strongly pseudoconvex Stein domain with boundary of class $\mathscr{C}^k$ for some $k \in \{2, 3, \dots\}$ in a complex manifold X, there exists a bounded linear operator $T : \mathscr{C}^0_{(0,1)}(\overline{\Omega}) \to \mathscr{C}^0(\overline{\Omega})$ satisfying the following properties:*

(i) *If $f \in \mathscr{C}^0_{0,1}(\overline{\Omega}) \cap \mathscr{C}^1_{0,1}(\Omega)$ and $\overline{\partial} f = 0$, then $\overline{\partial}(Tf) = f$.*
(ii) *If $f \in \mathscr{C}^0_{0,1}(\overline{\Omega}) \cap \mathscr{C}^r_{0,1}(\Omega)$ for some $r \in \{1, \dots, k\}$ then*

$$\|Tf\|_{\mathscr{C}^{l,1/2}(\overline{\Omega})} \le C_{l,\Omega} \|f\|_{\mathscr{C}^l_{0,1}(\overline{\Omega})}, \quad l = 0, 1, \dots, r.$$

Moreover, the constants $C_{l,\Omega}$ may be chosen uniformly for all domains sufficiently $\mathscr{C}^k$ close to $\overline{\Omega}$.

The kernel method led to a variety of applications to function theory on strongly pseudoconvex domains. In particular, G. Henkin (1969) [92], N. Kerzman (1971) [100], and I. Lieb (1969) [110] proved the Mergelyan property for strongly pseudoconvex domains with sufficiently smooth boundary, and J. E. Fornæss (1976) [48] improved this to domains with $\mathscr{C}^2$ boundary. Subsequently, J. E. Fornæss and A. Nagel (1977) [49] showed that the Mergelyan property holds in the presence of transverse holomorphic vector fields near the set of weakly pseudoconvex boundary points (the so-called *degeneracy set*); this holds in particular for any bounded pseudoconvex domain with real analytic boundary in $\mathbb{C}^2$. F. Beatrous and M. Range (1980) [15] proved for holomorphically convex domains $\Omega \Subset \mathbb{C}^n$ with $\mathscr{C}^2$ boundaries that a function $f \in \mathscr{A}(\Omega)$ can be uniformly approximated by functions in $\mathscr{O}(\overline{\Omega})$ if it can be approximated on a neighborhood of the degeneracy set. This result appeared earlier in the thesis of F. Beatrous (1978).

On the other hand, K. Diederich and J. E. Fornæss (1976) [42] found an example of a $\mathscr{C}^\infty$ smooth pseudoconvex domain $\Omega \subset \mathbb{C}^2$ for which the Mergelyan property fails. Their example is based on the presence of a Levi-flat hypersurface in $b\Omega$ having an annular leaf with infinitesimally nontrivial holonomy. This phenomenon was further explored by D. Barrett [13] who showed in 1992 that the Bergman projection on certain Diederich–Fornæss worm domains does not preserve smoothness as measured by Sobolev norms. In 1996, M. Christ [37] obtained a substantially stronger result to the effect that the Bergman projection on such domains Ω does not even preserve $\mathscr{C}^\infty(\Omega)$; this provided the first example of smoothly bounded pseudoconvex domains in $\mathbb{C}^n$ on which the Bell–Ligocka Condition R fails.

In 2008, F. Forstnerič and C. Laurent-Thiebaut proved the Mergelyan property for smoothly bounded pseudoconvex domains $\Omega \Subset \mathbb{C}^n$ whose degeneracy set consisting of weakly pseudoconvex boundary points $A \subset b\Omega$ is of the form $A = \{z \in M : \rho(z) \le 0\}$, where ρ is a strongly plurisubharmonic function in a neighborhood of A, $M \subset \mathbb{C}^n$ is a Levi-flat hypersurface whose Levi foliation is defined by a closed 1-form, and A is the closure of its relative interior in M (see [65, Theorem 1.9]). The paper [65] provides several sufficient conditions for a foliation to be defined by a closed 1-form. This condition implies in particular that every leaf of M is topologically closed and has trivial holonomy map. On the other hand, in the worm hypersurface of Diederich and Fornæss [42] the foliation contains a leaf with nontrivial holonomy to which other leaves spirally approach. In [65] it was shown under the same hypotheses that the $\overline\partial$-Neumann operator on Ω is regular. See E. Straube and M. Sucheston [158, 159] for related results.

We begin with the proof of the Mergelyan property on strongly pseudoconvex domains, taking for granted the existence of bounded solution operators for the $\overline\partial$-equation in Theorem 23. The proof we present here is similar to Sakai's proof [146] discussed already in the proof of Theorem 6 (see Remark 2).

Theorem 24 *Let X be a Stein manifold, and let $\Omega \subset X$ be a relatively compact strongly pseudoconvex domain of class $\mathscr{C}^k$ for $k \in \{2, 3, \ldots\}$. Then for any $f \in \mathscr{C}^k(\overline\Omega) \cap \mathscr{O}(\Omega)$ ($k \in \mathbb{Z}_+$) there exists a sequence of functions $f_m \in \mathscr{O}(\overline\Omega)$ such that* $\lim_{m\to\infty} \|f_m - f\|_{\mathscr{C}^k(\overline\Omega)} = 0$.

Proof Let $\rho \in \mathscr{C}^2(U)$ be a defining function for Ω in an open set $U \supset \overline\Omega$, so $\Omega = \{\rho < 0\}$ and $d\rho \ne 0$ on $b\Omega = \{\rho = 0\}$. For small $t > 0$, set $\Omega_t = \{\rho < t\} \subset U$ and $\overline\Omega_t = \{\rho \le t\}$. We cover $b\Omega$ by finitely many pairs of open sets $W_j \subset V_j$, $j = 1, \ldots l$, with flows $\phi_{j,t}(z)$ of holomorphic vector fields defined on V_j, such that

$$\phi_{j,t}(W_j \cap \Omega_t) \subset \Omega \quad \text{for all small } t > 0. \tag{22}$$

Such vector fields are obtained easily in local coordinates, using constant vector fields pointing into Ω with a suitable scaling. Set $W_0 = \Omega$, and let $\{\chi_j\}_{j=0}^l$ be a smooth partition of unity with respect to the cover $\{W_j\}_{j=0}^l$. Choose $m_0 \in \mathbb{N}$ such that $\overline\Omega_{1/m_0} \subset \bigcup_{j=0}^l W_j$ and (22) holds for all $0 \le t \le 1/m_0$. Note that the functions χ_j have bounded $\mathscr{C}^{k+1}(\overline\Omega_{1/m_0})$ norms. By Whitney's theorem (see Theorem 46) we may assume that f is extended to a $\mathscr{C}^k$-smooth function on $\overline\Omega_{1/m_0}$. For any integer $m \ge m_0$ we set

$$U_{m,0} = \Omega, \quad U_{m,j} = \Omega_{1/m} \cap W_j \text{ for } j = 1, \ldots, l, \tag{23}$$

$$f_{m,0} = f \text{ on } U_{m,0} = \Omega, \quad f_{m,j}(z) = f(\phi_{j,1/m}(z)), \; z \in U_{m,j}, \; j = 1, \ldots, l. \tag{24}$$

Consider the function

$$g_m = \sum_{j=0}^{l} \chi_j \, f_{m,j} \in \mathscr{C}^k(\overline{\Omega}_{1/m}).$$

From the definition of the functions $f_{m,j}$ (24) it follows that

$$\| f_{m,j} - f \|_{\mathscr{C}^k(\overline{U}_{m,j})} = \omega(1/m), \quad j = 1, \ldots, l, \tag{25}$$

where $\omega(1/m) \to 0$ as $m \to \infty$ (here $\omega(1/m)$ is proportional to the modulus of continuity of the top derivative of f), and hence $\|g_m - f\|_{\mathscr{C}^k(\overline{\Omega}_{1/m})} = \omega(1/m)$.

We now estimate the $\mathscr{C}^k$-norm of $\overline{\partial} g_m$. Since $\sum_{j=0}^{l} \overline{\partial} \chi_j = 0$, we have that

$$\overline{\partial} g_m = \sum_{j=0}^{l} f_{m,j} \, \overline{\partial} \chi_j = \sum_{j=0}^{l} (f_{m,j} - f) \, \overline{\partial} \chi_j,$$

and it follows from (25) that $\|\overline{\partial} g_m\|_{\mathscr{C}^k(\overline{\Omega}_{1/m})} = \omega(1/m)$.

As explained above, there are bounded linear operators $T_m : \mathscr{C}^k_{(0,1)}(\overline{\Omega}_{1/m}) \to \mathscr{C}^k(\overline{\Omega}_{1/m})$, with bounds independent of $m \geq m_0$ and satisfying $\overline{\partial} T_m(\alpha) = \alpha$ for every $\overline{\partial}$-closed $(0,1)$-form α on $\overline{\Omega}_{1/m}$. Setting $f_m = g_m - T_m(\overline{\partial} g_m) \in \mathscr{O}(\Omega_{1/m})$ we get that $\|f_m - f\|_{\mathscr{C}^k(\overline{\Omega}_{1/m})} = \omega(1/m)$, and this completes the proof. □

The next result gives $\mathscr{C}^k$-approximation on strongly admissible sets.

Theorem 25 *Let X be a complex manifold. Assume that $\Omega \Subset X$ is a strongly pseudoconvex Stein domain of class $\mathscr{C}^k$ for $k \in \{2, 3, \ldots\}$, and that $S = \overline{\Omega} \cup M \subset X$ is a strongly admissible set. Given $f \in \mathscr{C}(S) \cap \mathscr{A}(\Omega)$ there is a sequence $f_j \in \mathscr{O}(S)$ such that $\lim_{j \to \infty} \|f_j - f\|_{\mathscr{C}(S)} = 0$. Furthermore, if M is a totally real manifold of class $\mathscr{C}^k$ and $f \in \mathscr{C}^k(S)$, we may choose $f_j \in \mathscr{O}(S)$ such that* $\lim_{j \to \infty} \|f_j - f\|_{\mathscr{C}^k(S)} = 0$.

Proof We follow the proof of Theorem 24, but cover also M with the W_j's. By the theorem of Whitney and Glaeser (see Theorem 47 in the Appendix and the remark following it), we may extend $T_m(\overline{\partial} g_m)$ to $\mathscr{C}^k$ functions h_m on some neighborhood of S, converging to 0 in the $\mathscr{C}^k$-norm. Hence, $\tilde{f}_m := g_m - h_m$ is holomorphic on $\Omega_{1/m}$ and $\tilde{f}_m \to f$ in $\mathscr{C}^k(\overline{\Omega})$ as $m \to \infty$, and the result follows from Theorem 20. □

6.3 Mergelyan Approximation in L^2-Spaces

In his thesis from 2015, S. Gubkin [88] investigated Mergelyan approximation in L^2 spaces of holomorphic functions on pseudoconvex domains in $\mathbb{C}^n$:

$$H^2(\Omega) = \mathscr{O}(\Omega) \cap L^2(\Omega).$$

The following theorem generalizes both his main results [88, Theorems 4.2.2 and 4.3.3]; in the first one the domain is assumed to have $\mathscr{C}^\infty$-smooth boundary, and in the second one it is assumed to admit a $\mathscr{C}^2$ plurisubharmonic defining function. We only assume that the closure of the domain is a Stein compact.

Theorem 26 *Assume that X is a Stein manifold and $\Omega \Subset X$ is a relatively compact pseudoconvex domain with $\mathscr{C}^1$ boundary whose closure $\overline{\Omega}$ is a Stein compact. Then for any $f \in H^2(\Omega)$ there exists a sequence $f_j \in \mathscr{O}(\overline{\Omega})$ such that* $\lim_{j\to\infty} \|f_j - f\|_{L^2(\Omega)} = 0$.

Proof As in the proof of Theorem 24, we find an open cover $\{W_j\}_{j=0}^l$ of $\overline{\Omega}_{1/m_0}$ for some $m_0 \in \mathbb{N}$ such that (22) holds. (This only requires that $b\Omega$ is of class $\mathscr{C}^1$.) Let $\{\chi_j\}_{j=0}^l$ be a smooth partition of unity subordinate to $\{W_j\}_{j=0}^l$. Given an integer $m \geq m_0$ we define the cover $\{U_{m,j}\}_{j=0}^l$ and the functions $(f_{m,j})_{j=0}^l$ by (23) and (24), respectively. Consider the function

$$g_m = \sum_{j=0}^{l} \chi_j \, f_{m,j} \in L^2(\Omega_{1/m}).$$

Fix $\delta > 0$. Since $\|f\|_{L^2(\Omega)} < \infty$, there exists a compact subset $K \subset \Omega$ such that

$$\|f\|_{L^2(\Omega\setminus K)} < \delta. \tag{26}$$

Choose a compact set $K' \subset \Omega$ such that

$$K \cup \operatorname{supp}(\chi_0) \subset \mathring{K}'. \tag{27}$$

Note that $g_m \to f$ in sup-norm on K' as $m \to \infty$. Furthermore, (22) and (23) imply $\phi_{j,1/m}(U_{m,j}\setminus K') \subset \Omega \setminus K$ for all big enough m, and hence (24) and (26) give

$$\|f_{m,j}\|_{L^2(U_{m,j}\setminus K')} < 2\delta \quad \text{for all } m \text{ big enough and all } j = 1, \ldots, l. \tag{28}$$

(The factor 2 comes from a change of variable; note that $\phi_{j,t} \to \mathrm{Id}$ as $t \to 0$.) Since this holds for every $\delta > 0$, we see that $\lim_{m\to\infty} \|g_m - f\|_{L^2(\Omega)} = 0$.

Next, we need to estimate $\overline{\partial} g_m$ on $\Omega_{1/m}$. We have that

$$\overline{\partial} g_m = \sum_{j=0}^{l} f_{m,j}\, \overline{\partial}\chi_j = \sum_{j=0}^{l} (f_{m,j} - f)\, \overline{\partial}\chi_j,$$

where the second expression holds on Ω. It follows that $\lim_{m\to\infty} \|\overline{\partial} g_m\|_{L^2(K')} = 0$. On $\Omega \setminus K'$ we have in view of (27) that $\overline{\partial} g_m = \sum_{j=1}^{l} f_{m,j}\, \overline{\partial}\chi_j$, and hence (28) gives

$$\|\overline{\partial} g_m\|_{L^2(\Omega_{1/m}\setminus K')} < C_0\delta$$

for some constant $C_0 > 0$ depending only on the partition of unity $\{\chi_j\}$. Since $\delta > 0$ was arbitrary, it follows that $\lim_{m\to\infty} \|\overline{\partial} g_m\|_{L^2(\Omega_{1/m})} = 0$.

Set $\alpha_m = \overline{\partial} g_m$, and let Ω' be a pseudoconvex domain with $\overline{\Omega} \subset \Omega' \subset \Omega_{1/m}$. By Hörmander, there exists a constant $C > 0$, independent of m and the choice of Ω', such that there exists a solution h_m to the equation $\overline{\partial} h_m = \alpha_m$ with $\|h_m\|_{L^2(\Omega')} \le C \cdot \|\alpha_m\|_{L^2(\Omega')}$. Setting $f_m = g_m - h_m$ we get that $\lim_{m\to\infty} \|f_m - f\|_{L^2(\Omega)} = 0$. □

Remark 8 A simple example of a pseudoconvex domain on which the L^2 Mergelyan property fails is the Hartogs triangle $H = \{(z,w) \in \mathbb{C}^2 : |w| < |z| < 1\}$. The holomorphic function $f(z,w) = w/z$ on H is bounded by one, and it cannot be approximated in any natural sense by holomorphic functions in neighborhoods of $\overline{H}$ since its restriction to horizontal slices $w = const$ has winding number -1. Note that $\overline{H}$ is not a Stein compact. One can also see that the L^2 Mergelyan property fails on the Diederich–Fornæss worm domain [42]. □

We shall consider further topics in L^2-approximation theory in Section 8.

6.4 Carleman Approximation in Several Variables

Carleman approximation on the totally real affine subspace $M = \mathbb{R}^n \subset \mathbb{C}^n$ was proved by S. Scheinberg [147] in 1976. Such spaces are obviously polynomially convex, and, although less obviously so, they satisfy the following condition (compare with Definition 2). For any compact set $C \subset \mathbb{C}^n$ we set

$$h(C) := \overline{\widehat{C} \setminus C}.$$

Definition 6 A closed set $M \subset \mathbb{C}^n$ has the *bounded exhaustion hulls property* if for any polynomially convex compact set $K \subset \mathbb{C}^n$ there exists $R > 0$ such that for any compact set $L \subset M$ we have that

$$h(K \cup L) \subset \mathbb{B}^n(0, R). \tag{29}$$

Clearly, it suffices to test this condition on any increasing sequence of compact sets K_j increasing to $\mathbb{C}^n$. This notion extends in an obvious way to closed sets in an arbitrary complex manifold X, replacing polynomial hulls by $\mathscr{O}(X)$-convex hulls. For closed sets M in $\mathbb{C}$, this notion is equivalent to the one in Definition 2, and to the condition that $\mathbb{CP}^1 \setminus M$ is locally connected at infinity. (This is precisely the condition under which Arakelian's Theorem 10 holds.)

To see that $M = \mathbb{R}^n$ has bounded exhaustion hulls in $\mathbb{C}^n$, we consider compact sets of the form

$$K_r = \big\{z \in \mathbb{C}^n : |x_j| \le r,\ |y_j| \le r,\ j = 1, \dots, n\big\}.$$

Let us first look at a point $\tilde{z} = \tilde{x} + \imath \tilde{y} \in \mathbb{C}^n \setminus \mathbb{R}^n$ with $|\tilde{x}_j| > (\sqrt{n} + 1)r$ for some j. Consider the pluriharmonic polynomial

$$f(z) = -\Re((z - \tilde{x})^2) = \sum_{i=1}^{n} \big(y_i^2 - (x_i - \tilde{x}_i)^2\big), \qquad z \in \mathbb{C}^n.$$

A simple calculation shows that $f(z) < 0$ holds for any point $z \in K_r$, and we clearly have $f \le 0$ on $\mathbb{R}^n$ and $f(\tilde{z}) = (\tilde{y})^2 > 0$. This shows that

$$h(K_r \cup \mathbb{R}^n) \subset \big\{z \in \mathbb{C}^n : |x_j| \le (\sqrt{n} + 1)r,\quad j = 1, \dots, n\big\}.$$

Clearly we also have $h(K_r \cup \mathbb{R}^n) \subset \{z \in \mathbb{C}^n : |y_j| \le r,\ j = 1, \dots, n\}$, and (29) follows.

By using Theorem 20 it is easy to prove the following result, which by the argument just given implies Scheinberg's result in [147]. Fix a norm on the jet-space $\mathscr{J}^k(X)$, and denote it by $|\cdot|_{\mathscr{C}^k(x)}$. Recall that an unbounded closed set M in a Stein manifold X is called $\mathscr{O}(X)$-convex if M is exhausted by an increasing sequence of compact $\mathscr{O}(X)$-convex sets.

Theorem 27 (P. E. Manne (1993), [117]) *Let X be a Stein manifold. If $M \subset X$ is a closed totally real submanifold of class $\mathscr{C}^k$ that is holomorphically convex and has bounded exhaustion hulls, then M admits $\mathscr{C}^k$-Carleman approximation by entire functions.*

Proof For simplicity of exposition we give the proof in the case $X = \mathbb{C}^n$. Since M has bounded exhaustion hulls, there exists a normal exhaustion $\{K_j\}_{j\in\mathbb{N}}$ of $\mathbb{C}^n$ by polynomially convex compact sets such that $K_j \cup M$ is polynomially convex for each $j \in \mathbb{N}$. Choose a sequence $m_j \in \mathbb{N}$ such that $m_j < m_{j+1}$ and $K_j \subset \mathbb{B}^n(0, m_j)$ for each j. Set $M_j = M \cap \overline{\mathbb{B}^n(0, m_j)}$, and choose a function $\chi_j \in \mathscr{C}_0^\infty(\mathbb{B}^n(0, m_{j+1}))$ with $\chi_j \equiv 1$ near $\overline{\mathbb{B}^n(0, m_j)}$. To prove the theorem we proceed by induction, making the induction hypothesis that we have found $f_j \in \mathscr{C}^k(M) \cap \mathscr{O}(K_j \cup M_j)$ such that

$$|f_j - f|_{\mathscr{C}^k(x)} < \epsilon(x)/2, \qquad x \in M.$$

It will be clear from the induction step how to achieve this for $j = 1$. Theorems 20 and 18 furnish a sequence $g_{j,m} \in \mathscr{O}(K_{j+1} \cup M_{j+2})$ such that

$$\|g_{j,m} - f_j\|_{\mathscr{C}^k(K_j \cup M_{j+2})} \to 0 \text{ as } m \to \infty.$$

It follows that $f_{j+1} = g_{j,m} + (1 - \chi_{j+1})(f_j - g_{j,m})$ will reproduce the induction hypothesis for sufficiently large m, and we may furthermore achieve $\|f_{j+1} - f_j\|_{K_j} < 2^{-j}$. It follows that f_j converges uniformly on compacts in X to an entire function approximating f to the desired precision. □

Prior to Manne's result, H. Alexander [5] generalized Carleman's theorem [31] to smooth unbounded curves in $\mathbb{C}^n$ in 1979. By a fundamental work of G. Stolzenberg [153], such a curve is always polynomially convex and has bounded exhaustion hulls. In 2002 P. M. Gauthier and E. Zeron [80] improved Alexander's result to include locally rectifiable curves with trivial topology.

The situation is rather different for higher dimensional totally real manifolds. In 2009, E. F. Wold [177] gave an example of a $\mathscr{C}^\infty$ smooth totally real manifold $M \subset \mathbb{C}^3$ which is polynomially convex, but fails to have bounded exhaustion hulls. In 2011, P. E. Manne, N. Øvrelid, and E. F. Wold [119] showed that a totally real submanifold $M \subset \mathbb{C}^n$ admits $\mathscr{C}^1$ Carleman approximation only if M has bounded exhaustion hulls. Motivated by the problem of proving that the product of two totally real Carleman continua is again a Carleman continuum, B. Magnusson and E. F. Wold [114] gave in 2016 a very simple proof that a closed set admits $\mathscr{C}^0$ Carleman approximation only if it has bounded exhaustion hulls. Hence, we have the following characterization of closed totally real submanifolds which admit Carleman approximation.

Theorem 28 *Let M be a closed totally real submanifold of class $\mathscr{C}^k$ in a Stein manifold X. Then, M admits $\mathscr{C}^k$-Carleman approximation by entire functions if and only if M is $\mathscr{O}(X)$-convex and has bounded exhaustion hulls.*

On the other hand, for any closed totally real submanifold M in a Stein manifold there always exists some Stein open neighborhood Ω of M with respect to which M admits Carleman approximation, see P. Manne [118].

Problem 1 Let $E \subset \mathbb{C}^n$ be a closed polynomially convex subset with the bounded exhaustion hulls property (see Definition 6).

(a) Suppose that $k \in \mathbb{Z}_+$ and $f \in \mathscr{C}^k(\mathbb{C}^n)$ is holomorphic in $\mathring{E}$ and $\overline{\partial}$-flat to order k along E. Is f uniformly approximable on E by entire functions? A positive answer in dimension $n = 1$ is given by Arakelian's Theorem 10.
(b) Suppose further that any f as in part (a) is approximable uniformly on every compact $K \subset E$ by entire functions. Does it follow that E is an Arakelian set?

7 Approximation of Manifold-Valued Maps

We now apply results of the previous sections to approximation problems of Runge, Mergelyan, and Carleman type for maps to complex manifolds more general than Euclidean spaces. Such problems arise naturally in applications of complex analysis to geometry, dynamics, and other fields. With the exception of Runge's theorem which leads to Oka theory and the concept of Oka manifold (see Section 7.1), this area is fairly unexplored and offers interesting problems.

The most natural generalization of Runge's theorem to manifold-valued maps pertains to maps from Stein manifolds (and Stein spaces) to *Oka manifolds*; see Theorem 29. This class of manifolds was introduced in 2009 F. Forstnerič [59] after having proved that all natural Oka properties that had been considered in the literature, which a given complex manifold Y might or might not have, are pairwise equivalent. (See also [106].) The simplest one, which is commonly used as the definition of the class of Oka manifolds, is given by Definition 7 below. Since a comprehensive account of this subject is available in the monograph [62] and the introductory surveys [61, 64], we only give a brief outline in Section 7.1, focusing on the approximation theorem in line with the topic of this survey.

In Section 7.2 we consider the Mergelyan approximation problem for maps $K \to Y$ from a Stein compact K in a complex manifold X to another manifold Y. Assuming that the map is of class $\mathscr{A}(K, Y)$, the main question is whether it is approximable uniformly on K by maps holomorphic in open neighborhoods of K. (The remaining question of approximability by entire maps $X \to Y$ is the subject of Oka theory discussed in Section 7.1.) If this holds for every $f \in \mathscr{A}(K, Y)$, we say that the space $\mathscr{A}(K, Y)$ enjoys the *Mergelyan property*. Thanks to a Stein neighborhood theorem due to E. Poletsky (see Theorem 32), it is possible to show for many classes of Stein compacts K that the Mergelyan property for functions on K implies the Mergelyan property for maps $K \to Y$ to an arbitrary complex manifold Y.

In Section 7.3 we present some recent results on Carleman and Arakelian type approximation of manifold-valued maps.

7.1 *Runge Theorem for Maps from Stein Spaces to Oka Manifolds*

Oka theory concerns the existence, approximation, and interpolation results for holomorphic maps from Stein manifolds and, more generally, Stein spaces, to complex manifolds. To avoid topological obstructions one considers globally defined continuous or smooth maps, and the main question is whether they can be deformed to holomorphic maps, often with additional approximation and interpolation conditions. Thus, Oka theory may be understood as the theory of homotopy principle in complex analysis, a point of view emphasized in the monographs [62, 86].

The classical aspect of Oka theory is known as the *Oka–Grauert theory*. It originates in K. Oka's paper [133] from 1939 where he proved that a holomorphic line bundle $E \to X$ over a Stein manifold X is holomorphically trivial if it is topologically trivial; the converse is obvious. This is equivalent to the problem of constructing a holomorphic section $X \to E$ without zeros, granted a continuous section without zeros. (Oka only considered the case when X is a domain of holomorphy in $\mathbb{C}^n$ since the notion of a Stein manifold was introduced only in 1951 [152]; however, the same proof applies to Stein manifolds and, more generally, to Stein spaces.) It follows that holomorphic line bundles $E_1 \to X$, $E_2 \to X$ over a Stein manifold are holomorphically equivalent if they are topologically equivalent; it suffices to apply Oka's theorem to the line bundle $E_1^{-1} \otimes E_2$. In particular, any holomorphic line bundle over an open Riemann surface X is holomorphically trivial. A cohomological proof of this result is obtained by applying the long exact sequence of cohomology groups to the exponential sheaf sequence $0 \to \mathbb{Z} \to \mathscr{O}_X \to \mathscr{O}_X^* \to 0$, where $\mathscr{O}_X^*$ is the sheaf of nonvanishing holomorphic functions and the map $\mathscr{O}_X \to \mathscr{O}_X^*$ is given by $f \mapsto e^{2\pi \mathrm{i} f}$ (see, e.g., [62, Sect. 5.2]).

In 1958, Oka's theorem was extended by H. Grauert [83] to much more general fiber bundles with complex Lie group fibers over Stein spaces; see also H. Cartan [33] for an exposition. Grauert's results apply in particular to holomorphic vector bundles of arbitrary rank over Stein spaces and show that their holomorphic classification agrees with the topological classification. The cohomological point of view is still possible by considering nonabelian cohomology groups with values in a Lie group. Surveys of Oka–Grauert theory can be found in the paper [108] by J. Leiterer and in the monograph [62] by F. Forstnerič.

The main ingredient in the proof of Grauert's theorem is a parametric version of the Oka–Weil approximation theorem for maps from Stein manifolds to complex homogeneous manifolds. More precisely, given a compact $\mathscr{O}(X)$-convex set K in a Stein space X and a continuous map $f \colon X \to Y$ to a complex homogeneous manifold Y such that f is holomorphic in an open neighborhood of K, it is possible to deform f by a homotopy $f_t \colon X \to Y$ $(t \in [0, 1])$ to a holomorphic map f_1 such that every map f_t in the homotopy is holomorphic in a neighborhood of K and close to the initial map $f = f_0$ on K. Analogous results hold for families of maps $f_p \colon X \to Y$ depending continuously on a parameter p in a compact Hausdorff space P, where the homotopy is fixed for values of $p \in P_0$ in a closed subset P_0 of P for which the map $f_p \colon X \to Y$ is holomorphic on all on X. In other words, Theorem 19 holds with the target $\mathbb{C}$ replaced by any complex homogeneous manifold Y, provided that all maps $f_p \colon X \to Y$ $(p \in P)$ in the family are defined and continuous on all of X. This point of view on Grauert's theorem is explained in [62, Sects. 5.3 and 8.2].

After some advances during 1960s, most notably those of O. Forster and K. J. Ramspott [52, 53], a major extension of the Oka–Grauert theory was made by M. Gromov [87] in 1989. He showed in particular that the existence of a dominating holomorphic spray on a complex manifold Y implies all forms of the h-principle, also called the *Oka principle* in this context, for holomorphic maps from any Stein manifold to Y. The subject was brought into an axiomatic form by F. Forstnerič who introduced the class of *Oka manifolds* (see [55, 57, 59, 60] and the monograph [62]).

Definition 7 A complex manifold Y is an *Oka manifold* if every holomorphic map $K \to Y$ from a neighborhood of any compact convex set $K \subset \mathbb{C}^n$ for any $n \in \mathbb{N}$ can be approximated uniformly on K by entire maps $\mathbb{C}^n \to Y$.

The following version of the Oka–Weil for maps from Stein spaces to Oka manifolds is a special case of [62, Theorem 5.4.4].

Theorem 29 (Runge Theorem for Maps to Oka Manifolds) *Assume that X is a Stein space and Y is an Oka manifold. Let* dist *denote a Riemannian distance function on Y. Given a compact $\mathscr{O}(X)$-convex subset K of X and a continuous map $f: X \to Y$ which is holomorphic in a neighborhood of K, there exists for every $\epsilon > 0$ a homotopy of continuous maps $f_t: X \to Y$ $(t \in [0, 1])$ such that $f_0 = f$, for every t the map f_t is holomorphic on a neighborhood of K and satisfies* $\sup_{x\in K} \operatorname{dist}(f_t(x), f(x)) < \epsilon$*, and the map f_1 is holomorphic on X.*

A complex manifold Y which satisfies the conclusion of Theorem 29 for every triple (X, K, f) is said to satisfy the *basic Oka property with approximation* (see [62, p. 258]). A more general version of this result (see [62, Theorem 5.4.4]) includes the parametric case, as well as interpolation (or jet interpolation) on a closed complex subvariety X_0 of X provided all maps $f_p: X \to Y$ $(p \in P)$ in a given continuous compact family are holomorphic on X_0, or in a neighborhood of X_0 when considering jet interpolation. Since a compact convex set in $\mathbb{C}^n$ is $\mathscr{O}(\mathbb{C}^n)$-convex, the condition that Y be an Oka manifold is clearly necessary in Theorem 29.

The class of Oka manifolds contains all complex homogeneous manifolds, but also many nonhomogeneous ones. For example, if the tangent bundle TY of a complex manifold Y is pointwise generated by $\mathbb{C}$-*complete* holomorphic vector fields on Y (such a manifold is called *flexible* [9]), then Y is an Oka manifold [62, Proposition 5.6.22]. For examples and properties of Oka manifolds, see [62, Chaps. 5–7].

Recently, two new characterizations of the class of Oka manifolds have been found by Y. Kusakabe. In his first paper [104], Kusakabe showed that a complex manifold Y is Oka if (and only if) for any Stein manifold X, the mapping space $\mathscr{O}(X, Y)$ is $\mathbb{C}$-connected. In his second paper [105], he showed that Y is Oka if (and only if) Y satisfies Gromov's Condition Ell_1 [87]. This condition means that for every holomorphic map $f : X \to Y$ there exists a dominating holomorphic spray $F : X \times \mathbb{C}^N \to Y$ with $F(\cdot, 0) = f$, where the domination property means that for any fixed $x \in X$ the differential of the map $F(x, \cdot) : \mathbb{C}^N \to Y$ is surjective at $0 \in \mathbb{C}^N$. Kusakabe's second result implies that a complex manifold Y is Oka if and only if every point $y_0 \in Y$ has a Zariski open Oka neighborhood [105, Theorem 1.4].

In another recent direction, L. Studer proved a homotopy theorem for Oka property and extended its validity to Oka pairs of sheaves [160], generalizing the work of Forster and Ramspott [53].

Theorem 29 has a partial analogue in the algebraic category, concerning maps from affine algebraic varieties to algebraically subelliptic manifolds. For the definition of the latter class, see [62, Definition 5.6.13(e)]. The following result is [62, Theorem 6.15.1]; the original reference is [56, Theorem 3.1].

Theorem 30 *Assume that X is an affine algebraic variety, Y is an algebraically subelliptic manifold, and $f_0\colon X \to Y$ is a (regular) algebraic map. Given a compact $\mathscr{O}(X)$-convex subset K of X, an open set $U \subset X$ containing K, and a homotopy $f_t\colon U \to Y$ of holomorphic maps $(t \in [0,1])$, there exists for every $\epsilon > 0$ an algebraic map $F\colon X \times \mathbb{C} \to Y$ such that $F(\cdot\,,0) = f_0$ and*

$$\sup_{x\in K,\ t\in[0,1]} \operatorname{dist}(F(x,t),\ f_t(x)) < \epsilon.$$

In particular, a holomorphic map $X \to Y$ which is homotopic to an algebraic map can be approximated uniformly on compacts in X by algebraic maps $X \to Y$.

Simple examples show that Theorem 30 does not hold in the absolute form, i.e., there are examples of holomorphic maps which are not homotopic to algebraic maps (see [62, Examples 6.15.7 and 6.15.8]).

By [62, Proposition 6.4.5], the class of algebraically subelliptic manifolds contains all algebraic manifolds which are Zariski locally affine (such manifolds are said to be of *Class* A_0, see [62, Definition 6.4.4]), and all complements of closed algebraic subvarieties of codimension at least two in such manifolds. In particular, every complex Grassmannian is algebraically subelliptic, so Theorem 30 includes as a special case the result of W. Kucharz [103, Theorem 1] from 1995. Another paper on this topic is due to J. Bochnak and W. Kucharz [22].

In conclusion, we mention another interesting Runge type approximation theorem of a rather different type, due to A. Gournay [82]. A smooth almost complex manifold (M, J) is said to satisfy the *double tangent property* if through almost every point $p \in M$ and almost every 2-jet of J-holomorphic discs at p, there exists a J-holomorphic map $u\colon \mathbb{CP}^1 \to M$ having this jet as its second jet at $0 \in \mathbb{CP}^1$.

Theorem 31 (A. Gournay (2012), [82]) *Let (M, J) be a compact almost complex manifold satisfying the double tangent property and let R be a compact Riemann surface. Then, for every open set $U \subset R$ and every compact $K \subset U$, every J-holomorphic map $u\colon U \to M$ which continuously extends to R can be approximated uniformly on K by J-holomorphic maps from R to M.*

7.2 Mergelyan Theorem for Manifold-Valued Maps

In this section, we consider the question for which compact sets K in a complex manifold X does the approximability of functions in $\mathscr{A}^r(K)$ $(r \in \mathbb{Z}_+)$ by functions in $\mathscr{O}(K)$ imply the analogous result for maps to an arbitrary complex manifold Y. Such approximation problems arise naturally in many applications.

Recall that $\mathscr{A}(K,Y)$ denotes the set of all continuous maps $K \to Y$ which are holomorphic in $\mathring{K}$, and that if $r \in \mathbb{N}$, then $\mathscr{A}^r(K,Y)$ is the set of all maps $f \in \mathscr{A}(K,Y)$ which admit a $\mathscr{C}^r$ extension to an open neighborhood of K in X. We say that the mapping space $\mathscr{A}(K,Y)$ has the *Mergelyan property* if

$$\overline{\mathscr{O}}(K,Y) = \mathscr{A}(K,Y),$$

that is, every continuous map $K \to Y$ that is holomorphic in the interior $\mathring{K}$ is a uniform limit of maps that are holomorphic in open neighborhoods of K in X.

Lemma 3 *Assume that X is a complex manifold and $K \subset X$ is a compact set satisfying $\overline{\mathscr{O}}(K) = \mathscr{A}(K)$. Let Y be a complex manifold, and let $f \in \mathscr{A}(K,Y)$. Then $f \in \overline{\mathscr{O}}(K,Y)$ if one of the following conditions hold:*

(a) *The image $f(K) \subset Y$ has a Stein neighborhood in Y.*
(b) *The graph $G_f = \big\{(x, f(x)) : x \in K\big\}$ has a Stein neighborhood in $X \times Y$.*

Proof We will give a proof of (b); the proof of (a) is essentially the same. Assume that $V \subset X \times Y$ is a Stein neighborhood of G_f. By the Remmert–Bishop–Narasimhan theorem (see [62, Theorem 2.4.1]) there is a biholomorphic map $\phi\colon V \to \Sigma \subset \mathbb{C}^N$ onto a closed complex submanifold of a Euclidean space. By the Docquier–Grauert theorem (see [62, Theorem 3.3.3]) there is a neighborhood $\Omega \subset \mathbb{C}^N$ of Σ and a holomorphic retraction $\rho\colon \Omega \to \Sigma$. Assuming that $\overline{\mathscr{O}}(K) = \mathscr{A}(K)$, we can approximate the map $\phi \circ f\colon K \to \Sigma \subset \mathbb{C}^N$ as closely as desired uniformly on K by a holomorphic map $G\colon U \to \Omega \subset \mathbb{C}^N$ from an open neighborhood $U \subset X$ of K. The map $g = pr_Y \circ \phi^{-1} \circ \rho \circ G : U \to Y$ then approximates f on K. □

Given a compact set K in a complex manifold X and a complex manifold Y, let

$$\overline{\mathscr{O}}_{\mathrm{loc}}(K,Y)$$

denote the set of all continuous maps $f\colon K \to Y$ which are locally approximable by holomorphic maps, in the sense that every point $x \in K$ has an open neighborhood $U \subset X$ such that $f|_{K\cap\overline{U}} \in \overline{\mathscr{O}}(K \cap \overline{U})$. Clearly,

$$\mathscr{O}(K,Y) \subset \overline{\mathscr{O}}(K,Y) \subset \overline{\mathscr{O}}_{\mathrm{loc}}(K,Y) \subset \mathscr{A}(K,Y).$$

When $Y = \mathbb{C}$, we simply write $\mathscr{O}(K) \subset \overline{\mathscr{O}}(K) \subset \overline{\mathscr{O}}_{\mathrm{loc}}(K) \subset \mathscr{A}(K)$. We say that the space $\mathscr{A}(K,Y)$ has the *local Mergelyan property* if

$$\overline{\mathscr{O}}_{\mathrm{loc}}(K,Y) = \mathscr{A}(K,Y). \tag{30}$$

The following theorem was proved by E. Poletsky [137, Theorem 3.1].

Theorem 32 (Poletsky (2013), [137]) *Let K be a Stein compact in a complex manifold X, and let Y be a complex manifold. For every $f \in \overline{\mathscr{O}}_{\mathrm{loc}}(K,Y)$, the graph* *of f on K is a Stein compact in $X \times Y$. In particular, if $\mathscr{A}(K, Y)$ has the local Mergelyan property* (30)*, then the graph of every map $f \in \mathscr{A}(K, Y)$ is a Stein compact in $X \times Y$.*

Poletsky's proof uses the technique of *fusing plurisubharmonic functions*. Roughly speaking, we approximate a collection of plurisubharmonic functions $\rho_j : U_j \to \mathbb{R}$ on open sets $U_j \subset X \times Y$ covering the graph of f by a plurisubharmonic function ρ on $U = \bigcup_j U_j$, in the sense that the sup-norm $\|\rho - \rho_j\|_{U_j}$ for each j is estimated in terms of $\max_{i,j} \|\rho_i - \rho_j\|_{U_i \cap U_j}$ and a certain positive constant which depends on a strongly plurisubharmonic function τ in a Stein open neighborhood of K in X. This fusing procedure is rather similar to the proof of Y.-T. Siu's theorem [151] given by J.-P. Demailly [41] and Colţoiu [40]. (Demailly's proof can also be found in [62, Sect. 3.2].) The functions ρ_j alluded to above are of the form $|f_j(x) - y|^2$, where (x, y) is a local holomorphic coordinate on $U_j = V_j \times W_j$ with $V_j \subset X$ and $W_j \subset Y$, and $f_j \in \mathscr{O}(U_j, Y)$ is a holomorphic map which approximates f on $U_j \cap K$. (Such local approximations exist by the hypothesis of the theorem.) By this technique, one finds strongly plurisubharmonic exhaustion functions on arbitrarily small open neighborhoods of the graph of f in $X \times Y$; by Grauert's theorem [84] such neighborhoods are Stein.

In the special case when the set K in Theorem 32 is the closure of a relatively compact strongly pseudoconvex Stein domain, the existence of a Stein neighborhood basis of the graph of any map $f \in \mathscr{A}(K, Y)$ was first proved by F. Forstnerič [58] in 2007. His proof uses the method of gluing holomorphic sprays.

Theorem 32 and Lemma 3 give the following corollary.

Corollary 5 *Let K be a Stein compact in a complex manifold X. If $\mathscr{A}(K) = \overline{\mathscr{O}}(K)$, then $\overline{\mathscr{O}}_{\rm loc}(K, Y) = \overline{\mathscr{O}}(K, Y)$ holds for any complex manifold Y.*

Proof Note that $\overline{\mathscr{O}}(K, Y) \subset \overline{\mathscr{O}}_{\rm loc}(K, Y)$. Assume now that $f \in \overline{\mathscr{O}}_{\rm loc}(K, Y)$. By Theorem 32, the graph of f on K admits an open Stein neighborhood in $X \times Y$. Assuming that $\mathscr{A}(K) = \overline{\mathscr{O}}(K)$, Lemma 3(b) shows that $f \in \overline{\mathscr{O}}(K, Y)$. □

In light of Theorem 32 and Corollary 5, it is natural to ask when does the space $\mathscr{A}(K, Y)$ enjoy the local Mergelyan property (30). To this end, we introduce the following property of a compact set in a complex manifold.

Definition 8 A compact set K in a complex manifold X enjoys the *strong local Mergelyan property* if for every point $x \in K$ and neighborhood $x \in U \subset X$ there is a neighborhood $x \in V \subset U$ such that $\mathscr{A}(K \cap \overline{V}) = \overline{\mathscr{O}}(K \cap \overline{V})$.

Remark 9 Clearly, the strong local Mergelyan property of K implies the local Mergelyan property $\mathscr{A}(K) = \overline{\mathscr{O}}_{\rm loc}(K)$ of the algebra $\mathscr{A}(K)$. However, the former property is ostensibly stronger since it asks for approximability of functions defined on small neighborhoods of points in K, and not only of functions in $\mathscr{A}(K)$. If K has empty interior, we have $\mathscr{A}(K) = \mathscr{C}(K)$ and the two properties are equivalent by the Tietze extension theorem for continuous functions. Theorem 14 due to A. Boivin and B. Jiang [26, Theorem 1] shows that, for a compact set K in a Riemann surface, the Mergelyan property $\mathscr{A}(K) = \overline{\mathscr{O}}(K)$ implies the strong local Mergelyan property of K. We do not know whether the same holds for compact sets in higher dimensional manifolds. It is obvious that every compact set with boundary of class $\mathscr{C}^1$ in any complex manifold has the strong local Mergelyan property. Note also that the strong local Mergelyan property for functions implies the same property for maps to an arbitrary complex manifold Y, for the simple reason that locally any map has range in a local chart of Y which is biholomorphic to an open subset of a Euclidean space. This is the main use of this property in the present paper. □

Problem 2 Let K be a compact set in a complex manifold X.

1. Does $\mathscr{A}(K) = \overline{\mathscr{O}}_{\rm loc}(K)$ imply the strong local Mergelyan property of K?
2. Does $\mathscr{A}(K) = \overline{\mathscr{O}}(K)$ imply the strong local Mergelyan property of K?

We have the following corollary to Theorem 32.

Corollary 6 *Let K be a compact set in a complex manifold X.*

(a) *If K has the strong local Mergelyan property (see Definition 8), then $\mathscr{A}(K, Y) = \overline{\mathscr{O}}_{\rm loc}(K, Y)$ holds for every complex manifold Y.*
(b) *If K is a Stein compact with the strong local Mergelyan property and $\mathscr{A}(K) = \overline{\mathscr{O}}(K)$, then $\mathscr{A}(K, Y) = \overline{\mathscr{O}}(K, Y)$ holds for every complex manifold Y.*
(c) *If K is a Stein compact with $\mathscr{C}^1$ boundary such that $\mathscr{A}(K) = \overline{\mathscr{O}}(K)$, then $\mathscr{A}(K, Y) = \overline{\mathscr{O}}(K, Y)$ holds for every complex manifold Y.*

Proof (a) Let $f \in \mathscr{A}(K, Y)$. Every point $x \in K$ has an open neighborhood $U_x \subset X$ such that $f(K \cap \overline{U}_x)$ is contained in a coordinate chart $W \subset Y$ biholomorphic to an open subset of $\mathbb{C}^n$, $n = \dim Y$. Since K is assumed to have the strong local Mergelyan property, there exists a compact relative neighborhood $K_x \subset K \cap U$ of the point x in K such that $f|_{K_x} \in \overline{\mathscr{O}}(K_x, W)$. (See Remark 9.) This means that $f \in \overline{\mathscr{O}}_{\rm loc}(K, Y)$, thereby proving (a). In case (b), Corollary 5 implies $\overline{\mathscr{O}}_{\rm loc}(K, Y) = \overline{\mathscr{O}}(K, Y)$, and together with part (a) we get $\mathscr{A}(K, Y) = \overline{\mathscr{O}}(K, Y)$. In case (c), the set K clearly has the strong local Mergelyan property, so the conclusion follows from (b). □

The following case concerning compact sets in Riemann surfaces may be of particular interest (see [63, Theorem 1.4]).

Corollary 7 *If K is a compact set in a Riemann surface X such that $\mathscr{A}(K) = \overline{\mathscr{O}}(K)$, then $\mathscr{A}(K, Y) = \overline{\mathscr{O}}(K, Y)$ holds for any complex manifold Y. This holds in particular if $X \setminus K$ has no relatively compact connected components.*

Proof Note that any compact set in a Riemann surface is a Stein compact (since every open Riemann surface is Stein according to H. Behnke and K. Stein [17]). According to Theorem 14, the hypothesis $\mathscr{A}(K) = \overline{\mathscr{O}}(K)$ implies that K has the strong local Mergelyan property, so the result follows from Corollary 6.

In the special case when $X \setminus K$ has no relatively compact connected components, we can give a simple proof as follows. By Theorem 5, every function $f \in \mathscr{A}(K)$ is a uniform limit on K of functions in $\mathscr{O}(X)$, hence $\mathscr{A}(K) = \overline{\mathscr{O}}(K)$. Fix a point

$x \in K$ and let $U \subset X$ be a coordinate neighborhood of x with a biholomorphic map $\phi \colon U \to \mathbb{D} \subset \mathbb{C}$. Pick a number $0 < r < 1$. The compact set $K' = K \cap \phi^{-1}(r\overline{\mathbb{D}})$ does not have any holes in U. (Indeed, any such would also be a hole of K in X, contradicting the hypothesis.) By Theorem 5 it follows that $\mathscr{A}(K') = \overline{\mathscr{O}}(K')$. This shows that K enjoys the strong local Mergelyan property, and hence the conclusion follows from Corollary 6. □

The following consequence of Corollary 7 and of the Oka principle (see Theorem 29) has been observed recently in [63, Theorem 1.2].

Corollary 8 (Mergelyan Theorem for Maps from Riemann Surfaces to Oka Manifolds) *If K is a compact set without holes in an open Riemann surface X and Y is an Oka manifold, then every continuous map $f \colon X \to Y$ which is holomorphic in $\mathring{K}$ can be approximated uniformly on K by holomorphic maps $X \to Y$ homotopic to f.*

It was shown by J. Winkelmann [175] in 1998 that Mergelyan's theorem also holds for maps from compact sets in $\mathbb{C}$ to the domain $\mathbb{C}^2 \setminus \mathbb{R}^2$; this result is not covered by Corollary 8. His proof can be adapted to give the analogous result for maps from any open Riemann surface to $\mathbb{C}^2 \setminus \mathbb{R}^2$.

Remark 10 The following claim was stated by E. Poletsky [137, Corollary 4.4]:
(*) *If K is a Stein compact in a complex manifold X and $\mathscr{A}(K)$ has the Mergelyan property, then $\mathscr{A}(K, Y)$ has the Mergelyan property for any complex manifold Y.*

The proof in [137] tacitly assumes that under the assumptions of the corollary the space $\mathscr{A}(K, Y)$ has the local Mergelyan property, but no explanation for this is given. Corollaries 6 and 7 above provide several sufficient conditions for this to hold. We do not know whether (*) is true for every Stein compact in a complex manifold of dimension > 1; compare with Remark 9 on p. 176. □

Corollary 9 *If $S = K \cup M$ is a strongly admissible set in a complex manifold X (see Definition 5), then $\mathscr{A}(S, Y) = \overline{\mathscr{O}}(S, Y)$ holds for any complex manifold Y. Furthermore, for each $r \in \mathbb{N}$, every map $f \in \mathscr{A}^r(S, Y)$ is a $\mathscr{C}^r(S, Y)$ limit of maps $U \to Y$ holomorphic in open neighborhoods $U \subset X$ of S.*

Proof It is clear from the definition of a strongly admissible set that for every point $x \in S$ and neighborhood $x \in U \subset X$ there is a smaller neighborhood $U_0 \Subset U$ of x such that the set $S_0 = \overline{U}_0 \cap S$ is also strongly admissible. By Theorem 25 we have that $\mathscr{A}(S) = \overline{\mathscr{O}}(S)$, and also $\mathscr{A}(S_0) = \overline{\mathscr{O}}(S_0)$ for any S_0 as above. This means that S has the strong local Mergelyan property. The conclusion now follows from Corollary 6. A similar argument applies to maps of class $\mathscr{A}^r(S, Y)$ for any $r \in \mathbb{N}$.
□

In the special case when the strongly admissible set $K = S$ is the closure of a relatively compact strongly pseudoconvex domain, Corollary 9 was proved by F. Forstnerič [58] in 2007. His proof is different from those above which rely on Poletsky's Theorem 32. Instead it uses the method of gluing sprays, which is essentially a nonlinear version of the $\overline{\partial}$-problem. In the same paper, Forstnerič showed that many natural mapping spaces $K \to Y$ carry the structure of a Banach, Hilbert of Fréchet manifold (see [58, Theorem 1.1] and also [62, Theorem 8.13.1]). The following special case of the cited result is relevant to the present discussion.

Theorem 33 *Let K be a compact strongly pseudoconvex domain with $\mathscr{C}^2$ boundary in a Stein manifold X. Then, for every $r \in \mathbb{Z}_+$ and any complex manifold Y the space $\mathscr{A}^r(K, Y)$ carries the structure of an infinite dimensional Banach manifold.*

Further and more precise approximation results for maps from compact strongly pseudoconvex domains to Oka manifolds were obtained by B. Drinovec Drnovšek and F. Forstnerič in [45].

The proof of Theorem 20 in Section 6.1 is easily generalized to give the following approximation result for sections of holomorphic submersions over admissible sets in complex spaces. This plays a major role in the constructions in Oka theory (in particular, in the proof of [62, Theorem 5.4.4]).

Theorem 34 *Assume that X and Z are complex spaces, $\pi\colon Z \to X$ is a holomorphic submersion, and X' is a closed complex subvariety of X containing its singular locus X_{sing}. Let $S = K \cup M$ be an admissible set in X (see Definition 5), where $M \subset X \setminus X'$ is a compact totally real submanifold of class $\mathscr{C}^k$ for some $k \in \mathbb{N}$. Given an open set $U \subset X$ containing K and a section $f\colon U \cup M \to Z|_{U\cup M}$ such that $f|_U$ is holomorphic and $f|_M \in \mathscr{C}^k(M)$, there exist for every $s \in \mathbb{N}$ a sequence of open sets $V_j \supset S$ in X and holomorphic sections $f_j\colon V_j \to Z|_{V_j}$ ($j \in \mathbb{N}$) such that f_j agrees with f to order s along $X' \cap V_j$ for each $j \in \mathbb{N}$, and $\lim_{j\to\infty} f_j|_S = f|_S$ in the $\mathscr{C}^k(S)$-topology.*

A version of this result, with some loss of derivatives on the totally real submanifold M (due to the use of Hörmander's L^2 method) and without the interpolation condition, is [55, Theorem 3.1]. (A proof also appears in [62, Theorem 3.8.1].) The case when $Z = X \times \mathbb{C}$ (i.e., for functions) and without loss of derivatives was proved earlier by P. Manne [117] by using the convolution method (see Proposition 2 in Section 6.1). The general case is obtained from the special case for functions by following [62, proof of Theorem 3.8.1], noting also that the interpolation condition on the subvariety X' is easily achieved by a standard application of the Oka–Cartan theory. As always in results of this type, one begins by showing that the graph of the section admits a Stein neighborhood in Z; see [62, Lemma 3.8.3].

Another case of interest is when K is a compact set with empty interior, so $\mathscr{A}(K) = \mathscr{C}(K)$. The following result is due to E. L. Stout [157].

Theorem 35 *If K is a compact set in a complex space X such that $\mathscr{C}(K) = \overline{\mathscr{O}}(K)$ (hence $\mathring{K} = \varnothing$), then $\mathscr{C}(K, Y) = \overline{\mathscr{O}}(K, Y)$ holds for any complex manifold Y.*

Unlike in the previous results, the set K in Theorem 35 need not be a Stein compact. Special cases of Stout's theorem were obtained earlier by D. Chakrabarti (2007, 2008) [34, 35] who also obtained uniform approximation of continuous maps on arcs by pseudoholomorphic curves in almost complex manifolds.

Proof Choose a smooth embedding $\phi\colon Y \hookrightarrow \mathbb{R}^m$ for some $m \in \mathbb{N}$. Considering $\mathbb{R}^m$ as the real subspace of $\mathbb{C}^m$, the graph $Z = \{(y, \phi(y)) : y \in Y\} \subset Y \times \mathbb{R}^m \subset Y \times \mathbb{C}^m$ is a totally real submanifold of $Y \times \mathbb{C}^m$, so it has an open Stein neighborhood Ω in $Y \times \mathbb{C}^m$. Let $\pi : Y \times \mathbb{C}^m \to Y$ denote the projection onto the first factor. Given a continuous map $f\colon K \to Y$, the hypothesis of the theorem together with Lemma 3 implies that the continuous map $K \ni x \mapsto F(x) = (f(x), \phi(f(x))) \in \Omega$ can be approximated by holomorphic maps $G\colon U \to \Omega$ in open neighborhoods $U \subset X$ of K. The map $g = \pi \circ G\colon U \to Y$ then approximates f on K. □

7.3 Carleman and Arakelian Theorems for Manifold-Valued Maps

In Sections 3 and 6.4 we have considered Carleman and Arakelian type approximation in one and several variables, respectively. In this section, we present some applications and extensions of these results to manifold-valued maps.

The following result has been proved recently by B. Chenoweth.

Theorem 36 (Chenoweth (2019), [36]) *Let X be a Stein manifold and Y be an Oka manifold. If $K \subset X$ is a compact $\mathscr{O}(X)$-convex subset and $M \subset X$ is a closed totally real submanifold of class $\mathscr{C}^r$ $(r \in \mathbb{N})$ with bounded exhaustion hulls (see Definition 6) such that $K \cup M$ is $\mathscr{O}(X)$-convex, then for any $k \in \{0, 1, \dots, r\}$ the set $K \cup M$ admits $\mathscr{C}^k$-Carleman approximation of maps $f \in \mathscr{C}^k(X, Y)$ which are holomorphic on a neighborhood of K.*

This is proved by inductively applying Mergelyan's theorem for admissible sets (see Theorem 34), together with the Oka principle for maps from Stein manifolds to Oka manifolds (see [62, Theorem 5.4.4] which is a more precise version of Theorem 29 above). These two methods are intertwined at every step of the induction procedure. In view of Theorem 28 characterizing totally real submanifolds admitting Carleman approximation, the conditions in the theorem are optimal.

Carleman type approximation theorems have also been proved for some special classes of maps such as embeddings and automorphisms. Typically, proofs of such results combine methods of approximation theory with those from the Andersén–Lempert theory concerning holomorphic automorphisms of complex Euclidean spaces and, more generally, of Stein manifolds with the density property. Space limitation do not allow us to present this theory here; instead, we refer the reader to the recent survey in [62, Chapter 4].

We have already seen that Arakelian type approximation on closed sets with unbounded interior is considerably more difficult than Carleman approximation. In fact, we are not aware of a single result of this type on subsets of $\mathbb{C}^n$ for $n > 1$. However, the following extension of the classical one variable Arakelian's theorem (see Theorem 10) was proved by F. Forstnerič [63] in 2019.

Theorem 37 *If E is an Arakelian set in a domain $X \subset \mathbb{C}$ and Y is a compact complex homogeneous manifold, then every continuous map $X \to Y$ which is holomorphic in $\mathring{E}$ can be approximated uniformly on E by holomorphic maps $X \to Y$.*

The scheme of proof in [63] follows the proof of Theorem 10, but with improvements from Oka theory which are needed in the nonlinear setting. The proof does not apply to general Oka manifolds, not even to noncompact homogeneous manifolds. Note that the approximation problems of Arakelian type for maps to noncompact manifolds may crucially depend on the choice of the metrics on both spaces.

8 Weighted Approximation in L^2 Spaces

All approximation results considered so far were in one of the $\mathscr{C}^k$ topologies on the respective sets. We now present some results of a rather different kind, concerning approximation and density in weighted L^2 spaces of holomorphic functions.

Let Ω be a domain in $\mathbb{C}^n$, and let ϕ be a plurisubharmonic function on Ω. We denote by $L^2(\Omega, e^{-\phi})$ the space of measurable functions which are square integrable with respect to the measure $e^{-\phi} d\lambda$, where $d\lambda$ is the Lebesgue measure:

$$\|f\|_\phi^2 := \int_\Omega |f|^2 e^{-\phi} d\lambda < \infty.$$

By $H^2(\Omega, e^{-\phi})$ we denote the space of holomorphic functions on Ω with finite ϕ-norm:

$$H^2(\Omega, e^{-\phi}) = \big\{ f \in \mathscr{O}(\Omega) : \|f\|_\phi < \infty \big\}.$$

Note that if $\phi_1 \le \phi_2$, then $H^2(\Omega, e^{-\phi_1}) \subset H^2(\Omega, e^{-\phi_2})$ and the inclusion map is continuous, in fact, norm decreasing.

Let $z = (z_1, \ldots, z_n)$ be coordinates on $\mathbb{C}^n$ and $|z|^2 = \sum_{i=1}^n |z_i|^2$. Let $\phi_1 \le \phi_2 \le \cdots$ and ϕ be plurisubharmonic functions on $\mathbb{C}^n$ with $\phi_j \to \phi$ pointwise as $j \to \infty$. Set

$$\psi_j = \phi_j + \log(1 + |z|^2), \qquad \psi = \phi + \log(1 + |z|^2).$$

Assume in addition that $\int_K e^{-\phi_1} d\lambda < \infty$ for every compact set $K \subset \mathbb{C}^n$. The following theorem was proved by B. A. Taylor in 1971; see [161, Theorem 1.1].

Theorem 38 (Assumptions as Above.) *For every $f \in H^2(\mathbb{C}^n, e^{-\phi})$ there is a sequence $f_j \in H^2(\mathbb{C}^n, e^{-\psi_j})$ such that $\|f_j - f\|_\psi \to 0$ as $j \to \infty$.*

This result was improved in a recent paper by J. E. Fornæss and J. Wu [50].

Theorem 39 *Let $\phi_1 \le \phi_2 \le \cdots$ and ϕ be plurisubharmonic functions on $\mathbb{C}^n$ such that $\phi_j \to \phi$ pointwise. For any $\epsilon > 0$, let $\tilde{\phi}_j = \phi_j + \epsilon \log(1 + |z|^2)$ and $\tilde{\phi} = \phi + \epsilon \log(1 + |z|^2)$. Then $\bigcup_{j=1}^{\infty} H^2(\mathbb{C}^n, e^{-\tilde{\phi}_j})$ is dense in $H^2(\mathbb{C}^n, e^{-\tilde{\phi}})$.*

Question 1 Let $\phi_1 \le \phi_2 \le \cdots$ and ϕ be plurisubharmonic functions on $\Omega \subset \mathbb{C}^n$ such that $\phi_j \to \phi$. Is $\bigcup_{j=1}^{\infty} H^2(\Omega, e^{-\phi_j})$ dense in $H^2(\Omega, e^{-\phi})$? □

Recently, J. E. Fornæss and J. Wu [178] solved this problem in the case of $\Omega = \mathbb{C}$.

Theorem 40 *If $\phi_1 \le \phi_2 \le \cdots$ and ϕ are subharmonic functions on $\mathbb{C}$ such that $\phi_j \to \phi$ a.e. as $j \to \infty$, then $\bigcup_{j=1}^{\infty} H^2(\mathbb{C}, e^{-\phi_j})$ is dense in $H^2(\mathbb{C}, e^{-\phi})$.*

This problem has a rich history in dimension one. Here one considers more general weights w which are positive measurable functions on a domain $\Omega \subset \mathbb{C}$, and one defines for $1 \le p < \infty$ the weighted L^p-space of holomorphic functions:

$$H^p(\Omega, w) = \left\{ f \in \mathscr{O}(\Omega) : \int_{\Omega} |f|^p w d\lambda < \infty \right\}.$$

The so-called *completeness problem* is whether polynomials in $H^p(\Omega, w)$ are dense. There are two lines of investigation. One is about finding sufficient conditions on the domain and the weight in order for the polynomials to be dense in the weighted Hilbert space. Another one is to look at specific types of domains and ask the same question for the weight function. These questions go back to T. Carleman [30] who proved in 1923 that if Ω is a Jordan domain and $w \equiv 1$, then holomorphic polynomials are dense in $H^2(\Omega) = L^2(\Omega) \cap \mathscr{O}(\Omega)$. Carleman's result was extended by O. J. Farrell and A. I. Markuševič to Carathéodory domains (see [46, 126]). It is well known that this property need not hold for non-Carathéodory regions. The book by D. Gaier [70] (see in particular Chapter 1, Section 3) contains further results about L^2 polynomials approximation on some simply connected domains in the plane. For weight functions other than the identity, L. I. Hedberg proved in 1965 [91] that polynomials are dense when Ω is a Carathéodory domain, the weight function is continuous, and it satisfies some technical condition near the boundary. For certain non-Carathéodory domains, the weighted polynomial approximation is usually considered under the assumption that the weight w is essentially bounded and satisfies some additional conditions. For a more complete description of the history of this problem and many related references, see the survey by J. E. Brennan [28].

By using Hörmander's L^2 estimate for the $\overline{\partial}$-operator, B. A. Taylor [161] proved the following result which can be seen as a major breakthrough for general weighted approximation. (See also D. Wohlgelernter [176].)

Theorem 41 (B. A. Taylor (1971), Theorem 2 in [161]) *If ϕ is a convex function on $\mathbb{C}^n$ such that the space $H^2(\mathbb{C}^n, e^{-\phi})$ contains all polynomials, then polynomials are dense in $H^2(\Omega, e^{-\phi})$.*

In 1976 N. Sibony [150] generalized Taylor's result as follows. Given a domain $\Omega \subset \mathbb{C}^n$, we denote by $d_\Omega(z)$ the Euclidean distance of a point $z \in \Omega$ to $\mathbb{C}^n \setminus \Omega$. Write $\delta_0(z) = (1 + |z|^2)^{-1/2}$ and

$$\delta_\Omega(z) = \min\{d_\Omega(z), \delta_0(z)\}, \quad z \in \Omega.$$

Theorem 42 (N. Sibony (1976), [150]) *If Ω is an open convex domain in $\mathbb{C}^n$ and ϕ is a convex function on Ω satisfying*

$$\sup_{z \in \Omega} e^{-\phi(z)} \delta_\Omega^{-k}(z) < +\infty, \qquad k \in \mathbb{N},$$

then polynomials are dense in $H^p(\Omega, e^{-\phi})$ for all $1 \le p \le +\infty$.

In the same paper, Sibony also proved the analogous result for homogeneous plurisubharmonic weights.

Theorem 43 (N. Sibony (1976), [150]) *Let ϕ be a plurisubharmonic function on $\mathbb{C}^n$ which is complex homogeneous of order $\rho > 0$, that is, $\phi(uz) = |u|^\rho \phi(z)$ for all $u \in \mathbb{C}$ and $z \in \mathbb{C}^n$. Then, polynomials are dense in $H^2(\Omega, e^{-\phi})$.*

It is well known that every convex function is plurisubharmonic, but the converse is not true. In view of Theorem 41 it is therefore natural to ask the following question. Let ϕ be a plurisubharmonic function on a Runge domain $\Omega \subset \mathbb{C}^n$. Suppose that the restrictions of polynomials to Ω belong to $H^2(\Omega, e^{-\phi})$. Does it follow that polynomials are dense in $H^2(\Omega, e^{-\phi})$? Recently, S. Biard, J. E. Fornæss, and J. Wu [179] found a counterexample in the plane.

Theorem 44 *There is a subharmonic function ϕ on $\mathbb{C}$ such that all polynomials belong to $H^2(\mathbb{C}, e^{-\phi})$, but polynomials are not dense in $H^2(\mathbb{C}, e^{-\phi})$.*

They also proved the following positive result under additional conditions.

Theorem 45 *Let ϕ be plurisubharmonic on a neighborhood of $\overline{\Omega} \subset \mathbb{C}^n$, and suppose that $\overline{\Omega}$ is bounded, uniformly H-convex and polynomially convex. If $H^2(\Omega, e^{-\phi})$ contains all polynomials, then polynomials are dense in $H^2(\Omega, e^{-\phi})$.*

Recall that a compact set $K \subset \mathbb{C}^n$ is said to be *uniformly H-convex* if there exist a sequence $\varepsilon_j > 0$ converging to 0, a constant $c > 1$, and a sequence of pseudoconvex domains $D_j \subset \mathbb{C}^n$ such that $K \subset D_j$ and

$$\varepsilon_j \le \operatorname{dist}(K, \mathbb{C}^n \setminus D_j) \le c\varepsilon_j, \quad j = 1, 2, \ldots.$$

This terminology is due to E. M. Čirka [38] who showed that uniform H-convexity implies a Mergelyan-like approximation property for holomorphic functions; however, the condition was used in L^2 approximation results already by L. Hörmander and J. Wermer [97] in 1968 (see Remark 5). A related notion is that of a *strong Stein neighborhood basis* (which holds in particular for strongly hyperconvex domains);

we refer to the paper by S. Şahutoğlu [145]. It seems an open problem whether any of these conditions for the closure $K = \overline{D}$ of a smoothly bounded pseudoconvex domain $D \Subset \mathbb{C}^n$ implies the Mergelyan property for the algebra $\mathscr{A}(K)$.

9 Appendix: Whitney's Extension Theorem

Given a closed set K in a smooth manifold X, the notation $f \in \mathscr{C}^m(K)$ means that f is the restriction to K of a function in $\mathscr{C}^m(X)$.

Theorem 46 (Whitney (1934), [174]) *Let $\Omega \subset \mathbb{R}^n$ be a domain, and assume that there exists a constant $c \geq 1$ such that any two points $x, y \in \Omega$ can be joined by a curve in Ω of length less than $c|x-y|$. If $f \in \mathscr{C}^m(\Omega)$ is such that all its partial derivatives of order m extend continuously to $\overline{\Omega}$, then $f \in \mathscr{C}^m(\overline{\Omega})$.*

In fact, a much stronger extension theorem was proved by Whitney. To state it, we need to introduce some notation and terminology.

Let $K \subset \mathbb{R}^n$ be a compact set, and fix $m \in \mathbb{N}$. A collection $f = (f_\alpha)$ of functions $f_\alpha \in \mathscr{C}(K)$, where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$ is a multiindex with $|\alpha| = \alpha_1 + \cdots + \alpha_n \leq m$, is called an m-jet on K. Let $\mathscr{J}^m(K)$ denote the vector space of m-jets on K. Set

$$\|f\|_{m,K} = \max_{|\alpha| \leq m} \sup_{x \in K} |f_\alpha(x)|.$$

An m-jet $f = (f_\alpha) \in \mathscr{J}^m(K)$ is said to be a *Whitney function of class $\mathscr{C}^m$* on K if

$$f_\alpha(x) = \sum_{|\beta| \leq m-|\alpha|} \frac{f_{\alpha+\beta}(y)}{\beta!}(x-y)^\beta + o(|x-y|^{m-|\alpha|})$$

holds for all $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| \leq m$ and all $x, y \in K$. We denote by $\mathscr{J}_{\mathscr{W}}^m(K)$ the space of all Whitney functions of class $\mathscr{C}^m$ on K.

Theorem 47 (Whitney [174], Glaeser [81]) *Let K be a compact set in $\mathbb{R}^n$. Given $f \in \mathscr{J}^m(K)$, there exists $\tilde{f} \in \mathscr{C}^m(\mathbb{R}^n)$ such that $\mathscr{J}^m(\tilde{f})|_K = f$ if and only if f is a Whitney function of class $\mathscr{C}^m$, that is, $f \in \mathscr{J}_{\mathscr{W}}^m(K)$. Furthermore, there exists a linear extension operator $\Lambda : \mathscr{J}_{\mathscr{W}}^m(K) \to \mathscr{C}^m(\mathbb{R}^n)$ such that $\mathscr{J}^m \Lambda(f)|_K = f$ for each $f \in \mathscr{J}_{\mathscr{W}}^m(K)$, and for every compact set $L \subset \mathbb{R}^n$ with $K \subset L$ there is a constant $C > 0$ depending only on K, L, m, n such that*

$$\|\Lambda(f)\|_{m,L} \leq C\|f\|_{m,K}, \qquad f \in \mathscr{J}_{\mathscr{W}}^m(K). \tag{31}$$

A proof of Whitney's theorem, including the extensions and simplifications due to Glaeser [81], can be found in the monograph by Malgrange [116, Theorem 3.2 and Complement 3.5].

Remark 11 An inspection of the proof in [116] shows that, if the set K in Theorem 47 is the closure of a domain $\Omega \Subset \mathbb{R}^n$ with $\mathscr{C}^m$-smooth boundary, then there are extension operators for all domains sufficiently close to Ω with the same bound in (31). Furthermore, if Ω_j is a sequence of domains such that $\Omega_j \to \Omega$ in $\mathscr{C}^m$ topology as $j \to \infty$, we may fix a domain $\widetilde{\Omega}$ containing $\overline{\Omega}$ and smooth maps $\phi_j : \widetilde{\Omega} \to \mathbb{R}^n$ such that $\phi_j(\Omega_j) = \Omega$ and $\phi_j \to \mathrm{Id}$ in the $\mathscr{C}^m$-norm on $\widetilde{\Omega}$. □

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A Potapov-Type Approach to a Truncated Matricial Stieltjes-Type Power Moment Problem

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Abstract The paper gives a parametrization of the solution set of a matricial Stieltjes-type truncated power moment problem in the non-degenerate and degenerate cases. The key role plays the solution of the corresponding system of Potapov's fundamental matrix inequalities. The original matricial moment problem will be reformulated in a system of interpolation problems for distinguished classes of holomorphic $q \times q$ matrix-valued functions. A key instrument of our strategy is to use an appropriate synthesis of techniques from the theory of meromorphic matrix-valued functions with elements from the J-theory due to V. P. Potapov.

1 Introduction and Preliminaries

The starting point of studying power moment problems on semi-infinite intervals was the famous two parts memoir of T. J. Stieltjes [58, 59]. A complete theory of the treatment of power moment problems on semi-infinite intervals in the scalar case was developed by M. G. Krein in collaboration with A. A. Nudelman (see [50, Section 10], [51], [52, Chapter V]). What concerns an operator-theoretic treatment of the power moment problems named after Hamburger and Stieltjes and its interrelations, we refer the reader to Simon [57].

In the 1970s, V. P. Potapov developed a special approach to discuss matrix versions of classical interpolation and moment problems. The main idea of his method is based on transforming such problems into equivalent matrix inequalities

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with respect to the Löwner semi-ordering. Using this strategy, several matricial interpolation and moment problems could successfully be handled (see, e. g., [7, 8, 15, 16, 18, 19, 21, 22, 24–26, 37, 38, 42–49, 53, 60]). L. A. Sakhnovich enriched Potapov's method by unifying the particular instances of Potapov's procedure under the framework of one type of operator identities [10, 40, 55].

Matrix versions of the classical Stieltjes moment problem were studied by Adamyan/Tkachenko [1, 2], Andô [4], Bolotnikov [6, 7, 9], Bolotnikov/Sakhnovich [10], Chen/Hu [13], Chen/Li [14], Dyukarev [20, 21], Dyukarev/Katsnelson [24, 25], and Hu/Chen [39]. The considerations of this paper deal with the more general case of an arbitrary semi-infinite interval $[\alpha, \infty)$, where α is an arbitrarily given real number.

In order to formulate the moment problem, we are going to study, we first review some notation. Throughout this paper, let p and q be positive integers. Let $\mathbb{C}$, $\mathbb{R}$, $\mathbb{N}_0$, and $\mathbb{N}$ be the set of all complex numbers, the set of all real numbers, the set of all non-negative integers, and the set of all positive integers, respectively. For every choice of $\upsilon, \omega \in \mathbb{R} \cup \{-\infty, \infty\}$, let $\mathbb{Z}_{\upsilon,\omega}$ be the set of all integers k for which $\upsilon \leq k \leq \omega$ holds. If $\mathcal{X}$ is a non-empty set, then $\mathcal{X}^{p\times q}$ stands for the set of all $p \times q$ matrices each entry of which belongs to $\mathcal{X}$, and $\mathcal{X}^{p}$ is short for $\mathcal{X}^{p\times 1}$. If $(\Omega, \mathfrak{A})$ is a measurable space, then each countably additive mapping whose domain is $\mathfrak{A}$ and whose values belong to the set $\mathbb{C}^{q\times q}_{\geq}$ of all non-negative Hermitian complex $q \times q$ matrices is called a non-negative Hermitian $q \times q$ measure on $(\Omega, \mathfrak{A})$. By $\mathcal{M}^{q}_{\geq}(\Omega, \mathfrak{A})$ we denote the set of all non-negative Hermitian $q \times q$ measures on $(\Omega, \mathfrak{A})$. For the integration theory for non-negative Hermitian measures, we refer to [41, 54]. If $\mu = [\mu_{jk}]_{j,k=0}^{q}$ is a non-negative Hermitian $q \times q$ measure on a measurable space $(\Omega, \mathfrak{A})$ and if $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, then we use $\mathcal{L}^1(\Omega, \mathfrak{A}, \mu; \mathbb{K})$ to denote the set of all Borel-measurable functions $f : \Omega \to \mathbb{K}$ for which the integral exists, i. e., that $\int_\Omega |f| \mathrm{d}\tilde{\mu}_{jk} < \infty$ for every choice of j and k in $\mathbb{Z}_{1,q}$, where $\tilde{\mu}_{jk}$ is the variation of the complex measure μ_{jk}. If $f \in \mathcal{L}^1(\Omega, \mathfrak{A}, \mu; \mathbb{K})$, then let $\int_A f \mathrm{d}\mu := [\int_\Omega 1_A f \mathrm{d}\mu_{jk}]_{j,k=1}^{q}$ for all $A \in \mathfrak{A}$ and we will also write $\int_A f(\omega)\mu(\mathrm{d}\omega)$ for this integral.

Let $\mathfrak{B}_\mathbb{R}$ (resp. $\mathfrak{B}_\mathbb{C}$) be the σ-algebra of all Borel subsets of $\mathbb{R}$ (resp. $\mathbb{C}$). For all $\Omega \in \mathfrak{B}_\mathbb{R} \setminus \{\emptyset\}$, let $\mathfrak{B}_\Omega$ be the σ-algebra of all Borel subsets of Ω, let $\mathcal{M}^{q}_{\geq}(\Omega) := \mathcal{M}^{q}_{\geq}(\Omega, \mathfrak{B}_\Omega)$ and, for all $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, let $\mathcal{M}^{q}_{\geq,\kappa}(\Omega)$ be the set of all $\sigma \in \mathcal{M}^{q}_{\geq}(\Omega)$ such that for all $j \in \mathbb{Z}_{0,\kappa}$ the function $f_j : \Omega \to \mathbb{C}$ defined by $f_j(t) := t^j$ belongs to $\mathcal{L}^1(\Omega, \mathfrak{B}_\Omega, \sigma; \mathbb{C})$. If $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and if $\sigma \in \mathcal{M}^{q}_{\geq,\kappa}(\Omega)$, then we set

$$s_j^{[\sigma]} := \int_\Omega t^j \sigma(\mathrm{d}t) \qquad \text{for each } j \in \mathbb{Z}_{0,\kappa}.$$

The following matricial power moment problem lies in the background of our considerations:

Problem 1 ($\mathsf{MP}[\Omega; (s_j)_{j=0}^{m}, \leq]$) *Let $\Omega \in \mathfrak{B}_\mathbb{R} \setminus \{\emptyset\}$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^{m}$ be a sequence of complex $q \times q$ matrices. Describe the set $\mathcal{M}^{q}_{\geq}[\Omega; (s_j)_{j=0}^{m}, \leq]$ of* *all* $\sigma \in \mathscr{M}^q_{\geq,m}(\Omega)$ *for which the matrix* $s_m - s_m^{[\sigma]}$ *is non-negative Hermitian and for which, in the case* $m > 0$*, moreover* $s_j^{[\sigma]} = s_j$ *is fulfilled for all* $j \in \mathbb{Z}_{0,m-1}$.

Note that we also sometimes turn our attention to the following power moment problem:

Problem 2 ($\mathsf{MP}[\Omega; (s_j)_{j=0}^{\kappa}, =]$) *Let* $\Omega \in \mathfrak{B}_{\mathbb{R}} \setminus \{\emptyset\}$*, let* $\kappa \in \mathbb{N}_0 \cup \{\infty\}$*, and let* $(s_j)_{j=0}^{\kappa}$ *be a sequence of complex* $q \times q$ *matrices. Describe the set* $\mathscr{M}^q_{\geq}[\Omega; (s_j)_{j=0}^{\kappa}, =]$ *of all* $\sigma \in \mathscr{M}^q_{\geq,\kappa}(\Omega)$ *for which* $s_j^{[\sigma]} = s_j$ *is fulfilled for all* $j \in \mathbb{Z}_{0,\kappa}$.

The considerations of this paper are mostly concentrated on the case that the set Ω is a one-sided bounded and closed infinite interval of the real axis. Such moment problems are called to be of Stieltjes type.

In the case $\Omega = [\alpha, \infty)$ where α is an arbitrary given real number, in [31, 35], we have handled both matricial moment problems formulated above via Schur analysis methods. These moment problems were reformulated via a particular integral transform. A parametrization of the solution set was given by a certain linear fractional transformation of matrices, the generating matrix-valued function of which is a suitable matrix polynomial built explicitly by the prescribed data.

In this paper we treat the problem with the aid of V. P. Potapov's method of fundamental matrix inequalities (short FMI method). In the first step it is shown that the Stieltjes transforms of all solutions of the matricial moment problem are exactly the solutions of a coupled system of two fundamental matrix inequalities. This main step was carried out in detail in [36, Theorem 6.20]. For the convenience of the reader this result is repeated in Theorem 4.3. The main aim of this paper is to construct an explicit parametrization for the solution set of the system of the FMIs of V. P. Potapov. The particular feature of the problem under consideration is that we investigate the most general case which includes all possibilities of degeneracies. For the case of non-degenerate interpolation or moment problems a machinery for solving the FMI was developed (see, e. g., [47, 48]). For $\Omega := [0, \infty)$ the non-degenerate situation was handled by Yu. M. Dyukarev [20, 23] by use of the FMI method. The treatment of degenerate problems started with the pioneering work [18] (see also [19, Section 5.3]) of V. K. Dubovoj connected with the matricial Schur problem and was continued with the investigations of V. A. Bolotnikov [7, 8] in the context of degenerate truncated matricial moment problems.

A parametrization of the solution set of the moment problem [$\mathsf{MP}[[\alpha, \infty); (s_j)_{j=0}^{m}, \leq]$], where α is an arbitrarily given real number, by using Schur-type algorithms is given in [35]. In this paper, we use Potapov's method for solving Problem $\mathsf{MP}[[\alpha, \infty); (s_j)_{j=0}^{m}, \leq]$. The key role for solving the moment problem $\mathsf{MP}[[\alpha, \infty); (s_j)_{j=0}^{m}, \leq]$, where α is an arbitrarily given real number, is Theorem 4.3 below. It will turn out that the solution set of the moment problem (obtained via Stieltjes transformation) coincides with the solution set of a certain system of Potapov's fundamental matrix inequalities. The considerations in this paper are aimed to solve these inequalities. In Section 15, we give a parametrization of the solution set $\mathscr{M}^q_{\geq}[[\alpha, \infty); (s_j)_{j=0}^{2n+1}, \leq]$ of Problem $\mathsf{MP}[[\alpha, \infty); (s_j)_{j=0}^{2n+1}, \leq]$, where α is an arbitrarily given real number and where n is an arbitrarily given non-negative integer. Note that Problem $\mathsf{MP}[[\alpha, \infty); (s_j)_{j=0}^{2n}, \leq]$ can be discussed by similar methods (see [56]). Furthermore observe that a parametrization of the solution set of the moment problem $\mathsf{MP}[[\alpha, \infty); (s_j)_{j=0}^{m}, =]$, where α is an arbitrarily given real number, was worked out by using Schur algorithms in [31].

In Section 2, we recall necessary and sufficient conditions of solvability of the moment problems in question. In Section 3, we reformulate these problems in the language of certain matrix-valued functions. Section 4 is aimed at recalling that the solutions of the reformulated problem are exactly the solutions of the corresponding system of Potapov's fundamental matrix inequalities. Section 5 is aimed to give some identities for block Hankel matrices. In Section 6, we study special subspaces of $\mathbb{C}^q$, so-called Dubovoj subspaces against the background of particular generalized inverses of matrices (see Section 17). These objects turn out to be a basic tool to handle the degenerate case of the moment problem under consideration. Section 7 is aimed at realizing first steps on the way to the solution of the system of FMIs of V. P. Potapov. First the two matricial inequalities will be handled separately. In a second step (see Proposition 7.16) a coupling will be established between the two single FMIs of the system. The role of Section 7 is to provide a $2q \times 2q$ matrix polynomial $\Theta_{n,\alpha}$ which generates via Stieltjes transform by linear fractional transformation the set of solutions of the original moment problem. In Section 8 the set of parameters of this linear fractional transformation is discussed. More precisely, a class of ordered pairs of $q \times q$ matrix functions is studied which are meromorphic in $\mathbb{C} \setminus [\alpha, \infty)$. A closer analysis of the $2q \times 2q$ matrix polynomials $\Theta_{n,\alpha}$ and $\tilde{\Theta}_{n,\alpha}$ studied in Section 7 leads us to a particular class $\tilde{\mathfrak{W}}_{\tilde{J}_q,\alpha}$ of $\tilde{J}_q$-inner functions which is investigated in Section 9 and a distinguished subclass $\mathfrak{W}_{\tilde{J}_q,\alpha}$ of $\tilde{\mathfrak{W}}_{\tilde{J}_q,\alpha}$. In Section 10 we consider linear fractional transformations with generating matrix-valued function belonging to $\mathfrak{W}_{\tilde{J}_q,\alpha}$. In Section 11, we apply Proposition 10.1 to realize an important intermediate step on the way to the solution of the system of FMIs of Potapov type. Section 12 handles particular aspects of the degenerate case of the moment problem. In Section 13, we are able to construct a parametrization of the solution set of our moment problem (see Theorem 13.7). In the degenerate case the set of parameters used here depends on the initial data. In order to improve this situation we carry out a closer analysis of the phenomenon of degeneracy in Section 14. This leads us to a relevant classification of the cases of possible kinds of degeneracy. There arise two basically different situations of degeneracy which will be handled separately in Sections 15 and 16, respectively.

At the end of this section, let us introduce some further notations, which are useful for our considerations. We will write I_q for the identity matrix in $\mathbb{C}^{q \times q}$, whereas $0_{p \times q}$ is the null matrix belonging to $\mathbb{C}^{p \times q}$. If the size of the identity matrix or the null matrix is obvious, then we will also omit the indexes. The notations $\mathbb{C}_{\mathrm{H}}^{q \times q}$, $\mathbb{C}_{\geq}^{q \times q}$, and $\mathbb{C}_{>}^{q \times q}$ stand for the set of all Hermitian complex $q \times q$ matrices, the set of all non-negative Hermitian complex matrices, and the set of all positive Hermitian complex matrices, respectively. If A and B are complex $q \times q$ matrices, then we will write $A \leq B$ or $B \geq A$ to indicate that A and B are Hermitian matrices such that the matrix $B - A$ is non-negative Hermitian. For each $A \in \mathbb{C}^{p\times q}$, let $\mathscr{N}(A)$ be the null space of A, let $\mathscr{R}(A)$ be the column space of A, and let $\operatorname{rank} A$ be the rank of A. For each $A \in \mathbb{C}^{q\times q}$, we will use $\operatorname{Im} A$ to denote the imaginary part of A: $\operatorname{Im} A := \frac{1}{2\mathrm{i}}(A - A^*)$. Furthermore, for each $A \in \mathbb{C}^{p\times q}$, let $\|A\|_{\mathrm{S}}$ be the operator norm of A. For each $x \in \mathbb{C}^q$, we write $\|x\|_{\mathrm{E}}$ for the Euclidean norm of x. If $A \in \mathbb{C}^{q\times q}$, then $\det A$ stands for the determinant of A. We will often use the Moore–Penrose inverse of a complex $p \times q$ matrix A. This is the unique complex $q \times p$ matrix X such that the four equations $AXA = A$, $XAX = X$, $(AX)^* = AX$, and $(XA)^* = XA$ hold true (see, e. g., [19, Proposition 1.1.1]). As usual, we will write $A^\dagger$ for this matrix X.

If $n \in \mathbb{N}$, if $(p_j)_{j=1}^n$ is a sequence of positive integers, and if $x_j \in \mathbb{C}^{p_j\times q}$ for each $j \in \mathbb{Z}_{1,n}$, then let $\operatorname{col}(x_j)_{j=1}^n := \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$. If $n \in \mathbb{N}$, if $(q_k)_{k=1}^n$ is a sequence of positive integers, and if $y_k \in \mathbb{C}^{p\times q_k}$ for each $k \in \mathbb{Z}_{1,n}$, then let $\operatorname{row}(y_k)_{k=1}^n := [y_1, y_2, \ldots, y_n]$. If $n \in \mathbb{N}$, if $(p_j)_{j=1}^n$ and $(q_j)_{j=1}^n$ are sequences of positive integers, and if $A_j \in \mathbb{C}^{p_j\times q_j}$ for every choice of j in $\mathbb{Z}_{1,n}$, then let $\operatorname{diag}(A_1, A_2, \ldots, A_n) := [\delta_{jk} A_j]_{j,k=1}^n$, where δ_{jk} is the Kronecker delta: $\delta_{jk} := 1$ in the case $j = k$ and $\delta_{jk} := 0$ if $j \neq k$. We also use the notation $\operatorname{diag}(A_j)_{j=1}^n$ instead of $\operatorname{diag}(A_1, A_2, \ldots, A_n)$. If $\mathscr{M}$ is a non-empty subset of $\mathbb{C}^q$, then let $\mathscr{M}^\perp$ be the set of all vectors in $\mathbb{C}^q$ which are orthogonal to $\mathscr{M}$ (with respect to the Euclidean inner product). If $\mathscr{X}$, $\mathscr{Y}$, and $\mathscr{Z}$ are non-empty sets with $\mathscr{Z} \subseteq \mathscr{X}$ and if $f\colon \mathscr{X} \to \mathscr{Y}$ is a mapping, then $\operatorname{Rstr}_{\mathscr{Z}} f$ stands for the restriction of f onto $\mathscr{Z}$.

2 On the Solvability of Matricial Power Moment Problems

In this section, we recall necessary and sufficient conditions for the solvability of the Stieltjes moment problems $\mathsf{MP}[[\alpha, \infty); (s_j)_{j=0}^m, \leq]$ and $\mathsf{MP}[[\alpha, \infty); (s_j)_{j=0}^m, =]$, where α is an arbitrarily given real number and m is an arbitrarily given non-negative integer. First we introduce certain sets of sequences of complex $q \times q$ matrices, which are determined by the properties of particular block Hankel matrices built of them. For each $n \in \mathbb{N}_0$, let $\mathscr{H}_{q,2n}^{\geq}$ be the set of all sequences $(s_j)_{j=0}^{2n}$ of complex $q \times q$ matrices such that the block Hankel matrix $H_n := [s_{j+k}]_{j,k=0}^n$ is non-negative Hermitian. Furthermore, let $\mathscr{H}_{q,\infty}^{\geq}$ be the set of all sequences $(s_j)_{j=0}^\infty$ of complex $q \times q$ matrices such that, for all $n \in \mathbb{N}_0$, the sequence $(s_j)_{j=0}^{2n}$ belongs to $\mathscr{H}_{q,2n}^{\geq}$. The elements of the set $\mathscr{H}_{q,2\kappa}^{\geq}$, where $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, are called *Hankel non-negative definite* sequences. For all $n \in \mathbb{N}_0$, let $\mathscr{H}_{q,2n}^{\geq,\mathrm{e}}$ be the set of all sequences $(s_j)_{j=0}^{2n}$ of complex $q \times q$ matrices for which there are matrices $s_{2n+1} \in \mathbb{C}^{q\times q}$ and $s_{2n+2} \in \mathbb{C}^{q\times q}$ such that $(s_j)_{j=0}^{2(n+1)}$ belongs to $\mathscr{H}_{q,2(n+1)}^{\geq}$.

Furthermore, for all $n \in \mathbb{N}_0$, we will use $\mathscr{H}^{\geq,\mathrm{e}}_{q,2n+1}$ to denote the set of sequences $(s_j)_{j=0}^{2n+1}$ of complex $q \times q$ matrices for which there is some $s_{2n+2} \in \mathbb{C}^{q\times q}$ such that $(s_j)_{j=0}^{2(n+1)}$ belongs to $\mathscr{H}^{\geq}_{q,2(n+1)}$. For all $m \in \mathbb{N}_0$, the elements of the set $\mathscr{H}^{\geq,\mathrm{e}}_{q,m}$ are called *Hankel non-negative definite extendable* sequences. For technical reasons, we set $\mathscr{H}^{\geq,\mathrm{e}}_{q,\infty} := \mathscr{H}^{\geq}_{q,\infty}$. Observe that the solvability of the matricial Hamburger moment problems can be characterized by the introduced classes of sequences of complex $q \times q$ matrices:

Theorem 2.1 (See, e. g., [12, Theorem 3.2] or [26, Theorem 4.16]) *Let $n \in \mathbb{N}_0$ and let $(s_j)_{j=0}^{2n}$ be a sequence of complex $q \times q$ matrices. Then $\mathscr{M}^q_{\geq}[\mathbb{R}; (s_j)_{j=0}^{2n}, \leq] \neq \emptyset$ if and only if $(s_j)_{j=0}^{2n} \in \mathscr{H}^{\geq}_{q,2n}$.*

Theorem 2.2 (See [26, Theorem 4.17], [29, Theorem 6.6]) *Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $q \times q$ matrices. Then $\mathscr{M}^q_{\geq}[\mathbb{R}; (s_j)_{j=0}^{\kappa}, =] \neq \emptyset$ if and only if $(s_j)_{j=0}^{\kappa} \in \mathscr{H}^{\geq,\mathrm{e}}_{q,\kappa}$.*

Let $\alpha \in \mathbb{C}$, let $\kappa \in \mathbb{N} \cup \{\infty\}$, and let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $p \times q$ matrices. Then let the sequence $(s_{\alpha\triangleright j})_{j=0}^{\kappa-1}$ be defined by

$$s_{\alpha\triangleright j} := -\alpha s_j + s_{j+1} \qquad \text{for all } j \in \mathbb{Z}_{0,\kappa-1}. \tag{1}$$

The sequence $(s_{\alpha\triangleright j})_{j=0}^{\kappa-1}$ is called the *sequence generated from $(s_j)_{j=0}^{\kappa}$ by right-sided α-shifting*. (An analogous left-sided version is discussed in [30, Definition 2.1].) The sequence $(s_{\alpha\triangleright j})_{j=0}^{\kappa-1}$ is used to define further sets of sequences of complex matrices, which are useful to discuss the Stieltjes moment problems we consider. Let $\mathscr{K}^{\geq}_{q,0,\alpha} := \mathscr{H}^{\geq}_{q,0}$. For every choice of $n \in \mathbb{N}$, let $\mathscr{K}^{\geq}_{q,2n,\alpha} := \{(s_j)_{j=0}^{2n} \in \mathscr{H}^{\geq}_{q,2n} : (s_{\alpha\triangleright j})_{j=0}^{2(n-1)} \in \mathscr{H}^{\geq}_{q,2(n-1)}\}$. For all $m \in \mathbb{N}_0$, by $\mathfrak{S}_m(\mathbb{C}^{q\times q})$ we denote the set of all sequences $(s_j)_{j=0}^{m}$ of complex $q \times q$ matrices. Then we set $\mathscr{K}^{\geq}_{q,2n+1,\alpha} := \{(s_j)_{j=0}^{2n+1} \in \mathfrak{S}_{2n+1}(\mathbb{C}^{q\times q}) : \{(s_j)_{j=0}^{2n}, (s_{\alpha\triangleright j})_{j=0}^{2n}\} \subseteq \mathscr{H}^{\geq}_{q,2n}\}$. For all $m \in \mathbb{N}_0$, let $\mathscr{K}^{\geq,\mathrm{e}}_{q,m,\alpha}$ be the set of all sequences $(s_j)_{j=0}^{m}$ of complex $q \times q$ matrices for which there exists a complex $q \times q$ matrix s_{m+1} such that $(s_j)_{j=0}^{m+1}$ belongs to $\mathscr{K}^{\geq}_{q,m+1,\alpha}$. Obviously, we have $\mathscr{K}^{\geq,\mathrm{e}}_{q,2n,\alpha} = \{(s_j)_{j=0}^{2n} \in \mathscr{H}^{\geq}_{q,2n} : (s_{\alpha\triangleright j})_{j=0}^{2n-1} \in \mathscr{H}^{\geq,\mathrm{e}}_{q,2n-1}\}$ for all $n \in \mathbb{N}$ and $\mathscr{K}^{\geq,\mathrm{e}}_{q,2n+1,\alpha} = \{(s_j)_{j=0}^{2n+1} \in \mathscr{H}^{\geq,\mathrm{e}}_{q,2n+1} : (s_{\alpha\triangleright j})_{j=0}^{2n} \in \mathscr{H}^{\geq}_{q,2n}\}$ for all $n \in \mathbb{N}_0$.

Remark 2.3 Let $\alpha \in \mathbb{R}$ and let $m \in \mathbb{N}_0$. Then $\mathscr{K}^{\geq,\mathrm{e}}_{q,m,\alpha} \subseteq \mathscr{K}^{\geq}_{q,m,\alpha}$. Furthermore, if $(s_j)_{j=0}^{m} \in \mathscr{K}^{\geq}_{q,m,\alpha}$ (resp. $\mathscr{K}^{\geq,\mathrm{e}}_{q,m,\alpha}$), then we easily see that $(s_j)_{j=0}^{\ell} \in \mathscr{K}^{\geq}_{q,\ell,\alpha}$ (resp. $(s_j)_{j=0}^{\ell} \in \mathscr{K}^{\geq,\mathrm{e}}_{q,\ell,\alpha}$) holds true for all $\ell \in \mathbb{Z}_{0,m}$.

The essential feature of a sequence $(s_j)_{j=0}^{2n} \in \mathscr{K}^{\geq}_{q,2n,\alpha}$ is a specific interplay between the sequences $(s_j)_{j=0}^{2n} \in \mathscr{H}^{\geq}_{q,2n}$ and $(s_{\alpha\triangleright j})_{j=0}^{2n-2} \in \mathscr{H}^{\geq}_{q,2n-2}$. An analogous fact is also essential for a sequence $(s_j)_{j=0}^{2n+1} \in \mathscr{K}^{\geq}_{q,2n+1,\alpha}$.

In view of Remark 2.3, for all $\alpha \in \mathbb{R}$, let $\mathscr{K}^{\geq}_{q,\infty,\alpha}$ be the set of all sequences $(s_j)_{j=0}^{\infty}$ of complex $q \times q$ matrices such that $(s_j)_{j=0}^{m}$ belongs to $\mathscr{K}^{\geq}_{q,m,\alpha}$ for all $m \in \mathbb{N}_0$, and let $\mathscr{K}^{\geq,\mathrm{e}}_{q,\infty,\alpha} := \mathscr{K}^{\geq}_{q,\infty,\alpha}$. For all $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, we call a sequence $(s_j)_{j=0}^{\kappa}$ $[\alpha, \infty)$-*Stieltjes right-sided non-negative definite* (resp. $[\alpha, \infty)$-*Stieltjes right-sided non-negative definite extendable*) if it belongs to $\mathscr{K}^{\geq}_{q,\kappa,\alpha}$ (resp. to $\mathscr{K}^{\geq,\mathrm{e}}_{q,\kappa,\alpha}$). Note that left versions of these notions are used in [30, Definition 1.3].

Using the introduced sets of sequences of complex $q \times q$ matrices, we are able to recall solvability criteria of the problems $\mathsf{MP}[[\alpha, \infty); (s_j)_{j=0}^{m}, \leq]$ and $\mathsf{MP}[[\alpha, \infty); (s_j)_{j=0}^{m}, =]$:

Theorem 2.4 ([27, Theorem 1.4]) *Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^{m}$ be a sequence of complex $q \times q$ matrices. Then $\mathscr{M}^{q}_{\geq}[[\alpha, \infty); (s_j)_{j=0}^{m}, \leq] \neq \emptyset$ if and only if $(s_j)_{j=0}^{m} \in \mathscr{K}^{\geq}_{q,m,\alpha}$.*

Theorem 2.5 *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $q \times q$ matrices. Then $\mathscr{M}^{q}_{\geq}[[\alpha, \infty); (s_j)_{j=0}^{\kappa}, =] \neq \emptyset$ if and only if $(s_j)_{j=0}^{\kappa} \in \mathscr{K}^{\geq,\mathrm{e}}_{q,\kappa,\alpha}$.*

In the case $\kappa \in \mathbb{N}_0$, a proof of Theorem 2.5 is given in [27, Theorem 1.3]. If $\kappa = \infty$, then the asserted equivalence can be proved using the equation $\mathscr{M}^{q}_{\geq}[[\alpha, \infty); (s_j)_{j=0}^{\infty}, =] = \bigcap_{m=0}^{\infty} \mathscr{M}^{q}_{\geq}[[\alpha, \infty); (s_j)_{j=0}^{m}, =]$ and a matricial version of the Helly–Prohorov theorem (see [28, Satz 9]). We omit the details of the proof, the essential idea of which is originated in [3, proof of Theorem 2.1.1].

For the description of the solution set $\mathscr{M}^{q}_{\geq}[[\alpha, \infty); (s_j)_{j=0}^{m}, \leq]$ of Problem $\mathsf{MP}[[\alpha, \infty); (s_j)_{j=0}^{m}, \leq]$, it is essential that one can suppose extendable data without loss of generality:

Theorem 2.6 ([27, Theorem 5.2]) *Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^{m} \in \mathscr{K}^{\geq}_{q,m,\alpha}$. Then there is a unique sequence $(\tilde{s}_j)_{j=0}^{m} \in \mathscr{K}^{\geq,\mathrm{e}}_{q,m,\alpha}$ such that $\mathscr{M}^{q}_{\geq}[[\alpha, \infty); (\tilde{s}_j)_{j=0}^{m}, \leq] = \mathscr{M}^{q}_{\geq}[[\alpha, \infty); (s_j)_{j=0}^{m}, \leq]$.*

In [33] it was shown that the construction of the sequence $(\tilde{s}_j)_{j=0}^{m}$ occurring in Theorem 2.6 is a consequence of a general principle which also works analogously for the truncated matricial Hamburger moment problem.

3 Some Classes of Holomorphic Matrix-Valued Functions

The main goal of this section can be summarized as follows. Using particular integral representations we are going to reformulate the matricial moment problems $\mathsf{MP}[[\alpha, \infty); (s_j)_{j=0}^{m}, \leq]$ and $\mathsf{MP}[[\alpha, \infty); (s_j)_{j=0}^{m}, =]$ into equivalent interpolation problems for appropriately chosen classes of holomorphic $q \times q$ matrix-valued functions. The main tool is the following class of matrix-valued functions. For each $\alpha \in \mathbb{R}$, let $\mathscr{S}_{q;[\alpha,\infty)}$ be the set of all matrix-valued functions

$S\colon \mathbb{C}\setminus[\alpha,\infty)\to\mathbb{C}^{q\times q}$ which are holomorphic in $\mathbb{C}\setminus[\alpha,\infty)$ and which satisfy $\operatorname{Im}[S(\Pi_+)]\subseteq\mathbb{C}_{\geq}^{q\times q}$ as well as $S((-\infty,\alpha))\subseteq\mathbb{C}_{\geq}^{q\times q}$. In [32, Theorems 3.1 and 3.6, Proposition 2.16], integral representations of functions belonging to $\mathscr{S}_{q;[\alpha,\infty)}$ are proved. Furthermore, several characterizations of the class $\mathscr{S}_{q;[\alpha,\infty)}$ are given in [32, Section 4]. For each $\alpha\in\mathbb{R}$, let $\mathscr{S}_{0,q;[\alpha,\infty)}$ be the class of all $F\in\mathscr{S}_{q;[\alpha,\infty)}$ which satisfy $\sup_{y\in[1,\infty)} y\|F(\mathrm{i}y)\|_{\mathrm{S}}<\infty$. The functions belonging to $\mathscr{S}_{0,q;[\alpha,\infty)}$ admit a particular integral representation. Before we state this, let us note the following:

Remark 3.1 For every choice of $\alpha\in\mathbb{R}$ and $z\in\mathbb{C}\setminus[\alpha,\infty)$, the function $b_{\alpha,z}\colon[\alpha,\infty)\to\mathbb{C}$ given by $b_{\alpha,z}(t):=1/(t-z)$ is a bounded and continuous function which, in particular, belongs to $\mathscr{L}^1([\alpha,\infty),\mathfrak{B}_{[\alpha,\infty)},\sigma;\mathbb{C})$ for all $\sigma\in\mathscr{M}_{\geq}^{q}([\alpha,\infty))$.

Theorem 3.2 ([32, Theorem 5.1]) *Let $\alpha\in\mathbb{R}$.*

(a) If $S\in\mathscr{S}_{0,q;[\alpha,\infty)}$, then there is a unique $\sigma\in\mathscr{M}_{\geq}^{q}([\alpha,\infty))$ such that

$$S(z)=\int_{[\alpha,\infty)}\frac{1}{t-z}\sigma(\mathrm{d}t)\qquad \text{for each } z\in\mathbb{C}\setminus[\alpha,\infty). \tag{2}$$

(b) If $\sigma\in\mathscr{M}_{\geq}^{q}([\alpha,\infty))$ is such that $S\colon\mathbb{C}\setminus[\alpha,\infty)\to\mathbb{C}^{q\times q}$ can be represented via (2), then S belongs to $\mathscr{S}_{0,q;[\alpha,\infty)}$.

If $F\in\mathscr{S}_{0,q;[\alpha,\infty)}$ is given, then the unique $\sigma\in\mathscr{M}_{\geq}^{q}([\alpha,\infty))$ which fulfills the representation (2) of F is called the $[\alpha,\infty)$-*Stieltjes transform of* F. If $\sigma\in\mathscr{M}_{\geq}^{q}([\alpha,\infty))$ is given, then $F\colon\mathbb{C}\setminus[\alpha,\infty)\to\mathbb{C}^{q\times q}$ defined by (2) is said to be the $[\alpha,\infty)$-*Stieltjes transform of* σ. In view of Theorem 3.2, the moment problems $\mathsf{MP}[[\alpha,\infty);(s_j)_{j=0}^{m},\leq]$ and $\mathsf{MP}[[\alpha,\infty);(s_j)_{j=0}^{\kappa},=]$ admit reformulations in the language of $[\alpha,\infty)$-Stieltjes transforms:

Problem 3 ($\mathsf{S}[[\alpha,\infty);(s_j)_{j=0}^{m},\leq]$) *Let $\alpha\in\mathbb{R}$, let $m\in\mathbb{N}_0$, and let $(s_j)_{j=0}^{m}$ be a sequence of complex $q\times q$ matrices. Describe the set $\mathscr{S}_{0,q;[\alpha,\infty)}[(s_j)_{j=0}^{m},\leq]$ of all $F\in\mathscr{S}_{0,q;[\alpha,\infty)}$, the $[\alpha,\infty)$-Stieltjes measure of which belongs to $\mathscr{M}_{\geq}^{q}[[\alpha,\infty);(s_j)_{j=0}^{m},\leq]$.*

Problem 4 ($\mathsf{S}[[\alpha,\infty);(s_j)_{j=0}^{\kappa},=]$) *Let $\alpha\in\mathbb{R}$, let $\kappa\in\mathbb{N}_0\cup\{\infty\}$, and let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $q\times q$ matrices. Describe the set $\mathscr{S}_{0,q;[\alpha,\infty)}[(s_j)_{j=0}^{\kappa},=]$ of all $F\in\mathscr{S}_{0,q;[\alpha,\infty)}$, the $[\alpha,\infty)$-Stieltjes measure of which belongs to $\mathscr{M}_{\geq}^{q}[[\alpha,\infty);(s_j)_{j=0}^{\kappa},=]$.*

Following the classical line started by Stieltjes [58, 59] we investigate the reformulated problems in the sequel.

4 On the Equivalence of the Stieltjes Moment Problem to a System of Two Fundamental Matrix Inequalities of Potapov Type

In this section, we introduce the system of Potapov's fundamental matrices corresponding to the matricial Stieltjes moment problem $\mathsf{MP}[[\alpha, \infty); (s_j)_{j=0}^m, \leq]$. We will see that each solution of this moment problem fulfills necessarily the system of Potapov's fundamental matrix inequalities. First we are going to introduce further notations and, in particular, several block Hankel matrices which will play a key role in our considerations.

For each $n \in \mathbb{N}_0$, we set

$$T_{q,n} := [\delta_{j,k+1} I_q]_{j,k=0}^n, \quad v_{q,n} := \operatorname{col}(\delta_{j,0} I_q)_{j=0}^n, \quad \text{and} \quad \mathfrak{v}_{q,n} := \operatorname{col}(\delta_{n-j,0} I_q)_{j=0}^n, \tag{3}$$

where $\delta_{j,k}$ is again the Kronecker delta. Obviously, $T_{q,n}^* = [\delta_{j+1,k} I_q]_{j,k=0}^n$ for each $n \in \mathbb{N}_0$.

Remark 4.1 For each $n \in \mathbb{N}_0$, the matrix-valued functions $R_{T_{q,n}} : \mathbb{C} \to \mathbb{C}^{(n+1)q \times (n+1)q}$ and $R_{T_{q,n}^*} : \mathbb{C} \to \mathbb{C}^{(n+1)q \times (n+1)q}$ given by

$$R_{T_{q,n}}(z) := (I_{(n+1)q} - zT_{q,n})^{-1} \quad \text{and} \quad R_{T_{q,n}^*}(z) := (I_{(n+1)q} - zT_{q,n}^*)^{-1} \tag{4}$$

are well-defined matrix polynomials of degree n, which can be represented, for each $z \in \mathbb{C}$, via $R_{T_{q,n}}(z) = \sum_{j=0}^n z^j T_{q,n}^j$ and $R_{T_{q,n}^*}(z) = \sum_{j=0}^n z^j (T_{q,n}^*)^j$, respectively. In particular, $R_{T_{q,n}^*}(z) = [R_{T_{q,n}}(\overline{z})]^*$ for all $z \in \mathbb{C}$.

For technical reason, let $s_{-1} := 0_{p\times q}$.

Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and let $(s_j)_{j=0}^\kappa$ be a sequence of complex $p \times q$ matrices. For each $n \in \mathbb{N}_0$ with $2n \leq \kappa$, let $H_n := [s_{j+k}]_{j,k=0}^n$. If m and n are integers such that $-1 \leq m \leq n \leq \kappa$, then we set

$$y_{m,n} := \operatorname{col}(s_j)_{j=m}^n \qquad \text{and} \qquad z_{m,n} := \operatorname{row}(s_k)_{k=m}^n.$$

Let $u_0 := 0_{p\times q}$, $\mathfrak{u}_0 := 0_{p\times q}$, $w_0 := 0_{p\times q}$, and $\mathfrak{w}_0 := 0_{p\times q}$. For all $n \in \mathbb{N}$ with $n \leq \kappa + 1$, let $u_n := -y_{-1,n-1}$, and $w_n := z_{-1,n-1}$. Further, for each $n \in \mathbb{N}_0$ with $2n \leq \kappa$, let $\mathfrak{u}_n := \begin{bmatrix} -y_{n+1,2n} \\ 0_{p\times q} \end{bmatrix}$ and $\mathfrak{w}_n := [z_{n+1,2n}, 0_{p\times q}]$.

If a real number α is additionally given, then we continue to use the notation given by (1), and we set $H_{\alpha\triangleright n} := [s_{\alpha\triangleright j+k}]_{j,k=0}^n$ for each $n \in \mathbb{N}_0$ with $2n + 1 \leq \kappa$.

Notation 4.2 *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $(s_j)_{j=0}^\kappa$ be a sequence of complex $q \times q$ matrices. Further, let $\mathcal{G}$ be a subset of $\mathbb{C}$ with $\mathcal{G} \setminus \mathbb{R} \neq \emptyset$ and let $f : \mathcal{G} \to \mathbb{C}^{q\times q}$ be a matrix-valued function. Then, for each $n \in \mathbb{N}_0$ with $2n \leq \kappa$, let $P_{2n}^{[f]} : \mathcal{G} \setminus \mathbb{R} \to \mathbb{C}^{(n+2)q\times(n+2)q}$ be defined by*

$$P_{2n}^{[f]}(z) := \begin{bmatrix} H_n & \mathbf{b}_{2n}^{[f]}(z) \\ [\mathbf{b}_{2n}^{[f]}(z)]^* & \frac{f(z)-f^*(z)}{z-\overline{z}} \end{bmatrix} \tag{5}$$

where $\mathbf{b}_{2n}^{[f]}\colon \mathscr{G} \to \mathbb{C}^{(n+1)q\times q}$ *is defined by*

$$\mathbf{b}_{2n}^{[f]}(z) := R_{T_{q,n}}(z)\big[v_{q,n}f(z) - u_n\big]. \tag{6}$$

If $\kappa \geq 1$, *then, for each* $n \in \mathbb{N}_0$ *with* $2n+1 \leq \kappa$, *let* $P_{2n+1}^{[f]}\colon \mathscr{G} \setminus \mathbb{R} \to \mathbb{C}^{(n+2)q\times(n+2)q}$ *be given by*

$$P_{2n+1}^{[f]}(z) := \begin{bmatrix} H_{\alpha\triangleright n} & \mathbf{b}_{2n+1}^{[f]}(z) \\ [\mathbf{b}_{2n+1}^{[f]}(z)]^* & \frac{(z-\alpha)f(z)-[(z-\alpha)f(z)]^*}{z-\overline{z}} \end{bmatrix} \tag{7}$$

where $\mathbf{b}_{2n+1}^{[f]}\colon \mathscr{G} \to \mathbb{C}^{(n+1)q\times q}$ *is defined by*

$$\mathbf{b}_{2n+1}^{[f]}(z) := R_{T_{q,n}}(z)\big(v_{q,n}[(z-\alpha)f(z)] - (-\alpha u_n - y_{0,n})\big). \tag{8}$$

Furthermore, let $P_{-1}^{[f]}\colon \mathscr{G} \setminus \mathbb{R} \to \mathbb{C}^{q\times q}$ *be defined by*

$$P_{-1}^{[f]}(z) := \frac{(z-\alpha)f(z) - [(z-\alpha)f(z)]^*}{z-\overline{z}}.$$

With respect to the Stieltjes moment problem $\mathsf{MP}[[\alpha,\infty); (s_j)_{j=0}^m, \leq]$ if $\mathscr{G} = \mathbb{C} \setminus [\alpha,\infty)$, then the functions (5) and (7) are called the Potapov fundamental matrix-valued functions connected to the Stieltjes moment problem (generated by f). If these matrices are both non-negative Hermitian, then one says that the Potapov's fundamental matrix inequalities for the function f are fulfilled.

The following result indicates the key role of these functions in our concept.

Theorem 4.3 ([36, Theorem 6.20]) *Let* $\alpha \in \mathbb{R}$, *let* $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, *and let* $(s_j)_{j=0}^\kappa$ *be a sequence of complex* $q\times q$ *matrices. Let* $\mathscr{D}$ *be a discrete subset of* $\Pi_+ := \{z \in \mathbb{C}\colon \operatorname{Im} z \in (0,\infty)\}$ *and let* $S\colon \mathbb{C} \setminus [\alpha,\infty) \to \mathbb{C}^{q\times q}$ *be a holomorphic matrix-valued function. Then:*

(a) Let $n \in \mathbb{N}_0$ *be such that* $2n \leq \kappa$. *Then the following statements are equivalent:*

  (i) $S \in \mathscr{S}_{0,q;[\alpha,\infty)}[(s_j)_{j=0}^{2n}, \leq]$.

  (ii) $P_{2n-1}^{[S]}(z) \in \mathbb{C}_{\geq}^{(n+1)q\times(n+1)q}$ *and* $P_{2n}^{[S]}(z) \in \mathbb{C}_{\geq}^{(n+2)q\times(n+2)q}$ *for all* $z \in \Pi_+ \setminus \mathscr{D}$.

(b) Let $n \in \mathbb{N}_0$ *be such that* $2n+1 \leq \kappa$. *Then the following statements are equivalent:*

(iii) $S \in \mathscr{S}_{0,q;[\alpha,\infty)}[(s_j)_{j=0}^{2n+1}, \leq]$.

(iv) $\{P_{2n}^{[S]}(z), P_{2n+1}^{[S]}(z)\} \subseteq \mathbb{C}_{\geq}^{(n+2)q\times(n+2)q}$ *for all* $z \in \Pi_+ \setminus \mathscr{D}$.

We interpret now the meaning of each of the two FMIs. In this way, we are led to two truncated matricial Hamburger moment problems.

Remark 4.4 Let $n \in \mathbb{N}_0$ and let $(s_j)_{j=0}^{2n} \in \mathscr{H}_{q,2n}^{\geq}$. Consider a complex $q \times q$ matrix-valued function f holomorphic in Π_+. Then the matrix $P_{2n}^{[f]}(z)$ given via (5) is non-negative Hermitian for all $z \in \Pi_+$ if and only if f corresponds to a solution σ of the truncated matricial Hamburger moment problem $\mathsf{MP}[\mathbb{R}; (s_j)_{j=0}^{2n}, \leq]$, i. e. $f(z) = \int_{\mathbb{R}} (t-z)^{-1}\sigma(\mathrm{d}t)$ for all $z \in \Pi_+$ (see [47, 48]).

Remark 4.5 Let $n \in \mathbb{N}$ and let $(s_j)_{j=0}^{2n+1} \in \mathscr{K}_{q,2n+1,\alpha}^{\geq}$. Then the sequence $(\tilde{s}_j)_{j=0}^{2n}$ given by $\tilde{s}_j := -\alpha s_j + s_{j+1}$ belongs to $\mathscr{H}_{q,2n}^{\geq}$, i. e. the block Hankel matrix $\tilde{H}_n := [\tilde{s}_{j+k}]_{j,k=0}^{n}$ is non-negative Hermitian. Consider a complex $q \times q$ matrix-valued function f holomorphic in $\mathbb{C} \setminus [\alpha, \infty)$. Let $\tilde{f} : \Pi_+ \to \mathbb{C}^{q\times q}$ be defined by $\tilde{f}(z) := (z-\alpha)f(z) + s_0$. Since the matrix s_0 is non-negative Hermitian, we have

$$\frac{\tilde{f}(z) - [\tilde{f}(z)]^*}{z - \overline{z}} = \frac{(z-\alpha)f(z) - [(z-\alpha)f(z)]^*}{z - \overline{z}}.$$

With obvious notation, furthermore

$$-\alpha u_n - y_{0,n} = -\alpha \begin{bmatrix} 0_{q\times q} \\ -y_{0,n-1} \end{bmatrix} - \begin{bmatrix} s_0 \\ y_{1,n} \end{bmatrix} = \begin{bmatrix} -s_0 \\ \alpha y_{0,n-1} - y_{1,n} \end{bmatrix} = \begin{bmatrix} -s_0 \\ \alpha s_0 - s_1 \\ \alpha s_1 - s_2 \\ \vdots \\ \alpha s_{n-1} - s_n \end{bmatrix}$$

$$= \begin{bmatrix} -s_0 \\ -\tilde{s}_0 \\ -\tilde{s}_1 \\ \vdots \\ -\tilde{s}_{n-1} \end{bmatrix} = \begin{bmatrix} -s_0 \\ -\tilde{y}_{0,n-1} \end{bmatrix} = \begin{bmatrix} -s_0 \\ 0_{nq\times q} \end{bmatrix} + \begin{bmatrix} 0_{q\times q} \\ -\tilde{y}_{0,n-1} \end{bmatrix} = -v_{q,n}s_0 + \tilde{u}_n$$

holds true. Hence,

$$v_{q,n}[(z-\alpha)f(z)] - (-\alpha u_n - y_{0,n}) = v_{q,n}[(z-\alpha)f(z)] + v_{q,n}s_0 - \tilde{u}_n = v_{q,n}\tilde{f}(z) - \tilde{u}_n$$

and, in view of (8) and (6) with obvious notation, consequently $\mathbf{b}_{2n+1}^{[f]}(z) = \tilde{\mathbf{b}}_{2n}^{[\tilde{f}]}(z)$ for all $z \in \Pi_+$. Taking additionally into account $H_{\alpha \triangleright n} = \tilde{H}_n$, in view of (5) and (7) with obvious notation, then $P_{2n+1}^{[f]}(z) = \tilde{P}_{2n}^{[\tilde{f}]}(z)$ for all $z \in \Pi_+$ follows. According to Remark 4.4, thus the matrix $P_{2n+1}^{[f]}(z)$ is non-negative Hermitian for all $z \in \Pi_+$ if and only if $\tilde{f}$ corresponds to a solution $\tilde{\sigma}$ of the truncated matricial Hamburger moment problem $\mathsf{MP}[\mathbb{R}; (\tilde{s}_j)_{j=0}^{2n}, \leq]$, i. e.

$$(z-\alpha)f(z) = \tilde{f}(z) - s_0 = -s_0 + \int_{\mathbb{R}} \frac{1}{t-z}\tilde{\sigma}(\mathrm{d}t)$$

for all $z \in \Pi_+$.

Part (b) of Theorem 4.3 determines the direction of the subsequent considerations of this paper. Using it we want to derive a complete description of the set $\mathscr{S}_{0,q;[\alpha,\infty)}[(s_j)_{j=0}^{2n+1}, \leq]$.

In order to realize this we will apply the following result on non-negative Hermitian block matrices which can be found, e. g., in [19, Lemma 1.1.9].

Lemma 4.6 *Let $E \in \mathbb{C}^{(p+q)\times(p+q)}$ and let*

$$E = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{9}$$

be the block partition of E with $p \times p$ block a.

(a) The following statements are equivalent:

(i) $E \in \mathbb{C}_{\geq}^{(p+q)\times(p+q)}$.
(ii) $a \in \mathbb{C}_{\geq}^{p\times p}$, $\mathscr{R}(b) \subseteq \mathscr{R}(a)$, $c = b^*$, and $d - ca^\dagger b \in \mathbb{C}_{\geq}^{q\times q}$.

(b) The following statements are equivalent:

(iii) $E \in \mathbb{C}_{>}^{(p+q)\times(p+q)}$.
(iv) $a \in \mathbb{C}_{>}^{p\times p}$, $c = b^*$, and $d - ca^\dagger b \in \mathbb{C}_{>}^{q\times q}$.

Lemma 4.6 shows that the situation is much simpler if the left upper $p \times p$ block a in (9) is positive Hermitian. This case is called the non-degenerate case. In the first period V. P. Potapov and his associates studied mainly this situation. The key tool here was an appropriate factorization of the Schur complement (see Kovalishina [48], Dubovoj [18] or [19, Section 5.3]). Starting with V. K. Dubovoj [18, part IV] and continued by V. A. Bolotnikov [8, 9] the treatment of the degenerate case was handled. Having a closer view to part (a) of Lemma 4.6 it turns out to be useful to characterize the range condition $\mathscr{R}(b) \subseteq \mathscr{R}(a)$.

Lemma 4.7 *Let $a \in \mathbb{C}_{\mathrm{H}}^{p\times p}$ and $b \in \mathbb{C}^{p\times q}$. Then the following statements are equivalent:*

(i) $\mathscr{R}(b) \subseteq \mathscr{R}(a)$.
(ii) $aa^\dagger b = b$.
(iii) $(I_p - a^\dagger a)b = 0_{p\times q}$.

Proof (i)⇔(ii): This follows from Remark 17.1.

(ii)⇔(iii): In view of $a \in \mathbb{C}_{\mathrm{H}}^{p\times p}$ it follows from [19, Lemma 1.1.4] that $aa^\dagger = a^\dagger a$ which completes the proof.

An essential tool of our strategy is based on the use of a particular generalized inverse of matrices. Against this background it is important that it is possible to replace the Moore–Penrose inverse in Lemma 4.6 by this particular generalized inverse.

Lemma 4.8 *Let $a \in \mathbb{C}^{r\times p}$, let $b \in \mathbb{C}^{r\times q}$, and let $c \in \mathbb{C}^{s\times p}$ be such that $\mathscr{R}(b) \subseteq \mathscr{R}(a)$ and $\mathscr{N}(a) \subseteq \mathscr{N}(c)$. For each $x \in \mathbb{C}^{p\times r}$ fulfilling $axa = a$, then $cxb = ca^\dagger b$.*

Proof Because of $\mathscr{R}(b) \subseteq \mathscr{R}(a)$ we have $aa^\dagger b = b$, whereas $\mathscr{N}(a) \subseteq \mathscr{N}(c)$ implies $ca^\dagger a = c$. Consequently, $cxb = ca^\dagger axaa^\dagger b = ca^\dagger aa^\dagger b = ca^\dagger b$.

In the remaining part of this paper we are going to construct a full concept of constructing the general solution for the here considered system of FMIs of Potapov type. Our procedure is basically inspired by ideas of Yu. M. Dyukarev in the non-degenerate case and V. K. Dubovoj and V. A. Bolotnikov what concerns the degenerate situation. It should be mentioned that in [7, 23] the case $\alpha = 0$ was studied. However, in view of arbitrary $\alpha \in \mathbb{R}$ the concrete considerations are much more complicated.

In view of Theorems 2.4 and 2.6 we will suppose in our subsequent considerations that we work with a sequence $(s_j)_{j=0}^{2n+1} \in \mathscr{K}_{q,2n+1,\alpha}^{\geq,\mathrm{e}}$.

Remark 4.9 Let $\alpha \in \mathbb{R}$, let $n \in \mathbb{N}_0$, and let $(s_j)_{j=0}^{2n+1} \in \mathscr{K}_{q,2n+1,\alpha}^{\geq,\mathrm{e}}$. Further, let $\mathscr{G}$ be a non-empty subset of $\mathbb{C}$ and let $f : \mathscr{G} \to \mathbb{C}^{q\times q}$ be a matrix-valued function. Then:

(a) The matrix H_n is non-negative Hermitian, in particular we have $H_n^* = H_n$. Thus, if $z \in \mathscr{G}$, then Lemma 4.7 implies that the equations

$$H_n H_n^\dagger \mathbf{b}_{2n}^{[f]}(z) = \mathbf{b}_{2n}^{[f]}(z)$$

and

$$(I_{(n+1)q} - H_n^\dagger H_n)\mathbf{b}_{2n}^{[f]}(z) = 0_{(n+1)q\times q}$$

are equivalent.

(b) The matrix $H_{\alpha\triangleright n}$ is non-negative Hermitian. Thus, if $z \in \mathscr{G}$, then Lemma 4.7 implies that the equations

$$H_{\alpha\triangleright n} H_{\alpha\triangleright n}^\dagger \mathbf{b}_{2n+1}^{[f]}(z) = \mathbf{b}_{2n+1}^{[f]}(z)$$

and

$$(I_{(n+1)q} - H_{\alpha\triangleright n}^\dagger H_{\alpha\triangleright n})\mathbf{b}_{2n+1}^{[f]}(z) = 0_{(n+1)q\times q}$$

are equivalent.

Summarizing Theorem 4.3, Lemmas 4.6 and 4.7, and Remark 4.9 we obtain the following result which determines the direction of our further considerations.

Proposition 4.10 *Let* $\alpha \in \mathbb{R}$, *let* $n \in \mathbb{N}_0$, *and let* $(s_j)_{j=0}^{2n+1} \in \mathscr{K}_{q,2n+1,\alpha}^{\geq,\mathrm{e}}$. *Let* $S \in \mathbb{C} \setminus [\alpha, \infty) \to \mathbb{C}^{q\times q}$ *be holomorphic in* $\mathbb{C} \setminus [\alpha, \infty)$. *For* $z \in \mathbb{C} \setminus \mathbb{R}$ *let*

$$\Sigma_{2n}^{[S]}(z) := \frac{S(z) - [S(z)]^*}{z - \overline{z}} - \left[\mathbf{b}_{2n}^{[S]}(z)\right]^* H_n^\dagger \left[\mathbf{b}_{2n}^{[S]}(z)\right]$$

and

$$\Sigma_{2n+1}^{[S]}(z) := \frac{(z-\alpha)S(z) - [(z-\alpha)S(z)]^*}{z - \overline{z}} - \left[\mathbf{b}_{2n+1}^{[S]}(z)\right]^* H_{\alpha\triangleright n}^\dagger \left[\mathbf{b}_{2n+1}^{[S]}(z)\right].$$

Let $\mathscr{D}$ *be a discrete subset of* Π_+. *Then the following statements are equivalent:*

(i) $S \in \mathscr{S}_{0,q;[\alpha,\infty)}[(s_j)_{j=0}^{2n+1}, \leq]$.
(ii) For all $z \in \Pi_+ \setminus \mathscr{D}$ *the conditions*

$$(I_{(n+1)q} - H_n^\dagger H_n)\mathbf{b}_{2n}^{[S]}(z) = 0_{(n+1)q\times q}, \qquad \Sigma_{2n}^{[S]}(z) \in \mathbb{C}_{\geq}^{q\times q}$$

and

$$(I_{(n+1)q} - H_{\alpha\triangleright n}^\dagger H_{\alpha\triangleright n})\mathbf{b}_{2n+1}^{[S]}(z) = 0_{(n+1)q\times q}, \qquad \Sigma_{2n+1}^{[S]}(z) \in \mathbb{C}_{\geq}^{q\times q}$$

are satisfied.

A closer view to Proposition 4.10 shows that the situation is much more simpler if we have the so-called non-degenerate case that the block Hankel matrices H_n and $H_{\alpha\triangleright n}$ are both positive Hermitian. In this case the identities $H_n^\dagger H_n = I_{(n+1)q}$ and $H_{\alpha\triangleright n}^\dagger H_{\alpha\triangleright n} = I_{(n+1)q}$ are satisfied and the statement (ii) in Proposition 4.10 reduces to $\Sigma_{2n}^{[S]}(z) \in \mathbb{C}_{\geq}^{q\times q}$ and $\Sigma_{2n+1}^{[S]}(z) \in \mathbb{C}_{\geq}^{q\times q}$ for all $z \in \Pi_+ \setminus \mathscr{D}$.

5 Some Considerations on Block Hankel Matrices

The above considerations show that the moment problem under study is essentially governed by the interplay of several block Hankel matrices. For this reason, we will summarize in this section several important identities describing this interplay.

Remark 5.1 Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $q \times q$ matrices. If $n \in \mathbb{N}_0$ is such that $2n \leq \kappa$, then $H_n \in \mathbb{C}_{\mathrm{H}}^{(n+1)q\times(n+1)q}$ if and only if $\{s_j : j \in \mathbb{Z}_{0,2n}\} \subseteq \mathbb{C}_{\mathrm{H}}^{q\times q}$. Furthermore, if $\alpha \in \mathbb{R}$, if $\kappa \geq 1$, and if

$n \in \mathbb{N}_0$ is such that $2n+1 \leq \kappa$, then $\{H_n, H_{\alpha\triangleright n}\} \subseteq \mathbb{C}_{\mathrm{H}}^{(n+1)q\times(n+1)q}$ if and only if $\{s_j : j \in \mathbb{Z}_{0,2n+1}\} \subseteq \mathbb{C}_{\mathrm{H}}^{q\times q}$.

Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $p \times q$ matrices. For each $n \in \mathbb{N}_0$ with $2n+1 \leq \kappa$, let $K_n := [s_{j+k+1}]_{j,k=0}^{n}$ and, for each $n \in \mathbb{N}_0$ with $2n+2 \leq \kappa$, let $G_n := [s_{j+k+2}]_{j,k=0}^{n}$.

Remark 5.2 Let $n \in \mathbb{N}$ and let $(s_j)_{j=0}^{2n}$ be a sequence of complex $p \times q$ matrices. Then the block Hankel matrix H_n admits the block representations

$$H_n = \begin{bmatrix} H_{n-1} & y_{n,2n-1} \\ z_{n,2n-1} & s_{2n} \end{bmatrix}, \qquad H_n = \begin{bmatrix} s_0 & z_{1,n} \\ y_{1,n} & G_{n-1} \end{bmatrix}, \tag{10}$$

$$H_n = \begin{bmatrix} y_{0,n-1} & K_{n-1} \\ s_n & z_{n+1,2n} \end{bmatrix}, \qquad \text{and} \qquad H_n = \begin{bmatrix} z_{0,n-1} & s_n \\ K_{n-1} & y_{n+1,2n} \end{bmatrix}.$$

It seems to be useful to recall the well-known Lyapunov-type identities for block Hankel matrices. (These equations can be easily proved by straightforward calculation.)

Remark 5.3 Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $p \times q$ matrices.

(a) For each $n \in \mathbb{N}_0$ with $2n \leq \kappa$, then $H_n T_{q,n}^* - T_{p,n} H_n = u_n v_{q,n}^* - v_{p,n} w_n$ and $H_n T_{q,n} - T_{p,n}^* H_n = \mathfrak{u}_n \mathfrak{v}_{q,n}^* - \mathfrak{v}_{p,n} \mathfrak{w}_n$. In particular, if $p = q$ and if $s_j^* = s_j$ for each $j \in \mathbb{Z}_{0,\kappa}$, then $H_n T_{q,n}^* - T_{q,n} H_n = u_n v_{q,n}^* - v_{q,n} u_n^*$ and $H_n T_{q,n} - T_{q,n}^* H_n = \mathfrak{u}_n \mathfrak{v}_{q,n}^* - \mathfrak{v}_{q,n} \mathfrak{u}_n^*$ for each $n \in \mathbb{N}_0$ with $2n \leq \kappa$.

(b) For each $n \in \mathbb{N}_0$ with $2n+1 \leq \kappa$, we have $H_{\alpha\triangleright n} = -\alpha H_n + K_n$, $v_{p,n} v_{p,n}^* H_n = \left[R_{T_{p,n}}(\alpha)\right]^{-1} H_n - T_{p,n} H_{\alpha\triangleright n}$, and, in the case that $p = q$ and $s_j^* = s_j$ for each $j \in \mathbb{Z}_{0,\kappa}$ hold true, moreover $H_{\alpha\triangleright n} T_{q,n}^* - T_{q,n} H_{\alpha\triangleright n} = (-\alpha u_n - y_{0,n}) v_{q,n}^* - v_{q,n}(-\alpha u_n - y_{0,n})^*$ for each $n \in \mathbb{N}_0$ with $2n+1 \leq \kappa$.

(c) For each $n \in \mathbb{N}_0$ with $2n+2 \leq \kappa$, we have $H_{\alpha\triangleright n} T_{q,n} - T_{p,n}^* H_{\alpha\triangleright n} = (-\alpha \mathfrak{u}_n - y_{n+2,2n+2}) \mathfrak{v}_{q,n}^* - \mathfrak{v}_{p,n}(-\alpha \mathfrak{w}_n - z_{n+2,2n+2})$ and, in particular, if $p = q$ and if $s_j^* = s_j$ for each $j \in \mathbb{Z}_{0,\kappa}$, then $H_{\alpha\triangleright n} T_{q,n} - T_{q,n}^* H_{\alpha\triangleright n} = (-\alpha \mathfrak{u}_n - y_{n+2,2n+2}) \mathfrak{v}_{q,n}^* - \mathfrak{v}_{q,n}(-\alpha \mathfrak{u}_n - y_{n+2,2n+2})^*$ for each $n \in \mathbb{N}_0$ with $2n+2 \leq \kappa$.

(d) The equations $H_n v_{q,n} = y_{0,n}$ and $-T_{p,n} H_n v_{q,n} = u_n$ hold true for each $n \in \mathbb{N}_0$ with $2n \leq \kappa$.

Remark 5.4 Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and let $(s_j)_{j=0}^{\kappa}$ be a sequence of Hermitian complex $q \times q$ matrices. In view of Remark 5.3(a), it is readily checked that

$$\left[R_{T_{q,n}^*}(w)\right]^{-*} H_n T_{q,n}^* - T_{q,n} H_n \left[R_{T_{q,n}^*}(z)\right]^{-1} + (\overline{w} - z) T_{q,n} H_n T_{q,n}^*$$
$$= v_{q,n} v_{q,n}^* H_n T_{q,n}^* - T_{q,n} H_n v_{q,n} v_{q,n}^*$$

and

$$\left[R_{T_{q,n}}(z)\right]^{-1} H_n T_{q,n}^* - T_{q,n} H_n \left[R_{T_{q,n}}(\overline{z})\right]^{-*} = v_{q,n} v_{q,n}^* H_n T_{q,n}^* - T_{q,n} H_n v_{q,n} v_{q,n}^* \tag{11}$$

are fulfilled for every choice of $n \in \mathbb{N}_0$ with $2n \le \kappa$ and $w, z \in \mathbb{C}$.

Remark 5.5 Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N} \cup \{\infty\}$, and let $(s_j)_{j=0}^{\kappa}$ be a sequence of Hermitian complex $q \times q$ matrices. In view of Remarks 5.1 and 5.3, we have then

$$\begin{aligned}\left[R_{T_{q,n}}(z)\right]^{-1} H_{\alpha \triangleright n} T_{q,n}^* - T_{q,n} H_{\alpha \triangleright n} \left[R_{T_{q,n}}(\overline{z})\right]^{-*} \\ = v_{q,n} v_{q,n}^* H_n \left[R_{T_{q,n}}(\alpha)\right]^{-*} - \left[R_{T_{q,n}}(\alpha)\right]^{-1} H_n v_{q,n} v_{q,n}^*\end{aligned} \tag{12}$$

for all $n \in \mathbb{N}_0$ with $2n + 1 \le \kappa$ and all $z \in \mathbb{C}$. For all $n \in \mathbb{N}_0$ with $2n \le \kappa$, Remark 5.3 yields

$$\left[R_{T_{q,n}}(\alpha)\right]^{-1} H_n v_{q,n} = \left[R_{T_{q,n}}(\alpha)\right]^{-1} y_{0,n}, \left[R_{T_{q,n}}(\alpha)\right]^{-1} H_n v_{q,n} = \alpha u_n + y_{0,n},$$

$$z_{0,n} \left[R_{T_{q,n}}(\alpha)\right]^{-*} = v_{q,n}^* H_n \left[R_{T_{q,n}}(\alpha)\right]^{-*}, \text{ and } \alpha w_n + z_{0,n} = v_{q,n}^* H_n \left[R_{T_{q,n}}(\alpha)\right]^{-*}.$$

We will see that certain Schur complements play an essential role for our considerations. Let $L_0 := s_0$ and, for each $n \in \mathbb{N}$ with $2n \le \kappa$, furthermore $L_n := s_{2n} - z_{n,2n-1} H_{n-1}^{\dagger} y_{n,2n-1}$. For every choice of integers m and n with $0 \le m \le n \le \kappa - 1$, let $y_{\alpha \triangleright m,n} := \operatorname{col}(s_{\alpha \triangleright m+j})_{j=0}^{m-n}$ and $z_{\alpha \triangleright m,n} := \operatorname{row}(s_{\alpha \triangleright m+k})_{j=0}^{m-n}$. Let $L_{\alpha \triangleright 0} := s_{\alpha \triangleright 0}$ and, for each $n \in \mathbb{N}_0$ with $2n + 1 \le \kappa$, moreover $L_{\alpha \triangleright n} := s_{\alpha \triangleright 2n} - z_{\alpha \triangleright n,2n-1} H_{\alpha \triangleright n-1}^{\dagger} y_{\alpha \triangleright n,2n-1}$.

Remark 5.6 ([26, Remark 2.1]) Let $n \in \mathbb{N}$ and let $(s_j)_{j=0}^{2n}$ be a sequence of complex $q \times q$ matrices. In view of (10) and a well-known characterization of non-negative Hermitian block matrices (see, e. g., [19, Lemmas 1.1.9 and 1.1.7]), one can easily see that $(s_j)_{j=0}^{2n}$ belongs to $\mathscr{H}_{q,2n}^{\ge}$ if and only if the four conditions $(s_j)_{j=0}^{2(n-1)} \in \mathscr{H}_{q,2(n-1)}^{\ge}$, $\mathscr{R}(y_{n,2n-1}) \subseteq \mathscr{R}(H_{n-1})$, $s_{2n-1}^* = s_{2n-1}$, and $L_n \in \mathbb{C}_{\ge}^{q \times q}$ hold true. If $(s_j)_{j=0}^{2n}$ belongs to $\mathscr{H}_{q,2n}^{\ge}$, then $\operatorname{rank} H_n = \sum_{j=1}^{n} \operatorname{rank} L_j$.

Remark 5.7 Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N} \cup \{\infty\}$, and let $(s_j)_{j=0}^{\kappa} \in \mathscr{K}_{q,\kappa,\alpha}^{\ge}$. By virtue of Remark 5.1, one can easily check then that $s_j^* = s_j$ for each $k \in \mathbb{Z}_{0,\kappa}$ and $s_{\alpha \triangleright k}^* = s_{\alpha \triangleright k}$ for each $k \in \mathbb{Z}_{0,\kappa-1}$. Furthermore, for each $n \in \mathbb{N}_0$ with $2n \le \kappa$, from Remark 5.6 one can see that the matrices s_{2n}, H_n, and L_n are non-negative Hermitian and, for each $n \in \mathbb{N}_0$ with $2n + 1 \le \kappa$, the matrices $s_{\alpha \triangleright 2n}$, $H_{\alpha \triangleright n}$, and $L_{\alpha \triangleright n}$ are non-negative Hermitian as well.

Remark 5.8 Let $\alpha \in \mathbb{R}$ and let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$. According to the definition of $\mathscr{K}_{q,\kappa,\alpha}^{\ge,\mathrm{e}}$ and [27, Lemma 4.7], one can easily check that $\mathscr{K}_{q,\kappa,\alpha}^{\ge,\mathrm{e}} \subseteq \mathscr{K}_{q,\kappa,\alpha}^{\ge} \cap \mathscr{H}_{q,\kappa}^{\ge,\mathrm{e}}$. In particular, if $(s_j)_{j=0}^{\kappa}$ belongs to $\mathscr{K}_{q,\kappa,\alpha}^{\ge,\mathrm{e}}$, then, in view of Remark 2.3

for each $m \in \mathbb{Z}_{0,\kappa}$, the sequence $(s_j)_{j=0}^{m}$ belongs to $\mathscr{K}_{q,m,\alpha}^{\geq,\mathrm{e}} \cap \mathscr{H}_{q,\kappa}^{\geq,\mathrm{e}}$. Further, if $(s_j)_{j=0}^{\kappa} \in \mathscr{K}_{q,\kappa,\alpha}^{\geq,\mathrm{e}}$, then the definition of the sets $\mathscr{K}_{q,\kappa,\alpha}^{\geq,\mathrm{e}}$ and $\mathscr{H}_{q,\kappa}^{\geq,\mathrm{e}}$ and [27, Proposition 4.8 and Lemma 4.11] show that, for each $m \in \mathbb{Z}_{0,\kappa-1}$, the sequence $(s_{\alpha\triangleright j})_{j=0}^{m}$ belongs to $\mathscr{H}_{q,m}^{\geq,\mathrm{e}}$.

Remark 5.9 Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N} \cup \{\infty\}$, and let $(s_j)_{j=0}^{\kappa} \in \mathscr{K}_{q,\kappa,\alpha}^{\geq,\mathrm{e}}$. For each $m \in \mathbb{Z}_{0,\kappa}$, we have $(s_j)_{j=0}^{m} \in \mathscr{K}_{q,m,\alpha}^{\geq,\mathrm{e}}$. In view of Remarks 5.7 and 5.8, from [27, Lemmas 4.15 and 4.16] one can easily see that $\mathscr{N}(L_0) \subseteq \mathscr{N}(L_{\alpha\triangleright 0}) \subseteq \mathscr{N}(L_1) \subseteq \cdots \subseteq \mathscr{N}(L_n) \subseteq \mathscr{N}(L_{\alpha\triangleright n})$ and that $\mathscr{R}(L_0) \supseteq \mathscr{R}(L_{\alpha\triangleright 0}) \supseteq \mathscr{R}(L_1) \supseteq \cdots \supseteq \mathscr{R}(L_n) \supseteq \mathscr{R}(L_{\alpha\triangleright n})$ are valid for each $n \in \mathbb{N}_0$ with $2n+1 \leq \kappa$ and that $\mathscr{N}(L_0) \subseteq \mathscr{N}(L_{\alpha\triangleright 0}) \subseteq \mathscr{N}(L_1) \subseteq \cdots \subseteq \mathscr{N}(L_{\alpha\triangleright n-1}) \subseteq \mathscr{N}(L_n)$ as well as $\mathscr{R}(L_0) \supseteq \mathscr{R}(L_{\alpha\triangleright 0}) \supseteq \mathscr{R}(L_1) \supseteq \cdots \supseteq \mathscr{R}(L_{\alpha\triangleright n-1}) \supseteq \mathscr{R}(L_n)$ hold true for each $n \in \mathbb{N}$ with $2n \leq \kappa$.

The interplay between the null spaces and ranges of the sequences $L_0, L_1, L_2, \ldots$ and $L_{\alpha\triangleright 0}, L_{\alpha\triangleright 1}, L_{\alpha\triangleright 2}, \ldots$ which is described in Remark 5.9 is of extreme importance for our subsequent considerations.

6 Dubovoj Subspaces and Associated Generalized Inverses of Matrices

In this section, we explain one of the cornerstones of the concept which was developed in order to obtain a general method to solve Potapov's fundamental matrix inequality in the case of a degenerate information block. This method originates in the work of V. K. Dubovoj (see [18, part IV]) on his treatment of the matricial Schur problem. The basic feature of V. K. Dubovoj's method consists of appropriately splitting off the null space of the information block. This idea of V. K. Dubovoj was taken up and modified by V. A. Bolotnikov to handle the degenerate matricial Stieltjes problem. More precisely, he treated Problem $\mathsf{MP}[[0,\infty); (s_j)_{j=0}^{m}, \leq]$. V. A. Bolotnikov observed that V. K. Dubovoj's construction is essentially connected with the use of a particular type of generalized inverses of matrices. In this section, we strive for a systematic treatment of V. K. Dubovoj's method including V. A. Bolotnikov's modification taking into account the general case of arbitrary real α which requires to overcome some unforeseen technical difficulties.

Now we are going to present the machinery associated with the basic concept of this section. If $\mathscr{U}$ and $\mathscr{W}$ are subspaces of $\mathbb{C}^q$, then we write $\mathscr{U} + \mathscr{W}$ for the Minkowski sum of $\mathscr{U}$ and $\mathscr{W}$. To indicate that the Minkowski sum $\mathscr{U} + \mathscr{W}$ is a direct sum, i. e., that $\mathscr{U} \cap \mathscr{W} = \{0_{q\times 1}\}$ is fulfilled, we use the notation $\mathscr{U} \dotplus \mathscr{W}$. V. K. Dubovoj studied in [18] particular invariant subspaces to discuss the matricial Schur problem. Having in mind this, we give the following definition:

Definition 6.1 We call a subspace $\mathscr{D}$ of $\mathbb{C}^p$ a *Dubovoj subspace* corresponding to a given ordered pair (H, T) of complex $p \times p$ matrices if $T^*(\mathscr{D}) \subseteq \mathscr{D}$ and $\mathscr{N}(H) \dotplus \mathscr{D} = \mathbb{C}^p$ are fulfilled.

Now we are going to consider special Dubovoj subspaces adapted to our situation.

Notation 6.2 *Let* $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ *and let* $(s_j)_{j=0}^{\kappa}$ *be a sequence of complex* $p \times q$ *matrices. For each* $n \in \mathbb{N}_0$ *with* $2n \leq \kappa$, *let* $\mathscr{D}_n := \mathscr{R}(\operatorname{diag}(L_0, L_1, \ldots, L_n))$. *Furthermore, if* $\kappa \geq 1$, *then, for every choice of* $\alpha \in \mathbb{R}$ *and* $n \in \mathbb{N}_0$ *with* $2n+1 \leq \kappa$, *let* $\mathscr{D}_{\alpha \triangleright n} := \mathscr{R}(\operatorname{diag}(L_{\alpha \triangleright 0}, L_{\alpha \triangleright 1}, \ldots, L_{\alpha \triangleright n}))$.

Using the Kronecker delta, we set $V_{q,n} := [\delta_{j,k} I_q]_{\substack{j=0,\ldots,n\\ k=0,\ldots,n-1}}$ and $\mathfrak{V}_{q,n} := [\delta_{j,k+1} I_q]_{\substack{j=0,\ldots,n\\ k=0,\ldots,n-1}}$. In the following we often use the mapping $T_{q,n}$ given in (3).

Lemma 6.3 *Let* $\kappa \in \mathbb{N} \cup \{\infty\}$, *and let* $(s_j)_{j=0}^{\kappa} \in \mathscr{H}_{q,\kappa}^{\geq,\mathrm{e}}$. *For each* $n \in \mathbb{N}$ *with* $2n \leq \kappa$, *then* $T_{q,n}^*(\mathscr{D}_n) \subseteq \mathscr{D}_n$, $\mathfrak{V}_{q,n}^*(\mathscr{D}_n) \subseteq \mathscr{D}_{n-1}$, *and* $V_{q,n}^*(\mathscr{D}_n) \subseteq \mathscr{D}_{n-1}$.

Proof Because of $T_{q,n}^* \cdot \operatorname{diag}(L_0, L_1, \ldots, L_n) = \begin{bmatrix} 0_{nq \times q} & \operatorname{diag}(L_1, L_2, \ldots, L_n) \\ 0_{q \times q} & 0_{q \times nq} \end{bmatrix}$, we have $T_{q,n}^*(\mathscr{D}_n) \subseteq \mathscr{R}(\operatorname{diag}(L_1, L_2, \ldots, L_n, 0_{q \times q}))$. From Remark 5.6 we see that $\{L_0, L_1, \ldots, L_n\} \subseteq \mathbb{C}_{\geq}^{q \times q}$. Thus, using Remark 17.2 and [26, Proposition 2.13], we get $\mathscr{R}(L_j) = [\mathscr{N}(L_j)]^\perp \subseteq [\mathscr{N}(L_{j-1})]^\perp = \mathscr{R}(L_{j-1})$ for each $j \in \mathbb{Z}_{1,n}$, which implies $\mathscr{R}(\operatorname{diag}(L_1, L_2, \ldots, L_n, 0_{q \times q})) \subseteq \mathscr{R}(\operatorname{diag}(L_0, L_1, \ldots, L_{n-1}, 0_{q \times q})) \subseteq \mathscr{D}_n$. Consequently, $T_{q,n}^*(\mathscr{D}_n) \subseteq \mathscr{D}_n$. Obviously, we have $\mathfrak{V}_{q,n}^* \cdot \operatorname{diag}(L_0, L_1, \ldots, L_n) = [0_{nq \times q}, \operatorname{diag}(L_1, L_2, \ldots, L_n)]$ and $V_{q,n}^* \cdot \operatorname{diag}(L_0, L_1, \ldots, L_n) = (\operatorname{diag}(L_0, L_1, \ldots, L_{n-1}), 0_{nq \times q})$. Therefore, $\mathfrak{V}_{q,n}^*(\mathscr{D}_n) \subseteq \mathscr{R}(\operatorname{diag}(L_1, L_2, \ldots, L_n)) \subseteq \mathscr{R}(\operatorname{diag}(L_0, L_1, \ldots, L_{n-1})) = \mathscr{D}_{n-1}$ and $V_{q,n}^*(\mathscr{D}_n) \subseteq \mathscr{D}_{n-1}$.

If $n \in \mathbb{N}_0$ and if $(s_j)_{j=0}^{2n} \in \mathscr{H}_{q,2n}^{\geq,\mathrm{e}}$, then the existence of a Dubovoj subspace corresponding to $(H_n, T_{q,n})$ was proved in [8, Lemma 3.2], [60, Satz 1.24], and [22]. An explicit construction of such a subspace contains the following result:

Proposition 6.4 *Let* $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ *and let* $(s_j)_{j=0}^{\kappa} \in \mathscr{H}_{q,\kappa}^{\geq,\mathrm{e}}$. *For each* $n \in \mathbb{N}_0$ *with* $2n \leq \kappa$, *then* $\mathscr{D}_n$ *is a Dubovoj subspace for* $(H_n, T_{q,n})$, *where in particular* $\dim \mathscr{D}_n = \operatorname{rank} H_n$ *and* $\dim \mathscr{D}_n = \sum_{j=0}^{n} \operatorname{rank} L_j$.

Proof Let $n \in \mathbb{N}_0$ be such that $2n \leq \kappa$. Then $(s_j)_{j=0}^{2n} \in \mathscr{H}_{q,2n}^{\geq,\mathrm{e}}$. Furthermore, Lemma 6.3 shows that $T_{q,n}^*(\mathscr{D}_n) \subseteq \mathscr{D}_n$. Now we check inductively that

$$\dim \mathscr{D}_k = \dim \mathscr{R}(H_k) \tag{13}$$

holds true for each $k \in \mathbb{Z}_{0,n}$. Because of $L_0 = s_0 = H_0$, equation (13) is valid for $k = 0$. Thus, there is an $m \in \mathbb{Z}_{0,n}$ such (13) is fulfilled for each $k \in \mathbb{Z}_{0,m}$. We consider the case that $2(m+1) \leq \kappa$. Then from Notation 6.2 and (13) we obtain

$$\dim \mathscr{D}_{m+1} = \dim \mathscr{D}_m + \dim \mathscr{R}(L_{m+1}) = \dim \mathscr{R}(H_m) + \dim \mathscr{R}(L_{m+1}). \tag{14}$$

Since we know from Remark 5.6 that the right-hand side of (14) coincides with $\dim \mathscr{R}(H_{m+1})$, we see that (13) is true for $k = m + 1$ as well. Consequently, (13) holds for each $k \in \mathbb{Z}_{0,n}$. This implies $\dim \mathscr{D}_n + \dim \mathscr{N}(H_n) = \dim \mathbb{C}^{(n+1)q}$. Furthermore, (13) and Remark 5.6 show that $\dim \mathscr{D}_n = \sum_{j=0}^{n} \operatorname{rank} L_j$ holds true. It remains to prove that $\mathscr{D}_n \cap \mathscr{N}(H_n) \subseteq \{0_{(n+1)q\times 1}\}$. We consider an arbitrary $x \in \mathscr{D}_n \cap \mathscr{N}(H_n)$. Let $x = \operatorname{col}(x_j)_{j=0}^{n}$ be the $q \times 1$ block representation of x. Because of $x \in \mathscr{N}(H_n)$, from [26, Lemma A.2] we see that x_n belongs to $\mathscr{N}(L_n)$. Since we know from Remark 5.6 that L_n is non-negative Hermitian, we conclude $x_n \in \mathscr{R}(L_n)^\perp$. On the other hand, we have $x \in \mathscr{D}_n$, which implies $\operatorname{col}(x_j)_{j=0}^{n} \in \mathscr{R}(\operatorname{diag}(L_0, L_1, \dots, L_n))$ and, consequently, $x_n \in \mathscr{R}(L_n)$. Thus, $x_n \in \mathscr{R}(L_n) \cap \mathscr{R}(L_n)^\perp = \{0_{q\times 1}\}$, i. e., $x_n = 0_{q\times 1}$. Inductively, then $x_{n-j} = 0_{q\times 1}$ follows for each $j \in \mathbb{Z}_{0,n}$. Therefore, $\mathscr{D}_n \cap \mathscr{N}(H_n) \subseteq \{0_{(n+1)q\times 1}\}$.

For each $n \in \mathbb{N}_0$ and each $(s_j)_{j=0}^{2n} \in \mathscr{H}_{q,2n}^{\geq,\mathrm{e}}$, we will call $\mathscr{D}_n$ defined in Notation 6.2 the *canonical Dubovoj subspace corresponding to* $(H_n, T_{q,n})$.

In [60, Abschnitt 1.4], H. C. Thiele showed that $(s_j)_{j=0}^{2}$ given by $s_0 := 0$, $s_1 := 0$, and $s_2 := 1$ is a sequence belonging to $\mathscr{H}_{1,2}^{\geq}$ for which no Dubovoj subspace corresponding to $(H_1, T_{1,1})$ exists.

Remark 6.5 Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, let $(s_j)_{j=0}^{\kappa} \in \mathscr{H}_{q,\kappa}^{\geq,\mathrm{e}}$, and let $n \in \mathbb{N}_0$ be such that $2n \leq \kappa$. Let $\mathscr{D}_n$ be the canonical Dubovoj subspace corresponding to $(H_n, T_{q,n})$. In view of Proposition 6.4, one can easily see that $\dim \mathscr{D}_n \geq 1$ if and only if $s_0 \neq 0_{q\times q}$. Furthermore, it is readily checked that $\dim \mathscr{D}_n < (q + 1)n$ if and only if $\det H_n = 0$.

Remark 6.6 Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $(s_j)_{j=0}^{\kappa} \in \mathscr{K}_{q,\kappa,\alpha}^{\geq,\mathrm{e}}$. From Remark 5.8 and Proposition 6.4 one can see then that, for each $n \in \mathbb{N}_0$ with $2n \leq \kappa$, the subspace $\mathscr{D}_n$ of $\mathbb{C}^{(n+1)q}$ is a Dubovoj subspace corresponding to $(H_n, T_{q,n})$. Furthermore, if $\kappa \geq 1$, then for each $n \in \mathbb{N}_0$ with $2n + 1 \leq \kappa$, the subspace $\mathscr{D}_{\alpha\triangleright n}$ of $\mathbb{C}^{(n+1)q}$ is a Dubovoj subspace corresponding to $(H_{\alpha\triangleright n}, T_{q,n})$. To verify this one has to take into account that Remark 5.8 implies that for $m \in \mathbb{Z}_{0,\kappa-1}$ we have $(s_{\alpha\triangleright j})_{j=0}^{m} \in \mathscr{H}_{q,m}^{\geq,\mathrm{e}}$. This enables us to apply the preceding considerations to the sequence $(s_{\alpha\triangleright j})_{j=0}^{m}$ and the matrices $H_{\alpha\triangleright n}$ and $T_{q,n}$.

The following result contains important direct sum decompositions of the spaces $\mathbb{C}^{(n+1)q}$ and $\mathbb{C}^{nq}$, respectively.

Proposition 6.7 *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N} \cup \{\infty\}$, and let $(s_j)_{j=0}^{\kappa} \in \mathscr{K}_{q,\kappa,\alpha}^{\geq,\mathrm{e}}$. Then:*

(a) For each $n \in \mathbb{N}_0$ with $2n + 1 \leq \kappa$, then $T_{q,n}^*(\mathscr{D}_n) \subseteq \mathscr{D}_{\alpha\triangleright n} \subseteq \mathscr{D}_n$,

$$\mathscr{N}(H_n) \dotplus \mathscr{D}_n = \mathbb{C}^{(n+1)q}, \quad \text{and} \quad \mathscr{N}(H_{\alpha\triangleright n}) \dotplus \mathscr{D}_{\alpha\triangleright n} = \mathbb{C}^{(n+1)q}. \tag{15}$$

(b) For each $n \in \mathbb{N}$ with $2n \leq \kappa$, furthermore $\mathfrak{V}_{q,n}^*(\mathscr{D}_n) \subseteq \mathscr{D}_{\alpha\triangleright n-1}$, $V_{q,n}(\mathscr{D}_{\alpha\triangleright n-1}) \subseteq \mathscr{D}_n$,

$$\mathscr{N}(H_n) \dotplus \mathscr{D}_n = \mathbb{C}^{(n+1)q}, \quad \textit{and} \quad \mathscr{N}(H_{\alpha\triangleright n-1}) \dotplus \mathscr{D}_{\alpha\triangleright n-1} = \mathbb{C}^{nq}. \tag{16}$$

Proof According to Remark 5.8, we have $(s_j)_{j=0}^{m} \in \mathscr{K}_{q,m,\alpha}^{\geq,\mathrm{e}}$ for each $m \in \mathbb{Z}_{0,\kappa}$. Consequently, Remark 5.9 yields

$$\mathscr{R}(L_{j+1}) \subseteq \mathscr{R}(L_{\alpha\triangleright j}) \qquad \text{for each } j \in \mathbb{N}_0 \text{ with } 2j+2 \leq \kappa \tag{17}$$

and

$$\mathscr{R}(L_{\alpha\triangleright j}) \subseteq \mathscr{R}(L_j) \qquad \text{for each } j \in \mathbb{N}_0 \text{ with } 2j+1 \leq \kappa. \tag{18}$$

(a) Let $n \in \mathbb{N}_0$ be such that $2n+1 \leq \kappa$. Because of Remark 6.6 and the definition of a Dubovoj subspace, we get (15). In view of (18), we have $\mathscr{D}_{\alpha\triangleright n} = \mathscr{R}(\operatorname{diag}(L_{\alpha\triangleright j})_{j=0}^{n}) \subseteq \mathscr{R}(\operatorname{diag}(L_j)_{j=0}^{n})) = \mathscr{D}_n$. If $n = 0$, then $T_{q,n} = 0_{q\times q}$ and, consequently, $T_{q,n}^*(\mathscr{D}_n) \subseteq \mathscr{D}_{\alpha\triangleright n}$.

Now we assume that $n \geq 1$. In view of (17), then it is readily checked that

$$T_{q,n}^*(\mathscr{D}_n) = T_{q,n}^*\Big[\mathscr{R}\Big(\operatorname{diag}(L_j)_{j=0}^{n}\Big)\Big] \subseteq \mathscr{R}\Big(\operatorname{diag}\Big(\operatorname{diag}(L_{j+1})_{j=0}^{n-1}, 0_{q\times q}\Big)\Big) = \mathscr{D}_{\alpha\triangleright n}.$$

(b) Let $\kappa \geq 2$ and let $n \in \mathbb{N}$ such that $2n \leq \kappa$. Because of Remark 6.6 and the definition of a Dubovoj subspace, we get (16). From $\mathfrak{V}_{q,n}^* \cdot \operatorname{diag}(L_j)_{j=0}^{n} = [0_{nq\times q}, \operatorname{diag}(L_{j+1})_{j=0}^{n-1}]$ we conclude $\mathfrak{V}_{q,n}^*(\mathscr{D}_n) \subseteq \mathscr{R}(\operatorname{diag}(L_{j+1})_{j=0}^{n-1})$. Using (17), we obtain $\mathscr{R}(\operatorname{diag}(L_{j+1})_{j=0}^{n-1}) \subseteq \mathscr{R}(\operatorname{diag}(L_{\alpha\triangleright j})_{j=0}^{n-1}) = \mathscr{D}_{\alpha\triangleright n-1}$ and, consequently, $\mathfrak{V}_{q,n}^*(\mathscr{D}_n) \subseteq \mathscr{D}_{\alpha\triangleright n-1}$. Obviously, $V_{q,n} \cdot \operatorname{diag}(L_{\alpha\triangleright j})_{j=0}^{n-1} = \operatorname{diag}(\operatorname{diag}(L_{\alpha\triangleright j})_{j=0}^{n-1}, 0_{q\times q}) \cdot V_{q,n}$ and, hence, $V_{q,n}(\mathscr{D}_{\alpha\triangleright n-1}) = \mathscr{R}(\operatorname{diag}[\operatorname{diag}(L_{\alpha\triangleright j})_{j=0}^{n-1}, 0_{q\times q}])$. Since (18) implies $\mathscr{R}(\operatorname{diag}[\operatorname{diag}(L_{\alpha\triangleright j})_{j=0}^{n-1}, 0_{q\times q}]) \subseteq \mathscr{R}(\operatorname{diag}(L_j)_{j=0}^{n}) = \mathscr{D}_n$, we get $V_{q,n}(\mathscr{D}_{\alpha\triangleright n-1}) \subseteq \mathscr{D}_n$.

The proof of Proposition 6.7 shows why it is important for our subsequent considerations to assume that we start with a sequence $(s_j)_{j=0}^{\kappa} \in \mathscr{K}_{q,\kappa,\alpha}^{\geq,\mathrm{e}}$. This assumption enables us to apply Remark 5.9 which implies (17) and (18). Proposition 6.7 allows us to apply a particular generalized inverse of matrices. This will be explained now in detail.

Remark 6.8 If $A \in \mathbb{C}^{p\times q}$ and if $\mathscr{U}$ and $\mathscr{V}$ are subspaces of $\mathbb{C}^q$ and $\mathbb{C}^p$, respectively, such that $\mathscr{N}(A) \dotplus \mathscr{U} = \mathbb{C}^q$ and $\mathscr{R}(A) \dotplus \mathscr{V} = \mathbb{C}^p$ are fulfilled, then there is a unique $X \in \mathbb{C}^{q\times p}$ such that

$$AXA = A, \qquad XAX = X, \qquad \mathscr{R}(X) = \mathscr{U}, \qquad \text{and} \qquad \mathscr{N}(X) = \mathscr{V}$$

(see, e. g., [5, Chapter 2, Theorem 12(c)]), and we will use $A_{\mathscr{U},\mathscr{V}}^{(1,2)}$ to denote this matrix X.

Remark 6.9 Let $A \in \mathbb{C}^{q \times q}$ be invertible. Then $\mathscr{N}(A) = \{0_{q\times 1}\}$ and $\mathscr{R}(A) = \mathbb{C}^q$. Hence, $\mathscr{U} = \mathbb{C}^q$ and $\mathscr{V} = \{0_{q\times 1}\}$ is the only possible choice of subspaces $\mathscr{U}$ and $\mathscr{V}$ of $\mathbb{C}^q$ fulfilling $\mathscr{N}(A) \dotplus \mathscr{U} = \mathbb{C}^q$ and $\mathscr{R}(A) \dotplus \mathscr{V} = \mathbb{C}^q$. It is readily checked that for this choice $A^{(1,2)}_{\mathscr{U},\mathscr{V}} = A^{-1}$.

Remark 6.10 Let $A \in \mathbb{C}^{p\times q}$. Then $\mathscr{U} = [\mathscr{N}(A)]^\perp$ and $\mathscr{V} = [\mathscr{R}(A)]^\perp$ is a particular choice of subspaces $\mathscr{U}$ of $\mathbb{C}^q$ and $\mathscr{V}$ of $\mathbb{C}^p$ fulfilling $\mathscr{N}(A) \dotplus \mathscr{U} = \mathbb{C}^q$ and $\mathscr{R}(A) \dotplus \mathscr{V} = \mathbb{C}^p$. According to Remarks 6.8 and 17.2 then $X := A^{(1,2)}_{\mathscr{U},\mathscr{V}}$ satisfies $AXA = A$ and $XAX = X$ as well as $\mathscr{R}(X) = \mathscr{U} = \mathscr{R}(A^*)$ and $\mathscr{N}(X) = \mathscr{V} = \mathscr{N}(A^*)$. Consequently, $X = A^\dagger$ (see [5, Section 6, Ex. 38, p. 73]).

If A is a Hermitian complex $q \times q$ matrix and if $\mathscr{U}$ is a subspace of $\mathbb{C}^q$ with $\mathscr{N}(A) \dotplus \mathscr{U} = \mathbb{C}^q$, then $\mathscr{R}(A) \dotplus \mathscr{U}^\perp = \mathbb{C}^q$ and we will also write $A^-_{\mathscr{U}}$ for $A^{(1,2)}_{\mathscr{U},\mathscr{U}^\perp}$. (In Section 17, we turn our attention to the Hermitian case, in which a lot of special equations hold true.)

If $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and a sequence $(s_j)_{j=0}^{\kappa} \in \mathscr{H}^{\geq,\mathrm{e}}_{q,\kappa}$ are given, then, for each $n \in \mathbb{N}_0$ with $2n \leq \kappa$, let $H_n^- := H^{(1,2)}_{\mathscr{D}_n,\mathscr{D}_n^\perp}$, where $\mathscr{D}_n$ is given by Notation 6.2. (Note that Remark 5.8 shows that $\mathscr{K}^{\geq,\mathrm{e}}_{q,\kappa,\alpha} \subseteq \mathscr{H}^{\geq,\mathrm{e}}_{q,\kappa}$ holds true for each $\alpha \in \mathbb{R}$ and each $\kappa \in \mathbb{N}_0 \cup \{\infty\}$.) If $\alpha \in \mathbb{R}$, $\kappa \in \mathbb{N} \cup \{\infty\}$, and $(s_j)_{j=0}^{\kappa} \in \mathscr{K}^{\geq,\mathrm{e}}_{q,\kappa,\alpha}$ are given, then, for each $n \in \mathbb{N}_0$ with $2n + 1 \leq \kappa$, let $H^-_{\alpha\triangleright n} := H^{(1,2)}_{\mathscr{D}_{\alpha\triangleright n},\mathscr{D}^\perp_{\alpha\triangleright n}}$, where $\mathscr{D}_{\alpha\triangleright n}$ is also given by Notation 6.2. Now we are going to have a closer view on the matrices H_n^- and $H^-_{\alpha\triangleright n}$ and their interrelations.

Remark 6.11 Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and let $(s_j)_{j=0}^{\kappa} \in \mathscr{H}^{\geq,\mathrm{e}}_{q,\kappa}$. In view of Lemma 17.3, for each $n \in \mathbb{N}_0$ with $2n \leq \kappa$, then it is readily checked that $H_n^- \in \mathbb{C}_{\geq}^{(n+1)q\times(n+1)q}$,

$$H_n H_n^- H_n = H_n, \qquad \text{and} \qquad H_n^- H_n H_n^- = H_n^-. \tag{19}$$

Lemma 6.12 *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N} \cup \{\infty\}$, and let $(s_j)_{j=0}^{\kappa} \in \mathscr{K}^{\geq,\mathrm{e}}_{q,\kappa,\alpha}$, and let $n \in \mathbb{N}_0$ be such that $2n + 1 \leq \kappa$. Then the matrices H_n^- and $H^-_{\alpha\triangleright n}$ are both non-negative Hermitian and fulfill*

$$(H_n^-)^* = H_n^-, \qquad (H^-_{\alpha\triangleright n})^* = H^-_{\alpha\triangleright n}. \tag{20}$$

Furthermore, both equations in (19) as well as the following four identities hold true:

$$H_{\alpha\triangleright n} H^-_{\alpha\triangleright n} H_{\alpha\triangleright n} = H_{\alpha\triangleright n}, \qquad H^-_{\alpha\triangleright n} H_{\alpha\triangleright n} H^-_{\alpha\triangleright n} = H^-_{\alpha\triangleright n}, \tag{21}$$

$$H_n^- H_n H^-_{\alpha\triangleright n} = H^-_{\alpha\triangleright n}, \qquad \textit{and} \qquad H^-_{\alpha\triangleright n} H_n H_n^- = H^-_{\alpha\triangleright n}. \tag{22}$$

Proof The matrices H_n and $H_{\alpha\triangleright n}$ are both non-negative Hermitian. Lemma 17.3 yields then that H_n^- and $H^-_{\alpha\triangleright n}$ are both non-negative Hermitian and that the equations in (19), (20), and (21) hold true. In order to prove (22), we consider an arbitrary $n \in \mathbb{N}_0$ such that $2n + 1 \leq \kappa$. Taking into account $H_n^* = H_n$, Proposition 6.7, Lemma 17.3, and Remark 17.7, we conclude

$$\begin{aligned}\mathscr{R}(H_{\alpha\triangleright n}^-) &= \mathscr{R}(H_{\mathscr{D}_{\alpha\triangleright n}}^-) = \mathscr{D}_{\alpha\triangleright n}\\ &\subseteq \mathscr{D}_n = \mathscr{N}(I_{(n+1)q} - H_{\mathscr{D}_n}^- H_n) = \mathscr{N}(I_{(n+1)q} - H_n^- H_n).\end{aligned}$$

For every choice of $x \in \mathbb{C}^{(n+1)q}$, this implies $0 = (I_{(n+1)q} - H_n^- H_n)H_{\alpha\triangleright n}^- x$ and, consequently, $H_n^- H_n H_{\alpha\triangleright n}^- x = H_{\alpha\triangleright n}^- x$. Thus, the first equation in (22) is verified. Hence, $H_n^* = H_n$, (20), and the first equation in (22) yield $H_{\alpha\triangleright n}^- H_n H_n^- = (H_n^- H_n H_{\alpha\triangleright n}^-)^* = (H_{\alpha\triangleright n}^-)^* = H_{\alpha\triangleright n}^-$.

Remark 6.13 Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N} \cup \{\infty\}$, let $(s_j)_{j=0}^\kappa \in \mathscr{K}_{q,\kappa,\alpha}^{\geq,\mathrm{e}}$, and let $n \in \mathbb{N}_0$ be such that $2n + 1 \leq \kappa$. Then Remarks 5.8 and 5.7 yield $H_n^* = H_n$ and $H_{\alpha\triangleright n}^* = H_{\alpha\triangleright n}$. Thus,

$$H_n^\dagger H_n = H_n H_n^\dagger \qquad \text{and} \qquad H_{\alpha\triangleright n}^\dagger H_{\alpha\triangleright n} = H_{\alpha\triangleright n} H_{\alpha\triangleright n}^\dagger. \tag{23}$$

In view of Lemma 6.12, thus, $(I - H_n^\dagger H_n)H_n = 0$ and $(I - H_{\alpha\triangleright n}^\dagger H_{\alpha\triangleright n})H_{\alpha\triangleright n} = 0$ as well as

$$(I - H_n H_n^-)(I - H_n^\dagger H_n) = (I - H_n H_n^-)(I - H_n H_n^\dagger) = I - H_n H_n^-,$$

and, in view of (23), furthermore

$$(I - H_{\alpha\triangleright n} H_{\alpha\triangleright n}^-)(I - H_{\alpha\triangleright n}^\dagger H_{\alpha\triangleright n}) = I - H_{\alpha\triangleright n} H_{\alpha\triangleright n}^-.$$

Remark 6.14 Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N} \cup \{\infty\}$, and let $(s_j)_{j=0}^\kappa \in \mathscr{K}_{q,\kappa,\alpha}^{\geq,\mathrm{e}}$. Further, let $n \in \mathbb{N}_0$ be such that $2n + 1 \leq \kappa$. For every choice of $k \in \mathbb{N}_0$, Lemma 6.12, Proposition 6.7, and Lemma 17.9 yield then $H_n^- T_{q,n}^k (I_{(n+1)q} - H_n H_n^-) = 0$, $H_{\alpha\triangleright n}^- T_{q,n}^k (I_{(n+1)q} - H_n H_n^-) = 0$, and $H_{\alpha\triangleright n}^- T_{q,n}^k (I_{(n+1)q} - H_{\alpha\triangleright n} H_{\alpha\triangleright n}^-) = 0$. Moreover, $H_n^- T_{q,n}^k (I_{(n+1)q} - H_{\alpha\triangleright n} H_{\alpha\triangleright n}^-) = 0$ for all $k \in \mathbb{N}$.

Remark 6.15 Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N} \cup \{\infty\}$, and let $(s_j)_{j=0}^\kappa \in \mathscr{K}_{q,\kappa,\alpha}^{\geq,\mathrm{e}}$. In view of Remark 5.7, Lemma 6.12, and Remarks 4.1 and 6.14, it is readily checked that for each $n \in \mathbb{N}_0$ with $2n + 1 \leq \kappa$ and every choice of $\zeta \in \mathbb{C}$ and $k \in \mathbb{N}_0$, then $H_n^- R_{T_{q,n}}(\zeta) T_{q,n}^k (I_{(n+1)q} - H_n H_n^-) = 0$, $(I_{(n+1)q} - H_n^- H_n)(T_{q,n}^*)^k [R_{T_{q,n}}(\zeta)]^* H_n^- = 0$, $H_{\alpha\triangleright n}^- R_{T_{q,n}}(\zeta) T_{q,n}^k (I_{(n+1)q} - H_{\alpha\triangleright n} H_{\alpha\triangleright n}^-) = 0$, and $(I_{(n+1)q} - H_{\alpha\triangleright n}^- H_{\alpha\triangleright n})(T_{q,n}^*)^k [R_{T_{q,n}}(\zeta)]^* H_{\alpha\triangleright n}^- = 0$. Furthermore, for each $\zeta \in \mathbb{C}$ and each $k \in \mathbb{N}$, $H_n^- R_{T_{q,n}}(\zeta) T_{q,n}^k (I_{(n+1)q} - H_{\alpha\triangleright n} H_{\alpha\triangleright n}^-) = 0$ and $(I_{(n+1)q} - H_{\alpha\triangleright n}^- H_{\alpha\triangleright n})(T_{q,n}^*)^k [R_{T_{q,n}}(\zeta)]^* H_n^- = 0$.

Remark 6.16 Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N} \cup \{\infty\}$, and let $(s_j)_{j=0}^{\kappa} \in \mathcal{K}_{q,\kappa,\alpha}^{\geq,\mathrm{e}}$. In view of Remark 5.7, Lemma 6.12, and Remarks 4.1 and 6.15, it is readily checked that for each $n \in \mathbb{N}_0$ with $2n+1 \leq \kappa$ and every choice of $\zeta, \eta \in \mathbb{C}$, that $H_n^- R_{T_{q,n}}(\zeta)[R_{T_{q,n}^*}(\eta)]^{-*}(I - H_n H_n^-) = 0$, $(I - H_n^- H_n)[R_{T_{q,n}^*}(\eta)]^{-1}[R_{T_{q,n}}(\zeta)]^* H_n^- = 0$ as well as $H_{\alpha\triangleright n}^- R_{T_{q,n}}(\zeta)[R_{T_{q,n}^*}(\eta)]^{-*}(I - H_{\alpha\triangleright n} H_{\alpha\triangleright n}^-) = 0$, and $(I - H_{\alpha\triangleright n}^- H_{\alpha\triangleright n})[R_{T_{q,n}^*}(\eta)]^{-1}[R_{T_{q,n}}(\zeta)]^* H_{\alpha\triangleright n}^- = 0$ hold true.

Lemma 6.17 *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N} \cup \{\infty\}$, and let $(s_j)_{j=0}^{\kappa} \in \mathcal{K}_{q,\kappa,\alpha}^{\geq,\mathrm{e}}$. For each $n \in \mathbb{N}_0$ with $2n+1 \leq \kappa$, then $H_n^- R_{T_{q,n}}(\alpha)(v_{q,n}v_{q,n}^* H_n H_{\alpha\triangleright n}^- + T_{q,n}) = H_{\alpha\triangleright n}^-$ and*

$$H_{\alpha\triangleright n}^-\left[I_{(n+1)q} - H_n v_{q,n} v_{q,n}^* R_{T_{q,n}^*}(\alpha) H_n^-\right] = T_{q,n}^* R_{T_{q,n}^*}(\alpha) H_n^-. \tag{24}$$

Proof Let $n \in \mathbb{N}_0$ be such that $2n+1 \leq \kappa$. Because of $\mathcal{K}_{q,\kappa,\alpha}^{\geq,\mathrm{e}} \subseteq \mathcal{K}_{q,\kappa,\alpha}^{\geq}$, Remark 5.7 and Lemma 6.12 show that all the matrices H_n, H_n^-, $H_{\alpha\triangleright n}$, and $H_{\alpha\triangleright n}^-$ are Hermitian. Remark 5.3, Lemma 6.12, and Remark 6.15 yield

$$\begin{aligned} &H_n^- R_{T_{q,n}}(\alpha)(v_{q,n}v_{q,n}^* H_n H_{\alpha\triangleright n}^- + T_{q,n}) \\ &\quad = H_n^- R_{T_{q,n}}(\alpha)\left[\left(\left[R_{T_{q,n}}(\alpha)\right]^{-1} H_n - T_{q,n} H_{\alpha\triangleright n}\right) H_{\alpha\triangleright n}^- + T_{q,n}\right] \\ &\qquad = H_{\alpha\triangleright n}^- + H_n^- R_{T_{q,n}}(\alpha) T_{q,n}(I_{(n+1)q} - H_{\alpha\triangleright n} H_{\alpha\triangleright n}^-) = H_{\alpha\triangleright n}^-. \end{aligned}$$

This implies $[I_{(n+1)q} - H_n^- R_{T_{q,n}}(\alpha) v_{q,n} v_{q,n}^* H_n] H_{\alpha\triangleright n}^- = H_n^- R_{T_{q,n}}(\alpha) T_{q,n}$. Thus, in view of $\{H_n, H_n^-, H_{\alpha\triangleright n}, H_{\alpha\triangleright n}^-\} \subseteq \mathbb{C}_{\mathrm{H}}^{(n+1)q\times(n+1)q}$ and $[R_{T_{q,n}}(\alpha)]^* = R_{T_{q,n}^*}(\alpha)$, it follows (24).

Lemma 6.17 contains important coupling identities connecting the generalized inverses H_n^- and $H_{\alpha\triangleright n}^-$.

7 Construction of a Pair of Coupled $\tilde{J}_q$-Inner $2q \times 2q$ Matrix Polynomials

In this section we realize an important step on the way to the description of the solution set of the system of Potapov's FMI given in Theorem 4.3. We are going to construct a pair of $2q \times 2q$ matrix polynomials having the property that the linear fractional transformation generated by them can be used to parametrize the solution set of the first and second FMI of V. P. Potapov. After having done this we have to take into account the coupling between the two FMIs of V. P. Potapov. What concerns a former application of this strategy we refer to the papers [16, 17] in which the truncated matricial Hausdorff moment problem is studied by use of the FMI method of V. P. Potapov.

A key tool in our construction of $2q \times 2q$ matrix polynomials is an appropriate use of the particular generalized inverses H_n^- and $H_{\alpha \triangleright n}^-$ introduced in Section 6. These $2q \times 2q$ matrix polynomials will turn out to be closely related with the matrix

$$\tilde{J}_q := \begin{bmatrix} 0_{q\times q} & -\mathrm{i}I_q \\ \mathrm{i}I_q & 0_{q\times q} \end{bmatrix}.$$

which is obviously a $2q \times 2q$ signature matrix, i.e. $\tilde{J}_q^* = \tilde{J}_q$ and $\tilde{J}_q^2 = I_{2q}$ hold true. In particular we compute the right and left $\tilde{J}_q$-forms of the $2q \times 2q$ matrix polynomials under consideration. We modify the approach of V. A. Bolotnikov [7] who considered the particular case $\alpha = 0$. However, the calculations in the general case $\alpha \in \mathbb{R}$ are much more complicated.

Remark 7.1 For every choice of $A, B \in \mathbb{C}^{q\times q}$, we have $\left[\begin{smallmatrix} A \\ B \end{smallmatrix}\right]^*(-\tilde{J}_q)\left[\begin{smallmatrix} A \\ B \end{smallmatrix}\right] = -\mathrm{i}(B^*A - A^*B)$. In particular, $\left[\begin{smallmatrix} A \\ I_q \end{smallmatrix}\right]^*(-\tilde{J}_q)\left[\begin{smallmatrix} A \\ I_q \end{smallmatrix}\right] = 2\operatorname{Im} A$.

Remark 7.2 For $A \in \mathbb{C}^{q\times q}$ and $z \in \mathbb{C} \setminus \mathbb{R}$, we have

$$\frac{A - A^*}{z - \overline{z}} = \begin{bmatrix} A \\ I_q \end{bmatrix}^* \left(\frac{-\tilde{J}_q}{2\operatorname{Im} z}\right) \begin{bmatrix} A \\ I_q \end{bmatrix}.$$

In the following we often use the notation introduced in (3) and (4).

Remark 7.3 For each $n \in \mathbb{N}_0$ and each $A \in \mathbb{C}^{(n+1)q\times(n+1)q}$, by direct calculation the following identities can be verified:

$$[I_{(n+1)q}, A]\operatorname{diag}(v_{q,n}, v_{q,n})\tilde{J}_q = \mathrm{i}[A, -I_{(n+1)q}]\operatorname{diag}(v_{q,n}, v_{q,n}), \tag{25}$$

$$[A, -I_{(n+1)q}]\operatorname{diag}(v_{q,n}, v_{q,n})\tilde{J}_q = -\mathrm{i}[I_{(n+1)q}, A]\operatorname{diag}(v_{q,n}, v_{q,n}),$$

$$\tilde{J}_q[\operatorname{diag}(v_{q,n}, v_{q,n})]^*[I_{(n+1)q}, A]^* = -\mathrm{i}[\operatorname{diag}(v_{q,n}, v_{q,n})]^*[A, -I_{(n+1)q}]^*, \tag{26}$$

$$\tilde{J}_q[\operatorname{diag}(v_{q,n}, v_{q,n})]^*[A, -I_{(n+1)q}]^* = \mathrm{i}[\operatorname{diag}(v_{q,n}, v_{q,n})]^*[I_{(n+1)q}, A]^*,$$

$$[I_{(n+1)q}, A]\operatorname{diag}(v_{q,n}, v_{q,n})\tilde{J}_q[\operatorname{diag}(v_{q,n}, v_{q,n})]^*[I_{(n+1)q}, A]^* = \mathrm{i}(Av_{q,n}v_{q,n}^* - v_{q,n}v_{q,n}^*A^*), \tag{27}$$

$$\begin{aligned}[A, -I_{(n+1)q}]\operatorname{diag}(v_{q,n}, v_{q,n})\tilde{J}_q[\operatorname{diag}(v_{q,n}, v_{q,n})]^*[A, -I_{(n+1)q}]^* \\ = \mathrm{i}(Av_{q,n}v_{q,n}^* - v_{q,n}v_{q,n}^*A^*),\end{aligned} \tag{28}$$

and

$$\begin{aligned}[I_{(n+1)q}, A]\operatorname{diag}(v_{q,n}, v_{q,n})[\operatorname{diag}(v_{q,n}, v_{q,n})]^*[A, -I_{(n+1)q}]^* \\ = -(Av_{q,n}v_{q,n}^* - v_{q,n}v_{q,n}^*A^*).\end{aligned} \tag{29}$$

Now we introduce a $2q \times 2q$ matrix polynomial which will turn out to be intimately related with the first Potapov-type FMI occurring in Theorem 4.3.

Remark 7.4 Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $(s_j)_{j=0}^{\kappa} \in \mathscr{H}_{q,\kappa}^{\geq,\mathrm{e}}$. For each $n \in \mathbb{N}_0$ with $2n \leq \kappa$, Remark 4.1 shows that $U_{n,\alpha} : \mathbb{C} \to \mathbb{C}^{2q \times 2q}$ defined by

$$U_{n,\alpha}(\zeta) := I_{2q} + (\zeta - \alpha)[\operatorname{diag}(v_{q,n}, v_{q,n})]^*[T_{q,n}H_n, -I_{(n+1)q}]^* \\ \times R_{T_{q,n}^*}(\zeta)H_n^- R_{T_{q,n}}(\alpha)[I_{(n+1)q}, T_{q,n}H_n] \operatorname{diag}(v_{q,n}, v_{q,n}) \tag{30}$$

is a matrix polynomial of degree not greater than $n+1$, where $H_n^* = H_n$ implies that, for each $\zeta \in \mathbb{C}$, the matrix $U_{n,\alpha}(\zeta)$ admits the block representation

$$U_{n,\alpha}(\zeta) = \begin{bmatrix} A_n(\zeta) & B_n(\zeta) \\ C_n(\zeta) & D_n(\zeta) \end{bmatrix} \tag{31}$$

with

$$A_n(\zeta) := I_q + (\zeta - \alpha)v_{q,n}^* H_n T_{q,n}^* R_{T_{q,n}^*}(\zeta) H_n^- R_{T_{q,n}}(\alpha) v_{q,n}, \tag{32}$$

$$B_n(\zeta) := \quad + (\zeta - \alpha)v_{q,n}^* H_n T_{q,n}^* R_{T_{q,n}^*}(\zeta) H_n^- R_{T_{q,n}}(\alpha) T_{q,n} H_n v_{q,n}, \tag{33}$$

$$C_n(\zeta) := \quad - (\zeta - \alpha)v_{q,n}^* R_{T_{q,n}^*}(\zeta) H_n^- R_{T_{q,n}}(\alpha) v_{q,n}, \tag{34}$$

$$D_n(\zeta) := I_q - (\zeta - \alpha)v_{q,n}^* R_{T_{q,n}^*}(\zeta) H_n^- R_{T_{q,n}}(\alpha) T_{q,n} H_n v_{q,n}. \tag{35}$$

In the next step we compute the left and right $\tilde{J}_q$-forms of the $2q \times 2q$ matrix polynomial introduced in Remark 7.4.

Lemma 7.5 *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $(s_j)_{j=0}^{\kappa} \in \mathscr{K}_{q,\kappa,\alpha}^{\geq,\mathrm{e}}$. For all of $n \in \mathbb{N}_0$ with $2n \leq \kappa$ and all $z, w \in \mathbb{C}$, the function $U_{n,\alpha} : \mathbb{C} \to \mathbb{C}^{2q \times 2q}$ given by (30) fulfills*

$$\tilde{J}_q - U_{n,\alpha}(z)\tilde{J}_q U_{n,\alpha}^*(w) = -\mathrm{i}(z - \overline{w})[\operatorname{diag}(v_{q,n}, v_{q,n})]^*[T_{q,n}H_n, -I_{(n+1)q}]^* R_{T_{q,n}^*}(z) H_n^- \\ \times \left[R_{T_{q,n}^*}(w)\right]^* [T_{q,n}H_n, -I_{(n+1)q}] \operatorname{diag}(v_{q,n}, v_{q,n}). \tag{36}$$

Proof Let $n \in \mathbb{N}_0$ be such that $2n \leq \kappa$ and let $z, w \in \mathbb{C}$. Remark 5.7 yields $H_n^* = H_n$. Lemma 6.12 yields (20) and (19). Using (30), $\tilde{J}_q^2 = I_{2q}$, and (20), we conclude

$$
\begin{aligned}
&\tilde{J}_q - U_{n,\alpha}(z)\tilde{J}_q U_{n,\alpha}^*(w) \\
&= \tilde{J}_q - \Big\{ I_{2q} + (z-\alpha)[\operatorname{diag}(v_{q,n}, v_{q,n})]^* [T_{q,n}H_n, -I_{(n+1)q}]^* R_{T_{q,n}^*}(z) H_n^- \\
&\quad \times R_{T_{q,n}}(\alpha)[I_{(n+1)q}, T_{q,n}H_n] \operatorname{diag}(v_{q,n}, v_{q,n}) \Big\} \tilde{J}_q \Big\{ I_{2q} + (\overline{w}-\alpha)[\operatorname{diag}(v_{q,n}, v_{q,n})]^* \\
&\quad \times [I_{(n+1)q}, T_{q,n}H_n]^* \Big[R_{T_{q,n}}(\alpha)\Big]^* H_n^- \Big[R_{T_{q,n}^*}(w)\Big]^* [T_{q,n}H_n, -I_{(n+1)q}] \operatorname{diag}(v_{q,n}, v_{q,n}) \Big\} \\
&= S_1(z) + S_2(w) + S_3(z,w)
\end{aligned} \tag{37}
$$

where

$$
\begin{aligned}
S_1(z) &:= -(z-\alpha)[\operatorname{diag}(v_{q,n}, v_{q,n})]^* [T_{q,n}H_n, -I_{(n+1)q}]^* R_{T_{q,n}^*}(z) H_n^- \\
&\qquad \times R_{T_{q,n}}(\alpha)[I_{(n+1)q}, T_{q,n}H_n] \operatorname{diag}(v_{q,n}, v_{q,n}) \tilde{J}_q, \qquad (38) \\
S_2(w) &:= -(\overline{w}-\alpha)\tilde{J}_q[\operatorname{diag}(v_{q,n}, v_{q,n})]^* [I_{(n+1)q}, T_{q,n}H_n]^* \big[R_{T_{q,n}}(\alpha)\big]^* H_n^- \\
&\qquad \times \Big[R_{T_{q,n}^*}(w)\Big]^* [T_{q,n}H_n, -I_{(n+1)q}] \operatorname{diag}(v_{q,n}, v_{q,n}), \qquad (39)
\end{aligned}
$$

and

$$
\begin{aligned}
&S_3(z,w) \\
&:= -(z-\alpha)(\overline{w}-\alpha)[\operatorname{diag}(v_{q,n}, v_{q,n})]^* [T_{q,n}H_n, -I_{(n+1)q}]^* R_{T_{q,n}^*}(z) H_n^- R_{T_{q,n}}(\alpha) \\
&\quad \times [I_{(n+1)q}, T_{q,n}H_n] \operatorname{diag}(v_{q,n}, v_{q,n}) \tilde{J}_q [\operatorname{diag}(v_{q,n}, v_{q,n})]^* [I_{(n+1)q}, T_{q,n}H_n]^* \\
&\qquad \times \big[R_{T_{q,n}}(\alpha)\big]^* H_n^- \Big[R_{T_{q,n}^*}(w)\Big]^* [T_{q,n}H_n, -I_{(n+1)q}] \operatorname{diag}(v_{q,n}, v_{q,n}).
\end{aligned} \tag{40}
$$

Because of (38), (39), (25), (26), and Remark 4.1, we get then

$$
\begin{aligned}
S_1(z) &= -\mathrm{i}(z-\alpha)[\operatorname{diag}(v_{q,n}, v_{q,n})]^* [T_{q,n}H_n, -I_{(n+1)q}]^* R_{T_{q,n}^*}(z) \\
&\qquad \times H_n^- R_{T_{q,n}}(\alpha) \Big[R_{T_{q,n}^*}(w)\Big]^{-*} \Big[R_{T_{q,n}^*}(w)\Big]^* [T_{q,n}H_n, -I_{(n+1)q}] \operatorname{diag}(v_{q,n}, v_{q,n}),
\end{aligned} \tag{41}
$$

$$
\begin{aligned}
S_2(w) &= \mathrm{i}(\overline{w}-\alpha)[\operatorname{diag}(v_{q,n}, v_{q,n})]^* [T_{q,n}H_n, -I_{(n+1)q}]^* R_{T_{q,n}^*}(z) \Big[R_{T_{q,n}^*}(z)\Big]^{-1} \\
&\qquad \times \big[R_{T_{q,n}}(\alpha)\big]^* H_n^- \Big[R_{T_{q,n}^*}(w)\Big]^* [T_{q,n}H_n, -I_{(n+1)q}] \operatorname{diag}(v_{q,n}, v_{q,n}),
\end{aligned} \tag{42}
$$

and, according to (40), (27), and $H_n^* = H_n$, we have

$$S_3(z,w) = -\mathrm{i}(z-\alpha)(\overline{w}-\alpha)[\operatorname{diag}(v_{q,n}, v_{q,n})]^*[T_{q,n}H_n, -I_{(n+1)q}]^* R_{T_{q,n}^*}(z) H_n^- \\ \times R_{T_{q,n}}(\alpha)(T_{q,n}H_n v_{q,n} v_{q,n}^* - v_{q,n} v_{q,n}^* H_n T_{q,n}^*)\left[R_{T_{q,n}}(\alpha)\right]^* \\ \times H_n^- \left[R_{T_{q,n}^*}(w)\right]^* [T_{q,n}H_n, -I_{(n+1)q}] \operatorname{diag}(v_{q,n}, v_{q,n}). \tag{43}$$

In view of $H_n^* = H_n$, $R_{T_{q,n}^*}(\alpha) = [R_{T_{q,n}}(\alpha)]^*$, (11), (249), and (248), it follows

$$(z-\alpha)(\overline{w}-\alpha) R_{T_{q,n}}(\alpha)(T_{q,n}H_n v_{q,n} v_{q,n}^* - v_{q,n} v_{q,n}^* H_n T_{q,n}^*)\left[R_{T_{q,n}}(\alpha)\right]^* \\ = (z-\alpha)(\overline{w}-\alpha) R_{T_{q,n}}(\alpha)\Big(T_{q,n}H_n[R_{T_{q,n}}(\alpha)]^{-*} - \left[R_{T_{q,n}}(\alpha)\right]^{-1} H_n T_{q,n}^*\Big)\left[R_{T_{q,n}}(\alpha)\right]^* \\ = (z-\alpha)(\overline{w}-\alpha) R_{T_{q,n}}(\alpha) T_{q,n} H_n - (z-\alpha)(\overline{w}-\alpha) H_n T_{q,n}^* \left[R_{T_{q,n}}(\alpha)\right]^* \\ = -(z-\alpha)\left(R_{T_{q,n}}(\alpha)\left[R_{T_{q,n}^*}(w)\right]^{-*} - I_{(n+1)q}\right) H_n \\ + (\overline{w}-\alpha) H_n \left(\left[R_{T_{q,n}^*}(z)\right]^{-1}\left[R_{T_{q,n}}(\alpha)\right]^* - I_{(n+1)q}\right) \\ = -(z-\alpha) R_{T_{q,n}}(\alpha)\left[R_{T_{q,n}^*}(w)\right]^{-*} H_n \\ + (\overline{w}-\alpha) H_n \left[R_{T_{q,n}^*}(z)\right]^{-1}\left[R_{T_{q,n}}(\alpha)\right]^* + (z-\overline{w}) H_n.$$

Consequently, from (43) we get then

$$S_3(z,w) = -\mathrm{i}[\operatorname{diag}(v_{q,n}, v_{q,n})]^*[T_{q,n}H_n, -I_{(n+1)q}]^* R_{T_{q,n}^*}(z) H_n^- \\ \times \Big\{ -(z-\alpha) R_{T_{q,n}}(\alpha)\left[R_{T_{q,n}^*}(w)\right]^{-*} H_n + (\overline{w}-\alpha) H_n \left[R_{T_{q,n}^*}(z)\right]^{-1} [R_{T_{q,n}}(\alpha)]^* \\ + (z-\overline{w}) H_n \Big\} H_n^- \left[R_{T_{q,n}^*}(w)\right]^* [T_{q,n}H_n, -I_{(n+1)q}] \operatorname{diag}(v_{q,n}, v_{q,n}). \tag{44}$$

The combination of (37), (41), (42), and (44) provides

$$\tilde{J}_q - U_{n,\alpha}(z)\tilde{J}_q U_{n,\alpha}^*(w) = -\mathrm{i}[\operatorname{diag}(v_{q,n}, v_{q,n})]^*[T_{q,n}H_n, -I_{(n+1)q}]^* R_{T_{q,n}^*}(z) S(z,w) \\ \times \left[R_{T_{q,n}^*}(w)\right]^* [T_{q,n}H_n, -I_{(n+1)q}] \operatorname{diag}(v_{q,n}, v_{q,n}) \tag{45}$$

where

$$S(z,w) := (z-\alpha) H_n^- R_{T_{q,n}}(\alpha)\left[R_{T_{q,n}^*}(w)\right]^{-*}$$

$$- (\overline{w} - \alpha)\Big[R_{T^*_{q,n}}(z)\Big]^{-1}\Big[R_{T_{q,n}}(\alpha)\Big]^* H_n^- - (z-\alpha)H_n^- R_{T_{q,n}}(\alpha)\Big[R_{T^*_{q,n}}(w)\Big]^{-*} H_n H_n^-$$
$$+ (\overline{w} - \alpha)H_n^- H_n\Big[R_{T^*_{q,n}}(z)\Big]^{-1}\Big[R_{T_{q,n}}(\alpha)\Big]^* H_n^- + (z-\overline{w})H_n^- H_n H_n^-.$$

Thus, Remarks 6.16 and 6.11 yield $S(z, w) = (z-\overline{w})H_n^-$. Hence, (45) implies (36).

Lemma 7.6 *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $(s_j)_{j=0}^{\kappa} \in \mathscr{K}^{\geq,\mathrm{e}}_{q,\kappa,\alpha}$. Let $U_{n,\alpha}\colon \mathbb{C} \to \mathbb{C}^{2q\times 2q}$ be defined by (30). For each $n \in \mathbb{N}_0$ with $2n \leq \kappa$ and all $z, w \in \mathbb{C}$, then*

$$\tilde{J}_q - U^*_{n,\alpha}(w)\tilde{J}_q U_{n,\alpha}(z)$$
$$= \mathrm{i}(\overline{w}-z)[\operatorname{diag}(v_{q,n}, v_{q,n})]^*[I_{(n+1)q}, T_{q,n}H_n]^*\Big[R_{T_{q,n}}(\alpha)\Big]^* H_n^-\Big[R_{T^*_{q,n}}(w)\Big]^*\Big[R_{T_{q,n}}(\alpha)\Big]^{-1}$$
$$\times H_n\Big[R_{T^*_{q,n}}(\alpha)\Big]^{-1} R_{T^*_{q,n}}(z)H_n^- R_{T_{q,n}}(\alpha)[I_{(n+1)q}, T_{q,n}H_n]\operatorname{diag}(v_{q,n}, v_{q,n}). \tag{46}$$

Proof Let $n \in \mathbb{N}_0$ be such that $2n \leq \kappa$. Since $(s_j)_{j=0}^{\kappa} \in \mathscr{K}^{\geq,\mathrm{e}}_{q,\kappa,\alpha} \subseteq \mathscr{K}^{\geq}_{q,\kappa,\alpha}$ holds true, Remark 5.7 yields $H_n^* = H_n$ and from Lemma 6.12 we get (20). Now let $z, w \in \mathbb{C}$. In view of (30), $\tilde{J}_q^2 = I_{2q}$, and $H^*_{\alpha\triangleright n} = H_{\alpha\triangleright n}$, we conclude then

$$\begin{aligned}
&\tilde{J}_q - U^*_{n,\alpha}(w)\tilde{J}_q U_{n,\alpha}(z)\\
&= \tilde{J}_q - \Big\{I_{2q} + (\overline{w}-\alpha)[\operatorname{diag}(v_{q,n}, v_{q,n})]^*[I_{(n+1)q}, T_{q,n}H_n]^*\\
&\qquad \times \Big[R_{T_{q,n}}(\alpha)\Big]^* H_n^-\Big[R_{T^*_{q,n}}(w)\Big]^*[T_{q,n}H_n, -I_{(n+1)q}]\operatorname{diag}(v_{q,n}, v_{q,n})\Big\}\tilde{J}_q\\
&\quad \times \Big\{I_{2q} + (z-\alpha)[\operatorname{diag}(v_{q,n}, v_{q,n})]^*[T_{q,n}H_n, -I_{(n+1)q}]^* R_{T^*_{q,n}}(z)\\
&\qquad \times H_n^- R_{T_{q,n}}(\alpha)[I_{(n+1)q}, T_{q,n}H_n]\operatorname{diag}(v_{q,n}, v_{q,n})\Big\}\\
&= S_1(w) + S_2(z) + S_3(z, w)
\end{aligned} \tag{47}$$

where

$$S_1(w) := -(\overline{w}-\alpha)[\operatorname{diag}(v_{q,n}, v_{q,n})]^*[I_{(n+1)q}, T_{q,n}H_n]^*\Big[R_{T_{q,n}}(\alpha)\Big]^* H_n^-$$
$$\times \Big[R_{T^*_{q,n}}(w)\Big]^*[T_{q,n}H_n, -I_{(n+1)q}]\operatorname{diag}(v_{q,n}, v_{q,n})\tilde{J}_q,$$
$$S_2(z) := -(z-\alpha)\tilde{J}_q[\operatorname{diag}(v_{q,n}, v_{q,n})]^*[T_{q,n}H_n, -I_{(n+1)q}]^* R_{T^*_{q,n}}(z)$$
$$\times H_n^- R_{T_{q,n}}(\alpha)[I_{(n+1)q}, T_{q,n}H_n]\operatorname{diag}(v_{q,n}, v_{q,n}),$$

and

$$\begin{aligned}
&S_3(z,w)\\
&:= -(\overline{w}-\alpha)(z-\alpha)[\operatorname{diag}(v_{q,n},v_{q,n})]^*[I_{(n+1)q},T_{q,n}H_n]^*\left[R_{T_{q,n}}(\alpha)\right]^*H_n^-\left[R_{T_{q,n}^*}(w)\right]^*\\
&\quad\times[T_{q,n}H_n,-I_{(n+1)q}]\operatorname{diag}(v_{q,n},v_{q,n})\tilde{J}_q[\operatorname{diag}(v_{q,n},v_{q,n})]^*[T_{q,n}H_n,-I_{(n+1)q}]^*\\
&\qquad\times R_{T_{q,n}^*}(z)H_n^-R_{T_{q,n}}(\alpha)[I_{(n+1)q},T_{q,n}H_n]\operatorname{diag}(v_{q,n},v_{q,n}).
\end{aligned}$$

Keeping in mind the Remarks 7.3 and 4.1, we have

$$\begin{aligned}
S_1(w) &= \mathrm{i}(\overline{w}-\alpha)[\operatorname{diag}(v_{q,n},v_{q,n})]^*[I_{(n+1)q},T_{q,n}H_n]^*\left[R_{T_{q,n}}(\alpha)\right]^*H_n^-\\
&\quad\times\left[R_{T_{q,n}^*}(w)\right]^*\left[R_{T_{q,n}}(\alpha)\right]^{-1}R_{T_{q,n}}(\alpha)[I_{(n+1)q},T_{q,n}H_n]\operatorname{diag}(v_{q,n},v_{q,n}),
\end{aligned} \tag{48}$$

$$\begin{aligned}
S_2(z) &= -\mathrm{i}(z-\alpha)[\operatorname{diag}(v_{q,n},v_{q,n})]^*[I_{(n+1)q},T_{q,n}H_n]^*\left[R_{T_{q,n}}(\alpha)\right]^*\left[R_{T_{q,n}}(\alpha)\right]^{-*}\\
&\quad\times R_{T_{q,n}^*}(z)H_n^-R_{T_{q,n}}(\alpha)[I_{(n+1)q},T_{q,n}H_n]\operatorname{diag}(v_{q,n},v_{q,n}),
\end{aligned} \tag{49}$$

and, by virtue of (28) and $H_n^*=H_n$, furthermore

$$\begin{aligned}
S_3(z,w) &= -\mathrm{i}(\overline{w}-\alpha)(z-\alpha)[\operatorname{diag}(v_{q,n},v_{q,n})]^*[I_{(n+1)q},T_{q,n}H_n]^*\left[R_{T_{q,n}}(\alpha)\right]^*H_n^-\\
&\quad\times\left[R_{T_{q,n}^*}(w)\right]^*(T_{q,n}H_nv_{q,n}v_{q,n}^*-v_{q,n}v_{q,n}^*H_nT_{q,n}^*)R_{T_{q,n}^*}(z)\\
&\quad\times H_n^-R_{T_{q,n}}(\alpha)[I_{(n+1)q},T_{q,n}H_n]\operatorname{diag}(v_{q,n},v_{q,n}).
\end{aligned} \tag{50}$$

Remark 17.10 shows that

$$(z-\alpha)T_{q,n}^*R_{T_{q,n}^*}(z)=\left[R_{T_{q,n}^*}(\alpha)\right]^{-1}R_{T_{q,n}^*}(z)-I_{(n+1)q} \tag{51}$$

is valid and, because of $[R_{T_{q,n}^*}(w)]^*=R_{T_{q,n}}(\overline{w})$, that

$$(\overline{w}-\alpha)\left[R_{T_{q,n}^*}(w)\right]^*T_{q,n}=\left[R_{T_{q,n}^*}(w)\right]^*\left[R_{T_{q,n}}(\alpha)\right]^{-1}-I_{(n+1)q} \tag{52}$$

is also true. In view of (11), $[R_{T_{q,n}}(\alpha)]^{-*}=[R_{T_{q,n}^*}(\alpha)]^{-1}$, (51), and (52), we obtain

$$
\begin{aligned}
&(\overline{w}-\alpha)(z-\alpha)\Big[R_{T_{q,n}^*}(w)\Big]^*(T_{q,n}H_n v_{q,n}v_{q,n}^* - v_{q,n}v_{q,n}^* H_n T_{q,n}^*)R_{T_{q,n}^*}(z)\\
&= (\overline{w}-\alpha)(z-\alpha)\Big[R_{T_{q,n}^*}(w)\Big]^*\Big(T_{q,n}H_n\Big[R_{T_{q,n}^*}(\alpha)\Big]^{-1} - \big[R_{T_{q,n}}(\alpha)\big]^{-1}H_n T_{q,n}^*\Big)R_{T_{q,n}^*}(z)\\
&= (z-\alpha)\Big(\Big[R_{T_{q,n}^*}(w)\Big]^*\big[R_{T_{q,n}}(\alpha)\big]^{-1} - I_{(n+1)q}\Big)H_n\Big[R_{T_{q,n}^*}(\alpha)\Big]^{-1}R_{T_{q,n}^*}(z)\\
&\qquad - (\overline{w}-\alpha)\Big[R_{T_{q,n}^*}(w)\Big]^*\big[R_{T_{q,n}}(\alpha)\big]^{-1}H_n\Big(\Big[R_{T_{q,n}^*}(\alpha)\Big]^{-1}R_{T_{q,n}^*}(z) - I_{(n+1)q}\Big)\\
&= (\overline{w}-\alpha)\Big[R_{T_{q,n}^*}(w)\Big]^*\big[R_{T_{q,n}}(\alpha)\big]^{-1}H_n - (z-\alpha)H_n\Big[R_{T_{q,n}^*}(\alpha)\Big]^{-1}R_{T_{q,n}^*}(z)\\
&\qquad - (\overline{w}-z)\Big[R_{T_{q,n}^*}(w)\Big]^*\big[R_{T_{q,n}}(\alpha)\big]^{-1}H_n\Big[R_{T_{q,n}^*}(\alpha)\Big]^{-1}R_{T_{q,n}^*}(z),
\end{aligned}
$$

which together with (50) implies

$$
\begin{aligned}
&S_3(z,w)\\
&= -\mathrm{i}[\operatorname{diag}(v_{q,n},v_{q,n})]^*[I_{(n+1)q},T_{q,n}H_n]^*\big[R_{T_{q,n}}(\alpha)\big]^* H_n^-\\
&\qquad\times\Big\{(\overline{w}-\alpha)\Big[R_{T_{q,n}^*}(w)\Big]^*\big[R_{T_{q,n}}(\alpha)\big]^{-1}H_n - (z-\alpha)H_n\Big[R_{T_{q,n}^*}(\alpha)\Big]^{-1}R_{T_{q,n}^*}(z)\\
&\qquad\qquad - (\overline{w}-z)\Big[R_{T_{q,n}^*}(w)\Big]^*\big[R_{T_{q,n}}(\alpha)\big]^{-1}H_n\Big[R_{T_{q,n}^*}(\alpha)\Big]^{-1}R_{T_{q,n}^*}(z)\Big\}\\
&\qquad\times H_n^- R_{T_{q,n}}(\alpha)[I_{(n+1)q},T_{q,n}H_n]\operatorname{diag}(v_{q,n},v_{q,n}).
\end{aligned}
\tag{53}
$$

The combination of (47), (48), (49), and (53) provides us

$$
\begin{aligned}
\tilde{J}_q - U_{n,\alpha}^*(w)\tilde{J}_q U_{n,\alpha}(z) &= \mathrm{i}[\operatorname{diag}(v_{q,n},v_{q,n})]^*[I_{(n+1)q},T_{q,n}H_n]^*\big[R_{T_{q,n}}(\alpha)\big]^* S(z,w)\\
&\qquad\times R_{T_{q,n}}(\alpha)[I_{(n+1)q},T_{q,n}H_n]\operatorname{diag}(v_{q,n},v_{q,n})
\end{aligned}
\tag{54}
$$

where

$$
\begin{aligned}
S(z,w) := (\overline{w}-\alpha)H_n^-\Big[R_{T_{q,n}^*}(w)\Big]^*\big[R_{T_{q,n}}(\alpha)\big]^{-1} &- (z-\alpha)\big[R_{T_{q,n}}(\alpha)\big]^{-*}R_{T_{q,n}^*}(z)H_n^-\\
&- (\overline{w}-\alpha)H_n^-\Big[R_{T_{q,n}^*}(w)\Big]^*\big[R_{T_{q,n}}(\alpha)\big]^{-1}H_n H_n^-\\
&+ (z-\alpha)H_n^- H_n\Big[R_{T_{q,n}^*}(\alpha)\Big]^{-1}R_{T_{q,n}^*}(z)H_n^-\\
+ (\overline{w}-z)H_n^-\Big[R_{T_{q,n}^*}(w)\Big]^*\big[R_{T_{q,n}}(\alpha)\big]^{-1}H_n&\Big[R_{T_{q,n}^*}(\alpha)\Big]^{-1}R_{T_{q,n}^*}(z)H_n^-.
\end{aligned}
\tag{55}
$$

Because of $R_{T_{q,n}^*}(\zeta) = [R_{T_{q,n}}(\overline{\zeta})]^*$, which is true for each $\zeta \in \mathbb{C}$, Remark 6.16 shows that

$$H_n^-\Big[R_{T_{q,n}^*}(\zeta)\Big]^*\big[R_{T_{q,n}}(\alpha)\big]^{-1}H_nH_n^- = H_n^-\Big[R_{T_{q,n}^*}(\zeta)\Big]^*\big[R_{T_{q,n}}(\alpha)\big]^{-1} \tag{56}$$

and

$$H_n^-H_n\Big[R_{T_{q,n}^*}(\alpha)\Big]^{-1}R_{T_{q,n}^*}(\zeta)H_n^- = \big[R_{T_{q,n}}(\alpha)\big]^{-*}R_{T_{q,n}^*}(\zeta)H_n^- \tag{57}$$

are valid for all $\zeta \in \mathbb{C}$. Thus, from (55), (56), and (57) we get

$$S(z,w) = (\overline{w}-z)H_n^-\Big[R_{T_{q,n}^*}(w)\Big]^*\big[R_{T_{q,n}}(\alpha)\big]^{-1}H_n\Big[R_{T_{q,n}^*}(\alpha)\Big]^{-1}R_{T_{q,n}^*}(z)H_n^-. \tag{58}$$

Taking into account (54) and (58), we obtain (46).

The aim of our next considerations is to introduce a $2q \times 2q$ matrix polynomial which will turn out to be intimately related to the second FMI of Potapov type occurring in Theorem 4.3.

Remark 7.7 Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N} \cup \{\infty\}$, and let $(s_j)_{j=0}^{\kappa} \in \mathscr{K}_{q,\kappa,\alpha}^{\geq,\mathrm{e}}$. For each $n \in \mathbb{N}_0$ with $2n+1 \leq \kappa$, Remark 4.1 shows then that $\tilde{U}_{n,\alpha}\colon \mathbb{C} \to \mathbb{C}^{2q\times 2q}$ given by

$$\begin{aligned}\tilde{U}_{n,\alpha}(\zeta) &:= I_{2q}+(\zeta-\alpha)[\operatorname{diag}(v_{q,n}, v_{q,n})]^*\Big[\big[R_{T_{q,n}}(\alpha)\big]^{-1}H_n, -I_{(n+1)q}\Big]^*R_{T_{q,n}^*}(\zeta)\\ &\quad\times H_{\alpha\triangleright n}^-R_{T_{q,n}}(\alpha)\Big[I_{(n+1)q}, \big[R_{T_{q,n}}(\alpha)\big]^{-1}H_n\Big]\operatorname{diag}(v_{q,n}, v_{q,n})\end{aligned} \tag{59}$$

is a matrix polynomial of degree not greater that $n+1$, where $H_n^* = H_n$ shows that, for each $\zeta \in \mathbb{C}$, the matrix $\tilde{U}_{n,\alpha}(\zeta)$ admits the block representation

$$\tilde{U}_{n,\alpha}(\zeta) = \begin{bmatrix}\tilde{A}_n(\zeta) & \tilde{B}_n(\zeta)\\ \tilde{C}_n(\zeta) & \tilde{D}_n(\zeta)\end{bmatrix}$$

with

$$\begin{aligned}\tilde{A}_n(\zeta) &:= I_q + (\zeta-\alpha)v_{q,n}^*H_n\Big[R_{T_{q,n}^*}(\alpha)\Big]^{-1}R_{T_{q,n}^*}(\zeta)H_{\alpha\triangleright n}^-R_{T_{q,n}}(\alpha)v_{q,n},\\ \tilde{B}_n(\zeta) &:= \quad + (\zeta-\alpha)v_{q,n}^*H_n\Big[R_{T_{q,n}^*}(\alpha)\Big]^{-1}R_{T_{q,n}^*}(\zeta)H_{\alpha\triangleright n}^-H_nv_{q,n},\\ \tilde{C}_n(\zeta) &:= \quad - (\zeta-\alpha)v_{q,n}^*R_{T_{q,n}^*}(\zeta)H_{\alpha\triangleright n}^-R_{T_{q,n}}(\alpha)v_{q,n},\\ \tilde{D}_n(\zeta) &:= I_q - (\zeta-\alpha)v_{q,n}^*R_{T_{q,n}^*}(\zeta)H_{\alpha\triangleright n}^-H_nv_{q,n}.\end{aligned}$$

In the next step we compute the left and right $\tilde{J}_q$-forms of the matrix polynomial introduced in Remark 7.7.

Lemma 7.8 *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N} \cup \{\infty\}$, and let $(s_j)_{j=0}^{\kappa} \in \mathscr{K}_{q,\kappa,\alpha}^{\geq,\mathrm{e}}$. Let $\tilde{U}_{n,\alpha} \colon \mathbb{C} \to \mathbb{C}^{2q\times 2q}$ be given by (59). For all $n \in \mathbb{N}_0$ with $2n+1 \leq \kappa$ and all $z, w \in \mathbb{C}$, then*

$$\tilde{J}_q-\tilde{U}_{n,\alpha}(z)\tilde{J}_q\tilde{U}_{n,\alpha}^*(w) = -\mathrm{i}(z-\overline{w})[\operatorname{diag}(v_{q,n}, v_{q,n})]^*\Big[\big[R_{T_{q,n}}(\alpha)\big]^{-1}H_n, -I_{(n+1)q}\Big]^* \\ \times R_{T_{q,n}^*}(z)H_{\alpha\triangleright n}^{-}\Big[R_{T_{q,n}^*}(w)\Big]^*\Big[\big[R_{T_{q,n}}(\alpha)\big]^{-1}H_n, -I_{(n+1)q}\Big]\operatorname{diag}(v_{q,n}, v_{q,n}). \tag{60}$$

Proof Let $n \in \mathbb{N}_0$ be such that $2n+1 \leq \kappa$. Because of $(s_j)_{j=0}^{\kappa} \in \mathscr{K}_{q,\kappa,\alpha}^{\geq,\mathrm{e}} \subseteq \mathscr{K}_{q,\kappa,\alpha}^{\geq}$ and Remark 5.7, we have $H_n^* = H_n$ and $H_{\alpha\triangleright n}^* = H_{\alpha\triangleright n}$. Lemma 6.12 provides us (20) and (21). Let $z, w \in \mathbb{C}$. Taking into account (59), $\tilde{J}_q^2 = I_{2q}$, and (20), we get then

$$\begin{aligned} &\tilde{J}_q - \tilde{U}_{n,\alpha}(z)\tilde{J}_q\tilde{U}_{n,\alpha}^*(w) \\ &= \tilde{J}_q - \Bigg\{I_{2q} + (z-\alpha)[\operatorname{diag}(v_{q,n}, v_{q,n})]^*\Big[\big[R_{T_{q,n}}(\alpha)\big]^{-1}H_n, -I_{(n+1)q}\Big]^* \\ &\qquad \times R_{T_{q,n}^*}(z)H_{\alpha\triangleright n}^{-}R_{T_{q,n}}(\alpha)\Big[I_{(n+1)q}, \big[R_{T_{q,n}}(\alpha)\big]^{-1}H_n\Big]\operatorname{diag}(v_{q,n}, v_{q,n})\Bigg\}\tilde{J}_q \\ &\qquad \times \Bigg\{I_{2q} + (\overline{w}-\alpha)[\operatorname{diag}(v_{q,n}, v_{q,n})]^*\Big[I_{(n+1)q}, \big[R_{T_{q,n}}(\alpha)\big]^{-1}H_n\Big]^* \\ &\qquad\quad \times \big[R_{T_{q,n}}(\alpha)\big]^*H_{\alpha\triangleright n}^{-}\Big[R_{T_{q,n}^*}(w)\Big]^*\Big[\big[R_{T_{q,n}}(\alpha)\big]^{-1}H_n, -I_{(n+1)q}\Big]\operatorname{diag}(v_{q,n}, v_{q,n})\Bigg\} \\ &= \tilde{S}_1(z) + \tilde{S}_2(w) + \tilde{S}_3(z, w) \end{aligned} \tag{61}$$

where

$$\tilde{S}_1(z) := -(z-\alpha)[\operatorname{diag}(v_{q,n}, v_{q,n})]^*\Big[\big[R_{T_{q,n}}(\alpha)\big]^{-1}H_n, -I_{(n+1)q}\Big]^*R_{T_{q,n}^*}(z)H_{\alpha\triangleright n}^{-} \\ \times R_{T_{q,n}}(\alpha)\Big[I_{(n+1)q}, \big[R_{T_{q,n}}(\alpha)\big]^{-1}H_n\Big]\operatorname{diag}(v_{q,n}, v_{q,n})\tilde{J}_q, \tag{62}$$

$$\tilde{S}_2(w) := -(\overline{w}-\alpha)\tilde{J}_q[\operatorname{diag}(v_{q,n}, v_{q,n})]^*\Big[I_{(n+1)q}, \big[R_{T_{q,n}}(\alpha)\big]^{-1}H_n\Big]^*\big[R_{T_{q,n}}(\alpha)\big]^*H_{\alpha\triangleright n}^{-} \\ \times \Big[R_{T_{q,n}^*}(w)\Big]^*\Big[\big[R_{T_{q,n}}(\alpha)\big]^{-1}H_n, -I_{(n+1)q}\Big]\operatorname{diag}(v_{q,n}, v_{q,n}), \tag{63}$$

and

$$\begin{aligned}\tilde{S}_3(z,w) := -(z-\alpha)(\overline{w}-\alpha)[\operatorname{diag}(v_{q,n},v_{q,n})]^*\Big[\big[R_{T_{q,n}}(\alpha)\big]^{-1}H_n, -I_{(n+1)q}\Big]^*\\ \times R_{T^*_{q,n}}(z)H^-_{\alpha\triangleright n}R_{T_{q,n}}(\alpha)\Big[I_{(n+1)q}, \big[R_{T_{q,n}}(\alpha)\big]^{-1}H_n\Big]\operatorname{diag}(v_{q,n},v_{q,n})\tilde{J}_q\\ \times [\operatorname{diag}(v_{q,n},v_{q,n})]^*\Big[I_{(n+1)q}, \big[R_{T_{q,n}}(\alpha)\big]^{-1}H_n\Big]^*\big[R_{T_{q,n}}(\alpha)\big]^*H^-_{\alpha\triangleright n}\Big[R_{T^*_{q,n}}(w)\Big]^*\\ \times\Big[\big[R_{T_{q,n}}(\alpha)\big]^{-1}H_n, -I_{(n+1)q}\Big]\operatorname{diag}(v_{q,n},v_{q,n}).\end{aligned}\tag{64}$$

In view of (25), (26), and Remark 4.1, from (62) and (63) we obtain

$$\begin{aligned}\tilde{S}_1(z) = -\mathrm{i}(z-\alpha)[\operatorname{diag}(v_{q,n},v_{q,n})]^*\Big[\big[R_{T_{q,n}}(\alpha)\big]^{-1}H_n, -I_{(n+1)q}\Big]^*R_{T^*_{q,n}}(z)H^-_{\alpha\triangleright n}\\ \times R_{T_{q,n}}(\alpha)\Big[R_{T^*_{q,n}}(w)\Big]^{-*}\Big[R_{T^*_{q,n}}(w)\Big]^*\Big[\big[R_{T_{q,n}}(\alpha)\big]^{-1}H_n, -I_{(n+1)q}\Big]\operatorname{diag}(v_{q,n},v_{q,n}),\end{aligned}\tag{65}$$

$$\begin{aligned}\tilde{S}_2(w) = \mathrm{i}(\overline{w}-\alpha)[\operatorname{diag}(v_{q,n},v_{q,n})]^*\Big[\big[R_{T_{q,n}}(\alpha)\big]^{-1}H_n, -I_{(n+1)q}\Big]^*R_{T^*_{q,n}}(z)\Big[R_{T^*_{q,n}}(z)\Big]^{-1}\\ \times\big[R_{T_{q,n}}(\alpha)\big]^*H^-_{\alpha\triangleright n}\Big[R_{T^*_{q,n}}(w)\Big]^*\Big[\big[R_{T_{q,n}}(\alpha)\big]^{-1}H_n, -I_{(n+1)q}\Big]\operatorname{diag}(v_{q,n},v_{q,n}),\end{aligned}\tag{66}$$

and, because of (64), (27), and $H_n^* = H_n$, furthermore,

$$\begin{aligned}&\tilde{S}_3(z,w)\\ &= -\mathrm{i}(z-\alpha)(\overline{w}-\alpha)[\operatorname{diag}(v_{q,n},v_{q,n})]^*\Big[\big[R_{T_{q,n}}(\alpha)\big]^{-1}H_n, -I_{(n+1)q}\Big]^*R_{T^*_{q,n}}(z)H^-_{\alpha\triangleright n}\\ &\quad\times R_{T_{q,n}}(\alpha)\Big(\big[R_{T_{q,n}}(\alpha)\big]^{-1}H_nv_{q,n}v^*_{q,n} - v_{q,n}v^*_{q,n}H_n\big[R_{T_{q,n}}(\alpha)\big]^{-*}\Big)\big[R_{T_{q,n}}(\alpha)\big]^*\\ &\quad\times H^-_{\alpha\triangleright n}\Big[R_{T^*_{q,n}}(w)\Big]^*\Big[\big[R_{T_{q,n}}(\alpha)\big]^{-1}H_n, -I_{(n+1)q}\Big]\operatorname{diag}(v_{q,n},v_{q,n}).\end{aligned}\tag{67}$$

Using $H_n^* = H_n$, (12), (248), (249), and $R_{T^*_{q,n}}(\alpha) = [R_{T_{q,n}}(\alpha)]^*$, we infer

$$\begin{aligned}&(z-\alpha)(\overline{w}-\alpha)R_{T_{q,n}}(\alpha)\\ &\quad\times\Big(\big[R_{T_{q,n}}(\alpha)\big]^{-1}H_nv_{q,n}v^*_{q,n} - v_{q,n}v^*_{q,n}H_n\big[R_{T_{q,n}}(\alpha)\big]^{-*}\Big)\big[R_{T_{q,n}}(\alpha)\big]^*\\ &= (z-\alpha)(\overline{w}-\alpha)R_{T_{q,n}}(\alpha)\\ &\quad\times\Big(T_{q,n}H_{\alpha\triangleright n}\big[R_{T_{q,n}}(\alpha)\big]^{-*} - \big[R_{T_{q,n}}(\alpha)\big]^{-1}H_{\alpha\triangleright n}T^*_{q,n}\Big)\big[R_{T_{q,n}}(\alpha)\big]^*\end{aligned}$$

$$= (z-\alpha)(\overline{w}-\alpha)R_{T_{q,n}}(\alpha)T_{q,n}H_{\alpha\rhd n} - (z-\alpha)(\overline{w}-\alpha)H_{\alpha\rhd n}T_{q,n}^{*}\left[R_{T_{q,n}}(\alpha)\right]^{*}$$

$$= -(z-\alpha)\left(R_{T_{q,n}}(\alpha)\left[R_{T_{q,n}^{*}}(w)\right]^{-*} - I_{(n+1)q}\right)H_{\alpha\rhd n}$$

$$+ (\overline{w}-\alpha)H_{\alpha\rhd n}\left(\left[R_{T_{q,n}^{*}}(z)\right]^{-1}\left[R_{T_{q,n}}(\alpha)\right]^{*} - I_{(n+1)q}\right)$$

$$= -(z-\alpha)R_{T_{q,n}}(\alpha)\left[R_{T_{q,n}^{*}}(w)\right]^{-*}H_{\alpha\rhd n} + (\overline{w}-\alpha)H_{\alpha\rhd n}\left[R_{T_{q,n}^{*}}(z)\right]^{-1}\left[R_{T_{q,n}}(\alpha)\right]^{*}$$

$$+ (z-\overline{w})H_{\alpha\rhd n}.$$

Consequently, from (67) we get then

$$\tilde{S}_3(z,w) = -\mathrm{i}[\operatorname{diag}(v_{q,n}, v_{q,n})]^{*}\left[\left[R_{T_{q,n}}(\alpha)\right]^{-1}H_n, -I_{(n+1)q}\right]^{*}R_{T_{q,n}^{*}}(z)H_{\alpha\rhd n}^{-}$$
$$\times\Bigg\{-(z-\alpha)R_{T_{q,n}}(\alpha)\left[R_{T_{q,n}^{*}}(w)\right]^{-*}H_{\alpha\rhd n} + (\overline{w}-\alpha)H_{\alpha\rhd n}\left[R_{T_{q,n}^{*}}(z)\right]^{-1}\left[R_{T_{q,n}}(\alpha)\right]^{*}$$
$$+ (z-\overline{w})H_{\alpha\rhd n}\Bigg\}H_{\alpha\rhd n}^{-}\left[R_{T_{q,n}^{*}}(w)\right]^{*}\left[\left[R_{T_{q,n}}(\alpha)\right]^{-1}H_n, -I_{(n+1)q}\right]\operatorname{diag}(v_{q,n}, v_{q,n}). \tag{68}$$

The combination of (61), (65), (66), and (68) yields

$$\tilde{J}_q - \tilde{U}_{n,\alpha}(z)\tilde{J}_q\tilde{U}_{n,\alpha}^{*}(w) = -\mathrm{i}[\operatorname{diag}(v_{q,n}, v_{q,n})]^{*}\left[\left[R_{T_{q,n}}(\alpha)\right]^{-1}H_n, -I_{(n+1)q}\right]^{*}R_{T_{q,n}^{*}}(z)$$
$$\times\tilde{S}(z,w)\left[R_{T_{q,n}^{*}}(w)\right]^{*}\left[\left[R_{T_{q,n}}(\alpha)\right]^{-1}H_n, -I_{(n+1)q}\right]\operatorname{diag}(v_{q,n}, v_{q,n}) \tag{69}$$

where

$$\tilde{S}(z,w)$$
$$:= (z-\alpha)H_{\alpha\rhd n}^{-}R_{T_{q,n}}(\alpha)\left[R_{T_{q,n}^{*}}(w)\right]^{-*} - (\overline{w}-\alpha)\left[R_{T_{q,n}^{*}}(z)\right]^{-1}\left[R_{T_{q,n}}(\alpha)\right]^{*}H_{\alpha\rhd n}^{-}$$
$$- (z-\alpha)H_{\alpha\rhd n}^{-}R_{T_{q,n}}(\alpha)\left[R_{T_{q,n}^{*}}(w)\right]^{-*}H_{\alpha\rhd n}H_{\alpha\rhd n}^{-}$$
$$+ (\overline{w}-\alpha)H_{\alpha\rhd n}^{-}H_{\alpha\rhd n}\left[R_{T_{q,n}^{*}}(z)\right]^{-1}\left[R_{T_{q,n}}(\alpha)\right]^{*}H_{\alpha\rhd n}^{-} + (z-\overline{w})H_{\alpha\rhd n}^{-}H_{\alpha\rhd n}H_{\alpha\rhd n}^{-}. \tag{70}$$

From (70), Remark 6.16, and Lemma 6.12 we obtain $\tilde{S}(z,w) = (z-\overline{w})H_{\alpha\rhd n}^{-}$. Hence, taking into account (69), we get (60).

Lemma 7.9 *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N} \cup \{\infty\}$, and let $(s_j)_{j=0}^{\kappa} \in \mathcal{K}_{q,\kappa,\alpha}^{\geq,\mathrm{e}}$. For each $n \in \mathbb{N}_0$ with $2n+1 \leq \kappa$ and all $w, z \in \mathbb{C}$, then*

$$\begin{aligned}
&\tilde{J}_q - \tilde{U}_{n,\alpha}^*(w)\tilde{J}_q\tilde{U}_{n,\alpha}(z) \\
&\quad = \mathrm{i}(\overline{w} - z)[\operatorname{diag}(v_{q,n}, v_{q,n})]^*\Big[I_{(n+1)q}, \big[R_{T_{q,n}}(\alpha)\big]^{-1} H_n\Big]^*\big[R_{T_{q,n}}(\alpha)\big]^* \\
&\qquad \times H_{\alpha\triangleright n}^{-}\Big[R_{T_{q,n}^*}(w)\Big]^*\big[R_{T_{q,n}}(\alpha)\big]^{-1} H_{\alpha\triangleright n}\big[R_{T_{q,n}}(\alpha)\big]^{-*} R_{T_{q,n}^*}(z) H_{\alpha\triangleright n}^{-} \\
&\qquad \times R_{T_{q,n}}(\alpha)\Big[I_{(n+1)q}, \big[R_{T_{q,n}}(\alpha)\big]^{-1} H_n\Big] \operatorname{diag}(v_{q,n}, v_{q,n}).
\end{aligned} \tag{71}$$

Proof Let $n \in \mathbb{N}_0$ be such that $2n+1 \leq \kappa$. Because of $(s_j)_{j=0}^{\kappa} \in \mathcal{K}_{q,\kappa,\alpha}^{\geq,\mathrm{e}} \subseteq \mathcal{K}_{q,\kappa,\alpha}^{\geq}$ and Remark 5.7, we have $H_n^* = H_n$ and $H_{\alpha\triangleright n}^* = H_{\alpha\triangleright n}$. Lemma 6.12 shows that (20) holds true. Let $w, z \in \mathbb{C}$. In view of (59) and (20), we get

$$\begin{aligned}
&\tilde{J}_q - \tilde{U}_{n,\alpha}^*(w)\tilde{J}_q\tilde{U}_{n,\alpha}(z) \\
&= \tilde{J}_q - \Bigg\{I_{2q} + (\overline{w} - \alpha)[\operatorname{diag}(v_{q,n}, v_{q,n})]^*\Big[I_{(n+1)q}, \big[R_{T_{q,n}}(\alpha)\big]^{-1} H_n\Big]^*\big[R_{T_{q,n}}(\alpha)\big]^* \\
&\qquad \times H_{\alpha\triangleright n}^{-}\Big[R_{T_{q,n}^*}(w)\Big]^*\Big[\big[R_{T_{q,n}}(\alpha)\big]^{-1} H_n, -I_{(n+1)q}\Big] \operatorname{diag}(v_{q,n}, v_{q,n})\Bigg\}\tilde{J}_q \\
&\quad \times \Bigg\{I_{2q} + (z - \alpha)[\operatorname{diag}(v_{q,n}, v_{q,n})]^*\Big[\big[R_{T_{q,n}}(\alpha)\big]^{-1} H_n, -I_{(n+1)q}\Big]^* R_{T_{q,n}^*}(z) H_{\alpha\triangleright n}^{-} \\
&\qquad \times R_{T_{q,n}}(\alpha)\Big[I_{(n+1)q}, \big[R_{T_{q,n}}(\alpha)\big]^{-1} H_n\Big] \operatorname{diag}(v_{q,n}, v_{q,n})\Bigg\} \\
&= \tilde{S}_1(w) + \tilde{S}_2(z) + \tilde{S}_3(z, w),
\end{aligned} \tag{72}$$

where

$$\begin{aligned}
\tilde{S}_1(w) &:= -(\overline{w} - \alpha)[\operatorname{diag}(v_{q,n}, v_{q,n})]^*\Big[I_{(n+1)q}, \big[R_{T_{q,n}}(\alpha)\big]^{-1} H_n\Big]^*\big[R_{T_{q,n}}(\alpha)\big]^* H_{\alpha\triangleright n}^{-} \\
&\qquad \times \Big[R_{T_{q,n}^*}(w)\Big]^*\Big[\big[R_{T_{q,n}}(\alpha)\big]^{-1} H_n, -I_{(n+1)q}\Big] \operatorname{diag}(v_{q,n}, v_{q,n})\tilde{J}_q, \\
\tilde{S}_2(z) &:= -(z - \alpha)\tilde{J}_q[\operatorname{diag}(v_{q,n}, v_{q,n})]^*\Big[\big[R_{T_{q,n}}(\alpha)\big]^{-1} H_n, -I_{(n+1)q}\Big]^* R_{T_{q,n}^*}(z) H_{\alpha\triangleright n}^{-} \\
&\qquad \times R_{T_{q,n}}(\alpha)\Big[I_{(n+1)q}, \big[R_{T_{q,n}}(\alpha)\big]^{-1} H_n\Big] \operatorname{diag}(v_{q,n}, v_{q,n}),
\end{aligned}$$

and

$$\tilde{S}_3(z, w) := -(\overline{w} - \alpha)(z - \alpha)[\operatorname{diag}(v_{q,n}, v_{q,n})]^*\Big[I_{(n+1)q}, \big[R_{T_{q,n}}(\alpha)\big]^{-1} H_n\Big]^*$$

$$\times \left[R_{T_{q,n}}(\alpha)\right]^* H_{\alpha\triangleright n}^- \left[R_{T_{q,n}^*}(w)\right]^* \left[\left[R_{T_{q,n}}(\alpha)\right]^{-1} H_n, -I_{(n+1)q}\right] \operatorname{diag}(v_{q,n}, v_{q,n})$$
$$\times \tilde{J}_q [\operatorname{diag}(v_{q,n}, v_{q,n})]^* \left[\left[R_{T_{q,n}}(\alpha)\right]^{-1} H_n, -I_{(n+1)q}\right]^* R_{T_{q,n}^*}(z) H_{\alpha\triangleright n}^- R_{T_{q,n}}(\alpha)$$
$$\times \left[I_{(n+1)q}, \left[R_{T_{q,n}}(\alpha)\right]^{-1} H_n\right] \operatorname{diag}(v_{q,n}, v_{q,n}).$$

From Remark 7.3 we obtain

$$\tilde{S}_1(w) = \mathrm{i}(\overline{w} - \alpha)[\operatorname{diag}(v_{q,n}, v_{q,n})]^* \left[I_{(n+1)q}, \left[R_{T_{q,n}}(\alpha)\right]^{-1} H_n\right]^* \left[R_{T_{q,n}}(\alpha)\right]^* H_{\alpha\triangleright n}^-$$
$$\times \left[R_{T_{q,n}^*}(w)\right]^* \left[R_{T_{q,n}}(\alpha)\right]^{-1} R_{T_{q,n}}(\alpha) \left[I_{(n+1)q}, \left[R_{T_{q,n}}(\alpha)\right]^{-1} H_n\right] \operatorname{diag}(v_{q,n}, v_{q,n}), \tag{73}$$

$$\tilde{S}_2(z) = -\mathrm{i}(z - \alpha)[\operatorname{diag}(v_{q,n}, v_{q,n})]^* \left[I_{(n+1)q}, \left[R_{T_{q,n}}(\alpha)\right]^{-1} H_n\right]^* \left[R_{T_{q,n}}(\alpha)\right]^*$$
$$\times \left[R_{T_{q,n}}(\alpha)\right]^{-*} R_{T_{q,n}^*}(z) H_{\alpha\triangleright n}^- R_{T_{q,n}}(\alpha) \left[I_{(n+1)q}, \left[R_{T_{q,n}}(\alpha)\right]^{-1} H_n\right] \operatorname{diag}(v_{q,n}, v_{q,n}), \tag{74}$$

and, in view of (28) and $H_n^* = H_n$, furthermore

$$\tilde{S}_3(z, w)$$
$$= -\mathrm{i}(\overline{w}-\alpha)(z-\alpha)[\operatorname{diag}(v_{q,n}, v_{q,n})]^* \left[I_{(n+1)q}, \left[R_{T_{q,n}}(\alpha)\right]^{-1} H_n\right]^* \left[R_{T_{q,n}}(\alpha)\right]^* H_{\alpha\triangleright n}^-$$
$$\times \left[R_{T_{q,n}^*}(w)\right]^* \left(\left[R_{T_{q,n}}(\alpha)\right]^{-1} H_n v_{q,n} v_{q,n}^* - v_{q,n} v_{q,n}^* H_n \left[R_{T_{q,n}}(\alpha)\right]^{-*}\right) R_{T_{q,n}^*}(z)$$
$$\times H_{\alpha\triangleright n}^- R_{T_{q,n}}(\alpha) \left[I_{(n+1)q}, \left[R_{T_{q,n}}(\alpha)\right]^{-1} H_n\right] \operatorname{diag}(v_{q,n}, v_{q,n}). \tag{75}$$

With the aid of Remark 17.10 and $[R_{T_{q,n}^*}(\alpha)]^{-*} = [R_{T_{q,n}}(\alpha)]^{-1}$, we get (51) and (52) and, in view of (12) in Remark 5.5, (51), and (52),

$$(\overline{w} - \alpha)(z - \alpha)\left[R_{T_{q,n}^*}(w)\right]^*$$
$$\times \left(\left[R_{T_{q,n}}(\alpha)\right]^{-1} H_n v_{q,n} v_{q,n}^* - v_{q,n} v_{q,n}^* H_n \left[R_{T_{q,n}}(\alpha)\right]^{-*}\right) R_{T_{q,n}^*}(z)$$
$$= (\overline{w} - \alpha)(z - \alpha)\left[R_{T_{q,n}^*}(w)\right]^*$$
$$\times \left(T_{q,n} H_{\alpha\triangleright n} \left[R_{T_{q,n}}(\alpha)\right]^{-*} - \left[R_{T_{q,n}}(\alpha)\right]^{-1} H_{\alpha\triangleright n} T_{q,n}^*\right) R_{T_{q,n}^*}(z)$$
$$= (\overline{w} - \alpha)(z - \alpha)\left[R_{T_{q,n}^*}(w)\right]^* T_{q,n} H_{\alpha\triangleright n} \left[R_{T_{q,n}}(\alpha)\right]^{-*} R_{T_{q,n}^*}(z)$$
$$- (\overline{w} - \alpha)(z - \alpha)\left[R_{T_{q,n}^*}(w)\right]^* \left[R_{T_{q,n}}(\alpha)\right]^{-1} H_{\alpha\triangleright n} T_{q,n}^* R_{T_{q,n}^*}(z)$$

$$= (z-\alpha)\Big(\Big[R_{T^*_{q,n}}(w)\Big]^*\big[R_{T_{q,n}}(\alpha)\big]^{-1} - I_{(n+1)q}\Big)H_{\alpha\rhd n}\big[R_{T_{q,n}}(\alpha)\big]^{-*}R_{T^*_{q,n}}(z)$$
$$- (\overline{w}-\alpha)\Big[R_{T^*_{q,n}}(w)\Big]^*\big[R_{T_{q,n}}(\alpha)\big]^{-1}H_{\alpha\rhd n}\Big(\Big[R_{T^*_{q,n}}(\alpha)\Big]^{-1}R_{T^*_{q,n}}(z) - I_{(n+1)q}\Big)$$
$$= (\overline{w}-\alpha)\Big[R_{T^*_{q,n}}(w)\Big]^*\big[R_{T_{q,n}}(\alpha)\big]^{-1}H_{\alpha\rhd n} - (z-\alpha)H_{\alpha\rhd n}\big[R_{T_{q,n}}(\alpha)\big]^{-*}R_{T^*_{q,n}}(z)$$
$$- (\overline{w}-z)\Big[R_{T^*_{q,n}}(w)\Big]^*\big[R_{T_{q,n}}(\alpha)\big]^{-1}H_{\alpha\rhd n}\big[R_{T_{q,n}}(\alpha)\big]^{-*}R_{T^*_{q,n}}(z),$$

which, because of (75), implies

$$\begin{aligned}\tilde{S}_3(z,w) = &-\mathrm{i}[\operatorname{diag}(v_{q,n},v_{q,n})]^*\Big[I_{(n+1)q},\big[R_{T_{q,n}}(\alpha)\big]^{-1}H_n\Big]^*\big[R_{T_{q,n}}(\alpha)\big]^*H^-_{\alpha\rhd n}\\
&\times\Big\{(\overline{w}-\alpha)\Big[R_{T^*_{q,n}}(w)\Big]^*\big[R_{T_{q,n}}(\alpha)\big]^{-1}H_{\alpha\rhd n}-(z-\alpha)H_{\alpha\rhd n}\big[R_{T_{q,n}}(\alpha)\big]^{-*}R_{T^*_{q,n}}(z)\\
&\quad- (\overline{w}-z)\Big[R_{T^*_{q,n}}(w)\Big]^*\big[R_{T_{q,n}}(\alpha)\big]^{-1}H_{\alpha\rhd n}\big[R_{T_{q,n}}(\alpha)\big]^{-*}R_{T^*_{q,n}}(z)\Big\}\\
&\times H^-_{\alpha\rhd n}R_{T_{q,n}}(\alpha)\Big[I_{(n+1)q},\big[R_{T_{q,n}}(\alpha)\big]^{-1}H_n\Big]\operatorname{diag}(v_{q,n},v_{q,n}).\end{aligned} \tag{76}$$

Combining (72), (73), (74), and (76), we see that

$$\begin{aligned}&\tilde{J}_q - \tilde{U}^*_{n,\alpha}(w)\tilde{J}_q\tilde{U}_{n,\alpha}(z)\\
&= \mathrm{i}[\operatorname{diag}(v_{q,n},v_{q,n})]^*\Big[I_{(n+1)q},\big[R_{T_{q,n}}(\alpha)\big]^{-1}H_n\Big]^*\big[R_{T_{q,n}}(\alpha)\big]^*\tilde{S}(z,w)\\
&\quad\times R_{T_{q,n}}(\alpha)\Big[I_{(n+1)q},\big[R_{T_{q,n}}(\alpha)\big]^{-1}H_n\Big]\operatorname{diag}(v_{q,n},v_{q,n})\end{aligned} \tag{77}$$

is valid, where

$$\begin{aligned}&\tilde{S}(z,w)\\
&:= (\overline{w}-\alpha)H^-_{\alpha\rhd n}\Big[R_{T^*_{q,n}}(w)\Big]^*\big[R_{T_{q,n}}(\alpha)\big]^{-1} - (z-\alpha)\big[R_{T_{q,n}}(\alpha)\big]^{-*}R_{T^*_{q,n}}(z)H^-_{\alpha\rhd n}\\
&\quad- (\overline{w}-\alpha)H^-_{\alpha\rhd n}\Big[R_{T^*_{q,n}}(w)\Big]^*\big[R_{T_{q,n}}(\alpha)\big]^{-1}H_{\alpha\rhd n}H^-_{\alpha\rhd n}\\
&\quad+ (z-\alpha)H^-_{\alpha\rhd n}H_{\alpha\rhd n}\big[R_{T_{q,n}}(\alpha)\big]^{-*}R_{T^*_{q,n}}(z)H^-_{\alpha\rhd n}\\
&\quad+ (\overline{w}-z)H^-_{\alpha\rhd n}\Big[R_{T^*_{q,n}}(w)\Big]^*\big[R_{T_{q,n}}(\alpha)\big]^{-1}H_{\alpha\rhd n}\big[R_{T_{q,n}}(\alpha)\big]^{-*}R_{T^*_{q,n}}(z)H^-_{\alpha\rhd n}.\end{aligned} \tag{78}$$

Since $[R_{T_{q,n}^*}(\eta)]^* = R_{T_{q,n}}(\overline{\eta})$ holds true for all $\eta \in \mathbb{C}$, from Remark 6.16 we infer that

$$H_{\alpha \triangleright n}^{-}\Big[R_{T_{q,n}^*}(\zeta)\Big]^*\Big[R_{T_{q,n}}(\alpha)\Big]^{-1}H_{\alpha \triangleright n}H_{\alpha \triangleright n}^{-} = H_{\alpha \triangleright n}^{-}\Big[R_{T_{q,n}^*}(\zeta)\Big]^*\Big[R_{T_{q,n}}(\alpha)\Big]^{-1} \tag{79}$$

and

$$H_{\alpha \triangleright n}^{-}H_{\alpha \triangleright n}\Big[R_{T_{q,n}}(\alpha)\Big]^{-*}R_{T_{q,n}^*}(\zeta)H_{\alpha \triangleright n}^{-} = \Big[R_{T_{q,n}}(\alpha)\Big]^{-*}R_{T_{q,n}^*}(\zeta)H_{\alpha \triangleright n}^{-} \tag{80}$$

are fulfilled for each $\zeta \in \mathbb{C}$. Using (79), (80), and (78), we get

$$\tilde{S}(z,w) = (\overline{w} - z)H_{\alpha \triangleright n}^{-}\Big[R_{T_{q,n}^*}(w)\Big]^*\Big[R_{T_{q,n}}(\alpha)\Big]^{-1}H_{\alpha \triangleright n}\Big[R_{T_{q,n}}(\alpha)\Big]^{-*}R_{T_{q,n}^*}(z)H_{\alpha \triangleright n}^{-}$$

and, because of (77), then (71) follows.

Our next goal is to establish an appropriate coupling between both Potapov-type FMIs occurring in Theorem 4.3. The key observation to realize this aim is based on the following remark.

Remark 7.10 Let $A \in \mathbb{C}_{\mathrm{H}}^{q \times q}$. Then the matrices $B := \begin{bmatrix} I_q & 0_{q\times q} \\ A & I_q \end{bmatrix}$ and $C := \begin{bmatrix} I_q & A \\ 0_{q\times q} & I_q \end{bmatrix}$ fulfill $B^*\tilde{J}_q B = \tilde{J}_q$, $C^*\tilde{J}_q C = \tilde{J}_q$, $B\tilde{J}_q B^* = \tilde{J}_q$, and $C\tilde{J}_q C^* = \tilde{J}_q$.

More precisely, because right multiplication by a $\tilde{J}_q$-unitary matrix does not influence the image of linear fractional transformations with $\tilde{J}_q$-contractive matrices, the desired coupling will be produced by right multiplication of the matrix polynomials introduced in Remarks 7.4 and 7.7 by appropriately chosen $\tilde{J}_q$-unitary matrices of block triangular type. The following choices of $\tilde{J}_q$-unitary matrices are of importance for our subsequent considerations.

Remark 7.11 Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N} \cup \{\infty\}$, and let $(s_j)_{j=0}^{\kappa} \in \mathcal{K}_{q,\kappa,\alpha}^{\geq,\mathrm{e}}$. For each $n \in \mathbb{N}_0$ with $2n+1 \leq \kappa$, in view of the Remarks 7.10 and 5.7 and Lemma 6.12, it is readily checked that

$$B_{n,\alpha} := \left[\begin{array}{c:c} I_q & v_{q,n}^* H_n H_{\alpha \triangleright n}^{-} H_n v_{q,n} \\ \hdashline 0_{q\times q} & I_q \end{array}\right] \tag{81}$$

and

$$\tilde{B}_{n,\alpha} := \left[\begin{array}{c:c} I_q & 0_{q\times q} \\ \hdashline -v_{q,n}^* R_{T_{q,n}^*}(\alpha) H_n^{-} R_{T_{q,n}}(\alpha) v_{q,n} & I_q \end{array}\right] \tag{82}$$

are $\tilde{J}_q$-unitary matrices, i.e., that $B_{n,\alpha}\tilde{J}_q B_{n,\alpha}^* = \tilde{J}_q$, $B_{n,\alpha}^*\tilde{J}_q B_{n,\alpha} = \tilde{J}_q$, $\tilde{B}_{n,\alpha}\tilde{J}_q\tilde{B}_{n,\alpha}^* = \tilde{J}_q$, and $\tilde{B}_{n,\alpha}^*\tilde{J}_q\tilde{B}_{n,\alpha} = \tilde{J}_q$ hold true.

Remark 7.12 Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N} \cup \{\infty\}$, and let $(s_j)_{j=0}^{\kappa} \in \mathscr{K}_{q,\kappa,\alpha}^{\geq,\mathrm{e}}$. In view of (81) and (82), for each $n \in \mathbb{N}_0$ with $2n+1 \leq \kappa$, then

$$[I_{(n+1)q}, T_{q,n}H_n]\operatorname{diag}(v_{q,n}, v_{q,n})B_{n,\alpha}$$
$$= \left[I_{(n+1)q}, (v_{q,n}v_{q,n}^* H_n H_{\alpha\triangleright n}^- + T_{q,n})H_n\right]\operatorname{diag}(v_{q,n}, v_{q,n})$$

and

$$\left[I_{(n+1)q}, \left[R_{T_{q,n}}(\alpha)\right]^{-1}H_n\right]\operatorname{diag}(v_{q,n}, v_{q,n})\tilde{B}_{n,\alpha}$$
$$=\left[R_{T_{q,n}}(\alpha)\right]^{-1}\left[\left[I_{(n+1)q} - H_n v_{q,n}v_{q,n}^* R_{T_{q,n}^*}(\alpha)H_n^-\right]R_{T_{q,n}}(\alpha), H_n\right]\operatorname{diag}(v_{q,n}, v_{q,n}).$$

Remark 7.13 Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N} \cup \{\infty\}$, and let $(s_j)_{j=0}^{\kappa} \in \mathscr{K}_{q,\kappa,\alpha}^{\geq,\mathrm{e}}$. In view of Remark 7.12 and Lemma 6.17, it is readily checked that, for each $n \in \mathbb{N}_0$ with $2n+1 \leq \kappa$, then

$$H_n^- R_{T_{q,n}}(\alpha)[I_{(n+1)q}, T_{q,n}H_n]\operatorname{diag}(v_{q,n}, v_{q,n})B_{n,\alpha}$$
$$= \left[H_n^- R_{T_{q,n}}(\alpha), H_{\alpha\triangleright n}^- H_n\right]\operatorname{diag}(v_{q,n}, v_{q,n}).$$

Remark 7.14 Let $\alpha \in \mathbb{R}$ and let $n \in \mathbb{N}_0$. According to Remark 4.1, the matrix-valued functions $\Omega_{q,n,\alpha}\colon \mathbb{C} \to \mathbb{C}^{2(n+1)q\times 2(n+1)q}$ and $\tilde{\Omega}_{q,n,\alpha}\colon \mathbb{C} \to \mathbb{C}^{2(n+1)q\times 2(n+1)q}$ given by

$$\Omega_{q,n,\alpha}(\zeta) := \left[\begin{array}{c|c} (\zeta-\alpha)T_{q,n}^* & [R_{T_{q,n}^*}(\alpha)]^{-1} \\ \hline -(\zeta-\alpha)I_{(n+1)q} & -(\zeta-\alpha)I_{(n+1)q} \end{array}\right]\operatorname{diag}\Big(R_{T_{q,n}^*}(\zeta), R_{T_{q,n}^*}(\zeta)\Big) \tag{83}$$

and

$$\tilde{\Omega}_{q,n,\alpha}(\zeta) := \left[\begin{array}{c|c} (\zeta-\alpha)T_{q,n}^* & (\zeta-\alpha)[R_{T_{q,n}^*}(\alpha)]^{-1} \\ \hline -I_{(n+1)q} & -(\zeta-\alpha)I_{(n+1)q} \end{array}\right]\operatorname{diag}\Big(R_{T_{q,n}^*}(\zeta), R_{T_{q,n}^*}(\zeta)\Big) \tag{84}$$

are both matrix polynomials of degree $n+1$.

The following result is connected with the right multiplication of the matrix polynomials introduced in Remarks 7.4 and 7.7, respectively, by the $\tilde{J}_q$-unitary matrices given in (81) and (82), respectively.

Lemma 7.15 *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N} \cup \{\infty\}$, and let $(s_j)_{j=0}^{\kappa} \in \mathscr{K}_{q,\kappa,\alpha}^{\geq,\mathrm{e}}$. Let $n \in \mathbb{N}_0$ be such that $2n+1 \leq \kappa$. In view of (83) and (84), let $\Theta_{n,\alpha} : \mathbb{C} \to \mathbb{C}^{2q\times 2q}$ and $\tilde{\Theta}_{n,\alpha} : \mathbb{C} \to \mathbb{C}^{2q\times 2q}$ be given by*

$$\Theta_{n,\alpha}(\zeta) := I_{2q} + [\operatorname{diag}(v_{q,n}, v_{q,n})]^* \cdot \operatorname{diag}(H_n, I_{(n+1)q}) \cdot \Omega_{q,n,\alpha}(\zeta) \\ \times \operatorname{diag}(H_n^-, H_{\alpha\triangleright n}^-) \cdot \operatorname{diag}\big(R_{T_{q,n}}(\alpha), H_n\big) \cdot \operatorname{diag}(v_{q,n}, v_{q,n}) \tag{85}$$

and

$$\tilde{\Theta}_{n,\alpha}(\zeta) := I_{2q} + [\operatorname{diag}(v_{q,n}, v_{q,n})]^* \cdot \operatorname{diag}(H_n, I_{(n+1)q}) \cdot \tilde{\Omega}_{q,n,\alpha}(\zeta) \\ \times \operatorname{diag}(H_n^-, H_{\alpha\triangleright n}^-) \cdot \operatorname{diag}\big(R_{T_{q,n}}(\alpha), H_n\big) \cdot \operatorname{diag}(v_{q,n}, v_{q,n}). \tag{86}$$

Then $\Theta_{n,\alpha}$ and $\tilde{\Theta}_{n,\alpha}$ are matrix polynomials of degree not greater than $n+1$ and, for each $\zeta \in \mathbb{C}$, the representations

$$\Theta_{n,\alpha}(\zeta) = U_{n,\alpha}(\zeta) B_{n,\alpha} \qquad \text{and} \qquad \tilde{\Theta}_{n,\alpha}(\zeta) = \tilde{U}_{n,\alpha}(\zeta) \tilde{B}_{n,\alpha} \tag{87}$$

hold true, where $U_{n,\alpha} : \mathbb{C} \to \mathbb{C}^{2q\times 2q}$ and $\tilde{U}_{n,\alpha} : \mathbb{C} \to \mathbb{C}^{2q\times 2q}$ are defined by (30) and (59), and where $B_{n,\alpha}$ and $\tilde{B}_{n,\alpha}$ are given by (81) and (82), respectively. If

$$\Theta_{n,\alpha} = [\Theta_{n,\alpha}^{(j,k)}]_{j,k=1}^{2} \qquad \text{and} \qquad \tilde{\Theta}_{n,\alpha} = [\tilde{\Theta}_{n,\alpha}^{(j,k)}]_{j,k=1}^{2} \tag{88}$$

are the $q \times q$ block representations of $\Theta_{n,\alpha}$ and $\tilde{\Theta}_{n,\alpha}$, respectively, for each $\zeta \in \mathbb{C}$, then

$$\Theta_{n,\alpha}^{(1,1)}(\zeta) = I_q + (\zeta - \alpha) v_{q,n}^* H_n T_{q,n}^* R_{T_{q,n}^*}(\zeta) H_n^- R_{T_{q,n}}(\alpha) v_{q,n}, \tag{89}$$

$$\Theta_{n,\alpha}^{(1,2)}(\zeta) = v_{q,n}^* H_n \Big[R_{T_{q,n}^*}(\alpha) \Big]^{-1} R_{T_{q,n}^*}(\zeta) H_{\alpha\triangleright n}^- H_n^- v_{q,n}, \tag{90}$$

$$\Theta_{n,\alpha}^{(2,1)}(\zeta) = -(\zeta - \alpha) v_{q,n}^* R_{T_{q,n}^*}(\zeta) H_n^- R_{T_{q,n}}(\alpha) v_{q,n}, \tag{91}$$

$$\Theta_{n,\alpha}^{(2,2)}(\zeta) = I_q - (\zeta - \alpha) v_{q,n}^* R_{T_{q,n}^*}(\zeta) H_{\alpha\triangleright n}^- H_n v_{q,n}, \tag{92}$$

$$\tilde{\Theta}_{n,\alpha}^{(1,1)}(\zeta) = I_q + (\zeta - \alpha) v_{q,n}^* H_n T_{q,n}^* R_{T_{q,n}^*}(\zeta) H_n^- R_{T_{q,n}}(\alpha) v_{q,n}, \tag{93}$$

$$\tilde{\Theta}_{n,\alpha}^{(1,2)}(\zeta) = (\zeta - \alpha) v_{q,n}^* H_n \Big[R_{T_{q,n}^*}(\alpha) \Big]^{-1} R_{T_{q,n}^*}(\zeta) H_{\alpha\triangleright n}^- H_n v_{q,n}, \tag{94}$$

$$\tilde{\Theta}_{n,\alpha}^{(2,1)}(\zeta) = -v_{q,n}^* R_{T_{q,n}^*}(\zeta) H_n^- R_{T_{q,n}}(\alpha) v_{q,n}, \tag{95}$$

and

$$\tilde{\Theta}_{n,\alpha}^{(2,2)}(\zeta) = I_q - (\zeta - \alpha) v_{q,n}^* R_{T_{q,n}^*}(\zeta) H_{\alpha \triangleright n}^- H_n v_{q,n}. \tag{96}$$

Proof Remark 7.14 shows that $\Theta_{n,\alpha}$ and $\tilde{\Theta}_{n,\alpha}$ are matrix polynomials of degree not greater than $n + 1$. Let $\zeta \in \mathbb{C}$. Because of the Remarks 5.8 and 5.7, we have $H_n^* = H_n$ and $H_{\alpha \triangleright n}^* = H_{\alpha \triangleright n}$. Using (85) and (83), one can easily check that (89), (90), (91), and (92) hold true. From (86), (84), and (88) we infer that (93), (94), (95), and (96) are valid. Let

$$\Phi_{n,\alpha} = [\Phi_{n,\alpha}^{(j,k)}]_{j,k=1}^2 \qquad \text{and} \qquad \tilde{\Phi}_{n,\alpha} = [\tilde{\Phi}_{n,\alpha}^{(j,k)}]_{j,k=1}^2 \tag{97}$$

be the $q \times q$ block representations of

$$\Phi_{n,\alpha} := U_{n,\alpha} B_{n,\alpha} \qquad \text{and} \qquad \tilde{\Phi}_{n,\alpha} := \tilde{U}_{n,\alpha} \tilde{B}_{n,\alpha}. \tag{98}$$

By virtue of (31)–(35), (81), and (89), then

$$\Phi_{n,\alpha}^{(1,1)}(\zeta) = I_q + (\zeta - \alpha) v_{q,n}^* H_n T_{q,n}^* R_{T_{q,n}^*}(\zeta) H_n^- R_{T_{q,n}}(\alpha) v_{q,n} = \Theta_{n,\alpha}^{(1,1)}(\zeta), \tag{99}$$

follows, whereas (31)–(35), (81), and (91) show that

$$\Phi_{n,\alpha}^{(2,1)}(\zeta) = -(\zeta - \alpha) v_{q,n}^* R_{T_{q,n}^*}(\zeta) H_n^- R_{T_{q,n}}(\alpha) v_{q,n} = \Theta_{n,\alpha}^{(2,1)}(\zeta). \tag{100}$$

From (98), (31)–(35), (81), Lemma 6.17, Remark 17.10, and (90) we conclude

$$\begin{aligned}
\Phi_{n,\alpha}^{(1,2)}(\zeta) &= \left[I_q + (\zeta - \alpha) v_{q,n}^* H_n T_{q,n}^* R_{T_{q,n}^*}(\zeta) H_n^- R_{T_{q,n}}(\alpha) v_{q,n} \right] v_{q,n}^* H_n H_{\alpha \triangleright n}^- H_n v_{q,n} \\
&\quad + (\zeta - \alpha) v_{q,n}^* H_n T_{q,n}^* R_{T_{q,n}^*}(\zeta) H_n^- R_{T_{q,n}}(\alpha) T_{q,n} H_n v_{q,n} \\
&= v_{q,n}^* H_n \left[H_{\alpha \triangleright n}^- + (\zeta - \alpha) T_{q,n}^* R_{T_{q,n}^*}(\zeta) H_n^- R_{T_{q,n}}(\alpha) \left(v_{q,n} v_{q,n}^* H_n H_{\alpha \triangleright n}^- + T_{q,n} \right) \right] H_n v_{q,n} \\
&= v_{q,n}^* H_n \left[I_{(n+1)q} + (\zeta - \alpha) T_{q,n}^* R_{T_{q,n}^*}(\zeta) \right] H_{\alpha \triangleright n}^- H_n v_{q,n} \\
&= v_{q,n}^* H_n \left[R_{T_{q,n}^*}(\alpha) \right]^{-1} R_{T_{q,n}^*}(\zeta) H_{\alpha \triangleright n}^- H_n v_{q,n} = \Theta_{n,\alpha}^{(1,2)}(\zeta)
\end{aligned}$$

and, using additionally (92) instead of (90), furthermore

$$\begin{aligned}
\Phi_{n,\alpha}^{(2,2)}(\zeta) &= -(\zeta-\alpha)v_{q,n}^* R_{T_{q,n}^*}(\zeta) H_n^- R_{T_{q,n}}(\alpha) v_{q,n} v_{q,n}^* H_n H_{\alpha\triangleright n}^- H_n v_{q,n} \\
&\qquad + I_q - (\zeta-\alpha) v_{q,n}^* R_{T_{q,n}^*}(\zeta) H_n^- R_{T_{q,n}}(\alpha) T_{q,n} H_n v_{q,n} \\
&= I_q - (\zeta-\alpha) v_{q,n}^* R_{T_{q,n}^*}(\zeta) H_n^- R_{T_{q,n}}(\alpha)(v_{q,n} v_{q,n}^* H_n H_{\alpha\triangleright n}^- + T_{q,n}) H_n v_{q,n} \\
&= I_q - (\zeta-\alpha) v_{q,n}^* R_{T_{q,n}^*}(\zeta) H_{\alpha\triangleright n}^- H_n v_{q,n} = \Theta_{n,\alpha}^{(2,2)}(\zeta).
\end{aligned}$$

Consequently, taking additionally into account (99), (100), (88), (98), and (97), we obtain the first equation in (87). From (98), Remark 7.7, (82), (97), Lemma 6.17, Remark 17.10, and (93) we obtain

$$\begin{aligned}
&\tilde{\Phi}_{n,\alpha}^{(1,1)}(\zeta) \\
&= I_q + (\zeta-\alpha) v_{q,n}^* H_n \left[R_{T_{q,n}^*}(\alpha)\right]^{-1} R_{T_{q,n}^*}(\zeta) H_{\alpha\triangleright n}^- R_{T_{q,n}}(\alpha) v_{q,n} \\
&\quad -(\zeta-\alpha) v_{q,n}^* H_n \left[R_{T_{q,n}^*}(\alpha)\right]^{-1} R_{T_{q,n}^*}(\zeta) H_{\alpha\triangleright n}^- H_n v_{q,n} v_{q,n}^* R_{T_{q,n}^*}(\alpha) H_n^- R_{T_{q,n}}(\alpha) v_{q,n} \\
&= I_q + (\zeta-\alpha) v_{q,n}^* H_n \left[R_{T_{q,n}^*}(\alpha)\right]^{-1} R_{T_{q,n}^*}(\zeta) H_{\alpha\triangleright n}^- \\
&\quad \times \left[I_{(n+1)q} - H_n v_{q,n} v_{q,n}^* R_{T_{q,n}^*}(\alpha) H_n^-\right] R_{T_{q,n}}(\alpha) v_{q,n} \\
&= I_q + (\zeta-\alpha) v_{q,n}^* H_n \left[R_{T_{q,n}^*}(\alpha)\right]^{-1} R_{T_{q,n}^*}(\zeta) T_{q,n}^* R_{T_{q,n}^*}(\alpha) H_n^- R_{T_{q,n}}(\alpha) v_{q,n} \\
&= I_q + (\zeta-\alpha) v_{q,n}^* H_n T_{q,n}^* R_{T_{q,n}^*}(\zeta) H_n^- R_{T_{q,n}}(\alpha) v_{q,n} = \tilde{\Theta}_{n,\alpha}^{(1,1)}(\zeta),
\end{aligned} \tag{101}$$

whereas (98), Remark 7.7, (82), (97), and (94) show that

$$\tilde{\Phi}_{n,\alpha}^{(1,2)}(\zeta) = (\zeta-\alpha) v_{q,n}^* H_n \left[R_{T_{q,n}^*}(\alpha)\right]^{-1} R_{T_{q,n}^*}(\zeta) H_{\alpha\triangleright n}^- H_n v_{q,n} = \tilde{\Theta}_{n,\alpha}^{(1,2)}(\zeta). \tag{102}$$

Using (98), Remark 7.7, (82), (97), Lemma 6.17, Remark 4.1, and (95), we get

$$\begin{aligned}
&\tilde{\Phi}_{n,\alpha}^{(2,1)}(\zeta) \\
&= -(\zeta-\alpha) v_{q,n}^* R_{T_{q,n}^*}(\zeta) H_{\alpha\triangleright n}^- R_{T_{q,n}}(\alpha) v_{q,n} \\
&\qquad - \left[I_q - (\zeta-\alpha) v_{q,n}^* R_{T_{q,n}^*}(\zeta) H_{\alpha\triangleright n}^- H_n v_{q,n}\right] v_{q,n}^* R_{T_{q,n}^*}(\alpha) H_n^- R_{T_{q,n}}(\alpha) v_{q,n} \\
&= -v_{q,n}^* R_{T_{q,n}^*}(\zeta) \Bigg\{ (\zeta-\alpha) H_{\alpha\triangleright n}^- \left[I_{(n+1)q} - H_n v_{q,n} v_{q,n}^* R_{T_{q,n}^*}(\alpha) H_n^-\right] \\
&\qquad + \left[R_{T_{q,n}^*}(\zeta)\right]^{-1} R_{T_{q,n}^*}(\alpha) H_n^- \Bigg\} R_{T_{q,n}}(\alpha) v_{q,n}
\end{aligned}$$

$$= -v_{q,n}^* R_{T_{q,n}^*}(\zeta)\Big((\zeta-\alpha)T_{q,n}^* R_{T_{q,n}^*}(\alpha)H_n^- + \Big[R_{T_{q,n}^*}(\zeta)\Big]^{-1} R_{T_{q,n}^*}(\alpha)H_n^-\Big) R_{T_{q,n}}(\alpha)v_{q,n}$$
$$= -v_{q,n}^* R_{T_{q,n}^*}(\zeta)(I_{(n+1)q} - \alpha T_{q,n}^*) R_{T_{q,n}^*}(\alpha) H_n^- R_{T_{q,n}}(\alpha) v_{q,n}$$
$$= -v_{q,n}^* R_{T_{q,n}^*}(\zeta) H_n^- R_{T_{q,n}}(\alpha) v_{q,n} = \tilde{\Theta}_{n,\alpha}^{(2,1)}(\zeta) \tag{103}$$

and, in view of (96), furthermore

$$\tilde{\Phi}_{n,\alpha}^{(2,2)}(\zeta) = I_q - (\zeta-\alpha) v_{q,n}^* R_{T_{q,n}^*}(\zeta) H_{\alpha \triangleright n}^- H_n v_{q,n} = \tilde{\Theta}_{n,\alpha}^{(2,2)}(\zeta).$$

Thus, (101), (102), (103), (88), (98), and (97) imply the second equation in (87).

The following result marks one of the crucial points of the whole paper. It describes the desired coupling between the two single Potapov matrix inequalities of our system. For analogous results we refer the reader to [16, Proposition 6.10] and [17, Proposition 6.10].

Proposition 7.16 *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N} \cup \{\infty\}$, and let $(s_j)_{j=0}^{\kappa} \in \mathscr{K}_{q,\kappa,\alpha}^{\geq,\mathrm{e}}$. For each $n \in \mathbb{N}_0$ with $2n+1 \leq \kappa$, the functions $\Theta_{n,\alpha}\colon \mathbb{C} \to \mathbb{C}^{2q\times 2q}$ given by (85) and $\tilde{\Theta}_{n,\alpha}\colon \mathbb{C} \to \mathbb{C}^{2q\times 2q}$ given by (86) fulfill for each $\zeta \in \mathbb{C} \setminus \{\alpha\}$ the identity*

$$\tilde{\Theta}_{n,\alpha}(\zeta) = \operatorname{diag}\big((\zeta-\alpha)I_q, I_q\big) \cdot \Theta_{n,\alpha}(\zeta) \cdot \operatorname{diag}\Big((\zeta-\alpha)^{-1} I_q, I_q\Big).$$

Proof In view of (83), (84), (85), and (86), the assertion follows by direct computation (see also [34, Lemma 9.13]).

It should be mentioned that assuming $\alpha = 0$ and positive Hermitian information blocks H_n and $H_{\alpha\triangleright n}$ matrix polynomials closely related to $\Theta_{n,\alpha}$ and $\tilde{\Theta}_{n,\alpha}$ were introduced by Yu. M. Dyukarev in [23]. He also computed their $\tilde{J}_q$-forms and observed some coupling relation between them. Now we obtain a generalization of a result which, for the special case $\alpha = 0$, corresponds to Bolotnikov [7, Lemma 4.2].

Lemma 7.17 *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N} \cup \{\infty\}$, and let $(s_j)_{j=0}^{\kappa} \in \mathscr{K}_{q,\kappa,\alpha}^{\geq,\mathrm{e}}$. For each $n \in \mathbb{N}_0$ with $2n+1 \leq \kappa$ and every choice of $z \in \mathbb{C}$ and $w \in \mathbb{C}$, then*

$$\tilde{J}_q - \Theta_{n,\alpha}(z)\tilde{J}_q \Theta_{n,\alpha}^*(w) = -\mathrm{i}(z-\overline{w})[\operatorname{diag}(v_{q,n}, v_{q,n})]^*[T_{q,n}H_n, -I_{(n+1)q}]^*$$
$$\times R_{T_{q,n}^*}(z) H_n^- \Big[R_{T_{q,n}^*}(w)\Big]^* [T_{q,n}H_n, -I_{(n+1)q}] \operatorname{diag}(v_{q,n}, v_{q,n})$$

and

$$\tilde{J}_q - \tilde{\Theta}_{n,\alpha}(z)\tilde{J}_q\tilde{\Theta}_{n,\alpha}^*(w) = -\mathrm{i}(z-\overline{w})[\operatorname{diag}(v_{q,n}, v_{q,n})]^* \Big[\big[R_{T_{q,n}}(\alpha)\big]^{-1} H_n, -I_{(n+1)q}\Big]^*$$
$$\times R_{T_{q,n}^*}(z) H_{\alpha\triangleright n}^- \Big[R_{T_{q,n}^*}(w)\Big]^* \Big[\big[R_{T_{q,n}}(\alpha)\big]^{-1} H_n, -I_{(n+1)q}\Big] \operatorname{diag}(v_{q,n}, v_{q,n}).$$

Proof Let $n \in \mathbb{N}_0$ be such that $2n+1 \leq \kappa$ and let $z, w \in \mathbb{C}$. Using Lemma 7.15, Remark 7.11, and Lemma 7.5, we get

$$\begin{aligned}\tilde{J}_q - \Theta_{n,\alpha}(z)\tilde{J}_q\Theta_{n,\alpha}^*(w) &= \tilde{J}_q - U_{n,\alpha}(z)B_{n,\alpha}\tilde{J}_q B_{n,\alpha}^* U_{n,\alpha}^*(w)\\ &= \tilde{J}_q - U_{n,\alpha}(z)\tilde{J}_q^* U_{n,\alpha}^*(w)\\ &= -\mathrm{i}(z-\overline{w})[\operatorname{diag}(v_{q,n}, v_{q,n})]^*\\ &\qquad [T_{q,n}H_n, -I_{(n+1)q}]^* R_{T_{q,n}^*}(z)H_n^-\\ &\qquad \times \left[R_{T_{q,n}^*}(w)\right]^*[T_{q,n}H_n, -I_{(n+1)q}]\operatorname{diag}(v_{q,n}, v_{q,n}).\end{aligned}$$

Analogously, from Lemma 7.15, Remark 7.11, and Lemma 7.8 we conclude similarly

$$\begin{aligned}&\tilde{J}_q - \tilde{\Theta}_{n,\alpha}(z)\tilde{J}_q\tilde{\Theta}_{n,\alpha}^*(w) = -\mathrm{i}(z-\overline{w})[\operatorname{diag}(v_{q,n}, v_{q,n})]^*\left[\left[R_{T_{q,n}}(\alpha)\right]^{-1}H_n, -I_{(n+1)q}\right]^*\\ &\times R_{T_{q,n}^*}(z)H_{\alpha\triangleright n}^-\left[R_{T_{q,n}^*}(w)\right]^*\left[\left[R_{T_{q,n}}(\alpha)\right]^{-1}H_n, -I_{(n+1)q}\right]\operatorname{diag}(v_{q,n}, v_{q,n}).\end{aligned}$$

Remark 7.18 Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N} \cup \{\infty\}$, and let $(s_j)_{j=0}^{\kappa} \in \mathscr{K}_{q,\kappa,\alpha}^{\geq,\mathrm{e}}$. For each $n \in \mathbb{N}_0$ with $2n+1 \leq \kappa$ and each $z \in \mathbb{C} \setminus \mathbb{R}$, then Lemmas 6.12 and 7.17 show that

$$\frac{1}{2\operatorname{Im} z}\left[\tilde{J}_q - \Theta_{n,\alpha}(z)\tilde{J}_q\Theta_{n,\alpha}^*(z)\right] \geq 0 \quad \text{and} \quad \frac{1}{2\operatorname{Im} z}\left[\tilde{J}_q - \tilde{\Theta}_{n,\alpha}(z)\tilde{J}_q\tilde{\Theta}_{n,\alpha}^*(z)\right] \geq 0.$$

Lemma 7.19 *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N} \cup \{\infty\}$, and let $(s_j)_{j=0}^{\kappa} \in \mathscr{K}_{q,\kappa,\alpha}^{\geq,\mathrm{e}}$. For each $n \in \mathbb{N}_0$ with $2n+1 \leq \kappa$, then the following statements hold true:*

(a) $\tilde{J}_q - \Theta_{n,\alpha}(x)\tilde{J}_q\Theta_{n,\alpha}^*(x) = 0$ *and* $\tilde{J}_q - \tilde{\Theta}_{n,\alpha}(x)\tilde{J}_q\tilde{\Theta}_{n,\alpha}^*(x) = 0$ *hold true for all* $x \in \mathbb{R}$.

(b) For all $z \in \mathbb{C}$, the matrices $\Theta_{n,\alpha}(z)$ and $\tilde{\Theta}_{n,\alpha}(z)$ are both non-singular and fulfill

$$\Theta_{n,\alpha}^{-1}(z) = \tilde{J}_q\Theta_{n,\alpha}^*(\overline{z})\tilde{J}_q \qquad \text{and} \qquad \tilde{\Theta}_{n,\alpha}^{-1}(z) = \tilde{J}_q\tilde{\Theta}_{n,\alpha}^*(\overline{z})\tilde{J}_q. \tag{104}$$

(c) For every choice of z and w in $\mathbb{C}$, the equations

$$\tilde{J}_q - \Theta_{n,\alpha}^{-*}(z)\tilde{J}_q\Theta_{n,\alpha}^{-1}(w) = \tilde{J}_q\left[\tilde{J}_q - \Theta_{n,\alpha}(\overline{z})\tilde{J}_q\Theta_{n,\alpha}^*(\overline{w})\right]\tilde{J}_q$$

and

$$\tilde{J}_q - \tilde{\Theta}_{n,\alpha}^{-*}(z)\tilde{J}_q\tilde{\Theta}_{n,\alpha}^{-1}(w) = \tilde{J}_q\left[\tilde{J}_q - \tilde{\Theta}_{n,\alpha}(\overline{z})\tilde{J}_q\tilde{\Theta}_{n,\alpha}^*(\overline{w})\right]\tilde{J}_q$$

hold true.

Proof

(a) Use Lemma 7.17.

(b) We know from Lemma 7.15 that $\Theta_{n,\alpha}$ and $\tilde{\Theta}_{n,\alpha}$ are matrix polynomials. Consequently, $\Theta^{\vee}_{n,\alpha}\colon \mathbb{C} \to \mathbb{C}^{2q\times 2q}$ and $\tilde{\Theta}^{\vee}_{n,\alpha}\colon \mathbb{C} \to \mathbb{C}^{2q\times 2q}$ defined by $\Theta^{\vee}_{n,\alpha}(\zeta) := \Theta^{*}_{n,\alpha}(\overline{\zeta})$ and $\tilde{\Theta}^{\vee}_{n,\alpha}(\zeta) := \tilde{\Theta}^{*}_{n,\alpha}(\overline{\zeta})$ are matrix polynomials as well. Thus, $F := \tilde{J}_q - \Theta_{n,\alpha}\tilde{J}_q\Theta^{\vee}_{n,\alpha}$ and $\tilde{F} := \tilde{J}_q - \tilde{\Theta}_{n,\alpha}\tilde{J}_q\tilde{\Theta}^{\vee}_{n,\alpha}$ are holomorphic in $\mathbb{C}$. For each $x \in \mathbb{R}$, we see from part (a) that $F(x) = \tilde{J}_q - \Theta_{n,\alpha}(x)\tilde{J}_q\Theta^{\vee}_{n,\alpha}(x) = \tilde{J}_q - \Theta_{n,\alpha}(x)\tilde{J}_q\Theta^{*}_{n,\alpha}(\overline{x}) = 0$ and, analogously, that $\tilde{F}(x) = 0$. The identity theorem for holomorphic functions implies $\Theta_{n,\alpha}(z)\tilde{J}_q\Theta^{*}_{n,\alpha}(\overline{z}) = \Theta_{n,\alpha}(z)\tilde{J}_q\Theta^{\vee}_{n,\alpha}(z) = \tilde{J}_q - F(z) = \tilde{J}_q$ and, analogously, $\tilde{\Theta}_{n,\alpha}(z)\tilde{J}_q\tilde{\Theta}^{*}_{n,\alpha}(\overline{z}) = \tilde{J}_q$ for all $z \in \mathbb{C}$. Because of $\tilde{J}_q^2 = I_{2q}$, we get $\Theta_{n,\alpha}(z)\tilde{J}_q\Theta^{*}_{n,\alpha}(\overline{z})\tilde{J}_q = \tilde{J}_q^2 = I_{2q}$ and $\tilde{\Theta}_{n,\alpha}(z)\tilde{J}_q\tilde{\Theta}^{*}_{n,\alpha}(\overline{z})\tilde{J}_q = \tilde{J}_q^2 = I_{2q}$ for each $z \in \mathbb{C}$. For all $z \in \mathbb{C}$, then $\det\Theta_{n,\alpha}(z) \neq 0$ and $\det\tilde{\Theta}_{n,\alpha}(z) \neq 0$ and (104) follows.

(c) Use part (b) and $\tilde{J}_q^2 = I_{2q}$.

Lemma 7.20 *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}\cup\{\infty\}$, and let $(s_j)_{j=0}^{\kappa} \in \mathscr{K}^{\geq,\mathrm{e}}_{q,\kappa,\alpha}$. For each $n \in \mathbb{N}_0$ such that $2n+1 \leq \kappa$ and every choice of z and w in $\mathbb{C}$, then*

$$\begin{aligned}\tilde{J}_q - \Theta^{-*}_{n,\alpha}(z)\tilde{J}_q\Theta^{-1}_{n,\alpha}(w) &= -\mathrm{i}(\overline{z}-w)[\operatorname{diag}(v_{q,n}, v_{q,n})]^*[I_{(n+1)q}, T_{q,n}H_n]^* \\ &\quad\times R_{T^*_{q,n}}(\overline{z})H_n^- R_{T_{q,n}}(w)[I_{(n+1)q}, T_{q,n}H_n]\operatorname{diag}(v_{q,n}, v_{q,n})\end{aligned}$$

and

$$\begin{aligned}\tilde{J}_q - \tilde{\Theta}^{-*}_{n,\alpha}(z)\tilde{J}_q\tilde{\Theta}^{-1}_{n,\alpha}(w) &= -\mathrm{i}(\overline{z}-w)[\operatorname{diag}(v_{q,n}, v_{q,n})]^*[I_{(n+1)q}, [R_{T_{q,n}}(\alpha)]^{-1}H_n]^* \\ &\quad\times R_{T^*_{q,n}}(\overline{z})H^-_{\alpha\triangleright n}R_{T_{q,n}}(w)[I_{(n+1)q}, [R_{T_{q,n}}(\alpha)]^{-1}H_n]\operatorname{diag}(v_{q,n}, v_{q,n}). \end{aligned} \tag{105}$$

Proof Let $n \in \mathbb{N}_0$ be such that $2n+1 \leq \kappa$ and let $z, w \in \mathbb{C}$. Using Lemma 7.19(b), Lemma 7.19(c), Lemma 7.17, and Remark 7.3, we obtain

$$\begin{aligned}&\tilde{J}_q - \Theta^{-*}_{n,\alpha}(z)\tilde{J}_q\Theta^{-1}_{n,\alpha}(w) \\ &= \tilde{J}_q\Big[\tilde{J}_q - \Theta_{n,\alpha}(\overline{z})\tilde{J}_q\Theta^{*}_{n,\alpha}(\overline{w})\Big]\tilde{J}_q \\ &= \tilde{J}_q\Big\{-\mathrm{i}(\overline{z}-w)[\operatorname{diag}(v_{q,n}, v_{q,n})]^*[T_{q,n}H_n, -I_{(n+1)q}]^* R_{T^*_{q,n}}(\overline{z})H_n^- \\ &\qquad\times\Big[R_{T^*_{q,n}}(\overline{w})\Big]^*[T_{q,n}H_n, -I_{(n+1)q}]\operatorname{diag}(v_{q,n}, v_{q,n})\Big\}\tilde{J}_q\end{aligned}$$

$$
\begin{aligned}
&= -\mathrm{i}(\overline{z} - w)\big(\mathrm{i}[\operatorname{diag}(v_{q,n}, v_{q,n})]^*[I_{(n+1)q}, T_{q,n}H_n]^*\big)R_{T_{q,n}^*}(\overline{z})H_n^- \\
&\quad \times R_{T_{q,n}}(w)\big((-\mathrm{i}) \cdot [I_{(n+1)q}, T_{q,n}H_n]\operatorname{diag}(v_{q,n}, v_{q,n})\big) \\
&= -\mathrm{i}(\overline{z} - w)[\operatorname{diag}(v_{q,n}, v_{q,n})]^*[I_{(n+1)q}, T_{q,n}H_n]^* R_{T_{q,n}^*}(\overline{z})H_n^- \\
&\quad \times R_{T_{q,n}}(w)[I_{(n+1)q}, T_{q,n}H_n]\operatorname{diag}(v_{q,n}, v_{q,n})
\end{aligned}
$$

and analogously (105).

Lemma 7.21 *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N} \cup \{\infty\}$, and let $(s_j)_{j=0}^{\kappa} \in \mathscr{K}_{q,\kappa,\alpha}^{\geq,\mathrm{e}}$. For each $n \in \mathbb{N}_0$ with $2n+1 \leq \kappa$ and all $w, z \in \mathbb{C}$, then*

$$
\begin{aligned}
\tilde{J}_q - \Theta_{n,\alpha}^*(w)\tilde{J}_q\Theta_{n,\alpha}(z) &= \mathrm{i}(\overline{w} - z)B_{n,\alpha}^*[\operatorname{diag}(v_{q,n}, v_{q,n})]^*[I_{(n+1)q}, T_{q,n}H_n]^* \\
\times \big[R_{T_{q,n}}(\alpha)\big]^* H_n^- \Big[R_{T_{q,n}^*}(w)\Big]^* &\big[R_{T_{q,n}}(\alpha)\big]^{-1} H_n \Big[R_{T_{q,n}^*}(\alpha)\Big]^{-1} R_{T_{q,n}^*}(z)H_n^- R_{T_{q,n}}(\alpha) \\
&\times [I_{(n+1)q}, T_{q,n}H_n]\operatorname{diag}(v_{q,n}, v_{q,n})B_{n,\alpha} \qquad (106)
\end{aligned}
$$

and

$$
\begin{aligned}
&\tilde{J}_q - \tilde{\Theta}_{n,\alpha}^*(w)\tilde{J}_q\tilde{\Theta}_{n,\alpha}(z) = \mathrm{i}(\overline{w} - z)\tilde{B}_{n,\alpha}^*[\operatorname{diag}(v_{q,n}, v_{q,n})]^*\Big[I_{(n+1)q}, \big[R_{T_{q,n}}(\alpha)\big]^{-1}H_n\Big]^* \\
&\times\big[R_{T_{q,n}}(\alpha)\big]^* H_{\alpha\triangleright n}^- \Big[R_{T_{q,n}^*}(w)\Big]^* \big[R_{T_{q,n}}(\alpha)\big]^{-1} H_{\alpha\triangleright n}\Big[R_{T_{q,n}^*}(\alpha)\Big]^{-1} R_{T_{q,n}^*}(z)H_{\alpha\triangleright n}^- R_{T_{q,n}}(\alpha) \\
&\times \Big[I_{(n+1)q}, \big[R_{T_{q,n}}(\alpha)\big]^{-1}H_n\Big]\operatorname{diag}(v_{q,n}, v_{q,n})\tilde{B}_{n,\alpha}. \qquad (107)
\end{aligned}
$$

Proof Let $n \in \mathbb{N}_0$ be such that $2n+1 \leq \kappa$. From Remark 7.11 and Lemma 7.15 we get

$$
\tilde{J}_q - \Theta_{n,\alpha}^*(w)\tilde{J}_q\Theta_{n,\alpha}(z) = B_{n,\alpha}^*\Big[\tilde{J}_q - U_{n,\alpha}^*(w)\tilde{J}_q U_{n,\alpha}(z)\Big]B_{n,\alpha} \qquad (108)
$$

and, analogously,

$$
\tilde{J}_q - \tilde{\Theta}_{n,\alpha}^*(w)\tilde{J}_q\tilde{\Theta}_{n,\alpha}(z) = \tilde{B}_{n,\alpha}^*\Big[\tilde{J}_q - \tilde{U}_{n,\alpha}^*(w)\tilde{J}_q\tilde{U}_{n,\alpha}(z)\Big]\tilde{B}_{n,\alpha} \qquad (109)
$$

for every choice of z and w in $\mathbb{C}$. Using (108) and Lemma 7.6, we obtain (106). Because of (109) and Lemma 7.9, then (107) follows.

Lemma 7.22 *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N} \cup \{\infty\}$, and let $(s_j)_{j=0}^{\kappa} \in \mathscr{K}_{q,\kappa,\alpha}^{\geq,\mathrm{e}}$. For each $n \in \mathbb{N}_0$ with $2n+1 \leq \kappa$ and each $z \in \mathbb{C} \setminus \mathbb{R}$, then*

$$
\frac{1}{2\operatorname{Im} z}\Big[\tilde{J}_q - \Theta_{n,\alpha}^*(z)\tilde{J}_q\Theta_{n,\alpha}(z)\Big] \geq 0 \quad \textit{and} \quad \frac{1}{2\operatorname{Im} z}\Big[\tilde{J}_q - \tilde{\Theta}_{n,\alpha}^*(z)\tilde{J}_q\tilde{\Theta}_{n,\alpha}(z)\Big] \geq 0.
$$

Proof Let $n \in \mathbb{N}_0$ be such that $2n+1 \leq \kappa$. Because of $(s_j)_{j=0}^{\kappa} \in \mathcal{K}_{q,\kappa,\alpha}^{\geq,\mathrm{e}} \subseteq \mathcal{K}_{q,\kappa,\alpha}^{\geq}$ and Remark 5.7, we have $H_n \in \mathbb{C}_{\geq}^{(n+1)q\times(n+1)q}$ and $H_{\alpha\triangleright n} \in \mathbb{C}_{\geq}^{(n+1)q\times(n+1)q}$. Using Lemma 7.21 and Lemma 6.12, for all $z \in \mathbb{C}\setminus\mathbb{R}$, we get then

$$\begin{aligned}
&\frac{1}{2\operatorname{Im} z}\Big[\tilde{J}_q - \Theta_{n,\alpha}^*(z)\tilde{J}_q\Theta_{n,\alpha}(z)\Big]\\
&= \frac{1}{2\operatorname{Im} z}\Big\{\mathrm{i}(\overline{z}-z)B_{n,\alpha}^*[\operatorname{diag}(v_{q,n}, v_{q,n})]^*[I_{(n+1)q}, T_{q,n}H_n]^*\big[R_{T_{q,n}}(\alpha)\big]^*\\
&\qquad\times H_n^-\Big[R_{T_{q,n}^*}(z)\Big]^*\big[R_{T_{q,n}}(\alpha)\big]^{-1}H_n\Big[R_{T_{q,n}^*}(\alpha)\Big]^{-1}R_{T_{q,n}^*}(z)H_n^-\\
&\qquad\times R_{T_{q,n}}(\alpha)[I_{(n+1)q}, T_{q,n}H_n]\operatorname{diag}(v_{q,n}, v_{q,n})B_{n,\alpha}\Big\}\\
&= \bigg(\Big[R_{T_{q,n}^*}(\alpha)\Big]^{-1}R_{T_{q,n}^*}(z)H_n^-R_{T_{q,n}}(\alpha)[I_{(n+1)q}, T_{q,n}H_n]\operatorname{diag}(v_{q,n}, v_{q,n})B_{n,\alpha}\bigg)^*\\
&\times H_n\bigg(\Big[R_{T_{q,n}^*}(\alpha)\Big]^{-1}R_{T_{q,n}^*}(z)H_n^-R_{T_{q,n}}(\alpha)[I_{(n+1)q}, T_{q,n}H_n]\operatorname{diag}(v_{q,n}, v_{q,n})B_{n,\alpha}\bigg)\geq 0
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{2\operatorname{Im} z}\Big[\tilde{J}_q - \tilde{\Theta}_{n,\alpha}^*(z)\tilde{J}_q\tilde{\Theta}_{n,\alpha}(z)\Big]\\
&= \frac{1}{2\operatorname{Im} z}\Big\{\mathrm{i}(\overline{z}-z)\tilde{B}_{n,\alpha}^*[\operatorname{diag}(v_{q,n}, v_{q,n})]^*\Big[I_{(n+1)q}, \big[R_{T_{q,n}}(\alpha)\big]^{-1}H_n\Big]^*\big[R_{T_{q,n}}(\alpha)\big]^*\\
&\qquad\times H_{\alpha\triangleright n}^-\Big[R_{T_{q,n}^*}(z)\Big]^*\big[R_{T_{q,n}}(\alpha)\big]^{-1}H_{\alpha\triangleright n}\Big[R_{T_{q,n}^*}(\alpha)\Big]^{-1}R_{T_{q,n}^*}(z)H_{\alpha\triangleright n}^-\\
&\qquad\times R_{T_{q,n}}(\alpha)\Big[I_{(n+1)q}, \big[R_{T_{q,n}}(\alpha)\big]^{-1}H_n\Big]\operatorname{diag}(v_{q,n}, v_{q,n})\tilde{B}_{n,\alpha}\Big\}\\
&= \Big\{\Big[R_{T_{q,n}^*}(\alpha)\Big]^{-1}R_{T_{q,n}^*}(z)H_{\alpha\triangleright n}^-R_{T_{q,n}}(\alpha)\Big[I_{(n+1)q}, \big[R_{T_{q,n}}(\alpha)\big]^{-1}H_n\Big]\\
&\quad\times\operatorname{diag}(v_{q,n}, v_{q,n})\tilde{B}_{n,\alpha}\Big\}^*H_{\alpha\triangleright n}\Big\{\Big[R_{T_{q,n}^*}(\alpha)\Big]^{-1}R_{T_{q,n}^*}(z)H_{\alpha\triangleright n}^-R_{T_{q,n}}(\alpha)\\
&\quad\times\Big[I_{(n+1)q}, \big[R_{T_{q,n}}(\alpha)\big]^{-1}H_n\Big]\operatorname{diag}(v_{q,n}, v_{q,n})\tilde{B}_{n,\alpha}\Big\}\geq 0.
\end{aligned}$$

The combination of Lemma 7.19(a) with Lemma 7.22 shows in particular that the $2q\times 2q$ matrix polynomials $\Theta_{n,\alpha}$ and $\tilde{\Theta}_{n,\alpha}$ are both $\tilde{J}_q$-inner. Now we are going to derive factorizations of the Schur complements occurring in the Potapov-type fundamental matrices defined in Notation 4.2. In order to realize this aim we need a little preparation.

Lemma 7.23 *Let $\alpha \in \mathbb{R}$, let $n \in \mathbb{N}_0$, and let $(s_j)_{j=0}^{2n+1}$ be a sequence of complex $q \times q$ matrices. Let $\mathscr{G}$ be a non-empty subset of $\mathbb{C}$ and let $f : \mathscr{G} \to \mathbb{C}^{q\times q}$ be a matrix-valued function. Let $\mathbf{b}_{2n}^{[f]} : \mathscr{G} \to \mathbb{C}^{(n+1)q\times q}$ and $\mathbf{b}_{2n+1}^{[f]} : \mathscr{G} \to \mathbb{C}^{(n+1)q\times q}$ be given by (6) and (8). For each $z \in \mathscr{G}$, then*

$$\mathbf{b}_{2n}^{[f]}(z) = R_{T_{q,n}}(z)[I_{(n+1)q}, T_{q,n}H_n]\operatorname{diag}(v_{q,n}, v_{q,n})\begin{bmatrix} f(z) \\ I_q \end{bmatrix} \tag{110}$$

and

$$\mathbf{b}_{2n+1}^{[f]}(z) = R_{T_{q,n}}(z)\Big[I_{(n+1)q}, \big[R_{T_{q,n}}(\alpha)\big]^{-1}H_n\Big]\operatorname{diag}(v_{q,n}, v_{q,n})\begin{bmatrix} (z-\alpha)f(z) \\ I_q \end{bmatrix}. \tag{111}$$

Proof Consider an arbitrary $z \in \mathscr{G}$. According to Remark 5.3(d) we have $-T_{q,n}H_n v_{q,n} = u_n$. Hence,

$$[I_{(n+1)q}, T_{q,n}H_n]\operatorname{diag}(v_{q,n}, v_{q,n})\begin{bmatrix} f(z) \\ I_q \end{bmatrix} = [v_{q,n}, -u_n]\begin{bmatrix} f(z) \\ I_q \end{bmatrix} = v_{q,n}f(z) - u_n.$$

Taking additionally into account (6), we get then

$$\begin{aligned}\mathbf{b}_{2n}^{[f]}(z) &= R_{T_{q,n}}(z)\big[v_{q,n}f(z) - u_n\big] \\ &= R_{T_{q,n}}(z)[I_{(n+1)q}, T_{q,n}H_n]\operatorname{diag}(v_{q,n}, v_{q,n})\begin{bmatrix} f(z) \\ I_q \end{bmatrix}.\end{aligned}$$

According to Remark 5.3(d) we have $H_n v_{q,n} = y_{0,n}$ and $-T_{q,n}H_n v_{q,n} = u_n$. Furthermore, form (4) we see $[R_{T_{q,n}}(\alpha)]^{-1} = I_{(n+1)q} - \alpha T_{q,n}$. Thus, we obtain

$$\big[R_{T_{q,n}}(\alpha)\big]^{-1}H_n v_{q,n} = (I_{(n+1)q} - \alpha T_{q,n})H_n v_{q,n} = H_n v_{q,n} - \alpha T_{q,n}H_n v_{q,n} = y_{0,n} + \alpha u_n$$

and hence

$$\begin{aligned}\Big[I_{(n+1)q}, \big[R_{T_{q,n}}(\alpha)\big]^{-1}H_n\Big]\operatorname{diag}(v_{q,n}, v_{q,n}) &= \Big[v_{q,n}, \big[R_{T_{q,n}}(\alpha)\big]^{-1}H_n v_{q,n}\Big] \\ &= \big[v_{q,n}, y_{0,n} + \alpha u_n\big],\end{aligned}$$

implying

$$\Big[I_{(n+1)q}, \big[R_{T_{q,n}}(\alpha)\big]^{-1} H_n\Big] \operatorname{diag}(v_{q,n}, v_{q,n}) \begin{bmatrix} (z-\alpha) f(z) \\ I_q \end{bmatrix}$$
$$= (z-\alpha) v_{q,n} f(z) + y_{0,n} + \alpha u_n.$$

Taking additionally into account (8), we get then

$$\begin{aligned} \mathbf{b}_{2n+1}^{[f]}(z) &= R_{T_{q,n}}(z)\big(v_{q,n}[(z-\alpha) f(z)] - (-\alpha u_n - y_{0,n})\big) \\ &= R_{T_{q,n}}(z)\Big[I_{(n+1)q}, \big[R_{T_{q,n}}(\alpha)\big]^{-1} H_n\Big] \operatorname{diag}(v_{q,n}, v_{q,n}) \begin{bmatrix} (z-\alpha) f(z) \\ I_q \end{bmatrix}. \end{aligned}$$

Proposition 7.24 *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N} \cup \{\infty\}$, let $(s_j)_{j=0}^{\kappa} \in \mathscr{K}_{q,\kappa,\alpha}^{\geq,\mathrm{e}}$, and let $n \in \mathbb{N}_0$ with $2n+1 \leq \kappa$. Let $\mathscr{G}$ be a subset of $\mathbb{C}$ with $\mathscr{G} \setminus \mathbb{R} \neq \emptyset$ and let $f : \mathscr{G} \to \mathbb{C}^{q \times q}$ be a matrix-valued function. Let $\Sigma_{2n}^{[f]} : \mathscr{G} \setminus \mathbb{R} \to \mathbb{C}^{q \times q}$ and $\Sigma_{2n+1}^{[f]} : \mathscr{G} \setminus \mathbb{R} \to \mathbb{C}^{q \times q}$ be defined by*

$$\Sigma_{2n}^{[f]}(z) := \frac{f(z) - [f(z)]^*}{z - \overline{z}} - \Big[\mathbf{b}_{2n}^{[f]}(z)\Big]^* H_n^{\dagger} \Big[\mathbf{b}_{2n}^{[f]}(z)\Big] \tag{112}$$

and

$$\Sigma_{2n+1}^{[f]}(z) := \frac{(z-\alpha) f(z) - [(z-\alpha) f(z)]^*}{z - \overline{z}} - \Big[\mathbf{b}_{2n+1}^{[f]}(z)\Big]^* H_{\alpha \triangleright n}^{\dagger} \Big[\mathbf{b}_{2n+1}^{[f]}(z)\Big] \tag{113}$$

where $\mathbf{b}_{2n}^{[f]} : \mathscr{G} \to \mathbb{C}^{(n+1)q \times q}$ and $\mathbf{b}_{2n+1}^{[f]} : \mathscr{G} \to \mathbb{C}^{(n+1)q \times q}$ are given by (6) and (8). For each $z \in \mathscr{G} \setminus \mathbb{R}$ with $\mathscr{R}(\mathbf{b}_{2n}^{[f]}(z)) \subseteq \mathscr{R}(H_n)$ and $\mathscr{R}(\mathbf{b}_{2n+1}^{[f]}(z)) \subseteq \mathscr{R}(H_{\alpha \triangleright n})$, then

$$\Sigma_{2n}^{[f]}(z) = \begin{bmatrix} f(z) \\ I_q \end{bmatrix}^* \Theta_{n,\alpha}^{-*}(z) \left(\frac{-\tilde{J}_q}{2 \operatorname{Im} z} \right) \Theta_{n,\alpha}^{-1}(z) \begin{bmatrix} f(z) \\ I_q \end{bmatrix}$$

and

$$\begin{aligned} \Sigma_{2n+1}^{[f]}(z) &= \begin{bmatrix} (z-\alpha) f(z) \\ I_q \end{bmatrix}^* \tilde{\Theta}_{n,\alpha}^{-*}(z) \left(\frac{-\tilde{J}_q}{2 \operatorname{Im} z} \right) \tilde{\Theta}_{n,\alpha}^{-1}(z) \begin{bmatrix} (z-\alpha) f(z) \\ I_q \end{bmatrix} \\ &= \begin{bmatrix} f(z) \\ I_q \end{bmatrix}^* \Theta_{n,\alpha}^{-*}(z) \big[\operatorname{diag}\big((z-\alpha) I_q, I_q\big)\big]^* \\ &\quad \times \left(\frac{-\tilde{J}_q}{2 \operatorname{Im} z} \right) \big[\operatorname{diag}\big((z-\alpha) I_q, I_q\big)\big] \Theta_{n,\alpha}^{-1}(z) \begin{bmatrix} f(z) \\ I_q \end{bmatrix} \end{aligned}$$

Proof Consider an arbitrary $z \in \mathscr{G} \setminus \mathbb{R}$ with $\mathscr{R}(\mathbf{b}_{2n}^{[f]}(z)) \subseteq \mathscr{R}(H_n)$ and $\mathscr{R}(\mathbf{b}_{2n+1}^{[f]}(z)) \subseteq \mathscr{R}(H_{\alpha \triangleright n})$. Since H_n and $H_{\alpha \triangleright n}$ are non-negative Hermitian, we have according to Remark 17.2 then

$$\mathscr{N}(H_n) = \left[\mathscr{R}(H_n^*)\right]^{\perp} = [\mathscr{R}(H_n)]^{\perp} \subseteq \left[\mathscr{R}\left(\mathbf{b}_{2n}^{[f]}(z)\right)\right]^{\perp} = \mathscr{N}\left(\left[\mathbf{b}_{2n}^{[f]}(z)\right]^*\right)$$

and analogously $\mathscr{N}(H_{\alpha \triangleright n}) \subseteq \mathscr{N}([\mathbf{b}_{2n+1}^{[f]}(z)]^*)$. Remark 6.11 and Lemma 6.12 furthermore yield $H_n H_n^- H_n = H_n$ and $H_{\alpha \triangleright n} H_{\alpha \triangleright n}^- H_{\alpha \triangleright n} = H_{\alpha \triangleright n}$. By virtue of Lemma 4.8, then

$$\left[\mathbf{b}_{2n}^{[f]}(z)\right]^* H_n^{\dagger} \left[\mathbf{b}_{2n}^{[f]}(z)\right] = \left[\mathbf{b}_{2n}^{[f]}(z)\right]^* H_n^- \left[\mathbf{b}_{2n}^{[f]}(z)\right] \tag{114}$$

and

$$\left[\mathbf{b}_{2n+1}^{[f]}(z)\right]^* H_{\alpha \triangleright n}^{\dagger} \left[\mathbf{b}_{2n+1}^{[f]}(z)\right] = \left[\mathbf{b}_{2n+1}^{[f]}(z)\right]^* H_{\alpha \triangleright n}^- \left[\mathbf{b}_{2n+1}^{[f]}(z)\right] \tag{115}$$

follow. Remark 7.2 yields

$$\frac{f(z) - [f(z)]^*}{z - \overline{z}} = \begin{bmatrix} f(z) \\ I_q \end{bmatrix}^* \left(\frac{-\tilde{J}_q}{2\operatorname{Im} z}\right) \begin{bmatrix} f(z) \\ I_q \end{bmatrix} = -\frac{1}{2\operatorname{Im} z} \begin{bmatrix} f(z) \\ I_q \end{bmatrix}^* \tilde{J}_q \begin{bmatrix} f(z) \\ I_q \end{bmatrix} \tag{116}$$

and analogously

$$\frac{(z-\alpha) f(z) - [(z-\alpha) f(z)]^*}{z - \overline{z}} = -\frac{1}{2\operatorname{Im} z} \begin{bmatrix} (z-\alpha) f(z) \\ I_q \end{bmatrix}^* \tilde{J}_q \begin{bmatrix} (z-\alpha) f(z) \\ I_q \end{bmatrix}. \tag{117}$$

According to Lemma 7.23 we have (110) and consequently

$$\left[\mathbf{b}_{2n}^{[f]}(z)\right]^* = \begin{bmatrix} f(z) \\ I_q \end{bmatrix}^* \left[\operatorname{diag}(v_{q,n}, v_{q,n})\right]^* [I_{(n+1)q}, T_{q,n} H_n]^* \left[R_{T_{q,n}}(z)\right]^*.$$

Since Remark 4.1 yields $R_{T_{q,n}^*}(\overline{z}) = \left[R_{T_{q,n}}(z)\right]^*$, then

$$\left[\mathbf{b}_{2n}^{[f]}(z)\right]^* = \begin{bmatrix} f(z) \\ I_q \end{bmatrix}^* \left[\operatorname{diag}(v_{q,n}, v_{q,n})\right]^* [I_{(n+1)q}, T_{q,n} H_n]^* R_{T_{q,n}^*}(\overline{z}) \tag{118}$$

follows. Taking into account (118) and (110) we can conclude from Lemma 7.20 then

$$\begin{bmatrix} f(z) \\ I_q \end{bmatrix}^* \left[\tilde{J}_q - \Theta_{n,\alpha}^{-*}(z) \tilde{J}_q \Theta_{n,\alpha}^{-1}(z) \right] \begin{bmatrix} f(z) \\ I_q \end{bmatrix} = -\mathrm{i}(\overline{z} - z) \left[\mathbf{b}_{2n}^{[f]}(z) \right]^* H_n^- \left[\mathbf{b}_{2n}^{[f]}(z) \right].$$

Because of $-\mathrm{i}(\overline{z} - z) = -2 \operatorname{Im} z$, hence

$$\frac{1}{2 \operatorname{Im} z} \begin{bmatrix} f(z) \\ I_q \end{bmatrix}^* \left[\tilde{J}_q - \Theta_{n,\alpha}^{-*}(z) \tilde{J}_q \Theta_{n,\alpha}^{-1}(z) \right] \begin{bmatrix} f(z) \\ I_q \end{bmatrix} = -\left[\mathbf{b}_{2n}^{[f]}(z) \right]^* H_n^- \left[\mathbf{b}_{2n}^{[f]}(z) \right]. \tag{119}$$

Combining (112), (114), (116), and (119), we get

$$\begin{aligned}
\Sigma_{2n}^{[f]}(z) &= \frac{f(z) - [f(z)]^*}{z - \overline{z}} - \left[\mathbf{b}_{2n}^{[f]}(z) \right]^* H_n^- \left[\mathbf{b}_{2n}^{[f]}(z) \right] \\
&= -\frac{1}{2 \operatorname{Im} z} \begin{bmatrix} f(z) \\ I_q \end{bmatrix}^* \tilde{J}_q \begin{bmatrix} f(z) \\ I_q \end{bmatrix} + \frac{1}{2 \operatorname{Im} z} \begin{bmatrix} f(z) \\ I_q \end{bmatrix}^* \left[\tilde{J}_q - \Theta_{n,\alpha}^{-*}(z) \tilde{J}_q \Theta_{n,\alpha}^{-1}(z) \right] \begin{bmatrix} f(z) \\ I_q \end{bmatrix} \\
&= \begin{bmatrix} f(z) \\ I_q \end{bmatrix}^* \Theta_{n,\alpha}^{-*}(z) \left(\frac{-\tilde{J}_q}{2 \operatorname{Im} z} \right) \Theta_{n,\alpha}^{-1}(z) \begin{bmatrix} f(z) \\ I_q \end{bmatrix}.
\end{aligned}$$

According to Lemma 7.23 we have (111) and consequently

$$\begin{aligned}
&\left[\mathbf{b}_{2n+1}^{[f]}(z) \right]^* \\
&= \begin{bmatrix} (z - \alpha) f(z) \\ I_q \end{bmatrix}^* [\operatorname{diag}(v_{q,n}, v_{q,n})]^* \left[I_{(n+1)q}, \left[R_{T_{q,n}}(\alpha) \right]^{-1} H_n \right]^* \left[R_{T_{q,n}}(z) \right]^*.
\end{aligned}$$

Since Remark 4.1 yields $R_{T_{q,n}^*}(\overline{z}) = \left[R_{T_{q,n}}(z) \right]^*$, then

$$\left[\mathbf{b}_{2n+1}^{[f]}(z) \right]^* = \begin{bmatrix} (z-\alpha) f(z) \\ I_q \end{bmatrix}^* [\operatorname{diag}(v_{q,n}, v_{q,n})]^* \left[I_{(n+1)q}, \left[R_{T_{q,n}}(\alpha) \right]^{-1} H_n \right]^* R_{T_{q,n}^*}(\overline{z}) \tag{120}$$

follows. Taking into account (120) and (111) we can conclude from Lemma 7.20 then

$$\begin{aligned}
&\begin{bmatrix} (z - \alpha) f(z) \\ I_q \end{bmatrix}^* \left[\tilde{J}_q - \tilde{\Theta}_{n,\alpha}^{-*}(z) \tilde{J}_q \tilde{\Theta}_{n,\alpha}^{-1}(z) \right] \begin{bmatrix} (z - \alpha) f(z) \\ I_q \end{bmatrix} \\
&\qquad = -\mathrm{i}(\overline{z} - z) \left[\mathbf{b}_{2n+1}^{[f]}(z) \right]^* H_n^- \left[\mathbf{b}_{2n+1}^{[f]}(z) \right].
\end{aligned}$$

Because of $-\mathrm{i}(\overline{z} - z) = -2 \operatorname{Im} z$, hence

$$\frac{1}{2\operatorname{Im} z}\begin{bmatrix}(z-\alpha)f(z)\\ I_q\end{bmatrix}^* \Big[\tilde{J}_q - \tilde{\Theta}_{n,\alpha}^{-*}(z)\tilde{J}_q\tilde{\Theta}_{n,\alpha}^{-1}(z)\Big]\begin{bmatrix}(z-\alpha)f(z)\\ I_q\end{bmatrix}$$
$$= -\Big[\mathbf{b}_{2n+1}^{[f]}(z)\Big]^* H_n^- \Big[\mathbf{b}_{2n+1}^{[f]}(z)\Big]. \tag{121}$$

Combining (113), (115), (117), and (121), we get

$$\begin{aligned}
\Sigma_{2n+1}^{[f]}(z) &= \frac{(z-\alpha)f(z)-[(z-\alpha)f(z)]^*}{z-\overline{z}} - \Big[\mathbf{b}_{2n+1}^{[f]}(z)\Big]^* H_{\alpha\triangleright n}^- \Big[\mathbf{b}_{2n+1}^{[f]}(z)\Big]\\
&= -\frac{1}{2\operatorname{Im} z}\begin{bmatrix}(z-\alpha)f(z)\\ I_q\end{bmatrix}^* \tilde{J}_q \begin{bmatrix}(z-\alpha)f(z)\\ I_q\end{bmatrix}\\
&\quad + \frac{1}{2\operatorname{Im} z}\begin{bmatrix}(z-\alpha)f(z)\\ I_q\end{bmatrix}^* \Big[\tilde{J}_q - \tilde{\Theta}_{n,\alpha}^{-*}(z)\tilde{J}_q\tilde{\Theta}_{n,\alpha}^{-1}(z)\Big]\begin{bmatrix}(z-\alpha)f(z)\\ I_q\end{bmatrix}\\
&= \begin{bmatrix}(z-\alpha)f(z)\\ I_q\end{bmatrix}^* \tilde{\Theta}_{n,\alpha}^{-*}(z)\left(\frac{-\tilde{J}_q}{2\operatorname{Im} z}\right)\tilde{\Theta}_{n,\alpha}^{-1}(z)\begin{bmatrix}(z-\alpha)f(z)\\ I_q\end{bmatrix}.
\end{aligned} \tag{122}$$

Proposition 7.16 yields

$$\tilde{\Theta}_{n,\alpha}(z) = \operatorname{diag}\big((z-\alpha)I_q, I_q\big)\Theta_{n,\alpha}(z)\operatorname{diag}\Big((z-\alpha)^{-1}I_q, I_q\Big),$$

implying

$$\tilde{\Theta}_{n,\alpha}^{-1}(z)\operatorname{diag}\big((z-\alpha)I_q, I_q\big) = \operatorname{diag}\big((z-\alpha)I_q, I_q\big)\Theta_{n,\alpha}^{-1}(z).$$

Hence,

$$\operatorname{diag}\big((z-\alpha)I_q, I_q\big)\Theta_{n,\alpha}^{-1}(z)\begin{bmatrix}f(z)\\ I_q\end{bmatrix} = \tilde{\Theta}_{n,\alpha}^{-1}(z)\begin{bmatrix}(z-\alpha)f(z)\\ I_q\end{bmatrix}$$

and consequently

$$\begin{bmatrix}(z-\alpha)f(z)\\ I_q\end{bmatrix}^* \tilde{\Theta}_{n,\alpha}^{-*}(z) = \begin{bmatrix}f(z)\\ I_q\end{bmatrix}^* \Theta_{n,\alpha}^{-*}(z)\big[\operatorname{diag}\big((z-\alpha)I_q, I_q\big)\big]^*.$$

Taking additionally into account (122), we obtain

$$\Sigma_{2n+1}^{[f]}(z) = \begin{bmatrix}f(z)\\ I_q\end{bmatrix}^* \Theta_{n,\alpha}^{-*}(z)\big[\operatorname{diag}\big((z-\alpha)I_q, I_q\big)\big]^*$$

$$\times \left(\frac{-\tilde{J}_q}{2\operatorname{Im} z}\right)\left[\operatorname{diag}\bigl((z-\alpha)I_q, I_q\bigr)\right]\Theta_{n,\alpha}^{-1}(z)\begin{bmatrix} f(z) \\ I_q \end{bmatrix}.$$

Against the background of Proposition 4.10 it becomes clear from Proposition 7.24 which important role the matrix polynomials $\Theta_{n,\alpha}$ and $\tilde{\Theta}_{n,\alpha}$ play in our approach. The former investigations of Yu. M. Dyukarev [23] and V. A. Bolotnikov [8] suggest to study the linear fractional transformation of matrices generated by $\Theta_{n,\alpha}$.

8 Stieltjes Pairs of Meromorphic Matrix-Valued Functions

The main aim of this section is to study the domain of definition of the linear fractional transformation generated by $\Theta_{n,\alpha}$. Against this background we introduce the class of pairs of meromorphic matrix-valued functions, which will play the role of the set of parameters in the description of the solution set of the Stieltjes moment problem under consideration.

Definition 8.1 Let $\alpha \in \mathbb{R}$. Let ϕ and ψ be $q \times q$ matrix-valued functions meromorphic in $\mathbb{C} \setminus [\alpha, \infty)$. Then $[\phi; \psi]$ is called a $q \times q$ *Stieltjes pair in* $\mathbb{C} \setminus [\alpha, \infty)$ if there exists a discrete subset $\mathscr{D}$ of $\mathbb{C} \setminus [\alpha, \infty)$ such that the following three conditions are fulfilled:

(i) ϕ are ψ are holomorphic in $\mathbb{C} \setminus ([\alpha, \infty) \cup \mathscr{D})$.
(ii) $\operatorname{rank}\begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix} = q$ for each $z \in \mathbb{C} \setminus ([\alpha, \infty) \cup \mathscr{D})$.
(iii) For each $z \in \mathbb{C} \setminus (\mathbb{R} \cup \mathscr{D})$,

$$\begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix}^* \left(\frac{-\tilde{J}_q}{2\operatorname{Im} z}\right)\begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix} \geq 0 \tag{123}$$

and

$$\begin{bmatrix} (z-\alpha)\phi(z) \\ \psi(z) \end{bmatrix}^* \left(\frac{-\tilde{J}_q}{2\operatorname{Im} z}\right)\begin{bmatrix} (z-\alpha)\phi(z) \\ \psi(z) \end{bmatrix} \geq 0. \tag{124}$$

The set of all $q \times q$ Stieltjes pairs in $\mathbb{C} \setminus [\alpha, \infty)$ will be denoted by $\mathscr{P}^{(q,q)}_{-\tilde{J}_q, \geq}(\mathbb{C} \setminus [\alpha, \infty))$.

For a detailed treatment of the just introduced class, we refer to [35, Section 7]

Remark 8.2 Let $\alpha \in \mathbb{R}$, let $[\phi; \psi] \in \mathscr{P}^{(q,q)}_{-\tilde{J}_q, \geq}(\mathbb{C} \setminus [\alpha, \infty))$, and let g be a $q \times q$ matrix-valued function which is meromorphic in $\mathbb{C} \setminus [\alpha, \infty)$ such that

$\det g$ does not vanish identically. Then it is readily checked that $[\phi g; \psi g] \in \mathscr{P}^{(q,q)}_{-\tilde{J}_q,\geq}(\mathbb{C} \setminus [\alpha,\infty))$.

Remark 8.3 Two $q \times q$ Stieltjes pairs $[\phi_1; \psi_1]$ and $[\phi_2; \psi_2]$ in $\mathbb{C} \setminus [\alpha,\infty)$ are said to be *equivalent* if there exist a $q \times q$ matrix-valued function g which is meromorphic in $\mathbb{C} \setminus [\alpha,\infty)$ and a discrete subset $\mathscr{D}$ of $\mathbb{C} \setminus [\alpha,\infty)$ such that ϕ_1, ϕ_2, ψ_1, ψ_2, and g are holomorphic in $\mathbb{C} \setminus ([\alpha,\infty) \cup \mathscr{D})$ and that $\det g(z) \neq 0$ and $\phi_2(z) = \phi_1(z)g(z)$ and $\psi_2(z) = \psi_1(z)g(z)$ hold true for each $z \in \mathbb{C} \setminus ([\alpha,\infty) \cup \mathscr{D})$. It is readily checked that this generates an equivalence relation on $\mathscr{P}^{(q,q)}_{-\tilde{J}_q,\geq}(\mathbb{C} \setminus [\alpha,\infty))$. For each $[\phi; \psi] \in \mathscr{P}^{(q,q)}_{-\tilde{J}_q,\geq}(\mathbb{C} \setminus [\alpha,\infty))$, by $\langle[\phi; \psi]\rangle$ we denote the equivalence class generated by $[\phi; \psi]$.

Remark 8.4 Let $\alpha \in \mathbb{R}$ and let $[\phi; \psi] \in \mathscr{P}^{(q,q)}_{-\tilde{J}_q,\geq}(\mathbb{C} \setminus [\alpha,\infty))$. Using a classical result of complex analysis (see, e. g., [11, Theorem 11.46, p. 395]), one can prove that there is a $(\mathbb{C} \setminus \{0\})$-valued function g holomorphic in $\mathbb{C} \setminus [\alpha,\infty)$ such that $\tilde{\phi} := g\phi$ and $\tilde{\psi} := g\psi$ are holomorphic in $\mathbb{C} \setminus [\alpha,\infty)$. In particular, $[\tilde{\phi}; \tilde{\psi}]$ belongs to $\mathscr{P}^{(q,q)}_{-\tilde{J}_q,\geq}(\mathbb{C} \setminus [\alpha,\infty))$ with $\langle[\tilde{\phi}; \tilde{\psi}]\rangle = \langle[\phi; \psi]\rangle$.

Now we indicate that there is an intimate connection between the class $\mathscr{P}^{(q,q)}_{-\tilde{J}_q,\geq}(\mathbb{C} \setminus [\alpha,\infty))$ and the class $\mathscr{S}_{q;[\alpha,\infty)}$ introduced in Section 3.

Definition 8.5 Let $\alpha \in \mathbb{R}$. A pair $[\phi; \psi] \in \mathscr{P}^{(q,q)}_{-\tilde{J}_q,\geq}(\mathbb{C} \setminus [\alpha,\infty))$ is said to be a *proper $q \times q$ Stieltjes pair in* $\mathbb{C} \setminus [\alpha,\infty)$ if $\det \psi$ does not vanish identically in $\mathbb{C} \setminus [\alpha,\infty)$. The set of all proper $q \times q$ Stieltjes pairs in $\mathbb{C} \setminus [\alpha,\infty)$ will be denoted by $\tilde{\mathscr{P}}^{(q,q)}_{-\tilde{J}_q,\geq}(\mathbb{C} \setminus [\alpha,\infty))$.

Denote by $\mathfrak{O}_q$ and $\mathfrak{I}_q$ the constant $\mathbb{C}^{q\times q}$-valued functions in $\mathbb{C} \setminus [\alpha,\infty)$ with values $0_{q\times q}$ and I_q, respectively.

Proposition 8.6 ([35, Proposition 7.7]) *Let $\alpha \in \mathbb{R}$ and let $f \in \mathscr{S}_{q;[\alpha,\infty)}$. Then:*

(a) The pair $[f; \mathfrak{I}_q]$ belongs to $\tilde{\mathscr{P}}^{(q,q)}_{-\tilde{J}_q,\geq}(\mathbb{C} \setminus [\alpha,\infty))$.
(b) Let $g \in \mathscr{S}_{q;[\alpha,\infty)}$. Then $\langle[f; \mathfrak{I}_q]\rangle = \langle[g; \mathfrak{I}_q]\rangle$ if and only if $f = g$.

Proposition 8.6 shows that the class $\mathscr{P}^{(q,q)}_{-\tilde{J}_q,\geq}(\mathbb{C} \setminus [\alpha,\infty))$ can be interpreted as a projective extension of the class $\mathscr{S}_{q;[\alpha,\infty)}$.

Example 8.7 ([35, Example 7.8]) Let $\alpha \in \mathbb{R}$. Then Proposition 8.6 shows that $[\mathfrak{O}_q; \mathfrak{I}_q]$ belongs to $\tilde{\mathscr{P}}^{(q,q)}_{-\tilde{J}_q,\geq}(\mathbb{C} \setminus [\alpha,\infty))$. Furthermore, Remark 7.1 yields $[\mathfrak{I}_q; \mathfrak{O}_q] \in \mathscr{P}^{(q,q)}_{-\tilde{J}_q,\geq}(\mathbb{C} \setminus [\alpha,\infty))$.

Remark 8.8 ([35, Remark 7.9]) Let $\alpha \in \mathbb{R}$. Then Example 8.7 shows that the set $\tilde{\mathscr{P}}^{(q,q)}_{-\tilde{J}_q,\geq}(\mathbb{C} \setminus [\alpha,\infty))$ is non-empty.

Proposition 8.9 ([35, Proposition 7.10]) *Let $\alpha \in \mathbb{R}$. Further, let ϕ be a $q \times q$ matrix-valued function which is meromorphic in $\mathbb{C} \setminus [\alpha, \infty)$ such that the condition $[\phi; \mathfrak{I}_q] \in \mathscr{P}^{(q,q)}_{-\tilde{J}_q,\geq}(\mathbb{C} \setminus [\alpha, \infty))$ is satisfied. Then $\phi \in \mathscr{S}_{q;[\alpha,\infty)}$.*

The following results complement the statements of Propositions 8.6 and 8.9. We see now that the equivalence class of a proper element of $\mathscr{P}^{(q,q)}_{-\tilde{J}_q,\geq}(\mathbb{C} \setminus [\alpha, \infty))$ is always represented by a function belonging to $\mathscr{S}_{q;[\alpha,\infty)}$.

Proposition 8.10 ([35, Proposition 7.11]) *Let $\alpha \in \mathbb{R}$ and let $[\phi; \psi] \in \tilde{\mathscr{P}}^{(q,q)}_{-\tilde{J}_q,\geq}(\mathbb{C} \setminus [\alpha, \infty))$. Then:*

(a) The function $S := \phi\psi^{-1}$ belongs to $\mathscr{S}_{q;[\alpha,\infty)}$.
(b) $[S; \mathfrak{I}_q] \in \mathscr{P}^{(q,q)}_{-\tilde{J}_q,\geq}(\mathbb{C} \setminus [\alpha, \infty))$ and $\langle[\phi; \psi]\rangle = \langle[S; \mathfrak{I}_q]\rangle$.

A matrix-valued function $S\colon \Pi_+ \to \mathbb{C}^{q\times q}$ is called $q \times q$ *Schur function in Π_+* if S is both holomorphic and contractive in Π_+. The set of all $q \times q$ Schur functions in Π_+ will be denoted by $\mathscr{S}_{q\times q}(\Pi_+)$. We indicate now an interesting connection between the class $\mathscr{P}^{(q,q)}_{-\tilde{J}_q,\geq}(\mathbb{C} \setminus [\alpha, \infty))$ and the Schur class $\mathscr{S}_{q\times q}(\Pi_+)$.

Lemma 8.11 *Let $[\phi; \psi] \in \mathscr{P}^{(q,q)}_{-\tilde{J}_q,\geq}(\mathbb{C} \setminus [\alpha, \infty))$. Then the function $\det(\psi - \mathrm{i}\phi)$ does not vanish identically and the function $F := (\psi + \mathrm{i}\phi)(\psi - \mathrm{i}\phi)^{-1}$ is meromorphic in $\mathbb{C} \setminus [\alpha, \infty)$ and fulfills $\mathrm{Rstr}_{\Pi_+} F \in \mathscr{S}_{q\times q}(\Pi_+)$. Furthermore, there exists a discrete subset $\mathscr{D}$ of $\mathbb{C} \setminus [\alpha, \infty)$ such that ϕ, ψ, $(\psi - \mathrm{i}\phi)^{-1}$, and F are holomorphic in $\Pi_+ \cup [\mathbb{C} \setminus ([\alpha, \infty) \cup \mathscr{D})]$ and that $\det[\psi(z) - \mathrm{i}\phi(z)] \neq 0$ and*

$$F(z) = [\psi(z) + \mathrm{i}\phi(z)][\psi(z) - \mathrm{i}\phi(z)]^{-1} \tag{125}$$

hold true for each $z \in \mathbb{C} \setminus ([\alpha, \infty) \cup \mathscr{D})$. Moreover, for each $z \in \mathbb{C} \setminus ([\alpha, \infty) \cup \mathscr{D})$, the matrix-valued functions ϕ and ψ admit the representations

$$\phi(z) = \frac{\mathrm{i}}{2}[I_q - F(z)][\psi(z) - \mathrm{i}\phi(z)] \quad \text{and} \quad \psi(z) = \frac{1}{2}\big[I_q + F(z)\big][\psi(z) - \mathrm{i}\phi(z)]. \tag{126}$$

The proof of Lemma 8.11 is straightforward. (A detailed proof is given in [53, Satz 10.19].)

Proposition 8.12 *Let $[\phi; \psi] \in \mathscr{P}^{(q,q)}_{-\tilde{J}_q,\geq}(\mathbb{C} \setminus [\alpha, \infty))$. Then there exists a discrete subset $\mathscr{D}$ of $\mathbb{C} \setminus [\alpha, \infty)$ such that ϕ and ψ are holomorphic in $\mathbb{C} \setminus ([\alpha, \infty) \cup \mathscr{D})$ and that $\mathscr{R}(\phi(w)) = \mathscr{R}(\phi(z))$ and $\mathscr{R}(\psi(w)) = \mathscr{R}(\psi(z))$ as well as $\psi(w)\mathscr{N}(\phi(w)) = \psi(z)\mathscr{N}(\phi(z))$ and $\phi(w)\mathscr{N}(\psi(w)) = \phi(z)\mathscr{N}(\psi(z))$ hold true for every choice of z and w in $\mathbb{C} \setminus ([\alpha, \infty) \cup \mathscr{D})$.*

A detailed proof of Proposition 8.12 is given in [34, Proposition 10.15]. Since we do not use this result in our following considerations, we omit a proof here.

9 The Class $\tilde{\mathfrak{W}}_{\tilde{J}_q,\alpha}$

A closer view on the $2q \times 2q$ matrix polynomials introduced in Lemma 7.15 leads us to the consideration of the following object.

Notation 9.1 *Let $\alpha \in \mathbb{R}$. By $\tilde{\mathfrak{W}}_{\tilde{J}_q,\alpha}$ we denote the set of all $2q \times 2q$ matrix-valued functions Θ which are meromorphic in $\mathbb{C} \setminus [\alpha, \infty)$ and for which there exists a discrete subset $\mathscr{D}$ of $\mathbb{C} \setminus [\alpha, \infty)$ such that the following three conditions are fulfilled:*

(I) Θ is holomorphic in $\mathbb{C} \setminus ([\alpha, \infty) \cup \mathscr{D})$.
(II) $\Theta(z)\tilde{J}_q\Theta^*(z) \leq \tilde{J}_q$ for each $z \in \Pi_+ \setminus \mathscr{D}$.
(III) $\Theta(x)\tilde{J}_q\Theta^*(x) = \tilde{J}_q$ for each $x \in (-\infty, \alpha) \setminus \mathscr{D}$.

Let $\mathscr{G}$ be a non-empty open subset of $\mathbb{C}$ and let $f = [f_{jk}]_{\substack{j=1,\dots,p\\k=1,\dots,q}}$ be a $p \times q$ matrix-valued function which is meromorphic in $\mathscr{G}$. For every choice of j in $\mathbb{Z}_{1,p}$ and k in $\mathbb{Z}_{1,q}$, then let $\mathbb{H}_{f_{jk}}$ be the set of all $z \in \mathscr{G}$ in which f_{jk} is holomorphic and let $\mathbb{P}_{f_{jk}}$ be the set of all poles of f_{jk}. Furthermore, let $\mathbb{H}_f := \bigcap_{j=1}^{p} \bigcap_{k=1}^{q} \mathbb{H}_{f_{jk}}$ and let $\mathbb{P}_f := \bigcup_{j=1}^{p} \bigcup_{k=1}^{q} \mathbb{P}_{f_{jk}}$.

Observe that continuity arguments show that conditions (II) and (III) in Notation 9.1 can be replaced equivalently by the following conditions (I') and (II'), respectively:

(I') $\Theta(z)\tilde{J}_q\Theta^*(z) \leq \tilde{J}_q$ for each $z \in \Pi_+ \cap \mathbb{H}_\Theta$.
(II') $(-\infty, \alpha) \subseteq \mathbb{H}_\Theta$ and $\Theta(x)\tilde{J}_q\Theta^*(x) = \tilde{J}_q$ for each $x \in (-\infty, \alpha)$.

The following observation shows the importance of the class $\tilde{\mathfrak{W}}_{\tilde{J}_q,\alpha}$ for the purposes of this paper.

Remark 9.2 Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N} \cup \{\infty\}$, and let $(s_j)_{j=0}^{\kappa} \in \mathscr{K}^{\geq,\mathrm{e}}_{q,\kappa,\alpha}$. From Lemma 7.15, Remark 7.18, and Lemma 7.19 we see then that, for each $n \in \mathbb{N}_0$ with $2n + 1 \leq \kappa$, the functions $\hat{\Theta}_{n,\alpha} := \operatorname{Rstr}_{\mathbb{C}\setminus[\alpha,\infty)} \Theta_{n,\alpha}$ and $\hat{\tilde{\Theta}}_{n,\alpha} := \operatorname{Rstr}_{\mathbb{C}\setminus[\alpha,\infty)} \tilde{\Theta}_{n,\alpha}$ given by (85) and (86) are holomorphic in $\mathbb{C} \setminus [\alpha, \infty)$ and belong both to $\tilde{\mathfrak{W}}_{\tilde{J}_q,\alpha}$.

Suggested by Proposition 7.16 and Remark 9.2 we are led to a particular subclass of $\tilde{\mathfrak{W}}_{\tilde{J}_q,\alpha}$ which is introduced now.

Notation 9.3 *Let $\alpha \in \mathbb{R}$ and let the matrix-valued function $P_\alpha \colon \mathbb{C} \setminus [\alpha, \infty) \to \mathbb{C}^{2q\times 2q}$ be defined by $P_\alpha(z) := \operatorname{diag}((z-\alpha)I_q, I_q)$. Then let $\mathfrak{W}_{\tilde{J}_q,\alpha}$ be the set of all $\Theta \in \tilde{\mathfrak{W}}_{\tilde{J}_q,\alpha}$ for which $\tilde{\Theta} := P_\alpha \Theta P_\alpha^{-1}$ belongs to $\tilde{\mathfrak{W}}_{\tilde{J}_q,\alpha}$.*

Remark 9.4 Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N} \cup \{\infty\}$, and let $(s_j)_{j=0}^{\kappa} \in \mathscr{K}^{\geq,\mathrm{e}}_{q,\kappa,\alpha}$. From Remark 9.2 and Proposition 7.16 one can easily see that, for each $n \in \mathbb{N}_0$ with $2n + 1 \leq \kappa$, the matrix-valued function $\operatorname{Rstr}_{\mathbb{C}\setminus[\alpha,\infty)} \Theta_{n,\alpha}$ given by (85) is holomorphic in $\mathbb{C} \setminus [\alpha, \infty)$ and belongs to $\mathfrak{W}_{\tilde{J}_q,\alpha}$.

Lemma 9.5 *Let $\alpha \in \mathbb{R}$ and let $\Theta \in \mathfrak{W}_{\tilde{J}_q,\alpha}$. Then there is a discrete subset $\mathscr{D}$ of $\mathbb{C} \setminus [\alpha, \infty)$ such that Θ is holomorphic in $\mathbb{C} \setminus ([\alpha, \infty) \cup \mathscr{D})$ and that $\det \Theta(z) \neq 0$ holds true for each $z \in \mathbb{C} \setminus ([\alpha, \infty) \cup \mathscr{D})$. Furthermore, Θ^{-1} is meromorphic in $\mathbb{C} \setminus [\alpha, \infty)$ with $\mathbb{H}_{\Theta^{-1}} \supseteq \mathbb{C} \setminus ([\alpha, \infty) \cup \mathscr{D})$ and the identity $\Theta^{-1}(z) = \tilde{J}_q \Theta^*(\overline{z}) \tilde{J}_q$ holds true for each $z \in \mathbb{C} \setminus ([\alpha, \infty) \cup \mathscr{D})$.*

Proof Let $\mathbb{H}_\Theta{}^\vee := \{z \in \mathbb{C} \setminus [\alpha, \infty) : \overline{z} \in \mathbb{H}_\Theta\}$. Then $\Theta^\vee : \mathbb{H}_\Theta{}^\vee \to \mathbb{C}^{2q \times 2q}$ given by $\Theta^\vee(z) := \Theta^*(\overline{z})$ is meromorphic in $\mathbb{C} \setminus [\alpha, \infty)$ with $\mathbb{H}_{\Theta^\vee} = (\mathbb{H}_\Theta)^\vee$. Thus, $\Omega := \tilde{J}_q - \Theta \tilde{J}_q \Theta^\vee$ is meromorphic in $\mathbb{C} \setminus [\alpha, \infty)$ with $\mathbb{H}_\Omega \supseteq \mathbb{H}_\Theta \cap \mathbb{H}_{\Theta^\vee} = \mathbb{H}_\Theta$. Because of $\Theta \in \mathfrak{W}_{\tilde{J}_q,\alpha}$, there is a discrete subset $\mathscr{D}$ of $\mathbb{C} \setminus [\alpha, \infty)$ with $\mathbb{C} \setminus ([\alpha, \infty) \cup \mathscr{D}) \subseteq \mathbb{H}_\Theta$ such that Ω is holomorphic in $\mathbb{C} \setminus ([\alpha, \infty) \cup \mathscr{D})$ and that $\Omega(x) = \tilde{J}_q - \Theta(x) \tilde{J}_q \Theta^\vee(x) = \tilde{J}_q - \Theta(x) \tilde{J}_q \Theta^*(\overline{x}) = \tilde{J}_q - \Theta(x) \tilde{J}_q \Theta^*(x) = 0$ holds true for each $x \in (-\infty, \alpha) \setminus \mathscr{D}$. Consequently, the identity theorem for holomorphic functions shows that $\Theta(z) \tilde{J}_q \Theta^\vee(z) = \tilde{J}_q$ is valid for each $z \in \mathbb{H}_\Theta \cap \mathbb{H}_{\Theta^\vee}$, which implies $\Theta(z) \tilde{J}_q \Theta^*(\overline{z}) \tilde{J}_q = \Theta(z) \tilde{J}_q \Theta^\vee(z) \tilde{J}_q = \tilde{J}_q^2 = I_{2q}$ for each $z \in \mathbb{H}_\Theta \cap \mathbb{H}_{\Theta^\vee}$ and, in particular, for each $z \in \mathbb{C} \setminus ([\alpha, \infty) \cup \mathscr{D})$. The rest is plain.

10 On the Class $\mathfrak{W}_{\tilde{J}_q,\alpha}$ Under the View of Linear Fractional Transformations

We are interested in linear fractional transformations with generating matrix-valued function belonging to $\mathfrak{W}_{\tilde{J}_q,\alpha}$. The domain of these transformations is the class $\mathscr{P}^{(q,q)}_{-\tilde{J}_q,\geq}(\mathbb{C} \setminus [\alpha, \infty))$ introduced in Definition 8.1.

The following result plays a key role in our subsequent considerations.

Proposition 10.1 *Let $\alpha \in \mathbb{R}$, let $\Theta \in \mathfrak{W}_{\tilde{J}_q,\alpha}$, and let $\Theta = [\Theta_{jk}]_{j,k=1}^2$ be the $q \times q$ block representation of Θ. Then:*

(a) The function $\det \Theta$ does not vanish identically and the matrix-valued function Θ^{-1} is meromorphic in $\mathbb{C} \setminus [\alpha, \infty)$.

(b) Let f be a $q \times q$ matrix-valued function meromorphic in $\mathbb{C} \setminus [\alpha, \infty)$. Suppose that there is a discrete subset $\mathscr{D}$ of $\mathbb{C} \setminus [\alpha, \infty)$ such that f and Θ are holomorphic in $\mathbb{C} \setminus ([\alpha, \infty) \cup \mathscr{D})$, that $\det \Theta(z) \neq 0$ holds true for each $z \in \mathbb{C} \setminus ([\alpha, \infty) \cup \mathscr{D})$, and that

$$\begin{bmatrix} f(z) \\ I_q \end{bmatrix}^* \Theta^{-*}(z) \left(\frac{-\tilde{J}_q}{2 \operatorname{Im} z} \right) \Theta^{-1}(z) \begin{bmatrix} f(z) \\ I_q \end{bmatrix} \geq 0_{q \times q} \tag{127}$$

and

$$\begin{bmatrix} f(z) \\ I_q \end{bmatrix}^* \Theta^{-*}(z) \big[\operatorname{diag}\big((z - \alpha) I_q, I_q\big) \big]^*$$

$$\times \left(\frac{-\tilde{J}_q}{2\operatorname{Im} z}\right)\left[\operatorname{diag}\left((z-\alpha)I_q, I_q\right)\right]\Theta^{-1}(z)\begin{bmatrix} f(z) \\ I_q \end{bmatrix} \geq 0_{q\times q} \tag{128}$$

are fulfilled for each $z \in \mathbb{C} \setminus (\mathbb{R} \cup \mathscr{D})$. *For every such discrete subset* $\mathscr{D}$ *of* $\mathbb{C} \setminus [\alpha, \infty)$, *there exists a pair* $[\phi; \psi] \in \mathscr{P}^{(q,q)}_{-\tilde{J}_q,\geq}(\mathbb{C} \setminus [\alpha, \infty))$ *such that* ϕ *and* ψ *are holomorphic in* $\mathbb{C} \setminus ([\alpha, \infty) \cup \mathscr{D})$ *and that*

$$\det[\Theta_{21}(z)\phi(z) + \Theta_{22}(z)\psi(z)] \neq 0 \tag{129}$$

and

$$f(z) = [\Theta_{11}(z)\phi(z) + \Theta_{12}(z)\psi(z)][\Theta_{21}(z)\phi(z) + \Theta_{22}(z)\psi(z)]^{-1} \tag{130}$$

hold true for each $z \in \mathbb{C} \setminus ([\alpha, \infty) \cup \mathscr{D})$.

(c) Let $[\phi; \psi] \in \mathscr{P}^{(q,q)}_{-\tilde{J}_q,\geq}(\mathbb{C} \setminus [\alpha, \infty))$ *be such that* $\det(\Theta_{21}\phi + \Theta_{22}\psi)$ *does not vanish identically. Then there exists a discrete subset* $\mathscr{D}$ *of* $\mathbb{C} \setminus [\alpha, \infty)$ *such that the following three statements are valid:*

(I) The matrix-valued functions Θ, ϕ, *and* ψ *are holomorphic in* $\mathbb{C}\setminus([\alpha, \infty)\cup \mathscr{D})$.

(II) The inequalities $\det \Theta(z) \neq 0$ *and* (129) *hold true for each* $z \in \mathbb{C} \setminus ([\alpha, \infty) \cup \mathscr{D})$.

(III) The function

$$f := (\Theta_{11}\phi + \Theta_{12}\psi)(\Theta_{21}\phi + \Theta_{22}\psi)^{-1} \tag{131}$$

is holomorphic in $\mathbb{C} \setminus ([\alpha, \infty) \cup \mathscr{D})$, *the inequalities* (127) *and* (128) *hold true for each* $z \in \mathbb{C} \setminus (\mathbb{R} \cup \mathscr{D})$ *and* (130) *is fulfilled for each* $z \in \mathbb{C} \setminus ([\alpha, \infty) \cup \mathscr{D})$.

(d) For each $k \in \{1, 2\}$, *let* $[\phi_k; \psi_k] \in \mathscr{P}^{(q,q)}_{-\tilde{J}_q,\geq}(\mathbb{C} \setminus [\alpha, \infty))$ *be such that* $\det(\Theta_{21}\phi_k+\Theta_{22}\psi_k)$ *does not vanish identically. Then* $\langle[\phi_1; \psi_1]\rangle = \langle[\phi_2; \psi_2]\rangle$ *if and only if*

$$(\Theta_{11}\phi_1+\Theta_{12}\psi_1)(\Theta_{21}\phi_1+\Theta_{22}\psi_1)^{-1} = (\Theta_{11}\phi_2+\Theta_{12}\psi_2)(\Theta_{21}\phi_2+\Theta_{22}\psi_2)^{-1}.$$

Proof

(a) Use Lemma 9.5.

(b) Let $\mathscr{D}$ be a discrete subset of $\mathbb{C} \setminus [\alpha, \infty)$ such that f and Θ are holomorphic in $\mathbb{C} \setminus ([\alpha, \infty) \cup \mathscr{D})$, that $\det \Theta(z) \neq 0$ is valid for each $z \in \mathbb{C} \setminus ([\alpha, \infty) \cup \mathscr{D})$, and that (127) and (128) are fulfilled for each $z \in \mathbb{C} \setminus (\mathbb{R} \cup \mathscr{D})$. Then the functions Θ^{-1}, $\phi := [I_q, 0_{q\times q}]\Theta^{-1}[f; I_q]$, and $\psi := [0_{q\times q}, I_q]\Theta^{-1}[f; I_q]$ are holomorphic in $\mathbb{C} \setminus ([\alpha, \infty) \cup \mathscr{D})$ and, for each $z \in \mathbb{C} \setminus ([\alpha, \infty) \cup \mathscr{D})$, we have

$$\begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix} = \Theta^{-1}(z) \begin{bmatrix} f(z) \\ I_q \end{bmatrix}, \tag{132}$$

consequently,

$$\Theta_{11}(z)\phi(z) + \Theta_{12}(z)\psi(z) = [I_q, 0_{q\times q}]\Theta(z) \begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix} = f(z) \tag{133}$$

and, analogously, $\Theta_{21}(z)\phi(z) + \Theta_{22}(z)\psi(z) = I_q$. The last equation implies (129) as well as

$$q \geq \operatorname{rank} \begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix} \geq \operatorname{rank}\left([\Theta_{21}(z), \Theta_{22}(z)] \begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix} \right) = \operatorname{rank} I_q = q.$$

Hence, $\operatorname{rank} \left[\begin{smallmatrix} \phi(z) \\ \psi(z) \end{smallmatrix} \right] = q$ for all $z \in \mathbb{C} \setminus ([\alpha, \infty) \cup \mathscr{D})$. In view of (132) and (127), we get

$$\begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix}^* \left(\frac{-\tilde{J}_q}{2\operatorname{Im} z} \right) \begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix} = \begin{bmatrix} f(z) \\ I_q \end{bmatrix}^* \frac{\Theta^{-*}(z)(-\tilde{J}_q)\Theta^{-1}(z)}{2\operatorname{Im} z} \begin{bmatrix} f(z) \\ I_q \end{bmatrix} \geq 0$$

for each $z \in \mathbb{C} \setminus (\mathbb{R} \cup \mathscr{D})$. From (132) we obtain $\left[\begin{smallmatrix} (z-\alpha)\phi(z) \\ \psi(z) \end{smallmatrix} \right] = [\operatorname{diag}((z - \alpha)I_q, I_q)]\Theta^{-1}(z)\left[\begin{smallmatrix} f(z) \\ I_q \end{smallmatrix} \right]$ for each $z \in \mathbb{C} \setminus ([\alpha, \infty) \cup \mathscr{D})$, and, according to (128), consequently, (124) for each $z \in \mathbb{C} \setminus (\mathbb{R} \cup \mathscr{D})$. Thus, we proved that $[\phi; \psi]$ is a $q \times q$ Stieltjes pair in $\mathbb{C} \setminus [\alpha, \infty)$. From (133) and $\Theta_{21}(z)\phi(z) + \Theta_{22}(z)\psi(z) = I_q$ we infer (130) for each $z \in \mathbb{C} \setminus ([\alpha, \infty) \cup \mathscr{D})$.

(c) Since Θ belongs to $\mathfrak{W}_{\tilde{J}_q,\alpha}$, Lemma 9.5 shows that there is a discrete subset $\mathscr{D}_1$ of $\mathbb{C} \setminus [\alpha, \infty)$ such that Θ is holomorphic in $\mathbb{C} \setminus ([\alpha, \infty) \cup \mathscr{D}_1)$ and that $\det \Theta(z) \neq 0$ is valid for each $z \in \mathbb{C} \setminus ([\alpha, \infty) \cup \mathscr{D}_1)$. Because of $[\phi; \psi] \in \mathscr{P}^{(q,q)}_{-\tilde{J}_q,\geq}(\mathbb{C} \setminus [\alpha, \infty))$, there is a discrete subset $\mathscr{D}_2$ of $\mathbb{C} \setminus [\alpha, \infty)$ such that ϕ and ψ are holomorphic in $\mathbb{C} \setminus ([\alpha, \infty) \cup \mathscr{D}_2)$ and that (123) and (124) hold true for each $z \in \mathbb{C} \setminus (\mathbb{R} \cup \mathscr{D}_2)$. Since the meromorphic function $\det(\Theta_{21}\phi + \Theta_{22}\psi)$ does not vanish identically, there is a discrete subset $\mathscr{D}_3$ of $\mathbb{C} \setminus [\alpha, \infty)$ such that $\det(\Theta_{21}\phi + \Theta_{22}\psi)$ is holomorphic in $\mathbb{C} \setminus ([\alpha, \infty) \cup \mathscr{D}_3)$ and that $\det(\Theta_{21}\phi + \Theta_{22}\psi)(z) \neq 0$ holds true for all $z \in \mathbb{C} \setminus ([\alpha, \infty) \cup \mathscr{D}_3)$. Thus, the set $\mathscr{D} := \mathscr{D}_1 \cup \mathscr{D}_2 \cup \mathscr{D}_3$ is a discrete subset of $\mathbb{C} \setminus [\alpha, \infty)$ and we see that Θ, ϕ, and ψ are holomorphic in $\mathbb{C} \setminus ([\alpha, \infty) \cup \mathscr{D})$ and that the inequalities (123), and (124) hold true for each $z \in \mathbb{C} \setminus (\mathbb{R} \cup \mathscr{D})$. Furthermore, $\det \Theta(z) \neq 0$ and (129) are valid for all $z \in \mathbb{C} \setminus ([\alpha, \infty) \cup \mathscr{D})$. Consequently, f defined by (131) is holomorphic in $\mathbb{C} \setminus ([\alpha, \infty) \cup \mathscr{D})$ and (130) is valid for each $z \in \mathbb{C} \setminus ([\alpha, \infty) \cup \mathscr{D})$. Now we consider an arbitrary $z \in \mathbb{C} \setminus (\mathbb{R} \cup \mathscr{D})$. Because of part (a), (129), (130), and $\Theta = [\Theta_{jk}]_{j,k=1}^{2}$, we have

$$\Theta^{-1}(z)\begin{bmatrix} f(z) \\ I_q \end{bmatrix} = \begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix}[\Theta_{21}(z)\phi(z) + \Theta_{22}(z)\psi(z)]^{-1} \tag{134}$$

and, consequently,

$$\begin{aligned}\begin{bmatrix} f(z) \\ I_q \end{bmatrix}^* \frac{\Theta^{-*}(z)(-\tilde{J}_q)\Theta^{-1}(z)}{2\operatorname{Im} z}\begin{bmatrix} f(z) \\ I_q \end{bmatrix} &= [\Theta_{21}(z)\phi(z)+\Theta_{22}(z)\psi(z)]^{-*}\begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix}^* \\ &\times\left(\frac{-\tilde{J}_q}{2\operatorname{Im} z}\right)\begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix}[\Theta_{21}(z)\phi(z) + \Theta_{22}(z)\psi(z)]^{-1}. \end{aligned}\tag{135}$$

In view of (123), the matrix on the right-hand side of (135) is non-negative Hermitian. Thus, (127) holds true. Using (134), we get

$$\begin{aligned}&\big[\operatorname{diag}\big((z-\alpha)I_q, I_q\big)\big]\Theta^{-1}(z)\begin{bmatrix} f(z) \\ I_q \end{bmatrix} \\ &\quad= \begin{bmatrix} (z-\alpha)\phi(z) \\ \psi(z) \end{bmatrix}[\Theta_{21}(z)\phi(z) + \Theta_{22}(z)\psi(z)]^{-1},\end{aligned}$$

which implies

$$\begin{aligned}&\begin{bmatrix} f(z) \\ I_q \end{bmatrix}^* \Theta^{-*}(z)\big[\operatorname{diag}((z-\alpha)I_q, I_q)\big]^*\left(\frac{-\tilde{J}_q}{2\operatorname{Im} z}\right) \\ &\qquad\big[\operatorname{diag}\big((z-\alpha)I_q, I_q\big)\big]\Theta^{-1}(z)\begin{bmatrix} f(z) \\ I_q \end{bmatrix} \\ &= [\Theta_{21}(z)\phi(z) + \Theta_{22}(z)\psi(z)]^{-*}\begin{bmatrix} (z-\alpha)\phi(z) \\ \psi(z) \end{bmatrix}^* \\ &\qquad\times\left(\frac{-\tilde{J}_q}{2\operatorname{Im} z}\right)\begin{bmatrix} (z-\alpha)\phi(z) \\ \psi(z) \end{bmatrix}[\Theta_{21}(z)\phi(z) + \Theta_{22}(z)\psi(z)]^{-1}.\end{aligned}\tag{136}$$

Because of (124), the matrix on the right-hand side of (136) is non-negative Hermitian. Consequently, (128) is proved as well.

(d) In view of part (c) and Lemma 9.5, the proof of part (d) is straightforward.

11 On the Solutions of the Schur Complement Matrix Inequalities

In this section we realize an important intermediate step on the way to the determination of the set $\mathscr{S}_{0,q;[\alpha,\infty)}[(s_j)_{j=0}^{2n+1}, \leq]$ of all $[\alpha,\infty)$-Stieltjes transforms of solutions to the moment problem $\mathsf{MP}[[\alpha,\infty); (s_j)_{j=0}^{2n+1}, \leq]$. More precisely, by application of Proposition 10.1 we will see that the key to determine this set is given by the linear fractional transformation generated by the restriction onto $\mathbb{C}\setminus[\alpha,\infty)$ of the $2q\times 2q$ matrix polynomial $\Theta_{n,\alpha}$ introduced in Lemma 7.15. The set of parameters is given by the class $\mathscr{P}^{(q,q)}_{-\tilde{J}_q,\geq}(\mathbb{C}\setminus[\alpha,\infty))$.

Proposition 11.1 *Let $\alpha\in\mathbb{R}$, let $\kappa\in\mathbb{N}\cup\{\infty\}$, let $(s_j)_{j=0}^{\kappa}\in\mathscr{K}^{\geq,\mathrm{e}}_{q,\kappa,\alpha}$, and let $n\in\mathbb{N}_0$ with $2n+1\leq\kappa$. Let $[\hat{\Theta}^{(j,k)}_{n,\alpha}]_{j,k=1}^{2}$ be the $q\times q$ block representation of the restriction $\hat{\Theta}_{n,\alpha}$ of $\Theta_{n,\alpha}$ onto $\mathbb{C}\setminus[\alpha,\infty)$. Then:*

(a) Let $[\phi;\psi]\in\mathscr{P}^{(q,q)}_{-\tilde{J}_q,\geq}(\mathbb{C}\setminus[\alpha,\infty))$ be such that $\det(\hat{\Theta}^{(2,1)}_{n,\alpha}\phi+\hat{\Theta}^{(2,2)}_{n,\alpha}\psi)$ does not vanish identically. Then there exists a discrete subset $\mathscr{D}$ of $\mathbb{C}\setminus[\alpha,\infty)$ such that

$$f := (\hat{\Theta}^{(1,1)}_{n,\alpha}\phi+\hat{\Theta}^{(1,2)}_{n,\alpha}\psi)(\hat{\Theta}^{(2,1)}_{n,\alpha}\phi+\hat{\Theta}^{(2,2)}_{n,\alpha}\psi)^{-1}$$

is holomorphic in $\mathbb{C}\setminus([\alpha,\infty)\cup\mathscr{D})$ and the inequalities

$$\begin{bmatrix} f(z) \\ I_q \end{bmatrix}^* \hat{\Theta}^{-*}_{n,\alpha}(z)\left(\frac{-\tilde{J}_q}{2\operatorname{Im} z}\right)\hat{\Theta}^{-1}_{n,\alpha}(z)\begin{bmatrix} f(z) \\ I_q \end{bmatrix} \geq 0_{q\times q} \tag{137}$$

and

$$\begin{aligned}&\begin{bmatrix} f(z) \\ I_q \end{bmatrix}^* \hat{\Theta}^{-*}_{n,\alpha}(z)\big[\operatorname{diag}\big((z-\alpha)I_q, I_q\big)\big]^* \\ &\quad\times\left(\frac{-\tilde{J}_q}{2\operatorname{Im} z}\right)\big[\operatorname{diag}\big((z-\alpha)I_q, I_q\big)\big]\hat{\Theta}^{-1}_{n,\alpha}(z)\begin{bmatrix} f(z) \\ I_q \end{bmatrix} \geq 0_{q\times q}\end{aligned} \tag{138}$$

hold true for all $z\in\mathbb{C}\setminus(\mathbb{R}\cup\mathscr{D})$. For each $z\in\mathbb{C}\setminus(\mathbb{R}\cup\mathscr{D})$ with

$$\mathscr{R}\left(\mathbf{b}^{[f]}_{2n}(z)\right)\subseteq\mathscr{R}(H_n) \quad \textit{and} \quad \mathscr{R}\left(\mathbf{b}^{[f]}_{2n+1}(z)\right)\subseteq\mathscr{R}(H_{\alpha\triangleright n}), \tag{139}$$

furthermore

$$\Sigma^{[f]}_{2n}(z)\geq 0_{q\times q} \qquad \textit{and} \qquad \Sigma^{[f]}_{2n+1}(z)\geq 0_{q\times q} \tag{140}$$

are fulfilled.

(b) *Let* $f \in \mathscr{S}_{0,q;[\alpha,\infty)}[(s_j)_{j=0}^{2n+1}, \leq]$. *Then* (139) *holds true for all* $z \in \mathbb{C} \setminus \mathbb{R}$. *Furthermore there exists a pair* $[\phi; \psi] \in \mathscr{P}_{-\tilde{J}_q,\geq}^{(q,q)}(\mathbb{C} \setminus [\alpha, \infty))$ *such that* ϕ *and* ψ *are holomorphic in* $\mathbb{C} \setminus [\alpha, \infty)$ *and* $\det[\hat{\Theta}_{n,\alpha}^{(2,1)}(z)\phi(z) + \hat{\Theta}_{n,\alpha}^{(2,2)}(z)\psi(z)] \neq 0$ *and*

$$f(z) = \left[\hat{\Theta}_{n,\alpha}^{(1,1)}(z)\phi(z) + \hat{\Theta}_{n,\alpha}^{(1,2)}(z)\psi(z)\right]\left[\hat{\Theta}_{n,\alpha}^{(2,1)}(z)\phi(z) + \hat{\Theta}_{n,\alpha}^{(2,2)}(z)\psi(z)\right]^{-1}$$

are valid for each $z \in \mathbb{C} \setminus [\alpha, \infty)$.

(c) *For each* $k \in \{1, 2\}$, *let* $[\phi_k; \psi_k] \in \mathscr{P}_{-\tilde{J}_q,\geq}^{(q,q)}(\mathbb{C} \setminus [\alpha, \infty))$ *be such that* $\det(\hat{\Theta}_{n,\alpha}^{(2,1)}\phi_k + \hat{\Theta}_{n,\alpha}^{(2,2)}\psi_k)$ *does not vanish identically. Then* $\langle[\phi_1; \psi_1]\rangle = \langle[\phi_2; \psi_2]\rangle$ *if and only if*

$$\begin{aligned}(\hat{\Theta}_{n,\alpha}^{(1,1)}\phi_1 + \hat{\Theta}_{n,\alpha}^{(1,2)}\psi_1)&(\hat{\Theta}_{n,\alpha}^{(2,1)}\phi_1 + \hat{\Theta}_{n,\alpha}^{(2,2)}\psi_1)^{-1}\\ &= (\hat{\Theta}_{n,\alpha}^{(1,1)}\phi_2 + \hat{\Theta}_{n,\alpha}^{(1,2)}\psi_2)(\hat{\Theta}_{n,\alpha}^{(2,1)}\phi_2 + \hat{\Theta}_{n,\alpha}^{(2,2)}\psi_2)^{-1}.\end{aligned}$$

Proof First observe that, according to Remark 9.4, the matrix-valued function $\hat{\Theta}_{n,\alpha}$ is holomorphic in $\mathbb{C} \setminus [\alpha, \infty)$ and belongs to $\mathfrak{W}_{\tilde{J}_q,\alpha}$. Thus, we can apply Proposition 10.1 to $\Theta = \hat{\Theta}_{n,\alpha}$. Furthermore, observe that Lemma 7.19(b) yields $\det \hat{\Theta}_{n,\alpha}(z) \neq 0$ for all $z \in \mathbb{C} \setminus [\alpha, \infty)$.

(a) By virtue of Proposition 10.1(c), there exists a discrete subset $\mathscr{D}$ of $\mathbb{C} \setminus [\alpha, \infty)$ such that f is holomorphic in $\mathbb{C} \setminus ([\alpha, \infty) \cup \mathscr{D})$ and the inequalities (137) and (138) hold true for each $z \in \mathbb{C} \setminus (\mathbb{R} \cup \mathscr{D})$. Now consider an arbitrary $z \in \mathbb{C} \setminus (\mathbb{R} \cup \mathscr{D})$ with (139). Then, we can conclude (140) from Proposition 7.24 together with (137) and (138).

(b) By assumption f belongs to $\mathscr{S}_{0,q;[\alpha,\infty)}$ and the $[\alpha, \infty)$-Stieltjes measure σ of f belongs to $\mathscr{M}_{\geq}^{q}[[\alpha, \infty); (s_j)_{j=0}^{2n+1}, \leq]$. Since f is the $[\alpha, \infty)$-Stieltjes transform of σ we can apply [36, Proposition 4.9] to conclude that the matrices $P_{2n}^{[f]}(z)$ and $P_{2n+1}^{[f]}(z)$ are both non-negative Hermitian for all $z \in \mathbb{C} \setminus \mathbb{R}$. In view of (5), (7), (112), and (113), the application of Lemma 4.6(a) yields for all $z \in \mathbb{C} \setminus \mathbb{R}$ then (139) and (140). Hence, we can conclude from Proposition 7.24 with $\mathscr{G} = \mathbb{C} \setminus [\alpha, \infty)$ that (137) and (138) hold true for all $z \in \mathbb{C} \setminus \mathbb{R}$. Since f and $\hat{\Theta}_{n,\alpha}$ are holomorphic in $\mathbb{C} \setminus [\alpha, \infty)$ and $\det \hat{\Theta}_{n,\alpha}(z) \neq 0$ for all $z \in \mathbb{C} \setminus [\alpha, \infty)$, we can thus apply Proposition 10.1(b) with $\mathscr{D} = \emptyset$ to complete the proof of part (b).

(c) This is a direct consequence of Proposition 10.1(d).

12 On a Closer Analysis of the Range Conditions in Proposition 4.10

The content of this section is motivated by the wish to get a deeper understanding of the two range conditions occurring in Proposition 4.10. To realize this aim we construct an appropriate $2(n+1)q \times 2q$ matrix polynomial which contains all information about these two range conditions. Our strategy to extract this information is based on finding a factorization of the relevant $2(n+1)q \times 2q$ matrix polynomial as a product of four matrix polynomials having a simpler block structure. In order to prepare this factorization we still need some algebraic identities which form the content of the following two lemmas.

Lemma 12.1 *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N} \cup \{\infty\}$, and let $(s_j)_{j=0}^{\kappa} \in \mathcal{K}_{q,\kappa,\alpha}^{\geq,\mathrm{e}}$. For each $n \in \mathbb{N}_0$ with $2n+1 \leq \kappa$ and all $z \in \mathbb{C}$, then*

$$\begin{aligned}
&(I_{(n+1)q} - H_n^\dagger H_n) R_{T_{q,n}}(z)[I_{(n+1)q}, T_{q,n} H_n] \operatorname{diag}(v_{q,n}, v_{q,n}) \Theta_{n,\alpha}(z) \\
&\quad = \Big[I_{(n+1)q} + (z-\alpha)(I_{(n+1)q} - H_n^\dagger H_n) T_{q,n} R_{T_{q,n}}(z)(I_{(n+1)q} - H_n H_n^-) \Big] \\
&\quad \times (I_{(n+1)q} - H_n^\dagger H_n) R_{T_{q,n}}(\alpha)[I_{(n+1)q}, T_{q,n} H_n] \operatorname{diag}(v_{q,n}, v_{q,n}) B_{n,\alpha}
\end{aligned} \tag{141}$$

and

$$\begin{aligned}
&(I_{(n+1)q} - H_{\alpha\triangleright n}^\dagger H_{\alpha\triangleright n}) R_{T_{q,n}}(z)\Big[I_{(n+1)q}, \big[R_{T_{q,n}}(\alpha)\big]^{-1} H_n \Big] \operatorname{diag}(v_{q,n}, v_{q,n}) \tilde{\Theta}_{n,\alpha}(z) \\
&= \Big[I_{(n+1)q} + (z-\alpha)(I_{(n+1)q} - H_{\alpha\triangleright n}^\dagger H_{\alpha\triangleright n}) T_{q,n} R_{T_{q,n}}(z)(I_{(n+1)q} - H_{\alpha\triangleright n} H_{\alpha\triangleright n}^-) \Big] \\
&\times (I_{(n+1)q} - H_{\alpha\triangleright n}^\dagger H_{\alpha\triangleright n}) R_{T_{q,n}}(\alpha)\Big[I_{(n+1)q}, \big[R_{T_{q,n}}(\alpha)\big]^{-1} H_n \Big] \operatorname{diag}(v_{q,n}, v_{q,n}) \tilde{B}_{n,\alpha}.
\end{aligned} \tag{142}$$

Proof Let $n \in \mathbb{N}_0$ be such that $2n+1 \leq \kappa$ and let $z \in \mathbb{C}$. Remarks 5.8 and 5.7 yield $H_n^* = H_n$ and $H_{\alpha\triangleright n}^* = H_{\alpha\triangleright n}$. Using (87), (30), and $R_{T_{q,n}^*}(z) = [R_{T_{q,n}}(\overline{z})]^*$, we have

$$\begin{aligned}
&(I_{(n+1)q} - H_n^\dagger H_n) R_{T_{q,n}}(z)[I_{(n+1)q}, T_{q,n} H_n] \operatorname{diag}(v_{q,n}, v_{q,n}) \Theta_{n,\alpha}(z) \\
&= (I_{(n+1)q} - H_n^\dagger H_n) R_{T_{q,n}}(z)[I_{(n+1)q}, T_{q,n} H_n] \operatorname{diag}(v_{q,n}, v_{q,n}) U_{n,\alpha}(z) B_{n,\alpha} \\
&= (I_{(n+1)q} - H_n^\dagger H_n) R_{T_{q,n}}(z)[I_{(n+1)q}, T_{q,n} H_n] \operatorname{diag}(v_{q,n}, v_{q,n}) \\
&\qquad \times \Big\{ I_{2q} + (z-\alpha)[\operatorname{diag}(v_{q,n}, v_{q,n})]^* [T_{q,n} H_n, -I_{(n+1)q}]^* \\
&\qquad\qquad \times \big[R_{T_{q,n}}(\overline{z})\big]^* H_n^- R_{T_{q,n}}(\alpha)[I_{(n+1)q}, T_{q,n} H_n] \operatorname{diag}(v_{q,n}, v_{q,n}) \Big\} B_{n,\alpha} \\
&= (I_{(n+1)q} - H_n^\dagger H_n) \Phi(z) R_{T_{q,n}}(\alpha)[I_{(n+1)q}, T_{q,n} H_n] \operatorname{diag}(v_{q,n}, v_{q,n}) B_{n,\alpha}
\end{aligned} \tag{143}$$

where

$$\Phi(z) := R_{T_{q,n}}(z)\big[R_{T_{q,n}}(\alpha)\big]^{-1} + (z-\alpha)R_{T_{q,n}}(z)[I_{(n+1)q}, T_{q,n}H_n]\operatorname{diag}(v_{q,n}, v_{q,n}) \\ \times [\operatorname{diag}(v_{q,n}, v_{q,n})]^*[T_{q,n}H_n, -I_{(n+1)q}]^*\big[R_{T_{q,n}}(\overline{z})\big]^* H_n^-. \tag{144}$$

Taking into account equation (29) in Remark 7.3, $H_n^* = H_n$, and (11), we obtain

$$\begin{aligned} & R_{T_{q,n}}(z)[I_{(n+1)q}, T_{q,n}H_n]\operatorname{diag}(v_{q,n}, v_{q,n}) \\ & \qquad \times [\operatorname{diag}(v_{q,n}, v_{q,n})]^*[T_{q,n}H_n, -I_{(n+1)q}]^*\big[R_{T_{q,n}}(\overline{z})\big]^* \\ & = R_{T_{q,n}}(z)(v_{q,n}v_{q,n}^* H_n T_{q,n}^* - T_{q,n}H_n v_{q,n}v_{q,n}^*)\big[R_{T_{q,n}}(\overline{z})\big]^* \\ & = R_{T_{q,n}}(z)\Big(\big[R_{T_{q,n}}(z)\big]^{-1} H_n T_{q,n}^* - T_{q,n}H_n\big[R_{T_{q,n}}(\overline{z})\big]^{-*}\Big)\big[R_{T_{q,n}}(\overline{z})\big]^* \\ & = H_n T_{q,n}^*\big[R_{T_{q,n}}(\overline{z})\big]^* - R_{T_{q,n}}(z)T_{q,n}H_n. \end{aligned} \tag{145}$$

From (145), (144), (247), and the identity $R_{T_{q,n}}(z)T_{q,n} = T_{q,n}R_{T_{q,n}}(z)$ we get

$$\begin{aligned} \Phi(z) &= R_{T_{q,n}}(z)\big[R_{T_{q,n}}(\alpha)\big]^{-1} + (z-\alpha)\Big(H_n T_{q,n}^*\big[R_{T_{q,n}}(\overline{z})\big]^* - R_{T_{q,n}}(z)T_{q,n}H_n\Big)H_n^- \\ &= I_{(n+1)q} + (z-\alpha)T_{q,n}R_{T_{q,n}}(z) + (z-\alpha)H_n T_{q,n}^*\big[R_{T_{q,n}}(\overline{z})\big]^* H_n^- \\ &\quad - (z-\alpha)T_{q,n}R_{T_{q,n}}(z)H_n H_n^- \\ &= I_{(n+1)q} + (z-\alpha)T_{q,n}R_{T_{q,n}}(z)(I_{(n+1)q} - H_n H_n^-) + (z-\alpha)H_n T_{q,n}^*\big[R_{T_{q,n}}(\overline{z})\big]^* H_n^-. \end{aligned} \tag{146}$$

In view of Remark 6.13, we have

$$(z-\alpha)(I_{(n+1)q} - H_n^\dagger H_n)H_n T_{q,n}^*\big[R_{T_{q,n}}(\overline{z})\big]^* H_n^- = 0. \tag{147}$$

By virtue of (146), Remark 6.13, and (147), we conclude

$$\begin{aligned} & (I_{(n+1)q} - H_n^\dagger H_n)\Phi(z)R_{T_{q,n}}(\alpha)[I_{(n+1)q}, T_{q,n}H_n]\operatorname{diag}(v_{q,n}, v_{q,n})B_{n,\alpha} \\ & = (I_{(n+1)q} - H_n^\dagger H_n) \\ & \times \Big[I_{(n+1)q} + (z-\alpha)T_{q,n}R_{T_{q,n}}(z)(I_{(n+1)q} - H_n H_n^-) + (z-\alpha)H_n T_{q,n}^*[R_{T_{q,n}}(\overline{z})]^* H_n^-\Big] \\ & \times R_{T_{q,n}}(\alpha)[I_{(n+1)q}, T_{q,n}H_n]\operatorname{diag}(v_{q,n}, v_{q,n})B_{n,\alpha} \end{aligned}$$

$$
\begin{aligned}
&= \Big\{ (I_{(n+1)q} - H_n^\dagger H_n) \\
&\quad + (z-\alpha)(I_{(n+1)q} - H_n^\dagger H_n) T_{q,n} R_{T_{q,n}}(z)(I_{(n+1)q} - H_n H_n^-)(I_{(n+1)q} - H_n^\dagger H_n) \\
&\quad + (z-\alpha)(I_{(n+1)q} - H_n^\dagger H_n) H_n T_{q,n}^* \big[R_{T_{q,n}}(\overline{z})\big]^* H_n^- \Big\} \\
&\quad \times R_{T_{q,n}}(\alpha)[I_{(n+1)q}, T_{q,n} H_n] \operatorname{diag}(v_{q,n}, v_{q,n}) B_{n,\alpha} \\
&= \Big[I_{(n+1)q} + (z-\alpha)(I_{(n+1)q} - H_n^\dagger H_n) T_{q,n} R_{T_{q,n}}(z)(I_{(n+1)q} - H_n H_n^-) \Big] \\
&\quad \times (I_{(n+1)q} - H_n^\dagger H_n) R_{T_{q,n}}(\alpha)[I_{(n+1)q}, T_{q,n} H_n] \operatorname{diag}(v_{q,n}, v_{q,n}) B_{n,\alpha}
\end{aligned}
$$

and, in view of (143), consequently (141). Furthermore, using (87) and (59), we infer

$$
\begin{aligned}
&(I_{(n+1)q} - H_{\alpha \triangleright n}^\dagger H_{\alpha \triangleright n}) R_{T_{q,n}}(z) \Big[I_{(n+1)q}, \big[R_{T_{q,n}}(\alpha)\big]^{-1} H_n \Big] \operatorname{diag}(v_{q,n}, v_{q,n}) \tilde{\Theta}_{n,\alpha}(z) \\
&= (I_{(n+1)q} - H_{\alpha \triangleright n}^\dagger H_{\alpha \triangleright n}) R_{T_{q,n}}(z) \Big[I_{(n+1)q}, \big[R_{T_{q,n}}(\alpha)\big]^{-1} H_n \Big] \operatorname{diag}(v_{q,n}, v_{q,n}) \\
&\quad \times \tilde{U}_{n,\alpha}(z) \tilde{B}_{n,\alpha} \\
&= (I_{(n+1)q} - H_{\alpha \triangleright n}^\dagger H_{\alpha \triangleright n}) R_{T_{q,n}}(z) \Big[I_{(n+1)q}, \big[R_{T_{q,n}}(\alpha)\big]^{-1} H_n \Big] \operatorname{diag}(v_{q,n}, v_{q,n}) \\
&\quad \times \Big\{ I_{2q} + (z-\alpha)[\operatorname{diag}(v_{q,n}, v_{q,n})]^* \Big[\big[R_{T_{q,n}}(\alpha)\big]^{-1} H_n, -I_{(n+1)q} \Big]^* \big[R_{T_{q,n}}(\overline{z})\big]^* H_{\alpha \triangleright n}^- \\
&\qquad \times R_{T_{q,n}}(\alpha) \Big[I_{(n+1)q}, \big[R_{T_{q,n}}(\alpha)\big]^{-1} H_n \Big] \operatorname{diag}(v_{q,n}, v_{q,n}) \Big\} \tilde{B}_{n,\alpha} \\
&= (I_{(n+1)q} - H_{\alpha \triangleright n}^\dagger H_{\alpha \triangleright n}) \tilde{\Phi}(z) R_{T_{q,n}}(\alpha) \Big[I_{(n+1)q}, \big[R_{T_{q,n}}(\alpha)\big]^{-1} H_n \Big] \\
&\qquad \operatorname{diag}(v_{q,n}, v_{q,n}) \tilde{B}_{n,\alpha}
\end{aligned}
\tag{148}
$$

where

$$
\begin{aligned}
\tilde{\Phi}(z) &:= R_{T_{q,n}}(z) \big[R_{T_{q,n}}(\alpha)\big]^{-1} + (z-\alpha) R_{T_{q,n}}(z) \Big[I_{(n+1)q}, \big[R_{T_{q,n}}(\alpha)\big]^{-1} H_n \Big] \\
&\quad \times \operatorname{diag}(v_{q,n}, v_{q,n}) [\operatorname{diag}(v_{q,n}, v_{q,n})]^* \Big[\big[R_{T_{q,n}}(\alpha)\big]^{-1} H_n, -I_{(n+1)q} \Big]^* \\
&\qquad \big[R_{T_{q,n}}(\overline{z})\big]^* H_{\alpha \triangleright n}^- .
\end{aligned}
\tag{149}
$$

Because of identity (29) in Remark 7.3, $H_n^* = H_n$, and equation (12) in Remark 5.5, we obtain

$$\begin{aligned}
&R_{T_{q,n}}(z)\Big[I_{(n+1)q}, \big[R_{T_{q,n}}(\alpha)\big]^{-1}H_n\Big]\operatorname{diag}(v_{q,n}, v_{q,n})\\
&\qquad\times [\operatorname{diag}(v_{q,n}, v_{q,n})]^*\Big[\big[R_{T_{q,n}}(\alpha)\big]^{-1}H_n, -I_{(n+1)q}\Big]^*\big[R_{T_{q,n}}(\overline{z})\big]^*\\
&= R_{T_{q,n}}(z)\Big(v_{q,n}v_{q,n}^*H_n\big[R_{T_{q,n}}(\alpha)\big]^{-*} - \big[R_{T_{q,n}}(\alpha)\big]^{-1}H_n v_{q,n}v_{q,n}^*\Big)\big[R_{T_{q,n}}(\overline{z})\big]^*\\
&= R_{T_{q,n}}(z)\Big(\big[R_{T_{q,n}}(z)\big]^{-1}H_{\alpha\triangleright n}T_{q,n}^* - T_{q,n}H_{\alpha\triangleright n}\big[R_{T_{q,n}}(\overline{z})\big]^{-*}\Big)\big[R_{T_{q,n}}(\overline{z})\big]^*\\
&= H_{\alpha\triangleright n}T_{q,n}^*\big[R_{T_{q,n}}(\overline{z})\big]^* - R_{T_{q,n}}(z)T_{q,n}H_{\alpha\triangleright n},
\end{aligned}$$

which, in view of (149), (247), and the identity $R_{T_{q,n}}(z)T_{q,n} = T_{q,n}R_{T_{q,n}}(z)$, implies

$$\begin{aligned}
&\tilde{\Phi}(z)\\
&= R_{T_{q,n}}(z)\big[R_{T_{q,n}}(\alpha)\big]^{-1}+(z-\alpha)\Big(H_{\alpha\triangleright n}T_{q,n}^*\big[R_{T_{q,n}}(\overline{z})\big]^*-R_{T_{q,n}}(z)T_{q,n}H_{\alpha\triangleright n}\Big)H_{\alpha\triangleright n}^-\\
&= I_{(n+1)q} + (z-\alpha)T_{q,n}R_{T_{q,n}}(z) + (z-\alpha)H_{\alpha\triangleright n}T_{q,n}^*\big[R_{T_{q,n}}(\overline{z})\big]^*H_{\alpha\triangleright n}^-\\
&\qquad - (z-\alpha)T_{q,n}R_{T_{q,n}}(z)H_{\alpha\triangleright n}H_{\alpha\triangleright n}^-\\
&= I_{(n+1)q} + (z-\alpha)T_{q,n}R_{T_{q,n}}(z)(I_{(n+1)q} - H_{\alpha\triangleright n}H_{\alpha\triangleright n}^-)\\
&\qquad + (z-\alpha)H_{\alpha\triangleright n}T_{q,n}^*\big[R_{T_{q,n}}(\overline{z})\big]^*H_{\alpha\triangleright n}^-.
\end{aligned}\tag{150}$$

From Remark 6.13 we see that

$$(z-\alpha)(I_{(n+1)q} - H_{\alpha\triangleright n}^{\dagger}H_{\alpha\triangleright n})H_{\alpha\triangleright n}T_{q,n}^*\big[R_{T_{q,n}}(\overline{z})\big]^*H_{\alpha\triangleright n}^- = 0 \tag{151}$$

is true. Using (150), Remark 6.13, and (151), we get

$$\begin{aligned}
&(I_{(n+1)q}-H_{\alpha\triangleright n}^{\dagger}H_{\alpha\triangleright n})\tilde{\Phi}(z)R_{T_{q,n}}(\alpha)\Big[I_{(n+1)q}, \big[R_{T_{q,n}}(\alpha)\big]^{-1}H_n\Big]\operatorname{diag}(v_{q,n}, v_{q,n})\tilde{B}_{n,\alpha}\\
&= (I_{(n+1)q} - H_{\alpha\triangleright n}^{\dagger}H_{\alpha\triangleright n})\Big\{I_{(n+1)q} + (z-\alpha)T_{q,n}R_{T_{q,n}}(z)(I_{(n+1)q} - H_{\alpha\triangleright n}H_{\alpha\triangleright n}^-)\\
&\qquad\qquad + (z-\alpha)H_{\alpha\triangleright n}T_{q,n}^*\big[R_{T_{q,n}}(\overline{z})\big]^*H_{\alpha\triangleright n}^-\Big\}\\
&\qquad\times R_{T_{q,n}}(\alpha)\Big[I_{(n+1)q}, \big[R_{T_{q,n}}(\alpha)\big]^{-1}H_n\Big]\operatorname{diag}(v_{q,n}, v_{q,n})\tilde{B}_{n,\alpha}
\end{aligned}$$

$$
\begin{aligned}
&= \Big\{(I_{(n+1)q} - H_{\alpha\triangleright n}^{\dagger} H_{\alpha\triangleright n}) + (z-\alpha)(I_{(n+1)q} - H_{\alpha\triangleright n}^{\dagger} H_{\alpha\triangleright n}) T_{q,n} R_{T_{q,n}}(z) \\
&\qquad\qquad \times (I_{(n+1)q} - H_{\alpha\triangleright n} H_{\alpha\triangleright n}^{-})(I_{(n+1)q} - H_{\alpha\triangleright n}^{\dagger} H_{\alpha\triangleright n}) \\
&\qquad\quad + (z-\alpha)(I_{(n+1)q} - H_{\alpha\triangleright n}^{\dagger} H_{\alpha\triangleright n}) H_{\alpha\triangleright n} T_{q,n}^{*} \big[R_{T_{q,n}}(\overline{z})\big]^{*} H_{\alpha\triangleright n}^{-}\Big\} \\
&\quad \times R_{T_{q,n}}(\alpha)[I_{(n+1)q}, [R_{T_{q,n}}(\alpha)]^{-1} H_n] \operatorname{diag}(v_{q,n}, v_{q,n}) \tilde{B}_{n,\alpha} \\
&= \Big[I_{(n+1)q} + (z-\alpha)(I_{(n+1)q} - H_{\alpha\triangleright n}^{\dagger} H_{\alpha\triangleright n}) T_{q,n} R_{T_{q,n}}(z)(I_{(n+1)q} - H_{\alpha\triangleright n} H_{\alpha\triangleright n}^{-})\Big] \\
&\quad \times (I_{(n+1)q} - H_{\alpha\triangleright n}^{\dagger} H_{\alpha\triangleright n}) R_{T_{q,n}}(\alpha)\Big[I_{(n+1)q}, \big[R_{T_{q,n}}(\alpha)\big]^{-1} H_n\Big] \operatorname{diag}(v_{q,n}, v_{q,n}) \tilde{B}_{n,\alpha}.
\end{aligned} \tag{152}
$$

The combination of (148) and (152) provides us (142).

Lemma 12.2 *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N} \cup \{\infty\}$, and let $(s_j)_{j=0}^{\kappa} \in \mathscr{K}_{q,\kappa,\alpha}^{\geq,\mathrm{e}}$. For each $n \in \mathbb{N}_0$ such that $2n+1 \leq \kappa$, then*

$$
\begin{aligned}
&(I_{(n+1)q} - H_n^{\dagger} H_n) R_{T_{q,n}}(\alpha)[I_{(n+1)q}, T_{q,n} H_n] \operatorname{diag}(v_{q,n}, v_{q,n}) B_{n,\alpha} \\
&= (I_{(n+1)q} - H_n^{\dagger} H_n) R_{T_{q,n}}(\alpha)\big[I_{(n+1)q}, T_{q,n}(I_{(n+1)q} - H_{\alpha\triangleright n} H_{\alpha\triangleright n}^{-}) H_n\big] \operatorname{diag}(v_{q,n}, v_{q,n})
\end{aligned}
$$

and

$$
\begin{aligned}
&(I_{(n+1)q} - H_{\alpha\triangleright n}^{\dagger} H_{\alpha\triangleright n}) R_{T_{q,n}}(\alpha)\Big[I_{(n+1)q}, \big[R_{T_{q,n}}(\alpha)\big]^{-1} H_n\Big] \operatorname{diag}(v_{q,n}, v_{q,n}) \tilde{B}_{n,\alpha} \\
&\quad = (I_{(n+1)q} - H_{\alpha\triangleright n}^{\dagger} H_{\alpha\triangleright n})\big[(I_{(n+1)q} - H_n H_n^{-}) R_{T_{q,n}}(\alpha), H_n\big] \operatorname{diag}(v_{q,n}, v_{q,n}).
\end{aligned}
$$

Proof Let $n \in \mathbb{N}_0$ be such that $2n+1 \leq \kappa$. From Remarks 5.8 and 5.7 we get $H_n^{*} = H_n$ and $H_{\alpha\triangleright n}^{*} = H_{\alpha\triangleright n}$. Because of the Remarks 5.3 and 6.13, we have

$$
\begin{aligned}
&(I_{(n+1)q} - H_n^{\dagger} H_n) R_{T_{q,n}}(\alpha)(v_{q,n} v_{q,n}^{*} H_n H_{\alpha\triangleright n}^{-} + T_{q,n}) \\
&= (I_{(n+1)q} - H_n^{\dagger} H_n) R_{T_{q,n}}(\alpha)\Big[\Big([R_{T_{q,n}}(\alpha)]^{-1} H_n - T_{q,n} H_{\alpha\triangleright n}\Big) H_{\alpha\triangleright n}^{-} + T_{q,n}\Big] \\
&= (I_{(n+1)q} - H_n^{\dagger} H_n) H_n H_{\alpha\triangleright n}^{-} - (I_{(n+1)q} - H_n^{\dagger} H_n) R_{T_{q,n}}(\alpha) T_{q,n} H_{\alpha\triangleright n} H_{\alpha\triangleright n}^{-} \\
&\qquad + (I_{(n+1)q} - H_n^{\dagger} H_n) R_{T_{q,n}}(\alpha) T_{q,n} \\
&= (I_{(n+1)q} - H_n^{\dagger} H_n) R_{T_{q,n}}(\alpha) T_{q,n}(I_{(n+1)q} - H_{\alpha\triangleright n} H_{\alpha\triangleright n}^{-}).
\end{aligned} \tag{153}
$$

Applying Remark 7.12 and (153), we conclude

$$\begin{aligned}
&(I_{(n+1)q} - H_n^\dagger H_n) R_{T_{q,n}}(\alpha)[I_{(n+1)q}, T_{q,n} H_n] \operatorname{diag}(v_{q,n}, v_{q,n}) B_{n,\alpha} \\
&= (I_{(n+1)q} - H_n^\dagger H_n) R_{T_{q,n}}(\alpha) \Big[I_{(n+1)q}, (v_{q,n} v_{q,n}^* H_n H_{\alpha\triangleright n}^- + T_{q,n}) H_n\Big] \operatorname{diag}(v_{q,n}, v_{q,n}) \\
&= \Big[(I_{(n+1)q} - H_n^\dagger H_n) R_{T_{q,n}}(\alpha), (I_{(n+1)q} - H_n^\dagger H_n) R_{T_{q,n}}(\alpha)(v_{q,n} v_{q,n}^* H_n H_{\alpha\triangleright n}^- + T_{q,n}) H_n\Big] \\
&\qquad \times \operatorname{diag}(v_{q,n}, v_{q,n}) \\
&= (I_{(n+1)q} - H_n^\dagger H_n) R_{T_{q,n}}(\alpha) \big[I_{(n+1)q}, T_{q,n}(I_{(n+1)q} - H_{\alpha\triangleright n} H_{\alpha\triangleright n}^-) H_n\big] \operatorname{diag}(v_{q,n}, v_{q,n}).
\end{aligned}$$

Taking into account $H_n^* = H_n$, $H_{\alpha\triangleright n}^* = H_{\alpha\triangleright n}$, and Remark 5.3, we obtain $H_n v_{q,n} v_{q,n}^* = H_n [R_{T_{q,n}}(\alpha)]^{-*} - H_{\alpha\triangleright n} T_{q,n}^*$ and, hence,

$$\begin{aligned}
&I_{(n+1)q} - H_n v_{q,n} v_{q,n}^* \big[R_{T_{q,n}}(\alpha)\big]^* H_n^- \\
&= I_{(n+1)q} - \Big(H_n \big[R_{T_{q,n}}(\alpha)\big]^{-*} - H_{\alpha\triangleright n} T_{q,n}^*\Big) \big[R_{T_{q,n}}(\alpha)\big]^* H_n^- \\
&= I_{(n+1)q} - H_n H_n^- + H_{\alpha\triangleright n} T_{q,n}^* \big[R_{T_{q,n}}(\alpha)\big]^* H_n^- .
\end{aligned} \tag{154}$$

Let $P := I_{(n+1)q} - H_{\alpha\triangleright n}^\dagger H_{\alpha\triangleright n}$. From (154) and Remark 6.13 we see that

$$\begin{aligned}
&P\Big(I_{(n+1)q} - H_n v_{q,n} v_{q,n}^* \big[R_{T_{q,n}}(\alpha)\big]^* H_n^-\Big) R_{T_{q,n}}(\alpha) \\
&= P\Big(I_{(n+1)q} - H_n H_n^- + H_{\alpha\triangleright n} T_{q,n}^* \big[R_{T_{q,n}}(\alpha)\big]^* H_n^-\Big) R_{T_{q,n}}(\alpha) \\
&= P(I_{(n+1)q} - H_n H_n^-) R_{T_{q,n}}(\alpha)
\end{aligned} \tag{155}$$

holds true. Using Remark 7.12 and (155), we obtain finally

$$\begin{aligned}
&P R_{T_{q,n}}(\alpha) \Big[I_{(n+1)q}, \big[R_{T_{q,n}}(\alpha)\big]^{-1} H_n\Big] \operatorname{diag}(v_{q,n}, v_{q,n}) \tilde{B}_{n,\alpha} \\
&= P R_{T_{q,n}}(\alpha) \big[R_{T_{q,n}}(\alpha)\big]^{-1} \\
&\qquad \times \Big[\Big[I_{(n+1)q} - H_n v_{q,n} v_{q,n}^* R_{T_{q,n}^*}(\alpha) H_n^-\Big] R_{T_{q,n}}(\alpha), H_n\Big] \operatorname{diag}(v_{q,n}, v_{q,n}) \\
&= \Big[P\Big[I_{(n+1)q} - H_n v_{q,n} v_{q,n}^* R_{T_{q,n}^*}(\alpha) H_n^-\Big] R_{T_{q,n}}(\alpha), P H_n\Big] \operatorname{diag}(v_{q,n}, v_{q,n}) \\
&= P\big[(I_{(n+1)q} - H_n H_n^-) R_{T_{q,n}}(\alpha), H_n\big] \operatorname{diag}(v_{q,n}, v_{q,n}).
\end{aligned}$$

The following lemma contains the announced factorization result for the relevant $2(n+1)q \times 2q$ matrix polynomial.

Lemma 12.3 *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N} \cup \{\infty\}$, and let $(s_j)_{j=0}^{\kappa} \in \mathscr{K}_{q,\kappa,\alpha}^{\geq,\mathrm{e}}$. Let $n \in \mathbb{N}_0$ be such that $2n+1 \leq \kappa$ and let the matrix-valued functions $P_{n,\alpha}$, $Q_{n,\alpha}$, and $S_{n,\alpha}$ be defined on $\mathbb{C}$ and, for every choice of $z \in \mathbb{C}$, be given by*

$$P_{n,\alpha}(z) := I_{(n+1)q} + (z-\alpha)(I_{(n+1)q} - H_n^\dagger H_n) T_{q,n} R_{T_{q,n}}(z)(I_{(n+1)q} - H_n H_n^-), \tag{156}$$

$$Q_{n,\alpha}(z) := I_{(n+1)q} + (z-\alpha)(I_{(n+1)q} - H_{\alpha\triangleright n}^\dagger H_{\alpha\triangleright n}) T_{q,n} R_{T_{q,n}}(z)(I_{(n+1)q} - H_{\alpha\triangleright n} H_{\alpha\triangleright n}^-), \tag{157}$$

and

$$S_{n,\alpha}(z) := I_{(n+1)q} - (z-\alpha)(I_{(n+1)q} - H_{\alpha\triangleright n}^\dagger H_{\alpha\triangleright n}) R_{T_{q,n}}(\alpha) T_{q,n}(I_{(n+1)q} - H_{\alpha\triangleright n} H_{\alpha\triangleright n}^-). \tag{158}$$

For each $z \in \mathbb{C}$, then

$$\begin{aligned}
&\operatorname{diag}\big[P_{n,\alpha}(z), Q_{n,\alpha}(z)\big] \\
&\quad\times \left[\begin{array}{c:c} I_{(n+1)q} & 0_{(n+1)q\times(n+1)q} \\ \hdashline (z-\alpha)(I_{(n+1)q} - H_{\alpha\triangleright n}^\dagger H_{\alpha\triangleright n})(I_{(n+1)q} - H_n H_n^-) & S_{n,\alpha}(z) \end{array}\right] \\
&\quad\times \left[\begin{array}{c:c} I_{(n+1)q} & (I_{(n+1)q} - H_n^\dagger H_n) R_{T_{q,n}}(\alpha) T_{q,n}(I_{(n+1)q} - H_{\alpha\triangleright n} H_{\alpha\triangleright n}^-) \\ \hdashline 0_{(n+1)q\times(n+1)q} & I_{(n+1)q} \end{array}\right] \\
&\quad\times \operatorname{diag}\Big((I_{(n+1)q} - H_n^\dagger H_n) R_{T_{q,n}}(\alpha) v_{q,n}, (I_{(n+1)q} - H_{\alpha\triangleright n}^\dagger H_{\alpha\triangleright n}) H_n v_{q,n}\Big) \\
&= \left[\begin{array}{c} (I_{(n+1)q} - H_n^\dagger H_n) R_{T_{q,n}}(z)[I_{(n+1)q}, T_{q,n} H_n] \operatorname{diag}(v_{q,n}, v_{q,n}) \Theta_{n,\alpha}(z) \\ \hdashline (I_{(n+1)q} - H_{\alpha\triangleright n}^\dagger H_{\alpha\triangleright n}) R_{T_{q,n}}(z)[I_{(n+1)q}, [R_{T_{q,n}}(\alpha)]^{-1} H_n] \operatorname{diag}(v_{q,n}, v_{q,n}) \\ \times \tilde{\Theta}_{n,\alpha}(z) \operatorname{diag}((z-\alpha) I_q, I_q) \end{array}\right].
\end{aligned} \tag{159}$$

Proof Let $z \in \mathbb{C}$. Obviously, the matrix on the left-hand side of (159) coincides with

$$R_{n,\alpha}(z) := \operatorname{diag}\big(P_{n,\alpha}(z), Q_{n,\alpha}(z)\big) \begin{bmatrix} \Psi_{n,\alpha}^{(1,1)}(z) & \Psi_{n,\alpha}^{(1,2)}(z) \\ \Psi_{n,\alpha}^{(2,1)}(z) & \Psi_{n,\alpha}^{(2,2)}(z) \end{bmatrix} \operatorname{diag}(v_{q,n}, v_{q,n})$$

where

$$\Psi_{n,\alpha}^{(1,1)}(z) := (I_{(n+1)q} - H_n^\dagger H_n) R_{T_{q,n}}(\alpha), \tag{160}$$

$$\begin{aligned}\Psi_{n,\alpha}^{(1,2)}(z) &:= (I_{(n+1)q} - H_n^\dagger H_n) R_{T_{q,n}}(\alpha) T_{q,n} \\ &\quad\times (I_{(n+1)q} - H_{\alpha\triangleright n} H_{\alpha\triangleright n}^-)(I_{(n+1)q} - H_{\alpha\triangleright n}^\dagger H_{\alpha\triangleright n}) H_n,\end{aligned} \tag{161}$$

$$\begin{aligned}\Psi_{n,\alpha}^{(2,1)}(z) &:= (z-\alpha)(I_{(n+1)q} - H_{\alpha\triangleright n}^\dagger H_{\alpha\triangleright n}) \\ &\quad\times (I_{(n+1)q} - H_n H_n^-)(I_{(n+1)q} - H_n^\dagger H_n) R_{T_{q,n}}(\alpha),\end{aligned} \tag{162}$$

and

$$\Psi_{n,\alpha}^{(2,2)}(z) := (z-\alpha)(I_{(n+1)q} - H_{\alpha\triangleright n}^{\dagger} H_{\alpha\triangleright n})(I_{(n+1)q} - H_n H_n^{-})(I_{(n+1)q} - H_n^{\dagger} H_n) \\ \times R_{T_{q,n}}(\alpha) T_{q,n}(I_{(n+1)q} - H_{\alpha\triangleright n} H_{\alpha\triangleright n}^{-})(I_{(n+1)q} - H_{\alpha\triangleright n}^{\dagger} H_{\alpha\triangleright n}) H_n \\ + S_{n,\alpha}(z)(I_{(n+1)q} - H_{\alpha\triangleright n}^{\dagger} H_{\alpha\triangleright n}) H_n. \tag{163}$$

Because of (161) and Remark 6.13, we have

$$\Psi_{n,\alpha}^{(1,2)}(z) = (I_{(n+1)q} - H_n^{\dagger} H_n) R_{T_{q,n}}(\alpha) T_{q,n}(I_{(n+1)q} - H_{\alpha\triangleright n} H_{\alpha\triangleright n}^{-}) H_n. \tag{164}$$

Furthermore, (162) and Remark 6.13 yield

$$\Psi_{n,\alpha}^{(2,1)}(z) = (z-\alpha)(I_{(n+1)q} - H_{\alpha\triangleright n}^{\dagger} H_{\alpha\triangleright n})(I_{(n+1)q} - H_n H_n^{-}) R_{T_{q,n}}(\alpha). \tag{165}$$

From Remarks 6.13 and 6.15 and (158) we conclude

$$\begin{aligned} &(z-\alpha)(I_{(n+1)q} - H_{\alpha\triangleright n}^{\dagger} H_{\alpha\triangleright n})(I_{(n+1)q} - H_n H_n^{-})(I_{(n+1)q} - H_n^{\dagger} H_n) \\ &\quad \times R_{T_{q,n}}(\alpha) T_{q,n}(I_{(n+1)q} - H_{\alpha\triangleright n} H_{\alpha\triangleright n}^{-}) \\ &= (z-\alpha)(I_{(n+1)q} - H_{\alpha\triangleright n}^{\dagger} H_{\alpha\triangleright n})(I_{(n+1)q} - H_n H_n^{-}) \\ &\quad \times R_{T_{q,n}}(\alpha) T_{q,n}(I_{(n+1)q} - H_{\alpha\triangleright n} H_{\alpha\triangleright n}^{-}) \\ &= (z-\alpha)(I_{(n+1)q} - H_{\alpha\triangleright n}^{\dagger} H_{\alpha\triangleright n}) R_{T_{q,n}}(\alpha) T_{q,n}(I_{(n+1)q} - H_{\alpha\triangleright n} H_{\alpha\triangleright n}^{-}) \\ &\quad - (z-\alpha)(I_{(n+1)q} - H_{\alpha\triangleright n}^{\dagger} H_{\alpha\triangleright n}) H_n H_n^{-} R_{T_{q,n}}(\alpha) T_{q,n}(I_{(n+1)q} - H_{\alpha\triangleright n} H_{\alpha\triangleright n}^{-}) \\ &= (z-\alpha)(I_{(n+1)q} - H_{\alpha\triangleright n}^{\dagger} H_{\alpha\triangleright n}) R_{T_{q,n}}(\alpha) T_{q,n}(I_{(n+1)q} - H_{\alpha\triangleright n} H_{\alpha\triangleright n}^{-}) \\ &= I_{(n+1)q} - S_{n,\alpha}(z). \end{aligned} \tag{166}$$

Combining (163) and (166), we obtain

$$\begin{aligned} &\Psi_{n,\alpha}^{(2,2)}(z) \\ &= (I_{(n+1)q} - S_{n,\alpha}(z))(I_{(n+1)q} - H_{\alpha\triangleright n}^{\dagger} H_{\alpha\triangleright n}) H_n + S_{n,\alpha}(z)(I_{(n+1)q} - H_{\alpha\triangleright n}^{\dagger} H_{\alpha\triangleright n}) H_n \\ &= (I_{(n+1)q} - H_{\alpha\triangleright n}^{\dagger} H_{\alpha\triangleright n}) H_n. \end{aligned} \tag{167}$$

By virtue of Lemma 12.1, (156), Lemma 12.2, (160), and (164), we get

$$\begin{aligned} &(I_{(n+1)q} - H_n^{\dagger} H_n) R_{T_{q,n}}(z) [I_{(n+1)q}, T_{q,n} H_n] \operatorname{diag}(v_{q,n}, v_{q,n}) \Theta_{n,\alpha}(z) \\ &= \Big[I_{(n+1)q} + (z-\alpha)(I_{(n+1)q} - H_n^{\dagger} H_n) T_{q,n} R_{T_{q,n}}(z)(I_{(n+1)q} - H_n H_n^{-}) \Big] \\ &\quad \times (I_{(n+1)q} - H_n^{\dagger} H_n) R_{T_{q,n}}(\alpha) [I_{(n+1)q}, T_{q,n} H_n] \operatorname{diag}(v_{q,n}, v_{q,n}) B_{n,\alpha} \end{aligned}$$

$$
\begin{aligned}
&= P_{n,\alpha}(z)(I_{(n+1)q} - H_n^\dagger H_n) R_{T_{q,n}}(\alpha)[I_{(n+1)q}, T_{q,n} H_n] \operatorname{diag}(v_{q,n}, v_{q,n}) B_{n,\alpha} \\
&= P_{n,\alpha}(z)(I_{(n+1)q} - H_n^\dagger H_n) R_{T_{q,n}}(\alpha) \\
&\qquad \times [I_{(n+1)q}, T_{q,n}(I_{(n+1)q} - H_{\alpha\triangleright n} H_{\alpha\triangleright n}^-) H_n] \operatorname{diag}(v_{q,n}, v_{q,n}) \\
&= P_{n,\alpha}(z)\left[\Psi_{n,\alpha}^{(1,1)}(z), \Psi_{n,\alpha}^{(1,2)}(z)\right] \operatorname{diag}(v_{q,n}, v_{q,n}).
\end{aligned}
\tag{168}
$$

Similarly, Lemma 12.1, (157), Lemma 12.2, (165), and (167) provide us

$$
\begin{aligned}
&(I_{(n+1)q} - H_{\alpha\triangleright n}^\dagger H_{\alpha\triangleright n}) R_{T_{q,n}}(z)\left[I_{(n+1)q}, \left[R_{T_{q,n}}(\alpha)\right]^{-1} H_n\right] \\
&\qquad \times \operatorname{diag}(v_{q,n}, v_{q,n}) \tilde{\Theta}_{n,\alpha}(z) \operatorname{diag}\big((z-\alpha) I_q, I_q\big) \\
&= \left[I_{(n+1)q} + (z-\alpha)(I_{(n+1)q} - H_{\alpha\triangleright n}^\dagger H_{\alpha\triangleright n}) T_{q,n} R_{T_{q,n}}(z)(I_{(n+1)q} - H_{\alpha\triangleright n} H_{\alpha\triangleright n}^-)\right] \\
&\qquad \times (I_{(n+1)q} - H_{\alpha\triangleright n}^\dagger H_{\alpha\triangleright n}) R_{T_{q,n}}(\alpha)\left[I_{(n+1)q}, \left[R_{T_{q,n}}(\alpha)\right]^{-1} H_n\right] \\
&\qquad \times \operatorname{diag}(v_{q,n}, v_{q,n}) \tilde{B}_{n,\alpha} \operatorname{diag}\big((z-\alpha) I_q, I_q\big) \\
&= Q_{n,\alpha}(z)(I_{(n+1)q} - H_{\alpha\triangleright n}^\dagger H_{\alpha\triangleright n}) R_{T_{q,n}}(\alpha)\left[I_{(n+1)q}, \left[R_{T_{q,n}}(\alpha)\right]^{-1} H_n\right] \\
&\qquad \times \operatorname{diag}(v_{q,n}, v_{q,n}) \tilde{B}_{n,\alpha} \operatorname{diag}\big((z-\alpha) I_q, I_q\big) \\
&= Q_{n,\alpha}(z)(I_{(n+1)q} - H_{\alpha\triangleright n}^\dagger H_{\alpha\triangleright n})\big[(I_{(n+1)q} - H_n H_n^-) R_{T_{q,n}}(\alpha), H_n\big] \\
&\qquad \times \operatorname{diag}(v_{q,n}, v_{q,n}) \operatorname{diag}\big((z-\alpha) I_q, I_q\big) \\
&= Q_{n,\alpha}(z)(I_{(n+1)q} - H_{\alpha\triangleright n}^\dagger H_{\alpha\triangleright n}) \\
&\qquad \times \big[(z-\alpha)(I_{(n+1)q} - H_n H_n^-) R_{T_{q,n}}(\alpha), H_n\big] \operatorname{diag}(v_{q,n}, v_{q,n}) \\
&= Q_{n,\alpha}(z)\left[\Psi_{n,\alpha}^{(2,1)}(z), \Psi_{n,\alpha}^{(2,2)}(z)\right] \operatorname{diag}(v_{q,n}, v_{q,n}).
\end{aligned}
\tag{169}
$$

Since $R_{n,\alpha}(z)$ coincides with the matrix on the left-hand side of (159) from (168) and (169), equation (159) follows.

Taking into account our knowledge on the role of the $2q \times 2q$ matrix polynomials $\Theta_{n,\alpha}$ and $\tilde{\Theta}_{n,\alpha}$ introduced in Lemma 7.15 the application of Lemma 12.3 leads us now to a deeper understanding of the two range conditions occurring in Proposition 4.10. If $\mathscr{G}$ is a non-empty subset of $\mathbb{C}$ and if $f : \mathscr{G} \to \mathbb{C}$ is a function, then let $\mathscr{N}_f := \{z \in \mathscr{G} : f(z) = 0\}$.

Proposition 12.4 *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N} \cup \{\infty\}$, let $(s_j)_{j=0}^{\kappa} \in \mathscr{K}_{q,\kappa,\alpha}^{\geq,\mathrm{e}}$, and let $n \in \mathbb{N}_0$ be such that $2n+1 \leq \kappa$. Then:*

(a) The set $\mathscr{N}_{\det P_{n,\alpha}} \cup \mathscr{N}_{\det Q_{n,\alpha}} \cup \mathscr{N}_{\det S_{n,\alpha}}$ is finite.
(b) Let $x \in \mathbb{C}^{q\times q}$ and let $y \in \mathbb{C}^{q\times q}$. Then the following statements are equivalent:

(i) *For each* $z \in \mathbb{C} \setminus (\mathcal{N}_{\det P_{n,\alpha}} \cup \mathcal{N}_{\det Q_{n,\alpha}} \cup \mathcal{N}_{\det S_{n,\alpha}})$, *the equations*

$$(I_{(n+1)q} - H_n^\dagger H_n) R_{T_{q,n}}(z) [I_{(n+1)q}, T_{q,n} H_n] \operatorname{diag}(v_{q,n}, v_{q,n}) \Theta_{n,\alpha}(z) \begin{bmatrix} x \\ y \end{bmatrix} = 0 \tag{170}$$

and

$$(I_{(n+1)q} - H_{\alpha \triangleright n}^\dagger H_{\alpha \triangleright n}) R_{T_{q,n}}(z) \Big[I_{(n+1)q}, \big[R_{T_{q,n}}(\alpha)\big]^{-1} H_n \Big] \times \operatorname{diag}(v_{q,n}, v_{q,n}) \tilde{\Theta}_{n,\alpha}(z) \operatorname{diag}\big((z-\alpha) I_q, I_q\big) \begin{bmatrix} x \\ y \end{bmatrix} = 0 \tag{171}$$

hold true.

(ii) *There exists a number* $z \in \mathbb{C} \setminus (\mathcal{N}_{\det P_{n,\alpha}} \cup \mathcal{N}_{\det Q_{n,\alpha}} \cup \mathcal{N}_{\det S_{n,\alpha}})$ *such that* (170) *and* (171) *are valid.*

(iii) *The equations*

$$(I_{(n+1)q} - H_n^\dagger H_n) R_{T_{q,n}}(\alpha) v_{q,n} x = 0 \tag{172}$$

and

$$(I_{(n+1)q} - H_{\alpha \triangleright n}^\dagger H_{\alpha \triangleright n}) H_n v_{q,n} y = 0 \tag{173}$$

are fulfilled.

Proof

(a) By virtue of Remark 4.1, (156), (157), and (158), we see that $P_{n,\alpha}$, $Q_{n,\alpha}$, and $S_{n,\alpha}$ are matrix polynomials with $P_{n,\alpha}(\alpha) = I_{(n+1)q}$, $Q_{n,\alpha}(\alpha) = I_{(n+1)q}$, and $S_{n,\alpha}(\alpha) = I_{(n+1)q}$. In particular, $\det P_{n,\alpha}$, $\det Q_{n,\alpha}$, and $\det S_{n,\alpha}$ are polynomials which do not vanish identically. In view of the fundamental theorem of algebra, the proof of part (a) is complete.

(b) For each $z \in \mathbb{C}$, Lemma 12.3 provides us

$$\begin{aligned} &\begin{bmatrix} (I_{(n+1)q} - H_n^\dagger H_n) R_{T_{q,n}}(z) [I_{(n+1)q}, T_{q,n} H_n] \operatorname{diag}(v_{q,n}, v_{q,n}) \Theta_{n,\alpha}(z) \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \\ \hdashline (I_{(n+1)q} - H_{\alpha \triangleright n}^\dagger H_{\alpha \triangleright n}) R_{T_{q,n}}(z) [I_{(n+1)q}, [R_{T_{q,n}}(\alpha)]^{-1} H_n] \\ \times \operatorname{diag}(v_{q,n}, v_{q,n}) \tilde{\Theta}_{n,\alpha}(z) \operatorname{diag}((z-\alpha) I_q, I_q) \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \end{bmatrix} \\ &= \begin{bmatrix} (I_{(n+1)q} - H_n^\dagger H_n) R_{T_{q,n}}(z) [I_{(n+1)q}, T_{q,n} H_n] \operatorname{diag}(v_{q,n}, v_{q,n}) \Theta_{n,\alpha}(z) \\ \hdashline (I_{(n+1)q} - H_{\alpha \triangleright n}^\dagger H_{\alpha \triangleright n}) R_{T_{q,n}}(z) [I_{(n+1)q}, [R_{T_{q,n}}(\alpha)]^{-1} H_n] \\ \times \operatorname{diag}(v_{q,n}, v_{q,n}) \tilde{\Theta}_{n,\alpha}(z) \operatorname{diag}((z-\alpha) I_q, I_q) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ &= K_{n,\alpha}(z), \end{aligned} \tag{174}$$

where

$$
\begin{aligned}
K_{n,\alpha}(z) &:= \operatorname{diag}\bigl(P_{n,\alpha}(z), Q_{n,\alpha}(z)\bigr)\\
&\times \left[\begin{array}{c:c} I_{(n+1)q} & 0_{(n+1)q\times(n+1)q} \\ \hdashline (z-\alpha)(I_{(n+1)q}-H_{\alpha\triangleright n}^{\dagger}H_{\alpha\triangleright n})(I_{(n+1)q}-H_nH_n^{-}) & S_{n,\alpha}(z)\end{array}\right]\\
&\times \left[\begin{array}{c:c} I_{(n+1)q} & (I_{(n+1)q}-H_n^{\dagger}H_n)R_{T_{q,n}}(\alpha)T_{q,n}(I_{(n+1)q}-H_{\alpha\triangleright n}H_{\alpha\triangleright n}^{-}) \\ \hdashline 0_{(n+1)q\times(n+1)q} & I_{(n+1)q}\end{array}\right]\\
&\times \left[\begin{array}{c} (I_{(n+1)q}-H_n^{\dagger}H_n)R_{T_{q,n}}(\alpha)v_{q,n}x \\ \hdashline (I_{(n+1)q}-H_{\alpha\triangleright n}^{\dagger}H_{\alpha\triangleright n})H_nv_{q,n}y\end{array}\right].
\end{aligned}
\tag{175}
$$

(i)⇒(ii): This implication is trivial.

(ii)⇒(iii): According to (ii), there exists a number $z \in \mathbb{C} \setminus (\mathscr{N}_{\det P_{n,\alpha}} \cup \mathscr{N}_{\det Q_{n,\alpha}} \cup \mathscr{N}_{\det S_{n,\alpha}})$ such that (170) and (171) hold true. Using (170), (171), and (174), we get $K_{n,\alpha}(z) = 0_{2(n+1)q\times q}$. Because of $z \in \mathbb{C} \setminus (\mathscr{N}_{\det P_{n,\alpha}} \cup \mathscr{N}_{\det Q_{n,\alpha}} \cup \mathscr{N}_{\det S_{n,\alpha}})$, the first three factors of the matrix product on the right-hand side of equation (175) are non-singular matrices. Thus, (175) implies (172) and (173).

(iii)⇒(i): Taking into account (175), (172), (173), and (174), we conclude that

$$
\begin{aligned}
0_{2(n+1)q\times q} &= K_{n,\alpha}(z)\\
&= \left[\begin{array}{c} (I_{(n+1)q}-H_n^{\dagger}H_n)R_{T_{q,n}}(z)[I_{(n+1)q}, T_{q,n}H_n]\operatorname{diag}(v_{q,n}, v_{q,n})\Theta_{n,\alpha}(z)\left[\begin{smallmatrix} x \\ y\end{smallmatrix}\right] \\ \hdashline (I_{(n+1)q}-H_{\alpha\triangleright n}^{\dagger}H_{\alpha\triangleright n})R_{T_{q,n}}(z)[I_{(n+1)q}, [R_{T_{q,n}}(\alpha)]^{-1}H_n] \\ \times \operatorname{diag}(v_{q,n}, v_{q,n})\tilde{\Theta}_{n,\alpha}(z)\operatorname{diag}((z-\alpha)I_q, I_q)\left[\begin{smallmatrix} x \\ y\end{smallmatrix}\right]\end{array}\right]
\end{aligned}
$$

and, consequently, (170) and (171) hold true for each $z \in \mathbb{C}$.

13 On a First Description of the Set $\mathscr{S}_{0,q;[\alpha,\infty)}[(s_j)_{j=0}^{2n+1}, \leq]$

The main goal of this section is to derive a parametrization of the set $\mathscr{S}_{0,q;[\alpha,\infty)}[(s_j)_{j=0}^{2n+1}, \leq]$ on the basis of the linear fractional transformation generated by the $2q \times 2q$ matrix polynomial $\Theta_{n,\alpha}$ given by (85). The role of parameters will be played by a particular subclass of $\mathscr{P}^{(q,q)}_{-\tilde{J}_q,\geq}(\mathbb{C} \setminus [\alpha,\infty))$ which depends on the sequence $(s_j)_{j=0}^{2n+1}$ (see Notation 13.3 below). More precisely, from the set $\mathscr{P}^{(q,q)}_{-\tilde{J}_q,\geq}(\mathbb{C} \setminus [\alpha,\infty))$ we select those pairs which are compatible with the range conditions contained in Proposition 4.10. To prepare this we start with two technical lemmas.

Lemma 13.1 *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N} \cup \{\infty\}$, and let $(s_j)_{j=0}^{\kappa} \in \mathscr{K}_{q,\kappa,\alpha}^{\geq,\mathrm{e}}$. Let ϕ and ψ be $q \times q$ matrix-valued functions which are meromorphic in $\mathbb{C} \setminus [\alpha, \infty)$. Let $n \in \mathbb{N}_0$ be such that $2n + 1 \leq \kappa$ and let $\Theta_{n,\alpha}\colon \mathbb{C} \to \mathbb{C}^{2q\times 2q}$ be defined by (85). Let $\hat{\Theta}_{n,\alpha} := \mathrm{Rstr}_{\mathbb{C}\setminus[\alpha,\infty)} \Theta_{n,\alpha}$ and let*

$$\hat{\Theta}_{n,\alpha} = [\hat{\Theta}_{n,\alpha}^{(j,k)}]_{j,k=1}^{2} \tag{176}$$

be the $q \times q$ block representation of $\hat{\Theta}_{n,\alpha}$. Let $\tilde{\phi} := \hat{\Theta}_{n,\alpha}^{(1,1)}\phi + \hat{\Theta}_{n,\alpha}^{(1,2)}\psi$ and $\tilde{\psi} := \hat{\Theta}_{n,\alpha}^{(2,1)}\phi + \hat{\Theta}_{n,\alpha}^{(2,2)}\psi$. Furthermore, let $z \in (\mathbb{H}_\phi \cap \mathbb{H}_\psi) \setminus \mathbb{R}$ be such that (123) and

$$(I_{(n+1)q} - H_n^\dagger H_n) R_{T_{q,n}}(\alpha) v_{q,n} \phi(z) = 0 \tag{177}$$

hold true. Then $\mathscr{N}(\tilde{\psi}(z)) \subseteq \mathscr{N}(\tilde{\phi}(z))$. Moreover, if

$$\operatorname{rank} \begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix} = q \tag{178}$$

is valid, then $\det \tilde{\psi}(z) \neq 0$ *is fulfilled.*

Proof Because of $\mathscr{K}_{q,\kappa,\alpha}^{\geq,\mathrm{e}} \subseteq \mathscr{K}_{q,\kappa,\alpha}^{\geq}$ and Lemma 6.12, the equations in (20) are true.

We consider an arbitrary $y \in \mathscr{N}(\tilde{\psi}(z))$. Because of Remark 7.1, we have then

$$y^* \begin{bmatrix} \tilde{\phi}(z) \\ \tilde{\psi}(z) \end{bmatrix}^* \tilde{J}_q \begin{bmatrix} \tilde{\phi}(z) \\ \tilde{\psi}(z) \end{bmatrix} y = \mathrm{i} y^* \Big[\tilde{\psi}^*(z)\tilde{\phi}(z) - \tilde{\phi}^*(z)\tilde{\psi}(z)\Big] y = 0. \tag{179}$$

Obviously, $\Theta_{n,\alpha}(z) = \hat{\Theta}_{n,\alpha}(z)$. By (176) and the definition of $\tilde{\phi}$ and $\tilde{\psi}$, we get

$$\hat{\Theta}_{n,\alpha}(z) \begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix} = \begin{bmatrix} \tilde{\phi}(z) \\ \tilde{\psi}(z) \end{bmatrix}. \tag{180}$$

Using (180) and (179), we conclude

$$y^* \begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix}^* \Big[\tilde{J}_q - \Theta_{n,\alpha}^*(z)\tilde{J}_q\Theta_{n,\alpha}(z)\Big] \begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix} y = -y^* \begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix}^* (-\tilde{J}_q) \begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix} y. \tag{181}$$

Because of Lemma 6.12, Remark 4.1, Lemma 7.21, (181), and (123), we obtain

$$\begin{aligned}
0 &\leq \Bigg\| \sqrt{H_n}\Big[R_{T_{q,n}^*}(\alpha)\Big]^{-1} R_{T_{q,n}^*}(z) H_n^- R_{T_{q,n}}(\alpha) \\
&\qquad \times [I_{(n+1)q}, T_{q,n}H_n] \operatorname{diag}(v_{q,n}, v_{q,n}) B_{n,\alpha} \begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix} y \Bigg\|_{\mathrm{E}}^2 \\
&= y^* \begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix}^* B_{n,\alpha}^* [\operatorname{diag}(v_{q,n}, v_{q,n})]^* [I_{(n+1)q}, T_{q,n}H_n]^* \big[R_{T_{q,n}}(\alpha)\big]^* \\
&\qquad \times H_n^- \Big[R_{T_{q,n}^*}(z)\Big]^* \big[R_{T_{q,n}}(\alpha)\big]^{-1} H_n \Big[R_{T_{q,n}^*}(\alpha)\Big]^{-1} R_{T_{q,n}^*}(z) H_n^- \\
&\qquad \times R_{T_{q,n}}(\alpha) [I_{(n+1)q}, T_{q,n}H_n] \operatorname{diag}(v_{q,n}, v_{q,n}) B_{n,\alpha} \begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix} y \\
&= y^* \begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix}^* \frac{1}{\mathrm{i}(\overline{z} - z)} \Big[\tilde{J}_q - \Theta_{n,\alpha}^*(z) \tilde{J}_q \Theta_{n,\alpha}(z)\Big] \begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix} y \\
&= -y^* \begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix}^* \left(\frac{-\tilde{J}_q}{2 \operatorname{Im} z} \right) \begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix} y \leq 0
\end{aligned}$$

and, consequently,

$$\begin{aligned}
&\sqrt{H_n}\Big[R_{T_{q,n}^*}(\alpha)\Big]^{-1} R_{T_{q,n}^*}(z) H_n^- R_{T_{q,n}}(\alpha) \\
&\qquad \times [I_{(n+1)q}, T_{q,n}H_n] \operatorname{diag}(v_{q,n}, v_{q,n}) B_{n,\alpha} \begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix} y = 0. \qquad (182)
\end{aligned}$$

Multiplying equation (182) from the left by $\sqrt{H_n}$ and using Remark 7.13, we get

$$H_n \Big[R_{T_{q,n}^*}(\alpha)\Big]^{-1} R_{T_{q,n}^*}(z) \big[H_n^- R_{T_{q,n}}(\alpha), H_{\alpha \triangleright n}^- H_n\big] \operatorname{diag}(v_{q,n}, v_{q,n}) \begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix} y = 0$$

and, setting $X := H_n [R_{T_{q,n}^*}(\alpha)]^{-1} R_{T_{q,n}^*}(z) H_{\alpha \triangleright n}^- H_n$, hence

$$\Big[H_n \Big[R_{T_{q,n}^*}(\alpha)\Big]^{-1} R_{T_{q,n}^*}(z) H_n^- R_{T_{q,n}}(\alpha), X\Big] \operatorname{diag}(v_{q,n}, v_{q,n}) \begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix} y = 0. \qquad (183)$$

Because of (180), $\Theta_{n,\alpha}(z) = \hat{\Theta}_{n,\alpha}(z)$, Lemma 7.15, and $v_{q,n}^* R_{T_{q,n}}(\alpha) v_{q,n} = I_q$, we have

$$
\begin{aligned}
\tilde{\phi}(z)y &= [I_q, 0_{q\times q}] \begin{bmatrix} \tilde{\phi}(z) \\ \tilde{\psi}(z) \end{bmatrix} y = [I_q, 0_{q\times q}] \Theta_{n,\alpha}(z) \begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix} y \\
&= \Big[I_q + (z-\alpha) v_{q,n}^* H_n T_{q,n}^* R_{T_{q,n}^*}(z) H_n^- R_{T_{q,n}}(\alpha) v_{q,n}, v_{q,n}^* X v_{q,n} \Big] \begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix} y \\
&= v_{q,n}^* \Big[R_{T_{q,n}}(\alpha) + (z-\alpha) H_n T_{q,n}^* R_{T_{q,n}^*}(z) H_n^- R_{T_{q,n}}(\alpha), X \Big] \\
&\qquad \operatorname{diag}(v_{q,n}, v_{q,n}) \begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix} y.
\end{aligned}
\tag{184}
$$

From Remark 17.10 we see that (51) holds true, which implies

$$
\begin{aligned}
&(z-\alpha) H_n T_{q,n}^* R_{T_{q,n}^*}(z) H_n^- R_{T_{q,n}}(\alpha) \\
&\qquad = H_n \left(\Big[R_{T_{q,n}^*}(\alpha) \Big]^{-1} R_{T_{q,n}^*}(z) - I_{(n+1)q} \right) H_n^- R_{T_{q,n}}(\alpha) \\
&\qquad = H_n \Big[R_{T_{q,n}^*}(\alpha) \Big]^{-1} R_{T_{q,n}^*}(z) H_n^- R_{T_{q,n}}(\alpha) - H_n H_n^- R_{T_{q,n}}(\alpha).
\end{aligned}
\tag{185}
$$

Taking (184), (185), and (183) into account, we obtain

$$
\begin{aligned}
\tilde{\phi}(z)y &= v_{q,n}^* \Big[R_{T_{q,n}}(\alpha) + H_n \Big[R_{T_{q,n}^*}(\alpha) \Big]^{-1} R_{T_{q,n}^*}(z) H_n^- R_{T_{q,n}}(\alpha) - H_n H_n^- R_{T_{q,n}}(\alpha), X \Big] \\
&\qquad \times \operatorname{diag}(v_{q,n}, v_{q,n}) \begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix} y \\
&= v_{q,n}^* \big[R_{T_{q,n}}(\alpha) - H_n H_n^- R_{T_{q,n}}(\alpha), 0_{(n+1)q\times(n+1)q} \big] \operatorname{diag}(v_{q,n}, v_{q,n}) \begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix} y.
\end{aligned}
\tag{186}
$$

Thus, using (186), Remark 6.13, and (177), we infer

$$
\begin{aligned}
\tilde{\phi}(z)y &= \Big[v_{q,n}^* R_{T_{q,n}}(\alpha) v_{q,n} - v_{q,n}^* H_n H_n^- R_{T_{q,n}}(\alpha) v_{q,n}, 0_{q\times q} \Big] \begin{bmatrix} \phi(z)y \\ \psi(z)y \end{bmatrix} \\
&= v_{q,n}^* (I_{(n+1)q} - H_n H_n^-) R_{T_{q,n}}(\alpha) v_{q,n} \phi(z) y \\
&= v_{q,n}^* (I_{(n+1)q} - H_n H_n^-)(I_{(n+1)q} - H_n^\dagger H_n) R_{T_{q,n}}(\alpha) v_{q,n} \phi(z) y = 0
\end{aligned}
$$

and, consequently, $y \in \mathcal{N}(\tilde{\phi}(z))$. Hence $\mathcal{N}(\tilde{\psi}(z)) \subseteq \mathcal{N}(\tilde{\phi}(z))$ is proved.

Now we suppose (178). We consider again an arbitrary $y \in \mathcal{N}(\tilde{\psi}(z))$. Then we already know that $y \in \mathcal{N}(\tilde{\phi}(z))$. In view of Lemma 7.19(b) and (180), we get then

$$\begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix} y = \left[\Theta_{n,\alpha}(z)\right]^{-1} \hat{\Theta}_{n,\alpha}(z) \begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix} y = \left[\Theta_{n,\alpha}(z)\right]^{-1} \begin{bmatrix} \tilde{\phi}(z)y \\ \tilde{\psi}(z)y \end{bmatrix} = 0.$$

Because of (178), this implies $y = 0_{q\times 1}$, and hence, $\det \tilde{\psi}(z) \neq 0$.

Lemma 13.2 *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N} \cup \{\infty\}$, let $(s_j)_{j=0}^{2n+1} \in \mathcal{K}^{\geq,\mathrm{e}}_{q,2n+1,\alpha}$, let $n \in \mathbb{N}_0$ with $2n+1 \leq \kappa$, and let $[\phi; \psi] \in \mathcal{P}^{(q,q)}_{-\tilde{J}_q,\geq}(\mathbb{C} \setminus [\alpha, \infty))$ be such that $(I_{(n+1)q} - H_n^\dagger H_n) R_{T_{q,n}}(\alpha) v_{q,n}\phi = 0_{(n+1)q\times q}$. Let $\Theta_{n,\alpha} \colon \mathbb{C} \to \mathbb{C}^{2q\times 2q}$ be defined by (85) and let $[\hat{\Theta}^{(j,k)}_{n,\alpha}]_{j,k=1}^2$ be the $q\times q$ block partition of the restriction $\hat{\Theta}_{n,\alpha}$ of $\Theta_{n,\alpha}$ onto $\mathbb{C} \setminus [\alpha, \infty)$. Then there is a discrete subset $\mathcal{D}$ of $\mathbb{C} \setminus [\alpha, \infty)$ such that ϕ and ψ are holomorphic in $\mathbb{C} \setminus ([\alpha, \infty) \cup \mathcal{D})$ and that*

$$\det\left[\hat{\Theta}^{(2,1)}_{n,\alpha}(z)\phi(z) + \hat{\Theta}^{(2,2)}_{n,\alpha}(z)\psi(z)\right] \neq 0 \tag{187}$$

holds true for each $z \in \mathbb{C} \setminus (\mathbb{R} \cup \mathcal{D})$.

Proof Use Definition 8.1 and Lemma 13.1.

Against the background of Proposition 4.10 we introduce now a particular subclass of $\mathcal{P}^{(q,q)}_{-\tilde{J}_q,\geq}(\mathbb{C} \setminus [\alpha, \infty))$ which is well adapted to the sequence $(s_j)_{j=0}^{2n+1}$.

Notation 13.3 *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N} \cup \{\infty\}$, and let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $q\times q$ matrices. For each $n \in \mathbb{N}_0$ with $2n+1 \leq \kappa$, let $\mathcal{P}^{(q,q)}_{-\tilde{J}_q,\geq}[\mathbb{C} \setminus [\alpha, \infty), (s_j)_{j=0}^{2n+1}]$ be the set of all $[\phi; \psi] \in \mathcal{P}^{(q,q)}_{-\tilde{J}_q,\geq}(\mathbb{C} \setminus [\alpha, \infty))$ such that*

$$(I_{(n+1)q} - H_n^\dagger H_n) R_{T_{q,n}}(\alpha) v_{q,n}\phi = 0 \tag{188}$$

and

$$(I_{(n+1)q} - H_{\alpha\triangleright n}^\dagger H_{\alpha\triangleright n}) H_n v_{q,n}\psi = 0. \tag{189}$$

Remark 13.4 If in the situation of Notation 13.3 we have $\det H_n \neq 0$ and $\det H_{\alpha\triangleright n} \neq 0$, then $\mathcal{P}^{(q,q)}_{-\tilde{J}_q,\geq}[\mathbb{C} \setminus [\alpha, \infty), (s_j)_{j=0}^{2n+1}] = \mathcal{P}^{(q,q)}_{-\tilde{J}_q,\geq}(\mathbb{C} \setminus [\alpha, \infty))$.

Remark 13.5 Let $\alpha \in \mathbb{R}$, let $(s_j)_{j=0}^{2n+1}$ be a sequence of complex $q\times q$ matrices, let $[\phi; \psi] \in \mathcal{P}^{(q,q)}_{-\tilde{J}_q,\geq}[\mathbb{C} \setminus [\alpha, \infty), (s_j)_{j=0}^{2n+1}]$, and let g be a $q\times q$ matrix-valued function which is meromorphic in $\mathbb{C} \setminus [\alpha, \infty)$ such that $\det g$ does not vanish identically. Then it is readily checked that $[\phi g; \psi g] \in \mathcal{P}^{(q,q)}_{-\tilde{J}_q,\geq}[\mathbb{C} \setminus [\alpha, \infty), (s_j)_{j=0}^{2n+1}]$.

Now we are going to derive an important characterization of the elements of the set $\mathscr{P}^{(q,q)}_{-\tilde{J}_q,\geq}[\mathbb{C}\setminus[\alpha,\infty),(s_j)_{j=0}^{2n+1}]$.

Proposition 13.6 *Let $\alpha\in\mathbb{R}$, let $\kappa\in\mathbb{N}\cup\{\infty\}$, let $(s_j)_{j=0}^{\kappa}\in\mathscr{K}^{\geq,\mathrm{e}}_{q,\kappa,\alpha}$, and let $n\in\mathbb{N}_0$ be such that $2n+1\leq\kappa$. Let $\Theta_{n,\alpha}\colon\mathbb{C}\to\mathbb{C}^{2q\times 2q}$ be defined by (85), let (176) be the $q\times q$ block representation of the restriction $\hat{\Theta}_{n,\alpha}$ of $\Theta_{n,\alpha}$ onto $\mathbb{C}\setminus[\alpha,\infty)$ and let $\hat{R}_{T_{q,n}}$ be the restriction of $R_{T_{q,n}}$ onto $\mathbb{C}\setminus[\alpha,\infty)$. Let $[\phi;\psi]\in\mathscr{P}^{(q,q)}_{-\tilde{J}_q,\geq}(\mathbb{C}\setminus[\alpha,\infty))$ be such that $\det(\hat{\Theta}^{(2,1)}_{n,\alpha}\phi+\hat{\Theta}^{(2,2)}_{n,\alpha}\psi)$ does not vanish identically and let $\hat{S}_{n,\alpha}:=(\hat{\Theta}^{(1,1)}_{n,\alpha}\phi+\hat{\Theta}^{(1,2)}_{n,\alpha}\psi)(\hat{\Theta}^{(2,1)}_{n,\alpha}\phi+\hat{\Theta}^{(2,2)}_{n,\alpha}\psi)^{-1}$. Further let $\hat{\mathscr{E}}:\mathbb{C}\setminus[\alpha,\infty)\to\mathbb{C}$ be given by $\hat{\mathscr{E}}(z):=z$. Then the following statements (a) and (b) are equivalent:*

(a) The following two equations hold true:

$$(I_{(n+1)q}-H_n^{\dagger}H_n)\hat{R}_{T_{q,n}}[I_{(n+1)q},T_{q,n}H_n]\operatorname{diag}(v_{q,n},v_{q,n})\begin{bmatrix}\hat{S}_{n,\alpha}\\ I_q\end{bmatrix}=0 \tag{190}$$

and

$$(I_{(n+1)q}-H_{\alpha\triangleright n}^{\dagger}H_{\alpha\triangleright n})\hat{R}_{T_{q,n}}\Big[I_{(n+1)q},\big[R_{T_{q,n}}(\alpha)\big]^{-1}H_n\Big]\\ \times\operatorname{diag}(v_{q,n},v_{q,n})\begin{bmatrix}(\hat{\mathscr{E}}-\alpha)\hat{S}_{n,\alpha}\\ I_q\end{bmatrix}=0. \tag{191}$$

(b) $[\phi;\psi]\in\mathscr{P}^{(q,q)}_{-\tilde{J}_q,\geq}[\mathbb{C}\setminus[\alpha,\infty),(s_j)_{j=0}^{2n+1}]$.

Proof The proof is partitioned into twelve steps.

(I) Since $\det(\hat{\Theta}^{(2,1)}_{n,\alpha}\phi+\hat{\Theta}^{(2,2)}_{n,\alpha}\psi)$ does not vanish identically, there is a discrete subset $\mathscr{D}$ of $\mathbb{C}\setminus[\alpha,\infty)$ such that the conditions (i), (ii), and (iii) of Definition 8.1 hold true and that (187) is fulfilled for each $z\in\mathbb{C}\setminus([\alpha,\infty)\cup\mathscr{D})$.

(II) In view of condition (i) of Definition 8.1, $\hat{S}_{n,\alpha}:=(\hat{\Theta}^{(1,1)}_{n,\alpha}\phi+\hat{\Theta}^{(1,2)}_{n,\alpha}\psi)(\hat{\Theta}^{(2,1)}_{n,\alpha}\phi+\hat{\Theta}^{(2,2)}_{n,\alpha}\psi)^{-1}$, and (187), the function $\hat{S}_{n,\alpha}$ admits the representation

$$\hat{S}_{n,\alpha}(z)=\Big[\hat{\Theta}^{(1,1)}_{n,\alpha}(z)\phi(z)+\hat{\Theta}^{(1,2)}_{n,\alpha}(z)\psi(z)\Big]\Big[\hat{\Theta}^{(2,1)}_{n,\alpha}(z)\phi(z)+\hat{\Theta}^{(2,2)}_{n,\alpha}(z)\psi(z)\Big]^{-1} \tag{192}$$

for all $z\in\mathbb{C}\setminus([\alpha,\infty)\cup\mathscr{D})$. Because of Definition 8.1(i), (187), (176), and (192), we get

$$\hat{\Theta}_{n,\alpha}(z)\begin{bmatrix}\phi(z)\\ \psi(z)\end{bmatrix}\Big[\hat{\Theta}^{(2,1)}_{n,\alpha}(z)\phi(z)+\hat{\Theta}^{(2,2)}_{n,\alpha}(z)\psi(z)\Big]^{-1}$$

$$= \begin{bmatrix} [\hat{\Theta}_{n,\alpha}^{(1,1)}(z)\phi(z) + \hat{\Theta}_{n,\alpha}^{(1,2)}(z)\psi(z)][\hat{\Theta}_{n,\alpha}^{(2,1)}(z)\phi(z) + \hat{\Theta}_{n,\alpha}^{(2,2)}(z)\psi(z)]^{-1} \\ [\hat{\Theta}_{n,\alpha}^{(2,1)}(z)\phi(z) + \hat{\Theta}_{n,\alpha}^{(2,2)}(z)\psi(z)][\hat{\Theta}_{n,\alpha}^{(2,1)}(z)\phi(z) + \hat{\Theta}_{n,\alpha}^{(2,2)}(z)\psi(z)]^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} \hat{S}_{n,\alpha}(z) \\ I_q \end{bmatrix} \tag{193}$$

for each $z \in \mathbb{C} \setminus ([\alpha, \infty) \cup \mathscr{D})$. Let $\tilde{\Theta}_{n,\alpha} : \mathbb{C} \to \mathbb{C}^{2q \times 2q}$ given by (86). Taking into account (187), Proposition 7.16, and (193), for each $z \in \mathbb{C} \setminus ([\alpha, \infty) \cup \mathscr{D})$, this implies

$$\begin{aligned} &\left[\operatorname{Rstr}_{\mathbb{C}\setminus[\alpha,\infty)} \tilde{\Theta}_{n,\alpha}(z)\right] \begin{bmatrix} [\hat{\mathscr{E}}(z) - \alpha]\phi(z) \\ \psi(z) \end{bmatrix} \left[\hat{\Theta}_{n,\alpha}^{(2,1)}(z)\phi(z) + \hat{\Theta}_{n,\alpha}^{(2,2)}(z)\psi(z)\right]^{-1} \\ &= \left[\operatorname{diag}\left(\left[\hat{\mathscr{E}}(z) - \alpha\right] I_q, I_q\right)\right] \hat{\Theta}_{n,\alpha}(z) \begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix} \left[\hat{\Theta}_{n,\alpha}^{(2,1)}(z)\phi(z) + \hat{\Theta}_{n,\alpha}^{(2,2)}(z)\psi(z)\right]^{-1} \\ &= \left[\operatorname{diag}\left(\left[\hat{\mathscr{E}}(z) - \alpha\right] I_q, I_q\right)\right] \begin{bmatrix} \hat{S}_{n,\alpha}(z) \\ I_q \end{bmatrix} = \begin{bmatrix} [\hat{\mathscr{E}}(z) - \alpha]\hat{S}_{n,\alpha}(z) \\ I_q \end{bmatrix}. \end{aligned} \tag{194}$$

(III) Since the functions $\hat{\mathscr{E}}$ and $\hat{R}_{T_{q,n}}$ are holomorphic in $\mathbb{C} \setminus [\alpha, \infty)$, statement (a) is equivalent to the following statement:

(c) There exists a discrete subset $\tilde{\mathscr{D}}$ of $\mathbb{C} \setminus [\alpha, \infty)$ such that $\hat{S}_{n,\alpha}$ is holomorphic in $\mathbb{C} \setminus ([\alpha, \infty) \cup \tilde{\mathscr{D}})$ and that, for each $z \in \mathbb{C} \setminus ([\alpha, \infty) \cup \tilde{\mathscr{D}})$, the following two equations hold true:

$$(I_{(n+1)q} - H_n^\dagger H_n)\hat{R}_{T_{q,n}}(z)[I_{(n+1)q}, T_{q,n} H_n] \operatorname{diag}(v_{q,n}, v_{q,n}) \begin{bmatrix} \hat{S}_{n,\alpha}(z) \\ I_q \end{bmatrix} = 0 \tag{195}$$

and

$$\begin{aligned} (I_{(n+1)q} - H_{\alpha\triangleright n}^\dagger H_{\alpha\triangleright n})\hat{R}_{T_{q,n}}(z)\left[I_{(n+1)q}, [R_{T_{q,n}}(\alpha)]^{-1} H_n\right] \\ \times \operatorname{diag}(v_{q,n}, v_{q,n}) \begin{bmatrix} [\hat{\mathscr{E}}(z) - \alpha]\hat{S}_{n,\alpha}(z) \\ I_q \end{bmatrix} = 0. \end{aligned} \tag{196}$$

(IV) In this step of the proof, we suppose (c). We are going to prove that the following statement holds true:

(d) There is a discrete subset $\hat{\mathscr{D}}$ of $\mathbb{C} \setminus [\alpha, \infty)$ such that ϕ and ψ are holomorphic in $\mathbb{C} \setminus ([\alpha, \infty) \cup \hat{\mathscr{D}})$ and that, for all $z \in \mathbb{C} \setminus ([\alpha, \infty) \cup \hat{\mathscr{D}})$, the following two equations hold true:

$$(I_{(n+1)q} - H_n^{\dagger} H_n) \hat{R}_{T_{q,n}}(z)[I_{(n+1)q}, T_{q,n} H_n] \operatorname{diag}(v_{q,n}, v_{q,n}) \hat{\Theta}_{n,\alpha}(z) \times \begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix} \left[\hat{\Theta}_{n,\alpha}^{(2,1)}(z)\phi(z) + \hat{\Theta}_{n,\alpha}^{(2,2)}(z)\psi(z) \right]^{-1} = 0 \tag{197}$$

and

$$(I_{(n+1)q} - H_{\alpha \triangleright n}^{\dagger} H_{\alpha \triangleright n}) \hat{R}_{T_{q,n}}(z) \left[I_{(n+1)q}, [R_{T_{q,n}}(\alpha)]^{-1} H_n \right] \operatorname{diag}(v_{q,n}, v_{q,n}) \tilde{\Theta}_{n,\alpha}(z) \times \begin{bmatrix} [\hat{\mathscr{E}}(z) - \alpha]\phi(z) \\ \psi(z) \end{bmatrix} \left[\hat{\Theta}_{n,\alpha}^{(2,1)}(z)\phi(z) + \hat{\Theta}_{n,\alpha}^{(2,2)}(z)\psi(z) \right]^{-1} = 0. \tag{198}$$

First we observe that $\mathscr{D}_{\#} := \mathscr{D} \cup \tilde{\mathscr{D}}$ is a discrete subset of $\mathbb{C} \setminus [\alpha, \infty)$. Since (195) and (196) are valid for each $z \in \mathbb{C} \setminus ([\alpha, \infty) \cup \mathscr{D}_{\#})$ and since (II) shows that (193) and (194) are fulfilled for each $z \in \mathbb{C} \setminus ([\alpha, \infty) \cup \mathscr{D}_{\#})$, we get that (197) and (198) hold true for each $z \in \mathbb{C} \setminus ([\alpha, \infty) \cup \mathscr{D}_{\#})$. Setting $\hat{\mathscr{D}} = \mathscr{D}_{\#}$, statement (d) is proved.

(V) In this step of the proof, we suppose (d). We are going to prove that (c) holds true. Obviously, $\mathscr{D}_{\square} := \mathscr{D} \cup \hat{\mathscr{D}}$ is a discrete subset of $\mathbb{C} \setminus [\alpha, \infty)$. According to (I) and (II), we get (187), (193), and (194) for each $z \in \mathbb{C} \setminus ([\alpha, \infty) \cup \mathscr{D}_{\square})$. Using these arguments and (197) and (198), we see that (195) and (196) are fulfilled for each $z \in \mathbb{C} \setminus ([\alpha, \infty) \cup \mathscr{D}_{\square})$. Consequently, statement (c) holds true with $\tilde{\mathscr{D}} = \mathscr{D}_{\square}$.

(VI) Now we verify that statement (d) implies the following statement:

(e) There is a discrete subset $\tilde{\mathscr{D}}_{\#}$ of $\mathbb{C} \setminus [\alpha, \infty)$ such that the functions ϕ and ψ are holomorphic in $\mathbb{C} \setminus ([\alpha, \infty) \cup \tilde{\mathscr{D}}_{\#})$ and that, for each $z \in \mathbb{C} \setminus ([\alpha, \infty) \cup \tilde{\mathscr{D}}_{\#})$, the following two equations hold true:

$$(I_{(n+1)q} - H_n^{\dagger} H_n) \hat{R}_{T_{q,n}}(z)[I_{(n+1)q}, T_{q,n} H_n] \operatorname{diag}(v_{q,n}, v_{q,n}) \hat{\Theta}_{n,\alpha}(z) \begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix} = 0 \tag{199}$$

and

$$(I_{(n+1)q} - H_{\alpha \triangleright n}^{\dagger} H_{\alpha \triangleright n}) \hat{R}_{T_{q,n}}(z) \left[I_{(n+1)q}, [R_{T_{q,n}}(\alpha)]^{-1} H_n \right] \times \operatorname{diag}(v_{q,n}, v_{q,n}) \left[\operatorname{Rstr}_{\mathbb{C} \setminus [\alpha, \infty)} \tilde{\Theta}_{n,\alpha}(z) \right] \begin{bmatrix} [\hat{\mathscr{E}}(z) - \alpha]\phi(z) \\ \psi(z) \end{bmatrix} = 0. \tag{200}$$

Let us assume that (d) is fulfilled. Because of (I), we know that $\tilde{\mathscr{D}}_{\square} := \mathscr{D} \cup \hat{\mathscr{D}}$ is a discrete subset of $\mathbb{C} \setminus [\alpha, \infty)$. From (I) and (d) we see that (187), (197), and (198) are valid for each $\mathbb{C} \setminus ([\alpha, \infty) \cup \tilde{\mathscr{D}}_{\square})$, which implies (199) and (200) for each $z \in \mathbb{C}([\alpha, \infty) \cup \tilde{\mathscr{D}}_{\square})$. Consequently, (e) holds true with $\tilde{\mathscr{D}}_{\#} = \tilde{\mathscr{D}}_{\square}$.

(VII) Now we show that (e) implies (d). Let (e) be fulfilled. Obviously, $\hat{\mathscr{D}}_\# := \tilde{\mathscr{D}}_\# \cup \mathscr{D}$ is a discrete subset of $\mathbb{C} \setminus [\alpha, \infty)$. Because of (I) and (e), we know that (187), (199), and (200) are valid for each $z \in \mathbb{C} \setminus ([\alpha, \infty) \cup \hat{\mathscr{D}}_\#)$. Consequently, (197) and (198) hold true for each $z \in \mathbb{C} \setminus ([\alpha, \infty) \cup \hat{\mathscr{D}}_\#)$. Hence, (d) is fulfilled with $\hat{\mathscr{D}} = \hat{\mathscr{D}}_\#$.

(VIII) Since $\hat{R}_{T_{q,n}}$ is the restriction of $R_{T_{q,n}}$ onto $\mathbb{C} \setminus [\alpha, \infty)$, we see that (e) is equivalent to the following statement:

(f) There is a discrete subset $\mathscr{D}'$ of $\mathbb{C} \setminus [\alpha, \infty)$ such that ϕ and ψ are holomorphic in $\mathbb{C} \setminus ([\alpha, \infty) \cup \mathscr{D}')$ and that, for all $z \in \mathbb{C} \setminus ([\alpha, \infty) \cup \mathscr{D}')$, the following two equations are true:

$$(I_{(n+1)q} - H_n^\dagger H_n) R_{T_{q,n}}(z) [I_{(n+1)q}, T_{q,n} H_n] \operatorname{diag}(v_{q,n}, v_{q,n}) \Theta_{n,\alpha}(z) \begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix} = 0 \tag{201}$$

and

$$(I_{(n+1)q} - H_{\alpha \triangleright n}^\dagger H_{\alpha \triangleright n}) R_{T_{q,n}}(z) \Big[I_{(n+1)q}, [R_{T_{q,n}}(\alpha)]^{-1} H_n \Big] \times \operatorname{diag}(v_{q,n}, v_{q,n}) \tilde{\Theta}_{n,\alpha}(z) \big[\operatorname{diag}\big((z - \alpha) I_q, I_q\big) \big] \begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix} = 0. \tag{202}$$

(IX) Let $P_{n,\alpha}$, $Q_{n,\alpha}$, and $S_{n,\alpha}$ be the matrix-valued functions defined (on $\mathbb{C}$) by (156), (157), and (158). According to Proposition 12.4(a), we see that $\mathscr{N} := \mathscr{N}_{\det P_{n,\alpha}} \cup \mathscr{N}_{\det Q_{n,\alpha}} \cup \mathscr{N}_{\det S_{n,\alpha}}$ is a finite and, in particular, discrete subset of $\mathbb{C}$.

(X) By virtue of (IX), we know that $\mathscr{N}$ is a discrete subset of $\mathbb{C}$. We suppose now (f). Then $\mathscr{N}' := \mathscr{N} \cup \mathscr{D}'$ is a discrete subset of $\mathbb{C}$, too. From Proposition 12.4(b) we see then that the following statement holds true:

(g) There is a discrete subset $\mathscr{D}''$ of $\mathbb{C} \setminus [\alpha, \infty)$ such that ϕ and ψ are holomorphic in $\mathbb{C} \setminus ([\alpha, \infty) \cup \mathscr{D}'')$ and that

$$(I_{(n+1)q} - H_n^\dagger H_n) R_{T_{q,n}}(\alpha) v_{q,n} \phi(z) = 0, \quad (I_{(n+1)q} - H_{\alpha \triangleright n}^\dagger H_{\alpha \triangleright n}) H_n v_{q,n} \psi(z) = 0 \tag{203}$$

are fulfilled for each $z \in \mathbb{C} \setminus ([\alpha, \infty) \cup \mathscr{D}'')$.

(XI) Conversely, now we suppose (g). We are going to prove (f). From (IX) we see that $\mathscr{N}$ is a discrete subset of $\mathbb{C}$. Hence, $\tilde{\mathscr{N}} := \mathscr{N} \cap (\mathbb{C} \setminus [\alpha, \infty))$ and $\mathscr{D}'_\square := \mathscr{D}'' \cup \tilde{\mathscr{N}}$ are discrete subsets of $\mathbb{C} \setminus [\alpha, \infty)$. Because of (g), the functions ϕ and ψ are holomorphic in $\mathbb{C} \setminus ([\alpha, \infty) \cup \mathscr{D}'_\square)$ and (203) is valid for each $z \in \mathbb{C}([\alpha, \infty) \cup \mathscr{D}'_\square)$. Let us consider an arbitrary $z \in \mathbb{C} \setminus ([\alpha, \infty) \cup \mathscr{D}'_\square)$. From (203) we get then that $x := \phi(z)$ and $y := \psi(z)$ fulfill (172) and (173). Consequently, Proposition 12.4 yields then that (170) and (171) hold true. Thus, we see that (201) and (202) are true. Hence, (f) is valid with $\mathscr{D}' = \mathscr{D}'_{\square}$.

(XII) In view of Notation 13.3, (g) and (b) are equivalent.

From (III)–(VIII) and (X)–(XII) we see that the statements (a) and (b) are equivalent.

Now we are able to prove one of the central results of this paper. Namely, we obtain a full description of the set $\mathscr{S}_{0,q;[\alpha,\infty)}[(s_j)_{j=0}^{2n+1}, \leq]$.

Theorem 13.7 *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N} \cup \{\infty\}$, let $(s_j)_{j=0}^{\kappa} \in \mathscr{K}^{\geq,\mathrm{e}}_{q,\kappa,\alpha}$, and let $n \in \mathbb{N}_0$ be such that $2n+1 \leq \kappa$. Let $[\hat{\Theta}_{n,\alpha}^{(j,k)}]_{j,k=1}^{2}$ be the $q \times q$ block representation of the restriction $\hat{\Theta}_{n,\alpha}$ of $\Theta_{n,\alpha}$ onto $\mathbb{C} \setminus [\alpha, \infty)$. Then:*

(a) For each $[\phi; \psi] \in \mathscr{P}^{(q,q)}_{-\tilde{J}_q,\geq}[\mathbb{C} \setminus [\alpha,\infty), (s_j)_{j=0}^{2n+1}]$, the function $\det(\hat{\Theta}_{n,\alpha}^{(2,1)}\phi + \hat{\Theta}_{n,\alpha}^{(2,2)}\psi)$ does not vanish identically in $\mathbb{C} \setminus [\alpha, \infty)$ and

$$\hat{S}_{n,\alpha} := (\hat{\Theta}_{n,\alpha}^{(1,1)}\phi + \hat{\Theta}_{n,\alpha}^{(1,2)}\psi)(\hat{\Theta}_{n,\alpha}^{(2,1)}\phi + \hat{\Theta}_{n,\alpha}^{(2,2)}\psi)^{-1}$$

belongs to the class $\mathscr{S}_{0,q;[\alpha,\infty)}[(s_j)_{j=0}^{2n+1}, \leq]$.

(b) For each $S \in \mathscr{S}_{0,q;[\alpha,\infty)}[(s_j)_{j=0}^{2n+1}, \leq]$, there exists a pair $[\phi; \psi] \in \mathscr{P}^{(q,q)}_{-\tilde{J}_q,\geq}[\mathbb{C} \setminus [\alpha,\infty), (s_j)_{j=0}^{2n+1}]$ consisting of two in $\mathbb{C} \setminus [\alpha, \infty)$ holomorphic $q \times q$ matrix-valued functions ϕ and ψ such that (187) and

$$S(z) = \left[\hat{\Theta}_{n,\alpha}^{(1,1)}(z)\phi(z) + \hat{\Theta}_{n,\alpha}^{(1,2)}(z)\psi(z)\right]\left[\hat{\Theta}_{n,\alpha}^{(2,1)}(z)\phi(z) + \hat{\Theta}_{n,\alpha}^{(2,2)}(z)\psi(z)\right]^{-1} \tag{204}$$

hold true for each $z \in \mathbb{C} \setminus [\alpha, \infty)$.

(c) Let $[\phi_1; \psi_1], [\phi_2; \psi_2] \in \mathscr{P}^{(q,q)}_{-\tilde{J}_q,\geq}[\mathbb{C} \setminus [\alpha,\infty), (s_j)_{j=0}^{2n+1}]$. Then $\langle[\phi_1; \psi_1]\rangle = \langle[\phi_2; \psi_2]\rangle$ if and only if

$$(\hat{\Theta}_{n,\alpha}^{(1,1)}\phi_1 + \hat{\Theta}_{n,\alpha}^{(1,2)}\psi_1)(\hat{\Theta}_{n,\alpha}^{(2,1)}\phi_1 + \hat{\Theta}_{n,\alpha}^{(2,2)}\psi_1)^{-1} = (\hat{\Theta}_{n,\alpha}^{(1,1)}\phi_2 + \hat{\Theta}_{n,\alpha}^{(1,2)}\psi_2)(\hat{\Theta}_{n,\alpha}^{(2,1)}\phi_2 + \hat{\Theta}_{n,\alpha}^{(2,2)}\psi_2)^{-1}.$$

Proof

(a) Let $[\phi; \psi] \in \mathscr{P}^{(q,q)}_{-\tilde{J}_q,\geq}[\mathbb{C} \setminus [\alpha,\infty), (s_j)_{j=0}^{2n+1}]$. According to Notation 13.3, we have $[\phi; \psi] \in \mathscr{P}^{(q,q)}_{-\tilde{J}_q,\geq}(\mathbb{C} \setminus [\alpha,\infty))$ as well as (188) and (189). In view of (188), we see from Lemma 13.2 that there is a discrete subset $\mathscr{D}_1$ of $\mathbb{C} \setminus [\alpha, \infty)$ such that ϕ and ψ are holomorphic in $\mathbb{C} \setminus ([\alpha, \infty) \cup \mathscr{D}_1)$ and that (187) holds true for each $z \in \mathbb{C} \setminus (\mathbb{R} \cup \mathscr{D}_1)$. In particular, the function $\det(\hat{\Theta}_{n,\alpha}^{(2,1)}\phi + \hat{\Theta}_{n,\alpha}^{(2,2)}\psi)$ does not vanish identically in $\mathbb{C} \setminus [\alpha, \infty)$. Thus, we can apply Proposition 11.1(a) to obtain the existence of a discrete subset $\mathscr{D}_2$ of $\mathbb{C} \setminus [\alpha, \infty)$ such that $\hat{S}_{n,\alpha}$ is holomorphic in $\mathbb{C} \setminus ([\alpha, \infty) \cup \mathscr{D}_2)$. Proposition 13.6 provides us (190) and (191). Because of Lemma 7.23, then $(I_{(n+1)q} - H_n^\dagger H_n)\mathbf{b}_{2n}^{[\hat{S}_{n,\alpha}]}(z) = 0_{(n+1)q\times q}$ and $(I_{(n+1)q} - H_{\alpha\triangleright n}^\dagger H_{\alpha\triangleright n})\mathbf{b}_{2n+1}^{[\hat{S}_{n,\alpha}]}(z) = 0_{(n+1)q\times q}$ for all $z \in \mathbb{C} \setminus ([\alpha, \infty) \cup \mathscr{D}_2)$. The matrices H_n and $H_{\alpha\triangleright n}$ are non-negative Hermitian. Thus we can apply Lemma 4.7 to obtain $\mathscr{R}(\mathbf{b}_{2n}^{[\hat{S}_{n,\alpha}]}(z)) \subseteq \mathscr{R}(H_n)$ and $\mathscr{R}(\mathbf{b}_{2n+1}^{[\hat{S}_{n,\alpha}]}(z)) \subseteq \mathscr{R}(H_{\alpha\triangleright n})$ for all $z \in \mathbb{C} \setminus ([\alpha, \infty) \cup \mathscr{D}_2)$. Using again Proposition 11.1(a), we can conclude that the Schur complements $\Sigma_{2n}^{[\hat{S}_{n,\alpha}]}(z)$ and $\Sigma_{2n+1}^{[\hat{S}_{n,\alpha}]}(z)$ are both non-negative Hermitian for all $z \in \mathbb{C} \setminus (\mathbb{R} \cup \mathscr{D}_2)$. Taking additionally into account (5) and (7), then Lemma 4.6(a) yields that the matrices $P_{2n}^{[\hat{S}_{n,\alpha}]}(z)$ and $P_{2n+1}^{[\hat{S}_{n,\alpha}]}(z)$ are both non-negative Hermitian for all $z \in \mathbb{C} \setminus (\mathbb{R} \cup \mathscr{D}_2)$. Obviously, $\mathscr{D}_3 := \mathscr{D}_2 \cap \Pi_+$ is a discrete subset of Π_+ and the restriction $f_{n,\alpha}$ of $\hat{S}_{n,\alpha}$ onto $\Pi_+ \setminus \mathscr{D}_3$ is holomorphic in $\Pi_+ \setminus \mathscr{D}_3$. For each $w \in \Pi_+ \setminus \mathscr{D}_3$, then the matrices $P_{2n}^{[f_{n,\alpha}]}(w)$ and $P_{2n+1}^{[f_{n,\alpha}]}(w)$ are both non-negative Hermitian. Thus, [36, Theorem 6.5] provides us that there is a unique $S \in \mathscr{S}_{0,q;[\alpha,\infty)}[(s_j)_{j=0}^{2n+1}, \leq]$ such that the restriction of S onto $\Pi_+ \setminus \mathscr{D}_3$ coincides with $f_{n,\alpha}$. Consequently, for each $w \in \Pi_+ \setminus \mathscr{D}_3$, we have $S(w) = f_{n,\alpha}(w) = \hat{S}_{n,\alpha}(w)$. Since S is holomorphic in $\mathbb{C} \setminus [\alpha, \infty)$, we get $S = \hat{S}_{n,\alpha}$, implying $\hat{S}_{n,\alpha} \in \mathscr{S}_{0,q;[\alpha,\infty)}[(s_j)_{j=0}^{2n+1}, \leq]$.

(b) Now we consider an arbitrary $S \in \mathscr{S}_{0,q;[\alpha,\infty)}[(s_j)_{j=0}^{2n+1}, \leq]$. From Proposition 11.1(b) we see that $\mathscr{R}(\mathbf{b}_{2n}^{[S]}(z)) \subseteq \mathscr{R}(H_n)$ and $\mathscr{R}(\mathbf{b}_{2n+1}^{[S]}(z)) \subseteq \mathscr{R}(H_{\alpha\triangleright n})$ hold true for all $z \in \mathbb{C} \setminus \mathbb{R}$. Since the matrices H_n and $H_{\alpha\triangleright n}$ are both non-negative Hermitian, we can infer form Lemma 4.7 then $(I_{(n+1)q} - H_n^\dagger H_n)\mathbf{b}_{2n}^{[S]}(z) = 0_{(n+1)q\times q}$ and $(I_{(n+1)q} - H_{\alpha\triangleright n}^\dagger H_{\alpha\triangleright n})\mathbf{b}_{2n+1}^{[S]}(z) = 0_{(n+1)q\times q}$ for all $z \in \mathbb{C} \setminus \mathbb{R}$. Taking into account that S is holomorphic in $\mathbb{C} \setminus [\alpha, \infty)$, then $(I_{(n+1)q} - H_n^\dagger H_n)\mathbf{b}_{2n}^{[S]} = 0_{(n+1)q\times q}$ and $(I_{(n+1)q} - H_{\alpha\triangleright n}^\dagger H_{\alpha\triangleright n})\mathbf{b}_{2n+1}^{[S]} = 0_{(n+1)q\times q}$ follow. Consequently, Lemma 7.23 yields

$$(I_{(n+1)q} - H_n^\dagger H_n)\hat{R}_{T_{q,n}}[I_{(n+1)q}, T_{q,n}H_n]\operatorname{diag}(v_{q,n}, v_{q,n})\begin{bmatrix} S \\ I_q \end{bmatrix} = 0_{(n+1)q\times q} \tag{205}$$

and

$$(I_{(n+1)q} - H_{\alpha\triangleright n}^\dagger H_{\alpha\triangleright n})\hat{R}_{T_{q,n}}\Big[I_{(n+1)q}, [R_{T_{q,n}}(\alpha)]^{-1}H_n\Big] \times \operatorname{diag}(v_{q,n}, v_{q,n})\begin{bmatrix} (\hat{\mathscr{E}} - \alpha)S \\ I_q \end{bmatrix} = 0_{(n+1)q\times q}, \tag{206}$$

where $\hat{R}_{T_{q,n}}$ is the restriction of $R_{T_{q,n}}$ onto $\mathbb{C} \setminus [\alpha, \infty)$ and $\hat{\mathscr{E}} : \mathbb{C} \setminus [\alpha, \infty) \to \mathbb{C}$ is given by $\hat{\mathscr{E}}(z) := z$. Using again Proposition 11.1(b) we obtain the existence of a pair $[\phi; \psi] \in \mathscr{P}^{(q,q)}_{-\tilde{J}_q,\geq}(\mathbb{C} \setminus [\alpha, \infty))$ such that ϕ and ψ are holomorphic in $\mathbb{C} \setminus [\alpha, \infty)$ and (187) and (204) hold true for all $z \in \mathbb{C} \setminus [\alpha, \infty)$. In particular, the function $\det(\hat{\Theta}^{(2,1)}_{n,\alpha}\phi + \hat{\Theta}^{(2,2)}_{n,\alpha}\psi)$ does not vanish identically and $S = (\hat{\Theta}^{(1,1)}_{n,\alpha}\phi + \hat{\Theta}^{(1,2)}_{n,\alpha}\psi)(\hat{\Theta}^{(2,1)}_{n,\alpha}\phi + \hat{\Theta}^{(2,2)}_{n,\alpha}\psi)^{-1}$. Taking additionally into account (205) and (206), we thus can apply Proposition 13.6 to infer $[\phi; \psi] \in \mathscr{P}^{(q,q)}_{-\tilde{J}_q,\geq}[\mathbb{C} \setminus [\alpha, \infty), (s_j)_{j=0}^{2n+1}]$.

(c) According to Notation 13.3, we have $[\phi_1; \psi_1], [\phi_2; \psi_2] \in \mathscr{P}^{(q,q)}_{-\tilde{J}_q,\geq}(\mathbb{C} \setminus [\alpha, \infty))$. In view of part (a), we furthermore know that, for each $k \in \{1, 2\}$, the function $\det(\hat{\Theta}^{(2,1)}_{n,\alpha}\phi_k + \hat{\Theta}^{(2,2)}_{n,\alpha}\psi_k)$ does not vanish identically in $\mathbb{C} \setminus [\alpha, \infty)$. Consequently, the application of Proposition 11.1(c) provides us the asserted equivalence.

If we consider Theorem 13.7 for the non-degenerate case $\det H_n \neq 0$ and $\det H_{\alpha \triangleright n} \neq 0$, then Remark 13.4 implies

$$\mathscr{P}^{(q,q)}_{-\tilde{J}_q,\geq}[\mathbb{C} \setminus [\alpha, \infty), (s_j)_{j=0}^{2n+1}] = \mathscr{P}^{(q,q)}_{-\tilde{J}_q,\geq}(\mathbb{C} \setminus [\alpha, \infty)).$$

Hence Theorem 13.7 provides a satisfactory description of the class $\mathscr{S}_{0,q;[\alpha,\infty)}[(s_j)_{j=0}^{2n+1}, \leq]$ which corresponds to that one which was obtained by Dyukarev [23, Theorem 2]. In the following we want to have a closer look at the degenerate situation. In this case Theorem 13.7 tells us that the set $\mathscr{P}^{(q,q)}_{-\tilde{J}_q,\geq}[\mathbb{C} \setminus [\alpha, \infty), (s_j)_{j=0}^{2n+1}]$ of parameters depends on the given sequence $(s_j)_{j=0}^{2n+1}$ of moments. Our subsequent considerations are aimed to look for a more transparent alternate description of the set $\mathscr{S}_{0,q;[\alpha,\infty)}[(s_j)_{j=0}^{2n+1}, \leq]$.

14 A Pair of Subspaces of $\mathbb{C}^q$ Which Describes the Degeneracy of the Moment Problem MP$[[\alpha, \infty); (s_j)_{j=0}^{2n+1}, \leq]$

Our strategy to realize the goal formulated at the end of the preceding section is based on a closer view of the nature of degeneracy of the moment problem under consideration. For this reason, we introduce a pair of linear subspaces of $\mathbb{C}^q$ which contains all information which is necessary to handle degeneracy. If $\mathscr{U}$ is a subspace of $\mathbb{C}^q$, then by $P_{\mathscr{U}}$ we denote the complex $q \times q$ matrix which represents the orthogonal projection onto $\mathscr{U}$, with respect to the standard basis of $\mathbb{C}^q$ i. e., $P_{\mathscr{U}}$ is the unique complex $q \times q$ matrix which fulfills the three conditions $P_{\mathscr{U}}^2 = P_{\mathscr{U}}$, $P_{\mathscr{U}}^* = P_{\mathscr{U}}$, and $\mathscr{R}(P_{\mathscr{U}}) = \mathscr{U}$. In this case, we have $\mathscr{N}(P_{\mathscr{U}}) = \mathscr{U}^{\perp}$.

Lemma 14.1 *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N} \cup \{\infty\}$, let $(s_j)_{j=0}^{\kappa} \in \mathscr{K}_{q,\kappa,\alpha}^{\geq,\mathrm{e}}$, and let $n \in \mathbb{N}_0$ be such that $2n+1 \leq \kappa$. Then:*

(a) The sets

$$\mathscr{U}_{n,\alpha} := \left[\mathscr{N}\left((I_{(n+1)q} - H_n^\dagger H_n) R_{T_{q,n}}(\alpha) v_{q,n}\right)\right]^\perp \tag{207}$$

and

$$\mathscr{V}_{n,\alpha} := \left[\mathscr{N}\left((I_{(n+1)q} - H_{\alpha\triangleright n}^\dagger H_{\alpha\triangleright n}) H_n v_{q,n}\right)\right]^\perp. \tag{208}$$

are orthogonal subspaces of $\mathbb{C}^q$ with $\dim \mathscr{U}_{n,\alpha} = \operatorname{rank}[(I_{(n+1)q} - H_n^\dagger H_n) R_{T_{q,n}}(\alpha) v_{q,n}]$ and $\dim \mathscr{V}_{n,\alpha} = \operatorname{rank}[(I_{(n+1)q} - H_{\alpha\triangleright n}^\dagger H_{\alpha\triangleright n}) H_n v_{q,n}]$.

(b) Let $A \in \mathbb{C}^{q\times p}$. Then $(I_{(n+1)q} - H_n^\dagger H_n) R_{T_{q,n}}(\alpha) v_{q,n} A = 0$ if and only if $P_{\mathscr{U}_{n,\alpha}} A = 0$. Moreover, $(I_{(n+1)q} - H_{\alpha\triangleright n}^\dagger H_{\alpha\triangleright n}) H_n v_{q,n} A = 0$ if and only if $P_{\mathscr{V}_{n,\alpha}} A = 0$.

Proof (a) Because of Remarks 5.8 and 5.7, we have $H_n^* = H_n$ and $H_{\alpha\triangleright n}^* = H_{\alpha\triangleright n}$. In particular, $H_n^\dagger H_n = H_n H_n^\dagger$. Obviously, $(H_n^\dagger H_n)^* = H_n^\dagger H_n$ and $(H_{\alpha\triangleright n}^\dagger H_{\alpha\triangleright n})^* = H_{\alpha\triangleright n}^\dagger H_{\alpha\triangleright n}$. Thus,

$$\mathscr{U}_{n,\alpha} = \mathscr{R}\left(\left[(I_{(n+1)q} - H_n^\dagger H_n) R_{T_{q,n}}(\alpha) v_{q,n}\right]^*\right) \tag{209}$$

and

$$\mathscr{V}_{n,\alpha} = \mathscr{R}\left(\left[(I_{(n+1)q} - H_{\alpha\triangleright n}^\dagger H_{\alpha\triangleright n}) H_n v_{q,n}\right]^*\right) = \mathscr{R}\left(v_{q,n}^* H_n (I_{(n+1)q} - H_{\alpha\triangleright n}^\dagger H_{\alpha\triangleright n})\right). \tag{210}$$

In particular, $\dim \mathscr{U}_{n,\alpha} = \operatorname{rank}((I_{(n+1)q} - H_n^\dagger H_n) R_{T_{q,n}}(\alpha) v_{q,n})$ and $\dim \mathscr{V}_{n,\alpha} = \operatorname{rank}((I_{(n+1)q} - H_{\alpha\triangleright n}^\dagger H_{\alpha\triangleright n}) H_n v_{q,n})$ hold true. Let $f \in \mathscr{U}_{n,\alpha}$ and $g \in \mathscr{V}_{n,\alpha}$ be arbitrary chosen. According to (209) and (210), there are $x, y \in \mathbb{C}^{(n+1)q}$ such that $f = [(I_{(n+1)q} - H_n^\dagger H_n) R_{T_{q,n}}(\alpha) v_{q,n}]^* x$ and $g = v_{q,n}^* H_n (I_{(n+1)q} - H_{\alpha\triangleright n}^\dagger H_{\alpha\triangleright n}) y$. By virtue of the Remarks 5.3 and 6.13, we have

$$\begin{aligned}
f^* g &= x^* (I_{(n+1)q} - H_n^\dagger H_n) R_{T_{q,n}}(\alpha) v_{q,n} v_{q,n}^* H_n (I_{(n+1)q} - H_{\alpha\triangleright n}^\dagger H_{\alpha\triangleright n}) y \\
&= x^* (I_{(n+1)q} - H_n^\dagger H_n) R_{T_{q,n}}(\alpha) \left(\left[R_{T_{q,n}}(\alpha)\right]^{-1} H_n - T_{q,n} H_{\alpha\triangleright n}\right) (I_{(n+1)q} - H_{\alpha\triangleright n}^\dagger H_{\alpha\triangleright n}) y \\
&= x^* (I_{(n+1)q} - H_n^\dagger H_n) H_n (I_{(n+1)q} - H_{\alpha\triangleright n}^\dagger H_{\alpha\triangleright n}) y = 0.
\end{aligned}$$

Consequently, the subspaces $\mathscr{U}_{n,\alpha}$ and $\mathscr{V}_{n,\alpha}$ are orthogonal.

(b) Use the equations $\mathscr{N}((I_{(n+1)q} - H_n^\dagger H_n) R_{T_{q,n}}(\alpha) v_{q,n}) = \mathscr{U}_{n,\alpha}^\perp = \mathscr{N}(P_{\mathscr{U}_{n,\alpha}})$ and $\mathscr{N}((I_{(n+1)q} - H_{\alpha\triangleright n}^\dagger H_{\alpha\triangleright n}) H_n v_{q,n}) = \mathscr{V}_{n,\alpha}^\perp = \mathscr{N}(P_{\mathscr{V}_{n,\alpha}})$.

The linear subspaces introduced in (207) and (208) provide the key instruments to realize our aim. This will be explained now. Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N} \cup \{\infty\}$, let $(s_j)_{j=0}^{\kappa} \in \mathscr{K}_{q,\kappa,\alpha}^{\geq,\mathrm{e}}$, and let n be a non-negative integer with $2n+1 \leq \kappa$. According to Lemma 14.1, the non-negative integers

$$m := \operatorname{rank}\Big[(I_{(n+1)q} - H_n^{\dagger} H_n) R_{T_{q,n}}(\alpha) v_{q,n}\Big] \tag{211}$$

and

$$\ell := \operatorname{rank}\Big[(I_{(n+1)q} - H_{\alpha\triangleright n}^{\dagger} H_{\alpha\triangleright n}) H_n v_{q,n}\Big] \tag{212}$$

fulfill $m + \ell \leq q$. In particular, $0 \leq m \leq q$ and $0 \leq \ell \leq q$. We consider separately the following three cases:

(I) $m + \ell = 0$, i. e., $m = 0$ and $\ell = 0$.
(II) $1 \leq m + \ell \leq q - 1$.
(III) $m + \ell = q$.

Our next consideration is dedicated to verify that the case (I) coincides with the above considered non-degenerate situation $\det H_n \neq 0$ and $\det H_{\alpha\triangleright n} \neq 0$.

Lemma 14.2 *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N} \cup \{\infty\}$, let $(s_j)_{j=0}^{\kappa} \in \mathscr{K}_{q,\kappa,\alpha}^{\geq,\mathrm{e}}$, and let $n \in \mathbb{N}_0$ with $2n+1 \leq \kappa$. Suppose that*

$$(I_{(n+1)q} - H_n^{\dagger} H_n) R_{T_{q,n}}(\alpha) v_{q,n} = 0_{(n+1)q\times q} \tag{213}$$

and

$$(I_{(n+1)q} - H_{\alpha\triangleright n}^{\dagger} H_{\alpha\triangleright n}) H_n v_{q,n} = 0_{(n+1)q\times q} \tag{214}$$

are fulfilled. For each $x \in \mathscr{N}(H_{\alpha\triangleright n})$, then $v_{q,n}^{*}[R_{T_{q,n}}(\alpha)]^{*} x = 0_{q\times 1}$ and $[R_{T_{q,n}}(\alpha)]^{*} T_{q,n}^{*} x \in \mathscr{N}(H_{\alpha\triangleright n})$.

Proof First observe that the matrices $s_0, s_1, \ldots, s_{2n+1}$ are Hermitian. Furthermore, the matrices $(I_{(n+1)q} - H_n^{\dagger} H_n)$ and $(I_{(n+1)q} - H_{\alpha\triangleright n}^{\dagger} H_{\alpha\triangleright n})$ correspond to orthogonal projections and thus are both Hermitian. Remark 5.3(d) yields $H_n v_{q,n} = y_{0,n}$, implying $v_{q,n}^{*} H_n^{*} = z_{0,n}$. Hence, from (213) and (214) we obtain

$$v_{q,n}^{*}\big[R_{T_{q,n}}(\alpha)\big]^{*}(I_{(n+1)q} - H_n^{\dagger} H_n) = 0_{q\times(n+1)q} \tag{215}$$

and

$$z_{0,n}(I_{(n+1)q} - H_{\alpha\triangleright n}^{\dagger} H_{\alpha\triangleright n}) = 0_{q\times(n+1)q}. \tag{216}$$

In view of (4) and (3), we have

$$\begin{aligned}\big[R_{T_{q,n+1}}(\alpha)\big]^{-1}\begin{bmatrix}H_n\\ z_{n+1,2n+1}\end{bmatrix} &= (I_{(n+2)q}-\alpha T_{q,n+1})\begin{bmatrix}H_n\\ z_{n+1,2n+1}\end{bmatrix}\\ &= \begin{bmatrix}H_n\\ z_{n+1,2n+1}\end{bmatrix}-\alpha\begin{bmatrix}0_{q\times(n+1)q}\\ H_n\end{bmatrix}=\begin{bmatrix}z_{0,n}\\ K_n\end{bmatrix}-\alpha\begin{bmatrix}0_{q\times(n+1)q}\\ H_n\end{bmatrix}\\ &= \begin{bmatrix}z_{0,n}\\ -\alpha H_n+K_n\end{bmatrix}=\begin{bmatrix}z_{0,n}\\ H_{\alpha\triangleright n}\end{bmatrix}\end{aligned}\tag{217}$$

and hence

$$[H_n, y_{n+1,2n+1}]\big[R_{T_{q,n+1}}(\alpha)\big]^{-*}=[y_{0,n}, H_{\alpha\triangleright n}].\tag{218}$$

Consider an arbitrary $x \in \mathcal{N}(H_{\alpha\triangleright n})$. According to (216), then $z_{0,n}x = 0_{q\times 1}$. Hence, $\left[\begin{smallmatrix}z_{0,n}\\ H_{\alpha\triangleright n}\end{smallmatrix}\right]x = 0_{(n+2)q\times 1}$. Because of (217), consequently $\left[\begin{smallmatrix}H_n\\ z_{n+1,2n+1}\end{smallmatrix}\right]x = 0_{(n+2)q\times 1}$, implying $H_n x = 0_{(n+1)q\times 1}$. From (215), then $v_{q,n}^*[R_{T_{q,n}}(\alpha)]^*x = 0_{q\times 1}$ follows. In view of (3), thus $f := [R_{T_{q,n}}(\alpha)]^*x$ satisfies $g = 0_{q\times 1}$, where $f = \left[\begin{smallmatrix}g\\ h\end{smallmatrix}\right]$ is the block representation of f with $q\times 1$ block g. Taking into account (4) and (3), we can infer $[R_{T_{q,n+1}}(\alpha)]^{-*} = \left[\begin{smallmatrix}[R_{T_{q,n}}(\alpha)]^{-*} & *\\ 0_{q\times(n+1)q} & I_q\end{smallmatrix}\right]$. Consequently, $[R_{T_{q,n+1}}(\alpha)]^{-*}\left[\begin{smallmatrix}f\\ 0_{q\times 1}\end{smallmatrix}\right] = \left[\begin{smallmatrix}[R_{T_{q,n}}(\alpha)]^{-*}f\\ 0_{q\times 1}\end{smallmatrix}\right] = \left[\begin{smallmatrix}x\\ 0_{q\times 1}\end{smallmatrix}\right]$. Using additionally (3) and (218), we get

$$\begin{aligned}H_{\alpha\triangleright n}\begin{bmatrix}h\\ 0_{q\times 1}\end{bmatrix} &= [y_{0,n}, H_{\alpha\triangleright n}]\begin{bmatrix}g\\ h\\ 0_{q\times 1}\end{bmatrix}=[H_n, y_{n+1,2n+1}]\big[R_{T_{q,n+1}}(\alpha)\big]^{-*}\begin{bmatrix}f\\ 0_{q\times 1}\end{bmatrix}\\ &= [H_n, y_{n+1,2n+1}]\begin{bmatrix}x\\ 0_{q\times 1}\end{bmatrix}=H_n x = 0_{(n+1)q\times 1}.\end{aligned}$$

From (3) and Remark 4.1 we can conclude $T_{q,n}^* f = \left[\begin{smallmatrix}h\\ 0_{q\times 1}\end{smallmatrix}\right]$ and $T_{q,n}^*\big[R_{T_{q,n}}(\alpha)\big]^* = [R_{T_{q,n}}(\alpha)]^* T_{q,n}^*$. Hence, $\left[\begin{smallmatrix}h\\ 0_{q\times 1}\end{smallmatrix}\right] = T_{q,n}^* f = T_{q,n}^*[R_{T_{q,n}}(\alpha)]^*x = [R_{T_{q,n}}(\alpha)]^*T_{q,n}^*x$, which completes the proof.

Lemma 14.3 *Let $\alpha\in\mathbb{R}$, let $\kappa\in\mathbb{N}\cup\{\infty\}$, let $(s_j)_{j=0}^{\kappa}\in\mathcal{K}_{q,\kappa,\alpha}^{\geq,\mathrm{e}}$, and let $n\in\mathbb{N}_0$ with $2n+1\leq\kappa$. Suppose that (213) and (214) are fulfilled. Then* $\det H_{\alpha\triangleright n}\neq 0$.

Proof Consider an arbitrary $x \in \mathcal{N}(H_{\alpha\triangleright n})$. Repeated application of Lemma 14.2 provides then $v_{q,n}^*[R_{T_{q,n}}(\alpha)]^*([R_{T_{q,n}}(\alpha)]^*T_{q,n}^*)^{\ell}x = 0_{q\times 1}$ and $([R_{T_{q,n}}(\alpha)]^*T_{q,n}^*)^{\ell}x\in\mathcal{N}(H_{\alpha\triangleright n})$ for $\ell = 0,1,2\dots,n$. Since Remark 4.1 yields $T_{q,n}^*\big[R_{T_{q,n}}(\alpha)\big]^* = [R_{T_{q,n}}(\alpha)]^*T_{q,n}^*$, we obtain $v_{q,n}^*([R_{T_{q,n}}(\alpha)]^*)^{\ell+1}(T_{q,n}^*)^{\ell}x =$

$0_{q\times 1}$ for all $\ell \in \mathbb{Z}_{0,n}$. Let $x = \operatorname{col}(x_j)_{j=0}^{n}$ be the $q \times 1$ block representation of x. Because of (3), then $(T_{q,n}^*)^\ell x = \begin{bmatrix} x_\ell \\ \vdots \\ x_n \\ 0_{\ell q\times 1} \end{bmatrix}$ for all $\ell \in \mathbb{Z}_{1,n}$. From Remark 4.1 we can conclude $([R_{T_{q,n}}(\alpha)]^*)^{\ell+1} = \begin{bmatrix} I_q & * \\ 0_{nq\times q} & * \end{bmatrix}$ for all $\ell \in \mathbb{Z}_{0,n}$.

We are now going to show by mathematical induction $x_j = 0_{q\times 1}$ for $j = n, n-1, \dots, 0$. We have $0_{q\times 1} = v_{q,n}^*([R_{T_{q,n}}(\alpha)]^*)^{n+1}(T_{q,n}^*)^n x = v_{q,n}^*([R_{T_{q,n}}(\alpha)]^*)^{n+1} \begin{bmatrix} x_n \\ 0_{nq\times 1} \end{bmatrix} = v_{q,n}^* \begin{bmatrix} x_n \\ 0_{nq\times 1} \end{bmatrix} = x_n$. Now suppose that $x_n = x_{n-1} = \cdots = x_{j+1} = 0_{q\times 1}$ holds true for some $j \in \mathbb{Z}_{0,n-1}$. Then, $(T_{q,n}^*)^j x = \begin{bmatrix} x_j \\ \vdots \\ x_n \\ 0_{jq\times 1} \end{bmatrix} = \begin{bmatrix} x_j \\ 0_{nq\times 1} \end{bmatrix}$. Hence $0_{q\times 1} = v_{q,n}^*([R_{T_{q,n}}(\alpha)]^*)^{j+1}(T_{q,n}^*)^j x = v_{q,n}^*([R_{T_{q,n}}(\alpha)]^*)^{j+1} \begin{bmatrix} x_j \\ 0_{nq\times 1} \end{bmatrix} = v_{q,n}^* \begin{bmatrix} x_j \\ 0_{nq\times 1} \end{bmatrix} = x_j$ follows.

Thus, we have shown $x = 0_{(n+1)q\times 1}$. Consequently $\mathscr{N}(H_{\alpha\triangleright n}) = \{0_{(n+1)q\times 1}\}$, which completes the proof.

Let $\alpha \in \mathbb{R}$. For all $n \in \mathbb{N}_0$, let $\mathscr{K}_{q,2n+1,\alpha}^{>}$ be the set of all sequences $(s_j)_{j=0}^{2n+1}$ of complex $q \times q$ matrices for which the block Hankel matrices H_n and $H_{\alpha\triangleright n} = -\alpha H_n + K_n$ are both positive Hermitian.

Remark 14.4 Let $\alpha \in \mathbb{R}$ and let $n \in \mathbb{N}_0$. Then from [30, Proposition 2.20] it follows that $\mathscr{K}_{q,2n+1,\alpha}^{>} \subseteq \mathscr{K}_{q,2n+1,\alpha}^{\geq,\mathrm{e}}$.

Proposition 14.5 *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N} \cup \{\infty\}$, let $(s_j)_{j=0}^{\kappa} \in \mathscr{K}_{q,\kappa,\alpha}^{\geq,\mathrm{e}}$, and let $n \in \mathbb{N}_0$ with $2n+1 \leq \kappa$. Then the following statements are equivalent:*

(i) $\mathscr{U}_{n,\alpha} = \{0_{q\times 1}\}$ *and* $\mathscr{V}_{n,\alpha} = \{0_{q\times 1}\}$.
(ii) $(s_j)_{j=0}^{2n+1} \in \mathscr{K}_{q,2n+1,\alpha}^{>}$.

Proof From Lemma 14.1(a) we can easily conclude that (i) is equivalent to the following statement:

(iii) The equalities (213) and (214) are fulfilled.

Furthermore, we can infer that the sequence $(s_j)_{j=0}^{2n+1}$ belongs to $\mathscr{K}_{q,2n+1,\alpha}^{\geq}$, i. e. that the matrices H_n and $H_{\alpha\triangleright n}$ are both non-negative Hermitian. Consequently, (ii) is equivalent to the following statement:

(iv) $\det H_n \neq 0$ and $\det H_{\alpha\triangleright n} \neq 0$.

We are now going to show the equivalence of (iii) and (iv).

(iii)⇒(iv): In view of (iii), we can apply Lemma 14.3 to obtain $\det H_{\alpha\triangleright n} \neq 0$. According to Remark 5.9, we have $\mathscr{N}(L_0) \subseteq \mathscr{N}(L_{\alpha\triangleright 0}) \subseteq \mathscr{N}(L_1) \subseteq \cdots \subseteq \mathscr{N}(L_n) \subseteq \mathscr{N}(L_{\alpha\triangleright n})$. From [30, Lemma 4.11] we can conclude $\det H_n = \prod_{k=0}^{n} \det L_k$ and $\det H_{\alpha\triangleright n} = \prod_{k=0}^{n} \det L_{\alpha\triangleright k}$. Because of $\det H_{\alpha\triangleright n} \neq 0$, then

$\det L_{\alpha \triangleright n} \neq 0$ follows, implying $\mathscr{N}(L_{\alpha \triangleright n}) = \{0_{q \times 1}\}$. For all $k \in \mathbb{Z}_{0,n}$, hence $\mathscr{N}(L_k) = \{0_{q \times 1}\}$ and consequently $\det L_k \neq 0$. Thus, $\det H_n \neq 0$ follows as well.

(iv)⇒(iii): Because of (iv), the matrices H_n and $H_{\alpha \triangleright n}$ are both invertible with $H_n^{-1} = H_n^{\dagger}$ and $H_{\alpha \triangleright n}^{-1} = H_{\alpha \triangleright n}^{\dagger}$. Consequently, we have $I_{(n+1)q} - H_n^{\dagger} H_n = 0_{(n+1)q \times (n+1)q}$ and $I_{(n+1)q} - H_{\alpha \triangleright n}^{\dagger} H_{\alpha \triangleright n} = 0_{(n+1)q \times (n+1)q}$, implying (iii).

As a consequence of Proposition 14.5 we infer from the consideration at the end of Section 13 that the case (I) is already treated in Theorem 13.7. It still remains to consider the cases (II) and (III).

15 A Further Parametrization of the Solution Set of the Truncated Matricial Stieltjes Moment Problem in the Degenerate But Not Completely Degenerate Case

In this section, we state a parametrization of the solution set of the matricial truncated Stieltjes moment problem $\mathsf{S}[[\alpha, \infty); (s_j)_{j=0}^{2n+1}, \leq]$ in the degenerate but not completely degenerate cases. First we recall that, in view of Theorems 2.4 and 2.6, one can suppose that the given sequence $(s_j)_{j=0}^{2n+1}$ of complex $q \times q$ matrices belongs to the set $\mathscr{K}_{q,2n+1,\alpha}^{\geq,\mathrm{e}}$. More precisely we turn our attention to case (II).

Remark 15.1 Let $\alpha \in \mathbb{R}$. Let V and W be complex $q \times q$ matrices with $W^* V = I_q$. Then it is readily checked that the following statements hold true:

(a) If $[\phi; \psi] \in \mathscr{P}_{-\tilde{J}_q, \geq}^{(q,q)}(\mathbb{C} \setminus [\alpha, \infty))$, then the pair $[V\phi; W\psi]$ belongs to $\mathscr{P}_{-\tilde{J}_q, \geq}^{(q,q)}(\mathbb{C} \setminus [\alpha, \infty))$.

(b) Let $[\phi_1; \psi_1], [\phi_2; \psi_2] \in \mathscr{P}_{-\tilde{J}_q, \geq}^{(q,q)}(\mathbb{C} \setminus [\alpha, \infty))$. Then $\langle [V\phi_1; W\psi_1] \rangle = \langle [V\phi_2; W\psi_2] \rangle$ if and only if $\langle [\phi_1; \psi_1] \rangle = \langle [\phi_2; \psi_2] \rangle$.

Lemma 15.2 *Let $\alpha \in \mathbb{R}$ and let $r \in \mathbb{N}$ be such that $r < q$. Let U and V be complex $(q-r) \times (q-r)$ matrices with $\operatorname{rank} \begin{bmatrix} U \\ V \end{bmatrix} = q - r$ and $V^* U = 0_{(q-r) \times (q-r)}$. Let $\mathscr{U}$ (resp. $\mathscr{V}$) be the constant matrix-valued function (defined on $\mathbb{C} \setminus [\alpha, \infty)$) with value U (resp. V). Then:*

(a) If $[\phi; \psi] \in \mathscr{P}_{-\tilde{J}_r, \geq}^{(r,r)}(\mathbb{C} \setminus [\alpha, \infty))$, then the pair $[\phi^{\square}; \psi^{\square}]$ given by $\phi^{\square} := \operatorname{diag}(\phi, \mathscr{U})$ and $\psi^{\square} := \operatorname{diag}(\psi, \mathscr{V})$ belongs to $\mathscr{P}_{-\tilde{J}_q, \geq}^{(q,q)}(\mathbb{C} \setminus [\alpha, \infty))$.

(b) Let $[\phi_1; \psi_1], [\phi_2; \psi_2] \in \mathscr{P}_{-\tilde{J}_r, \geq}^{(r,r)}(\mathbb{C} \setminus [\alpha, \infty))$. For each $k \in \{1, 2\}$, let $\phi_k^{\square} := \operatorname{diag}(\phi_k, \mathscr{U})$ and $\psi_k^{\square} := \operatorname{diag}(\psi_k, \mathscr{V})$. Then $\langle [\phi_1^{\square}; \psi_1^{\square}] \rangle = \langle [\phi_2^{\square}; \psi_2^{\square}] \rangle$ if and only if $\langle [\phi_1; \psi_1] \rangle = \langle [\phi_2; \psi_2] \rangle$.

The proof of Lemma 15.2 is straightforward. We omit the details.

In the following, we will use again $P_{\mathscr{U}}$ to denote the complex $q \times q$ matrix which represents the orthogonal projection onto a given subspace $\mathscr{U}$ of $\mathbb{C}^q$ with respect to the standard basis of $\mathbb{C}^q$, i. e., for each subspace $\mathscr{U}$ of $\mathbb{C}^q$, the matrix $P_{\mathscr{U}}$ is the unique complex $q \times q$ matrix P which fulfills the three conditions $P^2 = P$, $P^* = P$, and $\mathscr{R}(P) = \mathscr{U}$.

Lemma 15.3 *Let m and ℓ be non-negative integers such that $r := q-(m+\ell)$ fulfills $1 \leq r \leq q-1$. Let $\mathscr{U}$ and $\mathscr{V}$ be orthogonal subspaces of $\mathbb{C}^q$ with $\dim \mathscr{U} = m$ and $\dim \mathscr{V} = \ell$. Then:*

(a) There exists a unitary complex $q \times q$ matrix W such that

$$W^* P_{\mathscr{U}} W = \begin{cases} \operatorname{diag}(0_{r\times r}, I_m, 0_{\ell\times\ell}), & \text{if } m \geq 1 \text{ and } \ell \geq 1 \\ \operatorname{diag}(0_{r\times r}, I_m), & \text{if } m \geq 1 \text{ and } \ell = 0 \end{cases} \tag{219}$$

and

$$W^* P_{\mathscr{V}} W = \begin{cases} \operatorname{diag}(0_{r\times r}, 0_{m\times m}, I_\ell), & \text{if } m \geq 1 \text{ and } \ell \geq 1 \\ \operatorname{diag}(0_{r\times r}, I_\ell), & \text{if } m = 0 \text{ and } \ell \geq 1 \end{cases}. \tag{220}$$

(b) Let $\alpha \in \mathbb{R}$ and let W be a unitary complex $q \times q$ matrix such that (219) and (220) are fulfilled.

(b1) If $[\tilde{\phi}; \tilde{\psi}] \in \mathscr{P}^{(q,q)}_{-\tilde{J}_q,\geq}(\mathbb{C} \setminus [\alpha, \infty))$ is such that

$$P_{\mathscr{U}} \tilde{\phi} = 0_{q\times q} \qquad \text{and} \qquad P_{\mathscr{V}} \tilde{\psi} = 0_{q\times q}, \tag{221}$$

then there exists a pair $[\phi; \psi] \in \mathscr{P}^{(r,r)}_{-\tilde{J}_r,\geq}(\mathbb{C} \setminus [\alpha, \infty))$ such that ϕ and ψ and the functions

$$\phi^{\square} := \begin{cases} W \cdot \operatorname{diag}(\phi, 0_{m\times m}, I_\ell), & \text{if } m \geq 1 \text{ and } \ell \geq 1 \\ W \cdot \operatorname{diag}(\phi, 0_{m\times m}), & \text{if } m \geq 1 \text{ and } \ell = 0 \\ W \cdot \operatorname{diag}(\phi, I_\ell), & \text{if } m = 0 \text{ and } \ell \geq 1 \end{cases} \tag{222}$$

and

$$\psi^{\square} := \begin{cases} W \cdot \operatorname{diag}(\psi, I_m, 0_{\ell\times\ell}), & \text{if } m \geq 1 \text{ and } \ell \geq 1 \\ W \cdot \operatorname{diag}(\psi, I_m), & \text{if } m \geq 1 \text{ and } \ell = 0 \\ W \cdot \operatorname{diag}(\psi, 0_{\ell\times\ell}), & \text{if } m = 0 \text{ and } \ell \geq 1 \end{cases} \tag{223}$$

fulfill the following three conditions:

(i) ϕ, ψ, $\phi^{\square}$, and $\psi^{\square}$ are holomorphic in Π_+.
(ii) $[\phi^{\square}; \psi^{\square}] \in \mathscr{P}^{(q,q)}_{-\tilde{J}_q,\geq}(\mathbb{C} \setminus [\alpha, \infty))$.
(iii) $\langle[\tilde{\phi}; \tilde{\psi}]\rangle = \langle[\phi^{\square}; \psi^{\square}]\rangle$.

(b2) *For each pair* $[\phi; \psi] \in \mathscr{P}^{(r,r)}_{-\tilde{J}_r,\geq}(\mathbb{C} \setminus [\alpha, \infty))$*, the functions* $\phi^{\square}$ *and* $\psi^{\square}$ *given by* (222) *and* (223) *fulfill (ii).*

(b3) *Let* $[\phi; \psi] \in \mathscr{P}^{(r,r)}_{-\tilde{J}_r,\geq}(\mathbb{C} \setminus [\alpha, \infty))$*. Let* $\phi^{\square}$ *and* $\psi^{\square}$ *be defined by* (222) *and* (223)*. Then every pair* $[\tilde{\phi}; \tilde{\psi}] \in \mathscr{P}^{(q,q)}_{-\tilde{J}_q,\geq}(\mathbb{C} \setminus [\alpha, \infty))$ *for which (iii) holds true fulfills* (221)*.*

Lemma 15.4 is substantially proved in [7, Lemma 5.2, p. 459/460]. (A detailed proof for the case that $m \geq 1$ and $\ell \geq 1$ is also given in [53, Lemma 11.7].)

Lemma 15.4 *Let* $\alpha \in \mathbb{R}$*, let* $\kappa \in \mathbb{N} \cup \{\infty\}$*, let* $(s_j)_{j=0}^{\kappa} \in \mathscr{K}^{\geq,\mathrm{e}}_{q,\kappa,\alpha}$*, and let* $n \in \mathbb{N}_0$ *be such that* $2n+1 \leq \kappa$*. Let* m *and* ℓ *be given by* (211) *and* (212) *and let* $r := q - (m + \ell)$*. Suppose* $r \geq 1$*. Let* $\mathscr{U}_{n,\alpha}$ *and* $\mathscr{V}_{n,\alpha}$ *be given by* (207) *and* (208)*. Then:*

(a) *There exists a unitary complex* $q \times q$ *matrix* W *such that* (219) *with* $\mathscr{U} = \mathscr{U}_{n,\alpha}$ *and* (220) *with* $\mathscr{V} = \mathscr{V}_{n,\alpha}$ *hold true.*

(b) *Let* W *be a unitary complex* $q \times q$ *matrix such that* (219) *with* $\mathscr{U} = \mathscr{U}_{n,\alpha}$ *and* (220) *with* $\mathscr{V} = \mathscr{V}_{n,\alpha}$ *are valid.*

(b1) *Let* $[\tilde{\phi}; \tilde{\psi}] \in \mathscr{P}^{(q,q)}_{-\tilde{J}_q,\geq}[\mathbb{C} \setminus [\alpha, \infty), (s_j)_{j=0}^{2n+1}]$*. Then there exists a pair* $[\phi; \psi] \in \mathscr{P}^{(r,r)}_{-\tilde{J}_r,\geq}(\mathbb{C} \setminus [\alpha, \infty))$ *such that the conditions (i)–(iii) of Lemma 15.3 hold true with* $\phi^{\square}$ *and* $\psi^{\square}$ *given by* (222) *and* (223)*.*

(b2) *If* $[\phi; \psi] \in \mathscr{P}^{(r,r)}_{-\tilde{J}_r,\geq}(\mathbb{C} \setminus [\alpha, \infty))$*, then* $\phi^{\square}$ *and* $\psi^{\square}$ *be given by* (222) *and* (223) *fulfill condition (ii) of Lemma 15.3.*

(b3) *Let* $[\phi; \psi] \in \mathscr{P}^{(r,r)}_{-\tilde{J}_r,\geq}(\mathbb{C} \setminus [\alpha, \infty))$ *and let* $\phi^{\square}$ *and* $\psi^{\square}$ *be given by* (222) *and* (223)*. If* $[\tilde{\phi}; \tilde{\psi}] \in \mathscr{P}^{(q,q)}_{-\tilde{J}_q,\geq}(\mathbb{C} \setminus [\alpha, \infty))$ *fulfills condition (iii) of Lemma 15.3, then* $[\tilde{\phi}; \tilde{\psi}]$ *belongs to* $\mathscr{P}^{(q,q)}_{-\tilde{J}_q,\geq}[\mathbb{C} \setminus [\alpha, \infty), (s_j)_{j=0}^{2n+1}]$*.*

Lemma 15.4 can be easily proved using Lemmas 15.3 and 14.1 (see also [34, Lemma 12.6]). A closer look at the construction of the unitary matrix W in Lemma 15.4 shows that this matrix depends on the sequence $(s_j)_{j=0}^{2n+1}$ of given moments. Now we obtain a parametrization of the solution set of the matricial truncated Stieltjes moment problem in the so-called degenerate, but not completely degenerate case. In the rest of this section, let $\Theta_{n,\alpha} \colon \mathbb{C} \to \mathbb{C}^{2q \times 2q}$ be defined by (85), let $\hat{\Theta}_{n,\alpha}$ be the restriction of $\Theta_{n,\alpha}$ onto $\mathbb{C} \setminus [\alpha, \infty)$, and let (176) be the $q \times q$ block partition of $\hat{\Theta}_{n,\alpha}$.

Theorem 15.5 *Let* $\alpha \in \mathbb{R}$*, let* $\kappa \in \mathbb{N} \cup \{\infty\}$*, let* $(s_j)_{j=0}^{\kappa} \in \mathscr{K}^{\geq,\mathrm{e}}_{q,\kappa,\alpha}$*, and let* $n \in \mathbb{N}_0$ *be such that* $2n+1 \leq \kappa$*. Let the integers* m *and* ℓ *be given by* (211) *and* (212) *and let* $r := q - (m + \ell)$*. Suppose* $r \geq 1$*. Let* $\mathscr{U}_{n,\alpha}$ *and* $\mathscr{V}_{n,\alpha}$ *be the subspaces of* $\mathbb{C}^q$ *which are defined in* (207) *and* (208)*. Let* W *be a unitary complex* $q \times q$ *matrix such that* (219) *with* $\mathscr{U} = \mathscr{U}_{n,\alpha}$ *and* (220) *with* $\mathscr{V} = \mathscr{V}_{n,\alpha}$ *hold true. Then:*

(a) *Let* $[\phi; \psi] \in \mathscr{P}^{(r,r)}_{-\tilde{J}_r,\geq}(\mathbb{C} \setminus [\alpha, \infty))$ *and let* $\phi^{\square}$ *and* $\psi^{\square}$ *be defined by* (222) *and* (223). *Then the function* $\det(\hat{\Theta}^{(2,1)}_{n,\alpha}\phi^{\square} + \hat{\Theta}^{(2,2)}_{n,\alpha}\psi^{\square})$ *does not vanish identically and the matrix-valued function* $S := (\hat{\Theta}^{(1,1)}_{n,\alpha}\phi^{\square} + \hat{\Theta}^{(1,2)}_{n,\alpha}\psi^{\square})(\hat{\Theta}^{(2,1)}_{n,\alpha}\phi^{\square} + \hat{\Theta}^{(2,2)}_{n,\alpha}\psi^{\square})^{-1}$ *belongs to the class* $\mathscr{S}_{0,q;[\alpha,\infty)}[(s_j)_{j=0}^{2n+1}, \leq]$.

(b) *For each* $S \in \mathscr{S}_{0,q;[\alpha,\infty)}[(s_j)_{j=0}^{2n+1}, \leq]$, *there exists a pair* $[\phi; \psi] \in \mathscr{P}^{(r,r)}_{-\tilde{J}_r,\geq}(\mathbb{C} \setminus [\alpha, \infty))$ *such that the function* $\det(\hat{\Theta}^{(2,1)}_{n,\alpha}\phi^{\square} + \hat{\Theta}^{(2,2)}_{n,\alpha}\psi^{\square})$ *does not vanish identically and that* S *fulfills*

$$S = (\hat{\Theta}^{(1,1)}_{n,\alpha}\phi^{\square} + \hat{\Theta}^{(1,2)}_{n,\alpha}\psi^{\square})(\hat{\Theta}^{(2,1)}_{n,\alpha}\phi^{\square} + \hat{\Theta}^{(2,2)}_{n,\alpha}\psi^{\square})^{-1} \tag{224}$$

where $\phi^{\square}$ *and* $\psi^{\square}$ *are given by* (222) *and* (223).

(c) *Let* $[\phi_1; \psi_1], [\phi_2; \psi_2] \in \mathscr{P}^{(r,r)}_{-\tilde{J}_r,\geq}(\mathbb{C} \setminus [\alpha, \infty))$. *For each* $k \in \{1, 2\}$, *let* $\phi_k^{\square}$ *be defined as in* (222) *where* ϕ *is replaced by* ϕ_k *and let* $\psi_k^{\square}$ *be defined as in* (223) *where* ψ *is replaced by* ψ_k. *Then* $\langle[\phi_1; \psi_1]\rangle = \langle[\phi_2; \psi_2]\rangle$ *is equivalent to*

$$\begin{aligned}(\hat{\Theta}^{(1,1)}_{n,\alpha}\phi_1^{\square} + \hat{\Theta}^{(1,2)}_{n,\alpha}\psi_1^{\square})(\hat{\Theta}^{(2,1)}_{n,\alpha}\phi_1^{\square} + \hat{\Theta}^{(2,2)}_{n,\alpha}\psi_1^{\square})^{-1}\\ = (\hat{\Theta}^{(1,1)}_{n,\alpha}\phi_2^{\square} + \hat{\Theta}^{(1,2)}_{n,\alpha}\psi_2^{\square})(\hat{\Theta}^{(2,1)}_{n,\alpha}\phi_2^{\square} + \hat{\Theta}^{(2,2)}_{n,\alpha}\psi_2^{\square})^{-1}.\end{aligned} \tag{225}$$

Proof Let us consider the case that $m \geq 1$ and $\ell \geq 1$ hold true. (If $m = 0$ or if $\ell = 0$, then the assertions can be proved analogously.)

(a) Let $[\phi; \psi] \in \mathscr{P}^{(r,r)}_{-\tilde{J}_r,\geq}(\mathbb{C} \setminus [\alpha, \infty))$. Parts (b3) and (b2) of Lemma 15.4 and Notation 13.3 yield $[\phi^{\square}; \psi^{\square}] \in \mathscr{P}^{(q,q)}_{-\tilde{J}_q,\geq}[\mathbb{C} \setminus [\alpha, \infty), (s_j)_{j=0}^{2n+1}]$. Applying Theorem 13.7(a), part (a) is proved.

(b) Let $S \in \mathscr{S}_{0,q;[\alpha,\infty)}[(s_j)_{j=0}^{2n+1}, \leq]$. According to Theorem 13.7(b), then there is a pair $[\phi_\#; \psi_\#] \in \mathscr{P}^{(q,q)}_{-\tilde{J}_q,\geq}[\mathbb{C} \setminus [\alpha, \infty), (s_j)_{j=0}^{2n+1}]$, where $\phi_\#$ and $\psi_\#$ are matrix-valued functions which are holomorphic in $\mathbb{C} \setminus [\alpha, \infty)$ and which fulfill

$$\det\Big[\hat{\Theta}^{(2,1)}_{n,\alpha}(z)\phi_\#(z) + \hat{\Theta}^{(2,2)}_{n,\alpha}(z)\psi_\#(z)\Big] \neq 0 \tag{226}$$

and

$$S(z) = \Big[\hat{\Theta}^{(1,1)}_{n,\alpha}(z)\phi_\#(z)+\hat{\Theta}^{(1,2)}_{n,\alpha}(z)\psi_\#(z)\Big]\Big[\hat{\Theta}^{(2,1)}_{n,\alpha}(z)\phi_\#(z)+\hat{\Theta}^{(2,2)}_{n,\alpha}(z)\psi_\#(z)\Big]^{-1} \tag{227}$$

for all $z \in \mathbb{C} \setminus [\alpha, \infty)$. In view of Notation 13.3 and Lemma 15.4(b1), there is a pair $[\phi; \psi] \in \mathscr{P}^{(r,r)}_{-\tilde{J}_r,\geq}(\mathbb{C} \setminus [\alpha, \infty))$ such that $[\phi^{\square}; \psi^{\square}] \in \mathscr{P}^{(q,q)}_{-\tilde{J}_q,\geq}(\mathbb{C} \setminus [\alpha, \infty))$ and $\langle[\phi_\#; \psi_\#]\rangle = \langle[\phi^{\square}; \psi^{\square}]\rangle$ hold true. Consequently, there are a discrete subset $\mathscr{D}$ of $\mathbb{C} \setminus [\alpha, \infty)$ and a $q \times q$ matrix-valued function

g which is meromorphic in $\mathbb{C} \setminus [\alpha, \infty)$ such that $\phi_\#$, $\psi_\#$, ϕ, ψ, and g are holomorphic in $\mathbb{C} \setminus ([\alpha, \infty) \cup \mathscr{D})$ and that $\det g(z) \neq 0$ as well as

$$\phi_\#(z) = \phi^\square(z) g(z) \qquad \text{and} \qquad \psi_\#(z) = \psi^\square(z) g(z) \tag{228}$$

hold true for each $z \in \mathbb{C} \setminus ([\alpha, \infty) \cup \mathscr{D})$. Therefore, for each $z \in \mathbb{C} \setminus ([\alpha, \infty) \cup \mathscr{D})$, it follows from (226) that $0 \neq \det[\hat{\Theta}_{n,\alpha}^{(2,1)}(z)\psi^\square(z) + \hat{\Theta}_{n,\alpha}^{(2,2)}(z)\psi^\square(z)] \cdot \det g(z)$. In particular, the function $\det(\hat{\Theta}_{n,\alpha}^{(2,1)}\phi^\square + \hat{\Theta}_{n,\alpha}^{(2,2)}\psi^\square)$ does not vanish identically in $\mathbb{C} \setminus [\alpha, \infty)$. Because of (227) and (228), for all $z \in \mathbb{C} \setminus ([\alpha, \infty) \cup \mathscr{D})$, we get furthermore

$$S(z) = \Big[\hat{\Theta}_{n,\alpha}^{(1,1)}(z)\phi^\square(z) + \hat{\Theta}_{n,\alpha}^{(1,2)}(z)\psi^\square(z)\Big]\Big[\hat{\Theta}_{n,\alpha}^{(2,1)}(z)\phi^\square(z) + \hat{\Theta}_{n,\alpha}^{(2,2)}(z)\psi^\square(z)\Big]^{-1}.$$

In particular, (224) holds true.

(c) The matrices $U := \operatorname{diag}(0_{m\times m}, I_\ell)$ and $V := \operatorname{diag}(I_m, 0_{\ell\times\ell})$ fulfill $\operatorname{rank}\left[\begin{smallmatrix} U \\ V \end{smallmatrix}\right] = m + \ell = q - r$ and $V^*U = 0_{(q-r)\times(q-r)}$. For each $k \in \{1, 2\}$, let $\phi_k^\# = \operatorname{diag}(\phi_k, U)$ and $\psi_k^\# = \operatorname{diag}(\psi_k, V)$. From Lemma 15.2 we see that $[\phi_1^\#; \psi_1^\#]$ and $[\phi_2^\#; \psi_2^\#]$ belong to $\mathscr{P}_{-\tilde{J}_q,\geq}^{(q,q)}(\mathbb{C} \setminus [\alpha, \infty))$ and that $\langle[\phi_1; \psi_1]\rangle = \langle[\phi_2; \psi_2]\rangle$ is equivalent to $\langle[\phi_1^\#; \psi_1^\#]\rangle = \langle[\phi_2^\#; \psi_2^\#]\rangle$. Obviously, $\phi_k^\square := W\phi_k^\#$ and $\psi_k^\square := W\psi_k^\#$ for each $k \in \{1, 2\}$. Taking into account $W^*W = I_q$ and Remark 15.1, we get that $[\phi_1^\square; \psi_1^\square]$ and $[\phi_2^\square; \psi_2^\square]$ belong to $\mathscr{P}_{-\tilde{J}_q,\geq}^{(q,q)}(\mathbb{C} \setminus [\alpha, \infty))$ and that $\langle[\phi_1^\#; \psi_1^\#]\rangle = \langle[\phi_2^\#; \psi_2^\#]\rangle$ is equivalent to $\langle[\phi_1^\square; \psi_1^\square]\rangle = \langle[\phi_2^\square; \psi_2^\square]\rangle$. Because of Lemma 15.3(b3), we obtain $[\phi_k^\square; \psi_k^\square] \in \mathscr{P}_{-\tilde{J}_q,\geq}^{(q,q)}[\mathbb{C} \setminus [\alpha, \infty), (s_j)_{j=0}^{2n+1}]$ for each $k \in \{1, 2\}$. Using Theorem 13.7(c), we see that $\langle[\phi_1^\square; \psi_1^\square]\rangle = \langle[\phi_2^\square; \psi_2^\square]\rangle$ and (225) are equivalent.

16 The Completely Degenerate Case

Finally we consider the so-called completely degenerate case (III). We will see that, in this situation, the problem under consideration has a unique solution.

Lemma 16.1 *Let $m, \ell \in \mathbb{N}$ be such that $m + \ell = q$. Let $\mathscr{U}$ and $\mathscr{V}$ be orthogonal subspaces of $\mathbb{C}^q$ with $\dim \mathscr{U} = m$ and $\dim \mathscr{V} = \ell$. Then:*

(a) There exists a unitary complex $q \times q$ matrix W such that

$$W^* P_{\mathscr{U}} W = \operatorname{diag}(I_m, 0_{\ell\times\ell}) \quad \textit{and} \quad W^* P_{\mathscr{V}} W = \operatorname{diag}(0_{m\times m}, I_\ell). \tag{229}$$

(b) Let W be a unitary complex $q \times q$ matrix such that (229) is fulfilled. Let $\phi_\#$ and $\psi_\#$ be the constant matrix-valued functions defined on $\mathbb{C} \setminus [\alpha, \infty)$ given by

$$\phi_\#(z) := W \cdot \operatorname{diag}(0_{m\times m}, I_\ell) \quad \text{and} \quad \psi_\#(z) := W \cdot \operatorname{diag}(I_m, 0_{\ell\times\ell}) \tag{230}$$

for all $z \in \mathbb{C} \setminus [\alpha, \infty)$. Then:

(b1) The pair $[\phi_\#; \psi_\#]$ belongs to $\mathscr{P}^{(q,q)}_{-\tilde{J}_q,\geq}(\mathbb{C} \setminus [\alpha, \infty))$. Furthermore, $P_{\mathscr{U}}\, \phi_\# = 0$ and $P_{\mathscr{V}}\, \psi_\# = 0$.

(b2) If $[\phi; \psi] \in \mathscr{P}^{(q,q)}_{-\tilde{J}_q,\geq}(\mathbb{C} \setminus [\alpha, \infty))$ fulfills

$$\langle[\phi; \psi]\rangle = \langle[\phi_\#; \psi_\#]\rangle, \tag{231}$$

then

$$P_{\mathscr{U}}\, \phi = 0 \qquad \text{and} \qquad P_{\mathscr{V}}\, \psi = 0. \tag{232}$$

(b3) If $[\phi; \psi]$ belongs to $\mathscr{P}^{(q,q)}_{-\tilde{J}_q,\geq}(\mathbb{C} \setminus [\alpha, \infty))$ and fulfills (232), then (231) is valid.

Proof

(a) Let $\{u_1, u_1, \ldots, u_m\}$ be an orthonormal basis of $\mathscr{U}$ and let $\{v_1, v_2, \ldots, v_\ell\}$ be an orthonormal basis of $\mathscr{V}$. Let $U := [u_1, u_2, \ldots, u_m]$, let $V := [v_1, v_2, \ldots, v_\ell]$, and let $W := [U, V]$. Because of $m + \ell = q$ and since $\mathscr{U}$ and $\mathscr{V}$ are orthogonal subspaces, the matrix W is unitary. Obviously, we have $P_{\mathscr{U}}\, U = U$, $P_{\mathscr{U}}\, V = 0$, $U^*U = I_m$, and $V^*U = 0$. Consequently, $W^* P_{\mathscr{U}}\, W = \operatorname{diag}(I_m, 0_{\ell\times\ell})$. Analogously, $P_{\mathscr{V}}\, U = 0$, $P_{\mathscr{V}}\, V = V$, $U^*V = 0$, and $V^*V = I_\ell$ imply the second equation in (229).

(b1) Clearly, the constant matrix-valued functions $\phi_\#$ and $\psi_\#$ are holomorphic in $\mathbb{C} \setminus [\alpha, \infty)$. Since the matrix W is non-singular, we have $\operatorname{rank}\left[\begin{smallmatrix}\phi_\#(z)\\ \psi_\#(z)\end{smallmatrix}\right] = \operatorname{rank}\left[\begin{smallmatrix}\operatorname{diag}(0_{m\times m}, I_\ell)\\ \operatorname{diag}(I_m, 0_{\ell\times\ell})\end{smallmatrix}\right] = m + \ell = q$ for each $z \in \mathbb{C} \setminus [\alpha, \infty)$. For every choice of $k \in \{0, 1\}$ and $z \in \mathbb{C} \setminus \mathbb{R}$, from Remark 7.1 and $W^*W = I_q$, we conclude

$$\begin{aligned}
&\begin{bmatrix}(z-\alpha)^k\phi_\#(z)\\ \psi_\#(z)\end{bmatrix}^* \left(\frac{-\tilde{J}_q}{2\operatorname{Im} z}\right)\begin{bmatrix}(z-\alpha)^k\phi_\#(z)\\ \psi_\#(z)\end{bmatrix}\\
&= \frac{-\mathrm{i}}{2\operatorname{Im} z}\Big(\psi_\#^*(z)\Big[(z-\alpha)^k\phi_\#(z)\Big] - \Big[(\overline{z}-\alpha)^k\phi_\#^*(z)\Big]\psi_\#(z)\Big)\\
&= \frac{-\mathrm{i}}{2\operatorname{Im} z}\big\{(z-\alpha)^k \cdot \operatorname{diag}(I_m, 0_{\ell\times\ell}) \cdot W^*W \cdot \operatorname{diag}(0_{m\times m}, I_\ell)\\
&\qquad - (\overline{z}-\alpha)^k \cdot \operatorname{diag}(0_{m\times m}, I_\ell) \cdot W^*W \cdot \operatorname{diag}(I_m, 0_{\ell\times\ell})\big\} = 0_{q\times q}.
\end{aligned}$$

In view of Definition 8.1, then $[\phi_\#; \psi_\#]$ belongs to $\mathscr{P}^{(q,q)}_{-\tilde{J}_q,\geq}(\mathbb{C} \setminus [\alpha, \infty))$. Further, from (229) we obtain $P_{\mathscr{U}}\,\phi_\# = I_q P_{\mathscr{U}}\, W \cdot \operatorname{diag}(0_{m\times m}, I_\ell) = WW^* P_{\mathscr{U}}\, W \cdot \operatorname{diag}(0_{m\times m}, I_\ell) = W \cdot \operatorname{diag}(I_m, 0_{\ell\times\ell}) \cdot \operatorname{diag}(0_{m\times m}, I_\ell) = 0_{q\times q}$ and, analogously, $P_{\mathscr{V}}\,\psi_\# = 0_{q\times q}$.

(b2) Let $[\phi; \psi] \in \mathscr{P}^{(q,q)}_{-\tilde{J}_q,\geq}(\mathbb{C} \setminus [\alpha, \infty))$ be such that (231) holds true. According to Remark 8.3, there are a discrete subset $\mathscr{D}$ of $\mathbb{C} \setminus [\alpha, \infty)$ and a matrix-valued function g meromorphic in $\mathbb{C} \setminus [\alpha, \infty)$ such that ϕ, ψ, and g are holomorphic in $\mathbb{C}\setminus([\alpha, \infty)\cup\mathscr{D})$ and that $\det g(z) \neq 0$ as well as $\phi(z) = W \cdot \operatorname{diag}(0_{m\times m}, I_\ell) \cdot g(z)$ and $\psi(z) = W \cdot \operatorname{diag}(I_m, 0_{\ell\times\ell}) \cdot g(z)$ hold true for each $z \in \mathbb{C}\setminus([\alpha, \infty)\cup\mathscr{D})$. Because of (229) and $WW^* = I_q$, for each $z \in \mathbb{C} \setminus ([\alpha, \infty) \cup \mathscr{D})$, we get

$$\begin{aligned} P_{\mathscr{U}}\,\phi(z) &= WW^* P_{\mathscr{U}}\, W \cdot \operatorname{diag}(0_{m\times m}, I_\ell) \cdot g(z) \\ &= W \cdot \operatorname{diag}(I_m, 0_{\ell\times\ell}) \cdot \operatorname{diag}(0_{m\times m}, I_\ell) \cdot g(z) = 0 \end{aligned}$$

and, analogously $P_{\mathscr{V}}\,\psi(z) = 0_{q\times q}$. This implies (232).

(b3) Let $[\phi; \psi] \in \mathscr{P}^{(q,q)}_{-\tilde{J}_q,\geq}(\mathbb{C} \setminus [\alpha, \infty))$ be such that (232) holds true. According to Lemma 8.11, we see that the function $\det(\psi - \mathrm{i}\phi)$ does not vanish identically. Let $F := (\psi + \mathrm{i}\phi)(\psi - \mathrm{i}\phi)^{-1}$. Lemma 8.11 shows that there is a discrete subset $\mathscr{D}$ of $\mathbb{C} \setminus [\alpha, \infty)$ such that the following three conditions are fulfilled:

(i) F is holomorphic in $\Pi_+ \cup [\mathbb{C} \setminus ([\alpha, \infty) \cup \mathscr{D})]$.
(ii) The matrix-valued functions ϕ, ψ, and $(\psi - \mathrm{i}\phi)^{-1}$ are holomorphic in $\Pi_+ \cup [\mathbb{C} \setminus ([\alpha, \infty) \cup \mathscr{D})]$.
(iii) $\det[\psi(z) - \mathrm{i}\phi(z)] \neq 0$ as well as (125) and (126) hold true for all $z \in \mathbb{C} \setminus ([\alpha, \infty) \cup \mathscr{D})$.

Obviously, because of (i), the functions $\tilde{\phi} := \frac{\mathrm{i}}{2}(I_q - F)W$ and $\tilde{\psi} := \frac{1}{2}(I_q + F)W$ are meromorphic in $\mathbb{C} \setminus [\alpha, \infty)$ and holomorphic in $\Pi_+ \cup [\mathbb{C} \setminus ([\alpha, \infty) \cup \mathscr{D})]$. In view of (ii), the functions ϕ, ψ, $\tilde{\phi}$, $\tilde{\psi}$, and $(\psi - \mathrm{i}\phi)^{-1} W$ are holomorphic in $\Pi_+ \cup [\mathbb{C} \setminus ([\alpha, \infty) \cup \mathscr{D})]$. From (iii) we see that

$$\tilde{\phi}(z) = \phi(z)[\psi(z) - \mathrm{i}\phi(z)]^{-1} W \quad \text{and} \quad \tilde{\psi}(z) = \psi(z)[\psi(z) - \mathrm{i}\phi(z)]^{-1} W \tag{233}$$

hold true for each $z \in \mathbb{C} \setminus ([\alpha, \infty) \cup \mathscr{D})$. In view of (ii), the matrix-valued functions $(\psi + \mathrm{i}\phi)$ and $(\psi - \mathrm{i}\phi)^{-1} W$ are meromorphic in $\mathbb{C} \setminus [\alpha, \infty)$. Since the matrix W is unitary, for each $z \in \mathbb{C} \setminus ([\alpha, \infty) \cup \mathscr{D})$, we have $\det([\psi(z) - \mathrm{i}\phi(z)]^{-1} W) \neq 0$ by (iii). Consequently, (233) and Remark 8.2 imply that $[\tilde{\phi}; \tilde{\psi}]$ belongs to $\mathscr{P}^{(q,q)}_{-\tilde{J}_q,\geq}(\mathbb{C} \setminus [\alpha, \infty))$. Moreover, from Remark 8.3 we get

$$\langle[\tilde{\phi}; \tilde{\psi}]\rangle = \langle[\phi; \psi]\rangle. \tag{234}$$

By virtue of $W^*W = I_q$, (233), (229), and (232), we conclude

$$\begin{aligned}(I_m, 0_{m\times\ell})(I_q - W^*FW) &= (I_m, 0_{m\times\ell})W^*(I_q - F)W = -2\mathrm{i}(I_m, 0_{m\times\ell})W^*\tilde{\phi}\\ &= -2\mathrm{i}(I_m, 0_{m\times\ell})W^*\phi(\psi - \mathrm{i}\phi)^{-1}W\\ &= -2\mathrm{i}(I_m, 0_{m\times\ell}) \cdot \operatorname{diag}(I_m, 0_{\ell\times\ell}) \cdot W^*\phi(\psi-\mathrm{i}\phi)^{-1}W\\ &= -2\mathrm{i}(I_m, 0_{m\times\ell})W^*P_{\mathscr{U}}\,\phi(\psi - \mathrm{i}\phi)^{-1}W\\ &= -2\mathrm{i}(I_m, 0_{m\times\ell})W^*0_{q\times q}(\psi - \mathrm{i}\phi)^{-1}W = 0_{m\times q}\end{aligned} \tag{235}$$

and, analogously,

$$(0_{\ell\times m}, I_\ell)(I_q + W^*FW) = 0_{\ell\times q}. \tag{236}$$

Because of (i), we see that $G := W^*FW$ is a matrix-valued function which is meromorphic in $\mathbb{C} \setminus [\alpha, \infty)$ and holomorphic in $\Pi_+ \cup [\mathbb{C} \setminus ([\alpha, \infty) \cup \mathscr{D})]$. From (235) and (236) we obtain $G(w) = \operatorname{diag}(I_m, -I_\ell)$ for each $w \in \Pi_+$. Hence, $G = \operatorname{diag}(I_m, -I_\ell)$ by the identity theorem for holomorphic functions. Thus, since the matrix W is unitary, this implies $F = W \cdot \operatorname{diag}(I_m, -I_\ell) \cdot W^*$. Then

$$\tilde{\phi} = \frac{\mathrm{i}}{2}(I_q - F)W = \frac{\mathrm{i}}{2}\big[I_q - W \cdot \operatorname{diag}(I_m, -I_\ell) \cdot W^*\big]W = W \cdot \operatorname{diag}(0_{m\times m}, \mathrm{i}I_\ell) \tag{237}$$

and, analogously, $\tilde{\psi} = W \cdot \operatorname{diag}(I_m, 0_{\ell\times\ell})$. Since $\tilde{\phi}$ and $\tilde{\psi}$ are holomorphic in $\Pi_+ \cup [\mathbb{C} \setminus ([\alpha, \infty) \cup \mathscr{D})]$, the matrix-valued functions $\phi_\square := \tilde{\phi} \cdot \operatorname{diag}(I_m, -\mathrm{i}I_\ell)$ and $\psi_\square := \tilde{\psi} \cdot \operatorname{diag}(I_m, -\mathrm{i}I_\ell)$ are holomorphic in $\Pi_+ \cup [\mathbb{C} \setminus ([\alpha, \infty) \cup \mathscr{D})]$. From $\det(I_m, -\mathrm{i}I_\ell) \neq 0$, Remark 8.2, Remark 8.3, and (234) we get

$$[\phi_\square; \psi_\square] \in \mathscr{P}^{(q,q)}_{-\tilde{J}_q, \geq}(\mathbb{C} \setminus [\alpha, \infty)) \quad \text{and} \quad \langle[\phi_\square; \psi_\square]\rangle = \langle[\tilde{\phi}; \tilde{\psi}]\rangle = \langle[\phi; \psi]\rangle. \tag{238}$$

Because of (237) and (230), we have $\phi_\square = \tilde{\phi} \cdot \operatorname{diag}(I_m, -\mathrm{i}I_\ell) = W \cdot \operatorname{diag}(0_{m\times m}, \mathrm{i}I_\ell) \cdot \operatorname{diag}(I_m, -\mathrm{i}I_\ell) = \phi_\#$. Analogously, $\tilde{\psi} = W \cdot \operatorname{diag}(I_m, 0_{\ell\times\ell})$ and (230) imply $\psi_\square = \psi_\#$. Thus, (231) follows from (238).

Lemma 16.2 *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N} \cup \{\infty\}$, and let $(s_j)_{j=0}^{\kappa} \in \mathscr{K}^{\geq,\mathrm{e}}_{q,\kappa,\alpha}$. Let $n \in \mathbb{N}_0$ be such that $2n+1 \leq \kappa$. Suppose that the integers m and ℓ given by (211) and (212) fulfill $m + \ell = q$, $m \geq 1$, and $\ell \geq 1$. Let $\mathscr{U}_{n,\alpha}$ and $\mathscr{V}_{n,\alpha}$ be given by (207) and (208). Then:*

(a) There exists a unitary complex $q \times q$ matrix W such that

$$W^* P_{\mathscr{U}_{n,\alpha}} W = \operatorname{diag}(I_m, 0_{\ell\times\ell}) \quad \text{and} \quad W^* P_{\mathscr{V}_{n,\alpha}} W = \operatorname{diag}(0_{m\times m}, I_\ell). \tag{239}$$

(b) Let W be a unitary complex $q \times q$ matrix such that (239) holds true. Furthermore, let $\phi_\square$ and $\psi_\square$ be the matrix-valued functions defined on $\mathbb{C} \setminus [\alpha, \infty)$ given by $\phi_\square(z) := W \cdot \operatorname{diag}(0_{m\times m}, I_\ell)$ and $\psi_\square(z) := W \cdot \operatorname{diag}(I_m, 0_{\ell\times\ell})$ for all $z \in \mathbb{C} \setminus [\alpha, \infty)$. Then:

(b1) $[\phi_\square; \psi_\square] \in \mathscr{P}^{(q,q)}_{-\tilde{J}_q,\geq}(\mathbb{C} \setminus [\alpha, \infty))$.

(b2) Each pair $[\phi; \psi] \in \mathscr{P}^{(q,q)}_{-\tilde{J}_q,\geq}(\mathbb{C} \setminus [\alpha, \infty))$ with

$$\langle[\phi; \psi]\rangle = \langle[\phi_\square; \psi_\square]\rangle \tag{240}$$

belongs to $\mathscr{P}^{(q,q)}_{-\tilde{J}_q,\geq}[\mathbb{C} \setminus [\alpha, \infty), (s_j)_{j=0}^{2n+1}]$.

(b3) Each $[\phi; \psi] \in \mathscr{P}^{(q,q)}_{-\tilde{J}_q,\geq}[\mathbb{C} \setminus [\alpha, \infty), (s_j)_{j=0}^{2n+1}]$ fulfills (240).

Using Lemmas 14.1 and 16.1 the proof is straightforward (see also [34, Lemma 12.9]).

Remark 16.3 Let W be a non-singular complex $q \times q$ matrix and let $\mathscr{W}$ be the constant function with value W defined on $\mathbb{C} \setminus [\alpha, \infty)$. Then the pairs $[0_{q\times q}; \mathscr{W}]$ and $[\mathscr{W}; 0_{q\times q}]$ belong to $\mathscr{P}^{(q,q)}_{-\tilde{J}_q,\geq}(\mathbb{C} \setminus [\alpha, \infty))$. Furthermore we have:

(a) Each pair $[\phi; \psi] \in \mathscr{P}^{(q,q)}_{-\tilde{J}_q,\geq}(\mathbb{C} \setminus [\alpha, \infty))$ with $\langle[\phi; \psi]\rangle = \langle[0_{q\times q}; \mathscr{W}]\rangle$ fulfills $\phi = 0_{q\times q}$. Conversely, if $[\phi; \psi] \in \mathscr{P}^{(q,q)}_{-\tilde{J}_q,\geq}(\mathbb{C} \setminus [\alpha, \infty))$ is such that $\phi = 0_{q\times q}$ holds true, then $\det \psi$ does not vanish identically and $\langle[\phi; \psi]\rangle = \langle[0_{q\times q}; \mathscr{W}]\rangle$ is valid.

(b) Each pair $[\phi; \psi] \in \mathscr{P}^{(q,q)}_{-\tilde{J}_q,\geq}(\mathbb{C} \setminus [\alpha, \infty))$ with $\langle[\phi; \psi]\rangle = \langle[\mathscr{W}; 0_{q\times q}]\rangle$ fulfills $\psi = 0_{q\times q}$. Conversely, if $[\phi; \psi] \in \mathscr{P}^{(q,q)}_{-\tilde{J}_q,\geq}(\mathbb{C} \setminus [\alpha, \infty))$ is such that $\psi = 0_{q\times q}$ holds true, then $\det \phi$ does not vanish identically and $\langle[\phi; \psi]\rangle = \langle[\mathscr{W}; 0_{q\times q}]\rangle$ is valid.

Lemma 16.4 *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N} \cup \{\infty\}$, and let $(s_j)_{j=0}^{\kappa} \in \mathscr{K}^{\geq,\mathrm{e}}_{q,\kappa,\alpha}$. Let $n \in \mathbb{N}_0$ be such that $2n + 1 \leq \kappa$. Suppose that ℓ given by (212) fulfills $\ell = q$. Then m given by (211) fulfills $m = 0$ and:*

(a) $\mathscr{V}_{n,\alpha}$ defined by (208) fulfills $\mathscr{V}_{n,\alpha} = \mathbb{C}^q$ and, in particular, $P_{\mathscr{V}_{n,\alpha}} = I_q$.
(b) Let W be a non-singular complex $q \times q$ matrix and let $\mathscr{W}$ be the constant function with value W defined on $\mathbb{C} \setminus [\alpha, \infty)$. Then $\begin{bmatrix} \mathscr{W} \\ 0_{q\times q} \end{bmatrix}$ belongs to $\mathscr{P}^{(q,q)}_{-\tilde{J}_q,\geq}(\mathbb{C} \setminus [\alpha, \infty))$ and each pair $[\phi; \psi] \in \mathscr{P}^{(q,q)}_{-\tilde{J}_q,\geq}(\mathbb{C} \setminus [\alpha, \infty))$ with $\langle[\phi; \psi]\rangle = \langle[\mathscr{W}; 0_{q\times q}]\rangle$ belongs to the class $\mathscr{P}^{(q,q)}_{-\tilde{J}_q,\geq}[\mathbb{C} \setminus [\alpha, \infty), (s_j)_{j=0}^{2n+1}]$.

(c) If $[\phi;\psi] \in \mathscr{P}^{(q,q)}_{-\tilde{J}_q,\geq}(\mathbb{C}\setminus[\alpha,\infty))$ *fulfills* (189)*, then* $\langle[\phi;\psi]\rangle = \langle[\mathscr{W};0_{q\times q}]\rangle$ *holds true.*

Using Lemma 14.1 and Remark 16.3 the proof is straightforward (see also [34, Lemma 12.11]).

Lemma 16.5 *Let* $\alpha \in \mathbb{R}$*, let* $\kappa \in \mathbb{N}\cup\{\infty\}$*, and let* $(s_j)_{j=0}^{\kappa} \in \mathscr{K}^{\geq,\mathrm{e}}_{q,\kappa,\alpha}$*. Let* $n \in \mathbb{N}_0$ *be such that* $2n+1 \leq \kappa$*. Suppose that* m *given by* (211) *fulfills* $m = q$*. Let* $\mathscr{U}_{n,\alpha}$ *be defined by* (207)*. Then* ℓ *given by* (212) *fulfills* $\ell = 0$ *and the following statements hold true:*

(a) $\mathscr{U}_{n,\alpha} = \mathbb{C}^q$ *and, in particular,* $P_{\mathscr{U}_{n,\alpha}} = I_q$*.*
(b) Let W *be a non-singular complex* $q\times q$ *matrix and let* $\mathscr{W}$ *be the constant function with value* W *defined on* $\mathbb{C}\setminus[\alpha,\infty)$*. Then* $[0_{q\times q};\mathscr{W}] \in \mathscr{P}^{(q,q)}_{-\tilde{J}_q,\geq}(\mathbb{C}\setminus[\alpha,\infty))$ *and each pair* $[\phi;\psi] \in \mathscr{P}^{(q,q)}_{-\tilde{J}_q,\geq}(\mathbb{C}\setminus[\alpha,\infty))$ *with* $\langle[\phi;\psi]\rangle = \langle[0_{q\times q};\mathscr{W}]\rangle$ *belongs to the class* $\mathscr{P}^{(q,q)}_{-\tilde{J}_q,\geq}[\mathbb{C}\setminus[\alpha,\infty),(s_j)_{j=0}^{2n+1}]$*.*
(c) If $[\phi;\psi] \in \mathscr{P}^{(q,q)}_{-\tilde{J}_q,\geq}(\mathbb{C}\setminus[\alpha,\infty))$ *fulfills* (188)*, then* $\langle[\phi;\psi]\rangle = \langle[0_{q\times q};\mathscr{W}]\rangle$ *holds true.*

The proof is straightforward (see also [34, Lemma 12.12]).

Theorem 16.6 *Let* $\alpha \in \mathbb{R}$*, let* $\kappa \in \mathbb{N}\cup\{\infty\}$*, let* $(s_j)_{j=0}^{\kappa} \in \mathscr{K}^{\geq,\mathrm{e}}_{q,\kappa,\alpha}$*, and let* $n \in \mathbb{N}_0$ *be such that* $2n+1 \leq \kappa$*. Suppose that the integers* m *and* ℓ *given by* (211) *and* (212) *fulfill* $m+\ell = q$*. Then:*

(a) Suppose $m \geq 1$ *and* $\ell \geq 1$*. Let* W *be a unitary complex* $q\times q$ *matrix such that the equations in* (239) *hold true where* $\mathscr{U}_{n,\alpha}$ *and* $\mathscr{V}_{n,\alpha}$ *are given by* (207) *and* (208)*. Let* $\mathbb{U} := [U, 0_{q\times\ell}]$ *and* $\mathbb{V} := [0_{q\times m}, V]$ *be built with the* $q\times m$ *block* U *and the* $q\times\ell$ *block* V *from the block partition* $W = [U,V]$ *of* W*. Then the function* $\det(\hat{\Theta}^{(2,1)}_{n,\alpha}\mathbb{V} + \hat{\Theta}^{(2,2)}_{n,\alpha}\mathbb{U})$ *does not vanish identically and* $\mathscr{S}_{0,q;[\alpha,\infty)}[(s_j)_{j=0}^{2n+1},\leq]$ *consists of exactly one element, namely the matrix-valued function* $S := (\hat{\Theta}^{(1,1)}_{n,\alpha}\mathbb{V} + \hat{\Theta}^{(1,2)}_{n,\alpha}\mathbb{U})(\hat{\Theta}^{(2,1)}_{n,\alpha}\mathbb{V} + \hat{\Theta}^{(2,2)}_{n,\alpha}\mathbb{U})^{-1}$*.*
(b) Suppose $m = 0$*. Then the function* $\det(\hat{\Theta}^{(2,1)}_{n,\alpha})$ *does not vanish identically and* $\mathscr{S}_{0,q;[\alpha,\infty)}[(s_j)_{j=0}^{2n+1},\leq]$ *consists of exactly one element, namely the matrix-valued function* $S := \hat{\Theta}^{(1,1)}_{n,\alpha}(\hat{\Theta}^{(2,1)}_{n,\alpha})^{-1}$*.*
(c) Suppose $\ell = 0$*. Then the function* $\det(\hat{\Theta}^{(2,2)}_{n,\alpha})$ *does not vanish identically and* $\mathscr{S}_{0,q;[\alpha,\infty)}[(s_j)_{j=0}^{2n+1},\leq]$ *consists of exactly one element, namely the matrix-valued function* $S := \hat{\Theta}^{(1,2)}_{n,\alpha}(\hat{\Theta}^{(2,2)}_{n,\alpha})^{-1}$*.*

Using Lemmas 16.2, 16.4, and 16.5 and Theorem 13.7 the proof is straightforward (see also [34, Theorem 12.13]).

Remark 16.7 Under the assumptions of Theorem 16.6, we see from Theorem 16.6, [27, Theorems 6.5 and 6.4], [26, Definition 4.10], and [32, Theorem 5.1] that S given in Theorem 16.6 is exactly the $[\alpha,\infty)$-Stieltjes transform of the restriction onto $\mathfrak{B}_{[\alpha,\infty)}$ of the completely degenerate non-negative Hermitian measure corresponding to $(s_j)_{j=0}^{2n+1}$.

17 A Particular Generalized Inverse of a Complex Matrix

In this section, we state some useful identities for the particular generalized inverse of a Hermitian complex matrix, which is introduced in Remark 6.8.

Remark 17.1 Let $A \in \mathbb{C}^{p\times q}$ and let $B \in \mathbb{C}^{p\times r}$. Then $\mathscr{R}(B) \subseteq \mathscr{R}(A)$ if and only if $AA^\dagger B = B$.

Remark 17.2 If $A \in \mathbb{C}^{p\times q}$, then $\mathscr{N}(A) = \mathscr{R}(A^*)^\perp$ and $\mathscr{R}(A) = \mathscr{N}(A^*)^\perp$.

Lemma 17.3 *Let A be a Hermitian complex $q \times q$ matrix and let $\mathscr{U}$ be a subspace of $\mathbb{C}^q$ such that $\mathscr{N}(A) \dotplus \mathscr{U} = \mathbb{C}^q$. Then $(A^-_{\mathscr{U}})^* = A^-_{\mathscr{U}}$, $\mathscr{R}(A^-_{\mathscr{U}}) = \mathscr{U}$, $\mathscr{N}(A^-_{\mathscr{U}}) = \mathscr{U}^\perp$,*

$$AA^-_{\mathscr{U}}A = A, \qquad\qquad A^-_{\mathscr{U}}AA^-_{\mathscr{U}} = A^-_{\mathscr{U}}, \tag{241}$$

$$\dim \mathscr{R}(A^-_{\mathscr{U}}) = \operatorname{rank} A, \qquad \textit{and} \qquad \dim \mathscr{N}(A^-_{\mathscr{U}}) = q - \operatorname{rank} A. \tag{242}$$

In particular, if A is non-negative Hermitian, then $A^-_{\mathscr{U}}$ is non-negative Hermitian, too.

Proof By definition of $A^-_{\mathscr{U}}$, we have (241) as well as $\mathscr{R}(A^-_{\mathscr{U}}) = \mathscr{U}$ and $\mathscr{N}(A^-_{\mathscr{U}}) = \mathscr{U}^\perp$. In view of $\mathscr{N}(A) \dotplus \mathscr{U} = \mathbb{C}^q$, then (242) follows. Since $A^* = A$ is supposed, (241) implies $A(A^-_{\mathscr{U}})^*A = A$ and $(A^-_{\mathscr{U}})^*A(A^-_{\mathscr{U}})^* = (A^-_{\mathscr{U}})^*$. Moreover, $\mathscr{N}((A^-_{\mathscr{U}})^*) = \mathscr{R}(A^-_{\mathscr{U}})^\perp = \mathscr{U}^\perp$ and $\mathscr{R}((A^-_{\mathscr{U}})^*) = \mathscr{N}(A^-_{\mathscr{U}})^\perp = (\mathscr{U}^\perp)^\perp = \mathscr{U}$. Consequently, $(A^-_{\mathscr{U}})^* = A^{(1,2)}_{\mathscr{U},\mathscr{U}^\perp} = A^-_{\mathscr{U}}$.

Remark 17.4 Let $A \in \mathbb{C}^{q\times q}_{\mathrm{H}}$ and let $\mathscr{U}$ be a subspace of $\mathbb{C}^q$ such that $\mathscr{N}(A) \dotplus \mathscr{U} = \mathbb{C}^q$. Then $AA^\dagger = A^\dagger A$ and, in view of Lemma 17.3, one can easily check that

$$(A^-_{\mathscr{U}}A)^* = AA^-_{\mathscr{U}}, \ (AA^-_{\mathscr{U}})^* = A^-_{\mathscr{U}}A, \ (A^-_{\mathscr{U}}A)^2 = A^-_{\mathscr{U}}A, \ (AA^-_{\mathscr{U}})^2 = AA^-_{\mathscr{U}},$$

$$\mathscr{R}(A^-_{\mathscr{U}}A) = \mathscr{U}, \ \mathscr{R}(AA^-_{\mathscr{U}}) = \mathscr{R}(A), \ \mathscr{N}(AA^-_{\mathscr{U}}) = \mathscr{U}^\perp, \ \mathscr{N}(A^-_{\mathscr{U}}A) = \mathscr{N}(A),$$

$$\dim \mathscr{R}(A^-_{\mathscr{U}}A) = \dim \mathscr{R}(AA^-_{\mathscr{U}}) = \operatorname{rank} A,$$

$$\dim \mathscr{N}(AA^-_{\mathscr{U}}) = \dim \mathscr{N}(A^-_{\mathscr{U}}A) = q - \operatorname{rank} A,$$

$$A^\dagger AA^-_{\mathscr{U}}A = A^\dagger A = AA^\dagger = AA^\dagger A^-_{\mathscr{U}}A, \ \text{and} \ AA^-_{\mathscr{U}}AA^\dagger = AA^\dagger = A^\dagger A = AA^-_{\mathscr{U}}A^\dagger A.$$

Parts of the following result are already contained in [7, Lemma 2.3].

Lemma 17.5 *Let A be a Hermitian complex $q \times q$ matrix and let $\mathscr{U}$ be a subspace of $\mathbb{C}^q$ such that $\mathscr{N}(A) \dotplus \mathscr{U} = \mathbb{C}^q$. Then*

$$(I_q - AA^-_{\mathscr{U}})^* = I_q - A^-_{\mathscr{U}} A, \qquad (I_q - AA^-_{\mathscr{U}})^2 = I_q - AA^-_{\mathscr{U}},$$
$$\mathscr{N}(I_q - AA^-_{\mathscr{U}}) = \mathscr{R}(A), \qquad \mathscr{R}(I_q - AA^-_{\mathscr{U}}) = \mathscr{U}^\perp,$$
$$(I_q - AA^-_{\mathscr{U}})AA^\dagger = 0 \qquad (I_q - AA^-_{\mathscr{U}})A^\dagger A = 0,$$
$$(I_q - AA^-_{\mathscr{U}})(I_q - AA^\dagger) = I_q - AA^-_{\mathscr{U}}, \quad \textit{and} \quad (I_q - AA^-_{\mathscr{U}})(I_q - A^\dagger A) = I_q - AA^-_{\mathscr{U}}.$$

Proof The first two equations follow from Remark 17.4. From Lemma 17.3 we know that (241) is true. Using (241), the equation $\mathscr{N}(I_q - AA^-_{\mathscr{U}}) = \mathscr{R}(A)$ can be easily checked by straightforward calculations. In order to prove $\mathscr{R}(I_q - AA^-_{\mathscr{U}}) = \mathscr{U}^\perp$, one shows that (241) implies $\mathscr{R}(I_q - AA^-_{\mathscr{U}}) = \mathscr{N}(AA^-_{\mathscr{U}})$ and one applies the equation $\mathscr{N}(AA^-_{\mathscr{U}}) = \mathscr{U}^\perp$ stated in Remark 17.4. Because of $A^* = A$, we have $AA^\dagger = A^\dagger A$. Thus, from (241) we easily see that the remaining equations hold true.

Lemma 17.6 *Let A be a Hermitian complex $q \times q$ matrix and let $\mathscr{U}$ be a subspace of $\mathbb{C}^q$ such that $\mathscr{N}(A) \dotplus \mathscr{U} = \mathbb{C}^q$. Then*

$$\mathscr{R}(I_q - A^-_{\mathscr{U}} A) = \mathscr{N}(A^-_{\mathscr{U}} A) = \mathscr{N}(A), \quad \mathscr{N}(I_q - A^-_{\mathscr{U}} A) = \mathscr{R}(A^-_{\mathscr{U}} A) = \mathscr{U},$$
$$\dim \mathscr{R}(A^-_{\mathscr{U}} A) = \operatorname{rank} A, \quad A^\dagger AA^-_{\mathscr{U}} A = A^\dagger A = AA^\dagger = AA^\dagger A^-_{\mathscr{U}} A,$$
$$\textit{and} \quad (I_q - A^\dagger A)A^-_{\mathscr{U}} A = A^-_{\mathscr{U}} A - A^\dagger A = A^-_{\mathscr{U}} A - AA^\dagger = (I_q - AA^\dagger)A^-_{\mathscr{U}} A.$$

Proof Because of Lemma 17.3, we get (241). From (241) we obtain $\mathscr{N}(A^-_{\mathscr{U}} A) = \mathscr{N}(A)$ and $\mathscr{R}(A^-_{\mathscr{U}} A) = \mathscr{R}(A^-_{\mathscr{U}}) = \mathscr{U}$. In particular, $\dim \mathscr{R}(A^-_{\mathscr{U}} A) = \dim \mathscr{U} = \dim \mathbb{C}^q - \dim \mathscr{N}(A) = \operatorname{rank} A$. Because of $A^* = A$, we have $AA^\dagger = A^\dagger A$. Therefore, (241) shows that $A^\dagger AA^-_{\mathscr{U}} A = A^\dagger A = AA^\dagger$ and $A^\dagger AA^-_{\mathscr{U}} A = AA^\dagger$. Thus, the remaining equations immediately follow.

Remark 17.7 Let A be a Hermitian complex $q \times q$ matrix and let $\mathscr{U}$ be a subspace of $\mathbb{C}^q$ such that $\mathscr{N}(A) \dotplus \mathscr{U} = \mathbb{C}^q$. In view of the Lemmas 17.6 and 17.3, it is readily checked that

$$(I_q - A^-_{\mathscr{U}} A)^* = I_q - AA^-_{\mathscr{U}}, \qquad (I_q - A^-_{\mathscr{U}} A)^2 = I_q - A^-_{\mathscr{U}} A,$$
$$\mathscr{R}(I_q - A^-_{\mathscr{U}} A) = \mathscr{N}(A), \qquad \mathscr{N}(I_q - A^-_{\mathscr{U}} A) = \mathscr{U},$$
$$A^\dagger A(I_q - A^-_{\mathscr{U}} A) = 0, \qquad AA^\dagger(I_q - A^-_{\mathscr{U}} A) = 0,$$
$$(I_q - A^\dagger A)(I_q - A^-_{\mathscr{U}} A) = I_q - A^-_{\mathscr{U}} A, \quad \text{and} \quad (I_q - AA^\dagger)(I_q - A^-_{\mathscr{U}} A) = I_q - A^-_{\mathscr{U}} A.$$

Remark 17.8 Let $T \in \mathbb{C}^{q \times q}$ and let $\mathscr{U}$ and $\mathscr{V}$ be subspaces of $\mathbb{C}^q$ with $T^*(\mathscr{U}) \subseteq \mathscr{V} \subseteq \mathscr{U}$. Then it is readily checked that $(T^*)^k(\mathscr{U}) \subseteq \mathscr{V} \subseteq \mathscr{U}$ and $T^k(\mathscr{V}^\perp) \subseteq \mathscr{U}^\perp \subseteq \mathscr{V}^\perp$ is valid for each $k \in \mathbb{N}$ and that $(T^*)^\ell(\mathscr{V}) \subseteq \mathscr{U}$ and $T^\ell(\mathscr{U}^\perp) \subseteq$

$\mathscr{U}^{\perp} \subseteq \mathscr{V}^{\perp}$ for each $\ell \in \mathbb{N}_0$ hold true (see also [7, Corollary 3.3], where a special case is discussed).

The following lemma is a generalization of [7, Lemma 4.1], where special pairs of block Hankel matrices are considered.

Lemma 17.9 *Let A and B be Hermitian complex $q \times q$ matrices and let $T \in \mathbb{C}^{q\times q}$. Suppose that $\mathscr{U}$ and $\mathscr{V}$ are subspaces of $\mathbb{C}^q$ such that $\mathscr{N}(A) \dotplus \mathscr{U} = \mathbb{C}^q$ and $\mathscr{N}(B) \dotplus \mathscr{V} = \mathbb{C}^q$ and $T^*(\mathscr{U}) \subseteq \mathscr{V} \subseteq \mathscr{U}$ hold true. Then*

$$A_{\mathscr{U}}^{-} T^{\ell}(I_q - AA_{\mathscr{U}}^{-}) = 0 \qquad \textit{and} \qquad B_{\mathscr{V}}^{-} T^{\ell}(I_q - AA_{\mathscr{U}}^{-}) = 0 \tag{243}$$

for each $\ell \in \mathbb{N}_0$ and, for each $k \in \mathbb{N}$, furthermore

$$A_{\mathscr{U}}^{-} T^{k}(I_q - BB_{\mathscr{V}}^{-}) = 0 \qquad \textit{and} \qquad B_{\mathscr{V}}^{-} T^{k}(I_q - BB_{\mathscr{V}}^{-}) = 0. \tag{244}$$

Proof Lemma 17.5 yields $\mathscr{R}(I_q - AA_{\mathscr{U}}^{-}) = \mathscr{U}^{\perp}$. Hence, from Remark 17.8 we conclude

$$T^{\ell}\big(\mathscr{R}(I_q - AA_{\mathscr{U}}^{-})\big) = T^{\ell}(\mathscr{U}^{\perp}) \subseteq \mathscr{U}^{\perp} \subseteq \mathscr{V}^{\perp} \tag{245}$$

for each $\ell \in \mathbb{N}_0$. Since we know from Lemma 17.3 that $\mathscr{N}(A_{\mathscr{U}}^{-}) = U^{\perp}$ is valid, it follows $T^{\ell}(I_q - AA_{\mathscr{U}}^{-})x \in \mathscr{N}(A_{\mathscr{U}}^{-})$ for each $\ell \in \mathbb{N}_0$ and each $x \in \mathbb{C}^q$. Consequently, the first equation in (243) is fulfilled for each $\ell \in \mathbb{N}_0$. Lemma 17.3 yields $\mathscr{N}(B_{\mathscr{V}}^{-}) = \mathscr{V}^{\perp}$. Thus, we obtain from (245) that $T^{\ell}(I_q - AA_{\mathscr{U}}^{-})x \in \mathscr{N}(B_{\mathscr{V}}^{-})$ is fulfilled for every choice of ℓ in $\mathbb{N}_0$ and x in $\mathbb{C}^q$. Therefore, the second equation in (243) is proved for each $\ell \in \mathbb{N}_0$. Lemma 17.5 provides us $\mathscr{R}(I_q - BB_{\mathscr{V}}^{-}) = \mathscr{V}^{\perp}$. Hence, Remark 17.8 yields

$$T^{k}\big(\mathscr{R}(I_q - BB_{\mathscr{V}}^{-})\big) = T^{k}(\mathscr{V}^{\perp}) \subseteq \mathscr{U}^{\perp} \subseteq \mathscr{V}^{\perp} \tag{246}$$

for each $k \in \mathbb{N}$. Since Lemma 17.3 shows that $\mathscr{N}(A_{\mathscr{U}}^{-}) = \mathscr{U}^{\perp}$ is true, we obtain then $T^{k}(I_q - BB_{\mathscr{V}}^{-})x \in \mathscr{N}(A_{\mathscr{U}}^{-})$ for every choice of $k \in \mathbb{N}$ and x in $\mathbb{C}^q$. Consequently, the first equation in (244) is true for each $k \in \mathbb{N}$. Using (246) and the equation $\mathscr{N}(B_{\mathscr{V}}^{-}) = \mathscr{V}^{\perp}$, which is proved in Lemma 17.3, we get $T^{k}(I_q - BB_{\mathscr{V}}^{-})x \in \mathscr{N}(B_{\mathscr{V}}^{-})$ for each $k \in \mathbb{N}$ and each $x \in \mathbb{C}^q$. Thus, the second equation in (244) is verified for each $k \in \mathbb{N}$ as well.

Now we state some more or less known identities for the matrix-valued functions defined in Remark 4.1.

Remark 17.10 Let $n \in \mathbb{N}_0$ and let $w, z \in \mathbb{C}$. Then one can easily see that the equations

$$R_{T_{q,n}}(z)(I_{(n+1)q} - wT_{q,n}) = (I_{(n+1)q} - wT_{q,n})R_{T_{q,n}}(z),$$

$$R_{T^*_{q,n}}(z)(I_{(n+1)q} - wT^*_{q,n}) = (I_{(n+1)q} - wT^*_{q,n})R_{T^*_{q,n}}(z),$$
$$R_{T_{q,n}}(z) - R_{T_{q,n}}(w) = (z-w)R_{T_{q,n}}(w)T_{q,n}R_{T_{q,n}}(z),$$
$$R_{T^*_{q,n}}(z) - R_{T^*_{q,n}}(w) = (z-w)R_{T^*_{q,n}}(z)T^*_{q,n}R_{T^*_{q,n}}(w),$$
$$\left[R_{T_{q,n}}(w)\right]^{-1} - \left[R_{T_{q,n}}(z)\right]^{-1} = (z-w)T_{q,n},$$
$$R_{T_{q,n}}(z) + (w-z)R_{T_{q,n}}(z)T_{q,n}R_{T_{q,n}}(w) = R_{T_{q,n}}(w),$$
$$zR_{T_{q,n}}(z) + (w-z)R_{T_{q,n}}(z)R_{T_{q,n}}(w) = wR_{T_{q,n}}(w),$$
$$(z-\overline{w})\left[R_{T^*_{q,n}}(w)\right]^*T_{q,n}R_{T_{q,n}}(z) = R_{T_{q,n}}(z) - \left[R_{T^*_{q,n}}(w)\right]^*,$$
$$(z-w)T_{q,n}R_{T_{q,n}}(z) = R_{T_{q,n}}(z)\left[R_{T_{q,n}}(w)\right]^{-1} - I_{(n+1)q}, \tag{247}$$
$$(z-w)T^*_{q,n}R_{T^*_{q,n}}(z) = \left[R_{T^*_{q,n}}(w)\right]^{-1}R_{T^*_{q,n}}(z) - I_{(n+1)q}, \tag{248}$$

and

$$(z-w)R_{T_{q,n}}(z)T_{q,n} = R_{T_{q,n}}(z)\left[R_{T_{q,n}}(w)\right]^{-1} - I_{(n+1)q} \tag{249}$$

hold true. Furthermore, for each $\ell \in \mathbb{N}_0$, it is readily checked that

$$T^\ell_{q,n}R_{T_{q,n}}(z)=R_{T_{q,n}}(z)T^\ell_{q,n}, \quad R_{T_{q,n}}(z)T^\ell_{q,n}R_{T_{q,n}}(w)=R_{T_{q,n}}(w)T^\ell_{q,n}R_{T_{q,n}}(z),$$

and

$$(T^*_{q,n})^\ell R_{T^*_{q,n}}(z)=R_{T^*_{q,n}}(z)(T^*_{q,n})^\ell, \quad R_{T^*_{q,n}}(z)(T^*_{q,n})^\ell R_{T^*_{q,n}}(w)=R_{T^*_{q,n}}(w)(T^*_{q,n})^\ell R_{T^*_{q,n}}(z).$$

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Formulas and Inequalities for Some Special Functions of a Complex Variable

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Abstract The presented formulas and inequalities are based on interconnection of convolution and hypergeometric properties. Some known transformations and a direct coefficient technique are combined to analyze and structure a parameterized product identity involving three Gauss hypergeometric functions. A convolution presentation of this identity is a generalization of the Bateman and Kapteyn integrals for Bessel functions as well as of the addition theorem for the confluent hypergeometric functions. These results and integral properties of weighted convolutions lead to multiparameter weighted norm inequalities for generalized hypergeometric functions and special functions of hypergeometric type, in particular, for Bessel and Whittaker functions, and Laguerre and Hermite polynomials.

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1 Introduction

Various integral formulas, product identities, and inequalities in terms of the generalized hypergeometric and related special functions play an important role for numerous applications [1, 2, 5, 7–9, 21–34]. Such results focused on the Gauss hypergeometric functions, confluent hypergeometric functions, and Bessel functions are of particular interest from historical, theoretical, and practical point of view.

In the paper, our approach to derive formulas and inequalities is based on interconnection of convolution and hypergeometric properties. First, we combine some known transformations and a direct coefficient technique to analyze and

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structure a parameterized product identity involving three Gauss hypergeometric functions of the type ${}_2F_1(\alpha, \mu; 2\mu; z)$ (Section 2). Then we present this identity in a convolution form, which turns out to be a generalization of the well-known Bateman and Kapteyn convolution formulas for Bessel functions and of the addition theorem for the confluent hypergeometric functions. These results and the earlier theorem on weighted convolutions [14] (Section 3) allow us to derive the multiparameter weighted norm inequalities for some generalized hypergeometric functions and special functions of hypergeometric type, in particular, for Bessel and Whittaker functions, and Laguerre and Hermite polynomials (Sections 4–6, see also [11]).

We use the standard hypergeometric notation (see, e.g., [9, v.I, Chs. 2, 6], [1, Chs. 6, 13, 15]). The Pochhammer symbol $(\alpha)_n$ stands for the shifted factorial:

$$(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1) \ \text{ for } \ n \geq 1 \ \text{ and } \ (\alpha)_0 = 1. \tag{1}$$

For any $j, k = 0, 1, \ldots$, the generalized hypergeometric function ${}_jF_k$ of one complex variable is defined by the power series:

$${}_jF_k(\mu_1, \ldots, \mu_j; \nu_1, \ldots, \nu_k; z) = \sum_{n=0}^{\infty} \frac{\prod_{1\leq l\leq j}(\mu_l)_n}{\prod_{1\leq l\leq k}(\nu_l)_n} \cdot \frac{z^n}{n!}, \tag{2}$$

provided that $(\nu_l)_n \neq 0$ $(n \geq 1, l \leq k)$.
For a given power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $\alpha > 0$, the α-convolution of f denoted by $f_{*\alpha}$ is defined by the formula [12]:

$$f_{*\alpha}(z) = \sum_{n=0}^{\infty} \frac{a_n}{(\alpha)_n} z^n. \tag{3}$$

As usual, $B(\alpha, \beta)$ and $\Gamma(z)$ stand for the beta and gamma functions.

2 An Identity for Three Gauss Hypergeometric Functions

Our analysis of the identity for Gauss hypergeometric functions presented in Proposition 1 involves Euler's transformation [9, v.I], [1, Ch.15, #15.3.3]:

$${}_1F_0(\mu+\nu-\gamma; -; z)\, {}_2F_1(\gamma-\mu, \gamma-\nu; \gamma; z) \ = \ {}_2F_1(\mu, \nu; \gamma; z), \tag{4}$$

two algebraic functions in the ${}_2F_1$ form [9, v.I], [1, Ch. 15, #15.1.13–14]:

$$\left[\frac{1+(1-z)^{1/2}}{2}\right]^{1-2\mu} = {}_2F_1(\mu-1/2, \mu; 2\mu; z)$$
$$= (1-z)^{1/2}\,{}_2F_1(\mu+1/2, \mu; 2\mu; z), \tag{5}$$

and two simple hypergeometric formulas:

$${}_2F_1(2\mu+1, \mu; 2\mu; z) = (1-z)^{-\mu-1}(1-z/2), \tag{6}$$

$${}_2F_1(-3, 3\nu+1; 6\nu+2; z) = 1-3z/2+(3\nu+2)z^2/(4\nu+2)-(\nu+1)z^3/(8\nu+4). \tag{7}$$

Proposition 1 *The identity*

$${}_2F_1(\alpha, \mu; 2\mu; z)\,{}_2F_1(\beta, \nu; 2\nu; z) = {}_2F_1(\alpha+\beta, \gamma; 2\gamma; z), \tag{8}$$

where nonzero α *and* β *are linear functions of* μ *and* ν*:*

$$\alpha = a_1\mu + b_1\nu + c_1 \ \ \textit{and} \ \ \beta = a_2\mu + b_2\nu + c_2 \ \ (a_k, b_k, c_k \textit{ are constants}), \tag{9}$$

holds for any μ, ν, z, *and some* $\gamma = \gamma(\mu, \nu)$ ($2\mu \neq 0, -1, \ldots,$ *if* $\alpha \neq 2\mu$; $2\nu \neq 0, -1, \ldots,$ *if* $\beta \neq 2\nu$; $|z| < 1$) *if and only if* (α, β, γ) *is either one of the following eight combinations of parameters or symmetric to one of those:*

$$\begin{aligned}
&1.\ (2\mu+1, 2\nu, \mu+\nu);\ \ 2.\ (2\mu, -\mu+\nu, \nu);\ \ 3.\ (2\mu,\ 2\nu, \mu+\nu);\\
&4.\ (\mu+1/2, \nu-1/2, \mu+\nu-1/2);\ \ 5.\ (\mu-1/2, \nu-1/2, \mu+\nu-1/2);\\
&6.\ (2\mu, -1, \mu-1);\ \ 7.\ (-1,\ -2,\ 3\nu+1);\ \ 8.\ (-1,\ -1,\ \infty).
\end{aligned} \tag{10}$$

Proof First, by a direct coefficient method we show that combinations of parameters given in (10) or symmetric to one of those describe all the necessary conditions for identity (8). Then we use Euler's transformation (4) and formulas (5), (6), and (7) to complete the proof.

The equality of coefficients for z^2 in (8) implies that

$$\gamma = \left[\frac{(\alpha^2+\alpha)\mu}{2\mu+1} + \frac{(\beta^2+\beta)\nu}{2\nu+1} + \alpha\beta\right] \Big/ \left[\frac{\alpha^2+\alpha}{2\mu+1} + \frac{\beta^2+\beta}{2\nu+1}\right]. \tag{11}$$

The equality of coefficients for z^4 in (8) and formula (11) imply that:

$$
\begin{aligned}
&-\alpha(\alpha+1)^2(\alpha+2)(\mu+1)(2\mu+3)(2\nu+1)^2(2\nu+3)\\
&-\beta(\beta+1)^2(\beta+2)(\nu+1)(2\nu+3)(2\mu+1)^2(2\mu+3)\\
&+2(\alpha+1)(\beta+1)(\alpha\mu+\beta\nu)(2\mu+1)(2\mu+3)(2\nu+1)(2\nu+3)\\
&+(\alpha\beta-2)[\alpha(\alpha+1)(2\mu+1)(2\mu+3)(2\nu+1)^2(2\nu+3)\\
&+\beta(\beta+1)(2\nu+1)(2\nu+3)(2\mu+1)^2(2\mu+3)]\\
&+\alpha(\alpha+1)^2(\beta+1)(\mu^2-5\mu\nu-\mu-3\nu)(2\mu+3)(2\nu+1)(2\nu+3)\\
&+\beta(\beta+1)^2(\alpha+1)(\nu^2-5\mu\nu-3\mu-\nu)(2\nu+3)(2\mu+1)(2\mu+3)\\
&+\alpha(\alpha+1)(\alpha+2)(\alpha+3)(\mu+2)(2\mu+1)(2\nu+1)^2(2\nu+3)\\
&+\beta(\beta+1)(\beta+2)(\beta+3)(\nu+2)(2\nu+1)(2\mu+1)^2(2\mu+3)\\
&+(\alpha+1)(\beta+1)(\mu-\nu)[(\beta+2)(\beta+3)(\nu+2)(2\nu+1)(2\mu+1)(2\mu+3)\\
&-(\alpha+2)(\alpha+3)(\mu+2)(2\mu+1)(2\nu+1)(2\nu+3)]=0.
\end{aligned}
\tag{12}
$$

We use formulas (9) for α and β. Equation (12) with $\nu=0$ implies that $-4a_2^3(a_1+2a_2)\mu^7+O(\mu^6)=0$ as $\mu\to\infty$. Also (12) with $\mu=0$ implies that $-4b_1^3(b_2+2b_1)\nu^7+O(\nu^6)=0$ as $\nu\to\infty$. Hence

$$a_2(a_1+2a_2)=b_1(b_2+2b_1)=0. \tag{13}$$

With $\mu=\nu=0$ in (12) we obtain:

$$c_1^2(1-c_1^2)+c_2^2(1-c_2^2)+3c_1c_2[c_1^2+c_2^2+c_1+c_2]=0. \tag{14}$$

If $\nu=-3/2$ in (12), then for any μ:

$$(\beta+1)(\beta+2)(\beta+3)[\beta(2\mu+1)+(\alpha+1)(\mu+3/2)](2\mu+1)(2\mu+3)=0, \tag{15}$$

where $\alpha=a_1\mu-3b_1/2+c_1$ and $\beta=a_2\mu-3b_2/2+c_2$.

Equation (12) with $\nu=-1/2$ implies that for any μ:

$$\beta(\beta+1)^2[(\alpha+1)(3/2-\mu)-(\beta+2)(2\mu+1)](2\mu+1)(2\mu+3)=0, \tag{16}$$

where $\alpha=a_1\mu-b_1/2+c_1$ and $\beta=a_2\mu-b_2/2+c_2$.

Now let $\mu=-3/2$ in (12), then for any ν:

$$(\alpha+1)(\alpha+2)(\alpha+3)[\alpha(2\nu+1)+(\beta+1)(\nu+3/2)](2\nu+1)(2\nu+3)=0, \tag{17}$$

where $\alpha=-3a_1/2+b_1\nu+c_1$ and $\beta=-3a_2/2+b_2\nu+c_2$.

Equation (12) with $\mu = -1/2$ implies that for any ν:

$$\alpha(\alpha+1)^2[(\beta+1)(3/2-\nu)-(\alpha+2)(2\nu+1)](2\nu+1)(2\nu+3)=0, \tag{18}$$

where $\alpha = a_1\mu - b_1/2 + c_1$ and $\beta = a_2\mu - b_2/2 + c_2$.

The left-hand side in (15) can be equal to 0 for any μ only if at least one of its first four factors equals 0 for any μ. The left-hand side in (16) can be equal to 0 for any μ only if at least one of its first three factors equals 0 for any μ. Using (13) we conclude that (15) and (16) hold only if $a_2 \neq 0$ and $a_1 + 2a_2 = 0$, or $a_2 = 0$. One can make the similar observations for (17) and (18) for any ν.

We show that there is the only admissible combination of parameters (α, β), namely:

$$\alpha = 2\mu, \beta = -\mu + \nu, \tag{19}$$

if $a_2 \neq 0$. Indeed, in this case only the bracketed factors in (15) and (16) can be equal to 0 for any μ, which is possible only if the following four equations hold true:

$$\begin{aligned}
&-3b_2 + 2c_2 + a_1 - 3b_1/2 + c_1 + 1 = 0,\\
&-3b_2 + 2c_2 - 9b_1/2 + 3c_1 + 3 = 0,\\
&2a_1 + b_1/2 - c_1 - 5 + b_2 - 2c_2 = 0,\\
&-3b_1/2 + 3c_1 - 1 + b_2 - 2c_2 = 0.
\end{aligned} \tag{20}$$

Equations in (20) imply that

$$b_1 = 0,\ a_1 = 2c_1 + 2 = 0,\ b_2 = 1 + c_2, \ \text{ and } \ c_2 = 3c_1. \tag{21}$$

Using (21) we obtain that in the considered case $\beta = (3c_1+1)\nu + 9c_1/2 + 3/2$ and $\alpha = -2c_1 - 3$ in (17). Hence the left-hand side in (17) can be equal to 0 for any ν only if c_1 equals 0, -1, or $-1/2$. It follows that only $c_1 = 0$ satisfies equation (14). Thus, if $a_2 \neq 0$ then $a_1 = 2, a_2 = -1, b_1 = c_1 = c_2 = 0, b_2 = 1$ and we obtain (19).

If $b_1 \neq 0$ then we have the case that is symmetric to (19).

Now let $a_2 = b_1 = 0$. If the bracketed factor in (15) equals 0, then $a_1 = 0$ and the first two equations in (20) imply that $c_1 = -1$ and $c_2 = 3b_2/2$. If the bracketed factor in (16) equals 0, then $a_1 = 0$ and the last two equations in (19) imply that $c_1 = -1$ and $c_2 = b_2/2 - 2$. Hence and from (15) it follows that

$$(1 + c_2 - 3b_2/2)(2 + c_2 - 3b_2/2)(3 + c_2 - 3b_2/2) = 0, \tag{22}$$

or

$$a_1 = 0,\ c_2 = 3b_2/2, \text{ and } c_1 = -1. \tag{23}$$

Also from (16) it follows that

$$(c_2 - b_2/2)(1 + c_2 - b_2/2) = 0, \tag{24}$$

or

$$a_1 = 0,\ 4 + 2c_2 = b_2, \text{ and } c_1 = -1. \tag{25}$$

Equations (22)–(25) lead to the following pairs (b_2, c_2):

$$(1, 1/2),\ (2, 1),\ (3, 3/2),\ (0, -1),\ (1, -1/2),\ (2, 0); \tag{26}$$

and, if $a_1 = 0$ and $c_1 = -1$:

$$(-1, -3/2),\ (-1, -5/2),\ (0, -2),\ (1, -3/2),\ (-2, -3). \tag{27}$$

In a similar way we use (17) and (18) to find the possible pairs (a_1, c_1):

$$(1, 1/2),\ (2, 1),\ (3, 3/2),\ (0, -1),\ (1, -1/2),\ (2, 0); \tag{28}$$

and, if $b_2 = 0$ and $c_2 = -1$:

$$(-1, -3/2),\ (-1, -5/2),\ (0, -2),\ (1, -3/2),\ (-2, -3). \tag{29}$$

Equation (14) and the possible values of c_1 and c_2 in (26)–(29) leave us with the following pairs (c_1, c_2):

$$(1/2, -1/2),\ (1, 0),\ (-1, -1),\ (-1, 0),\ (-1/2, -1/2),\ (0, 0),\ (-1, -2) \tag{30}$$

and the pairs which are symmetric to one of them. Then we match pairs (c_1, c_2) given in (30) and the corresponding values of a_1 and b_2 given in (26) and (28). We obtain the following pairs of $\alpha = a_1\mu + c_1$ and $\beta = b_2\nu + c_2$:

$$\begin{gathered} (2\mu + 1, 2\nu),\ (2\mu,\ 2\nu),\ (2\mu, -1), (\mu + 1/2, \nu - 1/2), \\ (\mu - 1/2, \nu - 1/2), (-1,\ -2),\ (-1,\ -1) \end{gathered} \tag{31}$$

and the pairs which are symmetric to one of them.

Formulas (19), (31), and (11) imply that the triples (α, β, γ) described in (10) and those symmetric to them give the necessary conditions for parameters in identity (8). It remains to show that for these triples of parameters identity (8) reduces to the known or easily obtainable results. Indeed, cases 1 and 6, $(2\mu + 1, 2\nu, \mu + \nu)$ and $(2\mu, -1, \mu - 1)$, are based on formula (6):

$$\begin{aligned} {}_2F_1(2\mu + 1, \mu; 2\mu; z)\, {}_2F_1(2\nu, \nu; 2\nu; z) &= (1 - z)^{-\mu-1}(1 - z/2)(1 - z)^{-\nu} \\ &= {}_2F_1(2\mu + 2\nu + 1, \mu + \nu; 2\mu + 2\nu; z) \end{aligned}$$

$$\text{and} \qquad {}_2F_1(2\mu,\mu;2\mu;z)\,{}_2F_1(-1,\nu;2\nu;z) = (1-z)^{-\mu}(1-z/2)$$
$$= {}_2F_1(2\mu-1,\mu-1;2\mu-2;z).$$

Case 2, $(2\mu, -\mu+\nu, \nu)$, is a special case of Euler's transformation (4):

$$ {}_1F_0(\mu;-;z)\,{}_2F_1(-\mu+\nu,\nu;2\nu;z) = {}_2F_1(\mu+\nu,\nu;2\nu;z).$$

Case 3, $(2\mu, 2\nu, \mu+\nu)$, is based on the product of two binomial series:

$$ {}_2F_1(2\mu,\mu;2\mu;z)\,{}_2F_1(2\nu,\nu;2\nu;z) = (1-z)^{-\mu-\nu} = {}_2F_1(2\mu+2\nu,\mu+\nu;2\mu+2\nu;z).$$

Cases 4 and 5, $(\mu+1/2, \nu-1/2, \mu+\nu-1/2)$ and $(\mu-1/2, \nu-1/2, \mu+\nu-1/2)$, are based on formulas (5):

$$ {}_2F_1(\mu+1/2,\mu;2\mu;z)\,{}_2F_1(\nu-1/2,\nu;2\nu;z)$$
$$= (1-z)^{-1/2}[(1+(1-z)^{1/2})/2]^{1-2(\mu+\nu-1/2)} = {}_2F_1(\mu+\nu,\mu+\nu-1/2;2\mu+2\nu-1;z)$$

$$\text{and} \qquad {}_2F_1(\mu-1/2,\mu;2\mu;z)\,{}_2F_1(\nu-1/2,\nu;2\nu;z)$$
$$= [(1+(1-z)^{1/2})/2]^{1-2(\mu+\nu-1/2)} = {}_2F_1(\mu+\nu-1,\mu+\nu-1/2;2\mu+2\nu-1;z).$$

Case 7, $(-1, -2, 3\nu+1)$ is based on formula (7):

$$ {}_2F_1(-1,\mu;2\mu;z)\,{}_2F_1(-2,\nu;2\nu;z) = (1-z/2)(1-z+(\nu+1)z^2/(4\nu+2))$$
$$= 1-3z/2+(3\nu+2)z^2/(4\nu+2)-(\nu+1)z^3/(8\nu+4) = {}_2F_1(-3,3\nu+1;6\nu+2;z).$$

Case 8, $(-1, -1, \infty)$ is generated by the limit:

$$\lim_{\gamma\to\infty} {}_2F_1(-2,\gamma;2\gamma;z) = (1-z+z^2/4) = {}_2F_1(-1,\mu;2\mu;z)\,{}_2F_1(-1,\nu;2\nu;z).$$

□

Remark 1 If $\alpha = 0$ and $\beta \neq 0$, or $\beta = 0$ and $\alpha \neq 0$ then identity (8) is trivial with $\gamma = \nu$ or $\gamma = \mu$, respectively.

Remark 2 The known quadratic transformation [9, v. I, p.111]:

$$ {}_2F_1(a,b;2b;z) = (1-z/2)^{-a}\,{}_2F_1(a/2,a/2+1/2;b+1/2;w),$$

where $w = [z/(2-z)]^2$ allows us to present identity (8) in the following equivalent form:

$$ {}_2F_1(a, a+1/2; c; w)\, {}_2F_1(b, b+1/2; d; w) = {}_2F_1(a+b, a+b+1/2; \lambda; w), $$

where $a = \alpha/2, b = \beta/2, c = \mu + 1/2, d = \nu + 1/2, \lambda = \gamma + 1/2$. One can use this equivalent presentation for an alternative proof of Proposition 1. A standard approach involving two differential equations and the local exponents of both sides of (8) does not seem to lead to a simpler proof of Proposition 1.

Corollary 1 *Let (α, β, γ) be any admissible combination of parameters for identity (8) given in Proposition 1. Then for any complex ω, z, and $u, v > 0$, the following convolution formulas hold:*

$$ \begin{aligned} &B(u, v)\, {}_2F_2(\alpha+\beta, \gamma; u+v, 2\gamma; z) \\ &= \int_0^1 t^{u-1}(1-t)^{v-1}\, {}_2F_2(\alpha, \mu; u, 2\mu; zt)\, {}_2F_2(\beta, \nu; v, 2\nu; z(1-t))dt, \end{aligned} \tag{32} $$

$$ \begin{aligned} &B(u, v)\, {}_2F_3[\alpha+\beta, \gamma; (u+v)/2, (u+v+1)/2, 2\gamma; \omega z^2] \\ &= \int_0^1 t^{u-1}(1-t)^{v-1}\, {}_2F_3[\alpha, \mu; u/2, (u+1)/2, 2\mu; \omega(zt)^2] \\ &\times\, {}_2F_3[\beta, \nu; v/2, (v+1)/2, 2\nu; \omega(z(1-t))^2]dt. \end{aligned} \tag{33} $$

Formula (32) is a generalization of the well-known Bateman integrals for Bessel functions [8, v. II, pp. 354–355]. In particular, it contains the famous integral obtained by H. Bateman in 1905 which is the basis for hundreds of applications [6], [8, v. II, p. 354, (24)], [28] (case 1 in (10)). These Bateman integrals are implied by (32), where (α, β, γ) is any of the first five admissible combinations of parameters in Proposition 1 (or it is symmetric to one of them) and parameters u and v are equal to α and β, respectively (see Corollary 4 in Section 4). Both formulas (32) and (33) are generalizations of the Kapteyn integrals for Bessel and trigonometric functions [34, p. 380, (1); p. 381, (2)], [21, Ch. 2] (see Section 4). Also formula (32) is a generalization of the integral addition theorem for the confluent hypergeometric functions ${}_1F_1$ [9, v.I, p. 271 (15)], which corresponds to the case $\gamma = (\alpha + \beta)/2$ (case 3 in (10), see Section 5).

Corollary 1 is implied by Proposition 1 and the following simple lemmas which in turn are implied by the definitions (1)–(3).

Lemma 1 ([17]) *Let $f(z) = {}_jF_k(\omega z)$ and $g(z) = {}_jF_k(\omega z^2)$ (ω is a constant). Then $f_{*\alpha}(z) = {}_jF_{k+1}(\omega z)$, where the additional parameter equals α, and $g_{*\alpha}(z) = {}_jF_{k+2}(\omega z^2/4)$, where the additional parameters are equal to $\alpha/2$ and $(\alpha+1)/2$.*

Lemma 2 ([12, 13]) *Let $f_{*\alpha}(z)$ and $g_{*\beta}(z)$ be analytic in a disk $D_r = \{z : |z| < r\}$, where f and g are some power series and $\alpha, \beta > 0$. Then the $(\alpha+\beta)$-convolution $(fg)_{*(\alpha+\beta)}(z)$ is analytic in D_r and the integral formula*

$$B(\alpha,\beta)(fg)_{*(\alpha+\beta)}(z) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} f_{*\alpha}(zt) g_{*\beta}(z(1-t))dt \tag{34}$$

holds for any $z \in D_r$.

3 General Convolution Inequalities

The weighted convolution inequality presented in Theorem 1 below is obtained in [14] as a limit case of the discrete inequality for complex vectors and binomial weights. The weighted convolution integral on the left-hand side of inequality (35) can be expressed in terms of various special functions. This inequality being combined with the suitable convolution formulas is a source of many weighted norm inequalities for generalized hypergeometric functions and special functions of hypergeometric type. Some applications of Theorem 1, its discrete predecessor and special cases are given in [10–13] (case $\tau = 2$) and [14–17] (any $\tau \le \min(p, q)$). The probability connections and further generalizations are discussed in [18, 19]. It is proved in [19] that inequality (35) is a special case of the inequality for multiple convolutions of complex-valued functions with respect to Dirichlet probability measure on the standard simplex.

Theorem 1 ([14]) *Let $\phi(x)$ and $\psi(x)$ be complex-valued continuous functions on $[0, 1]$. Then for any numbers $\alpha, \beta, \lambda > 0$, $p > 1$ $(1/p + 1/q = 1)$, and $\tau \in (0, \min(p, q)]$, the following inequality holds:*

$$\left[\int_0^1 \left|\int_0^1 \phi(xt)\psi(x(1-t))\,\frac{t^{\alpha-1}(1-t)^{\beta-1}}{B(\alpha,\beta)}dt\right|^\tau \frac{x^{\alpha+\beta-1}(1-x)^{\lambda-1}}{B(\alpha+\beta,\lambda)}dx\right]^{1/\tau}$$
$$\le \left[\int_0^1 |\phi(x)|^p\,\frac{x^{\alpha-1}(1-x)^{\beta+\lambda-1}}{B(\alpha,\beta+\lambda)}dx\right]^{1/p}\left[\int_0^1 |\psi(x)|^q\,\frac{x^{\beta-1}(1-x)^{\alpha+\lambda-1}}{B(\beta,\alpha+\lambda)}dx\right]^{1/q}. \tag{35}$$

The equality in (35), provided that ϕ and ψ are not identically 0, holds if and only if $\phi(x) = \phi(0)e^{i\theta x}$ and $\psi(x) = \psi(0)e^{i\theta x}$ for $x \in [0, 1]$ (θ is real).

Remark 3 The limit of (35) as $\lambda \to 0$ corresponds to the integral Hölder inequality [20]:

$$\left|\int_0^1 U(t)V(t)dt\right| \le \left[\int_0^1 |U(t)|^p dt\right]^{1/p}\left[\int_0^1 |V(t)|^q dt\right]^{1/q},$$

where $U(t) = \phi(t)[t^{\alpha-1}(1-t)^{\beta-1}]^{1/p}$ and $V(t) = \psi(1-t)[t^{\alpha-1}(1-t)^{\beta-1}]^{1/q}$. Indeed, $\lambda(1-x)^{\lambda-1}$ and $\lambda\Gamma(\lambda)$ give the delta function $\delta(1-x)$ and 1 correspondingly if λ approaches 0.

Remark 4 Inequality (35) holds for any measurable functions ϕ and ψ on [0, 1], provided that the integrals in (35) are finite.

When it is convenient we will use the weighted L^p norm notation. For any $\alpha,\ \beta,\ p > 0$, real y, and a complex-valued function $f(t)$ on [0, 1], let

$$\|f\|_{[p;\alpha,\beta,y]} = \left[\int_0^1 |f(t)|^p \, \frac{t^{\alpha-1}(1-t)^{\beta-1}e^{yt}}{B(\alpha,\beta)} dt\right]^{1/p}, \tag{36}$$

provided that the integral in (36) is finite.

Theorem 2 given in terms of the α-convolutions (3) and L^p norms (36) is implied by Theorem 1. Inequality (37) in Theorem 2 is especially convenient for dealing with various generalized hypergeometric functions.

Theorem 2 ([17]) *Given $\alpha, \beta > 0$, let $f(z) = a_0 + a_1 z + \ldots$ and $g(z) = b_0 + b_1 z + \ldots$ be power series such that convolutions $f_{*\alpha}(z)$ and $g_{*\beta}(z)$ are analytic in a disk $D_r = \{z : |z| < r\}$. Then for any $\lambda > 0$, real y, $p > 1$ $(1/p + 1/q = 1)$, $\tau \in (0, \min(p,q)]$, and $\zeta \in D_r$, the following inequality holds:*

$$\|(fg)_{*(\alpha+\beta)}(\zeta t)\|_{[\tau;\alpha+\beta,\lambda,y\tau]} \le \|f_{*\alpha}(\zeta t)\|_{[p;\alpha,\beta+\lambda,yp]} \cdot \|g_{*\beta}(\zeta t)\|_{[q;\beta,\alpha+\lambda,yq]}. \tag{37}$$

The equality in (37), provided that f and g are not identically 0 and $\zeta \neq 0$, holds if and only if $f(z) = f(0)[1 + (y + i\theta)z/\zeta]^{-\alpha}$ and $g(z) = g(0)[1 + (y + i\theta)z/\zeta]^{-\beta}$ (θ is real). The limit of (37) as $\lambda \to 0$ corresponds to the integral Hölder inequality.

Corollary 2 *For any numbers $u, v, \lambda > 0$, $p > 1$ $(1/p + 1/q = 1)$, $\tau \in (0, \min(p,q)]$, real y, complex z, and any combination of the admissible parameters (α, β, γ) for identity (8) that are given in Proposition 1, the following inequalities hold:*

$$\begin{aligned}&\|{}_2F_2(\alpha+\beta,\gamma;u+v,2\gamma;zx)\|_{[\tau;u+v,\lambda,y\tau]}\\ &\le \|{}_2F_2(\alpha,\mu;u,2\mu;zx)\|_{[p;u,v+\lambda,yp]} \cdot \|{}_2F_2(\beta,\nu;v,2\nu;zx)\|_{[q;v,u+\lambda,yq]},\end{aligned} \tag{38}$$

$$\begin{aligned}&\|{}_2F_3[\alpha+\beta,\gamma;(u+v)/2,(u+v+1)/2,2\gamma;zx^2]\|_{[\tau;u+v,\lambda,y\tau]}\\ &\le \|{}_2F_3[\alpha,\mu;u/2,(u+1)/2,2\mu;zx^2]\|_{[p;u,v+\lambda,yp]}\\ &\times \|{}_2F_3[\beta,\nu;v/2,(v+1)/2,2\nu;zx^2]\|_{[q;v,u+\lambda,yq]}.\end{aligned} \tag{39}$$

The equality in (38) holds if and only if ${}_2F_2(\alpha,\mu;u,2\mu;zx) = e^{(-y+i\theta)x}$ and ${}_2F_2(\beta,\nu;v,2\nu;zx) = e^{(-y+i\theta)x}$ for $x \in [0,1]$ (θ is real).

Inequality (38) in Corollary 2 is implied by Proposition 1 and inequality (37), or alternatively by Theorem 1, where $\phi(x) = {}_2F_2(\alpha, \mu; u, 2\mu; zx)e^{yx}$ and $\psi(x) = {}_2F_2(\beta, \nu; v, 2\nu; zx)e^{yx}$, and formula (32). The proof of inequality (39) involving formula (33) is similar.

Corollary 3 *For any $u, v, \lambda > 0$, $\gamma \neq 0, -1, \ldots$, $p > 1$ $(1/p + 1/q = 1)$, $\tau \in (0, \min(p, q)]$, real y, and complex μ, ν, z, the following inequalities hold:*

$$\begin{aligned}
&\|{}_2F_2(\mu, \nu; u + v, \gamma; zx)\|_{[\tau;u+v,\lambda,y\tau]} \\
&\le \|{}_1F_1(\mu + \nu - \gamma; u; zx)\|_{[p;u,v+\lambda,yp]} \cdot \|{}_2F_2(\gamma - \mu, \gamma - \nu; v, \gamma; zx)\|_{[q;v,u+\lambda,yq]}, \\
&\|{}_1F_1(\gamma - v; \gamma; zx)\|_{[\tau;u+v,\lambda,y\tau]} \\
&\le [{}_1F_1(u; u + v + \lambda; (y + \Re z)p)]^{1/p} \cdot \|{}_1F_1(\gamma - u - v; \gamma; zx)\|_{[q;v,u+\lambda,yq]}.
\end{aligned} \tag{40}$$

Corollary 3 (see [16, 17]) is implied by Euler's transformation (4), inequality (37), and the integral representation for the confluent hypergeometric functions [9, v.I, p. 255], [1, Ch. 13, # 13.2.1]:

$$B(u, v)\ {}_1F_1(u; u + v; z) = \int_0^1 x^{u-1}(1 - x)^{v-1}e^{zx}dx \quad (u, v > 0).$$

4 Inequalities for Bessel Functions

We use Proposition 1 and Theorem 1 as well as Corollaries 1 and 2 to obtain some weighted convolution inequalities for Bessel functions. It is well known that certain integrals involving Bessel functions are used for numerous applications [8, 21, 22, 28, 34]. The Bessel function of the first kind of order α is denoted by $J_\alpha(z)$ [9, v.II, Ch. 7], [1, Ch. 9, 9.1.69]. It can be presented in the ${}_0F_1$-form,

$$J_\alpha(z) = \frac{(z/2)^\alpha}{\Gamma(\alpha + 1)}{}_0F_1(-, \alpha + 1; -(z/2)^2), \tag{41}$$

or in the ${}_1F_1$-form,

$$J_\alpha(z) = \frac{(z/2)^\alpha}{\Gamma(\alpha + 1)}e^{-iz}{}_1F_1(\alpha + 1/2, 2\alpha + 1; 2iz). \tag{42}$$

Corollary 4 *For any complex z, real y, $\lambda > 0$, $p > 1$ $(1/p + 1/q = 1)$, and $\tau \in (0, \min(p, q)]$, the following inequality holds:*

$$
\left[\int_0^1 |J_{\mu+\nu+\gamma}(zx)|^\tau \frac{x^{\alpha+\beta+(\mu+\nu)(1-\tau)-\gamma\tau-1}(1-x)^{\lambda-1}e^{y\tau x}}{B(\alpha+\beta+\mu+\nu,\lambda)}dx\right]^{1/\tau}
$$

$$
\leq K_1\left|\frac{z}{2}\right|^\gamma \left[\int_0^1 |J_\mu(zx)|^p \frac{x^{\alpha+\mu(1-p)-1}(1-x)^{\beta+\nu+\lambda-1}e^{ypx}}{B(\alpha+\mu,\beta+\nu+\lambda)}dx\right]^{1/p} \tag{43}
$$

$$
\times\left[\int_0^1 |J_\nu(zx)|^q \frac{x^{\beta+\nu(1-q)-1}(1-x)^{\alpha+\mu+\lambda-1}e^{yqx}}{B(\beta+\nu,\alpha+\mu+\lambda)}dx\right]^{1/q},
$$

where

$$
K_1 = \Gamma(\mu+1)\Gamma(\nu+1)/\Gamma(\mu+\nu+\gamma+1)
$$

and parameters μ, ν, α, β, and γ satisfy any of the following five sets of conditions or those which are symmetric to one of them:

$$
\begin{aligned}
&1.\ \mu > 0,\ \nu > -1,\ \alpha = 0,\ \beta = 1,\ \gamma = 0;\\
&2.\ \mu,\ \nu > 0,\ \alpha = \beta = \gamma = 0;\\
&3.\ \mu,\ \nu > -1/2,\ \alpha = \mu+1,\ \beta = \nu+1,\ \gamma = 1/2;\\
&4.\ \mu > -1/2,\ \nu > -1,\ \alpha = \mu+1,\ \beta = \nu+2,\ \gamma = 1/2;\\
&5.\ \nu > \mu > -1/2,\ \alpha = \mu+1,\ \beta = \gamma = -\mu.
\end{aligned} \tag{44}
$$

Proof We use inequality (38) in Corollary 2 with the first five admissible combinations of parameters (α, β, γ) for identity (8) given in Proposition 1. We set $u = \alpha$, $v = \beta$ to reduce ${}_2F_2$ to ${}_1F_1$ and replace y with $y - \Im z$. Then we use definition (42) to present the corresponding Bessel functions in the ${}_1F_1$ terms and we cancel factors e^{iz}. Finally we replace $z, \alpha, \beta, \mu, \nu$, and γ with $2iz$, $\alpha+\mu$, $\beta+\nu$, $\mu+1/2$, $\nu+1/2$, and $\mu+\nu+\gamma+1/2$, respectively. □

Corollary 5 *The convolution formula for Bessel functions*

$$
\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}J_\mu(zx)J_\nu(z(1-x))dx = A\left(\frac{z}{2}\right)^{-\gamma} J_{\mu+\nu+\gamma}(z), \tag{45}
$$

where α and β are some linear functions of μ and ν and

$$
A = \Gamma(\mu+\nu+\gamma+1)B(\alpha+\mu,\beta+\nu)/[\Gamma(\mu+1)\Gamma(\nu+1)],
$$

holds for any complex z if and only if real parameters μ, ν, α, β, and γ satisfy any of the five sets of conditions given in (44) or those which are symmetric to one of them.

Proof We use formula (32) in Corollary 1 with the first five admissible combinations of parameters (α, β, γ) for identity (8) given in Proposition 1 and we set $u = \alpha$, $v = \beta$. Definition (42) implies that any representation (45) is equivalent to formula (32) with positive $u = \alpha$ and $v = \beta$. To complete the proof we use (42) and replace z, α, β, μ, ν, and γ with $2iz$, $\alpha + \mu$, $\beta + \nu$, $\mu + 1/2$, $\nu + 1/2$, and $\mu + \nu + \gamma + 1/2$, respectively. □

Corollary 5 gives a combined Bateman convolution formula for Bessel functions [8, v. II, pp. 354–355, formulas (24), (26), (28), (29), (30)] (see also [4]). We can directly combine Corollary 5 and Theorem 1 with $\phi(x) = J_\mu(zx)e^{yx}$ and $\psi(x) = J_\nu(zx)e^{yx}$, if both α and β in (45) are positive. The cases 3, 4, and 5 with negative μ in (44) satisfy this requirement.

Corollary 6 *For any $\lambda > 0$, $p > 1$ $(1/p + 1/q = 1)$, $\tau \in (0, \min(p, q)]$, real y, and complex z, the following inequality holds:*

$$\left[\int_0^1 |J_{\mu+\nu+\gamma}(zx)|^\tau \frac{x^{\alpha+\beta-\gamma\tau-1}(1-x)^{\lambda-1}e^{y\tau x}}{B(\alpha+\beta,\lambda)}dx\right]^{1/\tau}$$
$$\le K_2\left|\frac{z}{2}\right|^\gamma \left[\int_0^1 |J_\mu(zx)|^p \frac{x^{\alpha-1}(1-x)^{\beta+\lambda-1}e^{ypx}}{B(\alpha,\beta+\lambda)}dx\right]^{1/p} \tag{46}$$
$$\times \left[\int_0^1 |J_\nu(zx)|^q \frac{x^{\beta-1}(1-x)^{\alpha+\lambda-1}e^{yqx}}{B(\beta,\alpha+\lambda)}dx\right]^{1/q},$$

where

$$K_2 = \frac{\Gamma(\mu+1)\Gamma(\nu+1)B(\alpha,\beta)}{\Gamma(\mu+\nu+\gamma+1)B(\alpha+\mu,\beta+\nu)}$$

and parameters μ, ν, α, β, and γ satisfy any of the following three sets of conditions or those which are symmetric to one of them:

$$\begin{aligned} &1.\ \mu,\ \nu > -1/2,\ \alpha = \mu+1,\ \beta = \nu+1,\ \gamma = 1/2;\\ &2.\ \mu > -1/2,\ \nu > -1,\ \alpha = \mu+1,\ \beta = \nu+2,\ \gamma = 1/2;\\ &3.\ \nu > \mu > -1/2,\ \alpha = \mu+1,\ \beta = \gamma = -\mu. \end{aligned} \tag{47}$$

Finally we mention that the typical Kapteyn integrals which involve both the Bessel and trigonometric functions [34, p. 380, (1) and p. 381, (2)], [3, 4], [21, Ch. 2]:

$$\int_0^1 J_0(zx)\cos(z(1-x))dx = J_0(z),\ \int_0^1 J_0(zx)\sin(z(1-x))dx = J_1(z)$$

are the special cases of representation (45). These integrals correspond to cases 4 and 3 in (44), respectively, as $\cos(z) = \sqrt{\pi z/2} J_{-1/2}(z)$ and $\sin(z) = \sqrt{\pi z/2} J_{1/2}(z)$. Namely, we take $\mu = 0, \alpha = 1, \beta = 3/2, \gamma = 1/2$ in both cases, and $\nu = -1/2$ for the first integral and $\nu = 1/2$ for the second one. At the same time, one can use representation (41) and formulas: $\cos(z) = {}_0F_1(-; 1/2; -z^2/4)$ and $\sin(z) = z\ {}_0F_1(-; 3/2; -z^2/4)$ to see that the above Kapteyn integrals are the special cases of (33) with $u = 1$ and $u = 2$, respectively, and $\omega = -1/4, \alpha = 2, v = \mu = \beta = 1, \nu = 1/2, \gamma = 3/2$ for both cases (see case 3 in (10)).

5 Inequalities for Laguerre and Hermite Polynomials

We have a relatively simple case of formula (32) in Corollary 1 if 2γ equals $\alpha + \beta$ (case 3 in (10)). It results in the integral addition theorem for the confluent hypergeometric functions, namely:

$$B(u, v)\ {}_1F_1(\mu + \nu; u + v; z) \\ = \int_0^1 t^{u-1}(1-t)^{v-1}\ {}_1F_1(\mu; u; zt)\,{}_1F_1(\nu; v; z(1-t))dt. \tag{48}$$

The related case of inequality (38) is given in Corollary 7 (see [16, 17]).

Corollary 7 *For any numbers* $u, v, \lambda > 0$, $p > 1$ $(1/p + 1/q = 1)$, $\tau \in (0, \min(p, q)]$, *real* y, *and complex* μ, ν, z, *the following inequality holds:*

$$\|{}_1F_1(\mu + \nu; u + v; zx)\|_{[\tau; u+v, \lambda, y\tau]} \\ \le \|{}_1F_1(\mu; u; zx)\|_{[p; u, v+\lambda, yp]} \cdot \|{}_1F_1(\nu; v; zx)\|_{[q; v, u+\lambda, yq]}. \tag{49}$$

The equality in (49) *holds if and only if* ${}_1F_1(\mu; u; zx) = e^{(-y+i\theta)x}$ *and* ${}_1F_1(\nu; v; zx) = e^{(-y+i\theta)x}$ *for* $x \in [0, 1]$ (θ *is real*).

Inequality (49) is a convenient tool for those special functions of hypergeometric type that can be expressed in a ${}_1F_1$-form with the mutually independent parameters, for example, for Laguerre and Hermite polynomials. The Laguerre polynomials $L_n^\alpha(z)$ are defined by the formula:

$$L_n^\alpha(z) = \frac{1}{n!} e^z z^{-\alpha} \frac{d^n}{dz^n}(e^{-z} z^{n+\alpha}) = \sum_{k=0}^n (-1)^k \binom{n+\alpha}{n-k} \frac{z^k}{k!},$$

which can be presented as

$$L_n^\alpha(z) = L_n^\alpha(0)\, {}_1F_1(-n; \alpha+1; z), \tag{50}$$

where $L_n^\alpha(0) = (\alpha+1)_n/n!$ [9, v.II, 10.12], [1, Ch. 13, 13.6].

Corollary 8 *For any $m, n = 0, 1, 2, \ldots$, $\alpha, \beta > -1, \lambda > 0$, $p > 1$ $(1/p + 1/q = 1)$, $\tau \in (0, \min(p, q)]$, real y, and complex z, the following inequality holds*

$$\begin{aligned}
&\left[\int_0^1 |L_{m+n}^{\alpha+\beta+1}(zt)|^\tau \, \frac{t^{\alpha+\beta+1}(1-t)^{\lambda-1}e^{y\tau x}}{B(\alpha+\beta+2,\lambda)} dt\right]^{1/\tau} \\
&\le K_3 \left[\int_0^1 |L_m^\alpha(zt)|^p \, \frac{t^\alpha(1-t)^{\beta+\lambda}e^{ypx}}{B(\alpha+1,\beta+\lambda+1)} dt\right]^{1/p} \\
&\times \left[\int_0^1 |L_n^\beta(zt)|^q \, \frac{t^\beta(1-t)^{\alpha+\lambda}e^{yqx}}{B(\beta+1,\alpha+\lambda+1)} dt\right]^{1/q},
\end{aligned} \tag{51}$$

where

$$K_3 = \frac{(\alpha+1)_m(\beta+1)_n m! n!}{(\alpha+\beta+2)_{m+n}(m+n)!}.$$

Corollary 8 is implied by formula (50) and Corollary 7. Alternatively, to prove inequality (51) one can use Theorem 1 and the known convolution integral for Laguerre polynomials [8, v. II, p. 293, (7)]:

$$\int_0^1 t^\alpha(1-t)^\beta L_m^\alpha(zt)L_n^\beta[z(1-t)]dt = \binom{m+n}{m} B(\alpha+m+1, \beta+n+1) L_{m+n}^{\alpha+\beta+1}(z), \tag{52}$$

where $\alpha, \beta > -1$. Note that (52) is implied by (48) and (50).

Corollary 7 leads to some inequalities involving both the Hermite and Laguerre polynomials [11]. The Hermite polynomials $H_n(z)$ [9, v.II, 10.13], [1, Ch. 13, 13.6],

$$H_n(z) = (-1)^n e^{z^2} \frac{d^n}{dz^n}(e^{-z^2}),$$

can be presented in the ${}_1F_1$-form:

$$\begin{aligned}
H_{2n}(z) &= (-1)^n \frac{(2n)!}{n!}\, {}_1F_1(-n; 1/2; z^2), \\
H_{2n+1}(z) &= (-1)^n 2 \frac{(2n+1)!}{n!} z\, {}_1F_1(-n; 3/2; z^2).
\end{aligned} \tag{53}$$

Corollary 9 *For any $m, n = 0, 1, 2, \ldots$, $\lambda > 0$, $p > 1$ $(1/p + 1/q = 1)$, $\tau \in (0, \min(p, q)]$, real y, and complex number z, the following inequalities hold*

$$\left[\int_0^1 |L_{n+m}^0(z^2t)|^\tau \; (1-t)^{\lambda-1} e^{y\tau t} dt\right]^{1/\tau}$$
$$\le K_4 \left[\int_0^1 |H_{2n}(zt)|^p \; (1-t^2)^{\lambda-1/2} e^{ypt^2} dt\right]^{1/p} \tag{54}$$
$$\times \left[\int_0^1 |H_{2m}(zt)|^q \; (1-t^2)^{\lambda-1/2} e^{yqt^2} dt\right]^{1/q},$$

$$|z| \left[\int_0^1 |L_{n+m}^1(z^2t)|^\tau \; t(1-t)^{\lambda-1} e^{y\tau t} dt\right]^{1/\tau}$$
$$\le K_5 \left[\int_0^1 |H_{2n}(zt)|^p \; (1-t^2)^{\lambda+1/2} e^{ypt^2} dt\right]^{1/p} \tag{55}$$
$$\times \left[\int_0^1 |H_{2m+1}(zt)|^q \; t^{2-q}(1-t^2)^{\lambda-1/2} e^{yqt^2} dt\right]^{1/q},$$

$$|z|^2 \left[\int_0^1 |L_{n+m}^2(z^2t)|^\tau \; t^2(1-t)^{\lambda-1} e^{y\tau t} dt\right]^{1/\tau}$$
$$\le K_6 \left[\int_0^1 |H_{2n+1}(zt)|^p \; t^{2-p}(1-t^2)^{\lambda+1/2} e^{ypt^2} dt\right]^{1/p} \tag{56}$$
$$\times \left[\int_0^1 |H_{2m+1}(zt)|^q \; t^{2-q}(1-t^2)^{\lambda+1/2} e^{yqt^2} dt\right]^{1/q},$$

where

$$K_4 = \frac{2^{1-2n-2m}\Gamma(\lambda+1)}{\sqrt{\pi}\lambda^{1/\tau}\Gamma(\lambda+1/2)(1/2)_n(1/2)_m},$$

$$K_5 = \frac{2^{1/q-2n-2m}\Gamma(\lambda+2)(n+m+1)}{\sqrt{\pi}[\lambda(\lambda+1)]^{1/\tau}(\lambda+1/2)^{1/p}\Gamma(\lambda+1/2)(1/2)_n(3/2)_m},$$

$$K_6 = \frac{2^{1/\tau-2n-2m-1}\Gamma(\lambda+3)(n+m+1)(n+m+2)}{\sqrt{\pi}[\lambda(\lambda+1)(\lambda+2)]^{1/\tau}\Gamma(\lambda+3/2)(3/2)_n(3/2)_m}.$$

Corollary 9 is implied by formulas (50) and (53) and inequality (49). Note that inequalities (54)–(56) can be presented in terms of the Laguerre polynomials L_n^0, $L_n^{\pm 1/2}$, L_n^1, and L_n^2.

6 Inequalities for Whittaker Functions

The Whittaker functions $M_{\gamma,\mu}(z)$ are defined by the formula [9, v.I, 6.9], [1, Ch. 13,#13.1.32]:

$$M_{\nu,\mu}(z) = z^{\mu+1/2}e^{-z/2}\,{}_1F_1(\mu-\nu+1/2; 2\mu+1; z), \quad |\arg z| < \pi. \tag{57}$$

Corollary 10 *For any* $\alpha, \beta > -1/2, \lambda > 0$*,* $p > 1$ $(1/p + 1/q = 1)$*,* $\tau \in (0, \min(p, q)]$*, real* y*, and complex number* z $(|\arg z| < \pi)$*, the following inequality holds*

$$\begin{aligned}
&\left[\int_0^1 |M_{\mu+\nu,\alpha+\beta+1/2}(zt)|^\tau \frac{t^{(2-\tau)(\alpha+\beta+1)-1}(1-t)^{\lambda-1}e^{y\tau x}}{B(2\alpha+2\beta+2,\lambda)}dt\right]^{1/\tau}\\
&\le \left[\int_0^1 |M_{\mu,\alpha}(zt)|^p \frac{t^{(2-p)(\alpha+1/2)-1}(1-t)^{2\beta+\lambda}e^{ypx}}{B(2\alpha+1,2\beta+\lambda+1)}dt\right]^{1/p}\\
&\times \left[\int_0^1 |M_{\nu,\beta}(zt)|^q \frac{t^{(2-q)(\beta+1/2)}(1-t)^{2\alpha+\lambda}e^{yqx}}{B(2\beta+1,2\alpha+\lambda+1)}dt\right]^{1/q}.
\end{aligned} \tag{58}$$

Corollary 10 is implied by definition (57) and Corollary 7.

Formulas (48) and (57) imply the known formula [8, v. II, p. 402, (7)]:

$$\begin{aligned}
&\int_0^1 t^{\alpha-1/2}(1-t)^{\beta-1/2}M_{\mu,\alpha}(zt)M_{\nu,\beta}(z(1-t))dt\\
&= B(2\alpha+1,2\beta+1)M_{\mu+\nu,\alpha+\beta+1/2}(z) \ \ (\alpha,\beta > -1/2).
\end{aligned} \tag{59}$$

Corollary 11 *For any* $\alpha, \beta > -1/2, \lambda > 0$*,* $p > 1$ $(1/p + 1/q = 1)$*,* $\tau \in (0, \min(p, q)]$*, real* y, μ, ν*, and complex number* z $(|\arg z| < \pi)$*, the following inequality holds:*

$$\begin{aligned}
&B(2\alpha+1,2\beta+1)\left[\int_0^1 |M_{\mu+\nu,\alpha+\beta+1/2}(zt)|^\tau \frac{t^{\alpha+\beta}(1-t)^{\lambda-1}e^{y\tau t}}{B(\alpha+\beta+1,\lambda)}dt\right]^{1/\tau}\\
&\le \left[\int_0^1 |M_{\mu,\alpha}(zt)|^p \frac{t^{\alpha-1/2}(1-t)^{\beta+\lambda-1/2}e^{ypt}}{B(\alpha+1/2,\beta+\lambda+1/2)}dt\right]^{1/p}\\
&\times \left[\int_0^1 |M_{\gamma,\beta}(zt)|^q \frac{t^{\beta-1/2}(1-t)^{\alpha+\lambda-1/2}e^{yqt}}{B(\beta+1/2,\alpha+\lambda+1/2)}dt\right]^{1/q}.
\end{aligned} \tag{60}$$

Corollary 11 is implied by formula (59) and Theorem 1 with $\phi(t) = e^t M_{\mu,\alpha}(zt)$, $\psi(t) = e^t M_{\nu,\beta}(zt)$, and $(\alpha + 1/2), (\beta + 1/2)$ instead of α, β.

Formulas (57) and (34), and Euler's transformation (4) imply the known convolution integral involving the Whittaker function [8, v. II, p. 402, (6)]:

$$\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}e^{(1-t)z/2}M_{\alpha+\beta,\mu}(zt)dt = B(\alpha+\mu+1/2,\beta)M_{\alpha,\mu}(z), \tag{61}$$

where $\alpha+\mu+1/2, \beta > 0$. Theorem 1 with $\phi(t) = e^t M_{\alpha+\beta,\mu}(zt)$ and $\psi(t) = e^{zt/2}$ as well as (61) and Corollary 3 lead to Corollary 12.

Corollary 12 *For any* $\lambda > 0$, $\mu > -1/2$, $p > 1$ $(1/p + 1/q = 1)$, $\tau \in (0, \min(p, q)]$, *real* y, *and complex* z $(|\arg z| < \pi)$, *the following inequalities hold:*

$$\begin{aligned}
&\left[\int_0^1 |M_{\alpha,\mu}(zt)|^\tau \frac{t^{\alpha+\beta-1}(1-t)^{\lambda-1}e^{y\tau t}}{B(\alpha+\beta,\lambda)}dt\right]^{1/\tau}\\
&\leq \frac{B(\alpha,\beta)\,[{}_1F_1(\beta;\alpha+\beta+\lambda;(y+\Re z/2)q)]^{1/q}}{B(\alpha+\mu+1/2,\beta)}\\
&\times\left[\int_0^1 |M_{\alpha+\beta,\mu}(zt)|^p \frac{t^{\alpha-1}(1-t)^{\beta+\lambda-1}e^{ypt}}{B(\alpha,\beta+\lambda)}dt\right]^{1/p} \quad (\alpha,\beta>0),
\end{aligned} \tag{62}$$

$$\begin{aligned}
&\left[\int_0^1 |M_{\alpha,\mu}(zt)|^\tau \frac{t^{\alpha+\beta+(1-\tau)(\mu+1/2)-1}(1-t)^{\lambda-1}e^{y\tau t}}{B(\alpha+\beta+\mu+1/2,\lambda)}dt\right]^{1/\tau}\\
&\leq [{}_1F_1(\beta;\alpha+\beta+\mu+\lambda+1/2;(y+3\Re z/2)p)]^{1/p}\\
&\times\left[\int_0^1 |M_{\alpha+\beta,\mu}(zt)|^q \frac{t^{\alpha+(1-q)(\mu+1/2)-1}(1-t)^{\beta+\lambda-1}e^{yqt}}{B(\alpha+\mu+1/2,\beta+\lambda)}dt\right]^{1/q},
\end{aligned} \tag{63}$$

where $\alpha > -\mu-1/2$, $\beta > \max(-\lambda, -\alpha-\mu-1/2)$.

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On the Means of the Non-trivial Zeros of the Riemann Zeta Function

Mehdi Hassani

Abstract In this paper we obtain asymptotic expansion of the sequence with general term $\mathscr{A}_n/\mathscr{G}_n$, where $\mathscr{A}_n$ and $\mathscr{G}_n$ are the arithmetic and geometric means of the numbers $\gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_n$ denoting consecutive ordinates of the imaginary parts of non-real zeros of the Riemann zeta function.

1 Introduction and Summary of the Results

Assume that $(a_n)_{n\geq 1}$ is a real sequence with $a_n > 0$. We denote the arithmetic and geometric means of the numbers $a_1, a_2, \ldots, a_n$ by $A(a_1, \ldots, a_n)$ and $G(a_1, \ldots, a_n)$, respectively. In this paper we are motivated by several results studying the ratio $A(a_1, \ldots, a_n)/G(a_1, \ldots, a_n)$. Stirling's approximation for $n!$ gives

$$\frac{A(1, \ldots, n)}{G(1, \ldots, n)} = \frac{\mathrm{e}}{2} + O\Big(\frac{\log n}{n}\Big).$$

We refer the reader to [4] for more details. The ratio $\mathrm{e}/2$ appears surprisingly in studying the ratio of the arithmetic to the geometric means of several number theoretic sequences, including the sequence of prime numbers. More precisely, in [1] we proved that

$$\frac{A(p_1, \ldots, p_n)}{G(p_1, \ldots, p_n)} = \frac{\mathrm{e}}{2} + O\Big(\frac{1}{\log n}\Big),$$

where p_n denotes the nth prime number. As a further example of this phenomenon, in [6] we showed that

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$$\frac{A(\varrho_1, \ldots, \varrho_{\phi(n)})}{G(\varrho_1, \ldots, \varrho_{\phi(n)})} = \frac{\mathrm{e}}{2} + O\Big(\frac{\log n \log\log n}{n}\Big),$$

where $\{\varrho_1, \ldots, \varrho_{\phi(n)}\}$ is the least positive reduced set of residues modulo n.

In continuation of our studies of the ratio A/G, in this paper we study the ratio $\mathscr{A}_n/\mathscr{G}_n$ with

$$\mathscr{A}_n := A(\gamma_1, \gamma_2, \ldots, \gamma_n), \quad \textit{and} \quad \mathscr{G}_n := G(\gamma_1, \gamma_2, \ldots, \gamma_n),$$

where $0 < \gamma_1 < \gamma_2 < \gamma_3 < \cdots$ denote consecutive ordinates of the imaginary parts of non-real zeros of the Riemann zeta function, which is defined by $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ for $\Re(s) > 1$, and extended by analytic continuation to the complex plane with a simple pole at $s = 1$. First we obtain an asymptotic expansion for $\mathscr{A}_n/\mathscr{G}_n$ as follows.

Theorem 1 *As $n \to \infty$ we have*

$$\frac{\mathscr{A}_n}{\mathscr{G}_n} = \frac{\mathrm{e}}{2}\left(1 - \frac{1}{2\log n} - \frac{\log\log n}{2\log^2 n} - \frac{1}{2\log^2 n} + O\Big(\frac{(\log\log n)^2}{\log^3 n}\Big)\right). \tag{1}$$

As an immediate corollary, we deduce that the ratio $\mathscr{A}_n/\mathscr{G}_n$ is strictly increasing for enough large values of n.

Corollary 1 *As $n \to \infty$ we have*

$$\frac{\mathscr{A}_{n+1}}{\mathscr{G}_{n+1}} - \frac{\mathscr{A}_n}{\mathscr{G}_n} = \frac{\mathrm{e}}{2}\left(\frac{1}{2n\log^2 n} + \frac{\log\log n}{n\log^3 n} + \frac{1}{2n\log^3 n}\right) + O\Big(\frac{(\log\log n)^2}{n\log^4 n}\Big).$$

We have used Maple software to do several computations running over the numbers γ_n, based on the tables of zeros of $\zeta(s)$ due to A. Odlyzko [7]. These computations, as pictured partially in Figure 1, suggest that the ratio $\mathscr{A}_n/\mathscr{G}_n$ is strictly increasing anywhere. Similar computations suggest that both of the

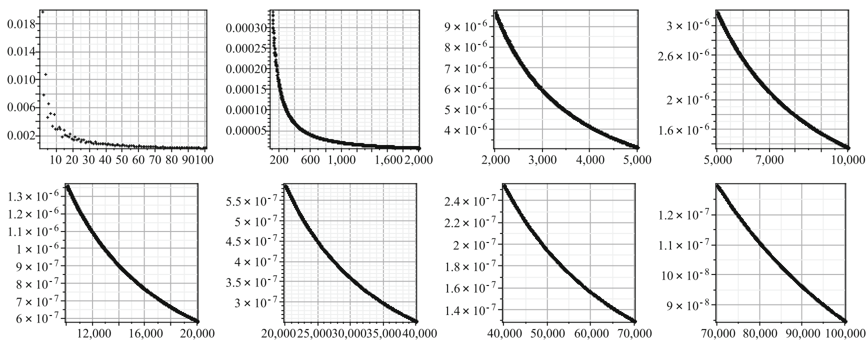

Fig. 1 Graphs of the points $\big(n, \frac{\mathscr{A}_{n+1}}{\mathscr{G}_{n+1}} - \frac{\mathscr{A}_n}{\mathscr{G}_n}\big)$ in several intervals from 1 to 10^5, with ending points 10^2, 2×10^3, 5×10^3, 10^4, 2×10^4, 4×10^4, 7×10^4, and 10^5.

sequences with general terms $\mathscr{A}_n$ and $\mathscr{G}_n$ are strictly decreasing. We formulate these observations in the following.

Conjecture 1 For any integer $n \geq 1$ we have

$$\mathscr{A}_{n+1} < \mathscr{A}_n, \quad \mathscr{G}_{n+1} < \mathscr{G}_n, \quad \textit{and} \quad \frac{\mathscr{A}_n}{\mathscr{G}_n} < \frac{\mathscr{A}_{n+1}}{\mathscr{G}_{n+1}}.$$

Remark 1 We note that the appearance of the similar limit value $\mathrm{e}/2$ in the above results is not trivial and a global property. As an example, we consider the asymptotic behaviour of the ratio under study for the values of the Euler function. By using the asymptotic expansions for $A(\phi(1), \ldots, \phi(n))$ and $G(\phi(1), \ldots, \phi(n))$ (see [9] for the arithmetic mean, and [2] for the geometric mean), we get

$$\frac{A(\phi(1), \ldots, \phi(n))}{G(\phi(1), \ldots, \phi(n))} = \frac{3\mathrm{e}}{\pi^2} \prod_p \left(1 - \frac{1}{p}\right)^{-\frac{1}{p}} + O\Big(\frac{\log n}{n}\Big),$$

where the product runs over all primes. This gives a limit value different from $\mathrm{e}/2$, for the case of Euler function. Another example, providing divergent limit value, regards to the function $d(n)$, which denotes the number of positive divisors of n. In [5] we showed that

$$\frac{A(d(1), \ldots, d(n))}{G(d(1), \ldots, d(n))} = r_0 (\log n)^{1-\log 2} \left(1 + \sum_{k=1}^{m} \frac{r_k}{\log^k n} + O\Big(\frac{1}{\log^{m+1} n}\Big)\right),$$

where

$$r_0 = 2^{-M} \prod_{\substack{p^\alpha \\ \alpha \geq 2}} \log \left(1 + \frac{1}{\alpha}\right)^{-\frac{1}{p^\alpha}},$$

is an absolute constant in terms of Meissel–Mertens constant M. The coefficients r_k are computable constants. More precisely, $r_1 = 2\gamma - 1 + (1 - \gamma) \log 2$.

To prove Theorem 1 we study the arithmetic and geometric means of the imaginary parts of non-real zeros of $\zeta(s)$ satisfying $0 < \Im(s) \leq T$, which we denote by $\mathscr{A}(T)$ and $\mathscr{G}(T)$, respectively. We have

$$\mathscr{A}(T) = \frac{1}{N(T)} \sum_{0<\gamma\leq T} \gamma, \quad \textit{and} \quad \mathscr{G}(T) = \Big(\prod_{0<\gamma\leq T} \gamma\Big)^{\frac{1}{N(T)}},$$

where $N(T)$ denotes the number of non-real zeros of $\zeta(s)$, in the rectangular region assigned by $0 < \Re(s) < 1$ and $0 < \Im(s) \leq T$. It is known [8] that

$$N(T) = M(T) + O(\log T), \tag{2}$$

where $M(T) = \alpha T \log T - \beta T$, with $\alpha = 1/(2\pi)$ and $\beta = (1 + \log(2\pi))/(2\pi)$. We obtain the following precise asymptotic expansions.

Theorem 2 *Let $\delta = \beta/\alpha$. For given integer $m \geq 1$, as $T \to \infty$, we have*

$$\mathscr{A}(T) = \frac{T}{2}\left(1 + \sum_{k=1}^{m} \frac{\delta^{k-1}}{2} \frac{1}{\log^k T} + O_m\Big(\frac{1}{\log^{m+1} T}\Big)\right), \tag{3}$$

$$\mathscr{G}(T) = \frac{T}{\mathrm{e}}\left(1 + \sum_{j=1}^{m} \frac{\eta_j}{\log^j T} + O_m\Big(\frac{1}{\log^{m+1} T}\Big)\right), \tag{4}$$

and

$$\frac{\mathscr{A}(T)}{\mathscr{G}(T)} = \frac{\mathrm{e}}{2}\left(1 + \sum_{\ell=1}^{m} \frac{\lambda_\ell}{\log^\ell T} + O_m\Big(\frac{1}{\log^{m+1} T}\Big)\right). \tag{5}$$

where the constants η_j and λ_ℓ are computable in terms of δ.

Remark 2 To compute the coefficients λ_ℓ in (5), we consider the relation (3) and forthcoming relation (10), then we apply the recurrence

$$\lambda_\ell = \lim_{T\to\infty} \Big(\frac{2\mathscr{A}(T)}{\mathrm{e}^{1+\log\mathscr{G}(T)}} - \sum_{i=0}^{\ell-1} \frac{\lambda_i}{\log^i T}\Big) \log^\ell T, \quad \lambda_0 := 1, \ell = 1, 2, \ldots, m.$$

Recalling that $\delta = \beta/\alpha$, some values of the coefficients λ_ℓ in terms of δ are

$$\begin{aligned}
\lambda_1 &= -\frac{1}{2},\\
\lambda_2 &= -\frac{1}{2}\delta,\\
\lambda_3 &= -\frac{1}{2}\delta^2 + \frac{1}{12},\\
\lambda_4 &= -\frac{1}{2}\delta^3 + \frac{1}{4}\delta - \frac{1}{24},\\
\lambda_5 &= -\frac{1}{2}\delta^4 + \frac{1}{2}\delta^2 - \frac{1}{6}\delta + \frac{1}{80},\\
\lambda_6 &= -\frac{1}{2}\delta^5 + \frac{5}{6}\delta^3 - \frac{5}{12}\delta^2 + \frac{1}{16}\delta - \frac{1}{360},\\
\lambda_7 &= -\frac{1}{2}\delta^6 + \frac{5}{4}\delta^4 - \frac{5}{6}\delta^3 + \frac{3}{16}\delta^2 - \frac{1}{60}\delta + \frac{1}{2016},
\end{aligned}$$

$$\lambda_8 = -\frac{1}{2}\delta^7 + \frac{7}{4}\delta^5 - \frac{35}{24}\delta^4 + \frac{7}{16}\delta^3 - \frac{7}{120}\delta^2 + \frac{1}{288}\delta - \frac{1}{13440},$$
$$\lambda_9 = -\frac{1}{2}\delta^8 + \frac{7}{3}\delta^6 - \frac{7}{3}\delta^5 + \frac{7}{8}\delta^4 - \frac{7}{45}\delta^3 + \frac{1}{72}\delta^2 - \frac{1}{1680}\delta + \frac{1}{103680}.$$

2 Proofs

To approximate $\mathscr{A}(T)$ and $\mathscr{G}(T)$ we provide a method to compute general averages of the form

$$\frac{1}{N(T)} \sum_{0<\gamma\leq T} g(\gamma),$$

where $g(t) \in C^1(a, b)$ is a non-negative function and $1 < a \leq b$. We have

$$\sum_{a<\gamma\leq b} g(\gamma) = \int_a^b g(t)\, dN(t) = g(b)N(b) - g(a)N(a) - \int_a^b g'(t)N(t)\, dt.$$

Let $a = \gamma_1 - \varepsilon$ with $\varepsilon \in (0, 1)$, and $b = T$. Hence

$$\sum_{0<\gamma\leq T} g(\gamma) = g(T)N(T) - \int_{\gamma_1-\varepsilon}^T g'(t)N(t)\, dt.$$

By using (2) and letting $\varepsilon \to 0^+$, we obtain

$$\sum_{0<\gamma\leq T} g(\gamma) = g(T)M(T) - \int_{\gamma_1}^T g'(t)M(t)\, dt + \mathscr{R}(T), \tag{6}$$

where

$$\mathscr{R}(T) \ll g(T)\log T + \int_{\gamma_1}^T \left|g'(t)\right| \log t\, dt.$$

The expansion

$$\frac{1}{1-x} = 1 + \sum_{k=1}^m x^k + O(x^{m+1}), \tag{7}$$

holds as $x \to 0$, for any fixed integer $m \geq 1$. Since

$$\frac{1}{N(T)} = \frac{1}{\alpha T \log T} \left(\frac{1}{1 - \frac{\delta}{\log T} + O\left(\frac{1}{T}\right)} \right),$$

we put in (7)

$$x = \frac{\delta}{\log T} + O\left(\frac{1}{T}\right),$$

for which we have

$$x^k = \frac{\delta^k}{\log^k T} + O\left(\frac{1}{T \log^{k-1} T}\right)$$

for any integer $k \geq 1$. Hence, for any fixed integer $m \geq 1$ we obtain

$$\frac{1}{N(T)} = \frac{1}{\alpha T \log T} \left(1 + \sum_{k=1}^{m} \frac{\delta^k}{\log^k T} + O_m\left(\frac{1}{\log^{m+1} T}\right) \right), \tag{8}$$

where O_m means that the constant in O-term depends (at most) on m. The relations (6) and (8) enable us to approximate the general averages.

2.1 Proof of (3)

By using (6) with $g(t) = t$ we get

$$\sum_{0<\gamma\leq T} \gamma = \alpha\, T \log T \left(\frac{T}{2} + \left(\frac{1}{4} - \frac{\delta}{2}\right) \frac{T}{\log T} + O(1) \right).$$

Let $\delta' = 1/4 - \delta/2$. Hence, the relation (8) gives

$$\mathscr{A}(T) = \frac{T}{2} + \delta' \frac{T}{\log T} + \frac{T}{2} \sum_{k=1}^{m} \frac{\delta^k}{\log^k T} + T \sum_{k=1}^{m} \frac{\delta' \delta^k}{\log^{k+1} T} + O_m\left(\frac{T}{\log^{m+1} T}\right),$$

and after simplifying

$$\mathscr{A}(T) = \frac{T}{2} + \left(\frac{\delta}{2} + \delta'\right) \frac{T}{\log T} + \sum_{k=2}^{m} \left(\frac{\delta^k}{2} + \delta' \delta^{k-1}\right) \frac{T}{\log^k T} + O_m\left(\frac{T}{\log^{m+1} T}\right),$$

for any integer $m \geq 1$. Considering $\delta/2 + \delta' = 1/4$ we obtain (3).

2.2 Proof of (4)

We have

$$\log \mathscr{G}(T) = \frac{1}{N(T)} \sum_{0<\gamma \le T} \log \gamma.$$

The relation (6) with $g(t) = \log t$ reads as

$$\sum_{0<\gamma \le T} \log \gamma = \alpha T \log^2 T - (\alpha+\beta) T \log T + (\alpha+\beta) T + O(\log^2 T). \tag{9}$$

Hence

$$\sum_{0<\gamma \le T} \log \gamma = \alpha T \log T \left(\log T - (1+\delta) + \frac{1+\delta}{\log T} + O\Big(\frac{\log T}{T}\Big) \right).$$

Let $\delta'' = 1 + \delta$. Thus, the relation (8) implies that

$$\log \mathscr{G}(T) = \log T - \delta'' + \frac{\delta''}{\log T} + \sum_{k=1}^{m} \Big(\frac{\delta^k}{\log^{k-1} T} - \frac{\delta'' \delta^k}{\log^k T} + \frac{\delta'' \delta^k}{\log^{k+1} T} \Big) + O_m\Big(\frac{1}{\log^m T}\Big).$$

Now, we assume that $m \ge 3$ is fixed integer. Hence

$$\log \mathscr{G}(T) = \log T + \big(\delta - \delta''\big) + \Big(\delta'' + \delta^2 - \delta''\delta\Big) \frac{1}{\log T} + \sum_{k=2}^{m-1} \Big(\delta^{k+1} - \delta''\delta^k + \delta''\delta^{k-1}\Big) \frac{1}{\log^k T} + O_m\Big(\frac{1}{\log^m T}\Big).$$

We note that $\delta - \delta'' = -1$ and $\delta'' + \delta^2 - \delta''\delta = 1$. Thus

$$\log \mathscr{G}(T) = \log T - 1 + \sum_{k=1}^{m-1} \frac{\delta^{k-1}}{\log^k T} + O_m\Big(\frac{1}{\log^m T}\Big),$$

for any fixed integer $m \ge 2$. Replacing m by $m+1$ gives

$$\log \mathscr{G}(T) = \log \frac{T}{\mathrm{e}} + S_m(T) + O_m\Big(\frac{1}{\log^{m+1} T}\Big), \tag{10}$$

for any fixed integer $m \geq 1$, where

$$S_m(T) = \sum_{k=1}^{m} \frac{\delta^{k-1}}{\log^k T}.$$

Let

$$y = S_m(T) + O_m\Big(\frac{1}{\log^{m+1} T}\Big).$$

As $T \to \infty$,

$$\mathrm{e}^y = \mathrm{e}^{S_m(T)}\left(1 + O_m\Big(\frac{1}{\log^{m+1} T}\Big)\right) = \mathrm{e}^{S_m(T)} + O_m\Big(\frac{1}{\log^{m+1} T}\Big).$$

Also

$$\mathrm{e}^{S_m(T)} = \prod_{k=1}^{m} \mathrm{e}^{\frac{\delta^{k-1}}{\log^k T}}.$$

Note that

$$\mathrm{e}^{\frac{\delta^{k-1}}{\log^k T}} = \sum_{i=0}^{\infty} \frac{1}{i!}\frac{\delta^{(k-1)i}}{\log^{ki} T} = 1 + \sum_{1 \leq i \leq \frac{m}{k}} \frac{1}{i!}\frac{\delta^{(k-1)i}}{\log^{ki} T} + O_m\Big(\frac{1}{\log^{m+1} T}\Big).$$

Hence, as $T \to \infty$, we get

$$\mathrm{e}^y = \prod_{k=1}^{m}\left(1 + \sum_{1 \leq i \leq \frac{m}{k}} \frac{1}{i!}\frac{\delta^{(k-1)i}}{\log^{ki} T} + O_m\Big(\frac{1}{\log^{m+1} T}\Big)\right).$$

Therefore

$$\mathrm{e}^y = 1 + \sum_{j=1}^{m} \frac{\eta_j}{\log^j T} + O_m\Big(\frac{1}{\log^{m+1} T}\Big), \tag{11}$$

where the coefficients η_j are computable constants in terms of δ. Finally, we observe that the relation (10) asserts that $\mathscr{G}(T) = (T/\mathrm{e})\mathrm{e}^y$. By using this relation and (11) we get (4).

2.3 Proof of (5)

We use (10) to write $1/\mathscr{G}(T) = (\mathrm{e}/T)\mathrm{e}^z$, where

$$z = -S_m(T) + O_m\Big(\frac{1}{\log^{m+1} T}\Big).$$

As $T \to \infty$,

$$\mathrm{e}^z = \mathrm{e}^{-S_m(T)}\left(1 + O_m\Big(\frac{1}{\log^{m+1} T}\Big)\right) = \mathrm{e}^{-S_m(T)} + O_m\Big(\frac{1}{\log^{m+1} T}\Big),$$

and

$$\mathrm{e}^{-S_m(T)} = \prod_{k=1}^{m}\left(1 + \sum_{1\le i\le \frac{m}{k}} \frac{(-1)^i}{i!}\frac{\delta^{(k-1)i}}{\log^{ki} T} + O_m\Big(\frac{1}{\log^{m+1} T}\Big)\right).$$

Hence

$$\mathrm{e}^z = 1 + \sum_{j=1}^{m}\frac{\kappa_j}{\log^j T} + O_m\Big(\frac{1}{\log^{m+1} T}\Big), \tag{12}$$

where the coefficients κ_j are computable constants in terms of δ. Thus, we obtain

$$\frac{1}{\mathscr{G}(T)} = \frac{\mathrm{e}}{T}\left(1 + \sum_{j=1}^{m}\frac{\kappa_j}{\log^j T} + O_m\Big(\frac{1}{\log^{m+1} T}\Big)\right).$$

By multiplying this relation and (3) we get (5).

2.4 Proof of Theorem 1

We read the sequences $\mathscr{A}_n$ and $\mathscr{G}_n$ as $\mathscr{A}_n = \mathscr{A}(\gamma_n)$ and $\mathscr{G}_n = \mathscr{G}(\gamma_n)$. Corollary 1.1 of [3] asserts that

$$\gamma_n = \frac{2\pi n}{\log n}\Big(1 + (1 + o(1))\frac{\log\log n}{\log n}\Big). \tag{13}$$

We apply the relation (13) and the expansion $\log(1+t) = t + O(t^2)$, which holds as $t \to 0$, with $t = (1+o(1))\frac{\log\log n}{\log n}$. So, as $n \to \infty$,

$$\log \gamma_n = (\log n)\,(1 + \mathscr{E}_1(n))\,,$$

where

$$\mathscr{E}_1(n) = -\frac{\log\log n}{\log n} + \frac{\log(2\pi)}{\log n} + (1 + o(1))\frac{\log\log n}{\log^2 n}.$$

We have $(1 + \mathscr{E}_1(n))^{-1} = 1 + \mathscr{E}_2(n)$, where

$$\mathscr{E}_2(n) = \frac{\log\log n}{\log n} - \frac{\log(2\pi)}{\log n} + O\left(\left(\frac{\log\log n}{\log n}\right)^2\right).$$

By putting the above relations in (5) we obtain

$$\frac{\mathscr{A}_n}{\mathscr{G}_n} = \frac{\mathrm{e}}{2}\left(1 + \sum_{\ell=1}^{m} \lambda_\ell \left(\frac{1 + \mathscr{E}_2(n)}{\log n}\right)^{\ell} + O_m\left(\frac{1}{\log^{m+1} n}\right)\right).$$

Taking $m = 2$ gives (1). This completes the proof of Theorem 1.

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Minimal Kernels and Compact Analytic Objects in Complex Surfaces

Samuele Mongodi and Giuseppe Tomassini

Abstract In this paper, we want to study the link between the presence of compact objects with some analytic structure and the global geometry of a weakly complete surface. We begin with a brief survey of some now classic results on the local geometry around a (complex) curve, which depends on the sign of its self-intersection and, in the flat case, on some more refined invariants (see the works of Grauert, Suzuki, Ueda). Then, we recall some results about the propagation of compact curves and the existence of holomorphic functions (from the works of Nishino and Ohsawa). With such considerations in mind, we give an overview of the classification results for weakly complete surfaces that we obtained in two joint papers with Slodkowski (see Mongodi et al. (Indiana Univ. Math. J., **67**(2), 899–935 (2018); Int. J. Math., **28**(8), 1750063, 16 (2017))) and we present some new results which stem from this somehow more local (or less global) viewpoint (see Sections 4.2, 4.3, and 5).

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1 Introduction

The Levi problem, in its broadest formulation, asks for geometric conditions to guarantee that a given complex space is (a modification of) a Stein space or, from another point of view, what are the geometric obstructions to the existence of "many" holomorphic functions.

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The geometric characteristics we look for are usually encoded in the existence of some particular defining function (for domains contained in an ambient space) or of some particular exhaustion function. As it is well known the class of functions that turns out to work well with holomorphic functions is the class of (*strictly*) *plurisubharmonic functions*. The original formulation of the Levi problem, i.e., that every domain in $\mathbb{C}^n$ with smooth pseudoconvex boundary is a domain of holomorphy, was solved by Oka [26, 27], Bremermann [1], Norguet [24]. A few years later, Grauert tackled and proved the generalization of the Levi problem to a complex manifold, proving that a complex manifold is Stein if and only if it admits a smooth strictly plurisubharmonic exhaustion function [6], and Narasimhan generalized the result to complex spaces [20, 21], allowing the exhaustion function to be just continuous.

A natural question is to ask what happens if we allow the exhaustion to be only plurisubharmonic, i.e., if we allow its Levi form to degenerate somewhere. The resulting spaces are called *weakly complete*. Quite obviously, the class of weakly complete spaces includes Stein spaces, but it is not limited to them, as any space which is proper over (i.e., admits a holomorphic proper surjective map onto) a Stein space is weakly complete. Grauert produced an example of a weakly complete space whose only holomorphic functions are the constants, thus showing that not all weakly complete spaces are holomorphically convex (see [22]). As a partial converse, a result by Ohsawa, generalized to complex spaces by Vajaitu and the second author (see [25, 34] and Section 3.2), shows that, in dimension 2, a weakly complete (complex) surface is holomorphically convex if and only if it admits a nonconstant holomorphic function.

At first sight, it seems that under the hypothesis of weakly completeness there is no way to tell "how many" holomorphic functions exist on our space. However, a closer look at some examples (see Section 4.1) reveals that we obtain more precise information as soon as we look at *how far* our space X is from being Stein: for example, if we have a proper surjective holomorphic map $p : X \to Y$ with positive-dimensional fibers onto a Stein space Y, then every plurisubharmonic function on X will be constant on each fiber $p^{-1}(y)$ for $y \in Y$ and every holomorphic function on Y will give, by pullback, a holomorphic function on X; on the other hand, in Grauert's example, X contains Levi-flat hypersurfaces whose Levi foliation has dense leaves, therefore $\mathcal{O}(X) = \mathbb{C}$.

In order to study the obstructions that prevent the existence of a strictly plurisubharmonic function on a complex manifold (or a complex space) X, Slodkowski and the second author introduced in [32] the concept of the minimal kernel Σ_X, as the set of points where no exhaustion function for X can be strictly plurisubharmonic (see Section 2). The crucial property of Σ_X is the following: the intersections of Σ_X with the regular level sets of a plurisubharmonic exhaustion function enjoy the so-called local maximum property and, in dimension 2, they are locally a union of complex discs. These compact sets, obtained by slicing Σ_X with the regular level sets of a plurisubharmonic exhaustion function, and their complex structure play a fundamental role in the classification theorem of weakly complete surfaces proved in [18] (see also Section 4.2). The other fundamental ingredient was the presence of a real-analytic plurisubharmonic exhaustion function, which allows us to create a bridge between the local geometry, the geometry of a level set, and the geometry of the whole surface.

In the quoted paper [18], on the classification of weakly complete surfaces, two kinds of *compact objects with an analytic structure* appeared, playing a significant role:

- compact complex curves, embedded in X,
- immersed complex curves, with compact closure in X.

They both force the plurisubharmonic functions to have degenerate Levi form, but they behave quite differently in terms of holomorphic functions. Also, their contributions to the global geometry of the surface do not immediately seem equivalent. In particular, by a result of Nishino (see [23] and Section 3.2), the presence of a "generic enough" compact curve forces the whole weakly complete surface to be a union of compact curves, hence holomorphically convex with Remmert reduction of dimension 1. On the other hand, immersed curves with compact closure can easily force any holomorphic function to be constant, as soon as the closure is "large enough" (e.g., contains a 3-dimensional stratum, as a real-analytic set, which is the case in the presence of a real-analytic plurisubharmonic exhaustion function), but they do not seem to necessarily "propagate" to the whole surface, without some additional hypothesis. When a real-analytic plurisubharmonic exhaustion exists, the presence of an immersed curve with compact closure forces the whole surface (possibly outside an analytic set) to be foliated in 3-dimensional Levi-flat hypersurfaces, in turn foliated with dense complex leaves. We called such surfaces "of Grauert type."

The aim of this paper is twofold.

On one side, we give an account of what is known about the geometry of weakly complete surfaces, through the study of the compact sets we mentioned above. We start by recalling the results by Grauert [7] and Suzuki [33] on the neighborhood of a "negative" curve (i.e., with self-intersection $(C^2)<0$) or a "positive" curve (i.e., with self-intersection $(C^2)>0$), respectively, and move on to describe the classification of curves with zero self-intersection obtained by Ueda [35]. Then, we study under which conditions a compact curve propagates to generate a family of compact curves, by recalling the works by Nishino [23] and Ohsawa [25]. The link between the presence of compact curves and the global geometry is made explicit, in terms of the classification results proved in [18]. We also briefly recall the results about the geometric structure of the Grauert-type surfaces, proved in [17]. Finally, using the result of [18] we give a slightly different proof of a statement by Brunella [3] on the non-existence of a real-analytic plurisubharmonic exhaustion function on some classes of surfaces.

On the other side, we present some new results which stem from this somehow more local (or less global) viewpoint.

We show that the classification result from [18] essentially holds when the hypotheses are true outside of a compact (see Theorem 4.8 and Corollary 4.10).

We give a classification theorem for coronae (see Section 4.3 for the precise definition), obtaining, also here, three cases (see Theorem 4.12): Grauert-type coronae, coronae which are proper over an open complex curve, coronae which are an increasing union of subcoronae with one strictly pseudoconvex boundary component. We show, by an example of Rossi [29], that the last case is not always the complement of a compact set in a (modification of a) Stein space.

We prove some results for weakly complete surfaces with a plurisubharmonic exhaustion function which is supposed to be only smooth, in the hypothesis that the minimal kernel coincides with the whole surfaces. In this case, our surface is either proper over an open complex curve or the regular level sets of the exhaustion are Levi flat and foliated by dense complex curves, as in Grauert-type surfaces (Theorem 5.4).

Finally, we introduce some generalizations of the notion of minimal kernel, due to Slodkowski [31], and employ one of them to give a condition under which the Levi problem for a pseudoconvex domain in a general complex manifold has a positive answer (see Theorem 5.14).

The content of the paper is organized in 4 sections. In Section 2, we recall the basics on the minimal kernel and its slices along the level sets of a plurisubharmonic exhaustion function. Section 3 is devoted to the study of the nature and the propagation of compact curves in a complex surface. In Section 4, we collect some classification results, known and new, and some examples. Finally, Section 5 contains the results on the smooth case, the description of the generalizations of the minimal kernel and its application to the Levi problem.

2 The Minimal Kernel and Its Slices

A (reduced, connected) complex space X is said *weakly complete* if there exists a (smooth, continuous) plurisubharmonic exhaustion function $\phi : X \longrightarrow \mathbb{R}$ which is plurisubharmonic; obviously, a compact space trivially satisfies the request, so we will further assume that X is noncompact.

We know, from the results of Grauert and Narasimhan, that when the exhaustion function can be taken to be everywhere strictly plurisubharmonic, the space is Stein; in general, it is a famous and hard problem to understand under which weaker conditions a space is a Stein space or a modification of a Stein space. Therefore, we study the obstructions that force a plurisubharmonic function to have a degenerate Levi form.

Definition 2.1 The $\mathcal{C}^k$ minimal kernel of X is the set of points $x \in X$ such that every $\mathcal{C}^k$ plurisubharmonic exhaustion function fails to be strictly plurisubharmonic at x and it is denoted by Σ_X^k. A $\mathcal{C}^k$ *minimal function* is a function $\phi \in \mathcal{C}^k(X)$ which is plurisubharmonic on X and strictly so exactly on $X \setminus \Sigma_X^k$.

We will usually employ the $\mathcal{C}^\infty$ minimal kernel and we will denote it simply by Σ_X; the definition and the main properties of the minimal kernel of a weakly complete space appeared first in [32]. We recall that a set $Y \subset X$ is called a *local maximum set* if for every $x \in Y$, there exists a neighborhood U of x with the following property: for every compact $K \subset U$ and every (smooth) plurisubharmonic function ψ defined in a neighborhood of K,

$$\max_{Y \cap K} \psi = \max_{Y \cap bK} \psi,$$

where the maximum over an empty set is understood to be $-\infty$. We will say that a set is a local maximum set or that it has the local maximum property.

Two significant results regarding the minimal kernel of a weakly complete space are the following.

Proposition 2.2 *Let $u : X \to \mathbb{R}$ be a smooth plurisubharmonic exhaustion function; then for every $c \in \mathbb{R}$*

$$\Sigma_X \cap \left\{x \in X \ : \ u(x) = c\right\}$$

is either empty or has the local maximum property.

See [32, Theorem 3.6] for the case when u is a minimal function, [18, Theorem 3.2] for the general case, [31, Lemma 4.8] for a further generalization.

Proposition 2.3 *Let X be a complex surface, $u : X \to \mathbb{R}$ be a smooth plurisubharmonic exhaustion function and Y be a connected component of the level set $\{x \in X \ : \ u(x) = c\}$ such that $Y \cap \Sigma_X \neq \emptyset$. Then, for every point $p \in Y_{reg} \cap \Sigma_X$, there exist an open neighborhood $U \subset X$ and local coordinates $z,\ w$ on U such that $U \cong \Delta_z \times \Delta_w$ and*

$$U \cap Y \cap \Sigma_X \cong \bigcup_{t \in T} \left\{(z, f_t(z)) \ : \ z \in \Delta_z\right\}$$

where each $f_t : \Delta_z \to \Delta_w$ is a holomorphic function.

See [32, Lemma 4.1] for the case when u is a minimal function, [18, Proposition 3.5] for the general case.

From these two results, there is an obvious relation between the presence of compact subspaces and the minimal kernel, at least in dimension 2.

In general, one can say that a compact subspace or an immersed complex space with compact closure belong to the minimal kernel; we do not know examples of spaces whose minimal kernel has a connected component which is not a union of the former.

One striking property of the minimal kernel is the following propagation result (see [32, Theorem 3.9]).

Theorem 2.4 *Let X be a weakly complete manifold of dimension ≥ 2 and $\phi : X \to \mathbb{R}$ be a $\mathcal{C}^2$ plurisubharmonic exhaustion function. Let $r > \min \phi$ and let Y be a connected component of $\{x \in X \ : \ \phi(x) = r\}$, relatively open in the latter and that does not contain local minimum points of ϕ.*

If Y is a local maximum set, there exists $s < r$ such that the topological boundary of the connected component K of $\{x \in X \ : \ s < \phi(x) < r\}$ containing Y is contained in $Y \cup \{x \in X \ : \ \phi(x) = s\}$. Then

(a) *K is a connected compact set with nonempty interior;*
(b) *the forms*

$$(\partial\bar{\partial}\phi)^{n-1} \wedge \partial\phi \wedge \bar{\partial}\phi, \quad (\partial\bar{\partial}\phi)^{n-1} \wedge \partial\phi, \quad (\partial\bar{\partial}\phi)^{n-1} \wedge \bar{\partial}\phi$$

vanish on K and $(\partial\bar{\partial}\phi)^n$ vanishes on $K \setminus Y$,
(c) *every level set*

$$\left\{x \in K \ : \ \phi(x) = t\right\},$$

for $s \leq t \leq r$, has the local maximum property.

An immediate consequence on Σ_X is the following, obtained by assuming that the level set considered is regular.

Corollary 2.5 *Let X be a weakly complete manifold of dimension ≥ 2 and $\phi : X \to \mathbb{R}$ be a C^2 plurisubharmonic exhaustion function. If there is a regular value r such that Σ_X contains a connected component of the corresponding level set, then there is $s < r$ such that Σ_X contains a connected component of the set $\left\{x \in X \ : \ s \leq \phi(x) \leq r\right\}$.*

Gaining an understanding about the geometry of Σ_X would give us insights on the kind of obstructions that prevent a weakly complete space from being (a modification of) a Stein space.

3 Compact Complex Curves

As we saw, in complex surfaces, the obstruction to having a strictly plurisubharmonic exhaustion is linked to the presence of complex curves (embedded compact curves or immersed curves with compact closure). In this section, we want to collect and comment some results about the presence of compact curves in complex surfaces; there are two types of results we are interested in:

- how the presence of a compact curve affects, in a neighborhood, the behavior of holomorphic and plurisubharmonic functions;
- which conditions guarantee the “propagation” of a compact curve.

In other words, what properties ensure that a compact curve cannot be contained in a weakly complete surface? What does the presence of the curve tell us about the surface? When does the curve belong to a family of complex curves, e.g., the fibers of a proper map?

3.1 The Neighborhood of a Compact Curve

Grauert Criterium

The first relevant result, in chronological order, is Grauert's investigation of *exceptional curves* or, more generally, *exceptional subspaces* of a complex space X.

Recall that a complex compact subspace Y of X is said *exceptional* if there is a proper holomorphic map $p : X \to X_0$ on a complex space X_0, such that $p(Y)$ is a point x_0 and $p_{|X \setminus Y} : X \setminus Y \to X_0 \setminus \{x_0\}$ is an isomorphism. By definition, exceptional subspaces are isolated.

A holomorphic vector bundle $\mathsf{F} \to X$ is said *negative* if the zero section $\mathbf{0}_{\mathsf{F}}$ of F has a strongly pseudoconvex neighborhood U, in particular U is holomorphically convex. By Cartan theorem on quotient spaces [4] a neighborhood $U' \Subset U$ of $\mathbf{0}_{\mathsf{F}}$ is proper on a Stein space U_0 and $\mathbf{0}_{\mathsf{F}}$ is exceptional in U.

We have the following (see [7, Satz 8])

Theorem (Grauert Criterion) *The normal bundle $\mathsf{N}_{Y/X}$ is negative (i.e., Y as 0-section of $\mathsf{N}_{Y/X}$ is exceptional) then Y is exceptional in X.*

If X is a nonsingular complex surface and C is a compact, nonsingular complex curve, then $\mathsf{N}_{C/X}$ is negative if and only if the self-intersection (C^2) of C is negative.

From Grauert criterium we deduce, in particular, that if $(C^2) < 0$, no compact (possibly singular) complex curve can be present near C.

The remaining cases $(C^2) > 0$, $(C^2) = 0$ were studied by Suzuki (see [33]) and Ueda (see [35]), respectively.

We say that C is *negative*, *positive*, *flat* if $(C^2) < 0$, $(C^2) > 0$, $(C^2) = 0$, respectively.

Positive Curves

The case of positive curves was studied by Suzuki; he obtained, as it may be expected, the opposite of Grauert's result (see [33, Proposition 2.2]).

Theorem (Suzuki) *If X is a nonsingular complex surface and $C \subset X$ is a compact nonsingular complex curve such that $(C^2) > 0$, then C has a fundamental system of strongly pseudoconcave neighborhoods. In particular, $\mathcal{O}(X) = \mathbb{C}$.*

In this case many compact complex curves can be present in an arbitrary neighborhood of C (e.g., $\mathbb{CP} \subset \mathbb{CP}^2$); however, no nonconstant holomorphic or plurisubharmonic function exists in a neighborhood of C, by pseudoconcavity.

Ueda's Paper

In [35], Ueda considers compact nonsingular complex curves C which are flat, i.e., such that $\mathsf{N}_{C/X}$ is topologically trivial. It is well known that $\mathsf{N}_{C/X}$ is represented by an element of $H^1(C, \mathbb{S}^1)$, the *Picard variety* Pic$_0(C)$.

The paper of Ueda is very deep. In order to state the main results of [35] we need some preliminary definitions

Definition 3.1 A line bundle $\mathsf{L} \to C$ is said to be of *finite order* m if L^{-m} is holomorphically trivial and m is the minimum integer with this property. If no such m exists, L is said to be of *infinite order*. The *order* of C is the order of its normal bundle $\mathsf{N}_{C/X}$. It is denoted by ord(C), $1 \leq \mathrm{ord}(C) \leq +\infty$.

The next fundamental concept is the *type* of the curve C.

Let $\{V_j\}_{1\leq j\leq m}$ be a covering of C by bidiscs $V_j = \{|z_j| < 1, |w_j| < 1\}$, where z_j and w_j are local holomorphic coordinates and $C_j := C \cap V_j = \{w_j = 0\}$. Let $U_j = V_j \cap C$ and $\mathcal{U} = \{U_j\}_{1\leq j\leq m}$. Then the coordinates $w_{(j)}$ can be chosen in such a way that $\mathsf{N}_{C/X}$ can be represented by a cocycle

$$t_{jk} = \left(w_j/w_k\right)_{|U_j\cap U_k} \in \mathrm{Z}^1(\mathcal{U}, \mathbb{S}) \tag{3.1}$$

Let $\{t_{jk}\}$, $\{w_j\}$ be fixed; then $w_j - t_{jk}w_k$ is vanishing on $U_j \cap U_k$ so $w_j - t_{jk}w_k = f_{jk}(z_j)w_j^{\nu+1}$ with $f_{jk}(z_j) \neq 0$ holomorphic. The system $\{w_j\}$ is then said of *type* ν. One check easily that $\{f_{jk}\}$ is a cocycle with values in $\mathsf{N}_{C/X}^{-\nu}$: $\{f_{jk}\} \in \mathrm{Z}^1(\mathcal{U}, \mathsf{N}_{C/X}^{-\nu})$.

The cocycle $\{f_{jk}\}$ is called the ν^{th} *obstruction* associated to the system $\{w_j\}$.

Definition 3.2 The curve C is said of *finite type* n (type$(C) = n$) if there exists a system $\{w_j\}$ of type n such that n^{th} obstruction associated to $\{w_j\}$ is not cohomologous to zero. The curve C is said of *infinite type* (type$(C) = +\infty$) if the obstruction associated to every system $\{w_j\}$ is cohomologous to zero.

The idea under the previous computations is to measure the degree of coincidence of the extension of $\mathsf{N}_{C/X}$ to a neighborhood of C as a flat bundle and the line bundle $[C]$, corresponding to the divisor of C.

It is a simple matter to prove that the previous definition is well posed; namely, that

1) if there exists a system of type n whose n^{th} obstruction is not cohomologous to zero, then no system $\{w_j\}$ exists of type $\nu > n$;
2) if there exists a system of type n whose n^{th} obstruction is not cohomologous to zero, for every system of type $\nu < n$, the ν^{th} obstruction is cohomologous to 0;
3) if the obstruction associated to every system $\{w_j\}$ is cohomologous to zero, then for every ν there exists a system of type ν;
4) the type of a curve does depend neither on the covering $\mathcal{U}$ nor on the cocycle $\{t_{jk}\}$.

Remark If the order is infinite, the type is infinite as well.

The type and the order allow us to divide compact nonsingular complex curves into four classes:

- if $\mathrm{ord}(C)$ is finite and $\mathrm{type}(C)$ is finite, we say that C belongs to Class α
- if $\mathrm{ord}(C)$ is finite, but $\mathrm{type}(C)$ is infinite, we say that C belongs to Class β'
- if $\mathrm{ord}(C)$ is infinite, then also $\mathrm{type}(C)$ is infinite and we identify two classes
  - if there exist a neighborhood V of C and a holomorphic function u on a covering manifold of V such that $|u|$ defines a single-valued function on V and the support of the divisor of u is C, then we say that C belongs to Class β''
  - otherwise, we say that C belongs to Class γ.

The situation near a curve C in Class α is summarized by the following results.

(α_1) For every $n' > \mathrm{type}(C)$ there exists a neighborhood V_0 of C and a strongly plurisubharmonic function $\Phi : V_0 \smallsetminus C \longrightarrow \mathbb{R}$ such that $\Phi(p) \sim \mathrm{dist}((p)^{-n'}$ as $p \to C$. It follows that, for $c \to +\infty$ the sets $\{\phi > c\} \cup C$ give a fundamental system of strongly pseudoconcave neighborhoods of C. In particular $\mathcal{O}(X) = \mathbb{C}$.

(α_2) Let V an open (connected) neighborhood of C and $\Psi : V \smallsetminus C \to \mathbb{R}$ a strongly plurisubharmonic function such that $\Psi(p) = o(\mathrm{dist}((p)^{-n''}), 0 < n'' < n$, as $p \to C$. Then Ψ is constant near C. In particular, if $f \in \mathcal{O}(V \smallsetminus C)$ and $\log^+ |f(p)| = o(\mathrm{dist}((p)^{-n''})$, then f is constant.

For the curves in Class β', we have the following.

(β') If $\mathrm{ord}(C) = m$, on a neighborhood V of C, there is a multivalued holomorphic function u such that $u^m \in \mathcal{O}(V)$ and whose divisor is $m\,C$.

Conversely, it is easy to show that if there is a nonconstant holomorphic function on a neighborhood of C, then C belongs to Class β'.

Finally, by definition, if C belongs to Class β''

(β'') on a neighborhood V of C, there is a multivalued holomorphic function u such that $|u|$ is single-valued on V and whose divisor is $m\,C$.

We analyze more closely what happens near a curve of Class β' or Class β''.

If $C \in$ Class $\beta' \cup$ Class β'', then there exists a multivalued holomorphic function in a neighborhood V of C. Let $\varepsilon > 0$ such that $V_\varepsilon := \{p \in V : |u(p)| > \varepsilon\} \Subset V$. For all r with $0 < r < \varepsilon$, the boundary $\mathrm{b}V_\varepsilon$ of V_ε is defined by the pluriharmonic function $\log |u| - \log \varepsilon = 0$, so it is Levi flat. The neighborhoods $V_\varepsilon{}'s$ are said *pseudoflat*.

Observe that, due to the presence of the nonconstant pluriharmonic function $\log |u|$ no neighborhood $W \Subset V$ can be strictly pseudoconcave. Moreover, if C belongs to Class β' for $|c| \le \varepsilon^m$ the curves $u^m = \mathrm{const}$ are compact and irreducible.

Suppose now that C is in Class β'' and let $\Sigma_r := \{p \in V : |u(p)| > \varepsilon\} \Subset V, 0 < r < \varepsilon$. Σ_r is Levi flat hence foliated by holomorphic curves. Then (see [35, Sections 2 and 3]) u defines a holomorphic foliation $\mathcal{F}$ on V such that every leaf of $\mathcal{F}$, except for C, is contained in some Σ_r and dense in it. It follows that every plurisubharmonic function on a neighborhood of a Σ_r is constant on Σ_r. Consequently, $\mathcal{O}(V) = \mathbb{C}$.

Curves Near C

Saving the same notations, let V be a fixed tubular neighborhood of C and Γ a 2-cycle. Since $\mathsf{N}_{C/X}$ is topologically trivial, $\Gamma \sim mC$ for some $m \in \mathbb{Z}$, hence $(\Gamma, C) = 0$. In particular, if $\Gamma \neq C$ is a compact, irreducible complex curve, then $\Gamma \cap C = \varnothing$, i.e., $\Gamma \subset V \setminus C$.

i) in view of (α_1), C does not belong to Class α
ii) if C belongs to Class β', then by (β'_1) the compact curves $u^m = \text{const}$ are the only ones belonging to $V \setminus C$
iii) by similar arguments one proves that if C is in Class β'', then in a neighborhood of C there is no compact complex curve other than C.

The situation concerning the curves belonging to Class γ is rather mysterious (see [35]).

Remark 3.3 There are at most countably many curves of Class α or Class β''; indeed, every such curve has an open neighborhood where no other curve is present, hence there can be at most countably many.

3.2 Propagation of Compact Curves

Nishino's Paper

In the paper [23] the existence of a nonconstant holomorphic function $f : X \to \mathbb{C}$ (and more generally of holomorphic maps $X \to R$ where R is a Riemann surface) is proved under the condition that X contains at least a *generic curve*. Let us recall some preliminary results proved in [23].

We start from the definition of a generic curve: morally, a curve S is called generic if it is locally "movable," i.e., if there are a neighborhood U of S and a holomorphic function $F \in \mathcal{O}(U)$ such that $\{x \in U \;:\; F(x) = 0\} = S$. However, the original definition by Nishino is more involved.

Let $S \subset X$ be a compact, nonsingular complex curve, $\{V_j\}_{1\leq j\leq m}$ a covering of S by bidiscs $V_j = \{|z_{(j)}| < 1, |w_{(j)}| < 1\}$, where $z_{(j)}, w_{(j)}$ are local holomorphic coordinates and $S_j := S \cap V_j = \{w_{(j)} = 0\}$. Such a covering is said *canonical*.

Let $(\mathcal{P}) = \{f_j\}_{1\leq m}$, $f_j : V_j \to \mathbb{C}$ meromorphic, be a datum for the additive Cousin problem on $V := \cup_{j=1}^m V_j$, so $f_{ij} = f_i - f_j \in \mathcal{O}(V_i \cap V_j)$ for all $i,\ j$, and let $(\mathcal{Z}) = \{g_j\}_{1\leq m}$, $g_j : V_j \to \mathbb{C}$ holomorphic, be a datum for the multiplicative Cousin problem on V, so $g_{ij} = g_i/g_j \in \mathcal{O}(V_i \cap V_j)$ for all $i,\ j$.

- $(\mathcal{P})$ is said to be *solvable* on X if there exist holomorphic functions $\phi_j \in \mathcal{O}(S_j)$ such that $f_{jk|S_j \cap S_k} = \phi_j - \phi_k$ for all $1 \leq j, k \leq m$.
- $(\mathcal{Z})$ is said to be *solvable* on X if there exist nowhere vanishing holomorphic functions $\psi_j \in \mathcal{O}^*(S_j)$ such that $g_{jk|S_j \cap S_k} = \psi_j / \phi_k$.

Definition 3.4 The complex curve S is said *generic* if

a) every datum $(\mathcal{P}) = \{f_j\}_{1 \leq m}$ on V which has S as the only pole (i.e., for every j, $S \cap V_j$ is the polar set of the meromorphic function f_j) is solvable on S;
b) every datum $(\mathcal{Z}) = \{g_j\}_{1 \leq m}$ on V which has S as the only zero (i.e., for every j, $S \cap V_j$ is the zero set of the holomorphic function g_j) is solvable on S;

About the interplay between generic curves and holomorphic or meromorphic functions, the main results of Nishino are the following

1) A generic curve S of a complex surface X is the zero set of a function f which is holomorphic on a neighborhood U of S. (see [23, Proposition 7]). In other words a generic curve propagates locally.
2) Assume that a domain $D \Subset X$ contains a non-countable family $\mathcal{F}$ of connected compact complex curves S such that $S \cap S' = \varnothing$ whenever $S \neq S'$. Then $\mathcal{F}$ contains at least one generic curve (see [23, Proposition 9]).

These results globalize via the study of normal families of compact curves.

3) Let X be a weakly complete or compact surface that contains at least one generic curve. Then there exist a Riemann surface R and a meromorphic map $f : X \to R$ with compact fibers.
4) If X is weakly complete and contains at least one generic curve, then it contains at least one nonconstant holomorphic function with compact fibers. In particular, X is holomorphically convex.

See [23, Section 5].

It is clear, *a posteriori*, that a curve is generic in the sense of Nishino only if it belongs to Ueda's Class β'. Moreover, using the solvability of the two Cousin's problems along S, it is easy to show that a generic curve has to be flat, of infinite type and finite order, hence it has to belong to Ueda's Class β'.

Ohsawa's Paper and Its Generalization

In [25] Ohsawa proved the following result (see [25, Proposition 1.4]).

Theorem 3.5 *Let X be a (connected) weakly complete nonsingular complex surface such that $\mathcal{O}(X) \supsetneq \mathbb{C}$. Then X is holomorphically convex.*

This result was generalized to weakly 1-complete complex surfaces in [34].

The main tool used by Ohsawa was an observation on the topology of the level sets of a holomorphic function, which holds in every dimension, not only for surfaces (see [25, Theorem 1.1]).

Proposition 3.6 *Let X be a weakly complete manifold, with a plurisubharmonic exhaustion function $\phi : X \to \mathbb{R}$, and let $f \in \mathcal{O}(X)$ be a nonconstant holomorphic function, then either*

i) $f^{-1}(z) \cap \{x \in X \,:\, \phi(x) < c\}$ *is empty or noncompact for all* $z \in \mathbb{C}$ *and* $c \in \mathbb{R}$

or

ii) $f^{-1}(z) \cap \{x \in X \,:\, \phi(x) < c\}$ *is compact for all* $z \in \mathbb{C}$ *and* $c \in \mathbb{R}$.

The key fact to get the first result from this observation is that if X is of dimension 2, the fibers of f are of dimension 1, hence they are Stein if and only if they are noncompact.

Observe that the condition $\mathcal{M}(X) \supsetneq \mathbb{C}$ for weakly complete surface X does not imply that X is holomorphically convex (take for X the surface U of the Example 4.4 below).

In higher dimension the existence of one holomorphic function is not enough to grant holomorphic convexity. As a trivial example take $Z = X \times Y$ where X is again the surface U of the Example 4.4 and Y is a Stein curve; we think the following statement could be a suitable generalization of Ohsawa's theorem in higher dimensions.

Conjecture Let Z be a weakly complete complex space of dimension $n + 1$, $n \geq 1$, and $f_1, \ldots, f_n \in \mathcal{O}(Z)$ analytically independent (i.e., $df_1 \wedge \ldots \wedge df_n \neq 0$) holomorphic functions. Then Z is holomorphically convex.

4 Weakly Complete Surfaces

In this section we look into the geometry of weakly complete surfaces. As we saw, in dimension two the minimal kernel carries a natural analytic structure and the presence of compact curves affects quite heavily the global geometry of the surface.

As we will see in the next pages, however, there are examples where the minimal kernel is not composed of compact curves, but of immersed complex curves with compact closure or, more precisely, of Levi-flat 3-dimensional hypersurfaces whose Levi foliation has dense complex leaves. This kind of phenomenon does not have the same "propagation" property as the presence of compact curves; therefore, we need some hypothesis that ensures us that we can extend this information from a level set to the whole surface. This is why we ask for the existence of a *real-analytic* plurisubharmonic exhaustion function and, under such hypothesis, we prove a classification result.

This hypothesis is not always verified, as an example taken from [3] shows. In that case, our classification holds where the exhaustion can be taken to be real analytic. We also study the case of coronae, where our classification result carries over with minimal modifications.

4.1 Examples and Remarks

We present some examples of weakly complete surfaces, studying their geometry and the presence of holomorphic and meromorphic functions.

Example 4.1 Let a_1, a_2 be complex numbers such that

$$0 < |a_1| \leq |a_2| < 1, \;\; a_1^k \neq a_2^l$$

for all $(k, l) \in \mathbb{N}^2 \setminus \{(0, 0)\}$ and define τ by $|a_1| = |a_2|^\tau$; by hypothesis $\tau \notin \mathbb{Q}$.

Consider on $\mathbb{C}^2 \setminus \{(0, 0)\}$ the equivalence relation $\sim$: $(z_1, z_2) \sim (a_1 z_1, a_2 z_2)$. The quotient space $\mathbb{C}^2 \setminus \{(0, 0)\}/\sim$ is the Hopf manifold $\mathcal{H}$. Let π denote the projection $\mathbb{C}^2 \setminus \{(0, 0)\} \to \mathcal{H}$. The complex lines $\mathbb{C}_{z_1} = \{z_2 = 0\}$, $\mathbb{C}_{z_2} = \{z_1 = 0\}$ project into complex compact curves C_1, C_2, respectively. We note that the curves C_1 and C_2 are the only compact complex curves in $\mathcal{H}$.

Let $X = \mathcal{H} \setminus C_2$. The function

$$\Phi(z_1, z_2) = \frac{|z_2|^{2\tau}}{|z_1|^2}$$

on $\mathbb{C}^2 \setminus \{(0, 0)\}$ is $\sim$-invariant and so defines a function $\phi : X \to \mathbb{R}_{\{\geq 0\}}$; ϕ is proper and $\log \phi$ is pluriharmonic on $X \setminus C_1$.The level sets $\{\phi = c\}$ contained in $X \setminus C_1$ are the projections of the sets $|z_1| = c|z_2|^\tau$, $c > 0$, and so foliated by the projections of the complex sets $\{z_1 = ce^{i\theta} z_2^\tau\}$ which are everywhere dense leaves, τ being irrational. If $f \in \mathcal{O}(X)$, then by the maximum principle, f is constant on a set $\{\phi = c\}$ which is of dimension 3 and this implies that f is constant on X. In the same way one shows that no strongly plurisubharmonic function exists on X. We also have $\mathcal{M}(\mathcal{H}) = \mathbb{C}$. Indeed let $f \in \mathcal{M}(\mathcal{H})$ and $\tilde{f} = f \circ \pi$: $\tilde{f} \in \mathcal{M}(\mathbb{C}^2 \setminus \{(0, 0)\})$ so it extends as meromorphic function on $\mathbb{C}^2$. Then $\tilde{f} = P/Q$ with $P, Q \in \mathcal{O}(\mathbb{C}^2)$. Set

$$P(z_1, z_2) = \sum_{j,k=0}^{\infty} P_{jk} z_1^j z_2^k, \quad Q(z_1, z_2) = \sum_{\alpha,\beta=0}^{\infty} Q_{\alpha\beta} z_1^\alpha z_2^\beta.$$

Because of the $\sim$-invariance

$$P(a_1 z_1, a_2 z_2) Q(z_1, z_2) = P(z_1, z_2) Q(a_1 z_1, a_2 z_2)$$

and from this we derive the identities

$$P_{jk} Q_{\alpha\beta} a_1^j a_2^k = P_{jk} Q_{\alpha\beta} a_1^\alpha a_2^\beta$$

for all $j, k, \alpha, \beta \in \mathbb{N}$. Since, by hypothesis, $a_1^k \neq a_2^l$ for all $(k, l) \in \mathbb{N}^2 \setminus \{(0, 0)\}$ we get $P_{jk} = 0$ for $j + k > 0$, $Q_{\alpha\beta} = 0$ for $\alpha + \beta > 0$, i.e., $P = \text{const}$, $Q = \text{const}$.

We can also prove that X does not carry meromorphic function. Indeed, let $f \in \mathcal{M}(X)$ and $\tilde{f} = f \circ \pi$: $\tilde{f} \in \mathcal{M}(\mathbb{C}^2 \setminus \{z_1 = 0\})$. Then, since $\mathbb{C}^2 \setminus \{z_1 = 0\}$ is Stein

$$\tilde{f}(z_1, z_2) = \sum_{j=0}^{\infty} \tilde{f}_j(z_1, z_2) z_1^{-j}$$

with $\tilde{f}_j(z_1 z_2)$ entire. Again, using the $\sim$-invariance of $\tilde{f}$, we conclude that $\tilde{f}_j = 0$ for $j > 0$.

The curve C_1 is the minimum set of ϕ and it is the only compact curve of X. Since $\mathcal{O}(X) = \mathbb{C}$ the line bundle associated to C_1 is not trivial.

Example 4.2 With the notation of the previous example, we consider $X_1 = \mathcal{H} \setminus (C_1 \cup C_2)$ with plurisubharmonic exhaustion function $\alpha = (\log \phi)^2$. X_1 is a weakly complete surface, obviously of Grauert type. Here, however, the plurisubharmonic function α_1 has a 3-dimensional minimum set, namely the quotient of the Levi-flat surface of $(\mathbb{C}^*)^2$ given by

$$H_0 = \{(z_1, z_2) \in (\mathbb{C}^*)^2 \ : \ |z_2|^\tau = |z_1|\}.$$

The pluriharmonic function on X_1 is, obviously, $\log(\phi)$, i.e., a befitting choice of the square root of α_1.

Another class of example is provided by total spaces of some complex line bundles over compact Riemann surfaces (see also [35]).

Example 4.3 Let M be a compact Riemann surface of genus $g > 0$. It is well known that every topologically trivial line bundle can be represented by a flat unimodular cocycle, i.e., an element of $H^1(M, \mathbb{S}^1)$.

Consider a line bundle $L \to M$ with trivialization given by the open covering $\{U_j\}_{j=1}^n$ and transition functions $\{\xi_{ij}\}_{i,j}$ which represent a cocycle $\xi \in H^1(M, \mathbb{S}^1)$. We can define a function $\alpha : L \to \mathbb{R}$ by defining it on each trivialization as $\alpha_j : U_j \times \mathbb{C}, \alpha_j(x, w) = |w|^2$. As $|\xi_{ij}| = 1$, these functions glue into $\alpha : L \to \mathbb{R}$, which is readily seen to be plurisubharmonic and exhaustive.

Now, consider $r > 0$ and the section $f_1 \in \Gamma(U_1, \xi)$ given by $f(x) \equiv r$ for all $x \in U_1$; taking all possible analytic continuations of f_1 as a section of the bundle L, we construct , for every chain $\{U_{j_k}\}_{k \in \mathbb{N}}$ with $j_0 = 1$ and $U_{j_k} \cap U_{j_{k+1}} \neq \varnothing$, the sections $f_k \equiv \xi_{j_k j_{k-1}} \xi_{j_{k-1} j_{k-2}} \cdots \xi_{j_1 j_0} r \in \Gamma(U_{j_k}, \xi)$. Representing ξ as a multiplicative homomorphism $\psi_\xi : \pi_1(M) \to \mathbb{S}^1$, it is easy to see that the graphs of such sections glue into a compact complex manifold if and only if $\psi(\pi_1(M))$ is contained in the roots of unity, i.e., if and only if $L^{\otimes n}$ is (analytically) trivial for some n, i.e., if and only if ξ (as an element of the group $H^1(M, \mathbb{S}^1)$) is unipotent.

If that is not the case, the graphs of such sections glue into an imbedded, nonclosed, complex manifold, contained in the Levi-flat hypersurface $\alpha^{-1}(r^2)$ and dense in it. The other leaves of the Levi foliation are obtained by the one constructed multiplying it by $e^{i\theta}$.

Finally, we have a pluriharmonic function $\chi : L \setminus M \to \mathbb{R}$ given by $\chi(p) = \log \alpha(p)$.

Example 4.4 Let $\Gamma \subset \mathbb{C} \times \mathbb{R}$ be the lattice generated by $e_1 = (0, 0, 1)$, $e_3 = (1, 0, 0)$, $e_4 = (0, 1, \sqrt{2})$ and $\mathbb{T}$ the real torus $\mathbb{C} \times \mathbb{R}/\Gamma$. $\mathbb{T}$ is foliated by complex curves which are everywhere dense. Let $\widetilde{\Gamma}$ be the lattice generated by $\tilde{e}_1 = (0, 0, 1, 0)$, $\tilde{e}_2 = (0, 0, 0, 1)$, $\tilde{e}_3 = (1, 0, 0, , 0)$, $\tilde{e}_4 = (0, 1, \sqrt{2}, 0)$ and $\widetilde{\mathbb{T}}$ the complex torus $\mathbb{C}^2/\widetilde{\Gamma}$: $\widetilde{\mathbb{T}}$ is the complexification of $\mathbb{T}$, which is a subfoliation of $\widetilde{\mathbb{T}}$. $\widetilde{\mathbb{T}}$ is not algebraic [30]. Following [8, Sections 5 and 6]) we consider the matrix $E = (E(\tilde{e}_{jk}))$ where $E(\tilde{e}_{12}) = 1$, $E(\tilde{e}_{21}) = -1$, $E(\tilde{e}_{jk}) = 0$ if $(j, k) \neq (1, 2), (2, 1)$ and define the Hermitian form H on $\mathbb{C}^2$ by $H(\zeta, \zeta') = E(i\zeta, \zeta') + iE(\zeta, \zeta')$. To H corresponds a line bundle $\mathsf{L} \longrightarrow \widetilde{\mathbb{T}}$ whose restriction to $\mathbb{T}$ is positive. It follows that there exist a (connected) weakly complete neighborhood U of $\mathbb{T}$ and a positive line bundle $\widetilde{\mathsf{L}} \longrightarrow U$ which extends L. By a theorem of Hironaka (see [19]) U embeds in some $\mathbb{P}^N$ (see [8, Sections 5 and 6]). In particular $\mathcal{M}(U) \neq \mathbb{C}$. On the other hand $\mathcal{O}(U) = \mathbb{C}$ since every holomorphic function in U must be constant on $\mathbb{T}$ whence on U.

Remark 4.5 In the first two examples one has $\mathcal{O}(X) = \mathcal{M}(X) = \mathbb{C}$, while for the third $\mathcal{O}(X) = \mathbb{C}$ but $\mathcal{M}(X)$ contains many meromorphic functions. Moreover, in the Example 4.1 $X' = X \smallsetminus C_1$ is a corona (see Section 4.3) for which $\mathcal{O}(X') = \mathbb{C}$.

4.2 Classification Results

In the previous examples the weakly complete surfaces have an exhaustion function which is real analytic, but this is not true in general. Indeed, an example by Brunella (see [3, Theorem 1] and Section 4.4) shows that there exist weakly complete surfaces which do not admit a real-analytic exhaustion function. Without the real-analyticity as a bridge from local to global, a "classification" of weakly complete surfaces in the general case seems to be very hard, so we restrict ourselves to the real-analytic case.

We consider a complex surface X endowed with a real-analytic, plurisubharmonic exhaustion function $\alpha : X \to \mathbb{R}$. In a joint paper with Slodkowski [18], we classified such complex surfaces. By passing to a desingularization, the theorem stated for a complex surface X covers also the case of singular spaces.

Before getting involved in the main results of [18], we point out some consequences of the results we presented in the previous section.

- Clearly, there are no obstruction for X to contain compact negative curves.
- No compact positive curve and no compact curve belonging to Class α are present on X. Indeed, by Suzuki (see section "Positive Curves") and Ueda (section "Ueda's Paper") such a curve should have strongly pseudoconcave neighborhoods and this would force $\alpha : X \to \mathbb{R}$ to be constant.

- If X contains a curve C of Class β', there exist uncountably many compact complex curves near C (see Section 3.1, and property (β'_1)). Then by Nishino results (see section "Nishino's Paper") X is proper over an open holomorphic curve.
- If X contains C belonging to Class β'', then there is a neighborhood of C which is of Grauert type (see section "Curves Near C")
- If $\mathcal{O}(X) \neq \mathbb{C}$, then, by Ohsawa's result, X is either a modification of a Stein space or, if it contains a non-negative curve, is proper over an open complex curve.

Question 1 Is it possible to have a weakly complete space that contains a compact curve belonging to Class γ? How do plurisubharmonic functions behave in the neighborhood of such curve?

We now turn our attention to the classification result for weakly complete complex surfaces X, endowed with a real-analytic plurisubharmonic exhaustion function $\alpha : X \to \mathbb{R}$.

Theorem 4.6 *Consider a nonsingular complex surface X as above. Then, one of the following three cases occurs:*

i) X is a modification of a Stein space of dimension 2
ii) X is proper over a (possibly singular) open Riemann surface
iii) the regular level sets of α are compact Levi-flat surfaces foliated with dense complex leaves.

A weakly complete surface X which carries a smooth plurisubharmonic exhaustion function φ whose regular level sets are Levi-flat hypersurfaces, foliated by dense complex leaves (along which the Levi form of φ degenerates) is said to be a space of *Grauert type* (see [17, 18]) as their structure generalizes an example by Grauert (see, for instance [22]);

Cases i) and ii) end up being holomorphically convex, with Remmert reductions of dimension 2 and 1, respectively, whereas, in the third case iii), no nonconstant holomorphic function exists: indeed, any holomorphic function would be constant along the complex leaves that foliate the regular levels of α, but then it would be constant on the whole level, which is of real dimension 3, so it would be constant on X.

The peculiar geometry of Grauert-type surfaces does not only affect holomorphic functions, but also plurisubharmonic functions. We have the following result.

Proposition 4.7 *Let X be a Grauert-type surface with a real-analytic plurisubharmonic exhaustion function $\alpha : X \to \mathbb{R}$ and let*

$$M = \left\{x \in X \ : \ \alpha(x) = \min_X \alpha\right\}.$$

Then, we have two possibilities:

iii-a) *M is a compact complex curve and there exists a proper pluriharmonic function χ on $X \setminus M$ such that every plurisubharmonic function on $X \setminus M$ is of the form $\gamma \circ \chi$*

iii-b) *M has real dimension 3 and there exist a double holomorphic covering map $\pi : X^* \to X$ and a proper pluriharmonic function $\chi^* : X^* \to \mathbb{R}$ such that every plurisubharmonic function on X^* is of the form $\gamma \circ \chi^*$.*

In both cases, γ is a convex, increasing real function.

In addition, it is also true that the set $\mathrm{Crt}(\alpha)$ of critical points of α has the same dimension of M.

The two previous statements are the content of the Main Theorem in [18]. Moreover, in [17], we analyze further the geometry of Grauert-type surfaces, obtaining the following results:

- the level sets of the pluriharmonic function χ (or χ^*) are connected, hence the function is somehow minimal,
- any compact curve not contained in M is negative in the sense of Grauert,
- there are examples of Grauert-type surfaces which do not possess any pluriharmonic function, but their double covering does.

We do not want to enter the details of the proofs, for which we refer the reader to the papers [17, 18] and the introductory note [16].

Here, we only want to remark that most of the methods used are of a "local" nature, i.e., they work in a neighborhood of a level of the exhaustion function. The main tool in our investigation was the minimal kernel Σ_X, defined in [32], and its intersections with the level sets of an exhaustion function.

As an example of the consequences of this "local" nature, we examine the case when all our hypotheses hold outside of a compact set.

We first observe that if X carries an exhaustion function ϕ which is plurisubharmonic away from a compact subset K, then X is weakly complete. Indeed, let c such that $K \subset \{\phi < c\}$ and take as a new exhaustion function $\psi := \max(\phi, c')$ with $c' > c$.

Then we deduce the following

Theorem 4.8 *Let X be a complex surface, $\phi : X \to \mathbb{R}$ an exhaustion function and $K \Subset X$ a compact set such that ϕ is real analytic and plurisubharmonic outside K. Then, one of the following cases occurs:*

1) *X is a modification of a Stein space*
2) *X is proper over a (possibly singular) open Riemann surface*
3) *$X \setminus K$ is of Grauert type.*

Proof By the above remark X is weakly complete so we can repeat verbatim the proof of [18, Theorem 4.4], restricting ourselves to $X \setminus K$. If we have a sequence $c_n \to +\infty$ such that $\{\phi = c_n\}$ is contained in $X \setminus K$ and is strictly pseudoconvex, then we obtain that X is a modification of a Stein space; if we have a regular level of ϕ in $X \setminus K$ containing a compact curve, then, by [18, Theorem 4.2], X would be foliated in compact complex curves. So, by [18, Corollary 4.3], all the regular level sets of ϕ contained in $X \setminus K$ are Levi-flat hypersurfaces foliated with dense complex leaves. □

One striking difference between the first two cases and the Grauert-type surfaces is the "propagation" of the information, when there is no real analyticity to act as a bridge from local to global.

If there exists $c \in \mathbb{R}$ such that the level set $\{x \in X : \alpha(x) = c\}$ is strictly pseudoconvex, then the sublevel set $\{x \in X : \alpha(x) < c\}$ is a modification of a Stein space; even more, if $\{x \in X : \alpha(x) = c\} \cap \Sigma_X = \emptyset$, then we can find a strictly pseudoconvex hypersurface, arbitrarily close to the level set of α; therefore, we can approximate the sublevel set with strictly pseudoconvex domains (which are modifications of Stein spaces), implying that also our sublevel is a modification of a Stein space.

If there exists $c \in \mathbb{R}$ such that the level set $\{x \in X : \alpha(x) = c\}$ contains uncountably many compact complex curves, or, equivalently, a generic compact complex curve (in the sense of Nishino, see section "Nishino's Paper"), then the whole manifold X is proper over an open complex curve; in this case, the information does not only extend to "fill the hole" in the sublevel set, but also to the whole of X.

If there exists $c \in \mathbb{R}$ such that $\{x \in X : \alpha(x) = c\}$ is of Grauert type, we are sure that no generic curves are present in X (otherwise X would be union of compact complex curves, by Nishino); we are also sure that all the level sets $\{x \in X : \alpha(x) = c'\}$ with $c' > c$ intersect Σ_X, otherwise the corresponding sublevel would be a modification of a Stein space containing a compact Levi-flat hypersurface, which is absurd. However, we are not able to say much about the level sets with $c' < c$.

This difference is at the core of the example by Brunella (see Section 4.4) of a weakly complete surface without real-analytic plurisubharmonic exhaustion functions; as a partial result, we note that we can exploit the existence of some particular pluriharmonic functions on a Grauert-type surface, in order to extend the information to a sublevel, given the appropriate topological condition.

By standard cohomologicaly techniques, we have the following extension result for pluriharmonic functions:

Proposition 4.9 *Suppose W is a complex manifold with $H^2_c(W, \mathbb{R}) = 0$, $U \subseteq W$ an open domain and $K \Subset U$ a compact set. Then every pluriharmonic function $\chi : U \setminus K \to \mathbb{R}$ has a pluriharmonic extension $\widetilde{\chi} : U \to \mathbb{R}$.*

Corollary 4.10 *Suppose X is a complex surface such that $H^2(X, \mathbb{R}) = 0$. If there exist an exhaustion function $\phi : X \to \mathbb{R}$ and $c \in \mathbb{R}$ such that*

1) $X_c = \{\phi \leq c\} \Subset X$ is connected
2) $X \setminus X_c$ is of Grauert type
3) ϕ is real analytic and plurisubharmonic on $X \setminus X_c$

then X is of Grauert type.

Proof For every $\epsilon > 0$ we can find $c' \in (c, c+\epsilon)$ such that $\{\phi = c'\}$ is a connected regular level, so, by [18, Corollary 4.3], we have a neighborhood V of $\{\phi = c'\}$ and a pluriharmonic function $\chi : V \to \mathbb{R}$, which is given by $\chi = \lambda \circ \phi$ on V.

Therefore, there exists $c'' \in (c, c')$ such that $\{c'' < \phi < c'\} \subseteq V$; we set $U = \{\phi < c'\}$ and $K = \{\phi \leq c''\}$. By duality $H^2_c(X, \mathbb{R}) = 0$, so Proposition 4.9 applies giving that χ extends to a pluriharmonic function on U. Its level sets in K are compact and Levi flat. If there is a compact complex curve in a regular level of χ, X is foliated in complex curves, by [18, Theorem 4.2], so, as $X \setminus K$ is of Grauert type, this is not the case. Hence, by [18, Corollary 4.3], the regular level sets of χ have dense complex leaves. □

This, in particular, implies that, in the situation described in Corollary 4.10, there is a global real-analytic plurisubharmonic exhaustion function.

4.3 Coronae of Dimension 2

Many of the results of [18] are "local," i.e., hold in a neighborhood of a level, and the topological condition needed on the plurisubharmonic function is the properness, i.e., the fact that the level sets are compact. Let X be a complex space with a smooth plurisubharmonic function $\phi : X \to (0, +\infty)$ such that

- $\inf_X \phi = 0$, $\sup_X \phi = +\infty$
- for every $0 < \epsilon \leq m < +\infty$ the subcorona

$$X_{\epsilon,m} = \{x \in X \ : \ \epsilon < \phi(x) < m\}$$

  is relatively compact in X.

Such a function is called a *corona exhaustion* and X is called a *corona*. We define Σ_X as the set of points $x \in X$ such that every smooth plurisubharmonic corona exhaustion is not strictly plurisubharmonic; it is easy to see that all the "local" results on the minimal kernel used or proved in [18] extend to this setting. We would like to obtain a classification theorem for coronae admitting a real-analytic plurisubharmonic corona exhaustion; one of the ingredients that use pseudoconvexity in a global way is the result by Nishino. We show how to adapt it to the case of a corona.

Lemma 4.11 *Let X be a complex surface, with a real-analytic plurisubharmonic corona exhaustion $\alpha : X \to (0, +\infty)$. Suppose $c \in \mathbb{R}$ is a regular value for α, $Y \subseteq \{x \in X \ : \alpha(x) = c\}$ a connected component containing a compact complex curve C and W a neighborhood of Y with a pluriharmonic function $\chi : W \to \mathbb{R}$ such that $Y = \{x \in W \ : \ \chi(x) = 0\}$.*

Then, X is proper over an open complex curve.

Proof We can assume that C is connected and that χ does not have critical points in W. By [18, Lemma 4.1], there exist a neighborhood V of C in W and a holomorphic function $f : V \to \mathbb{C}$ such that $C = \{x \in V \ : \ f(x) = 0\}$; therefore, the open set V is foliated in compact curves. This means that

$$\partial\bar{\partial}\alpha \wedge \partial\alpha \wedge \bar{\partial}\alpha = 0$$

on V, hence on X, by real analyticity. Therefore, every regular level set of α is Levi flat and foliated by immersed complex curves.

Let $\{a_n\}_{n\in\mathbb{N}}$, $\{b_n\}_{n\in\mathbb{N}}$ be sequences of regular values for α such that a_n decreases to 0 and b_n increases to $+\infty$; then the coronae

$$X_n = \{x \in X \ : \ a_n < \alpha(x) < b_n\}$$

have Levi-flat boundaries, therefore they give an exhaustion of X by relatively compact pseudoconvex sets.

We now apply [23, III.5.B] to obtain that X is proper over an open complex curve. □

In the previous Lemma, we actually proved that X is a *false corona*, meaning that it admits not only the function α, but also some other real-analytic plurisubharmonic function β which is exhaustive on X.

We derive the following structure theorem

Theorem 4.12 *Let X be a complex surface, $\alpha : X \to \mathbb{R}$ a real-analytic plurisubharmonic corona exhaustion function. Then the following three cases can occur:*

1) X is an increasing union of subcoronae with one strictly pseudoconvex boundary and Σ_X is contained in countably many level sets of α,
2) X is proper over a complex curve (therefore a false corona)
3) X is of Grauert type

Proof First, we assume that there exists $a \in \mathbb{R}$ such that $[a, +\infty) \subseteq \alpha(\Sigma_X)$; if every regular level $\{x \in X \ : \ \alpha(x) = b\}$, for $b \geq a$, contains a compact curve, then we have uncountably many compact curves in X and, by [23, Proposition 9 and 7], one of these curves has a neighborhood foliated with compact curves; therefore, reasoning as in the proof of Lemma 4.11, one regular level set is foliated and Levi flat, hence by [18, Lemma 3.7] it is the level set of a pluriharmonic function. Applying Lemma 4.11, we obtain that X is proper over an open (possibly singular) complex curve.

Let us suppose that there is a regular value $c \in \mathbb{R}$ for α such that the level set $Y = \{x \in X \ : \ \alpha(x) = c\}$ does not contain any compact complex curve and $Y \cap \Sigma_X \neq \emptyset$. Then, by [18, Theorem 3.6], $\Sigma_X \supseteq Y$, Y is Levi flat and $\partial\bar{\partial}\alpha \wedge \partial\alpha \wedge \bar{\partial}\alpha = 0$ on X. Hence, every regular level is Levi flat and, by [18, Lemma 3.7], is the (regular) level set of a pluriharmonic function; by the same reasoning used in the proof of [18, Corollary 3.8], Y is foliated with dense complex curves.

Therefore, if such c exists, no regular level set can contain a compact curve, otherwise it would propagate as we described earlier, by Lemma 4.11; so, every regular level is foliated with dense complex curves, i.e., X is of Grauert type.

The last remaining case to consider is when there exists a sequence of regular values that do not intersect Σ_X.

We note that if Σ_X intersects a regular level which does not contain a compact curve, then $\Sigma_X = X$; likewise, if there are uncountably many levels that contain a compact curve, then $\Sigma_X = X$. Therefore, there are at most countably many regular level intersecting Σ_X and they all contain a compact curve; let us denote by $\{c_n\}_{n\in\mathbb{N}}$ the sequence of associated regular values.

By [18, Proposition 3.5], $\Sigma_X \cap \{x \in X \ : \ \alpha(x) = c_n\}$ can locally be written as a union of holomorphic discs; reasoning as in [18, Theorem 3.6], we see that there are only two cases: either $\{x \in X \ : \ \alpha(x) = c_n\}$ is Levi flat and foliated in complex curves or $\Sigma_X \cap \{x \in X \ : \ \alpha(x) = c_n\}$ is the union of finitely many connected compact complex curves. Were the level set Levi flat, it would be the zero set of a pluriharmonic function, but as it contains at least one compact curve, this would imply $\Sigma_X = X$, which is not the case.

Therefore, $\Sigma_X \cap \{x \in X \ : \ \alpha(x) = c_n\}$ is a union of finitely many compact curves, for every n. Finally, if $d \in \mathbb{R}$ is a singular value of α such that $\Sigma_X \cap \{x \in X \ : \ \alpha(x) = d\} \neq \emptyset$, then for every p regular point in $\{x \in X \ : \ \alpha(x) = d\}$ there exists a neighborhood U such that $U \cap \Sigma_X \cap \{x \in X \ : \ \alpha(x) = d\}$ is a union of finitely many complex discs.

In conclusion, if $\Sigma_X \neq X$, then Σ_X is a countable union of compact curves and curves immersed in the singular levels, plus, maybe, the critical points of α; moreover, $\alpha(\Sigma_X)$ is countable and (as α is proper) closed. Therefore we can find intervals I_n of regular values, containing arbitrarily large real numbers, such that $\alpha(\Sigma_X) \cap I_n = \emptyset$, so we can write X as a union of open domains $\Omega_n = \{x \in X \ : \ \alpha(x) < t_n\}$ with a strictly pseudoconvex boundary such that $\Sigma_X \cap \Omega_n$ is a union of compact curves, immersed curves in singular levels and critical points of α. $\square$

One could ask whether, in cases 1 and 3, it could be possible to "fill the hole" and see the corona as a subdomain of a weakly complete surface. We show that, at least in the first case, it is not possible, in general.

We consider an example taken from Section 6 of [29]; we will briefly recall the construction. Let M be the complex manifold diffeomorphic to $\mathbb{C}^2 \setminus \{(0, 0)\}$ and endowed with the unique complex structure such that the 2-form

$$\phi = dz_1 \wedge dz_2 + \epsilon\partial\overline{\partial}\log(|z_1|^2 + |z_2|^2)$$

is holomorphic (such a complex structure exists and is unique by a theorem of Andreotti). A function $f : M \to \mathbb{C}$ is holomorphic if and only if $df \wedge \phi = 0$ and one can verify that the functions

$$v_1 = \frac{z_1^2}{2} - \frac{\epsilon z_1\overline{z}_2}{|z_1|^2 + |z_2|^2} \qquad v_2 = \frac{z_2^2}{2} - \frac{\epsilon z_2\overline{z}_1}{|z_1|^2 + |z_2|^2} \qquad v_3 = \frac{z_1 z_2}{2} - \frac{\epsilon z_2\overline{z}_2}{|z_1|^2 + |z_2|^2}$$

are holomorphic on M and satisfy $v_1^2 = v_2 v_3$. Hence, the map $v : M \to \mathbb{C}^3$ given by $v = (v_1, v_2, v_3)$ sends M on the quadratic cone $K = \{(x, y, z) \in \mathbb{C}^3 \;:\; x^2 = yz\}$ minus the origin; it is easy to check that $v(z_1, z_2) = v(w_1, w_2)$ if and only if $z_1 = \pm w_1, z_2 = \pm w_2$, hence v is a 2-to-1 covering map.

If one considers the blow-ups Q of $\mathbb{C}^2$ in (0, 0) and Q' of K in (0, 0, 0), then we have a 2-to-1 map from $Q \setminus Z$ to $Q' \setminus Z'$, where Z, Z' are the respective exceptional divisors; however, the induced map $M \mapsto Q' \setminus Z'$ is not holomorphic. We consider Q' as the total space of a line bundle over $\mathbb{CP}^1$, choosing coordinates $[x_1 : x_2]$ on $\mathbb{CP}^1$ and coordinates y_i for the fibers on $U_i = \{x_i \neq 0\}$; the transition function is then $y_2 = x_1^2 y_1$ on $U_1 \cap U_2$. We define new coordinates

$$\tilde{y}_1 = \frac{y_1}{2} - \frac{\epsilon \bar{x}_1}{1 + |x_1|^2} \text{ on } U_1$$

$$\tilde{y}_2 = \frac{y_2}{2} - \frac{\epsilon \bar{x}_2}{1 + |x_2|^2} \text{ on } U_2$$

with transition function $\tilde{y}_2 = (x_1^2 + \epsilon)\tilde{y}_1$. Let $\widetilde{Q}$ be the complex manifold obtained from Q' with this new choice of holomorphic coordinates, then the map $M \to \widetilde{Q}$ is holomorphic and $\widetilde{Q}$ is Stein. Indeed, $\widetilde{Q}$ can be embedded in $\mathbb{C}^3$ as the hypersurface $\{(w_1, w_2, w_3) \in \mathbb{C}^3 \;:\; w_3(w_3 + \epsilon) = w_1 w_2\}$.

The zero section Z' of Q' becomes a real-analytic submanifold A of $\widetilde{Q}$, which is no longer complex and, actually, totally real: in a chart U_i, it is the graph of one of the functions $f_\pm : \mathbb{C} \to \mathbb{C}$,

$$f_\pm(z) = \pm \frac{\epsilon \bar{z}}{1 + |z|^2} \,.$$

Therefore, the function $\phi(w) = \mathrm{dist}(w, A)^2$ is zero only on A, together with its gradient, and is strictly plurisubharmonic on $\widetilde{Q} \setminus A$; the pullback ψ to M is a real-analytic strictly plurisubharmonic function with no critical points on M. The level sets $\{p \in M \;:\; \psi(p) = t\}$ for t small enough are compact, as they bound a basis of neighborhoods of (0, 0) in $\mathbb{C}^2$, therefore every level set of ψ is compact.

Therefore, M is a corona with a real-analytic strictly plurisubharmonic corona exhaustion function and, by [29], M cannot be embedded as an open domain in a Stein space $\widetilde{M}$ such that the complement is compact.

Question 2 Is it true that, in case 1, Σ_X is the union of countably many curves? Are these curve negative in the sense of Grauert?

Question 3 Can one produce a similar example for case 3, i.e., for Grauert-type coronae?

4.4 Brunella's Example

In [3], Brunella gives an example of a family of weakly complete complex surfaces which do not admit any real-analytic plurisubharmonic exhaustion function; the main idea of his construction is the following.

Proposition 4.13 *Let us consider a compact complex surface S containing a compact curve C which belongs to Ueda's Class β'' and suppose that S does not admit any holomorphic foliation tangent to C and free of singularities along C. Then $S \setminus C$ is weakly complete, but does not admit any real-analytic plurisubharmonic exhaustion function.*

Proof By [35], there exist a neighborhood V of C in S and a function $u : V \to \mathbb{R}$ such that u is pluriharmonic and $C = \{p \in V \ : \ u(p) = 0\}$; moreover, the levels of u are Levi flat and foliated with dense leaves. We consider the function $-\log u$, defined on $V \setminus C$ and we extend it to $S \setminus C$ setting it constant outside of some set $\{p \in V \ : \ \log u(p) \geq c\}$; we can do so in a smooth way.

Therefore, $S \setminus C$ is weakly complete. On the other hand, were it possible to produce a real-analytic plurisubharmonic exhaustion function, we could apply our Theorem 4.6 and deduce that, as $S \setminus C$ contains some compact Levi-flat hypersurfaces with dense complex leaves, $S \setminus C$ has to be a Grauert-type surface. The leaves of the Levi foliation give a holomorphic foliation of $S \setminus C$ which is tangent to C and free of singularities along it. As this is impossible, we cannot have any real-analytic plurisubharmonic exhaustion function on $S \setminus C$. □

An extensive classification of foliation on surfaces was carried on in [13] and we also refer to [2] for the case of projective surfaces; Brunella in [3] considers the following explicit example. Let $C_0 \subset \mathbb{CP}^2$ be a smooth elliptic curve and define S as the blow-up of $\mathbb{CP}^2$ in 9 points belonging to C_0 and $C \subset S$ as the strict transform of C_0; obviously, $(C^2) = 0$. Moreover, if the 9 points are generic, then $C \subset S$ belongs to Ueda's Class β''.

From the classification of foliations mentioned above, we know that on S there is no foliation tangent to C with no singularities along it (see also Proposition 8 in [3]).

This example by Brunella tells us that the existence of real-analytic plurisubharmonic exhaustion is not ensured by weakly completeness; however, the surface $S \setminus C$ could still be of Grauert type, i.e., union of Levi-flat hypersurfaces, where the leaves of the Levi foliations are dense, but they do not constitute a holomorphic foliation of the whole $S \smallsetminus C$.

In $S \setminus C$, we have a compact set $\Sigma \Subset S \smallsetminus C$ such that, for every neighborhood $\Omega \Subset S \smallsetminus C$ of Σ, we can produce a smooth plurisubharmonic exhaustion which is real analytic outside Ω and vanished on Σ, for every neighborhood of Σ. The complement of Σ is the maximal pseudoflat neighborhood of C in S.

Question 4 What can we say about Σ? Does it have an interior? If so, what can we say about the interior? Is it weakly complete, holomorphically convex, Stein, none of the previous? Does its boundary carry any kind of analytic structure?

5 Minimal Kernels and the Structure of Complex Manifolds

The nature of the minimal kernel of a weakly complete manifold seems to be strictly linked to its geometry and to the presence of analytic objects, at least in dimension 2; as we recounted in the previous pages, precise global results are known only in the presence of a real-analytic exhaustion. However, there are some observations that can be made for a general weakly complete surface, for example, in the particular case when the minimal kernel is the whole surface.

Moreover, the notion of minimal kernel has been generalized by Slodkowski in [31] to an arbitrary complex manifold and to other classes of plurisubharmonic function, retrieving in the process also the notion of core of a domain; Slodkowski managed to show a decomposition theorem for these generalizations of the minimal kernel, where the components are pseudoconcave and every plurisubharmonic function in the considered class is constant along each such component.

We present yet another "kernel" and we employ the construction and the computations in [5] to show that, if such kernel is empty, a bounded, smooth, pseudoconvex domain is a modification of a Stein space.

5.1 Complex Surfaces with a Smooth Exhaustion

As already observed, the classification of weakly complete surfaces in the general case is very hard and it is not even clear that the three cases of Theorem 4.6 are the only possible.

We can nonetheless say something in some particular cases. In [14, 15], the following situation was studied: X is a weakly complete surface with a real-analytic plurisubharmonic exhaustion function and $\Omega \subset X$ is a domain with a smooth (in [14]), or continuous (in [15]), plurisubharmonic exhaustion function. Then, Ω falls into one of the three cases of Theorem 4.6, so it possesses a real-analytic plurisubharmonic exhaustion function.

If Σ_X is compact in X, it is quite easy and classical to prove that X is a modification of a Stein space (see [15, Lemma 2.1]). At the other end of the spectrum, we have the case when all the plurisubharmonic exhaustion functions on X have everywhere degenerate Levi form. Suppose X is a complex surface endowed with a smooth plurisubharmonic exhaustion $\phi : X \to \mathbb{R}$, moreover assume that $\Sigma_X = X$.

We recall that, given a k-form α on a vector space E, the *kernel* of α is defined as

$$\ker(\alpha) = \{v \in E \ : \ \alpha(v, v_2, \dots, v_k) = 0 \ \text{ for all } v_2, \dots, v_k \in E\} \ .$$

We have the following elementary lemma.

Lemma 5.1 *Let M be a real manifold, ω a k-form and α a 1-form; if $\alpha \wedge \omega = 0$, then for every $p \in M$ such that $\omega(p) \neq 0$ as a k-linear alternating form on T_pM, $\ker \alpha(p) \supset \ker \omega(p)$ as vector spaces in T_pM.*

Proof Let $X_0 \in \ker \omega(p)$ and take any $X_1, \dots, X_k \in T_pM$; denote by S the set of permutations σ of $\{0, \dots, n\}$ such that $\sigma(0) \neq 0$ and by T the set of permutations τ of $\{1, \dots, n\}$. Then

$$0 = (\alpha \wedge \omega)(p)[X_0, X_1, \dots, X_k] = \sum_{\tau \in T} (-1)^{|\tau|} \alpha(p)[X_0] \omega(p)[X_{\tau(1)}, \dots, X_{\tau(n)}] +$$

$$+ \sum_{\sigma} \alpha(p)[X_{\sigma(0)}] \omega(p)[X_{\sigma(1)}, \dots, X_{\sigma(n)}] =$$

$$= \sum_{\tau \in T} (-1)^{|\tau|} \alpha(p)[X_0] \omega(p)[X_{\tau(1)}, \dots, X_{\tau(n)}] = n! \alpha(p)[X_0] \omega(p)[X_1, \dots, X_n]$$

therefore, as $X_1, \dots, X_n$ are generic and $\omega(p) \neq 0$, we need to have $\alpha(p)[X_0] = 0$, so $\ker \alpha(p) \supset \ker \omega(p)$. □

We have an analogue of [18, Lemma 5.3] for the smooth case.

Lemma 5.2 *Let W be a complex manifold of complex dimension 2 and $\beta : W \to \mathbb{R}$ a smooth function such that*

$$\partial\bar{\partial}\beta \wedge \partial\beta = \partial\bar{\partial}\beta \wedge \bar{\partial}\beta = 0 \ .$$

Suppose $|d\beta| \neq 0$ on W, then there exists $\mu : W \to \mathbb{R}$ such that

$$\partial\bar{\partial}\beta = \mu \partial\beta \wedge \bar{\partial}\beta \ .$$

Proof Applying Lemma 5.1, we have that

$$\ker(\partial\bar{\partial}\beta) \subseteq \ker(\partial\beta) \cap \ker(\bar{\partial}\beta) = \ker(\partial\beta \wedge \bar{\partial}\beta)$$

whenever $\partial\bar{\partial}\beta \neq 0$. Therefore, as they both are real $(1,1)$-forms, they differ by a smooth function $\mu : W \to \mathbb{R}$. □

Remark 5.3 If β is plurisubharmonic, then μ is non-negative and there exists $\theta : \mathbb{R} \to \mathbb{R}$ such that $\theta \circ \mu$ is plurisubharmonic; moreover, the levels of β are Levi flat and, by the last observation, μ is constant on the leaves of the Levi foliation, therefore also the levels of μ are Levi flat.

We are now in the position to state and prove our result about the structure of X.

Theorem 5.4 *Let X be a complex surface with a smooth plurisubharmonic exhaustion function ϕ; suppose that $\Sigma_X = X$. Then, either X is proper on an open Riemann surface or the connected components of the regular level sets of ϕ are Levi flat and foliated in dense complex curves (i.e., X is of Grauert type). In the latter case, whenever $r, s \in \mathbb{R}$ are such that (r, s) is an interval of regular values and $W \subset \{r < \phi < s\}$ is such that $\phi|_W$ is proper and has connected level sets, there exists a pluriharmonic function $\chi : W \to \mathbb{R}$ such that $\chi = \lambda \circ \phi$.*

Proof As $\Sigma_X = X$, the Levi form of ϕ is degenerate everywhere on X, so every connected component of a regular level of ϕ is a local maximum set, hence by [32, Theorem 3.9], the forms $\partial\bar{\partial}\phi \wedge \partial\phi$ and $\partial\bar{\partial}\phi \wedge \bar{\partial}\phi$ vanish. We consider a connected component W of $\{p \in X : r < \phi(p) < s\}$ such that all the level sets of ϕ contained in W are regular and connected.

By Lemma 5.2 and the subsequent Remark, we have a non-negative function μ on W such that $\partial\bar{\partial}\phi = \mu\partial\phi \wedge \bar{\partial}\phi$.

Suppose now that $d\mu \wedge d\phi$ does not vanish identically; we want to show that the generic levels of ϕ and μ intersect transversally.

We note that $\ker(d\mu \wedge d\phi) \supset \ker(\partial\phi \wedge \bar{\partial}\phi)$.

We define $F : W \to \mathbb{R}^2$ by $F(p) = (\phi(p), \mu(p))$; obviously, $d\mu \wedge d\phi(p) \neq 0$ if and only if $\mathrm{rk}\, DF(p) = 2$. Therefore, there is an open set $U \subset W$ such that DF has rank 2, i.e., $F(U)$ is an open set in $\mathbb{R}^2$. By Sard's theorem, there are uncountably many regular values of F in U. Let $c \in \mathbb{R}^2$ be such a regular value; then,

$$\big\{p \in W \ : \ F(p) = c\big\} = C$$

is a 2-dimensional compact manifold (as the level sets of ϕ are compact), whose tangent space at every point is $\ker(\partial\phi) \cap \ker(\bar{\partial}\phi)$, i.e., a complex subspace of the tangent space of X; therefore, C is a compact complex curve.

This produces uncountably many compact complex curves. By Nishino, we have that X is foliated by compact curves.

If $d\mu \wedge d\phi = 0$, then μ is constant on the level sets of ϕ, which are connected; therefore, we can define $m : \mathbb{R} \to \mathbb{R}$ such that $m \circ \phi = \mu$. An easy check in a coordinate patch shows us that m is smooth.

We set $\chi = \lambda \circ \phi$ and we compute

$$\partial\bar{\partial}\chi = (\lambda'' \circ \phi)\partial\phi \wedge \bar{\partial}\phi + (\lambda' \circ \phi)\partial\bar{\partial}\phi = (\lambda'' \circ \phi)\partial\phi \wedge \bar{\partial}\phi + (\lambda' \circ \phi)\mu\partial\phi \wedge \bar{\partial}\phi =$$

$$= (\lambda'' \circ \phi)\partial\phi \wedge \bar{\partial}\phi + (\lambda' \circ \phi)(m \circ \phi)\partial\phi \wedge \bar{\partial}\phi \ .$$

Let $t = \phi(p)$, then χ is pluriharmonic if

$$\lambda''(t) + \lambda'(t)m(t) = 0. \tag{5.1}$$

We can find a solution such that $\lambda' > 0$. For such a choice of λ, χ is pluriharmonic on W.

If there is any compact complex curve in W, by the argument in [18, Section 4], we propagate the curve and, by Nishino's theorem we conclude again that X is proper on an open Riemann surface.

If there are no compact complex curves, we have that, on the manifold $Y = \{\chi = \chi(p)\}$ for $p \in W$, the Levi foliation is defined by the form $d^c\chi$, which is closed. Hence, the argument used in the proof of [18, Corollary 3.8] implies that Y, which does not contain any compact curve, is foliated in dense complex leaves. □

With no effort, we obtain also a "local" version of the previous result, employing the full statement of [32, Theorem 3.9].

Corollary 5.5 *Let X be a complex surface with a smooth plurisubharmonic exhaustion function ϕ. Suppose that $r \in \mathbb{R}$ is such that there exists a connected component Y of $\{p \in X \ : \ \phi(p) = r\}$ which is relatively open in the level set and is contained in Σ_X. Then, either X is proper on an open Riemann surface or there is $s < r$ such that for every regular value $t \in (s, r)$, the corresponding level set is Levi flat and foliated with dense complex curves.*

Moreover, recalling the homological condition used in Section 4, we have the following.

Corollary 5.6 *Suppose X is a complex surface of Grauert type, endowed with a smooth plurisubharmonic exhaustion ϕ, such that $H^2(X, \mathbb{R}) = 0$. Suppose that there exists $c \in \mathbb{R}$ such that the level sets of ϕ contained in $X \setminus \{\phi \leq c\}$ are connected and contain at least one regular point of ϕ. Then there exists a real-analytic plurisubharmonic exhaustion function on X.*

Proof We note that the regular values of ϕ are an open set in $\mathbb{R}$, because the set of critical points is closed and the function ϕ is proper, hence closed. So, by Sard's theorem, the set of critical values has measure 0, so we have a sequence of regular values growing to $+\infty$; moreover, the closure of the regular values is the image of ϕ.

For any interval $I = (a, b)$, constituted only of regular values, with $a > c$, we consider the set $W = \{a < \phi(p) < b\}$. Since X is of Grauert type, $\Sigma_X = X$ and, following the proof of Theorem 5.4, we have a pluriharmonic function $\chi : W \to \mathbb{R}$, which is obviously constant on (the connected components of) the level sets of ϕ. Moreover, such a function χ is obtained via a function $m : I \to \mathbb{R}$; the construction of the function m is local and works around every regular point, so we can extend it to the whole of $X \setminus \{\phi \leq c\}$ by continuity. Therefore, we obtain a function λ satisfying (5.1) and, subsequently, a pluriharmonic function $\chi : X \setminus \{\phi \leq c\} \to \mathbb{R}$.

As $H^2(X, \mathbb{R}) = 0$, by Proposition 4.9, we can extend χ as a pluriharmonic function on X and we obtain the desired exhaustion. □

5.2 The Singular Locus of an Admissible Class

In a series of papers, Harz, Shcherbina, and the second author introduced a concept analogous to the minimal kernel, dealing with bounded plurisubharmonic functions, see [10–12]. The *core* of a complex manifold X is defined as the set $\mathbf{c}(X)$ of points where every *bounded* plurisubharmonic function has a degenerate Levi form; as for the minimal kernel, such definition can be given in any regularity class, from continuous to smooth or real analytic.

The presence of the core of a domain in a complex manifold is obviously linked to the existence of strictly plurisubharmonic defining functions; as in the case of the minimal kernel, in dimension 2 the nature of the core is better understood and linked to the presence of some analytical objects, namely complex curves in the intersections of the core with level sets of a minimal function, where all the bounded plurisubharmonic functions are constant. For further details we refer the interested reader to the papers cited before or to the electronic preprint [9].

The existence of such objects is meaningful, as they, in some sense, explain the failure at being strictly plurisubharmonic because they force such functions to be constant; this point of view has been examined and studied in detail by Slodkowski in [31], where he generalizes the constructions of the minimal kernel and the core to what he calls singular locus of an admissible class of functions.

Definition 5.7 Let X be a complex manifold and let $\mathcal{U}$ be the set of all the open sets $U \subset X$ such that at least one between U and $X \setminus U$ is relatively compact. An admissible class $\mathcal{F}$ is the datum, for every $U \in \mathcal{U}$, of a set $\mathcal{F}(U)$ of continuous plurisubharmonic functions on U with the following properties:

(1) if $\phi \in \mathcal{F}(U)$ and $W \in \mathcal{U}$, $W \subset U$, then $\phi|_W \in \mathcal{F}(W)$
(2) if $\phi : X \to \mathbb{R}$, $U_1, \ldots, U_n \in \mathcal{U}$ form a covering of X and $\phi|_{U_i} \in \mathcal{F}(U_i)$ for $i = 1, \ldots, n$, then $\phi \in \mathcal{F}(X)$
(3) $\mathcal{F}(U)$ is a convex cone and contains every bounded smooth plurisubharmonic function on U
(4) if $\phi_1, \ldots, \phi_n \in \mathcal{F}(U)$ and $v : \mathbb{R}^n \to \mathbb{R}$ is smooth, convex, of at most linear growth and with non-negative partial derivatives, then $\phi = v(\phi_1, \ldots, \phi_n)$ belongs to $\mathcal{F}(U)$
(5) if $\phi \in \mathcal{F}(U)$ is strongly plurisubharmonic and if $\rho \in \mathcal{C}^\infty(U)$ with $\overline{\operatorname{supp} \rho} \subset U$, then there is $t > 0$ such that $\phi + t\rho \in \mathcal{F}(U)$
(6) for every sequence $\phi_n \in \mathcal{F}(X)$, there are positive numbers ϵ_n such that $\sum_n \epsilon_n \phi_n$ converges uniformly on compact subsets of X to a function in $\mathcal{F}(X)$.

Examples of admissible classes are the class of plurisubharmonic exhaustion function (of a given regularity), the class of bounded plurisubharmonic functions (of a given regularity), the class of all plurisubharmonic functions (of a given regularity); other examples can be obtained by adding any kind of growth condition.

Definition 5.8 Let $\mathcal{F}$ be an admissible class, then the singular locus $\Sigma^{\mathcal{F}}$ of $\mathcal{F}$ is defined as the set of points of X where no function of $\mathcal{F}(X)$ is strongly plurisubharmonic. A function $\phi \in \mathcal{F}(X)$ is called $\mathcal{F}$-*minimal* if it is strongly plurisubharmonic exactly on $X \setminus \Sigma^{\mathcal{F}}$.

As it is proved in [31, Section 4], the singular locus, if nonempty, is pseudoconcave, since its intersection with a ball in X is a local maximum set (see [31, Proposition 4.4 and Lemma 4.5]). Moreover, $\Sigma^{\mathcal{F}}$ can be decomposed in pseudoconcave parts where all the functions in $\mathcal{F}(X)$ are constant [31, Theorem 4.7]. For the minimal kernel, we can also prove that such pseudoconcave parts are compact (see [31, Theorem 5.2]).

We give the following definition (see [31, Section 5]).

Definition 5.9 Let X be a complex manifold, $k \in \mathbb{N} \cup \{\infty\}$ and consider the admissible class $\mathcal{F}$ such that, for $U \in \mathcal{U}$, $\mathcal{F}(U)$ is the set of all $\mathcal{C}^k$ plurisubharmonic functions on U. We define the *minimal kernel* of X as the singular locus of $\mathcal{F}$ and we denote it by Σ_X.

It is easy to see that, if X admits a $\mathcal{C}^k$ plurisubharmonic exhaustion function, then Σ_X coincides with the previously defined minimal kernel of a weakly complete space. The decomposition in pseudoconvex parts still holds; however, they may not be compact anymore.

5.3 A Levi Problem

Let M be a complex manifold.

Definition 5.10 For $U \subseteq M$ open domain with smooth boundary, we define the *boundary kernel* of U as

$$\mathrm{b}\Sigma_U = \overline{\Sigma}_U \cap \mathrm{b}U,$$

where the closure is taken with respect to the topology of M.

Definition 5.11 For $Y \subseteq M$ closed, let $\mathcal{U}(Y)$ be a basis of open neighborhoods of Y in M. We define the *psh kernel* of Y as

$$\Sigma_Y = \bigcap_{U \in \mathcal{U}(Y)} \Sigma_U.$$

The definition of some kind of minimal functions for the boundary kernel is not obvious and we do not know any results in this direction.

Question 5 Is there any kind of minimal function for the boundary kernel of a smoothly bounded domain?

In the case of the psh kernel of a closed set Y, a reasonable definition would be a plurisubharmonic function defined in a neighborhood Y which is strongly plurisubharmonic exactly outside of Σ_Y; this is equivalent to ask that there exists a neighborhood U of Y such that $\Sigma_U = \Sigma_Y$. If $\Sigma_Y = \emptyset$, the existence of such a function follows from the definition of the psh kernel. In general, we do not know if such a function exists.

Another interesting question is the following.

Question 6 Are there any relations between $\mathrm{b}\Sigma_U$ and $\Sigma_{\mathrm{b}U}$?

In general, they do not coincide, as the following example shows.

Example 5.12 Let M be the blow-up of $\mathbb{C}^2$ at the origin $\{z = w = 0\}$, $\pi : M \to \mathbb{C}^2$ the reduction map, $E = \pi^{-1}(0, 0)$ the exceptional divisor. Let $U = \{|z - 1|^2 + |w|^2 < 1\}$ and $\Omega = \pi^{-1}(U)$. Then, U is Stein, so Ω, which is biholomorphic to U, is Stein and $\phi := \psi \circ \pi$ with $\psi(z, w) = |z - 1|^2 + |w|^2 - 1$ is a defining function for Ω. It follows that $\mathrm{b}\Sigma_\Omega = \varnothing$. On the other hand $\mathrm{b}\Omega$ contains E, where every plurisubharmonic function has a degenerate Levi form, hence $E \subseteq \Sigma_{\mathrm{b}\Omega}$.

We do not know if the inclusion $\mathrm{b}\Sigma_U \subseteq \Sigma_{\mathrm{b}U}$ holds in general; however, if we consider the kernels Σ^0 given by continuous plurisubharmonic functions, we have the following result.

Lemma 5.13 *Suppose U is relatively compact in M. If there exists a function u defined in a neighborhood of $\overline{U}$ and plurisubharmonic there such that $U = \{u < 0\}$, then* $\mathrm{b}\Sigma^0_U \subseteq \Sigma^0_{\mathrm{b}U}$.

Proof Let us consider the open set V, neighborhood of bU in M and ψ continuous and plurisubharmonic on U. We can suppose that $\min_{\mathrm{b}U} \psi \geq 1$. Let $c < 0$ be such that $L = \{c \leq u \leq 0\}$ is compact in U and denote by S the level set $\{u = c\}$. .

Consider a function $\theta : (-\infty, 0] \to (-\infty, 0]$ satisfying

(1) $\theta(x) < 0$ if $x < 0$ and $\theta(0) = 0$
(2) $\theta \circ u$ is continuous and plurisubharmonic on $\mathrm{int}(L)$
(3) if $x \in \mathrm{int}(L)$ is a point of strong plurisubharmonicity for u, then it is also a point of strong plurisubharmonicity for $\tilde{u} = \theta \circ u$
(4) $\theta(c) < c - \max_L \psi$.

and let $\tilde{u} = \theta \circ u$. Then

$$(\psi + \tilde{u})(x) < u(x) \quad \forall x \in S$$

$$(\psi + \tilde{u})(x) \geq 1 > u(x) \quad \forall x \in \mathrm{b}U.$$

The function

$$v = \max\{u, \psi + \tilde{u}, c\}$$

is continuous and plurisubharmonic on U.

Now, if $x \notin \Sigma^0_{\mathrm{b}U}$, then we can find a ψ which is strongly plurisubharmonic in a neighborhood of x, which implies that v is strongly plurisubharmonic in the intersection of a neighborhood of $x \in \mathrm{b}U$ with U , so $x \notin \mathrm{b}\Sigma^0_{\Omega}$. □

The boundary kernel is clearly linked only with the geometry of the domain U near its boundary, whereas the psh kernel of the boundary bU also takes into account what happens *on the boundary*, as it is clearly outlined in Example 5.12. This leads us to conjecture that the conclusion of the previous Lemma should hold in greater generality.

Considering the greater quantity of information encoded in the psh kernel of the boundary, it is reasonable to think that its presence or absence have a strong role in determining the geometry of the domain. Indeed, we have the following result.

Theorem 5.14 *Let $\Omega \Subset M$ be a smoothly bounded, pseudoconvex domain; if $\Sigma_{\mathrm{b}\Omega} = \emptyset$, then Ω is a modification of a Stein space.*

Proof If $\Sigma_{\mathrm{b}\Omega}$ is empty, then there exists a strongly pseudoconvex function ψ defined in a neighborhood of bΩ, so, by [28], ψ can be taken to be $\mathcal{C}^\infty$. Since Ω is pseudoconvex, then there is a $\mathcal{C}^\infty$ function $\rho : M \to \mathbb{R}$ such that

(1) $\Omega = \{x \in M \ : \ \rho(x) < 0\}$
(2) $d\rho \neq 0$ on $\mathrm{b}\Omega = \{\rho = 0\}$
(3) $i\partial\bar\partial\rho(x)$ is positive semidefinite on $T^{1,0}_x\mathrm{b}\Omega \oplus T^{0,1}_x\mathrm{b}\Omega$ for every $x \in \mathrm{b}\Omega$.

We follow closely the approach and the computations of [5, Theorem 1].

We define $u = -(-\rho e^{-C\psi})^\alpha$, where C, α are positive real constants to be determined later. Up to shrinking U, we can suppose that ψ is bounded, so we can also assume $\psi \geq 0$.

In order to make the computations easier, we take an Hermitian metric ω on M, such that $i\partial\bar\partial\psi \geq \omega$ in a neighborhood of bΩ.

Following [5], we calculate

$$\bar\partial u = \alpha(-\rho e^{-C\psi})^{\alpha-1}(e^{-C\psi}\bar\partial\rho - C\rho e^{-C\psi}\bar\partial\psi)$$

and

$$\partial\bar\partial u = -\alpha(\alpha-1)(-\rho e^{-C\psi})^{\alpha-2}(e^{-C\psi}\partial\rho - C\rho e^{-C\psi}\partial\psi)\wedge$$

$$\wedge(e^{-C\psi}\bar\partial\rho - C\rho e^{-C\psi}\bar\partial\psi)+$$

$$+\alpha(-\rho e^{-C\psi})^{\alpha-1}(e^{-C\psi}\partial\bar\partial\rho - Ce^{-C\psi}\partial\psi\wedge\bar\partial\rho-$$

$$-C\rho e^{-C\psi}\partial\bar\partial\psi - Ce^{-C\psi}\partial\rho\wedge\bar\partial\psi + C^2\rho e^{-C\psi}\partial\psi\wedge\bar\partial\psi)\ .$$

We factor out the term $\alpha(-\rho e^{-C\psi})^{\alpha-2}e^{-2C\psi}$, which is always positive in $U \cap \Omega$, and we obtain

$$-(\alpha-1)\partial\rho\wedge\bar\partial\rho+2\alpha C\rho\mathsf{Re}\,(\partial\rho\wedge\partial\psi)-\alpha C^2\rho^2\partial\psi\wedge\bar\partial\psi-\rho\partial\bar\partial\rho+C\rho^2\partial\bar\partial\psi=$$

$$=C\rho^2(\partial\bar\partial\psi-C\alpha\partial\psi\wedge\bar\partial\psi)-\rho(\partial\bar\partial\rho-2\alpha C\mathsf{Re}\,(\partial\rho\wedge\bar\partial\psi))+(1-\alpha)\partial\rho\wedge\bar\partial\rho\ .$$

We note that

$$|2\alpha\rho\mathsf{Re}\,(\partial\rho\wedge\bar\partial\psi)|\le(\alpha^2|\partial\rho|^2+C^2\rho^2|\partial\psi|^2)\omega\ .$$

The pseudoconvexity of bΩ can be stated by saying that $i\partial\bar\partial\rho$ is positive semidefinite on ker $\partial\rho$ in $T_z^{1,0}M$ for $z\in b\Omega$; therefore, we can find $C_1>0$ such that

$$i\partial\bar\partial\rho\ge-C_1|\partial\rho|\omega$$

on bΩ. Now, if z is close enough to $b\Omega$, we can find a point $w\in b\Omega$ such that

$$i\partial\bar\partial\rho(z)\ge i\partial\bar\partial\rho(w)-C_2|\rho(z)|\omega$$

as $\rho(z)$ is comparable with the distance (with respect to ω) from z to w. Therefore, we have a constant $D>0$ such that

$$i\partial\bar\partial\rho(z)\ge-D(|\partial\rho|+|\rho|)\omega\ .$$

Therefore

$$C\rho^2(\partial\bar\partial\psi-C\alpha\partial\psi\wedge\bar\partial\psi)-\rho(\partial\bar\partial\rho-2\alpha C\mathsf{Re}\,(\partial\rho\wedge\bar\partial\psi))+(1-\alpha)\partial\rho\wedge\bar\partial\rho\ge$$

$$\ge C\rho^2(\partial\bar\partial\psi-C(1+\alpha)|\partial\psi|^2\omega)+(1-\alpha-\alpha^2)|\partial\rho|^2+D\rho|\partial\rho|\omega-D\rho^2\omega$$

and, as $D|\rho||\partial\rho|\le\delta D(|\partial\rho|^2+\delta^{-1}\rho^2))$, we get the lower bound

$$C\rho^2(\partial\bar\partial\psi-C(1+\alpha)|\partial\psi|^2\omega-C^{-1}D(1+\delta^{-1})\omega)+(1-\alpha-\alpha^2-D\delta)|\partial\rho|^2\omega\ .$$

As $0\le\psi\le m$, by replacing ψ with ψ^2/M, we have that $i\partial\bar\partial\psi\ge iM'\partial\psi\wedge\bar\partial\psi$, hence

$$\partial\bar\partial\psi-C(1+\alpha)|\partial\psi|^2\omega-\frac{D}{C}(1+\delta^{-1})\omega\ge(|\partial\psi|^2(M'-C(1+\alpha))-\frac{D}{C}(1+\delta^{-1}))\omega$$

and $|\partial\psi|^2$ can be supposed to be greater than a positive constant K along bΩ. We can choose M', C, α, and δ such that $C\sim\delta^{-1}\sim\alpha^{-2}$ and $M'>2C+2DK$ and, finally, α small enough so that $(1-\alpha-\alpha^2-D\delta)>0$.

Hence u is strictly plurisubharmonic in $U\cap\Omega$ and $\{u=0\}=b\Omega$. Therefore, $\{u=c\}$, for $c<0$ and small enough in absolute value, is compact in $U\cap\Omega$; so, we define an exhaustion function $\tilde u=\theta\circ\max\{u,c\}$, where θ is a convex, increasing real function. Then, Ω is a modification of a Stein space. $\square$

Remark 5.15 We note that $\Sigma_{b\Omega} = \emptyset$ is not a necessary condition for Ω to be a modification of a Stein space, as Example 5.12 shows.

The previous remark leads us to formulate a conjecture.

Conjecture Ω is a modification of a Stein space if and only if $b\Sigma_{\Omega} = \emptyset$.

Remark 5.16 If Ω is weakly complete and $b\Sigma_{\Omega} = \emptyset$, then Ω is a modification of a Stein space, by [15].

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On the Automorphic Group of an Entire Function

Ronen Peretz

Abstract This paper develops further the theory of the automorphic group of non-constant entire functions. This theory has already a long history that essentially started with two remarkable papers of Tatsujirô Shimizu that were published in 1931. The elements $\phi(z)$ of the group are defined by the automorphic equation $f(\phi(z)) = f(z)$, where $f(z)$ is entire. Tatsujirô Shimizu also refers to the functions of this group as those functions that are determined by $f^{-1} \circ f$. He proved many remarkable properties of those automorphic functions. He indicated how they induce a beautiful geometric structure on the complex plane. Those structures were termed by Tatsujirô Shimizu, the system of normal polygonal domains, and the more refined system of the fundamental domains of $f(z)$. The last system if exists tiles up the complex plane with remarkable geometric tiles that are conformally mapped to one another by the automorphic functions. In the Ph.D thesis of the author, those tiles were also called the system of the maximal domains of $f(z)$. One cannot avoid noticing the many similarities between this automorphic group and its accompanying geometric structures and analytic properties, and the more tame discrete groups that appear in the theory of hyperbolic geometry and also the arithmetic groups in number theory. This paper pursues further the theory initiated by Tatsujirô Shimizu, towards understanding global properties of the automorphic group, rather than just understanding the properties of the individual automorphic functions. We hope to be able in sequel papers to generalize arithmetic and analytic tools such as the Selberg trace formula, to this new setting.

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1 Some Background and the Contribution of Tatsujirô Shimizu

In 1931 Tatsujirô Shimizu published two remarkable papers having the titles: On the Fundamental Domains and the Groups for Meromorphic Functions. I and II. [14, 15]. Here are quotations from Shimizu's [14] that describe few of the main notions of the theory:

1. (p. 179) "We call an open domain of the Riemann surface of the inverse function $z = f^{-1}(w)$ of an integral or a meromorphic function $w = f(z)$ "a leaf" if it satisfies the following three conditions:

1) It covers almost all the whole w-plane without leaving any complementary domain.
2) It does not cover the w-plane more than once in any part.
3) Each part of the boundaries is common to certain domains of the surface which is exterior to the considered domain."

2. (p. 179) "To each leaf so defined on the Riemann surface of the meromorphic function $w = f(z)$ there corresponds an open domain on the z-plane which we call a "polygonal domain." In the polygonal domain of a meromorphic function the function is mono-valued and it takes all values except a set of values not forming a domain. We call such a leaf whose boundary consists of only accessible points from the inside of it "a leaf with accessible boundary" and the corresponding polygonal domain "a polygonal domain with accessible boundary." By mapping the sequence of leaves with accessible boundaries on the z-plane, the z-plane of the meromorphic function $w = f(z)$ is divided into a system of polygonal domains for the function, whose boundaries consist only of accessible points from the inside of each domain, respectively."

3. (p. 185) "I will here call that the z-plane is divided into "a system of normal polygonal domains", if the z-plane is divided into a system of polygonal domains whose boundaries consist of only accessible points from the inside of them so that an infinite number of the boundaries of different polygonal domains may not accumulate in the finite part of the z-plane.

Further we call that the z-plane is divided into "a system of fundamental domains" if the plane is divided into a system of normal polygonal domains so that the boundary of each normal polygonal domain may be all transformed into all the boundary of another normal polygonal domain by the transformation of the group of the function which I shall consider in section VII, that is, the transformation defined by $f(z') = f(z)$, where z runs over the boundary of some polygonal domain."

4. (p. 185) We show that:

Theorem 1 *"For any meromorphic function $f(z)$ we can divide the z-plane into a system of normal polygonal domains, that is, we can divide the Riemann surface for the inverse function of $f(z)$ into a system of leaves without leaving any elements* *(belonging to the surface) except point sets containing no domain such that each leaf covers almost all the w-plane except point sets not forming domains and the boundary of each leaf consists of only accessible points from the inside and further, when all the leaves thus obtained are mapped conformally on the z-plane, there exists no point set in the finite part of the z-plane which is a limiting set of an infinite number of boundaries of the images of the leaves."*

In this basic theorem of the theory, Shimizu demonstrates that any meromorphic function carries with it the geometric structure of a system of normal polygonal domains. However as he later on proves there are entire functions that have no system of fundamental polygonal domains. Gross constructed an entire function whose set of all asymptotic values is the whole of $\mathbb{C}$. Shimizu proves that Gross' function has no system of fundamental polygonal domains. Thus any meromorphic function induces those remarkable tilings of the complex plane by systems of normal polygonal domains. But there are entire functions for which the boundaries of the different tiles are mapped to one another (infinity included) by the automorphic functions, in a rather complicated manner.

A large portion of Tatsujirô Shimizu's papers was dedicated to understand the analytic and the geometric properties of the individual elements of the group that are defined by

$$f(z') = f(z). \tag{1}$$

In our paper we will call this defining equation, the automorphic equation of $f(z)$.

Remark 1 In our manuscript we will call this group of Shimizu, "the automorphic group of $f(z)$" and we will use the notation $\mathrm{Aut}(f)$ to designate it. The binary operation is composition of mappings.

Remark 2 As mentioned in Remark 1 we will use the term "automorphic function of $f(z)$" instead of Shimizu's "fundamental function with respect to $f(z)$".

For example, very simple such groups are $\mathrm{Aut}(z^n) = \{e^{2\pi ik/n}z|\, k = 0, \ldots, n-1\}$ and $\mathrm{Aut}(e^z) = \{z + 2\pi ik|\, k \in \mathbb{Z}\}$. Possible tilings of the complex plane that correspond to these groups (and functions) are $\Omega_j(z^n) = \{z \in \mathbb{C}|\, 2\pi ij < \arg z < 2\pi i(j+1)\}$, $j = 0, \ldots, n-1$ and $\Omega_j(e^z) = \{z \in \mathbb{C}|\, 2\pi j < \Im z < 2\pi(j+1)\}$, $j \in \mathbb{Z}$, respectively. However, these two examples are exceptional, having all of the automorphic functions entire. A remarkable property proved by Shimizu asserts that the only possible automorphic functions which are entire have the form $e^{i\theta}z + b$, where $\theta \in 2\pi \cdot \mathbb{Q}$. More complicated entire functions do not qualify being automorphic. In general those automorphic functions are multi-valued or "leaves" thereof with a complicated structure. Further research on this topic was carried on, for example, in [10] and [11]. Systems of fundamental domains (and their automorphic groups) for specific important functions, in particular in number theory (such as the Riemann Zeta function and the Gamma function), were computed in the past. It is clear that a lot of further research is needed in order to better understand the automorphic functions. In particular we clearly have to understand more global properties of the groups $\mathrm{Aut}(f)$ and of their induced normal (or fundamental if exist) polygonal domains $\{\Omega_j(f)\}_j$. For example, it is clearly important to understand if tools parallel to Selberg Trace Formula could be extended to the automorphic groups of entire functions.

Here is a brief summary of the results and the ideas in the paper. In Section 2 we use the Weierstrass representation as (generically) an infinite product for $f(w) - f(z)$. Here $w \in \mathbb{C}$ is the variable while the parameter z lies in $\mathbb{C} - f^{-1}(f(0))$. We have:

$$f(w) - f(z) = \exp\left(g(w,z)\right) \prod_{n=1}^{\infty} E\left(\frac{w}{\phi_{0n}(z)}, \lambda_n\right) =$$

$$= \exp\left(g(w,z)\right) \prod_{n=1}^{\infty} \left(1 - \frac{w}{\phi_{0n}(z)}\right) e^{Q_{\lambda_n}(w/\phi_{0n}(z))},$$

where if $\lambda_n > 0$, then:

$$Q_{\lambda_n}\left(\frac{w}{\phi_{0n}(z)}\right) = \left(\frac{w}{\phi_{0n}(z)}\right) + \frac{1}{2}\left(\frac{w}{\phi_{0n}(z)}\right)^2 + \ldots + \frac{1}{\lambda_n}\left(\frac{w}{\phi_{0n}(z)}\right)^{\lambda_n},$$

and $Q_0(w/\phi_{0n}(z)) \equiv 0$. Weierstrass representation parameters are the function $g(w,z)$ which is entire in w and z-holomorphic off $f^{-1}(f(0))$, and the non-negative integers λ_n that depend on z. Clearly $f(w) - f(z)$ is z-$\mathrm{Aut}(f)$ invariant. But this happens due to a complicated interaction of the infinite product and the exponential $\exp\left(g(w,z)\right)$. In the case that f has a finite order it follows that the infinite product as well as the exponential part are separately z-$\mathrm{Aut}(f)$ invariant. Thus in this case the behavior of the Weierstrass representation is tame, for the group invariance is not requiring any interaction between the two parts of the Weierstrass representation. This is proved in Proposition 3. We prove that the description of the Weierstrass representation of $f(w) - f(z)$ can be refined in that the exponential part has the form $exp(F(w, f(z)))$ where $F(w,t)$ is holomorphic in each variable separately. This depends among other things on Lemma 1. This lemma also implies the cycle relation, Corollary 3 and the chain relation, Corollary 4. However the proof of Lemma 1 follows by a result of Eremenko and Rubel, [3], which makes the use of the monodromy principle.

In Section 3 we indicate what conclusions can be reached when we have no monodromy. In particular the proof of Corollary 7 defines the mapping $T : \mathrm{Aut}(f) \to \mathbb{Z}$, which will later on be used in Section 6 (for example, in Corollary 19). Most of the results in this section deal with the arithmetic of the compositions of automorphic functions. The mapping T provides means to induce from any such a composition an appropriate factorization of a natural number over $\mathbb{Z}^+$. Thus we can use the multiplicative theory of numbers in order to deduce results on factorizations of an automorphic function into a composition of other automorphic functions. Theorem 2 gives the general picture by describing the finiteness of the decomposition of automorphic functions.

In Section 4 the cycle relation and the chain relation are discussed in the general case where no assumption on the finiteness of the order is assumed. Corollary 14 deals with the cycle relation while Corollary 15 and Corollary 16 deal with the chain relation.

In Section 5 we compute the example of the exponential function. We use our method of computation to arrive at a general result, Theorem 3 which indicates how to construct the entire function $f(w)$ from its automorphic group $\mathrm{Aut}(f)$. This construction depends on the assumption that has no justification at the moment, that we have some summation method for the infinite series:

$$\sum_{n=1}^{\infty} Q_{\lambda_n}\left(\frac{w}{\phi_{0n}(z)}\right).$$

The right summation method for this infinite sum of polynomials in w which are multi-valued functions of z is an open problem.

Section 6 gives among other things other types of reconstruction formulas both to $f(z)$ and to $f'(z)$ in terms of approximating automorphic functions. These are the automorphic functions of the partial sums of the power series expansion of $f(w)$. Proposition 4 gives the formulas: $f(z) = f(w) - \lim_{n\to\infty} a_n \prod_{j=1}^{n}(w - \phi_j(n)(z))$, and $f'(z) = \lim_{n\to\infty} a_n \prod_{j=1}^{n-1}(z - \phi_j^{(n)}(z))$. The remarkable thing here is that $a_n \to 0$ as a sequence of numbers while both products blow up but as sequences of functions, but just in the right pace so that the limits converge and reconstruct the function and its derivative. As mentioned above this section gives also properties of $\ker(T)$, where the mapping T was defined in Section 3 within the proof of Corollary 7. For example, Corollary 19 indicates relations between $\ker(T)$ and the automorphic group. In fact when f has a finite order as an entire function then $\ker(T) = \mathrm{Aut}_z(g(w,z))$, and all the automorphic functions in $\mathrm{Aut}(f) - \mathrm{Aut}_z(g(w,z))$ have infinite order in the sense of group members. We recall that $\mathrm{Aut}(f(z)) \subseteq \mathrm{Aut}_z(\exp(g(w,z))$. So $\mathrm{Aut}(f(z))$ is "trapped" between $\mathrm{Aut}_z(g(w,z))$ and $\mathrm{Aut}_z(\exp(g(w,z))$ and the automorphic functions of f outside the smaller group $\mathrm{Aut}_z(g(w,z))$ all have infinite order as group elements of the automorphic group of f, $\mathrm{Aut}(f)$.

In Section 7 we show how the function $g(w,z) - g(0,z)$, where $g(w,z)$ is the function that participates in the Weierstrass representation of $f(w) - f(z)$ is determined by negative moments of the automorphic functions. Theorem 4 determines $\frac{\partial^k g}{\partial w^k}(0,z)$ in terms of $\sum(\phi_{0n}(z))^{-k}$. In fact for $k = 1, 2, 3, \ldots$ we have the identities:

$$\frac{1}{k!}\frac{\partial^k g}{\partial w^k}(0,z) = -\frac{1}{k}\sum_{\begin{cases} n \\ \lambda_n \geq k \end{cases}} \left(\frac{1}{\phi_{0n}(z)}\right)^k.$$

The left-hand side is the $k+1$'st Maclaurin coefficient in the expansion of $g(w,z)-g(0,z)$. The right hand side is $(-1/k)$ multiplying the $(-k)$-moment of all the relevant automorphic functions of f. That explains the title of this section. The whole argument is based on the assumption that we have some summation method for the infinite series:

$$\sum_{n=1}^{\infty} Q_{\lambda_n}\left(\frac{w}{\phi_{0n}(z)}\right).$$

This assumption was also needed for deducing an essential part of Theorem 3 in Section 5. As mentioned in Section 5 this summation problem is an open problem.

In Section 8 an infinite product representation of $f'(w)$ in terms of automorphic functions is given. Proposition 5 shows that $\mathrm{Aut}(f) \subseteq \mathrm{Aut}(g)$ implies that $\exists\, G(w,z)$, entire in w and holomorphic in $z \in \mathbb{C} - g^{-1}(g(0)) - f^{-1}(f(0))$ such that $g(w)-g(z) = (f(w)-f(z)) \cdot G(w,z)$. In particular $\exists\, H(z)$, an entire function such that $g'(z) = H(z) \cdot f'(z)$. Lemma 3 points out to a relation between the fixed points of the non-identity automorphic functions ($\phi_{0n}(w) = w$ for $\phi_{0n} \not\equiv$ id.), and the zeros of $f'(z)$, i.e. $Z(f')$. Accordingly $Z(f')$ might contain on the top of these fixed-points also elements from the fiber $f^{-1}(f(0))$. Theorem 5 gives a formula for $Z(f')$ in terms of the fixed-point sets of the non-identity automorphic functions of $f(w)$. Theorem 6 uses the Laguerre Theorem on separation of zeros and the formula of Theorem 5 to show the reality and the separation property of $\mathrm{Fix}(\mathrm{Aut}(f))$ by $Z(f)$.

Section 9 deals with entire functions of the form $f(z) = P(z)e^{g(z)}$ where $P(z) \in \mathbb{C}[z]$ and where $g \in E$. Let $d := \deg p > 0$ and $Z(p) = \{\alpha_1, \ldots, \alpha_d\} \subseteq \mathbb{C}$. Then $\forall\, j = 1, \ldots, d$, α_j is a common zero of almost all the reciprocals of the automorphic functions of $f(z)$.

The main issue in Section 10 are formulas for the derivatives of the automorphic functions. Theorem 8 gives a kind of integral formula for $\sum_{|\phi_{0n}(z)|<R} \phi_{0n}^{(k)}(z)$. Proposition 6 gives a kind of a partial fractions expansion in terms of $1/(w-\phi_{0n}(z))$ to the w-logarithmic derivative of $f(w)-f(z)$. In Proposition 7 a parallel expansion is given for the z-logarithmic derivative of $f(w)-f(z)$ (recall that w is the function's variable while $z \notin f^{-1}(f(0))$ is a parameter).

In Section 11 we apply the Jensen Theorem to compute the absolute value of products of automorphic functions in terms of an integral of:

$$\log|f(|\phi_{0n}(z)|e^{i\theta}) - f(z)|d\theta$$

These products are further discussed in Section 13.

Section 14 deals with order and type estimates of $f(w)$ in terms of the convergence exponent of the automorphic group $\mathrm{Aut}(f)$. See Theorem 9. Some of the results are related to low order (less than 1) (Theorem 10). There are in this section also density estimates for $\mathrm{Aut}(f)$ for an entire $f(w)$ of a finite order.

Theorems 13 and 14 deal with entire functions of a finite and non-integral order and tie the convergence exponent of the $\mathrm{Aut}(f)$-orbits to this order.

Section 15 is preparing for a future research on extending scattering theory, Selberg Trace formula, etc...to the setting of the discrete groups $\mathrm{Aut}(f)$. Theorem 15 suggests what should be some of the counterparts of the classical theory in the setting of $\mathrm{Aut}(f)$. This is far from being final and conclusive!

In the short Section 16 we bring the basics of the notion of local groups. This notion is clearly relevant to the theory of the automorphic group of an entire function. The material is mostly taken from Terrence Tao's book [16].

In Section 17 we prove the remarkable identities

$$\lim_{j\to\infty} \sum_{\phi_{0n}(w)|<R_j} \phi_{0n}^{(k)}(w) \equiv 0, \quad \forall\, w \in \mathbb{C},$$

for certain sequences $0 < R_1 < R_2 < \ldots < R_n < \ldots (R_n \to \infty)$. Each of these sequences fits simultaneously all the values of $k \in \mathbb{Z}^+$. This is done for low order functions ($0 < \rho < \frac{1}{2}$). The main tools used in the proof are Wiman's-$\cos \pi\rho$ Theorem and our integral formulas for $\sum_{|\phi_{0n}(z)|<R} \phi_{0n}^{(k)}(z)$ in Theorem 8. See Theorem 16. Few examples are elaborated to demonstrate the sharpness of our results here.

In Section 18 we give in Theorem 17 some density estimates on $\{|\phi_{0n}(z)|\}_n$ for functions of a low order ($0 < \rho < \frac{1}{2}$). Again Wiman's-$\cos \pi\rho$ Theorem is a main tool in our proof.

Vieta type formulas for $\mathrm{Aut}(f)$, $0 \le \rho < 1$, are given in Section 19. Theorem 18 gives a formula for the Maclaurin coefficients of $f(w)$ in terms of $f(0) - f(z)$ and $\mathrm{Aut}(f)$.

A reasonable approach to try and extend the classical scattering theory results and the Selberg Trace formula to $\mathrm{Aut}(f)$ is to naturally embed the automorphic group in a larger group, the way $\mathrm{SL}_n(\mathbb{Z})$ is embedded in $\mathrm{GL}_n(\mathbb{R})$. In Section 20 and in Section 21 we try such an approach. Our initial embedding uses an ascending sequence of automorphic groups and a major problem is to try and understand what is the direct limit that is constructed. A typical example originates in the Tuen Wai NG construction of entire functions which have factorizations of unlimited number of prime factors, [17]. Theorem 19 gives the structure of the direct limit group of the ascending automorphic groups. That is done in a certain important case where a Tuen Wai NG function underlies the direct limit. Non-trivial consequences follow in Theorem 20 and in Theorem 21.

Section 22 gives continuity relations between the group $\mathrm{Aut}(f)$ and the sequence of groups $\mathrm{Aut}(f_n)$, where $f_n \to f$ uniformly on compact subsets of $\mathbb{C}$. The proof of that theorem (Theorem 25) is tricky. It uses the elementary Newton's identities for moments and symmetric functions of finite sets, and it uses one of the partial fraction expansions we found before for the w-logarithmic derivative $f'(w)/(f(w)-f(z))$, in Section 10, Proposition 6.

Those results are used in Section 23 to prove some results on the amenability of $\mathrm{Aut}(f)$. In Theorem 27 the assumptions are analytical while in Theorem 29 the assumptions are geometrical, i.e. they use the generations counting functions for systems of fundamental domains of $f(w)$. Some applications to cases where we have control on the growth of the generations counting functions are given in Corollary 21.

2 The Weierstrass Representation of the Automorphic Group of an Entire Function, and the Extra Properties in the Case of a Finite Order

We will use (and for no particular reasons) the following two books: [5], Chapter IV, page 56, and [12], Chapter 15, page 87. This material is classical.

Let $f(z)$ be a non-constant entire function, and let $\{\Omega_i\}$ be a normal system of maximal domains of $f(z)$ ("fundamental domains" in Shimizu's terminology), and $\{\phi_{ij}\}$ is the corresponding automorphic group. We view the difference $f(w) - f(z)$ as an entire function in w, and we view z as a complex parameter. We have the power series expansion $f(w) = a_0 + a_1 w + a_2 w^2 + \ldots$. Hence $f(w) - f(z) = a_1 w + a_2 w^2 + \ldots - (a_1 z + a_2 z^2 + \ldots)$. As a function of w, it has a zero at the origin, $w = 0$, if and only if $a_1 z + a_2 z^2 + \ldots = f(z) - f(0) = 0$ for the particular value z of the complex parameter. This is the case if $z = 0$ (the trivial case). In all other cases (where $f(z) - f(0) \neq 0$), the function $f(w) - f(z)$ (of w) does not vanish at the origin, $w = 0$. Hence the Weierstrass factorization theorem implies the following:

1) If $f(z) - f(0) \neq 0$, then there is a function $g(w, z)$, entire in w and there are non-negative integers $\lambda_n(w, z)$ which we will sometimes denote by λ_n, such that:

$$f(w) - f(z) = \exp\left(g(w, z)\right) \prod_{n=1}^{\infty} E\left(\frac{w}{\phi_{0n}(z)}, \lambda_n\right) =$$

$$= \exp\left(g(w, z)\right) \prod_{n=1}^{\infty} \left(1 - \frac{w}{\phi_{0n}(z)}\right) e^{Q_{\lambda_n}(w/\phi_{0n}(z))},$$

where if $\lambda_n > 0$, then:

$$Q_{\lambda_n}\left(\frac{w}{\phi_{0n}(z)}\right) = \left(\frac{w}{\phi_{0n}(z)}\right) + \frac{1}{2}\left(\frac{w}{\phi_{0n}(z)}\right)^2 + \ldots + \frac{1}{\lambda_n}\left(\frac{w}{\phi_{0n}(z)}\right)^{\lambda_n},$$

and $Q_0(w/\phi_{0n}(z)) \equiv 0$.

2) If $f(z) - f(0) = 0$, then there is a natural number m, and there is an entire function in w, $h(w, z)$ (depending on each zero of $f(z) - f(0)$) and there are non-negative numbers $\lambda_n'(z)$,such that:

$$f(w) - f(z) = w^m \exp\left(h(w, z)\right) \prod_{n=1, \phi_{0n}(z) \neq 0}^{\infty} E\left(\frac{w}{\phi_{0n}(z)}, \lambda_n'\right),$$

where z satisfies: (a) $f(z) - f(0) = 0$, (b) $\phi_{0n}(z) \neq 0$.

Next, we flip the roles of the variable w and the complex parameter z. We obtain:

1) If $f(w) - f(0) \neq 0$, then there is a function $g_1(z, w)$, entire in z and there are non-negative integers $\mu_n(z, w)$ which we will sometimes denote by μ_n, such that:

$$f(z) - f(w) = \exp\left(g_1(z, w)\right) \prod_{n=1}^{\infty} E\left(\frac{z}{\phi_{0n}(w)}, \mu_n\right) =$$

$$= \exp\left(g_1(z, w)\right) \prod_{n=1}^{\infty} \left(1 - \frac{z}{\phi_{0n}(w)}\right) e^{Q_{\mu_n}(z/\phi_{0n}(w))},$$

2) If $f(w) - f(0) = 0$, then with exactly the same values as in case 2 for $f(w) - f(z)$ above we have:

$$f(z) - f(w) = z^m \exp\left(h(z, w)\right) \prod_{n=1, \phi_{0n}(w) \neq 0}^{\infty} E\left(\frac{z}{\phi_{0n}(w)}, \lambda_n'\right),$$

where w satisfies: (a) $f(w) - f(0) = 0$, (b) $\phi_{0n}(w) \neq 0$.

Cases 1 are the generic cases (because cases 2 apply either to a discrete set of z or to a discrete set of w). By $f(w) - f(z) = -(f(z) - f(w))$ we obtain:

Proposition 1 *If $(f(w) - f(0))(f(z) - f(0)) \neq 0$, then:*

$$\exp\left(g(w, z)\right) \prod_{n=1}^{\infty} \left(1 - \frac{w}{\phi_{0n}(z)}\right) e^{Q_{\lambda_n}(w/\phi_{0n}(z))} =$$

$$= -\exp\left(g_1(z, w)\right) \prod_{n=1}^{\infty} \left(1 - \frac{z}{\phi_{0n}(w)}\right) e^{Q_{\lambda_n'}(z/\phi_{0n}(w))},$$

where $g(w, z)$ is entire in w, and $g_1(z, w)$ is entire in z. Moreover by the discussion in [14] that starts on page 229 we may assume that all the automorphic functions

ϕ_{0n} are holomorphic in the (interior) maximal domains Ω_i of the system we fixed. Hence $g(w,z)$ is z-holomorphic in the Ω_i's and $g_1(z,w)$ is w-holomorphic there.

Proposition 2 *If $(f(w)-f(0))(f(z)-f(0)) \neq 0$, then:*

$$\frac{\partial g(w,z)}{\partial w} + \sum_{n=1}^{\infty} \frac{(w/\phi_{0n}(z))^{\lambda_n}}{w-\phi_{0n}(z)} = \frac{\partial g_1(z,w)}{\partial w} + \sum_{n=1}^{\infty} \left(\frac{\phi'_{0n}(w)}{\phi_{0n}(w)}\right) z \frac{(z/\phi_{0n}(w))^{\lambda'_n}}{\phi_{0n}(w)-z}.$$

Proof Take the logarithm of the two sides in the identity of proposition 1, and then $\partial/\partial w$ both sides and simplify. We note that both $g(w,z)$ and $g_1(z,w)$ are holomorphic (usually not entire) in both variables in the appropriate domains of the $\mathbb{C} \times \{\mathbb{C} - \text{discrete set}\}$. □

Remark 3 If we $\partial/\partial w$ both sides of $f(\phi_{0n}(w)) = f(w)$, then we obtain $f'(\phi_{0n}(w))\phi'_{0n}(w) = f'(w)$ in the appropriate domain Ω. By Shimizu this domain is such that $\mathbb{C} - \Omega$ contains no continuum.

Remark 4 If we take $\partial/\partial z$ instead of $\partial/\partial w$, we get the symmetric identity:

$$\frac{\partial g(w,z)}{\partial z} + \sum_{n=1}^{\infty} \left(\frac{\phi'_{0n}(z)}{\phi_{0n}(z)}\right) w \frac{(w/\phi_{0n}(z))^{\lambda_n}}{\phi_{0n}(z)-w} =$$

$$= \frac{\partial g_1(z,w)}{\partial z} + \sum_{n=1}^{\infty} \frac{(z/\phi_{0n}(w))^{\lambda'_n}}{z-\phi_{0n}(w)}.$$

The entire w-functions $g(w,z)$ in the generic Weierstrass factorization of $f(w) - f(z)$ are special regarding their relation to the action of the automorphic group of $f(z)$ (note that here we take z as the variable).

Proposition 3 *If $f(z) - f(0) \neq 0$, then there is a function $g(w,z)$, entire in w and there are non-negative integers $\lambda_n = \lambda_n(w, \phi_{0n}(z))$ such that:*

$$f(w) - f(z) = \exp\left(g(w,z)\right) \prod_{n=1}^{\infty} \left(1 - \frac{w}{\phi_{0n}(z)}\right) e^{Q_{\lambda_n}(w/\phi_{0n}(z))}.$$

If $\forall\, n$ we have $\lambda_n = \lambda$, a constant value independent of n, then the function $\exp(g(w,z))$ *is z-invariant with respect to the action of the automorphic group of $f(z)$. This means that* $\exp\left(g(w,\phi_{ij}(z))\right) = \exp\left(g(w,z)\right)$ *for every element ϕ_{ij} in the automorphic group. Our assumption on the $\lambda_n \equiv \lambda$ is valid whenever the entire function $f(w)$ has a finite order.*

Proof The first part is just Weierstrass factorization theorem applied to $f(w) - f(z)$ (as an entire function of w). Let ϕ_{ij} be any automorphic function of $f(z)$. This means that $f(\phi_{ij}(z)) = f(z)$. We note that:

$$\prod_{n=1}^{\infty}\left(1-\frac{w}{\phi_{0n}(\phi_{ij}(z))}\right)e^{Q_{\lambda_n(w,\phi_{0n}(\phi_{ij}(z)))}(w/\phi_{0n}(\phi_{ij}(z)))}=$$

$$=\prod_{n=1}^{\infty}\left(1-\frac{w}{\phi_{0n}(z)}\right)e^{Q_{\lambda_n}(w/\phi_{0n}(z))},$$

because the left-hand side product is a product of a permutation of the factors of the right-hand side. This follows by the assumption on the $\lambda_n \equiv \lambda$, independent of n. By the convergence uniformly on compacta the two products are equal to one another. Hence the quotient function:

$$(f(w)-f(z))\Bigg/\prod_{n=1}^{\infty}\left(1-\frac{w}{\phi_{0n}(z)}\right)e^{Q_{\lambda_n}(w/\phi_{0n}(z))}$$

is invariant with respect to the action of the elements $\{\phi_{ij}\}$ of the automorphic group of $f(z)$. But by the Weierstrass identity above, this quotient function equals $\exp(g(w,z))$. □

We now indicate conclusions of this proposition which are of a different character than the ones above. We will use the following:

Lemma 1 *If $u(z)$ and $v(z)$ are non-constant entire functions and if* $\mathrm{Aut}(v)$ *is a subgroup of* $\mathrm{Aut}(u)$*, then there exists a function $h(w)$, holomorphic on the image of v (i.e., on $v(\mathbb{C})$) such that $u(z) = h(v(z))$.*

Proof That follows using the methods in [10], however, we will prove it using a result in [3]. Namely, we will make use of the result that appears on the last paragraph on page 334 and continues on the first paragraph on the next page, 335 in [3]. Thus we claim that $v \le u$, where the partial order is defined in [3] (where we indicated). To prove that we need to show that $v(z) = v(w)$ implies that $u(z) = u(w)$, $\forall\, z, w \in \mathbb{C}$. Thus assuming that $v(z) = v(w)$, it follows that $\exists\, \Phi \in \mathrm{Aut}(v)$, such that $w = \Phi(z)$. By an assumption we have, it follows that $\Phi \in \mathrm{Aut}(u)$. Hence it indeed follows that $u(z) = u(w)$, and we proved that $v \le u$. The claim in our lemma now follows by [3]. □

Using the Shimizu's [14] (or [10]) we can re-write Lemma 1 in a geometric manner:

Lemma 2 *If $u(z)$ and $v(z)$ are non-constant entire functions and if* $\mathrm{Aut}(v)$ *is a subgroup of* $\mathrm{Aut}(u)$*, then a normal system of maximal domains $\{\Omega_j\}$ of $v(z)$ is composed of maximal domains Ω_j each of which is tiled by some elements of the same normal system of maximal domains of $u(z)$.*

Specializing Lemma 2 to the setting of Proposition 3 we obtain:

Corollary 1 *Under the assumptions of Proposition 3: The normal system of maximal domains $\{\Omega_j\}$ of $f(z)$ which induces the Weierstrass factorization of*

$f(w) - f(z)$ is composed of maximal domains Ω_j, each of which is tiled by some elements of the same normal system of maximal domains of $\exp(g(w,z))$ *as a function of z and for a fixed w.*

Specializing Lemma 1 to the setting of Proposition 3 we obtain:

Corollary 2 *Under the assumptions of Proposition 3: There is a function $F(w, w_1)$, holomorphic in $\mathbb{C} \times f(\mathbb{C})$ such that* $\exp(g(w,z)) = \exp(F(w, f(z)))$.

So the Weierstrass factorization of $f(w) - f(z)$ that is described in Proposition 3 is special, and we add the extra information in the following:

Theorem 1 *Let f be a non-constant entire function of a finite order, and assuming that $f(z) - f(0) \neq 0$ we have the following expansion:*

$$f(w) - f(z) = \exp(F(w, f(z))) \prod_{n=1}^{\infty} \left(1 - \frac{w}{\phi_{0n}(z)}\right) e^{Q_\lambda(w/\phi_{0n}(z))}.$$

In particular, the canonical infinite product is in fact a holomorphic function of $(w, f(z))$, for all $w \in \mathbb{C}$ and $z \in (\mathbb{C} - Z(f - f(0)))$.

Corollary 3 *Let f be a non-constant entire function of a finite order, then we have the following cycle relation:*

$$\sum_{j=1}^{N} \exp\left(F(z_j, f(z_{j+1}))\right) \prod_{n=1}^{\infty} \left(1 - \frac{z_j}{\phi_{0n}(z_{j+1})}\right) e^{Q_\lambda(z_j/\phi_{0n}(z_{j+1}))} \equiv 0,$$

for any N independent variables: $z_1, \ldots, z_N$, where we agree that $z_{N+1} = z_1$.

Proof

<u>**Method 1:**</u> Using the identity in Theorem 1 we have the following:

$$f(z_1) = f(z_2) + \exp(F(z_1, f(z_2))) \prod_{n=1}^{\infty} \left(1 - \frac{z_1}{\phi_{0n}(z_2)}\right) e^{Q_\lambda(z_1/\phi_{0n}(z_2))},$$

$$f(z_2) = f(z_3) + \exp(F(z_2, f(z_3))) \prod_{n=1}^{\infty} \left(1 - \frac{z_2}{\phi_{0n}(z_3)}\right) e^{Q_\lambda(z_2/\phi_{0n}(z_3))},$$

$$\vdots$$

$$f(z_{N-1}) = f(z_N) + \exp(F(z_{N-1}, f(z_N))) \prod_{n=1}^{\infty} \left(1 - \frac{z_{N-1}}{\phi_{0n}(z_N)}\right) e^{Q_\lambda(z_{N-1}/\phi_{0n}(z_N))},$$

$$f(z_N) = f(z_1) + \exp(F(z_N, f(z_1))) \prod_{n=1}^{\infty} \left(1 - \frac{z_N}{\phi_{0n}(z_1)}\right) e^{Q_\lambda(z_N/\phi_{0n}(z_1))}.$$

We plug these identities successively each in its predecessor and eventually cancel out $f(z_1)$ from both sides of the equation.

Method 2: The cycle relation is merely the Weierstrass factorization of each term in the following telescopic identity:

$$(f(z_1)-f(z_2))+(f(z_2)-f(z_3))+\ldots+(f(z_{N-1})-f(z_N))+(f(z_N)-f(z_1)) \equiv 0.$$

□

Corollary 4 *Let f be a non-constant entire function of a finite order, then we have the following chain relation:*

$$f(z_1)-f(z_{N+1}) = \sum_{j=1}^{N} \exp\left(F(z_j, f(z_{j+1}))\right) \prod_{n=1}^{\infty} \left(1 - \frac{z_j}{\phi_{0n}(z_{j+1})}\right) e^{Q_\lambda(z_j/\phi_{0n}(z_{j+1}))}$$

for any $N+1$ independent variables $z_1, z_2, \ldots, z_{N+1}$.

Proof It is clear how to adopt any of the two methods of the proof we gave to Corollary 3. □

Corollary 5

$$\sum_{j=1}^{N} \exp\left(F(z_j, f(z_{j+1}))\right) \prod_{n=1}^{\infty} \left(1 - \frac{z_j}{\phi_{0n}(z_{j+1})}\right) e^{Q_\lambda(z_j/\phi_{0n}(z_{j+1}))} \equiv$$

$$\equiv \exp\left(F(z_1, f(z_{N+1}))\right) \prod_{n=1}^{\infty} \left(1 - \frac{z_1}{\phi_{0n}(z_{N+1})}\right) e^{Q_\lambda(z_1/\phi_{0n}(z_{N+1}))}$$

for any $N+1$ independent variables $z_1, z_2, \ldots, z_{N+1}$.

Remark 5 The results that begin in our Corollary 1 and end in Corollary 5 should be carefully interpreted, because $\exp(g(w, z))$ is entire in w, but is not known to be entire in z. This is because we are dealing only with those values of the parameter z for which $f(z) - f(0) \neq 0$. Thus a priori it is not clear what is the meaning of "elements of a normal system of maximal domains of $\exp(g(w, z)$ for a fixed w." This notion was defined by Shimizu only for meromorphic functions, but we do not know that $\exp(g(w, z))$ is meromorphic in z.

Remark 6 The cycle relation in Corollary 3 and the chain relation in Corollary 4 and in Corollary 5 resemble the fact that the value of a path integral is independent of the path that connects the two endpoints for a conservative field. In our setting one may think of $f(z)$ as the "potential" of the complicated Weierstrass products that appear within the sum. This resemblance originates in the elementary fact that the sum of telescopic series depends only on the initial and the terminal points.

3 Conclusions from Proposition 3 in Case We Have No Monodromy

The conclusions from Proposition 3 that were derived in the previous section originated in Lemmas 1 and 2. These, in turn, were based on a result of Eremenko and Rubel in [3]. However, their result used the monodromy principle that was available in their setting. What if we have no monodromy? Of course the conclusion of Proposition 3 is still valid but we have no composition relations between f and the Weierstrass factor $\exp(g(w, z))$. We will outline in this section what can we still conclude.

Corollary 6 *For any element $\phi_{ij}(z)$ of the automorphic group of $f(z)$, a non-constant entire function of a finite order, there exists an integer $n_{ij} \in \mathbb{Z}$ so that $g(w, \phi_{ij}(z)) = g(w, z) + 2\pi n_{ij} \cdot i$.*

Remark 7 It would be nice to compute $g(w, z)$ for different entire functions and to check the various identities we obtained. Later on we will carry such a computation for the exponential function.

Remark 8 The $g(w, z)$ function translates the group of automorphic functions (composition of mappings is its binary operation) into a subgroup of $(\mathbb{Z}, +)$.

Proof Let us take two automorphic functions ϕ_{ij} and $\phi_{\alpha,\beta}$. Then: $g(w, \phi_{ij}(z)) = g(w, z) + 2\pi n_{ij} \cdot i$ and $g(w, \phi_{\alpha\beta}(z)) = g(w, z) + 2\pi n_{\alpha\beta} \cdot i$.

$$g(w, \phi_{ij}(\phi_{\alpha,\beta}(z))) = g(w, \phi_{\alpha,\beta}(z)) + 2\pi n_{ij} \cdot i =$$

$$= (g(w, z) + 2\pi n_{\alpha\beta} \cdot i) + 2\pi n_{ij} \cdot i = g(w, z) + 2\pi(n_{\alpha\beta} + n_{ij}) \cdot i.$$

□

Corollary 7 *The automorphic group of a non-constant entire function of a finite order is homomorphic to a subgroup of $(\mathbb{Z}, +)$ (which is not always the trivial homomorphism).*

Proof Let f be a non-constant entire function. Let us denote by $\mathrm{Aut}(f)$ its group of automorphic functions. Let us define the mapping $T : \mathrm{Aut}(f) \to \mathbb{Z}$ by the formula suggested by Corollary 6, i.e.

$$T(\Phi) = \frac{1}{2\pi \cdot i} \left(g(w, \Phi(z)) - g(w, z)\right).$$

Here the complex numbers $w, z \in \mathbb{C}$ are completely arbitrary within the domain of the definition of $g(w, z)$. Then $\forall \Phi_1, \Phi_2 \in \mathrm{Aut}(f)$ we have the identity $T(\Phi_1 \circ \Phi_2) = T(\Phi_1) + T(\Phi_2)$ (by Remark 8). □

Corollary 8 *Let f be a non-constant entire function of a finite order. If $\Phi \in \mathrm{Aut}(f)$ is an element of a finite order, then $\Phi \in \ker(T)$.*

Proof The only finite subgroup of the infinite cyclic group $(\mathbb{Z}, +)$ is the trivial subgroup $\{0\}$. □

We deduce a family of functional-arithmetical identities from Corollary 7. For that we will use the obvious short notation for repeated composition of functions.

Definition 1 Let $h_j(z)$, $j = 1, \ldots, n$ be n complex valued functions for which the repeated composition makes sense. We will denote:

$$(h_1 \circ \ldots \circ h_n) = \bigcirc_{j=1}^{n} h_j.$$

Corollary 9 *Let f be a non-constant entire function of a finite order, and let $\Phi_1, \ldots, \Phi_n \in \mathrm{Aut}(f)$ $(n \geq 2)$. Then whenever the repeated composition makes sense we have the identity:*

$$g\left(w, \left(\bigcirc_{j=1}^{n} \Phi_j\right)(z)\right) - \sum_{j=1}^{n} g\left(w, \Phi_j(z)\right) + (n-1)g(w, z) \equiv 0,$$

$\forall\, (w, z) \in \mathbb{C} \times (\mathbb{C} - \text{a discrete set})$.

Proof The proof is inductive on $n \in \mathbb{Z}_{\geq 2}$. For $n = 2$ we have (using the map T in the proof of Corollary 7: $T(\Phi_1 \circ \Phi_2) = T(\Phi_1) + T(\Phi_2)$, i.e.:

$$\frac{1}{2\pi \cdot i} \left(g(w, (\Phi_1 \circ \Phi_2)(z)) - g(w, z)\right) =$$

$$= \frac{1}{2\pi \cdot i} \left(g(w, \Phi_1(z)) - g(w, z)\right) + \frac{1}{2\pi \cdot i} \left(g(w, \Phi_2(z)) - g(w, z)\right).$$

Hence:

$$g(w, (\Phi_1 \circ \Phi_2)(z)) - g(w, \Phi_1(z)) - g(w, \Phi_2(z)) + g(w, z) \equiv 0.$$

This completes the case $n = 2$. Similarly the case $n = 3$ follows in a very similar manner from: $T(\Phi_1 \circ \Phi_2 \circ \Phi_3) = T(\Phi_1) + T(\Phi_2) + T(\Phi_3)$, i.e.:

$$\begin{aligned} &\frac{1}{2\pi \cdot i} \left(g(w, (\Phi_1 \circ \Phi_2 \circ \Phi_3)(z)) - g(w, z)\right) = \\ &\quad = \frac{1}{2\pi \cdot i} \left(g(w, \Phi_1(z)) - g(w, z)\right) + \frac{1}{2\pi \cdot i} \left(g(w, \Phi_2(z)) - g(w, z)\right) \\ &\qquad + \frac{1}{2\pi \cdot i} \left(g(w, \Phi_3(z)) - g(w, z)\right). \end{aligned}$$

Hence:

$$g(w,(\Phi_1\circ\Phi_2\circ\Phi_3)(z))-g(w,\Phi_1(z))-g(w,\Phi_2(z))-g(w,\Phi_3(z))+2\cdot g(w,z)\equiv 0.$$

This completes the case $n=3$, etc... □

We have now a direct connection between composition arithmetic and lattice (integral) arithmetic. Here is a straightforward example. We might think of $|T(\Phi)|$ has the distance between $g(w,\Phi(z))$ and $g(w,z)$. Given two non-trivial elements $\Phi_1,\Phi_2\in \mathrm{Aut}(f)$, i.e. elements for which the corresponding distances are not 0 ($|T(\Phi_1)T(\Phi_2)|>0$) we can find an non-trivial element with a shorter distance. Here is a possible way to go about solving that:

Corollary 10 *Let f be a non-constant entire function of a finite order and let $\Phi_1,\Phi_2\in \mathrm{Aut}(f)$. Suppose that we have:*

$$m=\frac{1}{2\pi\cdot i}\left(g(w,\Phi_1(z))-g(w,z)\right),\;\; n=\frac{1}{2\pi\cdot i}\left(g(w,\Phi_2(z))-g(w,z)\right),$$

where $m\cdot n\neq 0$. Let $d=a\cdot m+b\cdot n=(m,n)$ the lcm of the integers m and n. Here d can be the positive or the negative lcm. Then if we define:

$$\Phi=\Phi_1^{\circ a}\circ\Phi_2^{\circ b},$$

then we have:

$$d=\frac{1}{2\pi\cdot i}\left(g(w,\Phi(z))-g(w,z)\right).$$

Proof We clearly define

$$\Phi_1^{\circ a}=\begin{cases}\Phi_1\circ\ldots\circ\Phi_1 & \text{if } a>0\\ (\Phi_1)^{-1}\circ\ldots\circ(\Phi_1)^{-1} & \text{if } (-a)>0\end{cases}.$$

We note that we have: $T(\Phi_1^{\circ a})=a\cdot T(\Phi_1)$. Hence $T(\Phi)=T(\Phi_1^{\circ a}\circ\Phi_2^{\circ b})=a\cdot T(\Phi_1)+b\cdot T(\Phi_2)=a\cdot m+b\cdot n=d$. □

Corollary 11 *Let f be a non-constant entire function of a finite order and let $\Phi_1,\Phi_2\in \mathrm{Aut}(f)$. Then we have:*

$$\frac{1}{2\pi\cdot i}\left(g(w,\Phi_1(z)^{\circ T(\Phi_2)})-g(w,z)\right)=\frac{1}{2\pi\cdot i}\left(g(w,\Phi_2(z)^{\circ T(\Phi_1)})-g(w,z)\right).$$

Proof This follows by the fact that:

$$T(\Phi_1^{\circ T(\Phi_2)})=T(\Phi_2)T(\Phi_1)=T(\Phi_1)T(\Phi_2)=T(\Phi_2^{\circ T(\Phi_1)}).$$

□

Corollary 12 *Let f be a non-constant entire function of a finite order and let $\Phi \in$ $\mathrm{Aut}(f)$ satisfy the condition that the number*

$$\frac{1}{2\pi \cdot i}\left(g(w, \Phi(z)) - g(w, z)\right)$$

is a prime number p. If $\Phi = \Psi^{\circ k}$ for some $\Psi \in \mathrm{Aut}(f)$, then either $T(\Psi)$ equals 1 or equals p.

Corollary 13 *Let f be a non-constant entire function of a finite order and let $\Phi \in$ $\mathrm{Aut}(f)$ satisfy the condition that the number*

$$of\,\frac{1}{2\pi \cdot i}\left(g(w, \Phi(z)) - g(w, z)\right)$$

is a prime number p. If $\Phi = \Phi_1 \circ \ldots \circ \Phi_n$, where $n \in \mathbb{Z}^+$, and where for $j = 1, \ldots, n$, $\Phi_j \in \mathrm{Aut}(f)$, then there is a single index k, between 1 and n such that $T(\Phi_k) = p$ while for $j \in \{1, \ldots, n\} - \{k\}$, $T(\Phi_j) = 1$.

Corollaries 10, 11, 12, and 13 are all particular cases of the principle that the arithmetic of composition of automorphic functions of a non-constant entire function has an analog in the arithmetic of the integers, $\mathbb{Z}$. We can describe the general principle in the following:

Theorem 2 (The Finiteness of the Decomposition of Automorphic Functions) *Let f be a non-constant entire function of a finite order and let $\Phi \in \mathrm{Aut}(f)$ satisfy the condition:*

$$\frac{1}{2\pi \cdot i}\left(g(w, \Phi(z)) - g(w, z)\right) = N \in \mathbb{Z} - \{0\}.$$

Then any decomposition of Φ into a composition of automorphic functions of f:

$$\Phi = \Phi_1 \circ \ldots \circ \Phi_n, \quad \Phi_1, \ldots, \Phi_n \in \mathrm{Aut}(f),$$

has the following properties:

1) n could be any natural number with no a priori upper bound.
2) We have the Diophantine identity:

$$\frac{1}{2\pi \cdot i}\left(g(w, \Phi(z)) - g(w, z)\right) = \prod_{j=1}^{n}\left(\frac{1}{2\pi \cdot i}\big(g(w, \Phi_j(z)) - g(w, z)\big)\right).$$

If we call an automorphic function Φ_j an arithmetical unit, if it satisfies:

$$\left|\frac{1}{2\pi \cdot i}\big(g(w, \Phi_j(z)) - g(w, z)\big)\right| = 1,$$

then in any such decomposition of Φ, the set:

$$\left\{ \frac{1}{2\pi \cdot i} \left(g(w, \Phi_j(z)) - g(w, z) \right) \neq \pm 1 \right\},$$

is a set of non-unit divisors of N whose product is N, and all the other factors belong to arithmetical units. In particular for any $\Phi \in \mathrm{Aut}(f)$, the number of different decompositions that differ in their non-units is bounded above by:

$$\sum m! \cdot |\{\{k_1, \ldots, k_m\} \,|\, k_1 \cdot \ldots \cdot k_m = N, \ |k_1|, \ldots, |k_m| > 1\}|.$$

The weights $m!$ must be present because of composition of functions, unlike multiplication of integers in a non-commutative binary operation.

4 The Cycle Relation and the Chain Relation in the General Case

The results in Section 2 dealt mostly with entire functions of a finite order. The key result was Proposition 3 and we assumed that $\lambda_n \equiv \lambda$ independent of n. This essentially is the assumption that f has a finite order. In this section we point out the results if this assumption is dropped out.

Corollary 14 *Let f be a non-constant entire function, then we have the following cycle relation:*

$$\sum_{j=1}^{N} \exp\left(g(z_j, z_{j+1})\right) \prod_{n=1}^{\infty} \left(1 - \frac{z_j}{\phi_{0n}(z_{j+1})}\right) e^{Q_{\lambda_n}(z_j/\phi_{0n}(z_{j+1}))} \equiv 0,$$

for any N independent variables: $z_1, \ldots, z_N$, where we agree that $z_{N+1} = z_1$.

Proof

Method 1: Using the first identity in Proposition 3 where no finite order assumption is needed, we have the following:

$$f(z_1) = f(z_2) + \exp\left(g(z_1, z_2)\right) \prod_{n=1}^{\infty} \left(1 - \frac{z_1}{\phi_{0n}(z_2)}\right) e^{Q_{\lambda_n}(z_1/\phi_{0n}(z_2))},$$

$$f(z_2) = f(z_3) + \exp\left(g(z_2, z_3)\right) \prod_{n=1}^{\infty} \left(1 - \frac{z_2}{\phi_{0n}(z_3)}\right) e^{Q_{\lambda_n}(z_2/\phi_{0n}(z_3))},$$

$$\vdots$$

$$f(z_{N-1}) = f(z_N) + \exp\left(g(z_{N-1}, z_N)\right) \prod_{n=1}^{\infty} \left(1 - \frac{z_{N-1}}{\phi_{0n}(z_N)}\right) e^{Q_{\lambda_n}(z_{N-1}/\phi_{0n}(z_N))},$$

$$f(z_N) = f(z_1) + \exp\left(g(z_N, z_1)\right) \prod_{n=1}^{\infty} \left(1 - \frac{z_N}{\phi_{0n}(z_1)}\right) e^{Q_{\lambda_n}(z_N/\phi_{0n}(z_1))}.$$

We plug these identities successively each in its predecessor and eventually cancel out $f(z_1)$ from both sides of the equation.

Method 2: The cycle relation is merely the Weierstrass factorization of each term in the following telescopic identity:

$$(f(z_1)-f(z_2))+(f(z_2)-f(z_3))+\ldots+(f(z_{N-1})-f(z_N))+(f(z_N)-f(z_1))\equiv 0.$$

□

Corollary 15 *Let f be a non-constant entire function, then we have the following chain relation:*

$$f(z_1) - f(z_{N+1}) = \sum_{j=1}^{N} \exp\left(g(z_j, z_{j+1})\right) \prod_{n=1}^{\infty} \left(1 - \frac{z_j}{\phi_{0n}(z_{j+1})}\right) e^{Q_{\lambda_n}(z_j/\phi_{0n}(z_{j+1}))}$$

for any $N+1$ independent variables $z_1, z_2, \ldots, z_{N+1}$.

Proof It is clear how to adopt any of the two methods of the proof we gave to Corollary 14. □

Corollary 16

$$\sum_{j=1}^{N} \exp\left(g(z_j, z_{j+1})\right) \prod_{n=1}^{\infty} \left(1 - \frac{z_j}{\phi_{0n}(z_{j+1})}\right) e^{Q_{\lambda_n}(z_j/\phi_{0n}(z_{j+1}))} \equiv$$

$$\equiv \exp\left(g(z_1, z_{N+1})\right) \prod_{n=1}^{\infty} \left(1 - \frac{z_1}{\phi_{0n}(z_{N+1})}\right) e^{Q_{\lambda_n}(z_1/\phi_{0n}(z_{N+1}))}$$

for any $N+1$ independent variables $z_1, z_2, \ldots, z_{N+1}$.

5 Examples (Mostly the Exponential Function) and the Role Played by the Assumption That We Have Some Summation Method for the Infinite Series: $\sum_{n=1}^{\infty} Q_{\lambda_n}\left(\frac{w}{\phi_{0n}(z)}\right)$, for the Reconstruction of f from Aut(f)

Let $f(z) = e^z$. We consider the following natural system of maximal domains of $f(z)$:

$$\{\Omega_n = \{z \in \mathbb{C} \,|\, 2\pi \cdot i \cdot n < \Im z < 2\pi \cdot i \cdot (n+1)\} \,|\, n \in \mathbb{Z}\}.$$

This induces the infinite cyclic automorphic group:

$$\mathrm{Aut}(e^z) = \{z + 2\pi \cdot i \cdot n \,|\, n \in \mathbb{Z}\} =< z + 2\pi \cdot i > .$$

The discrete exceptional set of z is the solution set of the equation $e^z - e^0 = 0$. So this is the discrete set $\{2\pi \cdot i \cdot n \,|\, n \in \mathbb{Z}\}$. Using the representation of Proposition 3 we clearly can choose the sequence $\lambda_n \equiv 1, \forall\, n \in \mathbb{Z}$. Thus for $z \notin \{2\pi \cdot i \cdot n \,|\, n \in \mathbb{Z}\}$, we have:

$$e^w - e^z = \exp\left(g(w,z)\right) \prod_{n \in \mathbb{Z}} \left(1 - \frac{w}{z + 2\pi \cdot i \cdot n}\right) e^{(w/(z+2\pi \cdot i \cdot n))}.$$

We can group together symmetric pairs n and $-n$, where $n \in \mathbb{Z}^+$. We compute the corresponding products:

$$\left(1 - \frac{w}{z + 2\pi \cdot i \cdot n}\right) e^{(w/(z+2\pi \cdot i \cdot n))} \times \left(1 - \frac{w}{z - 2\pi \cdot i \cdot n}\right) e^{(w/(z-2\pi \cdot i \cdot n))}$$

and we can write the final result in two forms as follows:

$$e^w - e^z = \exp\left(g(w,z)\right)\left(1 - \frac{w}{z}\right) e^{w/z} \prod_{n=1}^{\infty} \left(\frac{(z-w)^2 + 4\pi^2 n^2}{z^2 + 4\pi^2 n^2}\right) e^{2zw/(z^2+4\pi^2 n^2)} =$$

$$= \exp\left(g(w,z)\right)\left(1 - \frac{w}{z}\right) e^{w/z} \prod_{n=1}^{\infty} \left(1 - \frac{z^2 - (z-w)^2}{z^2 + 4\pi^2 n^2}\right) e^{2zw/(z^2+4\pi^2 n^2)},$$

$\forall\, w \in \mathbb{C}, \forall\, z \in \mathbb{C} - \{2\pi \cdot i \cdot n \,|\, n \in \mathbb{Z}\}$. Here $g(w,z)$ is entire in w and holomorphic in $z \notin \{2\pi \cdot i \cdot n \,|\, n \in \mathbb{Z}\}$. Next, we note that if we replace z by $z + 2\pi \cdot i \cdot k$ for some $k \in \mathbb{Z}$, then clearly:

$$\prod_{n\in\mathbb{Z}}\left(1-\frac{w}{(z+2\pi\cdot i\cdot k)+2\pi\cdot i\cdot n}\right)e^{(w/((z+2\pi\cdot i\cdot k)+2\pi\cdot i\cdot n))}=$$

$$=\prod_{n\in\mathbb{Z}}\left(1-\frac{w}{z+2\pi\cdot i\cdot(n+k)}\right)e^{(w/(z+2\pi\cdot i\cdot(n+k)))}=$$

$$=\prod_{n\in\mathbb{Z}}\left(1-\frac{w}{z+2\pi\cdot i\cdot n}\right)e^{(w/(z+2\pi\cdot i\cdot n))}.$$

Also $e^w-e^{z+2\pi\cdot i\cdot k}=e^w-e^z$. Hence the basic Weierstrass factorization:

$$e^w-e^z=\exp\left(g(w,z)\right)\prod_{n\in\mathbb{Z}}\left(1-\frac{w}{z+2\pi\cdot i\cdot n}\right)e^{(w/(z+2\pi\cdot i\cdot n))},$$

implies that indeed we have $\exp\left(g(w,z+2\pi\cdot i\cdot k)\right)=\exp\left(g(w,z)\right)$. Next, let us consider e^w-1. This entire function has simple zeros at $\{2\pi\cdot i\cdot n\,|\,n\in\mathbb{Z}\}$ and only there. So using the standard Weierstrass factorization we obtain an identity of the following form:

$$e^w-1=\exp\left(h(w)\right)\cdot w\cdot\prod_{n=1}^{\infty}\left(1-\frac{w^2}{4\pi^2n^2}\right).$$

This follows by taking the symmetric order of factors in:

$$e^w-1=\exp\left(h(w)\right)\cdot w\cdot\prod_{n\in\mathbb{Z}-\{0\}}\left(1-\frac{w}{2\pi\cdot i\cdot n}\right)e^{w/(2\pi\cdot i\cdot n)}.$$

Using this identity we obtain:

$$e^w-e^z=e^z(e^{w-z}-1)=e^z\exp\left(h(w-z)\right)\cdot(w-z)\prod_{n=1}^{\infty}\left(1+\frac{(w-z)^2}{4\pi^2n^2}\right)=$$

$$=e^z\exp\left(h(w-z)\right)\cdot(w-z)\prod_{n\in\mathbb{Z}-\{0\}}\left(1-\frac{w-z}{2\pi\cdot i\cdot n}\right)e^{(w-z)/(2\pi\cdot i\cdot n)}.$$

Thus we obtained two different identities:

$$e^w-e^z=\exp\left(g(w,z)\right)\prod_{n\in\mathbb{Z}}\left(1-\frac{w}{z+2\pi\cdot i\cdot n}\right)e^{w/(z+2\pi\cdot i\cdot n)}=$$

$$= e^z \exp\left(h(w-z)\right) \cdot (w-z) \prod_{n \in \mathbb{Z}-\{0\}} \left(1 - \frac{w-z}{2\pi \cdot i \cdot n}\right) e^{(w-z)/(2\pi \cdot i \cdot n)},$$

or by symmetric multiplication:

$$e^w - e^z = \exp\left(g(w,z)\right) \left(1 - \frac{w}{z}\right) e^{w/z} \prod_{n=1}^{\infty} \left(1 - \frac{z^2 - (w-z)^2}{z^2 + 4\pi^2 n^2}\right) e^{(2zw)/(z^2+4\pi^2 n^2)} =$$

$$= e^z \exp\left(h(w-z)\right) \cdot (w-z) \prod_{n=1}^{\infty} \left(1 + \frac{(w-z)^2}{4\pi^2 n^2}\right).$$

This is different from the unique factorization of polynomials. We have no uniqueness of product representation. A well-known phenomenon. Before proceeding to the computation of the Weierstrass factor $g(w,z)$, which is not trivial even for the exponential function, let us solve first the polynomial case. We start with the following quadratic $f(z) = z^2 + z$ and we note that $f(w) - f(z) = (w-z)(w+z+1)$, so that $\mathrm{Aut}(f) = \{z, -z-1\}$, and the product part is:

$$\left(1 - \frac{w}{z}\right)\left(1 - \frac{w}{-z-1}\right) = \left(\frac{z-w}{z}\right)\left(\frac{w+z+1}{z+1}\right) = \frac{f(z) - f(w)}{f(z)}.$$

Thus we get the representation:

$$f(w) - f(z) = (-1) \cdot f(z) \left(1 - \frac{w}{z}\right)\left(1 - \frac{w}{-z-1}\right) =$$

$$= (-1) \cdot f(z) \left(1 - \frac{w}{\phi_0(z)}\right)\left(1 - \frac{w}{\phi_1(z)}\right).$$

Now, let us consider a general polynomial: $f(z) = p_d(z) = a_d z^d + a_{d-1} z^{d-1} + \ldots + a_1 z + a_0$, $a_d \neq 0$. Then $\mathrm{Aut}(p_d) = \{\phi_0(z), \ldots, \phi_{d-1}(z)\}$, where $\phi_0(z) = z$. Clearly:

$$p_d(w) - p_d(z) = a_d \prod_{n=0}^{d-1} (w - \phi_n(z)) = (-1)^d a_d \left\{\prod_{n=0}^{n-1} \phi_n(z)\right\} \prod_{n=0}^{d-1} \left(1 - \frac{w}{\phi_n(z)}\right).$$

By $p_d(w) - p_d(z) = a_d \prod_{n=0}^{d-1} (w - \phi_n(z))$ it follows that the free term of this w-polynomial is given by $p_d(0) - p_d(z) = a_d \prod_{n=0}^{d-1} (0 - \phi_n(z)) = (-1)^d a_d \prod_{n=0}^{d-1} \phi_n(z)$. So we proved that the Weierstrass factorization representation of the automorphic group of a general monic polynomial is:

$$p_d(w) - p_d(z) = (p_d(0) - p_d(z)) \prod_{n=0}^{d-1} \left(1 - \frac{w}{\phi_n(z)}\right).$$

A full generalization of the quadratic case. Is that formula valid for any entire function? Unfortunately it is not the case. One might have falsely suspected at first that we can approximate an entire $f(z)$ by the polynomials $p_d(z)$ which are the partial sums of the power series expansion of f. Each p_d as the above simple Weierstrass factorization of $p_d(w) - p_d(z)$, and then when $d \to \infty$ we clearly have $p_d(w) - p_d(z) \to f(w) - f(z)$. We might have hoped that the automorphic functions ϕ_n^d converge when $d \to \infty$ to the automorphic functions ϕ_{0n} of f, and if we are lucky also

$$\lim_{d\to\infty} \prod_{n=0}^{d-1} \left(1 - \frac{w}{\phi_n^d(z)}\right) = \prod_{n=0}^{\infty} \left(1 - \frac{w}{\phi_{0n}(z)}\right),$$

thus proving that:

$$f(w) - f(z) = (f(0) - f(z)) \prod_{n=0}^{\infty} \left(1 - \frac{w}{\phi_{0n}(z)}\right).$$

However, this clearly is wrong for the last infinite product is usually divergent unless we multiply each term by the corresponding normalizing Weierstrass factor $\exp\left(Q_{\lambda_n}(w/\phi_{0n}(z))\right)$. This simple formula has a chance of being correct only if f is of order 0 and $\forall n$, $\lambda_n = 0$. For the sake of completeness let us give a concrete example which proves that this simplistic formula is wrong. If this formula were true for $f(z) = e^z$, we would have something like the following:

$$e^w - e^z = (1 - e^z) \prod_{n\in\mathbb{Z}} \left(1 - \frac{w}{z + 2\pi \cdot i \cdot n}\right) e^{w/(z+2\pi\cdot i\cdot n)} =$$

$$= (1 - e^z)\left(1 - \frac{w}{z}\right) e^{w/z} \prod_{n=1}^{\infty} \left(1 - \frac{z^2 - (z-w)^2}{z^2 + 4\pi^2 n^2}\right) e^{2zw/(z^2+4\pi^2 n^2)}.$$

If this was true then:

$$\frac{e^w - e^z}{w - z} = (e^z - 1)\frac{e^{w/z}}{z} \prod_{n=1}^{\infty} \left(1 - \frac{z^2 - (z-w)^2}{z^2 + 4\pi^2 n^2}\right) e^{2zw/(z^2+4\pi^2 n^2)}.$$

Taking the limits of both sides, when $w \to z$ we get:

$$e^z = (e^z - 1)\frac{e}{z} \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{z^2 + 4\pi^2 n^2}\right) e^{2z^2/(z^2+4\pi^2 n^2)}.$$

We note that the convergent infinite product has no zero, as to be expected. Thus it looks promising, till we specialize to $z = i\pi$:

$$-1 = \frac{-2}{i\pi} e \prod_{n=1}^{\infty} \left(1 + \frac{\pi^2}{4\pi^2 n^2 - \pi^2}\right) e^{-2\pi^2/(4\pi^2 n^2 - \pi^2)}.$$

That is nonsense, of course, because the left-hand side is a real number while the right-hand side is a pure imaginary number! Can we fix this wrong? Let us denote the partial sums of the power series expansions of $f(z) = e^z$ by:

$$p_d(z) = \sum_{n=0}^{d} \frac{z^n}{n!}.$$

Let us denote the automorphic functions of $p_d(z)$ by $\phi_n^d(z)$, $n = 0, \ldots, d-1$. Then we proved that:

$$p_d(w) - p_d(z) = (1 - p_d(z)) \prod_{n=0}^{d-1} \left(1 - \frac{w}{\phi_n^d(z)}\right).$$

The idea now is to mimic at the polynomial level the form of the Weierstrass factorization of the limiting function $e^w - e^z$. This means that we multiply the factors by the Weierstrass normalizing factors. The result is:

$$p_d(w) - p_d(z) = (1 - p_d(z)) \exp\left(-w \sum_{n=0}^{d-1} \left(\frac{1}{\phi_n^d(z)}\right)\right) \prod_{n=0}^{d-1} \left(1 - \frac{w}{\phi_n^d(z)}\right) e^{w/\phi_n^d(z)}.$$

At this point we take the limit $d \to \infty$ and assume that all the automorphic functions of the partial sums converge to those of e^z and that the finite normalized products of the p_d's converge to the Weierstrass canonical product of $e^w - e^z$. Here is what we get:

$$e^w - e^z = (1 - e^z)\left(1 - \frac{w}{z}\right) e^{w/z} \exp\left(-2zw \sum_{n=1}^{\infty} \left(\frac{1}{z^2 + 4\pi^2 n^2}\right)\right) \times \tag{2}$$

$$\times \prod_{n=1}^{\infty} \left(1 - \frac{z^2 - (z-w)^2}{z^2 + 4\pi^2 n^2}\right) e^{2zw/(z^2+4\pi^2 n^2)}.$$

In other words, this formula suggests the following identity using the notation of our Theorem 1:

$$\exp\left(F(w,e^z)\right) = (1-e^z)\exp\left(-2zw\sum_{n=1}^{\infty}\left(\frac{1}{z^2+4\pi^2n^2}\right)\right). \tag{3}$$

If true then it is interesting because it is not clear why the infinite sum of equation (3) is a holomorphic function of e^z. We can evaluate this infinite sum. The following formula is well-known:

$$2z\sum_{n=1}^{\infty}\left(\frac{1}{z^2-n^2}\right) = \pi\cot\pi z - \frac{1}{z}.$$

We make use of it. We let $z = iu$ below.

$$-2zw\sum_{n=1}^{\infty}\left(\frac{1}{z^2+4\pi^2n^2}\right) = -\left(\frac{z}{2\pi}\right)\left(\frac{w}{2\pi}\right)\sum_{n=1}^{\infty}\left(\frac{1}{(z/2\pi)^2+n^2}\right) =$$

$$= i\left(\frac{w}{2\pi}\right)\cdot 2\left(\frac{u}{2\pi}\right)\sum_{n=1}^{\infty}\left(\frac{1}{(u/2\pi)^2-n^2}\right) = i\left(\frac{w}{2}\right)\left\{\cot\left(\frac{u}{2}\right)-\left(\frac{2}{u}\right)\right\} =$$

$$= \left(\frac{w}{z}\right) - \left(\frac{w}{2}\right)\left(\frac{e^z+1}{e^z-1}\right).$$

The element w/z seems to be an obstacle for in order to make it a holomorphic function of e^z we might write it as $w/\log e^z$, which at least is not singular because $e^z \neq 1$. Plugging our result into equation (2) gives us finally the following interesting identity:

$$e^w - e^z = e^{w/z}(1-e^z)\left(1-\frac{w}{z}\right)e^{w/z}\exp\left(-\left(\frac{w}{2}\right)\left(\frac{e^z+1}{e^z-1}\right)\right)\times \tag{4}$$

$$\times\prod_{n=1}^{\infty}\left(1-\frac{z^2-(z-w)^2}{z^2+4\pi^2n^2}\right)e^{2zw/(z^2+4\pi^2n^2)}.$$

We recall that the last identity was derived using the idea outlined before equation (2), namely approximating the entire function $f(z)$ by a sequence of polynomials, the partial sums of its power series expansion, using the identity we proved for polynomials:

$$p_d(w) - p_d(z) = (p_d(0)-p_d(z))\prod_{n=0}^{d-1}\left(1-\frac{w}{\phi_n(z)}\right).$$

Then multiplying the last identity by the Weierstrass normalization factors that correspond to the Weierstrass expansion of f and letting $d \to \infty$ assuming we have convergence of the automorphic functions of the polynomials p_d to the automorphic functions of f, and also convergence of the finite products of the $p_d(w) - p_d(z)$ to the (generically) infinite product of $f(w) - f(z)$. Remarkably all of that actually works! We now give an independent proof of the identity (4) which does not rely on any of the above "convergences assumptions." Let us write our skeleton identity using the variables $i\pi w$ and $i\pi z$ instead of w and z:

$$e^{i\pi w} - e^{i\pi z} = \exp\left(g(i\pi w, i\pi z)\right)\left(1 - \frac{w}{z}\right)e^{w/z}\times$$

$$\times \prod_{n=1}^{\infty}\left(1 + \frac{z^2 - (z-w)^2}{4n^2 - z^2}\right)\exp\left(\frac{-2zw}{4n^2 - z^2}\right)$$

Now we use the cotangent fractional series expansion to compute:

$$\prod_{n=1}^{\infty}\exp\left(\frac{-2zw}{4n^2 - z^2}\right) = \exp\left(\left(\frac{\pi w}{2}\right)\cot\left(\frac{\pi z}{2}\right) - \left(\frac{w}{z}\right)\right).$$

Next we use the well-known expansion:

$$\frac{\pi z}{\sin(\pi z)} = \prod_{n=1}^{\infty}\left(\frac{n^2}{n^2 - z^2}\right),$$

to compute the infinite product:

$$\prod_{n=1}^{\infty}\left(1 + \frac{z^2 - (z-w)^2}{4n^2 - z^2}\right) = \prod_{n=1}^{\infty}\left(\frac{4n^2 - (z-w)^2}{4n^2}\right)\prod_{n=1}^{\infty}\left(\frac{4n^2}{4n^2 - z^2}\right) =$$

$$= \left(\frac{z}{z-w}\right)\frac{\sin(\pi(z-w)/2)}{\sin(\pi z/2)}.$$

Putting together the last three identities we proved gives:

$$e^{i\pi w} - e^{i\pi z} = \exp\left(g(i\pi w, i\pi z)\right)e^{w/z}\frac{\sin(\pi(z-w)/2)}{\sin(\pi z/2)}$$

$$\exp\left(\left(\frac{\pi w}{2}\right)\cot\left(\frac{\pi z}{2}\right) - \left(\frac{w}{z}\right)\right).$$

We solve for $\exp\left(g(i\pi w, i\pi z)\right)$ and replace $i\pi w$, $i\pi z$ by w and z, respectively. This gives:

$$\exp\left(g(w,z)\right) = (e^w - e^z)\frac{\sin(z/2i)}{\sin((z-w)/2i)}\exp\left(-\frac{w}{2i}\cot\left(\frac{z}{2i}\right)\right).$$

This concludes the proof of identity (4).

Using Proposition 1 we deduce that if $(e^w - 1)(e^z - 1) \neq 0$, then:

$$e^{w/z}(1-e^z)\left(1-\frac{w}{z}\right)e^{w/z}\exp\left(-\left(\frac{w}{2}\right)\left(\frac{e^z+1}{e^z-1}\right)\right)\times$$

$$\times\prod_{n=1}^{\infty}\left(1-\frac{z^2-(z-w)^2}{z^2+4\pi^2n^2}\right)e^{2zw/(z^2+4\pi^2n^2)} =$$

$$= -e^{z/w}(1-e^w)\left(1-\frac{z}{w}\right)e^{z/w}\exp\left(-\left(\frac{z}{2}\right)\left(\frac{e^w+1}{e^w-1}\right)\right)\times$$

$$\times\prod_{n=1}^{\infty}\left(1-\frac{w^2-(w-z)^2}{w^2+4\pi^2n^2}\right)e^{2wz/(w^2+4\pi^2n^2)}.$$

It is interesting to note that both sides are entire in w (left) and in z (right). That agrees with the Gronwall-Hahn Theorem. The left side is clearly z-holomorphic in $z \in \mathbb{C} - 2\pi i\mathbb{Z}$, and the right side is w-holomorphic in $w \in \mathbb{C} - 2\pi i\mathbb{Z}$. Thus both sides are entire in (w, z). The essential singularities of

$$\exp\left(-\left(\frac{w}{2}\right)\left(\frac{e^z+1}{e^z-1}\right)\right),$$

and of $e^{w/z}$ are somehow canceled out by the infinite product.

We end this section by pointing at two findings that seem to emerge from our computations. The first is the extent to which an entire function $f(z)$ is determined by a partial knowledge of its fibers. The notion of the fiber is very close to the notion of the automorphic group, namely $\forall\, w \in \mathbb{C}$ the fiber $f^{-1}(w) = \{z_j \mid f(z_j) = w\}$ is the discrete subset of $\mathbb{C}$ (we assume that f is non-constant) of all the f-pre-images of w. We note that if $z_0 \in f^{-1}(w)$, then $f^{-1}(w)$ is simply the $\mathrm{Aut}(f)$-orbit of z_0, i.e. we have the identity $f^{-1}(w) = \{\phi(z_0) \mid \phi \in \mathrm{Aut}(f)\}$. For a general function (not necessarily holomorphic or even continuous) the knowledge of the pairs $(w, f^{-1}(w))$ determines f (uniquely). The mere knowledge of all the fibers $f^{-1}(w)$, without knowing the w itself clearly does not determine f. This is very close to knowing the automorphic group of f (for that partitions $\mathbb{C}$ into the f-fibers without the knowledge of the w). So far for general functions f. Even if we know in advance that f is continuous, the automorphic group, i.e. the fibers $f^{-1}(w)$ do not determine f. If G is a continuous injection, then f and $G \circ f$ have identical family of fibers. But our case is very different from the continuous case. Our functions f are entire and hence are rigid. The case that shows how this holomorphic rigidity makes the difference is the case of polynomials. we already noted that if $p(z) = a_d z^d + \ldots + a_0$, $a_d \neq 0$, and if $\mathrm{Aut}(p) = \{\phi_0(z), \ldots, \phi_{d-1}(z)\}$, then

$$p(z) = p(0) + (-1)^{d+1} a_d \prod_{j=0}^{d-1} \phi_j(z).$$

Thus the product $\phi_0 \cdot \ldots \cdot \phi_{d-1}$, i.e. the product of the p-fiber determines the function $p(z)$ up to a multiplicative constant a_d different from 0 and an additive constant $p(0)$. Thus we do not have to know the fiber, just the product of its elements, a very partial information indeed, suffice to essentially reconstruct the function.

The second finding is closely related to the first one, but here we want to handle entire not necessarily polynomials. since in this case the group $\mathrm{Aut}(f)$ is usually infinite, it does not make sense to multiply its elements. Thus in this more complicated situation we ask the following: Given $\mathrm{Aut}(f)$ where f is a non-constant entire function, can we reconstruct the function f (up to minor parameters)? The way we outlined how to handle the case $f(z) = e^z$ might give us the way to solve this problem.

Theorem 3 *If $f(z) - f(0) \neq 0$, then there is a function $g(w, z)$, entire in w and there are non-negative integers $\lambda_n = \lambda_n(w, \phi_{0n}(z))$ such that:*

$$f(z) = f(w) - \exp\left(g(w, z)\right) \prod_{n=1}^{\infty} \left(1 - \frac{w}{\phi_{0n}(z)}\right) e^{Q_{\lambda_n}(w/\phi_{0n}(z))}.$$

Here as usual:

$$Q_\lambda\left(\frac{w}{\phi(z)}\right) = \left(\frac{w}{\phi(z)}\right) + \frac{1}{2}\left(\frac{w}{\phi(z)}\right)^2 + \ldots + \frac{1}{\lambda}\left(\frac{w}{\phi(z)}\right)^\lambda.$$

Here the non-negative integers λ_n are chosen so that the infinite product converges on uniformly on compact subsets of $\mathbb{C}$. For example, we might take the canonical product of the automorphic group $\{\phi_{0n}\}$. If we can sum up by some summation method for the infinite series:

$$\sum_{n=1}^{\infty} Q_{\lambda_n}\left(\frac{w}{\phi_{0n}(z)}\right),$$

to give a holomorphic sum, and that problem is at the moment an open problem, then:

$$\exp\left(g(w, z)\right) = \left(f(0) - f(z)\right) \exp\left(-\sum_{n=1}^{\infty} Q_{\lambda_n}\left(\frac{w}{\phi_{0n}(z)}\right)\right).$$

In this case we can reconstruct $f(z)$ *from* $\mathrm{Aut}(f)$ *by the formula:*

$$f(z) = \frac{f(0) \cdot L - f(w)}{L - 1},$$

where

$$L = \exp\left(-\sum_{n=1}^{\infty} Q_{\lambda_n}\left(\frac{w}{\phi_{0n}(z)}\right)\right) \cdot \prod_{n=1}^{\infty}\left(1 - \frac{w}{\phi_{0n}(z)}\right) e^{Q_{\lambda_n}(w/\phi_{0n}(z))},$$

and where $w \in \mathbb{C}$ *can be any number for which* $L \neq 1$.

6 Reconstruction Formulas for $f(z)$ and for $f'(z)$ in Terms of Approximating Automorphic Functions: Relations Between the Groups $\mathrm{Aut}(f)$ and $\mathrm{Aut}_z(g(w, z))$

The difference between the very simple reconstruction of a polynomial $p(z) = a_d z^d + \ldots + a_0$, $a_d \neq 0$ from its automorphic group $\mathrm{Aut}(p) = \{\phi_j(z) \mid j = 0, \ldots, d-1\}$, $p(z) = p(0) + (-1)^{d+1} a_d \prod_{j=0}^{d-1} \phi_j(z)$, on the one hand, and the reconstruction of a general non-constant entire function $f(z)$, in Theorem 3, on the other hand, gives the feeling of a possibility of a simpler reconstruction (in the general entire case). Indeed we can point at such a formula, seemingly simpler than the one in Theorem 3. However, the hidden complication is in the approximating procedure within that formula.

The setting is that we have an entire non-constant function $f(z)$ represented in terms of its Maclaurin's series: $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $\limsup_{n\to\infty} |a_n|^{1/n} = 0$. We denote the sequence of the partial sums by $f_n(z) = \sum_{k=0}^{n} a_k z^k$. Practically we consider those partial sums for which (in the notation above) $a_n \neq 0$. We denote the automorphic groups: $\mathrm{Aut}(f_n) = \{\phi_0^{(n)}(z), \ldots, \phi_{n-1}^{(n)}(z)\}$. These are all the solutions of the automorphic equation: $f_n(\phi_j^{(n)}(z)) = f_n(z)$, $j = 0, \ldots, n-1$. Thus $f_n(w) - f_n(z) = a_n \prod_{j=0}^{n-1}(w - \phi_j^{(n)}(z))$. The automorphic functions of $f_n(z)$ satisfy the Vieta identities:

$$(-1)^k a_n \sum_{0 \le i_1 < i_2 < \ldots < i_k \le n-1} \prod_{j=1}^{k} \phi_{i_j}^{(n)}(z) = \begin{cases} a_{n-k} & , k < n \\ a_0 - f_n(z) & , k = n \end{cases}.$$

As usual we denote $\mathrm{Aut}(f) = \{\phi_0(z), \phi_1(z), \phi_2(z), \ldots\}$, and we note that:

$$f(w) - f_n(w) = \sum_{k=n+1}^{\infty} a_k w^k, \quad f(z) - f_n(z) = \sum_{k=n+1}^{\infty} a_k z^k.$$

Hence:

$$f(w) - f(z) = (f(w) - f_n(w)) + (f_n(w) - f_n(z)) - (f(z) - f_n(z)) =$$

$$= \sum_{k=n+1}^{\infty} a_k(w^k - z^k) + a_n \prod_{j=0}^{n-1}(w - \phi_j^{(n)}(z)).$$

If we fix $R > 0$, then $\lim_{n\to\infty} \sum_{k=n+1}^{\infty} a_k(w^k - z^k) = 0$ uniformly in w, z. So that $f(w) - f(z) = \lim_{n\to\infty} a_n \prod_{j=0}^{n-1}(w - \phi_j^{(n)}(z))$ uniformly on compact subsets of $\mathbb{C}$. Thus it is straightforward to prove that we can divide by $(w - z)$ and take $w \to z$ and obtain: $f'(z) = \lim_{n\to\infty} a_n \prod_{j=1}^{n-1}(z - \phi_j^{(n)}(z))$. Finally let us fix N and replace the first N factors $\prod_{j=1}^{N-1}(z - \phi_j^{(n)}(z))$ when $n \to \infty$ by $\prod_{j=1}^{N-1}(z - \phi_j(z))$. We obtain:

$$\frac{f(w) - f(z)}{a_N \prod_{j=1}^{N-1}(z - \phi_j(z))} = \lim_{n\to\infty} \frac{a_n}{a_N} \prod_{j=N}^{n}(z - \phi_j^{(n)}(z)).$$

This proves the following:

Proposition 4 *We have* $f(w) - f(z) = \lim_{n\to\infty} a_n \prod_{j=0}^{n-1}(w - \phi_j^{(n)}(z))$ *uniformly on compact subsets of* $\mathbb{C}$. *Also* $f'(z) = \lim_{n\to\infty} a_n \prod_{j=1}^{n-1}(z - \phi_j^{(n)}(z))$ *uniformly on compact subsets of* $\mathbb{C}$ *and likewise for any fixed* N*:*

$$\frac{f(w) - f(z)}{a_N \prod_{j=1}^{N-1}(z - \phi_j(z))} = \lim_{n\to\infty} \frac{a_n}{a_N} \prod_{j=N}^{n}(z - \phi_j^{(n)}(z)).$$

Remark 9 The reconstruction formulas given in Proposition 4:

$$f(z) = f(w) - \lim_{n\to\infty} a_n \prod_{j=0}^{n-1}(w - \phi_j^{(n)}(z)),$$

and

$$f'(z) = \lim_{n\to\infty} a_n \prod_{j=1}^{n-1}(z - \phi_j^{(n)}(z)),$$

seem to be simpler than that in Theorem 3, but the cost lies in the sequential limit $\lim_{n\to\infty}$, which describes an auxiliary approximation of the entire functions by polynomials (the partial sums f_n).

We now bring few properties of the automorphic functions of a non-constant entire function f of a finite order. These are related to the results in Section 3. First it will be convenient to rephrase Proposition 3:

Corollary 17 *If f is a non-constant entire function of a finite order, then the automorphic group of $f(z)$ is a subgroup of the automorphic group of* $\exp(g(w,z))$ *as a function of z (for any fixed $w \in \mathbb{C}$). In symbols:* $\mathrm{Aut}(f(z)) \subseteq \mathrm{Aut}_z(\exp(g(w,z)),$ $\forall\, w \in \mathbb{C}$.

We noticed in Corollary 8 that any finite order element of $\mathrm{Aut}(f)$ (f a non-constant entire of a finite order) belongs to $\ker(T)$. Our next result gives a non-trivial characterization of the elements in $\ker(T)$.

Corollary 18 *Let $f(z)$ be a non-constant entire function of a finite order. Then:* $\ker(T) \equiv \mathrm{Aut}_z(g(w,z))$*, for any fixed $w \in \mathbb{C}$.*

Proof The automorphic function $\Phi \in \mathrm{Aut}(f(z))$ belongs to $\ker(T)$, where the homomorphism T was defined in Corollary 7 $\Longleftrightarrow T(\Phi) = \frac{1}{2\pi \cdot i}(g(w,\Phi(z)) - g(w,z)) = 0 \Longleftrightarrow g(w,\Phi(z)) = g(w,z)$, $\forall\, w \in \mathbb{C} \Longleftrightarrow \Phi \in \mathrm{Aut}_z(g(w,z))$, $\forall\, w \in \mathbb{C}$. □

Corollary 19 *Let f be a non-constant entire function of a finite order. Then:*

(a) *Any $\Phi \in \mathrm{Aut}(f(z)) - \mathrm{Aut}_z(g(w,z))$ is of an infinite order.*
(b) *Let $\Phi_{i_0 j_0} \in \mathrm{Aut}(f(z))$ be such that $|T(\Phi_{i0 j_0})| = \min\{|T(\Phi)|\,|\,\Phi \in \mathrm{Aut}(f(z)) - \ker(T)\}$. Then $T(\mathrm{Aut}(f(z))) =< \Phi_{i_0 j_0} >$. We have $\forall\, k \in \mathbb{Z}$, $T(\Phi_{i_0 j_0}^{\circ k}) = k \cdot T(\Phi_{i_0 j_0})$. If $\mathrm{Aut}(f(z)) - \ker(T) = \emptyset$, we agree to define $< \Phi_{i_0 j_0} >= \{0\}$.*
(c) *If $T(\Phi_{ij}) = T(\Phi_{\alpha\beta})$ then $\Phi_{ij} \circ \Phi_{\alpha\beta}^{-1} \in \mathrm{Aut}_z(g(w,z))$. This could be written as $g(w, \Phi_{ij} \circ \Phi_{\alpha\beta}^{-1}(z)) = g(w,z)$ or, equivalently as $g(w,\Phi_{ij}(z)) = g(w,\Phi_{\alpha\beta}(z))$.*

Remark 10 The automorphic group of a non-constant entire function of any order can contain elements of a finite order and elements of infinite order. For example, if $\mathrm{Aut}(f(z))$ contains elements of infinite order, then so does the group $\mathrm{Aut}(f(z^2))$ but this last group contains also the order 2 element $\Phi(z) = -z$.

Remark 11 We point out that the construction of a Dirichlet fundamental domain for Fuchsian groups could be used to construct a fundamental domain (maximal domain) for a non-constant entire function. Let $f(z)$ be a non-constant entire function, $z_0 \in \mathbb{C}$ a regular point of f (i.e., $f'(z_0) \neq 0$) and $\rho(\cdot,\cdot)$ a metric on $\mathbb{C}$. Mostly we have in mind the f-path metric, ρ_f that is induced by f. We recall what that is: let $z, w \in \mathbb{C}$ and let $\gamma : [0,1] \to \mathbb{C}$ be a continuous path from z to w. Thus $\gamma(0) = z$ and $\gamma(1) = w$. Then the length of γ is given by the standard length of the f-image path $f \circ \gamma$. We will denote this length by $l_f(\gamma)$.

$$l_f(\gamma) = \int_0^1 \left|f'(\gamma(t))\right| \left|\gamma'(t)\right| dt.$$

The f-path metric is given by: $\rho_f(z,w) = \inf_\gamma l_f(\gamma)$, where the infimum is taken over all the piecewise differentiable paths γ from z to w. The Dirichlet fundamental domain of $f(z)$, centered at z_0, with respect to the f-path metric is:

$$\{z \in \mathbb{C} \mid \rho_f(z,z_0) \le \rho_f(\Phi_{ij}(z), z_0)\ \forall\, \Phi_{ij} \in \mathrm{Aut}(f)\}.$$

An alternative way to define that, which avoids using the notion of the automorphic group of f is as follows:

$$\{z \in \mathbb{C} \mid \rho_f(z,z_0) \le \rho_f(w, z_0)\ \forall\, w \in f^{-1}(f(z))\}.$$

The interior of the set above is a domain (an open connected subset of $\mathbb{C}$), and the function f is one-to-one in this domain and the domain is maximal with respect to this property (of f being injective).

7 The Function $g(w,z) - g(0,z)$ Is Determined by the Negative Moments of the Elements in Aut$(f(z))$

Theorem 4 *Let f be a non-constant entire function. Let us denote* $\mathrm{Aut}(f) = \{\phi_{0n}(z) \mid n = 0,1,2,\ldots\}$ *($\phi_{00} \equiv$ id.) and let $g(w,z)$ be the function in the exponential of the Weierstrass (canonical) factorization of $f(w) - f(z)$. $g(w,z)$ is entire in w and holomorphic in $z \notin f^{-1}(f(0))$. Then for $k = 1,2,3,\ldots$ we have the identities:*

$$\frac{1}{k!}\frac{\partial^k g}{\partial w^k}(0,z) = -\frac{1}{k}\sum_{\begin{cases} n \\ \lambda_n \ge k \end{cases}} \left(\frac{1}{\phi_{0n}(z)}\right)^k.$$

The left-hand side is the $k+1$'st Maclaurin coefficient in the expansion of $g(w,z) - g(0,z)$. The right hand side is $-1/k$ multiplying the $-k$-moment of all the relevant automorphic functions of f. That explains the title of this section. Like in Theorem 3 we assume that we have some summation method for the infinite series:

$$\sum_{n=1}^{\infty} Q_{\lambda_n}\left(\frac{w}{\phi_{0n}(z)}\right).$$

Remark 12 Thus the proof below is based on a vague summability assumption! It is mostly supported by formal computational steps, and not justified. However, this already interesting sketch points to the fact that a true proof if exists will not be an easy one, and probably it will have to utilize summability theoretical arguments.

Proof By Theorem 3 we have the following identity:

$$\exp(g(w,z)) = (f(0) - f(z))\exp\left(-\sum_{n=1}^{\infty} Q_{\lambda_n}\left(\frac{w}{\phi_{0n}(z)}\right)\right).$$

It is assuming that we have a summability method for the infinite series:

$$\sum_{n=1}^{\infty} Q_{\lambda_n}\left(\frac{w}{\phi_{0n}(z)}\right),$$

which results in a holomorphic function. Here, as usual:

$$Q_{\lambda_n}\left(\frac{w}{\phi_{0n}(z)}\right) = \left(\frac{w}{\phi_{0n}(z)}\right) + \frac{1}{2}\left(\frac{w}{\phi_{0n}(z)}\right)^2 + \ldots + \frac{1}{\lambda_n}\left(\frac{w}{\phi_{0n}(z)}\right)^{\lambda_n}.$$

When we substitute $w = 0$ into the identity above, we obtain $\exp(g(0,z)) = f(0) - f(z)$. So we can rewrite our identity as follows:

$$\exp(g(w,z) - g(0,z)) = \exp\left(-\sum_{n=1}^{\infty} Q_{\lambda_n}\left(\frac{w}{\phi_{0n}(z)}\right)\right).$$

We deduce that there is an integer $N \in \mathbb{Z}$, such that:

$$g(w,z) - g(0,z) = 2\pi \cdot i \cdot N - \sum_{n=1}^{\infty} Q_{\lambda_n}\left(\frac{w}{\phi_{0n}(z)}\right).$$

If we plug in $w = 0$, we obtain $0 = 2\pi \cdot i \cdot N$, so that $N = 0$ and we have:

$$g(w,z) - g(0,z) = -\sum_{n=1}^{\infty} Q_{\lambda_n}\left(\frac{w}{\phi_{0n}(z)}\right).$$

On the other hand, $g(w,z)$ is an entire function in w and so it has a power series expansion that converges in the whole w-plane:

$$g(w,z) - g(0,z) = \sum_{n=1}^{\infty} \frac{1}{n!}\frac{\partial^n g}{\partial w^n}(0,z) \cdot w^n.$$

Hence we conclude that we have the following identity:

$$\sum_{n=1}^{\infty} \frac{1}{n!}\frac{\partial^n g}{\partial w^n}(0,z) \cdot w^n = -\sum_{n=1}^{\infty} Q_{\lambda_n}\left(\frac{w}{\phi_{0n}(z)}\right).$$

Let us write the series on the right-hand side, as power series in w. We recall that:

$$Q_{\lambda_n}\left(\frac{w}{\phi_{0n}(z)}\right)=\left(\frac{w}{\phi_{0n}(z)}\right)+\frac{1}{2}\left(\frac{w}{\phi_{0n}(z)}\right)^2+\ldots+\frac{1}{\lambda_n}\left(\frac{w}{\phi_{0n}(z)}\right)^{\lambda_n},$$

when $\lambda_n \in \mathbb{Z}^+$. Otherwise $Q_0(u) \equiv 0$. We would like to compute the coefficient of w^k on the right-hand side of our identity. It is the sum of all the elements of the following form:

$$\frac{1}{k}\left(\frac{w}{\phi_{0n}(z)}\right)^k,$$

provided, of course, that the condition $\lambda_n \geq k$ is fulfilled. Thus we obtain the following identity:

$$-\sum_{n=1}^{\infty} Q_{\lambda_n}\left(\frac{w}{\phi_{0n}(z)}\right)=-\sum_{k=1}^{\infty}\frac{1}{k}\sum_{\begin{cases} n \\ \lambda_n \geq k\end{cases}}\left(\frac{1}{\phi_{0n}(z)}\right)^k \cdot w^k.$$

By the uniqueness of the coefficients in a power series we conclude that for $k = 1, 2, 3, \ldots$ we have:

$$\frac{1}{k!}\frac{\partial^k g}{\partial w^k}(0,z)=-\frac{1}{k}\sum_{\begin{cases} n \\ \lambda_n \geq k\end{cases}}\left(\frac{1}{\phi_{0n}(z)}\right)^k.$$

Indeed we note that the sum on the right side of the last identity is the $(-k)$-moment of $\phi_{0n}(z)$, where, of course, $\lambda_n \geq k$. □

8 An Infinite Product Representation of $f'(w)$

In this section we will explore how the Weierstrass factorization of a non-constant entire function induces an infinite product representation on its derivative. This will reveal a connection between the zeros of the derivatives and the fixed-points of the elements of the automorphic group of the function. We begin with a peculiar division property of entire functions and of their derivatives which follows by a composition relation between these functions.

Proposition 5 *Let $f(z)$ and $g(z)$ be two non-constant entire functions. If* $\mathrm{Aut}(f(z)) \subseteq \mathrm{Aut}(g(z))$, *then $f'(z)$ divides $g'(z)$ over the algebra of entire* *functions, i.e.* $\exists\, H(z)$*, an entire function such that* $g'(z) = H(z) \cdot f'(z)$*. In fact* $\exists\, G(w, z)$*, entire in* w *and holomorphic in* $z \in \mathbb{C} - g^{-1}(g(0)) - f^{-1}(f(0))$ *such that* $g(w) - g(z) = (f(w) - f(z)) \cdot G(w, z)$*.*

Proof By Lemma 1 it follows that there exists a function $h(w)$, holomorphic on $f(\mathbb{C})$ such that $g(z) = h(f(z))$. Hence $g'(z) = h'(f(z)) \cdot f'(z)$ which proves the first assertion with the entire function $H(z) = h'(f(z))$. Next we denote $\mathrm{Aut}(f(z)) = \{\phi_{0n}(z)\}$, and $\mathrm{Aut}(g(z)) = \{\psi_{0n}(z)\}$. By the Weierstrass factorization theorem we have:

$$g(w) - g(z) = e^{L(w,z)} \prod_n \left(1 - \frac{w}{\psi_{0n}(z)}\right) e^{Q_{\delta_n}(w/\psi_{0n}(z))}, \quad g(0) - g(z) \neq 0,$$

$$f(w) - f(z) = e^{l(w,z)} \prod_n \left(1 - \frac{w}{\phi_{0n}(z)}\right) e^{Q_{\lambda_n}(w/\phi_{0n}(z))}, \quad f(0) - f(z) \neq 0,$$

where $\{\phi_{0m}(z)\} \subseteq \{\psi_{0m}(z)\}$. Hence:

$$g(w)-g(z) = (f(w)-f(z)) e^{L(w,z)-l(w,z)} \prod_{\psi_{0n} \notin \{\phi_{0m}\}} \left(1 - \frac{w}{\psi_{0n}(z)}\right) e^{Q_{\delta_n}(w/\psi_{0n}(z))} \times$$

$$\times \prod_n \left(1 - \frac{w}{\phi_{0n}(z)}\right) e^{(Q_{\delta_n} - Q_{\lambda_n})(w/\phi_{0n}(z))}.$$

□

We recall the reconstruction formula given in Proposition 4 for the derivative:

$$f'(z) = \lim_{n \to \infty} a_n \prod_{j=1}^{n-1} (z - \phi_j^{(n)}(z)),$$

This formula suggests a possible relation between the zeros of $f'(z)$ and the fixed points of the automorphic functions of $f(z)$. Indeed we will prove that this is the case.

Let $f(w)$ be a non-constant entire function. Let our z-parameter space be $\mathbb{C} - f^{-1}(f(0))$. We consider the Weierstrass factorization of $f(w) - f(z)$ as an entire function of w. By our choice of the parameter z, we have $f(0) - f(z) \neq 0$. Thus $0 \notin Z_w(f(w) - f(z))$ and hence:

$$f(w) - f(z) = e^{g(w,z)} \prod_n \left(1 - \frac{w}{\phi_{0n}(z)}\right) e^{Q_{\lambda_n}(w/\phi_{0n}(z))}.$$

Here $g(w,z)$ is entire in w and holomorphic for any $z \in \mathbb{C} - f^{-1}(f(0))$. Also $\mathrm{Aut}(f(w)) = \{\phi_{0n}(w)\}_n$, and we agree that $\phi_{00}(w) \equiv w$. The numbers $\lambda_n \in \mathbb{Z}^+ \cup \{0\}$ and:

$$Q_\lambda(u) = \begin{cases} u + u^2/2 + \ldots + u^\lambda/\lambda & , \lambda \in \mathbb{Z}^+ \\ 0 & , \lambda = 0 \end{cases}, \quad \lambda_0 = 0.$$

So:

$$f(w) - f(z) = \left(1 - \frac{w}{z}\right) e^{g(w,z)} \prod_{n \neq 0} \left(1 - \frac{w}{\phi_{0n}(z)}\right) e^{Q_{\lambda_n}(w/\phi_{0n}(z))} =$$

$$= \left(\frac{z-w}{z}\right) e^{g(w,z)} \prod_{n \neq 0} \left(1 - \frac{w}{\phi_{0n}(z)}\right) e^{Q_{\lambda_n}(w/\phi_{0n}(z))}.$$

Hence, assuming that $w \neq z$, we obtain:

$$\frac{f(w) - f(z)}{w - z} = -\frac{1}{z} e^{g(w,z)} \prod_{n \neq 0} \left(1 - \frac{w}{\phi_{0n}(z)}\right) e^{Q_{\lambda_n}(w/\phi_{0n}(z))}.$$

Assuming that $w \in \mathbb{C} - f^{-1}(f(0))$, and that z and w are close enough so that in the limit process $z \to w$, the non-negative integers λ_n do not change, we get:

$$f'(w) = \lim_{\substack{z \to w \\ z \notin f^{-1}(f(0))}} \frac{f(w) - f(z)}{w - z} = -\frac{1}{w} e^{g(w,w)} \prod_{n \neq 0} \left(1 - \frac{w}{\phi_{0n}(w)}\right) e^{Q_{\lambda_n}(w/\phi_{0n}(w))}.$$

This shows that the zero set of the derivative function, $Z(f'(w))$ originates in three possible locations:

a) The fiber $f^{-1}(f(0))$might contain zeros of $f'(w)$.
b) Any fixed-point w of a (non-identity) automorphic function ϕ_{0n} must be a zero of $f'(w)$. Thus:

$$\{w \in \mathbb{C} - f^{-1}(f(0)) \mid \exists n \neq 0,\ \phi_{0n}(w) = w\} \subseteq Z(f').$$

c) The zeros (if any) of the functions $e^{Q_{\lambda_n}(w/\phi_{0n}(w)}$ for $n \neq 0$ and off the fiber $f^{-1}(f(0))$. Thus:

$$\bigcup_{n \neq 0} Z\left(e^{Q_{\lambda_n}(w/\phi_{0n}(w)}\right) \cap \left(\mathbb{C} - f^{-1}(f(0))\right) \subseteq Z(f').$$

Let us look at any automorphic equation in the domain of the definition of the corresponding automorphic function: $f(\phi_{0n}(w)) = f(w)$. By differentiation (assuming that $\phi_{0n}(w)$ has a derivative there): $\phi'_{0n}(w) \cdot f'(\phi_{0n}(w)) = f'(w)$. This implies that if $f'(w) = 0$, then either $\phi'_{0n}(w) = 0$ or $f'(\phi_{0n}(w)) = 0$. If w is of type b, i.e. a fixed-point of the above automorphic function, $\phi_{0n}(w) = w$, then clearly $f'(\phi_{0n}(w)) = f'(w) = 0$ and a consideration of the order of this zero of f' implies that $\phi'_{0n}(w) \neq 0$. Thus w is a regular point of the automorphic function.

Remark 13 If $f'(\phi_{0n}(w)) = f'(w) = 0$, then $\phi'_{0n}(w) \neq 0$ and also for its inverse $(\phi_{0n}^{-1})'(\phi_{0n}(w)) \neq 0$.

If $f'(\phi_{0n}(w)) \neq 0$, then necessarily $\phi'_{0n}(w) = 0$. In this case (assuming it is not type b) we either have $f(w) = f(\phi_{0n}(w)) = f(0)$ (type a) or $e^{Q_{\lambda_n}(w/\phi_{0n}(w))} = 0$.
What Are the Type c Points? These are zeros of $e^{Q_{\lambda_n}(w/\phi_{0n}(w))}$ outside the fiber $f^{-1}(f(0))$. This implies that $\lambda_n > 0$ and that locally the function:

$$Q_{\lambda_n}\left(\frac{w}{\phi_{0n}(w)}\right) = \left(\frac{w}{\phi_{0n}(w)}\right) + \frac{1}{2}\left(\frac{w}{\phi_{0n}(w)}\right)^2 + \ldots + \frac{1}{\lambda_n}\left(\frac{w}{\phi_{0n}(w)}\right)^{\lambda_n},$$

is the logarithm of a function that vanishes at w. Hence:

$$Q_{\lambda_n}\left(\frac{w}{\phi_{0n}(w)}\right) = \left(\frac{w}{\phi_{0n}(w)}\right) + \frac{1}{2}\left(\frac{w}{\phi_{0n}(w)}\right)^2 + \ldots + \frac{1}{\lambda_n}\left(\frac{w}{\phi_{0n}(w)}\right)^{\lambda_n} =$$

$$= \log((w - w_0) \cdot h(w)).$$

But this implies that:

$$\lim_{w \to w_0}\left| Q_{\lambda_n}\left(\frac{w}{\phi_{0n}(w)}\right) = \left(\frac{w}{\phi_{0n}(w)}\right) + \frac{1}{2}\left(\frac{w}{\phi_{0n}(w)}\right)^2 + \ldots + \frac{1}{\lambda_n}\left(\frac{w}{\phi_{0n}(w)}\right)^{\lambda_n}\right| = +\infty.$$

So $\lim_{w \to w_0} \phi_{0n}(w) = 0$ and hence $\phi_{0n}(w_0) = 0$, which implies that the point $w_0 \in f^{-1}(f(0))$ in the forbidden fiber $f^{-1}(f(0))$. So type c points do not exist. We completed the proof of the following:

Lemma 3 *If $f(w)$ is a non-constant entire function, then:*

$$Z(f') = \bigcup_{\phi_{0n} \in \mathrm{Aut}(f) - \{id\}} \{w \in \mathbb{C} - f^{-1}(f(0)) \mid \phi_{0n}(w) = w\} \cup \left(Z(f') \cap f^{-1}(f(0))\right).$$

We can now prove the simple relation that exists between the zeros of the derivative and the fixed-point of the automorphic function.

Theorem 5 *Let $f(w)$ be a non-constant entire function. Then:*

$$Z(f') = \bigcup_{\phi_{0n} \in \mathrm{Aut}(f)-\{id\}} \{w \in \mathbb{C} \mid \phi_{0n}(w) = w\} := \mathrm{Fix}(\mathrm{Aut}(f)).$$

Proof Let $t \in \mathbb{C}$ be any complex number. It is clear that the number 0 that appears in the formula of Lemma 3 in $f^{-1}(f(0))$ has no special significance. Indeed we could have expanded $f(w) - f(z)$ in a Weierstrass product centered at t instead of 0 and obtain in Lemma 3 the t'th version:

$$Z(f') = \bigcup_{\phi_{0n} \in \mathrm{Aut}(f)-\{id\}} \{w \in \mathbb{C} - f^{-1}(f(t)) \mid \phi_{0n}(w) = w\} \cup \Big(Z(f') \cap f^{-1}(f(t))\Big).$$

The fibers $f^{-1}(f(t))$ are discrete subsets of $\mathbb{C}$ for any such a $t \in \mathbb{C}$. Even more is true, namely: $t_1 \neq t_2 \;\Leftrightarrow\; f^{-1}(f(t_1)) \cap f^{-1}(f(t_2)) = \emptyset$. Thus the claim of our theorem follows. $\square$

Theorem 6 *If f is a non-constant entire function with only real zeros, has genus 0 or 1, and is real on the real axis, then the points of* $\mathrm{Fix}(\mathrm{Aut}(f))$ *are real and are separated by the zeros of f, and the zeros of f are separated by the points of* $\mathrm{Fix}(\mathrm{Aut}(f))$.

Proof This follow by Laguerre's Theorem on Separation Zeros [12] (p. 89) and by Theorem 5. $\square$

9 Common Zeros of the Reciprocals of Almost All the Automorphic Functions

Definition 2 Let $f(z)$ be a non-constant entire function, let $\mathscr{P}$ be a property that an element in the automorphic group $\phi \in \mathrm{Aut}(f)$ can have or does not have. We say that the property $\mathscr{P}$ is common to almost all the automorphic functions of $f(z)$ if except for a finite number of them, all the automorphic functions $\phi \in \mathrm{Aut}(f)$ have the property $\mathscr{P}$.

We give in the current section a non-trivial such a property. The property will be: having a common zero for $1/\phi$, where $\phi \in \mathrm{Aut}(f)$.

Theorem 7 *Let $g(z)$ be an entire function. Let $p(z)$ be a polynomial, $d := \deg p > 0$ and $Z(p) = \{\alpha_1, \ldots, \alpha_d\} \subseteq \mathbb{C}$. Let $f(z) = p(z)e^{g(z)}$. Then $\forall\, j = 1, \ldots, d$, α_j is a common zero of almost all the reciprocals of the automorphic functions of $f(z)$.*

Proof Let us denote $\mathrm{Aut}(f) = \{\phi_n(z)\}_n$ and we consider a Weierstrass representation:

$$f(w) - f(z) = e^{g(w,z)} \prod_n \left(1 - \frac{w}{\phi_n(z)}\right) e^{Q_{\lambda_n}(w/\phi_n(z))}.$$

Thus we have:

$$p(w)e^{g(w)} - p(z)e^{g(z)} = e^{g(w,z)} \prod_n \left(1 - \frac{w}{\phi_n(z)}\right) e^{Q_{\lambda_n}(w/\phi_n(z))}.$$

For the sake of simplicity, let us assume that $f(0) \neq 0$. Consider any integer j, $1 \leq j \leq d$, and let us take the parameter value $z = \alpha_j$. This is a legitimate value of the parameter z for which the above Weierstrass representation holds true. The reason is that with this parameter we have $f(0) - f(z) = f(0) - f(\alpha_j) = f(0) - 0 = f(0) \neq 0$. This follows by: $f(\alpha_j) = p(\alpha_j)e^{g(\alpha_j)} = 0 \cdot e^{g(\alpha_j)} = 0$. Hence at least one n exists for which $\phi_n(\alpha_j) = \alpha_k$ for some $1 \leq k \leq d$. We note that $Z(f(w)) = Z(p(w)) = \{\alpha_1, \ldots, \alpha_d\}$, because the only solutions of $f(w) = 0$, i.e. $p(w)e^{g(w)} = 0$ are (exactly) the solutions of $p(w) = 0$ and vice versa. So in the Weierstrass product of $f(w) - f(\alpha_j) = f(w)$ there are exactly d factors. All the other factors are "phantom" factors, i.e.

$$1 - \frac{w}{\phi_n(\alpha_j)} \equiv 1.$$

Thus except for d factors we have:

$$\left|\frac{w}{\phi_n(\alpha_j)}\right| = 0.$$

This means that $|\phi_n(\alpha_j)| = \infty$, for all n, except for d of them. □

10 Sums of the Derivatives of the Automorphic Functions

Definition 3 Let $f(z)$ be a holomorphic function in some domain $\mathscr{D} \subseteq \mathbb{C}$. A differential monomial of f is a function of the form:

$$m_{n_1,\ldots,n_k;m_1,\ldots,m_k}(z) = a \cdot \left(f^{(n_1)}(z)\right)^{m_1} \ldots \left(f^{(n_k)}(z)\right)^{m_k}.$$

Here $k, n_1, \ldots, n_k, m_1 \ldots, m_k \in \mathbb{Z}^+$ and $a \in \mathbb{C}^\times$. The weight of the monomial $m_{n_1,\ldots,n_k;m_1,\ldots,m_k}(z)$ is $w(m_{n_1,\ldots,n_k;m_1,\ldots,m_k}(z)) = n_1 \cdot m_1 + \ldots + n_k \cdot m_k$.

In this section we will discuss the following result:

Theorem 8 *Let $f(z)$ be a non-constant entire function,* Aut$(f) = \{\phi_{0n}(z)\}_n$, $k \in \mathbb{Z}^+$ *and* $R > 0$. *Then there is an identity (independent of f) of the form:*

$$2\pi \cdot i \cdot \sum_{|\phi_{0n}(z)|<R} \phi_{0n}^{(k)}(z) = \sum_{j=1}^{k} m_j(z) \cdot \oint_{|w|=R} \frac{dw}{(f(w)-f(z))^j},$$

where $w(m_j(z)) = k$ *and in particular:* $m_1(z) = f^{(k)}(z)$ *and* $m_k(z) = (f'(z))^k$.

We start by writing explicit formulas for the results already obtained in Section 2, Proposition 2, and Remark 4.

Proposition 6 *Let* $f(w)$ *be a non-constant entire function,* $\mathrm{Aut}(f(z)) = \{\phi_{0n}(z)\}_n$ *and let us assume that* $f(z) - f(0) \neq 0$. *Then*

$$\frac{f'(w)}{f(w)-f(z)} = \frac{\partial g}{\partial w}(w,z) + \sum_n \left(\frac{w}{\phi_{0n}(z)}\right)^{\lambda_n} \left(\frac{1}{w-\phi_{0n}(z)}\right).$$

Here $g(w,z)$ *and the* λ_n *are the data of the Weierstrass presentation of* $f(w) - f(z)$ *(Proposition 3).*

Proof If $f(z) - f(0) \neq 0$, then there is a function $g(w,z)$, entire in w and there are non-negative integers $\lambda_n(w,z)$ which we will sometimes denote by λ_n, such that:

$$f(w) - f(z) = \exp\left(g(w,z)\right) \prod_{n=1}^{\infty} E\left(\frac{w}{\phi_{0n}(z)}, \lambda_n\right) =$$

$$= \exp\left(g(w,z)\right) \prod_{n=1}^{\infty} \left(1 - \frac{w}{\phi_{0n}(z)}\right) e^{Q_{\lambda_n}(w/\phi_{0n}(z))},$$

where if $\lambda_n > 0$, then:

$$Q_{\lambda_n}\left(\frac{w}{\phi_{0n}(z)}\right) = \left(\frac{w}{\phi_{0n}(z)}\right) + \frac{1}{2}\left(\frac{w}{\phi_{0n}(z)}\right)^2 + \ldots + \frac{1}{\lambda_n}\left(\frac{w}{\phi_{0n}(z)}\right)^{\lambda_n},$$

and $Q_0(w/\phi_{0n}(z)) \equiv 0$. Taking the logarithm of both sides of the identity we obtain:

$$\log\left(f(w) - f(z)\right) = \tag{5}$$

$$= g(w,z) + \sum_n \left(\log\left(1 - \frac{w}{\phi_{0n}(z)}\right) + Q_{\lambda_n}\left(\frac{w}{\phi_{0n}(z)}\right)\right).$$

We ∂w both sides of the last identity, equation (5) and obtain our result:

$$\frac{f'(w)}{f(w)-f(z)} = \frac{\partial g}{\partial w}(w,z) + \sum_n \left(\frac{w}{\phi_{0n}(z)}\right)^{\lambda_n} \left(\frac{1}{w-\phi_{0n}(z)}\right).$$

□

Proposition 7 *Let $f(w)$ be a non-constant entire function,* $\mathrm{Aut}(f(z)) = \{\phi_{0n}(z)\}_n$ *and let us assume that $f(z) - f(0) \neq 0$. Then*

$$\frac{f'(z)}{f(w)-f(z)} = -\frac{\partial g}{\partial z}(w,z) + \sum_n \left(\frac{w\phi_{0n}^{'}(z)}{\phi_{0n}(z)}\right)\left(\frac{w}{\phi_{0n}(z)}\right)^{\lambda_n}\left(\frac{1}{w-\phi_{0n}(z)}\right).$$

Here $g(w,z)$ and the λ_n are the data of the Weierstrass presentation of $f(w)-f(z)$ (Proposition 3).

Proof We ∂z both sides of equation (5) and obtain our result:

$$\frac{f'(z)}{f(w)-f(z)} = -\frac{\partial g}{\partial z}(w,z) + \sum_n \left(\frac{w\phi_{0n}^{'}(z)}{\phi_{0n}(z)}\right)\left(\frac{w}{\phi_{0n}(z)}\right)^{\lambda_n}\left(\frac{1}{w-\phi_{0n}(z)}\right).$$

□

Let us write the identity of Proposition 7 in the following form:

$$\frac{f'(z)}{f(w)-f(z)} + \frac{\partial g}{\partial z}(w,z) = \sum_n \frac{\phi_{0n}^{'}(z)}{(\phi_{0n}(z))^{\lambda_n+1}} \cdot \left(\frac{w^{\lambda_n+1}}{w-\phi_{0n}(z)}\right).$$

Let $R > 0$, $z \in \mathbb{C}$ be fixed so that the circle $|w| = R$ does not contain $\{\phi_{0n}(z)\}_n$. In that event we have $f(z) - f(w) \neq 0, \forall\, |w| = R$ (because $f(z) - f(w) = 0 \Leftrightarrow w = \phi_{0n}(z)$ for some $n \in \mathbb{Z}$). Thus we can path integrate our identity on $|w| = R$ and obtain:

$$\begin{aligned} f'(z)\oint_{|w|=R}\frac{dw}{f(w)-f(z)} + \oint_{|w|=R}\frac{\partial g}{\partial z}(w,z)dw \\ = \sum_n \frac{\phi_{0n}^{'}(z)}{(\phi_{0n}(z))^{\lambda_n+1}} \cdot \oint_{|w|=R}\frac{w^{\lambda_n+1}dw}{w-\phi_{0n}(z)}. \end{aligned}$$

Since the function $\partial g(w,z)/\partial z$ is entire in $w \in \mathbb{C}$, it follows by the Theorem of Cauchy that:

$$\oint_{|w|=R}\frac{\partial g}{\partial z}(w,z)dw = 0.$$

By the generalized argument principle we have:

$$\sum_n \frac{\phi_{0n}^{'}(z)}{(\phi_{0n}(z))^{\lambda_n+1}} \cdot \oint_{|w|=R}\frac{w^{\lambda_n+1}dw}{w-\phi_{0n}(z)} = 2\pi \cdot i \sum_{|\phi_{0n}(z)|<R} \phi_{0n}^{'}(z).$$

Putting what we have so far together:

$$f'(z)\oint_{|w|=R}\frac{dw}{f(w)-f(z)}=2\pi\cdot i\sum_{|\phi_{0n}(z)|<R}\phi_{0n}^{'}(z). \tag{6}$$

We just proved Theorem 8 for the case $k=1$.

Remark 14 If we add the assumption on z, that $f(z)-f(0)\neq 0$, then we have the Weierstrass presentation (Proposition 3):

$$f(w)-f(z)=\exp\left(g(w,z)\right)\prod_{n}\left(\frac{\phi_{0n}(z)-w}{\phi_{0n}(z)}\right)e^{Q_{\lambda_n}(w/\phi_{0n}(z))}.$$

In other words, $Z_w(f(w)-f(z))=\{\phi_{0n}(z)\}_n$. So $1/(f(w)-f(z))$ is meromorphic in w, it has no zeros, and its total set of poles are $\{\phi_{0n}(z)\}_n$. What is the residue:

$$\mathrm{Res}\left(\frac{1}{f(w)-f(z)},\phi_{0n}(z)\right)?$$

Let us assume for simplicity that all the zeros of $f(w)-f(z)$ are simple. Then this residue is:

$$\lim_{w\to\phi_{0n}(z)}(w-\phi_{0n}(z))\cdot\frac{1}{f(w)-f(z)}=\frac{1}{f'(\phi_{0n}(z))}.$$

We conclude that:

$$\oint_{|w|=R}\frac{dw}{f(w)-f(z)}=2\pi\cdot i\sum_{|\phi_{0n}(z)|<R}\frac{1}{f'(\phi_{0n}(z))}.$$

By equation (6):

$$f'(z)\left(2\pi\cdot i\sum_{|\phi_{0n}(z)|<R}\frac{1}{f'(\phi_{0n}(z))}\right)=2\pi\cdot i\sum_{|\phi_{0n}(z)|<R}\phi_{0n}^{'}(z).$$

By the automorphic equation $f(\phi_{0n}(z))=f(z)$ and the chain rule, we get: $\phi_{0n}^{'}(z)f'(\phi_{0n}(z))=f'(z)$. Hence:

$$\frac{1}{f'(\phi_{0n}(z))}=\frac{\phi_{0n}^{'}(z)}{f'(z)}.$$

This agrees with our equation (6). However, we proved equation (6) without the extra assumption on the simplicity of the zeros of $f(w)-f(z)$.

Let us show how to compute the next case, $k = 2$, of Theorem 8 and in fact any case follows just as simple using inductive argument. We do the obvious and apply the operator ∂z:

$$\left(\frac{f'(z)}{f(w) - f(z)}\right)'_z = \frac{f''(z)}{f(w) - f(z)} + \frac{(f'(z))^2}{(f(w) - f(z))^2}.$$

Hence using equation (6) we obtain:

$$f''(z) \oint_{|w|=R} \frac{dw}{f(w) - f(z)} + (f'(z))^2 \oint_{|w|=R} \frac{dw}{(f(w) - f(z))^2} = \tag{7}$$

$$= 2\pi \cdot i \sum_{|\phi_{0n}(z)|<R} \phi''_{0n}(z).$$

This proves the second case, $k = 2$ of Theorem 8. Let us do one more explicit computation and prove (explicitly) the third case, $k = 3$ too. Once more we do the obvious and apply the operator ∂z to the case $k = 2$:

$$\left(\frac{f''(z)}{f(w) - f(z)}\right)'_z + \left(\frac{(f'(z))^2}{(f(w) - f(z))^2}\right)'_z =$$

$$= \frac{f^{(3)}(z)}{f(w) - f(z)} + \frac{3f'(z)f''(z)}{(f(w) - f(z))^2} + \frac{(f'(z))^3}{(f(w) - f(z))^3}.$$

Using equation (7) we finally get:

$$f^{(3)}(z) \oint_{|w|=R} \frac{dw}{f(w) - f(z)} + 3f'(z)f''(z) \oint_{|w|=R} \frac{dw}{(f(w) - f(z))^2} +$$

$$\tag{8}$$

$$+(f'(z))^3 \oint_{|w|=R} \frac{dw}{(f(w) - f(z))^3} = 2\pi \cdot i \sum_{|\phi_{0n}(z)|<R} \phi'''_{0n}(z).$$

This proves the third case, $k = 3$ of Theorem 8. It is clear how to proceed inductively by repeatedly applying the operator ∂z and forming a simple weight calculation on the resulting differential monomials. □

11 An Application of Jensen's Theorem to the Automorphic Group of an Entire Function

Here is one of the most important theorems in analysis.

Theorem (Jensen's Theorem) *Let $f(z)$ be analytic for $|z| < R$. Suppose that $f(0)$ is not zero, and let $r_1, r_2, \ldots, r_n, \ldots$ be the moduli of the zeros of $f(z)$ in the disk $|z| < R$, arranged in a non-decreasing sequence. Then if $r_n \le r \le r_{n+1}$,*

$$\log \frac{r^n |f(0)|}{r_1 r_2 \ldots r_n} = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| d\theta,$$

where every zero is counted the number of its multiplicity.

Let $f(w)$ be a non-constant entire function, let $z \in \mathbb{C} - f^{-1}(f(0))$, and let us apply the Theorem of Jensen to the entire function $f(w) - f(z)$ of the variable w. Then indeed $f(0) - f(z)$ is not zero, and the parameter R in Jensen's Theorem can be an arbitrary positive number. The zero set of $f(w) - f(z)$ is the $\mathrm{Aut}(f(w))$-orbit of z. Thus: $Z(f(w) - f(z)) = \{\phi_{0n}(z)\}_n$ and we may assume that the moduli of the zeros are arranged in a non-decreasing order. Thus $|\phi_{00}(z)| \le |\phi_{01}(z)| \le |\phi_{02}(z)| \le \ldots$. In other words, in terms of the notation in Jensen's Theorem we have $r_j = |\phi_{0,j-1}(z)|$. We conclude that if $|\phi_{0n}(z)| \le r < |\phi_{0,n+1}(z)|$, then we have the following identity:

$$\log \frac{r^{n+1}|f(0) - f(z)|}{|\phi_{00}(z)\phi_{01}(z)\ldots\phi_{0n}(z)|} = \frac{1}{2\pi} \int_0^{2\pi} \log \left| f(re^{i\theta}) - f(z) \right| d\theta.$$

Equivalently:

$$\left| \prod_{j=0}^{n} \phi_{0j}(z) \right| = r^{n+1} |f(0) - f(z)| \exp \left\{ -\frac{1}{2\pi} \int_0^{2\pi} \log \left| f(re^{i\theta}) - f(z) \right| d\theta \right\}.$$

If we take (as is possible) $r = |\phi_{0n}(z)|$, then:

$$\left| \prod_{j=0}^{n-1} \phi_{0j}(z) \right| = |\phi_{0n}(z)|^n |f(0) - f(z)| \exp \left\{ -\frac{1}{2\pi} \int_0^{2\pi} \log \left| f(|\phi_{0n}(z)| e^{i\theta}) - f(z) \right| d\theta \right\}.$$

This gives a recursion between $|\phi_{0n}(z)|$, on the one hand, and the product of the previous terms $|\prod_{j=0}^{n-1} \phi_{0j}(z)|$, on the other hand. Does this determine $|\phi_{0n}(z)|$ uniquely? Let us consider the following function of r:

$$\psi_n(r) = r^n \, |h(0)| \exp \left\{ -\frac{1}{2\pi} \int_0^{2\pi} \log \left| h(re^{i\theta}) \right| d\theta \right\}.$$

Here $h(z)$ is a non-constant entire function such that $h(0) \neq 0$. If this function $\psi_n(r)$ turns out to be a strictly monotone function of r (in our case, necessarily increasing), then our recursion uniquely determines $|\phi_{0n}(z)|$ (up to multiplicity in r). We might go about as follows:

$$r^n \exp\left\{-\frac{1}{2\pi}\int_0^{2\pi} \log\left|h(re^{i\theta})\right| d\theta\right\} = e^{\log r^n} \exp\left\{-\frac{1}{2\pi}\int_0^{2\pi} \log\left|h(re^{i\theta})\right| d\theta\right\} =$$

$$= \exp\left\{\log r^n - \frac{1}{2\pi}\int_0^{2\pi} \log\left|h(re^{i\theta})\right| d\theta\right\} = \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \log\left(\frac{r^n}{|h(re^{i\theta})|}\right) d\theta\right\} =$$

$$= \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \log\left|\frac{(re^{i\theta})^n}{h(re^{i\theta})}\right| d\theta\right\}$$

The point is that $z^n/h(z)$ is not holomorphic for $|z| < r + \epsilon$ because of the zeros of $h(z)$ with moduli smaller than $r+\epsilon$, but, as in the proof of the Theorem of Jensen we divide out those zeros by dividing $h(z)$ by the corresponding finite Blaschke product $B_n(z)$, without changing the modulus of $h(z)$ on $|z| = r + \epsilon$. Thus we have for all the $z \in \mathbb{C}$, $|z| = r + \epsilon < |\phi_{0,n+1}(z)|$:

$$\left|\frac{h(z)}{B_n(z)}\right| = |h(z)|\,.$$

So on that circle of integration we have:

$$\left|\frac{z^n}{h(z)}\right| = \left|\frac{z^n B_n(z)}{h(z)}\right|,$$

and this function is analytic. The monotonicity now follows. A better approach: We have (write the recursion a bit different),

$$\left|\prod_{j=0}^{n-1} \phi_{0j}(z)\right|^{1/n} = \left|\phi_{0n}(z)\right| |f(0) - f(z)|^{1/n}$$
$$\times \exp\left\{-\frac{1}{2\pi n}\int_0^{2\pi} \log\left|f(|\phi_{0n}(z)|e^{i\theta}) - f(z)\right| d\theta\right\}.$$

So the right-hand side equals the geometric mean of the sequence $|\phi_{00}(z)|, \ldots, |\phi_{0,n-1}(z)|$ and this is a part of a non-decreasing sequence so those means are also non-decreasing.

12 A Computation of the Integral $\frac{1}{2\pi}\int_0^{2\pi}\log\left|f(re^{i\theta})-f(z)\right|d\theta$

Assuming $z \in \mathbb{C} - f^{-1}(f(0))$, i.e. $f(0) - f(z) \neq 0$, we have:

$$f(w) - f(z) = e^{g(w,z)} \prod_{n=0}^{\infty} \left(1 - \frac{w}{\phi_{0n}(z)}\right) e^{Q_{\lambda_n}(w/\phi_{0n}(z))}. \tag{9}$$

So

$$|f(w) - f(z)| = e^{\Re\, g(w,z)} \prod_{n=0}^{\infty} \left|1 - \frac{w}{\phi_{0n}(z)}\right| e^{\Re\, Q_{\lambda_n}(w/\phi_{0n}(z))}.$$

Hence

$$\log|f(w) - f(z)| = \Re\, g(w,z) + \sum_{n=0}^{\infty} \left\{\log\left|1 - \frac{w}{\phi_{0n}(z)}\right| + \Re\, Q_{\lambda_n}\left(\frac{w}{\phi_{0n}(z)}\right)\right\}.$$

The function $\Re\, g(w,z)$ is harmonic in all of the w-plane. Hence:

$$\frac{1}{2\pi}\int_0^{2\pi} \Re\, g(re^{i\theta}, z)d\theta = \Re\, g(0,z).$$

If we substitute $w = 0$ in equation (9) we get $f(0) - f(z) = e^{g(0,z)}$, so $|f(0) - f(z)| = e^{\Re\, g(0,z)}$ and $\log|f(0) - f(z)| = \Re\, g(0,z)$. Hence, we proved the following:

$$\frac{1}{2\pi}\int_0^{2\pi} \Re\, g(re^{i\theta}, z)d\theta = \log|f(0) - f(z)|.$$

Similarly (and in fact much simpler):

$$\frac{1}{2\pi}\int_0^{2\pi} \Re\, Q_{\lambda_n}\left(\frac{re^{i\theta}}{\phi_{0n}(z)}\right)d\theta = 0.$$

We are left with the computation of:

$$\frac{1}{2\pi}\int_0^{2\pi} \log\left|1 - \frac{re^{i\theta}}{\phi_{0n}(z)}\right| d\theta.$$

If $r < |\phi_{0n}(z)|$, then $1 - (re^{i\theta})/\phi_{0n}(z)$ never vanishes on $0 \le \theta < 2\pi$, and so the function $\log|1 - (re^{i\theta})/\phi_{0n}(z)|$ is harmonic for $|re^{i\theta}| < |\phi_{0n}(z)|$ and again the mean value property implies that:

$$\frac{1}{2\pi}\int_0^{2\pi} \log\left|1 - \frac{re^{i\theta}}{\phi_{0n}(z)}\right| d\theta = \log|1-0| = 0.$$

Thus we are left with computing the integral

$$\frac{1}{2\pi}\int_0^{2\pi} \log\left|1 - \frac{re^{i\theta}}{\phi_{0n}(z)}\right| d\theta,$$

for the case $|\phi_{0n}(z)| \le r$. In this case the integrand $\log|1 - w/\phi_{0n}(z)|$ is singular exactly in $w = \phi_{0n}(z)$ which lies within $|w| \le r$. Here is formula 15 on page 531 of the book [4]:

$$\int_0^{n\pi} \log\left(1 - 2a\cos\theta + a^2\right) d\theta = \begin{cases} 0 & , \ a^2 \le 1 \\ n\pi \log a^2 & , \ a^2 \ge 1 \end{cases}. \tag{10}$$

We need to evaluate:

$$\frac{1}{2\pi}\int_0^{2\pi} \log\left|1 - Re^{i\theta}\right| d\theta =^{(1\le R)} \frac{1}{2\pi}\int_0^{2\pi} \log\left|\frac{1}{R} - e^{i\theta}\right| d\theta + \log R =$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \log\left|a - e^{i\theta}\right| d\theta + \log R =^{(0<a=1/R\le 1)}$$

$$= \frac{1}{2\pi}\cdot\frac{1}{2}\int_0^{2\pi} \log\left(1 + a^2 - 2a\cos\theta\right) d\theta + \log R = \log R,$$

where in the last step we used the formula (10). The conclusion is:

$$\frac{1}{2\pi}\int_0^{2\pi} \log\left|1 - \frac{re^{i\theta}}{\phi_{0n}(z)}\right| d\theta = \begin{cases} 0 & , \ r < |\phi_{0n}(z)| \\ \log(r/|\phi_{0n}(z)|) & , \ r \ge |\phi_{0n}(z)| \end{cases}.$$

Finally we get:

$$\frac{1}{2\pi}\int_0^{2\pi} \log\left|f(re^{i\theta}) - f(z)\right| d\theta =$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \left\{\Re\, g(re^{i\theta}, z) + \sum_{n=0}^{\infty}\left(\log\left|1 - \frac{re^{i\theta}}{\phi_{0n}(z)}\right| + \Re\, Q_{\lambda_n}\left(\frac{re^{i\theta}}{\phi_{0n}(z)}\right)\right)\right\} d\theta =$$

$$= \log|f(0) - f(z)| + \sum_{|\phi_{0n}(z)|\le r} \log \frac{r}{|\phi_{0n}(z)|} + 0 =$$

$$= \log\left(|f(0) - f(z)| \times \prod_{|\phi_{0n}(z)|\le r} \frac{r}{|\phi_{0n}(z)|}\right).$$

We note that in fact our computation proved the Theorem of Jensen.

13 The Product of the Automorphic Functions

Let $f(w)$ be a non-constant entire function, and let $z \in \mathbb{C} - f^{-1}(f(0))$. Let $\mathrm{Aut}(f) = \{\phi_{0n}(z')\}_n$ and let us suppose that we arranged the automorphic functions in a non-decreasing order of their orbit at z. Thus: $|\phi_{00}(z)| \le |\phi_{01}(z)| \le \ldots$. Then we have the following identity:

$$\left|\prod_{j=0}^{n-1} \phi_{0j}(z)\right| = \left|\phi_{0n}(z)\right|^n |f(0) - f(z)|$$

$$\exp\left\{-\frac{1}{2\pi}\int_0^{2\pi} \log\left|f(\left|\phi_{0n}(z)\right| e^{i\theta}) - f(z)\right| d\theta\right\}.$$

One is tempted to compare this formula with the Vieta formula that corresponds to the special case of a non-constant polynomial of degree $d \ge 1$, $f(w) = p_d(w)$. We recall that if $p_d(w) = a_d w^d + \ldots + a_1 w + a_0$, where $a_d \in \mathbb{C} - \{0\}$, then: $p_d(w) - p_d(z) = a_d(w - \phi_{00}(z)) \ldots (w - \phi_{0,d-1}(z))$. Plugging into the last formula the value $w = 0$, we obtain:

$$\phi_{00}(z) \ldots \phi_{0,d-1}(z) = (-1)^d a_d^{-1} \left(p_d(0) - p_d(z)\right). \tag{11}$$

Taking absolute values we get:

$$\left|\phi_{00}(z) \ldots \phi_{0,d-1}(z)\right| = |a_d|^{-1} |p_d(0) - p_d(z)|.$$

Let N be an integer such that $0 \le N \le d - 1$. Then:

$$\left|\prod_{j=0}^{N-1} \phi_{0j}(z)\right| = \left|\prod_{k=N}^{d-1} \phi_{0k}(z)\right|^{-1} |a_d|^{-1} |p_d(0) - p_d(z)|.$$

We note that $\forall\, j, \phi_{0j}(z) \neq 0$ so that the expression on the right-hand side is defined. Comparing that to the more general expression:

$$\left|\prod_{j=0}^{N-1} \phi_{0j}(z)\right| = \left|\phi_{0N}(z)\right|^{N} |p_d(0) - p_d(z)|$$
$$\exp\left\{-\frac{1}{2\pi}\int_0^{2\pi} \log\left|p_d(\left|\phi_{0N}(z)\right| e^{i\theta}) - p_d(z)\right| d\theta\right\},$$

we deduce the following identity:

$$\exp\left\{-\frac{1}{2\pi}\int_0^{2\pi} \log\left|p_d(\left|\phi_{0N}(z)\right| e^{i\theta}) - p_d(z)\right| d\theta\right\} = \left|\prod_{k=N}^{d-1} \phi_{0k}(z)\right| \left|\phi_{0N}(z)\right|^{N} |a_d|.$$

If z has the smallest modulus in its Aut(p_d)-orbit and $N = 0$, we obtain the following interesting identity:

$$\exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \log\left|p_d(ze^{i\theta}) - p_d(z)\right| d\theta\right\} = \left|\prod_{k=0}^{d-1} \phi_{0k}(z)\right| |a_d|.$$

Using the identity in equation (11) this proves that:

$$\exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \log\left|p_d(ze^{i\theta}) - p_d(z)\right| d\theta\right\} = |p_d(0) - p_d(z)|.$$

This last identity could be seen to be true because of the mean value property of the harmonic function: $\log|p_d(w) - p_d(z)|$ in the disk $|w| < |z|$. This reproduces the Vieta identity and shows that we generalized it even in the special case of non-constant polynomials.

14 Consequences to Aut(f) That Follow from the Classical Theory of Entire Functions

Let f be a non-constant entire function of a single complex variable. The basic observation that the set of all the zeros of the entire function of w, $f(w) - f(z)$, where $z \in \mathbb{C}$ is a parameter, is the Aut(f)-orbit of z, plus the fact that for a fixed z, the functions $f(w)$ and $f(w) - f(z)$ differ by a constant suggest that the elements of $\text{Aut}(f(z)) = \{\phi_{0n}(z)\}_n$ share properties with the zeros of $f(w) + c$ for any (generic) constant. In this section we will repeatedly refer to the classical book [7].

Remark 15 For a fixed value of the parameter $z \in \mathbb{C}$ the w-entire functions $f(w)$ and $f(w) - f(z)$ have the same order ρ and the same type σ.

An immediate consequence that follows by the result on (page 16 of [7]) is:

Theorem 9 *Let $f(w)$ be a non-constant entire function, and let* $\mathrm{Aut}(f) = \{\phi_{0n}(z)\}_n$. *Then the convergence exponent of the sequence $\{\phi_{0n}(z)\}_n$ for any $z \in \mathbb{C}$ does not exceed the order of $f(w)$.*

Remark 16 This theorem is interesting only in the case of entire functions of a finite order $\rho < \infty$.

Using Theorem 7 on page 16 of [7] and the representation formula in the first equation in the proof of the Theorem of Wiman (page 72, [7]) we deduce some interesting consequences on the automorphic group of a non-constant entire function of order less than one.

Theorem 10 *Let $f(w)$ be a non-constant entire function of order less than one, and let* $\mathrm{Aut}(f) = \{\phi_{0n}(w)\}_n$. *Then $\forall\, z \in \mathbb{C}$, the convergence exponent of the sequence $\{\phi_{0n}(z)\}_n$ equals the order of f. In addition, if the order of f is not zero, then it is of maximal, minimal or of a normal type according to whether the upper density of any* $\mathrm{Aut}(f)$*-orbit of any $z \in \mathbb{C}$:*

$$\triangle\left(\{\phi_{0n}(z)\}_n\right) = \overline{\lim}_{r\to\infty} \frac{n_{\{\phi_{0n}(z)\}_n}(r)}{r^\rho},$$

equals infinity, zero or equals a number different from zero or infinity. Here we use the notation:

$$n_{\{\phi_{0n}(z)\}_n}(r) = \left|\{n \mid |\phi_{0n}(z)| < r\}\right|,$$

i.e. the counting function of the elements in the $\mathrm{Aut}(f)$*-orbit of z of modulus less than r. In particular the upper density of any* $\mathrm{Aut}(f)$*-orbit of any $z \in \mathbb{C}$ is independent of z.*

Proof $\forall\, z \in \mathbb{C}$, the order ρ and the type σ of the entire function of w, $f(w) - f(z)$ equal those of $f(w)$. Hence (like in the first equation in the proof above of the Theorem of Wiman) we have:

$$f(w) - f(z) = C \cdot w^m \prod_{k=1}^{\infty} \left(1 - \frac{w}{\phi_{0n}(z)}\right).$$

Here $C = C(z)$ is a function of z only, and $C(z) \neq 0\ \forall\, z \in \mathbb{C}$. So $f(w) - f(z)$ is a canonical product and Theorem 7 of [7] applies. □

We will quote few results from [2]. We will use those to extract information on the function $C(z)$ that appears in the proof of Theorem 10. On page 207 of [2]:

Theorem A.2 (Clunie) *Let $f(z)$ and $g(z)$ be entire with $g(0) = 0$. Let ρ satisfy $0 < \rho < 1$ and $c(\rho) = (1-\rho)^2/48$. Then for $R \geq 0$, $M(R, f \circ g) \geq M(c\rho M(\rho \cdot R, g), f)$.*

On page 208 of [2]:

Corollary A.1 (Pölya) *Let $f(z)$, $g(z)$ and $h(z)$ be entire functions with $h(z) = f(g(z))$. If $g(0) = 0$, then there exists an absolute constant c, $0 < c < 1$, such that for all $r > 0$ the following inequality holds:*

$$M(r, h) \geq M\left(c \cdot M\left(\frac{r}{2}, g\right), f\right).$$

When $g(0) \neq 0$, the corresponding inequality should read:

$$M(r, f \circ g) \geq M\left(c \cdot M\left(\frac{r}{2}, g\right) - |g(0)|, f\right).$$

The constant c can be chosen to be $1/8$.

On page 209 of [2]:

Theorem A.3 *If $f(z)$ and $g(z)$ are two entire functions such that $f \circ g$ is of a finite order (lower order), then either: (i) $g(z)$ is a polynomial and $f(z)$ is of a finite order (lower order), or (ii) $g(z)$ is not a polynomial but a function of a finite order (lower order) and $f(z)$ is of zero order (lower order).*

We conclude the following:

Remark 17 If $f(w)$ is an entire function of order less than one, and greater than zero, then $f(w)$ has infinitely many zeros.

A Proof of the claim in Remark 17 If $f(w)$ has no zeros, then $f(w) = e^{g(w)}$ for some entire function g. This corresponds in Theorem A.3. in [2] to case (i) with $e^w \circ g(w)$. Here $g(w)$ is a non-constant polynomial of degree $d \geq 1$. Hence the order of $f(w)$ equals to $d \geq 1$ and this contradicts the assumption that $f(w)$ has order less than one. If $f(a_1) = 0$, then $f(w) \cdot (w - a_1)^{-1}$ is a non-constant entire function of the same order as the order of $f(w)$. So if $f(w)$ had finitely many zeros, then we could have found a polynomial $p(w)$ such that $f(w)/p(w)$ had non zeros and was of the same order as that of $f(w)$. This contradicts the first part of our proof and thus the proof of the claim in Remark 17 is now completed. $\square$

Finally, the function $C(z)$ in the proof of Theorem 10 is entire because the infinite product is, and it never vanishes and since $f(w) - f(z)$ is symmetric in w and z, $C(z)$ must be a constant, i.e. independent of z (because its order is less than one).

Theorem 11 *Let $f(w)$ be a non-constant entire function of a finite order ρ and let* $\mathrm{Aut}(f) = \{\phi_{0n}(w)\}_n$. *Then $\forall\, z \in \mathbb{C}$, the upper density of the* $\mathrm{Aut}(f)$*-orbit of z satisfies:*

1. *If $f(w)$ is of type not greater than σ with respect to the order ρ, then*

$$\overline{\triangle}\left(\{\phi_{0n}(z)\}_n\right) = \overline{\lim}_{r\to\infty}\frac{n_{\{\phi_{0n}(z)\}_n}(r)}{r^\rho} \le \sigma \cdot e \cdot \rho.$$

The lower density:

$$\underline{\triangle}\left(\{\phi_{0n}(z)\}_n\right) = \underline{\lim}_{r\to\infty}\frac{n_{\{\phi_{0n}(z)\}_n}(r)}{r^\rho}$$

satisfies $\underline{\triangle}\left(\{\phi_{0n}(z)\}_n\right) \le \sigma\rho$.

2. *We have the following two identities for these densities:*

$$\overline{\triangle}\left(\{\phi_{0n}(z)\}_n\right) = \overline{\lim}_{n\to\infty}\frac{n}{|\{\phi_{0n}(z)\}_n|^\rho},\ \ \underline{\triangle}\left(\{\phi_{0n}(z)\}_n\right) = \underline{\lim}_{n\to\infty}\frac{n}{|\{\phi_{0n}(z)\}_n|^\rho}.$$

Proof Part 1 follows by Theorem 3 on page 19 of [6]. Part 2 follows by Problem 2 on page 17 of [6]. □

We can refine Part 1 of Theorem 11:

Theorem 12 *Let* $\rho > 0$, $A > 0$ *and let* $f(w)$ *be a non-constant entire function for which there exists a constant* $M > 0$ *such that* $|f(w)| \le M \cdot e^{A|z|^\rho}$, $\forall\, w \in \mathbb{C}$. *Let* $\mathrm{Aut}(f) = \{\phi_{0n}(w)\}_n$. *We assume that* $z \in \mathbb{C} - f^{-1}(f(0))$ *and that the automorphic functions are arranged so that* $|\phi_{00}(z)| \le |\phi_{01}(z)| \le |\phi_{02}(z)| \le \ldots$. *Then:*

$$(\rho\cdot e\cdot A)^{-1/\rho}\cdot M\left(\frac{1+e^{A|z|^\rho}}{f(0)-f(z)}\right)^{-1/n} n^{1/\rho} \le \left(\prod_{k=1}^n |\phi_{0k}(z)|\right)^{1/n} \le |\phi_{0n}(z)|,\ \ \forall n \in \mathbb{Z}^+.$$

If

$$M\left(\frac{1+e^{A|z|^\rho}}{f(0)-f(z)}\right) < 1,$$

then the left-hand side in the double inequality is $(\rho \cdot e \cdot A)^{-1/\rho} \cdot n^{1/\rho}$.

Proof We define an auxiliary non-constant entire function of w:

$$F(w) = \frac{f(w) - f(z)}{f(0) - f(z)}.$$

Then $F(0) = 1$, and

$$|F(w)| \le \frac{|f(w)|+|f(z)|}{|f(0)-f(z)|} \le \frac{Me^{A|w|^\rho}+Me^{A|z|^\rho}}{|f(0)-f(z)|} = \left(M\left(\frac{1+e^{A(|z|^\rho-|w|^\rho)}}{|f(0)-f(z)|}\right)\right)e^{A|w|^\rho} \le$$

$$\le \left(M\left(\frac{1+e^{A|z|^\rho}}{|f(0)-f(z)|}\right)\right)e^{A|w|^\rho} \le M_1 \cdot e^{A|w|^\rho},$$

where:

$$M_1 = \max\left\{1, M\left(\frac{1+e^{A|z|^\rho}}{|f(0)-f(z)|}\right)\right\}.$$

The function $F(w)$ satisfies the conditions of Proposition 1, in [13], with M instead of M_1. The result now follows. □

We recall the following result:

Theorem 1 ([6]) *The convergence exponent of the zero set of an entire function f of non-integer order is equal to the order of growth of f.*

This theorem implies:

Theorem 13 *Let $f(w)$ be a non-constant entire function of non-integer order ρ. Let the $\mathrm{Aut}(f)$-orbit of a point $z \in \mathbb{C}$ be $\{\phi_{0n}(z)\}_n\}$, then the convergence exponent of this orbit equals ρ.*

Next we have:

Theorem 2 ([6]) *If the order ρ of an entire function $f(z)$ is not an integer, then its type σ_f and the upper density of its zeros $\overline{\triangle}_f$ simultaneously are equal either to zero, or to infinity, or to positive numbers.*

This immediately implies:

Theorem 14 *If the order ρ of an entire function $f(w)$ is not an integer, then its type σ_f and the upper density of any $\mathrm{Aut}(f)$-orbit, $\{\phi_{0n}(z)\}_n$ ($z \in \mathbb{C}$) are equal either to zero, or to infinity, or to positive numbers.*

15 The Relations Between Scattering Theory and Automorphic Functions

The book [8] deals with a discrete subgroup Γ of the group of the fractional linear transformations,

$$w \to \frac{aw+b}{cw+d}, \quad ad-bc=1, \quad a,b,c,d \in \mathbb{R}.$$

The Riemannian metric $(dx^2+dy^2)\cdot y^{-2}$ is invariant under this group of motions. The invariant Dirichlet integral for functions is:

$$\iint (U_x^2+U_y^2)dxdy,$$

and the Laplace-Beltrami operator associated with this is:

$$L_0 = y^2 \cdot \triangle = y^2 \cdot (\partial_x^2 + \partial_y^2).$$

A function f defined on the Poincaré plane $\prod$, that is the upper half plane: $y > 0$, $-\infty < x < \infty$, $w = x + iy$ is called automorphic with respect to Γ (a discrete subgroup as above) if $f(\gamma w) = f(w)\ \forall\, \gamma \in \Gamma$. The Laplace-Beltrami operator L_0 maps automorphic functions into automorphic functions. A fundamental domain F of Γ is a sub-domain of the Poincaré plane such that every point of $\prod$ can be carried into a point of the closure $\overline{F}$ of F by a transformation in Γ and no point of F is carried into another point of F by such a transformation. $\overline{F}$ can be regarded as a manifold where those boundary points which can be mapped by a $\gamma \in \Gamma$ to each other are identified. The restriction of f (automorphic with respect to Γ) to the fundamental domain F has to satisfy the above-mentioned boundary relations imposed by $f(\gamma w) = f(w)$. These relations serve as boundary conditions for the operator L_0. In fact, they define L_0 as a self-adjoint operator acting on $L_2(F)$, the space of functions on F square integrable with respect to the invariant measure. Our setting is different but similar. We have a non-constant entire function $f(w)$, and its automorphic group $\mathrm{Aut}(f) = \{\phi_{0n}(w)\}_n$ which is a discrete group. Its elements are defined by all the maximal leaves of $f^{-1}(f(w))$. The function f is automorphic with respect to the discrete group $\mathrm{Aut}(f)$. The normal maximal domains of $f(w)$ are the parallels of the fundamental domains F of Γ. What can be the parallel of the Laplace-Beltrami operator L_0? It should be an operator that maps automorphic functions with respect to the discrete group $\mathrm{Aut}(f)$ into automorphic functions. We already know that the set of all the automorphic functions are the compositions $h \circ f$ where h is a non-constant function. Thus it is reasonable to take as a parallel to L_0 the right-shift operator induced by f, R_f, [11]. It has as its domain of definition the algebra E of all the non-constant entire functions and it maps them onto the sub-class $E \circ f = \{h \circ f \mid h \in E\}$, i.e. onto all the automorphic functions with respect to $\mathrm{Aut}(f)$. Thus:

$$R_f \,:\, E \to E \circ f, \quad R_f(h) = h \circ f.$$

The operator R_f is not only a linear operator, but it is also an algebra morphism. For let $h, g \in E$ and let $c \in \mathbb{C}$. Then:

$$R_f(c \cdot h) = (c \cdot h) \circ f = (c \cdot h)(f(w)) = c \cdot h(f(w)) = c \cdot (h \circ f) = c \cdot R_f(h),$$

and

$$\begin{aligned} R_f(g + h) &= (g + h) \circ f = (g + h)(f(w)) = g(f(w)) + h(f(w)) \\ &= (g \circ f) + (h \circ f) = R_f(g) + R_f(h), \end{aligned}$$

and similarly

$$R_f(g \cdot h) = (g \cdot h) \circ f = (g \cdot h)(f(w)) = g(f(w)) \cdot h(f(w))$$
$$= (g \circ f) \cdot (h \circ f) = R_f(g) \cdot R_f(h).$$

Another important property of the f right-shift operator is: R_f is an injective mapping on E, [11].

Also: the image of R_f, $R_f(E)$ is a closed subset of the topological space (E, τ_{cc}). The topology τ_{cc} is the topology of compact convergence on the space E. For each compact $K \subset \mathbb{C}$ and for each $\epsilon > 0$ and each $h \in E$ we define the corresponding open ball centered at h by the standard formula:

$$B_K(h, \epsilon) = \{g : \mathbb{C} \to \mathbb{C} \in E \,|\, |g(z) - h(z)| < \epsilon, \, \forall\, z \in K\}.$$

The family $\{B_K(h, \epsilon) \,|\, K \subset \mathbb{C} \text{ a compact}, h \in E, \epsilon > 0\}$ forms a sub-basis for the topology τ_{cc}. The sequence $g_n \in E$ converges to the limit $g \in E$ if and only if for restrictions to compacta we have $g_n|_K \to g|_K$ uniformly on K, $\forall$ compact $K \subset \mathbb{C}$.

Remark 18 The space (E, τ_{cc}) is a path connected space, [11]. We know that the mapping $R_f : E \to E$ is a continuous and an injective mapping [11]. Its image $R_f(E)$ is a closed subset of (E, τ_{cc}). Hence $R_f(E)$ is also open $\Leftrightarrow R_f(E) = E$. This is equivalent to $f(w) = aw + b$, for some $a \in \mathbb{C}^\times$ and some $b \in \mathbb{C}$.

Another interesting property of R_f is $\forall\, f \in E - \{aw + b \,|\, a \in \mathbb{C}^\times, b \in \mathbb{C}\}$ we have the identity $\partial R_f(E) = R_f(E)$. Thus the image $R_f(E)$ is its own boundary.

What can be the parallel of the hyperbolic metric? It should be a metric $d_f(\cdot, \cdot) : \mathbb{C}^2 \to \mathbb{R}_{\geq 0}$ which is invariant under the action of the automorphic group $\mathrm{Aut}(f)$. We give it with other facts that were mentioned above in the following:

Theorem 15 *Let $f \in E$. Then the parallel of the discrete group Γ of hyperbolic motions in the plane is* $\mathrm{Aut}(f)$. *The parallel of the Laplace-Beltrami operator, L_0, is the f right-shift operator. The parallel of the hyperbolic metric $(dx^2 + dy^2) \cdot y^{-2}$ is the f path-metric, $d_f(\cdot, \cdot) : \mathbb{C}^2 \to \mathbb{R}_{\geq 0}$ given by the following formula: $\forall\, a, b \in \mathbb{C}$,*

$$d_f(a, b) = \inf\{l_f(\gamma) \,|\, \gamma : [0, 1] \to \mathbb{C} \text{ is a smooth path from } \gamma(0) = a \text{ to } \gamma(1) = b$$

and

$l_f(\gamma)$ is the length of the path $f \circ \gamma : [0, 1] \to \mathbb{C}$ from $(f \circ \gamma)(0) = f(a)$ to $(f \circ \gamma)(1) = f(b)\}$.

We further have the following:

1) $R_f : E \to E \circ f$ *is continuous, injective, and surjective mapping in the topological space* (E, τ_{cc}).
2) R_f *is a linear operator on* E *which also preserves multiplication of functions.*
3) $R_f(E)$ *is closed in* (E, τ_{cc}), *and* $R_f(E) = E$ *if and only if* $f(w) = aw + b$, *for some* $a \in \mathbb{C}^\times$ *and* $b \in \mathbb{C}$.
4) $\forall\, f \in E - \{aw + b \,|\, a \in \mathbb{C}^\times, \, b \in \mathbb{C}\}$, $\partial R_f(E) = R_f(E)$.

Proof The only assertions we need to prove are those related to the metric d_f. Namely we need to prove two things:

(a) d_f is a metric on $\mathbb{C}$, (b) $\forall\, \phi \in \mathrm{Aut}(f)$ we have invariance of d_f, i.e. $\forall\, a, b \in \mathbb{C}$, $d_f(a, b) = d_f(\phi(a), \phi(b))$.

A Proof of (a) Let $a, b, c \in \mathbb{C}$, then clearly $d_f(a, b) = d_f(b, a)$ because each path $\gamma(t)$ from a to b induces the reverse path $\gamma(1 - t)$ from b to a and the f images of both are the same hence have equal length.

Also $d_f(a, a) = 0$ by using the constant path. Moreover, if $d_f(a, b) = 0$ then for each $\epsilon > 0$ there is a path γ from a to b such that the length of its f image $f \circ \gamma$ is smaller than $\epsilon > 0$. Since the entire function f is non-constant we can find a positive but small enough number $r > 0$ such that it has the following two properties:

(i) Any path $\gamma(t)$ from a to b must intersect the circle $\{w \in \mathbb{C} \mid |w - a| = r\}$.
(ii) $f^{'}(w)$ has no zero on that circle.

Since the circle is compact we have $\min\{|f^{'}(w)| \mid |w - a| = r\} = \delta > 0$. This implies that the length $l_f(\gamma)$ of the image path $f \circ \gamma$ is bounded from below by some fixed number $m(\delta) > 0$ and hence $d_f(a, b) \geq m(\delta > 0$ which contradicts the assumption $d_f(a, b) = 0$ unless $a = b$.

Finally, d_f satisfies the triangle inequality: $d_f(a, c) \leq d_f(a, b) + d_f(b, c)$, because the set of paths from a to c through b is a subset of the set of paths from a to c. This concludes the proof of (a).

A Proof of (b) We now prove the invariance of the metric $d_f(\cdot, \cdot)$ with respect to the discrete group $\mathrm{Aut}(f)$. This follows directly from the definitions of $d_f(\cdot, \cdot)$ and of $\mathrm{Aut}(f)$, namely:

$d_f(a, b) = \inf\{l_f(\gamma) \mid \gamma : [0, 1] \to \mathbb{C}$ is a smooth path from $\gamma(0) = a$ to $\gamma(1) = b$

and

$l_f(\gamma)$ is the length of the path $f \circ \gamma : [0, 1] \to \mathbb{C}$ from $(f \circ \gamma)(0) = f(a)$ to $(f \circ \gamma)(1) = f(b)\} =$

$= \inf\{l_f(\gamma) \mid \gamma : [0, 1] \to \mathbb{C}$ is a smooth path from $\gamma(0) = \phi(a)$ to $\gamma(1) = \phi(b)$ and

$l_f(\gamma)$ is the length of the path $f \circ \gamma : [0, 1] \to \mathbb{C}$ from $f(\phi(a)) = f(a)$ to

$$f(\phi(b)) = f(b)\} = d_f(\phi(a), \phi(b)).$$

□

The next obvious step is to look for the eigenvalues of the operator R_f. If we think of the f right composition operator R_f as a possible parallel of the Laplace-Beltrami operator, then it is natural to require about its eigenvalues, eigenfunctions and maybe try to come up with a kind of a Selberg-trace formula.

We recall that in our setting the underlying linear space, E contains all of the non-constant entire functions, and the right composition operators R_f of interest are those for which $f(w)$ is not an entire automorphism, i.e. $f(w) \neq a\cdot w+b, \forall\, a \in \mathbb{C}^{\times}$ and $\forall\, b \in \mathbb{C}$. It turns out that the result of this search is disappointing because the order of growth of entire functions imposes (within the class of functions of our interest) too much rigidity. Looking in the corresponding defining equation of the eigenvalues for R_f leads us to ask for which values of $\lambda \in \mathbb{C}$, the operator $R_f - \lambda \cdot I$ is not an invertible operator. Thus if we look for entire functions h that satisfy $(R_f - \lambda \cdot I)(h) = 0$, that is $h(f(w)) \equiv \lambda h(w)$, then because f is not affine and due to Theorem A.2. of Clunie, on page 207 of [2], and to Corollary A.1. of Pölya, on page 208 of [2], we deduce that the last equation can have only constant solutions $9w)$. If the value of the constant is not zero, then $\lambda = 1$, and if $h(w) \equiv 0$, then λ can be any complex number. However, non-zero constant function does not belong to E by its definition. This takes care of those cases in which the order of the growth of $h \circ f$ is larger than that of h.

If we adopt a different definition of eigenvalues, and we are interested in those λ for which $R_f - \lambda \cdot I$ is not injective on E, then we are led to consider the situation where $g, h \in E$ and: $(R_f - \lambda \cdot I)(h) = (R_f - \lambda \cdot I)(g)$, that is $h(f(w)) - g(f(w)) \equiv \lambda \cdot (h(w) - g(w))$, and again we deduce that $h(w) - g(w)$ must be a constant. If the constant difference between $h(w)$ and $g(w)$ is not zero, then $\lambda = 1$, and, of course if $h(w) \equiv g(w)$ then λ can be any complex number.

Remark 19 (1) An equation of the form $h(f(w)) \equiv \lambda h(w)$ resembles very much the equation that determines the automorphic functions of $h(w)$. When $\lambda = 1$ it is exactly that equation. By Theorem 10 of Shimizu, on page 237 of [15], the only possible automorphic functions which are also entire functions, are affine functions of a very special kind: $e^{\theta\pi} \cdot w + b$, where $\theta \in \mathbb{Q}$, and where $b \in \mathbb{C}$. This is consistent with our findings prior to this Remark 19.

Remark 20 We know that the operator R_f is injective. Hence $R_f - \lambda \cdot I$ is injective for $\lambda = 0$.

16 Local Groups

A good source for the theory of topological groups is the relatively new book [1]. It will be convenient to have the notion of local groups handy in our setting of the automorphic group. This is because the automorphic functions of an entire function are generically multivalued. Thus they naturally are defined and uniform on the corresponding Riemann surfaces. It can be useful occasionally to restrict them to a leaf, i.e. to a sub-domain of the complex plane whose complementary set contains no continuum. In those cases the different restricted automorphic functions might be defined on different sub-domains of the plane, that differ by small sets. Thus we might need the notion of a local group. We refer to [16] to definition 2.1.1 on page 26. In that book the need in local groups arise because the connection between Lie groups and Lie algebras are local. The only portion of the Lie group which is of importance in that respect is the portion that is close to the group identity 1. We will adopt here the notions and the notations from [16].

Definition 4 A local topological group $G = (G, \Omega, \Lambda, 1, \cdot, ()^{-1})$, or a local group for short, is a topological space G, equipped with an identity element $1 \in G$, a partially defined but continuous multiplication operation $\cdot : \Omega \to G$ for some domain $\Omega \subseteq G \times G$, and a partially defined but continuous inversion operation $()^{-1} : \Lambda \to G$, where $\Lambda \subseteq G$, obeying the following axioms:

(1) Ω is an open neighborhood of $G \times \{1\} \cup \{1\} \times G$, and Λ is an open neighborhood of 1.
(2) If $g, h, k \in G$ are such that $(g \cdot h) \cdot k$ and $g \cdot (h \cdot k)$ are both well-defined in G, then they are equal.
(3) For all $g \in G$, $g \cdot 1 = 1 \cdot g = g$.
(4) If $g \in G$ and g^{-1} are well-defined in G, then $g \cdot g^{-1} = g^{-1} \cdot g = 1$.

A local group is said to be symmetric if $\Lambda = G$, i.e. if every element of G has an inverse g^{-1} that is also in G. Clearly, every topological group is a local group. This the reason that sometimes the former are called global topological groups. A model class of examples of a local group comes from restricting a global group to an open neighborhood of the identity. Here is the definition from [16]:

Definition 5 If G is a local group, and U is an open neighborhood of the identity in G, then we define the restriction $G|_U$ of G to U to be the topological space U with domains:

$$\Omega|_U := \{(g, h) \in \Omega \mid g, h, g \cdot h \in U\} \text{ and } \Lambda|_U := \{g \in \Lambda \mid g, g^{-1} \in U\}$$

and with the group operations $\cdot$, $()^{-1}$ being the restriction of the group operations of G to $\Omega|_U$, $\Lambda|_U$, respectively. If U is symmetric (in the sense that g^{-1} is well-defined and lies in U, $\forall g \in U$, then this restriction $G|_U$ will also be symmetric. Sometimes the notation is abused and one refers to the local group $G|_U$ simply as U.

Remark 21 The natural question as to whether every local group arises as the restriction of a global group is not simple. The answer can be vaguely summarized as “essentially yes in certain circumstances, but not in general.” That is from [16].

Pushing forward a topological group via a homeomorphism near the identity: Let G be a global or local group and let $Phi : U \to V$ be a homeomorphism from a neighborhood U of the identity in G to a neighborhood V of the origin 0 in $\mathbb{R}^d$, such that $Phi(1) = 0$. Then we can define a local group $Phi_* G|_U$ which is the set V (viewed as a submanifold of $\mathbb{R}^d$) with the local group identity 0, the local group multiplication law $*$ defined by the formula: $x * y = \Phi(\Phi^{-1}(x) \cdot \Phi^{-1}(y))$ which is defined whenever $\Phi^{-1}(x)$, $\Phi^{-1}(y)$, $\Phi^{-1}(x) \cdot \Phi^{-1}(y)$ are well-defined and lie in U, and the local group inversion law $()^{*-1}$ defined by the formula: $x^{*-1} = \Phi(\Phi^{-1}(x)^{-1})$, defined whenever $\Phi^{-1}(x)$, $\Phi^{-1}(x)^{-1}$ are well-defined and lie in U. One easily verifies that $\Phi_* G|_U$ is a local group. Sometimes this group is denoted by $(V, *)$. It is different from the additive local group $(V, +)$ arising by the restriction of $(\mathbb{R}^d, +)$ to V.

Next we generalize the notion of homomorphism.

Definition 6 A continuous homomorphism $\Phi : G \to H$ between two local groups G, H is a continuous map from G to H with the following properties:

(i) $\Phi(1_G) = 1_H$, 1_G is the neutral element of G.
(ii) If $g \in G$ is such that g^{-1} is well-defined in G, then $\Phi(g)^{-1}$ is well-defined in H, and $\Phi(g)^{-1} = \Phi(g^{-1})$.
(iii) If $g, h \in G$ are such that $g \cdot h$ is well-defined in G, then $\Phi(g) \cdot \Phi(h)$ is well-defined in H and $\Phi(g) \cdot \Phi(h) = \Phi(g \cdot h)$.

17 The Sums of the k'th Derivatives of All the Elements of the Automorphic Group Aut(f), for Any $f \in E$ of Order $0 < \rho < \frac{1}{2}$, $k = 1, 2, 3, \ldots$

Theorem 16 *Let $f \in E$ have a positive order ρ, which is smaller than $\frac{1}{2}$, i.e. $0 < \rho < \frac{1}{2}$. Let* $\mathrm{Aut}(f) = \{\phi_{0n}\}_n$. *Then there exists a sequence of positive numbers, tending to infinity: $R_1 < R_2 < \ldots < R_n < \ldots (R_n \to \infty)$, such that $\forall\, k \in \mathbb{Z}^+$ we have the following identities:*

$$\lim_{j\to\infty} \sum_{|\phi_{0n}(w)|<R_j} \phi_{0n}^{(k)}(w) \equiv 0, \qquad \forall\, w \in \mathbb{C}.$$

Equivalently:

$$\lim_{j\to\infty} \frac{d^k}{dw^k} \left\{ \sum_{|\phi_{0n}(w)|<R_j} \phi_{0n}(w) \right\} \equiv 0, \qquad \forall\, w \in \mathbb{C}.$$

Proof We proved that $\forall\, f \in E$, regardless of its order ρ, we have the following infinitely many identities:
$\forall\, R > 0$ such that $\{|w| = R\} \cap \{\phi_{0n}(z)\}_n = \emptyset$,

$$\sum_{|\phi_{0n}(z)|<R} \phi_{0n}'(z) = \left(\frac{f'(z)}{2\pi i}\right) \oint_{|w|=R} \frac{dw}{f(w) - f(z)}, \tag{12}$$

$$\sum_{|\phi_{0n}(z)|<R} \phi''_{0n}(z) = \left(\frac{f''(z)}{2\pi i}\right) \oint_{|w|=R} \frac{dw}{f(w)-f(z)} + \tag{13}$$

$$+ \left(\frac{f'(z)^2}{2\pi i}\right) \oint_{|w|=R} \frac{dw}{(f(w)-f(z))^2},$$

$$\sum_{|\phi_{0n}(z)|<R} \phi'''_{0n}(z) = \left(\frac{f'''(z)}{2\pi i}\right) \oint_{|w|=R} \frac{dw}{f(w)-f(z)} + \tag{14}$$

$$+ \left(\frac{3f'(z)f''(z)}{2\pi i}\right) \oint_{|w|=R} \frac{dw}{(f(w)-f(z))^2} + \left(\frac{f'(z)^3}{2\pi i}\right) \oint_{|w|=R} \frac{dw}{(f(w)-f(z))^3},$$

etc.... In the identities above, the radius R of the circle of integration is such that for a fixed value of the parameter z, we have $f(w) - f(z) \neq 0 \; \forall |w| = R$. In other words, the radius R is chosen so that the circle of integration $|w| = R$ does not contain any element of the $\mathrm{Aut}(f)$-orbit of z, $\{\phi_{0n}(z)\}_n$. Also, since we used the one-dimensional Weierstrass product representation of $f(w) - f(z)$, the fixed value of the parameter z belongs to $\mathbb{C} - f^{-1}(f(0))$. We will prove that there exists a sequence of positive numbers tending to infinity: $R_1 < R_2 < R_3 < \ldots < R_n < \ldots (R_n \to \infty)$ such that $\lim_{j\to\infty} \sum_{|\phi_{0n}(z)|<R_j} \phi'_{0n}(z) = 0$ for any value of the parameter $z \notin f^{-1}(f(0))$. It will follow by continuity arguments that the identity is also true for the "forbidden values" of z, i.e. on the fiber $f^{-1}(f(0))$. Also it will be clear that a similar proof will apply for the second identity $\lim_{j\to\infty} \sum_{|\phi_{0n}(z)|<R_j} \phi''_{0n}(z) = 0 \; \forall w \in \mathbb{C}$, and another similar proof will apply for the third identity $\lim_{j\to\infty} \sum_{|\phi_{0n}(z)|<R_j} \phi'''_{0n}(z) = 0 \; \forall w \in \mathbb{C}$, and all with the same sequence of positive numbers $0 < R_1 < R_2 < R_3 < \ldots < R_n < \ldots (R_n \to \infty)$. That procedure of giving a separate proof for each identity will save us arguments about differentiations under the limit operator and inquiring whether formulas like this:

$$\lim_{j\to\infty} \sum_{|\phi_{0n}(z)|<R_j} \phi''_{0n}(z) = \left(\lim_{j\to\infty} \sum_{|\phi_{0n}(z)|<R_j} \phi'_{0n}(z)\right)', \quad \text{etc}\ldots$$

are valid. Up to this point restricting the value of the order ρ seem not come up. However, in order to prove that we have $\lim_{j\to\infty} \sum_{|\phi_{0n}(z)|<R_j} \phi'_{0n}(z) \equiv 0 \; \forall z \in \mathbb{C} - f^{-1}(f(0))$ we will need the assumption $0 < \rho < \frac{1}{2}$. That assumption will enable us to make the use of the $\cos \pi\rho$-Theorem of Wiman, where the power $\cos \pi\rho$ in the inequality of this theorem satisfies $0 < \cos \pi\rho < 1$, by the assumption $0 < \rho < \frac{1}{2}$. The value $\rho = 0$ is taken out due to another argument we need (an elementary one). A classical reference to the Wiman's $\cos \pi\rho$-Theorem is in the book [7], Theorem 30 on page 72. Using Wiman's $\cos \pi\rho$-Theorem, we conclude that there exists a sequence $r_1 < r_2 < r_3 < \ldots < r_n < \ldots (r_n \to \infty)$ such that for arbitrary $\epsilon > 0$ and $n > n_\epsilon$ we have:

$$m_f(r_n) > \left(M_f(r_n)\right)^{\cos\pi\rho-\epsilon},$$

where

$$m_f(r) = \min_{0\le\theta<2\pi} |f(re^{i\theta})| \text{ and } M_f(r) = \max_{0\le\theta<2\pi} |f(re^{i\theta})|.$$

One can get many more (uncountably many on each radius) such sequences $r_1 < r_2 < r_3 < \ldots < r_n < \ldots$ by perturbations. We recall that our first identity to be used is equation (12). We are going to prove that:

$$\lim_{j\to\infty} \left(\frac{f'(z)}{2\pi i}\right) \oint_{|w|=R_j} \frac{dw}{f(w)-f(z)} = 0,$$

for an appropriate Wiman's-type of a sequence $R_1 < R_2 < R_3 < \ldots < R_n < \ldots (R_n \to \infty)$. This will imply that we have:

$$\lim_{j\to\infty} \sum_{|\phi_{0n}(z)|<R_j} \phi_{0n}'(z) \equiv 0 \quad \forall z \in \mathbb{C} - f^{-1}(f(0)).$$

Using the flexibility in choosing Wiman's-type sequences (invoking perturbations), we are going to choose sequences of radii $\{R_j\}_j$ that satisfy the requirements $\{|w| = R_j\} \cap \{\phi_{0n}\}_n = \emptyset$, as well as the conclusion of the Wiman's $\cos\pi\rho$-Theorem, namely that:

$$m_f(R_j) > (M_f(R_j))^{\cos\pi\rho-\epsilon}.$$

By the triangle inequality we have:

$$\left|\left(\frac{f'(z)}{2\pi i}\right) \oint_{|w|=R_j} \frac{dw}{f(w)-f(z)}\right| \le \left(\frac{|f'(z)|}{2\pi}\right) \oint_{|w|=R_j} \frac{|dw|}{|(M_f(R_j))^{\cos\pi\rho} - |f(z)||},$$

for a large enough j. On the circle $|w| = R_j$ we have $w = R_j e^{i\theta}$, $dw = iR_j e^{i\theta} d\theta$, $|dw| = R_j d\theta$, $0 \le \theta < 2\pi$. We conclude that:

$$\left|\left(\frac{f'(z)}{2\pi i}\right) \oint_{|w|=R_j} \frac{dw}{f(w)-f(z)}\right| \le \frac{|f'(z)|R_j}{|(M_f(R_j))^{\cos\pi\rho} - |f(z)||}.$$

We recall that for a large j we have: $M_f(R_j) \approx e^{R_j^\rho}$, and since we have both $\rho > 0$ and $0 < \cos\pi\rho$, by our assumption on the order, $0 < \rho < \frac{1}{2}$, we conclude that:

$$\lim_{j\to\infty}\frac{|f'(z)|R_j}{|(M_f(R_j))^{\cos\pi\rho}-|f(z)||}=0.$$

Hence we proved our first identity on the restricted domain of the values of the parameter z:

$$\lim_{j\to\infty}\sum_{|\phi_{0n}(z)|<R_j}\phi_{0n}^{'}(z)\equiv 0\quad\forall z\in\mathbb{C}-f^{-1}(f(0)).$$

Now the theorem follows as explained above. □

Remark 22 Theorem 16 deals with the sums $\sum_{|\phi_{0n}(w)|<R_j}\phi_{0n}^{(k)}(w)$, $\forall w\in\mathbb{C}$ for values of k which are natural numbers. There is no claim for the value $k=0$, i.e. $\sum_{|\phi_{0n}(w)|<R_j}\phi_{0n}(w)$, $\forall w\in\mathbb{C}$. One might be expecting naively that $\lim_{j\to\infty}\sum_{|\phi_{0n}(w)|<R_j}\phi_{0n}(w)\equiv$ Const., $\forall w\in\mathbb{C}$, but this turns out to be false also within the restricted domain of the order ρ.

Examples The first two examples are of entire functions of order ρ, where ρ is off the domain $(0,\frac{1}{2})$.

(1) We consider the exponential function $f(w)=e^w$. Here $\rho=1$ and $\mathrm{Aut}(e^w)=\{w+2\pi in\}_{n\in\mathbb{Z}}$. Thus $\phi_{0n}^{'}(w)=1$ and hence we have

$$\sum_{|\phi_{0n}(w)|<R}\phi_{0n}^{'}(w)=|\{n\in\mathbb{Z}\,|\,|w+2\pi in|<R\}|\geq\left[\frac{R-|w|}{2\pi}\right]\to_{R\to\infty}\infty.$$

Thus clearly for the conclusions of Theorem 16 to hold, some assumptions on the order ρ are needed.

(2) Let us consider the case of a non-constant polynomial:

$$P_N(w)=a_Nw^N+\ldots+a_1w+a_0,\qquad a_N\neq 0,\quad N\geq 1.$$

Here $\rho=0$ and $\mathrm{Aut}(P_N(w))=\{w,\ldots,\phi_{N-1}(w)\}$. Hence:

$$P_N(w)-P_N(z)=a_N(w-z)\cdot\ldots\cdot(w-\phi_{N-1}(z))=$$

$$=a_N(w^N-(z+\ldots+\phi_{N-1}(z))w^{N-1}+\ldots+(-1)^N\cdot(z\cdot\ldots\cdot\phi_{N-1}(z))).$$

We conclude that:

$$z+\ldots+\phi_{N-1}(z)=\begin{cases}-\frac{a_{N-1}}{a_N} & \text{if } N>1\\ z & \text{if } N=1\end{cases}.$$

Thus in the simplest case $N=1$, $\sum_{|\phi_{0n}(z)|<R}\phi_{0n}^{'}(z)\to_{R\to\infty}1$.

Next we give an example that shows that the domain $0 < \rho < \frac{1}{2}$ is sharp for the conclusion of Theorem 16 to be valid. We already know that necessarily $0 < \rho$ (Example (2), the case $N = 1$). We now show that $\rho = \frac{1}{2}$ is out of the admissible domain.

(3) Let $f(w) = \cos\sqrt{w} = \frac{1}{2}(e^{i\sqrt{w}} + e^{-i\sqrt{w}})$. Since $\cos w$ is an even entire function, it follows that $f(w)$ is an entire function. We have $|f(w)| \leq \frac{1}{2}(|e^{i\sqrt{w}}| + |e^{-i\sqrt{w}}|) \leq \frac{1}{2}(e^{\sqrt{|w|}} + e^{\sqrt{|w|}}) = e^{\sqrt{|w|}}$. If $r > 0$, then: $f(-r) = \frac{1}{2}(e^{-\sqrt{r}} + e^{\sqrt{r}}) \geq \frac{1}{2}e^{\sqrt{r}}$. The two inequalities above prove that $\rho = \frac{1}{2}$. We next compute $\mathrm{Aut}(\cos\sqrt{w})$. For that purpose we first solve for θ the following equation: $e^{i\theta} + e^{-i\theta} = e^{i\psi} + e^{-i\psi}$. This gives the quadratic $e^{2i\theta} - (e^{i\psi} + e^{-i\psi})e^{i\theta} + 1 = 0$. We obtain the following solution:

$$e^{i\theta} = \begin{cases} e^{i\psi} & \text{if } + \\ e^{-i\psi} & \text{if } - \end{cases}.$$

Hence:

$$\begin{cases} i\theta = i\psi + 2\pi i k \\ i\theta = -i\psi + 2\pi i k, k \in \mathbb{Z} \end{cases}$$

So:

$$\begin{cases} \theta = \psi + 2\pi k \\ \theta = -\psi + 2\pi k, k \in \mathbb{Z} \end{cases}$$

In our case $\cos\sqrt{w} = \cos\sqrt{z}$, so $\theta = \sqrt{w}$, $\psi = \sqrt{z}$. Thus: $\sqrt{w} = \pm\sqrt{z} + 2\pi k$, $k \in \mathbb{Z}$. Squaring: $\phi_k(z) = z + 4\pi k\sqrt{z} + a\pi^2k^2,\ k \in \mathbb{Z}$. We proved that:

$$\mathrm{Aut}(\cos\sqrt{w}) = \left\{w + 4\pi k\sqrt{w} + 4\pi^2k^2 \mid k \in \mathbb{Z}\right\}.$$

In particular we have $f^{-1}(f(0)) = \{4\pi^2k^2 \mid k \in \mathbb{Z}\}$. Also we have:

$$\phi_k^{'}(z) = 1 + \frac{2\pi k}{\sqrt{z}}, \quad \phi_k^{''}(z) = -\frac{\pi k}{z\sqrt{z}}, \ldots.$$

So the identities: $\sum_{|\phi_k(z)|<R} \phi_k^{(j)}(z) \to_{R\to\infty} 0,\ \ j \geq 2$ are equivalent to:

$$\lim_{R\to\infty} \sum_{|z+4\pi k\sqrt{z}+4\pi^2k^2|<R} k = 0.$$

The identity: $\sum_{|\phi_k(z)|<R} \phi_k^{'}(z) \to_{R\to\infty} 0$ is equivalent to:

$$\lim_{R\to\infty} \sum_{|z+4\pi k\sqrt{z}+4\pi^2k^2|<R} \left(1+\frac{2\pi k}{\sqrt{z}}\right) = 0.$$

But this last identity is not consistent with the identities that correspond to $j \geq 2$. For the two identities together imply that:

$$lim_{R\to\infty} \sum_{|z+4\pi k\sqrt{z}+4\pi^2k^2|<R} 1 \equiv 0,$$

which is clearly false.

(4) Our last example is a straightforward application of Theorem 16.
Let $f(w) = \frac{1}{2}(\cos w^{1/4} + \cosh w^{1/4})$. Using the identities: $\cos w = \frac{1}{2}(e^{iw} + e^{-iw})$ and $\cosh w = \frac{1}{2}(e^w + e^{-w})$, we obtain the following power series representation of $f(w)$:

$$f(w) = \sum_{k=0}^{\infty} \frac{w^k}{(4k)!}.$$

This shows that $f(w)$ is an entire function. Next, by the triangle inequality we obtain:

$$|f(w)| \leq \frac{1}{2}\left(\frac{1}{2}\left(|e^{iw^{1/4}}| + |e^{-iw^{1/4}}|\right) + \frac{1}{2}\left(|e^{w^{1/4}}| + |e^{-w^{1/4}}|\right)\right) \leq$$

$$\leq \frac{1}{4}\left(e^{|w|^{1/4}} + e^{|w|^{1/4}} + e^{|w|^{1/4}} + e^{|w|^{1/4}}\right) = e^{|w|^{1/4}}.$$

Also, for $r > 0$ large enough we have:

$$|f(r)| = \frac{1}{2}\left|\cos r^{1/4} + \frac{e^{r^{1/4}} + e^{-r^{1/4}}}{2}\right| \geq \frac{1}{2}\left|-1 + \frac{1}{2}e^{r^{1/4}}\right| \geq \frac{1}{8}e^{r^{1/4}}.$$

The last two inequalities imply that $\rho = \frac{1}{4}$ for our entire function $f(w)$ and so Theorem 16 applies. Thus if for a fixed z the solutions of the following equation in the unknown w:

$$\cos w^{1/4} + \cosh w^{1/4} = \cos z^{1/4} + \cosh z^{1/4},$$

are given by $\{w\} = \{\phi_{0n}(z)\}_n$, then $\forall\, k \in \mathbb{Z}^+$ we have:

$$\lim_{j\to\infty} \sum_{|\phi_{0n}(z)|<R_j} \sum_{0n}^{(k)}(z) \equiv 0, \quad \forall\, z \in \mathbb{C},$$

for some Wiman's type sequence $0 < R_1 < R_2 < R_3 < \ldots < R_n < \ldots (R_n \to \infty)$. The task of actually computing the automorphic functions $\phi_{0n}(z)$ for this function $f(w)$ is probably not an easy task.

18 The Circular Density of the Orbits of the Automorphic Group Aut(f), for Any $f \in E$ of Order $0 < \rho < \frac{1}{2}$

Theorem 17 *Let $f \in E$ have a positive order ρ, which is smaller than $\frac{1}{2}$, i.e. $0 < \rho < \frac{1}{2}$. Let* $\mathrm{Aut}(f) = \{\phi_{0n}\}_n$. *For any $z \in \mathbb{C}$ we arrange the* $\mathrm{Aut}(f)$*-orbit of z, $Z(f(w) - f(z)) = \{\phi_{0n}(z)\}_n$ in a non-decreasing order of the moduli: $|\phi_{00}(z)| \le |\phi_{01}(z)| \le |\phi_{02}(z)| \le \ldots$ and for any r satisfying $|\phi_{0n}(z)| \le r \le |\phi_{0,n+1}(z)|$ we denote the corresponding index $n = n(r, z)$. If $|\phi_{0n}(z)| = |\phi_{0,n+1}(z)|$ we may denote anything within reason, for example $n(r, z) = n$* or $n + 1$. *Then we have the following asymptotic circular (or radial, if one prefers) density estimate:*

$$\lim_{r_j \to \infty} \Big(\big(n(r_j, z) + 1\big) \log r_j - r_j^{\rho} \cos \pi \rho \Big) = \infty,$$

for some Wiman's sequence $\{r_j\}_j$ and $\forall\, \mathbb{C}$.

Proof We have proved (see Section 11) that if $|\phi_{0n}(z)| \le r \le |\phi_{0,n+1}(z)|$, then by the Jensen's Theorem applied to $f(w) - f(z)$, where as usual $f(0) - f(z) \neq 0$, we have:

$$\left| \prod_{j=0}^{n} \phi_{0j}(z) \right| = r^{n+1} |f(0) - f(z)| \exp \left\{ -\frac{1}{2\pi} \int_0^{2\pi} \log \left| f(re^{i\theta}) - f(z) \right| d\theta \right\}.$$

Using Wiman's $\cos \pi \rho$-Theorem, as in the proof of Theorem 16, we conclude that there exists a sequence $r_1 < r_2 < r_3 < \ldots < r_n < \ldots (r_n \to \infty)$ such that for arbitrary $\epsilon > 0$ and $n > n_\epsilon$ we have:

$$m_f(r_n) > \big(M_f(r_n)\big)^{\cos \pi \rho - \epsilon}.$$

By the triangle inequality we have for any $r_j > 0$ that satisfies $(f(r_j e^{i\theta}) - f(z))(M_f(r_j)^{\cos \pi \rho} - |f(z)|) \neq 0$, the following estimate:

$$\log |f(r_j e^{i\theta}) - f(z)| \ge \log |m_f(r_j) - |f(z)|| \ge \log |M_f(r_j)^{\cos \pi \rho} - |f(z)||.$$

Hence:

$$\exp \left\{ -\frac{1}{2\pi} \int_0^{2\pi} \log \left| f(r_j e^{i\theta}) - f(z) \right| d\theta \right\} \le \exp \left\{ -\log |M_f(r_j)^{\cos \pi \rho} - |f(z)|| \right\}.$$

Since we have:

$$M_f(r_j)^{\cos \pi \rho} - |f(z)| \approx e^{r_j^\rho \cos \pi \rho},$$

for a large enough r_j, we conclude that for some $c = c(z)$, depending only on z, we have for $|\phi_{0,n(r_j,z)}(z)| \le r_j \le |\phi_{0,n(r_j,z)+1}(z)|$, the following estimate:

$$\left| \prod_{k=0}^{n(r_j,z)} \phi_{0k}(z) \right| \le r_j^{n(r_j,z)+1} e^{-r_j^\rho \cos \pi \rho} \cdot c.$$

But $0 < \rho$, so $f(w)$ is transcendental and hence:

$$\lim_{r_j \to \infty} \left| \prod_{k=0}^{n(r_j,z)} \phi_{0k}(z) \right| = +\infty.$$

Thus:

$$\lim_{r_j \to \infty} r_j^{n(r_j,z)+1} e^{-r_j^\rho \cos \pi \rho} = +\infty.$$

Since we have the elementary identity:

$$r_j^{n(r_j,z)+1} e^{-r_j^\rho \cos \pi \rho} = e^{(n(r_j,z)+1) \log r_j - r_j^\rho \cos \pi \rho},$$

it follows that:

$$\lim_{r_j \to \infty} \Big(\big(n(r_j, z) + 1\big) \log r_j - r_j^\rho \cos \pi \rho \Big) = +\infty$$

□

19 The Vieta Formulas for Aut(f), $f \in E$ of Order $0 \le \rho < 1$

For $f \in E$ of low order ρ, i.e. $0 \le \rho < 1$, the formulas for the symmetric functions of the reciprocals of the automorphic functions of f can be derived algebraically as easy as for polynomials. The reason is the fact that for those orders the Weierstrass canonical representations are exactly the factorization of $f(w) - f(z)$, because the Weierstrass factors reduce to the simplest form $(1 - u)$. This follows by the fact that there is no need in the Weierstrass auxiliary exponentials that cause the convergence of the infinite product. In low order, the infinite product converges automatically already at the level of $(1-u)$.

Theorem 18 *Let* $f(w) = \sum_{n=0}^{\infty} a_n w^n \in E$ *be of order* ρ*, where* $0 \le \rho < 1$*. Let the automorphic group of* f *be* $\mathrm{Aut}(f) = \{\phi_{0n}\}_n$*. Then for any* $z \in \mathbb{C} - f^{-1}(f(0))$ *and for any* $n \in \mathbb{Z}^+$ *we have:*

$$a_n = (-1)^n \left(f(0) - f(z)\right) \sum_{0 \le i_1 < i_2 < \ldots < i_n} \left(\prod_{j=1}^{n} \phi_{0 i_j}(z) \right)^{-1}.$$

Proof The assumption of low order, $0 \le \rho < 1$ implies that:

$$f(w) - f(z) = \left(f(0) - f(z)\right) \prod_n \left(1 - \frac{w}{\phi_{0n}(z)}\right) =$$

$$= \left(f(0) - f(z)\right) \sum_{n=0}^{\infty} (-1)^n \sum_{0 \le i_1 < i_2 < \ldots < i_n} \left(\prod_{j=1}^{n} \phi_{0 i_j}(z) \right)^{-1}.$$

□

Example We computed previously, in example 3 after Theorem 16, that for $f(w) = \cos\sqrt{w} \in E$, we have $\rho = \frac{1}{2}$, $\mathrm{Aut}(\cos\sqrt{w}) = \{w + 4\pi k \sqrt{w} + 4\pi^2 k^2 \mid k \in \mathbb{Z}\}$, $\cos\sqrt{w} = \sum_{n=0}^{\infty} (-1)^n \frac{w^n}{(2n)!}$, $a_n = \frac{(-1)^n}{(2n)!}$. The simplest case of Theorem 18 is the case $n = 1$:

$$\frac{-1}{2!} = (-1)^1 \left(\cos\sqrt{0} - \cos\sqrt{z}\right) \left(\frac{1}{z} + \frac{1}{z + 4\pi\sqrt{z} + 4\pi^2} + \frac{1}{z - 4\pi\sqrt{z} + 4\pi^2} + \right.$$

$$\left. + \frac{1}{z + 4\pi \cdot 2\sqrt{z} + 4\pi^2 2^2} + \frac{1}{z - 4\pi \cdot 2\sqrt{z} + 4\pi^2 2^2} + \ldots \right) =$$

$$= -\left(1 - \cos\sqrt{z}\right) \left(\frac{1}{z} + \sum_{k=1}^{\infty} \left(\frac{1}{z + 4\pi k\sqrt{z} + 4\pi^2 k^2} + \frac{1}{z - 4\pi k\sqrt{z} + 4\pi^2 k^2} \right) \right) =$$

$$= -\left(1 - \cos\sqrt{z}\right) \left(\frac{1}{z} + 2 \sum_{k=1}^{\infty} \frac{z + 4\pi^2 k^2}{(z - 4\pi^2 k^2)^2} \right).$$

Thus we obtain the following identity:

$$\frac{1}{2(1 - \cos\sqrt{z})} = \frac{1}{z} + 2 \sum_{k=1}^{\infty} \frac{z + 4\pi^2 k^2}{(z - 4\pi^2 k^2)^2}.$$

For instance, by substituting $z = \pi^2$ we obtain:

$$\frac{1}{4} = \frac{1}{\pi^2} + 2\sum_{k=1}^{\infty} \frac{\pi^2 + 4\pi^2 k^2}{(\pi^2 - 4\pi^2 k^2)^2},$$

$$\frac{\pi^2}{4} = 1 + 2\sum_{k=1}^{\infty} \frac{1 + 4k^2}{(1 - 4k^2)^2}.$$

20 Embedding the Automorphic Group Within a Larger Group

Let $f \in E$. If $g \in E$ then $\mathrm{Aut}(f) \subseteq \mathrm{Aut}(g \circ f)$. For $\phi \in \mathrm{Aut}(f) \Rightarrow f \circ \phi = f$, on the domain of definition of ϕ. Hence $g \circ (f \circ \phi) = g \circ f$, on the domain of definition of ϕ. Thus $(g \circ f) \circ \phi = g \circ f$, on the domain of definition of ϕ. This implies that $\phi \in \mathrm{Aut}(g \circ f)$. Using this observation we deduce that if $\{g_n\}_n$ is a sequence of elements in E, then it induces the following ascending sequence of discrete groups:

$$\mathrm{Aut}(f) \subseteq \mathrm{Aut}(g_1 \circ f) \subseteq \mathrm{Aut}(g_2 \circ g_1 \circ f) \subseteq \ldots.$$

Definition 7 Let (X_j, τ_j) be a sequence of topological spaces such that for any two indices i, j we have $X_i \cap \tau_j \subseteq \tau_i$. This means that for any $V \in \tau_j$ we have $X_i \cap V \in \tau_i$. We will define the direct limit topological space $\varinjlim(X_j, \tau_j) = (X, \tau)$ in this particular setting by:

$$\begin{cases} X = \bigcup_j X_j \\ \tau \ = \{U \subseteq X \mid \forall\, j,\ U \cap X_j \in \tau_j\} \end{cases}. \tag{15}$$

Remark 23 The definition above is the standard definition of the final topology on the set $X = \bigcup_j X_j$ with respect to the family of the inclusion mappings: $f_j : X_j \to X$, $f_j(x) = x$. Explicitly, a subset $U \subseteq X$ is open in the final topology if and only if $\forall\, j,\ f_j^{-1}(U)$ is open in (X_j, τ_j). For in our case we have $\forall\, U \subseteq X$ and for any index j, $f_j^{-1}(U) = U \cap X_j$. Thus a set $U \subseteq X$ is open in the final topology on X if and only if $\forall\, j,\ U \cap X_j \in \tau_j$. This is exactly the definition of the topology τ in equation (15). We note that $(\bigcup_j X_j, \tau)$ is a discrete topological space if and only if $\forall\, j,\ (X_j, \tau_j)$ is a discrete topological space. For $\forall\, x \in X$ we have by the definition of τ: $\{x\} \in \tau$ if and only if $\forall\, j,\ \{x\} \cap X_j \in \tau_j$. Clearly we have:

$$\{x\} \cap X_j = \begin{cases} \emptyset & \text{if } x \notin X_j \\ \{x\} & \text{if } x \in X_j \end{cases}.$$

The topological groups $\mathrm{Aut}(g_n \circ \ldots \circ g_1 \circ f)$ are discrete $\forall\, n \in \mathbb{Z}^+ \cup \{0\}$. Hence if we use the topology of Definition 7, the topological group

$$\bigcup_{n=0}^{\infty} \mathrm{Aut}(g_n \circ \ldots \circ g_1 \circ f)$$

is a discrete topological group. It will be useful to know when is it true that:

$$\bigcup_{n=0}^{\infty} \mathrm{Aut}(g_n \circ \ldots \circ g_1 \circ f) = \mathrm{Aut}(F),$$

for some $F \in E$? Let us assume that $\lim_{n\to\infty} g_n \circ \ldots \circ g_1 \circ f = F$ uniformly on compact subsets of $\mathbb{C}$. Will the answer to the question above be affirmative under this assumption? Let $\phi \in \bigcup_{n=0}^{\infty} \mathrm{Aut}(g_n \circ \ldots \circ g_1 \circ f)$. Then for some $N \in \mathbb{Z}^+ \cup \{0\}$ we have $\phi \in \mathrm{Aut}(g_N \circ \ldots \circ g_1 \circ f)$. By the ascending property mentioned above (prior to Definition 7) this implies that $\phi \in \mathrm{Aut}(g_k \circ \ldots \circ g_1 \circ f)$, $\forall\, k \geq N$. This is equivalent to $(g_k \circ \ldots \circ g_1 \circ f) \circ \phi = (g_k \circ \ldots \circ g_1 \circ f)$ on the domain of definition of ϕ, $\forall\, k \geq N$. Hence:

$$\lim_{k\to\infty} ((g_k \circ \ldots \circ g_1 \circ f) \circ \phi) = \lim_{k\to\infty} (g_k \circ \ldots \circ g_1 \circ f) = F,$$

uniformly on compact subsets of $\mathbb{C}$. But:

$$\lim_{k\to\infty} ((g_k \circ \ldots \circ g_1 \circ f) \circ \phi) = \left(\lim_{k\to\infty} (g_k \circ \ldots \circ g_1 \circ f)\right) \circ \phi = F \circ \phi,$$

uniformly on compact subsets of $\mathbb{C}$. Hence $F \circ \phi = F$ on the domain of definition of ϕ. Hence $\phi \in \mathrm{Aut}(F)$. We proved the following:

Proposition 8 *If* $\lim_{n\to\infty}(g_n \circ \ldots \circ g_1 \circ f) = F$ *uniformly on compact subsets of* $\mathbb{C}$*, then*

$$\bigcup_{n=0}^{\infty} \mathrm{Aut}(g_n \circ \ldots \circ g_1 \circ f) \subseteq \mathrm{Aut}(F).$$

Remark 24 We note that by a variant of Hurwitz Theorem, the limit function F in Proposition 8 is either in E or $F \equiv$ Const. in which case it makes sense to define $\mathrm{Aut}(F)$ as the set of all functions ϕ. For in that case $F \circ \phi = (\mathrm{Const.}) \circ \phi = \mathrm{Const.} = F$ for any ϕ, on its domain of definition. Thus in this case (when $F \equiv$ Const.) Proposition 8 is clearly true.

When trying to find if equality sign can hold in Proposition 8, we clearly must assume that $F \in E$, for there can be no equality if $F \equiv$ Const.. So we may assume that $\lim_{n\to\infty}(g_n \circ \ldots \circ g_1 \circ f) = F \in E$ uniformly on compact subsets of $\mathbb{C}$. Let

$\psi \in \mathrm{Aut}(F)$. Then $F \circ \psi = F$ on the domain of definition of ψ. Thus $(\lim_{n\to\infty}(g_n \circ \ldots \circ g_1 \circ f)) \circ \psi = \lim_{n\to\infty}(g_n \circ \ldots \circ g_1 \circ f)$. By the continuity argument:

$$\lim_{n\to\infty}((g_n \circ \ldots \circ g_1 \circ f) \circ \psi) = \lim_{n\to\infty}(g_n \circ \ldots \circ g_1 \circ f).$$

If $\forall k \in \mathbb{Z}^+$, $\psi \notin \mathrm{Aut}(g_k \circ \ldots \circ g_1 \circ f)$, then $(g_k \circ \ldots \circ g_1 \circ f) \circ \psi$ cannot be extended to become an entire function. At least there seems to be no reason for such an extension to exist, because ψ may not be entire. However, ψ is analytic on its domain of definition because it is a branch of $F^{-1}(F(w))$. So we have a sequence of functions $\{(g_n \circ \ldots \circ g_1 \circ f) \circ \psi\}n$, analytic on (at least) a fixed open subset of $\mathbb{C}$ (the domain of the definition of ψ). The complement of the domain of definition of ψ is a closed subset of $\mathbb{C}$ that contains no continuum (by a theorem of Julia). In that sense the open set is large. This sequence of analytic functions converges to a function F which can be extended to an entire function. Thus the limit process $\lim_{n\to\infty}((g_n \circ \ldots \circ g_1 \circ f) \circ \psi)$ desingularizes the full set of singularities that originated in the automorphic function $\psi \in \mathrm{Aut}(F)$. By results of Shimizu those singularities are branch points and cannot include poles or algebraic poles. Thus at each singular point the function $((g_n \circ \ldots \circ g_1 \circ f) \circ \psi)$ is many valued. If there is a corresponding Hurwitz principle that holds true, then the limit function is either multivalued at such a singular point, or a constant. We recall that the convergence is uniform on compact subsets of the domain of the definition of ψ. Hence $\psi \notin \mathrm{Aut}(g_k \circ \ldots \circ g_1 \circ f)$ $\forall k \in \mathbb{Z}^+$, but $\exists n_0$ such that for $n > n_0$, the function $(g_n \circ \ldots \circ g_1 \circ f) \circ \psi$ has no singularity and is analytic at a fixed singular point (a branch point) of ψ. So if the number of singular points of ψ is finite, then $\exists n_1$ such that for $n > n_1$, the function $(g_n \circ \ldots \circ g_1 \circ f) \circ \psi$ is an entire function. So we have a sequence of entire functions $\{(g_n \circ \ldots \circ g_1 \circ f) \circ \psi\}_{n>n_1}$ that converges uniformly on compact subsets of $\mathbb{C}$ to the entire function F. This gives a sequence of non-zero entire functions $\{((g_n \circ \ldots \circ g_1 \circ f) \circ \psi) - (g_n \circ \ldots \circ g_1 \circ f)\}_{n>n_1}$ that converges uniformly on compact subsets of $\mathbb{C}$ to the zero function $F \circ \psi - F \equiv 0$. This, of course, is a valid possibility by the Hurwitz Theorem. We recall the following:

Lemma ([3]) $\forall\, f, g \in E,\ \ \mathrm{Aut}(f) \subseteq \mathrm{Aut}(g) \Leftrightarrow \exists\, h \in E$ such that $g = h \circ f$.

We conclude that if $\bigcup_{n=0}^{\infty} \mathrm{Aut}(g_n \circ \ldots \circ g_1 \circ f) \subseteq \mathrm{Aut}(F)$ for some $F \in E$, then $\forall n \in \mathbb{Z}^+\ \exists\, F_n \in E$ such that $F = f_n \circ (g_n \circ \ldots \circ g_1 \circ f)$. By Proposition 8, this will be the case when $\lim_{n\to\infty}(g_n \circ \ldots \circ g_1 \circ f) = F$ uniformly on compact subsets of $\mathbb{C}$. Hence in this case we have $\lim_{n\to\infty} F_n = \mathrm{id.}$ uniformly on compact subsets of $\mathbb{C}$.

Remark 25 If $F_n \in E$ and $\lim_{n\to\infty} F_n = z$ uniformly on compact subsets of $\mathbb{C}$, then $\mathrm{Aut}(F_n) = \{z\} \cup \{\phi_{0k}^{(n)}\}_k$ where $\forall k$, $\lim_{n\to\infty} \phi_{0k}^{(n)} = \infty$. In that sense $\mathrm{Aut}(F_n) \to_{n\to\infty} \{z\}$.

We recall the following:

Theorem ([17]) *There exists a sequence of positive real numbers $\{c_n\}_{n=1}^{\infty}$ such that the sequence of functions $F_n(z) = (c_n e^z + z) \circ \ldots \circ (c_1 e^z + z)$ converges uniformly* *on compact subsets of* $\mathbb{C}$ *to an entire function* $F(z)$. *Furthermore, for each* $n \in \mathbb{Z}^+$, $F(z) = H_n \circ (c_n e^z + z) \circ \ldots \circ (c_1 e^z + z)$ *for some entire function* H_n. *Hence, there is no uniform bound on the number of prime factors* $c_n e^z + z$ *in different decompositions of* F *through transcendental entire functions.*

Remark 26 A similar result holds for factorization that go in the opposite direction, i.e. $(c_1 e^z + z) \circ \ldots \circ (c_n e^z + z)$.

As for the Riemann surface of the inverse functions that are the limits of factorizations of non-bounded number of factors, $F^{-1} = (c_1 e^z + z)^{-1} \circ (c_2 e^z + z)^{-1} \circ \ldots$. It contains the embedded copies of the Riemann surfaces of the factors nested one on the top of the other. One can outline the geometric construction of the Riemann surface of that is induced by $g_n \circ \ldots \circ g_1 \circ f$. if $\Gamma = \bigcup_{n=0}^{\infty} \mathrm{Aut}(g_n \circ \ldots \circ g_1 \circ f) = \mathrm{Aut}(F)$ for some $F \in E$, then the direct limit of those Riemann surfaces will be equal to the Riemann surface of F^{-1}. If, however, the discrete group $Gamma$ does not equal to an $\mathrm{Aut}(F)$ for some $F \in E$, then this direct limit of Riemann surfaces will not be a Riemann surface. This structure generalizes the Riemann surfaces.

21 Relations Between the Construction of the Direct System of the Automorphic Groups and Weierstrass Products

We recall that for $f \in E$, the elements of the automorphic group $\mathrm{Aut}(f)$ are the functions of $f^{-1} \circ f$. Let $g_1 \in E$ then $\mathrm{Aut}(f) \subseteq \mathrm{Aut}(g_1 \circ f)$. In fact the elements of $\mathrm{Aut}(g_1 \circ f)$ are the functions of $(g_1 \circ f)^{-1} \circ (g_1 \circ f) = f^{-1} \circ (g_1^{-1} \circ g_1) \circ f = f^{-1} \circ \mathrm{Aut}(g_1) \circ f$. We note that by taking the identity element id in $\mathrm{Aut}(g_1)$ we get $f^{-1} \circ \mathrm{id} \circ f = \mathrm{Aut}(f)$ which explains the relation $\mathrm{Aut}(f) \subseteq \mathrm{Aut}(g_1 \circ f)$. Both $f(w) - f(z)$ and $(g_1 \circ f)(w) - (g_1 \circ f)(z)$ of the variable w, with the parameter $z \in \mathbb{C} - f^{-1}(f(0))$ in the first and $z \in \mathbb{C} - (g_1 \circ f)^{-1}((g_1 \circ f)(0))$ in the second, have Weierstrass products representation that are based on the product:

$$\prod_{\phi_{0n} \in \mathrm{Aut}(f)} \left(1 - \frac{w}{\phi_{0n}(z)}\right),$$

for $f(w) - f(z)$ and on the product:

$$\prod_{\psi_{0n} \in \mathrm{Aut}(g_1 \circ f)} \left(1 - \frac{w}{\psi_{0n}(z)}\right),$$

for $(g_1 \circ f)(w) - (g_1 \circ f)(z)$. Since $\mathrm{Aut}(f) \subseteq \mathrm{Aut}(g_1 \circ f)$, any factor of the first product is also a factor of the second product. In that sense the first product divides the second one. We will denote that by standard notation:

$$\prod_{\phi_{0n}\in \mathrm{Aut}(f)}\left(1-\frac{w}{\phi_{0n}(z)}\right)\Bigg| \prod_{\psi_{0n}\in \mathrm{Aut}(g_1\circ f)}\left(1-\frac{w}{\psi_{0n}(z)}\right).$$

If we actually divide the full detailed second product by the full detailed first product, we obtain a Weierstrass product type representation for the meromorphic function of w,

$$\frac{(g_1\circ f)(w)-(g_1\circ f)(z)}{f(w)-f(z)}.$$

We will denote that by:

$$\frac{\prod_{\phi_{0n}\in \mathrm{Aut}(f)}\left(1-\frac{w}{\phi_{0n}(z)}\right)}{\prod_{\psi_{0n}\in \mathrm{Aut}(g_1\circ f)}\left(1-\frac{w}{\psi_{0n}(z)}\right)}\sim\frac{(g_1\circ f)(w)-(g_1\circ f)(z)}{f(w)-f(z)}.$$

We note that the meromorphic function of w, $\frac{(g_1\circ f)(w)-(g_1\circ f)(z)}{f(w)-f(z)}$, is in fact an entire function of w, because it has only removable singularities and not poles in the finite plane. Symbolically we have the following assignment:

$$f(w)-f(z)\to\prod_{\mathrm{Aut}(f)},\quad g_1(f(w))-g_1(f(z))\to\prod_{\mathrm{Aut}(g_1\circ f)},$$

$$\left\{\frac{g_1(f(w))-g_1(f(z))}{f(w)-f(z)}\right\}\to\prod_{\mathrm{Aut}(g_1\circ f)-\mathrm{Aut}(f)}.$$

Similarly we can go on:

$$g_2(g_1(f(w)))-g_2(g_1(f(z)))\to\prod_{\mathrm{Aut}(g_2\circ g_1\circ f)},$$

$$\left\{\frac{g_2(g_1(f(w)))-g_2(g_1(f(z)))}{g_1(f(w))-g_1(f(z))}\right\}\prod_{\mathrm{Aut}(g_2\circ g_1\circ f)-\mathrm{Aut}(g_1\circ f)},$$

$$\left\{\frac{g_2(g_1(f(w)))-g_2(g_1(f(z)))}{f(w)-f(z)}\right\}\prod_{\mathrm{Aut}(g_2\circ g_1\circ f)-\mathrm{Aut}(f)}.$$

We note the consistency:

$$\left\{\frac{g_2(g_1(f(w)))-g_2(g_1(f(z)))}{f(w)-f(z)}\right\}=\left\{\frac{g_2(g_1(f(w)))-g_2(g_1(f(z)))}{g_1(f(w))-g_1(f(z))}\right\}$$

$$\times \left\{ \frac{g_1(f(w)) - g_1(f(z))}{f(w) - f(z)} \right\} \rightarrow$$

$$\prod_{\mathrm{Aut}(g_2 \circ g_1 \circ f) - \mathrm{Aut}(f)} = \prod_{\mathrm{Aut}(g_2 \circ g_1 \circ f) - \mathrm{Aut}(g_1 \circ f)} \times \prod_{\mathrm{Aut}(g_1 \circ f) - \mathrm{Aut}(f)} .$$

If we denote union by plus: $+$, then it corresponds to multiplication. This is in agreement with the fact that minus: $-$ corresponded to division. In this notation we have:

$$\mathrm{Aut}(g_2 \circ g_1 \circ f) - \mathrm{Aut}(f) = (\mathrm{Aut}(g_2 \circ g_1 \circ f) - \mathrm{Aut}(g_1 \circ f)) + (\mathrm{Aut}(g_1 \circ f) - \mathrm{Aut}(f).$$

It is clear that in general we have:

Proposition 9 *If* $g_n, f \in E, \ \forall n \in \mathbb{Z}^+$, *then:*

$$(g_n \circ \ldots \circ g_1 \circ f)(w) - (g_n \circ \ldots \circ g_1 \circ f)(z) \rightarrow \mathrm{Aut}(g_n \circ \ldots \circ g_1 \circ f),$$

and $\forall n > m \geq 1$ *in* $\mathbb{Z}^+$*:*

$$\left\{ \frac{(g_n \circ \ldots \circ g_1 \circ f)(w) - (g_n \circ \ldots \circ g_1 \circ f)(z)}{(g_m \circ \ldots \circ g_1 \circ f)(w) - (g_m \circ \ldots \circ g_1 \circ f)(z)} \right\} \rightarrow \mathrm{Aut}(g_n \circ \ldots \circ g_1 \circ f)$$
$$- \mathrm{Aut}(g_m \circ \ldots \circ g_1 \circ f),$$

another suggestive assignment which is natural, is the exponential and the logarithmic notations:

$$\begin{cases} \log\left((g_n \circ \ldots \circ g_1 \circ f)(w) - (g_n \circ \ldots \circ g_1 \circ f)(z)\right) \rightarrow \mathrm{Aut}(g_n \circ \ldots \circ g_1 \circ f) \\ \exp\left(\mathrm{Aut}(g_n \circ \ldots \circ g_1 \circ f)\right) \rightarrow (g_n \circ \ldots \circ g_1 \circ f)(w) - (g_n \circ \ldots \circ g_1 \circ f)(z) \end{cases}$$

Next we note that the discrete group $\Gamma = \bigcup \mathrm{Aut}(g_n \circ \ldots \circ g_1 \circ f)$ can also be denoted by $\Gamma = \sum_{n=0}^{\infty} \mathrm{Aut}(g_n \circ \ldots \circ g_1 \circ f)$ and formally it should be assigned to the Weierstrass type product:

$$\prod_{\theta_{0n} \in \Gamma} \left(1 - \frac{w}{\theta_{0n}(z)}\right).$$

On the other hand, $\forall n \in \mathbb{Z}^+$ we have:

$$\mathrm{Aut}(g_n \circ \ldots \circ g_1 \circ f) = \sum_{k=1}^{n-1} (\mathrm{Aut}(g_{k+1} \circ \ldots \circ g_1 \circ f) - \mathrm{Aut}(g_k \circ \ldots \circ g_1 \circ f)) +$$
$$+ (\mathrm{Aut}(g_1 \circ f) - \mathrm{Aut}(f)) + \mathrm{Aut}(f).$$

That corresponds to:

$$(g_n \circ \ldots \circ g_1 \circ f)(w) - (g_n \circ \ldots \circ g_1 \circ f)(z) =$$

$$= \prod_{k=1}^{n-1} \left\{ \frac{(g_{k+1} \circ \ldots \circ g_1 \circ f)(w) - (g_{k+1} \circ \ldots \circ g_1 \circ f)(z)}{(g_k \circ \ldots \circ g_1 \circ f)(w) - (g_k \circ \ldots \circ g_1 \circ f)(z)} \right\} \times$$

$$\times \left\{ \frac{(g_1 \circ f)(w) - (g_1 \circ f)(z)}{f(w) - f(z)} \right\} \times (f(w) - f(z)).$$

If indeed $\Gamma = \mathrm{Aut}(F)$ for some $F \in E$, then:

$$F(w) - F(z) \to \prod_{\mathrm{Aut}(F)} = \prod_{\Gamma} \to$$

$$\to \prod_{k=1}^{n-1} \left\{ \frac{(g_{k+1} \circ \ldots \circ g_1 \circ f)(w) - (g_{k+1} \circ \ldots \circ g_1 \circ f)(z)}{(g_k \circ \ldots \circ g_1 \circ f)(w) - (g_k \circ \ldots \circ g_1 \circ f)(z)} \right\} \times$$

$$\times \left\{ \frac{(g_1 \circ f)(w) - (g_1 \circ f)(z)}{f(w) - f(z)} \right\} \times (f(w) - f(z)) =$$

$$= \lim_{n \to \infty} ((g_n \circ \ldots \circ g_1 \circ f)(w) - (g_n \circ \ldots \circ g_1 \circ f)(z)).$$

This implies:

Theorem 19 *Let $g_n, f \in E \;\; \forall n \in \mathbb{Z}^+$. Then there exists an $F \in E$ such that:*

$$\bigcup_{n=0}^{\infty} \mathrm{Aut}(g_n \circ \ldots \circ g_1 \circ f) = \mathrm{Aut}(F),$$

if and only if $G = \lim_{n \to \infty}(g_n \circ \ldots \circ g_1 \circ f)$ exists uniformly on compact subsets of $\mathbb{C}$ and is not a constant. In the case the limit $G \not\equiv$ Const. then $G \in E$ and we can take $F(w) = a \cdot G(w) + b \; \forall a \in \mathbb{C}^{\times}$ and $\forall b \in \mathbb{C}$.

Example If $F(z) = \lim_{n \to \infty}(c_n e^z + z) \circ \ldots \circ (c_1 e^z + z)$ is a Tuen Wai NG entire function, then we have the identity:

$$\mathrm{Aut}(F) = \bigcup_{n=1}^{\infty} \mathrm{Aut}((c_n e^z + z) \circ \ldots \circ (c_1 e^z + z)),$$

where the entire functions $c_k e^z + z$ $(c_k > 0)$ are primes. We also have the functional identity:

$$F(w) - F(z) =$$

$$= \prod_{n=1}^{\infty} \left\{ \frac{((c_{n+1}e^w + w) \circ \ldots \circ (c_1 e^w + w)) - ((c_{n+1}e^z + z) \circ \ldots \circ (c_1 e^z + z))}{((c_n e^w + w) \circ \ldots \circ (c_1 e^w + w)) - ((c_n e^z + z) \circ \ldots \circ (c_1 e^z + z))} \right\} \times$$

$$\times ((c_1 e^w + w) - (c_1 e^z + z)).$$

Theorem 20 *Let $g_n, f \in E \ \forall n \in \mathbb{Z}^+$. If the limit $G = \lim_{n\to\infty}(g_n \circ \ldots \circ g_1 \circ f)$ exists uniformly on compact subsets of $\mathbb{C}$ and is not a constant, then there exists a path metric $\rho : \mathbb{C} \times \mathbb{C} \to \mathbb{R}_{\geq 0}$ which is invariant for $\mathrm{Aut}(g_n \circ \ldots \circ g_1 \circ f)$, $\forall n = 0, 1, 2, \ldots$. In other words, $\forall n \geq 0$, $\forall \phi \in \mathrm{Aut}(g_n \circ \ldots \circ g_1 \circ f)$, ϕ is a ρ-isometry on a domain of definition of a leaf of ϕ, i.e. $\forall z, w$ we have: $\rho(\phi(z), \phi(w)) = \rho(z, w)$.*

Proof We have by the assumption on $\lim_{n\to\infty}(g_n \circ \ldots \circ g_1 \circ f)$ the containment:

$$\bigcup_{n=0}^{\infty} \mathrm{Aut}(g_n \circ \ldots \circ g_1 \circ f) \subseteq \mathrm{Aut}(G).$$

By the way, we do not need Theorem 19 for this. Now take the path metric on $\mathbb{C}$ induced by G, $\rho = \rho_G : \mathbb{C} \times \mathbb{C} \to \mathbb{R}_{\geq 0}$. We know that any G-automorphic function $\phi \in \mathrm{Aut}(G)$ is a ρ_G-isometry in the sense of the theorem. See Theorem 15. □

Theorem 21 *Let $h_n, g_n, f \in E \ \forall n \in \mathbb{Z}^+$. If $\lim_{n\to\infty}(g_n \circ \ldots \circ g_1 \circ f) = \lim_{n\to\infty}(h_n \circ \ldots \circ h_1 \circ f)$ exist uniformly on compact subsets of $\mathbb{C}$ and the limit function is not a constant, then:*

$$\bigcup_{n=0}^{\infty} \mathrm{Aut}(g_n \circ \ldots \circ g_1 \circ f) = \bigcup_{n=0}^{\infty} \mathrm{Aut}(h_n \circ \ldots \circ h_1 \circ f).$$

Proof Let G be the limit function of the two sequences $\{g_n \circ \ldots \circ g_1 \circ f\}_n$ and $\{h_n \circ \ldots \circ h_1 \circ f\}_n$ of functions in E. Then $G \in E$ and by Theorem 19 the unions of the automorphic groups, both, are equal to $\mathrm{Aut}(G)$. □

So far our construction gives under the appropriate conditions the identity:

$$\mathrm{Aut}(F) = \bigcup_{n=0}^{\infty} \mathrm{Aut}(g_n \circ \ldots \circ g_1 \circ f),$$

where the functions $F, g_n, f \in E$, $\forall n \in \mathbb{Z}_{\geq 0}$. Hence all the automorphic groups that are involved are discrete groups and are countable. Trivially any discrete group is a locally compact Hausdorff group. The compact subsets in a discrete group are the finite subsets, and the Haar measure up to a multiplication by a positive constant is the counting measure.

Definition 8 If H is a subgroup of the topological group G, then it induces two relations on G:
(a) The H-right relation: $\gamma_1 \sim_{H-\text{right}} \gamma_2 \Leftrightarrow \exists\, \delta \in H$ such that $\gamma_1 = \gamma_2 \cdot \delta$.
(b) The H-left relation: $\gamma_1 \sim_{H-\text{left}} \gamma_2 \Leftrightarrow \exists\, \delta \in H$ such that $\gamma_1 = \delta \cdot \gamma_2$.

Proposition 10 *Let H be a subgroup of the topological group G, then both H-right and H-left are equivalence relations on G.*

Proof This is straightforward from Definition 8. $\square$

Definition 9 We will denote the equivalence classes of G with respect to the equivalence relation H-right by $(G/H)_{\text{right}}$. Similarly $(G/H)_{\text{left}}$ will denote the family of equivalence classes with respect to H-left.

Theorem 22 *Let $h,\ f \in E$. Then:*

(a) $\forall\, \gamma_1, \gamma_2 \in \mathrm{Aut}(h \circ f)$ *we have* $\gamma_1 \sim_{\mathrm{Aut}(f)-\text{left}} \gamma_2 \Leftrightarrow f(\gamma_1) = f(\gamma_2)$.
(b) $\forall\, \gamma_1, \gamma_2 \in \mathrm{Aut}(h \circ f)$ *we have* $\gamma_1 \sim_{\mathrm{Aut}(f)-\text{right}} \gamma_2 \Leftrightarrow f(\gamma_1^{-1}) = f(\gamma_2^{-1})$.

Proof

(a) $\gamma_1 \sim_{\mathrm{Aut}(f)-\text{left}} \gamma_2 \Leftrightarrow \gamma_2 = \psi \circ \gamma_1$ for some $\psi \in \mathrm{Aut}(f) \Leftrightarrow f(\gamma_2) = f(\psi \circ \gamma_1) = (f \circ \psi) \circ \gamma_1 = f(\gamma_1)$.
(b) $\gamma_1 \sim_{\mathrm{Aut}(f)-\text{right}} \gamma_2 \Leftrightarrow \gamma_2 = \gamma_1 \circ \psi$ for some $\psi \in \mathrm{Aut}(f) \Leftrightarrow \gamma_2^{-1} = \psi^{-1} \circ \gamma_1^{-1} \Leftrightarrow \gamma_1^{-1} \sim_{\mathrm{Aut}(f)-\text{left}} \gamma_2^{-1} \Leftrightarrow f(\gamma_1^{-1}) = f(\gamma_2^{-2})$ where in the last step we made a use in (a).

$\square$

Theorem 23 *The cardinality of the equivalence classes in* $(\mathrm{Aut}(h \circ f)/\mathrm{Aut}(f))_{\text{left}}$ *and in* $(\mathrm{Aut}(h \circ f)/\mathrm{Aut}(f))_{\text{right}}$ *is equal to the cardinality of* $\mathrm{Aut}(f)$*, and hence are at most* $\aleph_0$.

Proof By Definition 8 it follows that for any $[\gamma] \in (\mathrm{Aut}(h \circ f)/\mathrm{Aut}(f))_{\text{left}}$ we have: $[\gamma] = \{\psi \circ \gamma \mid \psi \in \mathrm{Aut}(f)\}$. Since $\psi \circ \gamma = \psi_1 \circ \gamma \Leftrightarrow \psi = \psi_1$ it follows that the mapping:

$$[\gamma] \to \mathrm{Aut}(f), \quad \psi \circ \gamma \to \psi,$$

is a bijection. Hence $|[\gamma]| = |\mathrm{Aut}(f)|$. A similar argument works for the $\mathrm{Aut}(f)$-right equivalence relation. $\square$

Remark 27 If $f(z) \in E$ is a transcendental entire function, then the equivalence classes in both left and right $\mathrm{Aut}(f)$ equivalence relations have cardinality $\aleph_0$.

Theorem 24 *If $h,\ f \in E$, then both topological spaces* $(\mathrm{Aut}(h \circ f)/\mathrm{Aut}(f))_{\text{left}}$, $(\mathrm{Aut}(h \circ f)/\mathrm{Aut}(f))_{\text{right}}$ *are discrete and Hausdorff.*

Proof This follows by the following well-known facts:
If H is a subgroup of G, then (G/H) is discrete if and only if H is open in G.
(G/H) is Hausdorff if and only if H is closed in G.

In our case $\mathrm{Aut}(h \circ f)$ is a discrete group and hence $\mathrm{Aut}(f)$ is both open and closed in $\mathrm{Aut}(h \circ f)$. □

22 Continuity Properties of the Automorphic Groups

In this section we study the following: Let $f \in E$ and let $f_n \in E\ \forall n \in \mathbb{Z}^+$. Suppose that $\lim_{n\to\infty} f_n = f$ uniformly on compact subsets of $\mathbb{C}$. Is it true that the automorphic groups $\mathrm{Aut}(f_n)$ become closer to the automorphic group $\mathrm{Aut}(f)$? If the answer is affirmative, in what sense?

Clearly an attractive situation is the one in which $f_n(z) \in \mathbb{C}[z]$, i.e. the approximating sequence is a sequence of polynomials. For example, the partial sums of the power series expansion of f:

$$f_n(z) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}, \quad n \in \mathbb{Z}^+.$$

Since the functions that constitute $\mathrm{Aut}(f)$ are $f^{-1} \circ f$, it makes sense to find if in some sense the many valued functions f_n^{-1} approach f^{-1}. We recall once more the following well-known:

Theorem (The Generalized Argument Principle) *Let F be a meromorphic function in the simply connected domain D, a_j the zeros of F, b_k the poles of F in D and γ a closed curve in D avoiding the a_j, b_k. Then $\forall\, G \in C^{\omega}(D)$ we have:*

$$\sum_j G(a_j) \cdot n(\gamma, a_j) - \sum_k G(b_k) \cdot n(\gamma, b_k) = \frac{1}{2\pi i} \oint_\gamma G(z) \cdot \frac{F'(z)}{F(z)} dz.$$

Here we have, for any $a \notin \gamma$:

$$n(\gamma, a) = \frac{1}{2\pi i} \oint_\gamma \frac{dz}{z-a},$$

is the index or the winding number of the closed curve γ with respect to the point $a \notin \gamma$.

Theorem 25 *Let $f \in E$ and let $f_n \in E\ \forall n \in \mathbb{Z}^+$. Suppose that* $\lim_{n\to\infty} f_n = f$ *uniformly on compact subsets of $\mathbb{C}$. Let us denote the automorphic group's elements by* $\mathrm{Aut}(f) = \{\phi_{0k}\}_k$, $\mathrm{Aut}(f_n) = \{\phi_{0k}^{[n]}\}_k$. *Then for any $R > 0$ and for any $\epsilon > 0$, $\exists\, N = N(R, \epsilon)$ such that:*

(1) $\forall n > N(R, \epsilon)$, the number of $\phi_{0k}^{[n]}(z)$ for a fixed z, such that $|\phi_{0k}^{[n]}(z)| < R$ equals (counting with multiplicity) the number of those $\phi_{0k}(z)$ for which $|\phi_{0k}(z)| < R$.

(2) *The indexing of the* ϕ_{0k} *and of the* $\phi_{0k}^{[n]}$*, can be arranged, so that* $\forall\, n > N(R, \epsilon)$ *and* $\forall\, z$ *such that* $|\phi_{0k}(z)| < R$*, we have* $|\phi_{0k}(z) - \phi_{0k}^{[n]}(z)| < \epsilon$.

Proof We recall some elementary facts from the algebra of polynomials in one variable: Let $\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ be a finite sequence of complex numbers and let $\{\{\alpha_1^{[n]}, \alpha_2^{[n]}, \ldots, \alpha_m^{[n]}\}\}_n$ be an infinite sequence of finite sequences over $\mathbb{C}$ of the same length m as the first. We denote the moments by $m_k(\alpha_1, \ldots, \alpha_m) = \alpha_1^k + \ldots + \alpha_m^k$, and similarly $m_k(\alpha_1^{[n]}, \ldots, \alpha_m^{[n]}) = (\alpha_1^{[n]})^k + \ldots + (\alpha_m^{[n]})^k,\ k \in \mathbb{Z}^+$. If $\forall\, k$ we have:

$$\lim_{n\to\infty} m_k(\alpha_1^{[n]}, \ldots, \alpha_m^{[n]}) = m_k(\alpha_1, \ldots, \alpha_m),$$

then the indexing of the α_i and of the $\alpha_i^{[n]}$, $1 \le i \le m$ can be arranged, so that:

$$\lim_{n\to\infty} \alpha_i^{[n]} = \alpha_i, \quad 1 \le i \le m.$$

The reason for this is the following: Let us denote the symmetric functions of the sequence $\{\alpha_1, \ldots, \alpha_m\}$ by

$$S_k(\alpha_1, \ldots, \alpha_m) = \sum_{1\le i_1<\ldots<i_k\le m} \alpha_{i_1}\alpha_{i_2}\ldots\alpha_{i_k}, \ \ 1 \le k \le m.$$

Similarly $S_k(\alpha_1^{[n]}, \ldots, \alpha_m^{[n]})$, $1 \le k \le m$, denote the symmetric functions of the other sequences. Then each moment m_k can be written as a polynomial over $\mathbb{Q}$, with fixed coefficients for a given k, of the symmetric functions and vice versa. These are known as Newton's identities. They start as follows: $m_1 = S_1$, $m_2 = S_2 - S_1^2, \ldots$ and $S_1 = m_1$, $S_2 = m_1^2 - m_2 \ldots$. By the assumption $\lim_{n\to\infty} m_k(\alpha_1^{[n]}, \ldots, \alpha_m^{[n]}) = m_k(\alpha_1, \ldots, \alpha_m)$, it follows that $\lim_{n\to\infty} S_k(\alpha_1^{[n]}, \ldots, \alpha_m^{[n]}) = S_k(\alpha_1, \ldots, \alpha_m)$ for $1 \le k \le m$. We note that for the monic polynomials that have as their zero sets the negatives of these sequences we have:

$$P(w) = (w+\alpha_1)\ldots(w+\alpha_m) = w^m + S_1(\alpha)w^{m-1} + \ldots + S_k(\alpha)w^{m-k} + \ldots + S_m(\alpha),$$

$$P^{[n]}(w) = (w+\alpha_1^{[n]})\ldots(w+\alpha_m^{[n]}) = w^m + S_1(\alpha^{[n]})w^{m-1} + \ldots + S_k(\alpha^{[n]})w^{m-k}$$
$$+ \ldots + S_m(\alpha^{[n]}).$$

Hence $\lim_{n\to\infty} P^{[n]}(w) = P(w)$ uniformly on compact subsets of $\mathbb{C}$. Hence after an appropriate indexing the zeros of the $P^{[n]}$ (multiplicity included) approach as their limits when $n \to \infty$ the zeros of P and we proved the claimed fact.

Coming back to our entire functions, we recall that by using the Weierstrass representation as a canonical product of the function $f(w) - f(z)$ which is entire in w where z is a fixed parameter, and differentiating $\log(f(w) - f(z))$ with respect to w, we obtain:

$$\frac{f'(w)}{f(w)-f(z)} = \frac{\partial g}{\partial w}(w,z) + \sum_k \left(\frac{w}{\phi_{0k}(z)}\right)^{\lambda_k} \left(\frac{1}{w-\phi_{0k}}\right).$$

Hence we have (using the generalized argument principle):

$$\frac{1}{2\pi i}\oint_{|w|=R} \frac{f'(w)dw}{f(w)-f(z)} = \tag{16}$$

$$= \frac{1}{2\pi i}\oint_{|w|=R} \frac{\partial g}{\partial w}(w,z)dw + \sum_k \frac{1}{2\pi i}\oint_{|w|=R} \left(\frac{w}{\phi_{0k}(z)}\right)^{\lambda_k} \left(\frac{dw}{w-\phi_{0k}}\right).$$

We have:

$$\frac{\partial g}{\partial w}(w,z) \in C^{\omega}(\mathbb{C},w) \Rightarrow \frac{1}{2\pi i}\oint_{|w|=R} \frac{\partial g}{\partial w}(w,z)dw = 0,$$

$$\sum_k \frac{1}{2\pi i}\oint_{|w|=R} \left(\frac{w}{\phi_{0k}(z)}\right)^{\lambda_k} \left(\frac{dw}{w-\phi_{0k}}\right) = \sum_{|\phi_{0k}(z)|<R} \frac{1}{(\phi_{0k}(z))^{\lambda_k}} \cdot (\phi_{0k}(z))^{\lambda_k} =$$

$$= |\{k|\,|\phi_{0k}(z)| < R\}|.$$

Next we have:

$$\frac{1}{2\pi i}\oint_{|w|=R} w^l \cdot \frac{f'(w)dw}{f(w)-f(z)} = \tag{17}$$

$$= \frac{1}{2\pi i}\oint_{|w|=R} w^l \frac{\partial g}{\partial w}(w,z)dw + \sum_k \frac{1}{2\pi i}\oint_{|w|=R} w^l \left(\frac{w}{\phi_{0k}(z)}\right)^{\lambda_k} \left(\frac{dw}{w-\phi_{0k}}\right).$$

Once more, by the Cauchy Theorem:

$$\frac{1}{2\pi i}\oint_{|w|=R} w^l \frac{\partial g}{\partial w}(w,z)dw = 0,$$

and by the generalized argument principle:

$$\sum_k \frac{1}{2\pi i}\oint_{|w|=R} w^l \left(\frac{w}{\phi_{0k}(z)}\right)^{\lambda_k} \left(\frac{dw}{w-\phi_{0k}}\right) = \sum_{|\phi_{0k}(z)|<R} \phi_{0k}^l(z).$$

Hence we proved the following integral identity for the moments of the automorphic functions:

$$m_l(\phi_{0k}(z)||\phi_{0k}(z)| < R) = \frac{1}{2\pi i}\oint_{|w|=R} w^l \cdot \frac{f'(w)dw}{f(w)-f(z)}.$$

Similarly we have for any $n \in \mathbb{Z}^+$:

$$m_l(\phi_{0k}^{[n]}(z)||\phi_{0k}^{[n]}(z)| < R) = \frac{1}{2\pi i}\oint_{|w|=R} w^l \cdot \frac{f_n'(w)dw}{f_n(w)-f_n(z)}.$$

By the assumption: $\lim_{n\to\infty} f_n = f$ uniformly on compact subsets of $\mathbb{C}$ and by the Cauchy estimates: $\lim_{n\to\infty} f_n' = f'$ uniformly on compact subsets of $\mathbb{C}$. This implies that:

$$\lim_{n\to\infty}\frac{1}{2\pi i}\oint_{|w|=R} w^l \cdot \frac{f_n'(w)dw}{f_n(w)-f_n(z)} = \frac{1}{2\pi i}\oint_{|w|=R} w^l \cdot \frac{f'(w)dw}{f(w)-f(z)}.$$

We proved:

$$\lim_{n\to\infty} m_l(\phi_{0k}^{[n]}(z)||\phi_{0k}^{[n]}(z)| < R) = m_l(\phi_{0k}(z)||\phi_{0k}(z)| < R).$$

Now the assertions of our theorem follow by the first part of our proof. □

One way to interpret Theorem 25 is that the automorphic functions of the approximating functions f_n to the entire function $f \in E$ converge themselves to the automorphic functions of f. This convergence is very ordered and not chaotic. By that we mean that from a certain index n_0 and on it is unambiguous for certain of the automorphic functions of f which of the automorphic functions of f_n (for n large enough) correspond to them. This happens because when we fix the value z of the complex parameter in $f(w) - f(z)$ and in $f_n(w) - f_n(z)$ and consider the disk $B(0, R)$ and only those automorphic functions ϕ_{0k} of f, $\phi_{0k} \in \mathrm{Aut}(f)$ whose z-image lies inside that disk, i.e. $|\phi_{0k}(z)| < R$ and take in Theorem 25 the positive ϵ, small enough, then for values of the index $n > n_0$ it is clear which of the automorphic functions of f_n is the one that corresponds to a particular ϕ_{0k}. We changed the indices so that $|\phi_{0k}(z) - \phi_{0k}^{[n]}(z)| < \epsilon$. In other words, the values $\phi_{0k}^{[n]}(z)$ for $n > n_0$ (in Theorem 25 we denoted $n_0 = N(R, \epsilon)$) are trapped inside a small circle of a radius ϵ centered at $\phi_{0k}(z)$. The un-ambiguity follows because for a small enough $\epsilon > 0$, the disks $B(\phi_{0k}(z), \epsilon)$ for $|\phi_{0k}(z)| < R$ have disjoint closures. We can achieve this by choosing $\epsilon < \frac{1}{2}\min\{|\phi_{0,k_1}(z) - \phi_{0,k_2}(z)|||\phi_{0,k_1}(z)|, |\phi_{0,k_2}(z)| < R, \phi_{0,k_1}(z) \neq \phi_{0,k_2}(z)\}$. The minimum exists because the set $\{\phi_{0,k}(z)||\phi_{0,}(z)| < R\}$ is a finite set. The number ϵ should also be smaller than $\min\{R - |\phi_{0,k}(z)|||\phi_{0,k}(z)| < R\}$. Every value z of complex parameter determines such a configuration as the one described above. Thus those configurations (that geometrically look like an open disk of radius R punctured by finitely many small disks of radius ϵ that have disjoint closures and that stay away from $\partial B(0, R)$) are determined by three quantities:

$$(z, R, \epsilon) \in \left(\mathbb{C} - f^{-1}(f(0)) \cup \bigcup_n f_n^{-1}(f_n(0))\right) \times \mathbb{R}^+ \times (0, \delta(z, R)),$$

where we have the formula:

$$\delta(z, R) = \min\left\{\frac{1}{2}\min\{R - |\phi_{0,k}(z)| \,|\, |\phi_{0,k}(z)| < R\},\right.$$

$$\left.\frac{1}{2}\min\{|\phi_{0,k_1}(z) - \phi_{0,k_2}(z)| \,|\, |\phi_{0,k_1}(z)|, |\phi_{0,k_2}(z)| < R, \phi_{0,k_1}(z) \neq \phi_{0,k_2}(z)\}\right\}.$$

In the sequel we will be interested in such configurations determined by (z, R, ϵ) for which $R \to +\infty$ and $\epsilon \to 0^+$.

23 Amenability of the Automorphic Group

Let us assume that the sequence $\{f_n\}_n \subseteq E$ satisfies the following:

(a) $f_n \to f \in E$ uniformly on compact subsets of $\mathbb{C}$.
(b) The discrete groups $\mathrm{Aut}(f_n)$ are amenable.

Example For $f(z) = \sum_{j=0}^{\infty} a_j z^j \in E$ we can take $f_n(z) = \sum_{j=0}^{n} a_j z^j \in E$. Then $f_n(z) \in \mathbb{C}[z]$, polynomials, and hence for each n $\mathrm{Aut}(f_n)$ is a finite group (of order $\deg f_n$). Hence $\mathrm{Aut}(f_n)$ are amenable $\forall\, n \in \mathbb{Z}^+$, for which $f_n \in E$.

One might try to construct a Følner sequence in order to prove amenability of $\mathrm{Aut}(f)$, $f \in E$. Let us recall few notions and results.

Definition 10 A discrete group G satisfies the Følner condition if for every finite subset $A \subseteq G$ and every $\epsilon > 0$ there exists a finite nonempty subset $F \subseteq G$ such that $\forall\, a \in A$ we have:

$$\frac{|aF \bigtriangleup F|}{|F|} \leq \epsilon.$$

If G is locally compact we use the same definition but A is a compact subgroup, F is a Borel set with positive Haar measure and we use Haar measure instead of cardinality.

Example All finite (or compact in the locally compact case) groups satisfy the Følner condition, by simply taking $F = G$ ($aF \bigtriangleup F = aG \bigtriangleup G = \emptyset$).

Definition 11 For a discrete and countable (resp. locally compact) group G, a Følner sequence is a sequence $\{F_n\}$ of nonempty finite (resp. compact) subsets of G such that:

$$\frac{|gF_n \bigtriangleup F_n|}{|F_n|} \to_{n\to\infty} 0 \quad \left(\text{resp. } \frac{\mu(gF_n \bigtriangleup F_n)}{\mu(F_n)} \to_{n\to\infty} 0\right) \ \forall g \in G.$$

The following lemma is well-known.

Lemma 4 ([9]) *A group satisfies the Følner condition, if and only if it has a Følner sequence.*

Example The group $\mathbb{Z}$ has a Følner sequence, namely $F_n = \{-n, \ldots, n\}$.

The usefulness of Definition 10 comes from the following well-known theorem.

Theorem 26 ([9]) *A group satisfies the Følner condition, if and only if it is amenable.*

Coming back to our setting where $f \in E$, $f_n \in E$ are polynomials, we fix $z \in \mathbb{C}$. We take a sequence $0 < R_1 < R_2 < \ldots < R_n < \ldots (R_n \to \infty)$, and for each pair (z, R_n) we take an ϵ_n so that $0 < \epsilon_n < \delta(z, R_n)$ and $\epsilon_n \to 0^+$. We take $f_{m(n)}$ such that, using the notations of Theorem 25, $m(n) > N(R_n, \epsilon_n)$. We define a sequence of finite subsets of $\text{Aut}(f)$ by:

$$F_n = \{\phi_{0k} |\, |\phi_{0k}(z)| < R_n\}, \quad n \in \mathbb{Z}^+.$$

We fix an automorphic function $\phi_{0l} \in \text{Aut}(f)$ and we consider:

$$\frac{|\phi_{0l} \circ F_n \bigtriangleup F_n|}{|F_n|}.$$

By the choice $m(n) > N(R_n, \epsilon_n)$ there is (as explained after Theorem 25) a canonical bijection between F_n and $F_n(f_{m(n)}) = \{\phi_{0k}^{[m(n)]} |\, |\phi_{0k}^{[m(n)]}(z)| < R_n\}$. Moreover, if n is large enough, then $\phi_{0l} \in F_n$ and so it is canonically corresponding to $\phi_{0l}^{[m(n)]}$. Hence:

$$\frac{|\phi_{0l} \circ F_n \bigtriangleup F_n|}{|F_n|} = \frac{|\phi_{0l}^{[m(n)]} \circ F_n(f_{m(n)}) \bigtriangleup F_n(f_{m(n)})|}{|F_n(f_{m(n)})|}.$$

By $\phi_{0l} \in F_n$ we clearly have $\phi_{0l}^{[m(n)]} \in F_n(f_{m(n)})$ and in fact when $n \to \infty$, we have $\phi_{0l}^{[m(n)]}(z) \to \phi_{0l}(z)$. Thus $|\phi_{0l}^{[m(n)]}(z)|$ is bounded for $n \to \infty$ and gets closer as we please to $|\phi_{0l}(z)|$.

Theorem 27 *If*

$$\lim_{n\to\infty} \frac{|F_n(f_{m(n)})|}{|\text{Aut}(f_{m(n)})|} = 1,$$

then $\{F_n\}$ *is a Følner sequence and hence* $\mathrm{Aut}(f)$ *is amenable.*

Proof Clearly $\forall n \in \mathbb{Z}^+$ we have $|F_n(f_{m(n)})| \le |\mathrm{Aut}(f_{m(n)})|$ simply because $F_n(f_{m(n)}) \subseteq \mathrm{Aut}(f_{m(n)})$. Let us denote $|F_n(f_{m(n)})| = (1-\epsilon_n)|\mathrm{Aut}(f_{m(n)})|$. Then $0 \le \epsilon_n \le 1$, and by our assumption:

$$1 = \lim_{n\to\infty} \frac{|F_n(f_{m(n)})|}{|\mathrm{Aut}(f_{m(n)})|} = \lim_{n\to\infty}(1-\epsilon_n).$$

Thus $\lim_{n\to\infty}\epsilon_n = 0$. Clearly, we have the following straightforward estimate:

$$0 \le \frac{\phi_{0l}^{[m(n)]} \circ F_n(f_{m(n)}) \bigtriangleup F_n(f_{m(n)})|}{|F_n(f_{m(n)})|} \le \frac{2\epsilon_n}{1-\epsilon_n}.$$

Hence:

$$0 \le \frac{|\phi_{0l} \circ F_n \bigtriangleup F_n|}{|F_n|} \le \frac{2\epsilon_n}{1-\epsilon_n}.$$

This implies that:

$$\lim_{n\to\infty} \frac{|\phi_{0l} \circ F_n \bigtriangleup F_n|}{|F_n|} = 0,$$

and our theorem follows. □

We can give another condition on $f(w)$ that implies that $\mathrm{Aut}(f)$ is amenable. This time it is a geometrical condition. We start with the following:

Definition 12 Let $f \in E$. Suppose that the z-plane is tiled up by a system of fundamental domains $\{\Omega_j\}_j$ of the entire function $w = f(z)$. We say that two fundamental domains Ω_1 and Ω_2 are neighboring if $\Omega_1 \cap \Omega_2 = \emptyset$, $\partial\Omega_1 \cap \partial\Omega_2 \neq \emptyset$.

Definition 13 Let $f \in E$. Suppose that the z-plane is tiled up by a system of fundamental domains $\{\Omega_j\}_j$ of the entire function $w = f(z)$. Let Ω_0 be one of the fundamental domains in the system and let us denote by $G_1(\Omega_0)$ the family of all the neighboring domains of Ω_0. We will sometimes denote the members of $G_1(\Omega_0) = \{\Omega_{1j}\}_j$ and call $G_1(\Omega_0)$ the first generation about Ω_0.

Definition 14 Let $f \in E$. Suppose that the z-plane is tiled up by a system of fundamental domains $\{\Omega_j\}_j$ of the entire function $w = f(z)$. Let Ω_0 be one of the fundamental domains in the system. Let $n \in \mathbb{Z}_{\ge 2}$. The n'th generation about Ω_0 is denoted by $G_n(\Omega_0) = \{\Omega_{nj}\}_j$ and is defined recursively by the following recursive equation:

$$G_n(\Omega_0) = \bigcup_{\Omega\in G_{n-1}(\Omega_0)} G_1(\Omega) - \{\Omega_0\} \cup \bigcup_{j=1}^{n-1} G_j(\Omega_0).$$

The counting function of the generations about Ω_0 is defined by: $g(\Omega_0, n) = |G_n(\Omega_0)|$.

Examples

1) Let $f(z) = z^N$ for some $N \in \mathbb{Z}_{\geq 2}$. Then a natural system of fundamental domains are:

$$\Omega_j = \left\{ z \in \mathbb{C} | \frac{2\pi j}{N} < \arg z < \frac{2\pi(j+1)}{N} \right\}, \quad j = 0, 1, \ldots, N-1.$$

Then $\forall\, j\ G_1(\Omega_j) = \{\Omega_0, \Omega_1, \ldots, \Omega_{N-1}\} - \{\Omega_j\}$, and $G_n(\Omega_j) = \emptyset, \forall\, n > 1$. So:

$$g(\Omega_j, n) = \begin{cases} N-1 & \text{if } n = 1 \\ 0 & \text{if } n > 1 \end{cases}.$$

2) Let $f(z) = e^z$. A natural system of fundamental domains are:

$$\Omega_j = \{ z \in \mathbb{C} | \, 2\pi j < \Im z < 2\pi(j+1) \}, \quad j \in \mathbb{Z}.$$

Here we have: $G_n(\Omega_0) = \{\Omega_{-n}, \Omega_n\}$, and hence $g(\Omega_0, n) = 2$.

Theorem 28 *Let $f \in E$ have a system of fundamental domains. Let Ω_0 be a fundamental domain in the system and let $G_1(\Omega_0) = \{\Omega_{1j}\}_j$. Let denote (as usual) by $\phi_{0j} : \Omega_0 \to \Omega_{1j}$ the corresponding automorphic function of f. Then $\{\phi_{0j}\}_j$ is a generating set of the automorphic group,* Aut(f).

Proof This is immediate from the definitions. The automorphic function $\phi_{12} : \Omega_{11} \to \Omega_{12}$ is given by the composition: $\phi_{02} \circ \phi_{01}^{-1}$ which maps as follows:

$$\Omega_{11} \overset{\phi_{01}^{-1}}{\to} \Omega_0 \overset{\phi_{02}}{\to} \Omega_{12}.$$

If, for instance, the curve $a \curvearrowright b$ is common to $\partial\Omega_{12}$ and to $\partial\Omega_{24}$ and the automorphic function $\phi_{02} : \Omega_0 \to \Omega_{12}$ maps the curve $a' \curvearrowright b'$ which is common to $\partial\Omega_0$ and t $\partial\Omega_{13}$ to the curve $a \curvearrowright b$, then $\phi_{0(24)} : \Omega_0 \to \Omega_{24}$ is given by the composition: $\phi_{0(24)} = \phi_{03} \circ \phi_{02}$, etc… □

In particular we have:

Corollary 20 *Let $f \in E$ have a system of fundamental domains. If* Aut(f) *is not a finitely generated group, then for any system $\{\Omega_j\}_j$ of fundamental domains and for any j, the first generation $G_1(\Omega_j)$ is an infinite family.*

Remark 28 We recall that according to Shimizu's definition in [14], the boundaries of a fundamental system of an entire function have no accumulation point in the finite plane. Moreover, not every entire function has a system of fundamental domains. Gross constructed an entire function which has all the points of $\mathbb{C}$ as its asymptotic values. In [14] Shimizu proved that the Gross function has no system of fundamental domains.

Theorem 29 *Let $f \in E$ have a system of fundamental domains $\{\Omega_j\}_j$ having the property that $\forall\, j$ we have:*

$$\lim_{n\to\infty}\left\{\frac{g(\Omega_j, n)}{\sum_{m=1}^{n} g(\Omega_j, m)}\right\} = 0.$$

In particular the $g(\Omega_j, m)$ are always finite! Then $\mathrm{Aut}(f)$ *is amenable.*

Proof One can check that the finite sets of automorphic functions $\phi : \Omega_0 \to \Omega$ where $\Omega \in \bigcup_{k=1}^{n} G_k(\Omega_0)$, which we denote by F_n form a Følner sequence for $\mathrm{Aut}(f)$. □

Corollary 21 *Let $f \in E$ have a system of fundamental domains $\{\Omega_j\}_j$ such that $\forall\, j$ there is a polynomial $P_j(x)$ of degree d_j for which $g(\Omega_j, n) \in \Omega(P_j(n))$, i.e. there are two positive numbers $0 < c_j < C_j$ such that $\forall\, n \in \mathbb{Z}^+$, $c_j \cdot P_j(n) \le g(\Omega_j, n) \le C_j \cdot P_j(n)$. Then* $\mathrm{Aut}(f)$ *is amenable.*

Proof We will use the following well-known estimate of the moments of the natural numbers:

$$1^d + 2^d + \ldots + n^d = \frac{n^{d+1}}{d+1} + \frac{n^d}{2} + \frac{dn^{d-1}}{12} + \mathcal{O}(n^{d-3}).$$

By this estimate we obtain:

$$\lim_{n\to\infty} \frac{n^d}{1^d + 2^d + 3^d + \ldots + n^d} = 0.$$

By the assumption on the counting function $g(\Omega_j, n)$ and by Theorem 29 the result follows. □

Remark 29 Theorem 29 does not imply anything in the case where $g(\Omega_j, n) = \Omega(q^n)$, i.e. a geometric growth. For:

$$\lim_{n\to\infty} \frac{q^n}{1 + q + q^2 + \ldots + q^n} = 1.$$

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Integral Representations in Complex Analysis: From Classical Results to Recent Developments

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Abstract After a review of key classical results concerning integral kernels in multidimensional complex analysis and their numerous applications, we discuss some of the central open problems in the general case of weakly pseudoconvex domains. In the final section we describe some more recent results that suggest new approaches towards making progress on some of these problems.

1 Classical Results

Most of the results described in this section are covered in detail in [15, 37], and/or [27], to which the reader is referred to for any explanation of the basic terminology and concepts of multidimensional complex analysis.

1.1 Results Up to the 1940s

In dimension one, the Cauchy integral formula is a most fundamental tool. First generalizations to higher dimensions involved the case of polydiscs, or more general products of domains in $\mathbb{C}$, which was already used by Weierstrass in order to prove standard local properties of holomorphic functions in several variables, such as infinite differentiability and local power series expansion. These results, essentially, just involved an iteration of the one-dimensional formula, one variable at a time.

In the 1930s, A. Weil and, independently, S. Bergman introduced integral representations on so-called *polynomial polyhedra*, a special case of *analytic* polyhedra A. The latter are defined by finitely many holomorphic functions $h_1, \ldots, h_l$ on a domain D in $\mathbb{C}^n$ as follows:

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$$A = \{z \in D : |h_j(z)| < 1,\ j = 1, \ldots, l\},$$

with the requirement that A is relatively compact in D. Since every domain of holomorphy can be exhausted by an increasing sequence of analytic polyhedra, these domains played an important role in the development of the basic theory of holomorphic functions on such domains in the period 1935–1955. In particular, the Bergman–Weil integral formula was an important ingredient in K. Oka's 1942 solution of the Levi problem in dimension two.

A key feature of the integral representation formulas for product domains and analytic polyhedra is that integration is over a thin subset of the topological boundary bA which is known as the *distinguished boundary* of A.

Another natural generalization of a disc to higher dimensions is a ball

$$B(a, r) = \{z \in \mathbb{C}^n : \textstyle\sum_{j=1}^n |z_j - a_j|^2 < r^2\}.$$

Quite different techniques are needed in order to develop integral representation formulas in this case, or for the more general case of a smoothly bounded domain in $\mathbb{C}^n$.

The first and most general integral representation formula for such domains involves the Bochner–Martinelli kernel $K^{BM}(\zeta, z)$. It is defined by

$$K^{BM}(\zeta, z) = \frac{(n-1)!}{(2\pi i)^n} \frac{\sum_{j=1}^n (\overline{\zeta_j - z_j}) d\zeta_j \wedge (\wedge_{\nu \neq j} d\overline{\zeta_\nu} \wedge d\zeta_\nu)}{|\zeta - z|^{2n}}.$$

Note that in case $n = 1$, $K^{BM}(\zeta, z) = \frac{1}{(2\pi i)} \frac{(\overline{\zeta - z}) d\zeta}{|\zeta - z|^2} = \frac{1}{(2\pi i)} \frac{d\zeta}{\zeta - z}$ is exactly the familiar Cauchy kernel.

One has the following Bochner–Martinelli formula [3, 30]:
If $D \subset\subset C^n$ has (piecewise) differentiable boundary, then

$$f(z) = \int_{bD} f(\zeta) K^{BM}(\zeta, z)\ , z \in D,$$

for all $f \in \mathcal{O}(D) \cap C(\overline{D})$.

Note that for $n = 1$ this is exactly the standard Cauchy integral formula. Just as in dimension one, the Bochner–Martinelli kernel is closely related to harmonic analysis, in particular, it can be obtained quite directly from the Newtonian kernel, i.e., from the fundamental solution of the Laplacian.

The main advantage of the Bochner–Martinelli kernel and of the related integral representation formula is that it does not depend on the domain. On the other hand it has significant disadvantages. For example, when $n \geq 2$, $K^{BM}(\zeta, z)$ is NOT holomorphic in z or ζ. Furthermore, its singularity behaves the same way in **all** directions, and it ignores the complex geometry of the boundary. Also, it does not distinguish between $\partial/\partial z_j$ and $\partial/\partial \overline{z_j}$ derivatives in any essential way.

So, K^{BM} is not particularly useful for any *refined* constructions and estimations in *multidimensional* complex analysis.

Still, there are some interesting applications, such as one of the simplest proofs of the famous Hartogs extension theorem, as given both by Martinelli in 1942 and Bochner [3], independently, based on an idea of R. Fueter, as well as of the following important generalization.

Global CR - Extension Theorem *Suppose bD is connected, and $n \geq 2$. Let $f \in C^1(bD)$ be a CR-function (i.e., $\overline{\partial_b} f = 0$, that is, f satisfies the* **tangential** *Cauchy–Riemann equations). Then f extends to a holomorphic function on D, i.e., there exists $F \in \mathcal{O}(D) \cap C(\overline{D})$ with $F = f$ on bD.*

In the classical Hartogs theorem, the given function f was assumed to be holomorphic in an open neighborhood of the boundary bD.

Since the correct history of this result is not well known, I take the opportunity to recall the key contributions.

The concept of tangential Cauchy–Riemann equations on real submanifolds of $\mathbb{C}^n$ and of the corresponding notion of CR-function goes back to W. Wirtinger in 1926. The first version of the global CR-extension theorem was proved by Francesco Severi in 1931 in the case that bD and f are real analytic. In case bD is of class C^2 and strictly pseudoconvex, the result was proved by Hellmuth Kneser in 1936 (in dimension two). Kneser, in fact, proved a *local* CR-extension theorem which is somewhat stronger than the result proved 20 years later by Hans Lewy and that became widely known in the 1950s. The general differentiable case was first proved by Gaetano Fichera in 1957, and in 1961 Martinelli found the simplest proof by using directly the BM-kernel.

Unfortunately, the work of Wirtinger, Severi, Kneser, and Fichera and the 1961 proof of Martinelli have been largely overlooked in the literature for many years. On the other hand, since the late 1960s the global CR extension theorem had been widely attributed to S. Bochner, with reference to his 1943 paper, even going as far as suggesting that Bochner introduced the idea of CR-functions. However, there is NO such result in Bochner's 1943 paper, and furthermore there is NO evidence in Bochner's work in the 1940s and 1950s that he knew of CR functions at that time, and that he proved this result or anything similar to it. The reader interested in more details should consult [39, 40].

Given these rather limited results in higher dimensions, it is not surprising that the major developments in multidimensional complex analysis in the 1950s and 1960s did not use any integral representation formulas at all, except for the elementary polydisc case. In fact, in contrast to the dominant role of the Cauchy integral formula in the classical one variable theory, most of the widely known books of that period did not mention any integral representation formulas besides the polydisc case. Instead, the key developments during those decades relied on new techniques such as coherent analytic sheaves and, eventually, deep methods from partial differential equations.

1.2 Leray's New Kernel Construction

In order to develop integral kernels that could be applied to generalize results from classical complex analysis, such as refined approximation theorems and relevant estimates up to the boundary of a domain, one needed a kernel that is holomorphic in the parameter z, just like the classical Cauchy kernel. Furthermore, such a kernel would be needed for domains more general than product domains and analytic polyhedra. So a new method to construct kernels—explicitly using the complex structure—had to be found. Furthermore, since such a kernel holomorphic in z on D needs to have a singularity at points $\zeta \in bD$, in essence it will be necessary to require that the domain D is a domain of holomorphy, and hence pseudoconvex.

A key result was obtained by Jean Leray in 1956, with full details published in [24]. Leray introduced a general technique to construct a large class of kernels. He called them **Cauchy–Fantappié kernels**, in memory of his close friend, the Italian mathematician L. Fantappié, who had just died unexpectedly.

Given the importance of these kernels for all subsequent work up to the most recent times, we shall now describe Leray's construction, in a slightly modified version from the original one. Suppose D is a domain with smooth boundary. A $(1, 0)$ form $W(\zeta, z) = \sum_{j=1}^{n} w_j(\zeta, z)d\zeta_j$ of class C^1 on $bD \times D$ is called a (kernel) *generating form* for D if

$$\textstyle\sum_{j=1}^{n} w_j(\zeta, z)(\zeta_j - z_j) = 1 \text{ on } bD \times D.$$

Note that in dimension one there exists exactly one generating form, namely $\frac{d\varsigma}{\varsigma - z}$, while in higher dimensions, as we shall see, there are numerous possibilities.

Given such a form W, one defines the Cauchy–Fantappié kernel generated by W to be the $(n, n-1)$ form in ς (with coefficients depending on (ς, z))

$$\Omega(W) = \frac{1}{(2\pi i)^n} W \wedge (\overline{\partial_\zeta} W)^{n-1}.$$

In particular, the Bochner–Martinelli kernel discussed earlier is the (global) CF kernel generated by

$$W^{BM} = \textstyle\sum_{j=1}^{n} \dfrac{\overline{\zeta_j - z_j}}{|\zeta - z|^2} d\zeta_j.$$

By using the defining equation for generating forms, one readily proves that $d_\varsigma \Omega(W) = \overline{\partial_\varsigma} \Omega(W) = 0$. Most significantly, this then implies the Cauchy–Fantappié formula of Leray,

$$f(z) = \int_{bD} f(\zeta)\Omega(W)(\zeta, z)\,, z \in D,$$

for all $f \in \mathcal{O}(D) \cap C(\overline{D})$.

A simple important case involves (Euclidean) convex domains D with C^2 boundary. If r is a C^2 defining function for D (i.e., $D = \{z : r(z) < 0\}$ and $dr \neq 0$ on bD), convexity implies that

$$\Phi(\zeta, z) = \sum_j \frac{\partial r}{\partial \zeta_j}(\zeta)(\zeta_j - z_j) \neq 0 \text{ on } bD \times D.$$

Therefore $W^C = \partial r/\Phi$ is a generating form for D that is holomorphic in z. The corresponding CF - kernel $\Omega(W^C)(\zeta, z)$ is then **holomorphic in** $z \in D$ as well.

Its pull-back to the boundary is independent of the defining function r, so that this kernel, known as the *Cauchy–Leray kernel for convex domains*, is an intrinsically defined object for any smoothly bounded convex domain. Note that this includes a holomorphic Cauchy-type kernel for a ball.

Unfortunately, Leray did not pursue this line of research further. Since convex domains were viewed as much too special for the purposes of global complex analysis, and since in the 1950s questions related to boundary behavior of analytic objects were not at the forefront, this construction moved to the sidelines for the next decade.

1.3 Kernels for Strictly Pseudoconvex Domains

Things were very different about 10 years later. Most likely the emergence of PDE methods in multidimensional complex analysis, beginning in the early 1960s, had major influence. The work of J. J. Kohn [19], who obtained sharp optimal L^2 estimates for the $\overline{\partial}$-Neumann problem up to the boundary of strictly pseudoconvex domains, and subsequently of A. Andreotti and E. Vesentini [1] and L. Hörmander [16], signaled a more analytic approach to several complex variables. It thus appeared quite natural to search for Cauchy-type holomorphic kernels on strictly pseudoconvex domains. In particular, H. Grauert in Göttingen—quite aware of the potential deep applications—presented this problem to Enrique Ramirez, one of his doctoral students. Ramirez solved the problem [34], and even before his work was published, Grauert, in collaboration with I. Lieb, used the Ramirez kernel to prove sup-norm estimates for solutions of the Cauchy–Riemann equations [12]. Lieb [25] then used this result to prove a higher dimensional version of the classical 1952 approximation theorem of S. N. Mergelyan, as follows:

If D is strictly pseudoconvex, then every $f \in \mathcal{O}(D) \cap C(\overline{D})$ can be approximated uniformly on $\overline{D}$ by functions holomorphic in a neighborhood of $\overline{D}$.

Given that the late 1960s seemed ripe for such results, it is not surprising that essentially around the same time these results were also obtained, independently, by G. M. Henkin in Moscow [13, 14]. We remark that the above approximation theorem was also proved by N. Kerzman [17], who built upon the results of Grauert and Lieb, and, in particular, showed the usefulness of *local* kernels.

In hindsight, the *local* construction of the kernel should have been seen already by Leray. In fact, since near any boundary point a strictly pseudoconvex domain is locally biholomorphically equivalent to a convex domain, it is just an exercise to transport Leray's Cauchy kernel—locally—to a strictly pseudoconvex domain, thereby obtaining a generating form holomorphic in z for such domains, that is defined for $|\zeta - z| < \varepsilon$, where $\varepsilon > 0$ is sufficiently small. The main difficulty then is to use the local data to construct a **global holomorphic** generating form W^{HR} on $bD \times D$. Ramirez solved this problem by using deep results from coherent analytic sheaf theory, while Henkin used the global $\overline{\partial}$ methods developed by Hörmander, resulting in a technically simpler construction than the one of Ramirez. Still, the essential local features of the resulting kernels are identical. This kernel $\Omega(W^{HR})$ is generally known as the **Henkin–Ramirez (HR-) kernel.**

Since every domain with smooth boundary in $\mathbb{C}^1$ is trivially strictly pseudoconvex, the HR-kernel can be viewed as the optimal higher dimensional generalization of the classical Cauchy kernel. Just like in dimension one, the HR-kernel can readily be estimated, and in the 1970s and 1980s it has led to numerous major applications and new results in function theory on strictly pseudoconvex domains.

Furthermore, W. Koppelman [23] showed how to generalize the Cauchy–Fantappié kernels of Leray to differential forms of type $(0, q)$, or more generally, type (p, q). Given a generating form W and $0 \le q \le n-1$, one defines the double differential form

$$\Omega_q(W) = c_{n,q} W \wedge (\overline{\partial_\varsigma} W)^{n-q-1} \wedge (\overline{\partial_z} W)^q,$$

which is of type $(n, n-q-1)$ in ς and type $(0, q)$ in z. Here $c_{n,q}$ is a numerical constant that depends on n and q. Note that $\Omega_0(W) = \Omega(W)$ (the kernel introduced by Leray); for convenience one also sets $\Omega_{-1}(W) = \Omega_n(W) = 0$. The corresponding Bochner–Martinelli–Koppelman kernels $\Omega_q(W^{BM})$ lead to the following representation formula for forms $f \in C^1_{(0,q)}(\overline{D}), 0 \le q \le n$, on an arbitrary bounded domain D with piecewise C^1 boundary, where the integration is with respect to ς, and $z \in D$ is a parameter:

$$f(z) = \int_{bD} f \wedge \Omega_q(W^{BM}) - \overline{\partial_z} \int_D f \wedge \Omega_{q-1}(W^{BM}) - \int_D \overline{\partial} f \wedge \Omega_q(W^{BM}).$$

This formula gives an explicit solution for the $\overline{\partial}$−equation $\overline{\partial} u = f$ in case f is $\overline{\partial}$-closed with compact support, or for forms of type $(0, n)$, generalizing classical results in dimension one.

Koppelman also showed how the preceding representation formula needs to be adjusted if in the boundary integral the BM-generating form W^{BM} is replaced with another generating form W. Given such a W, one defines an integral operator

$$T_q^W : C_{(0,q)}(\overline{D}) \to C_{(0,q-1)}(D),$$

for $0 \leq q \leq n$ by

$$T_q^W f = \int_{bD} f \wedge \Omega_{q-1}(W, W^{BM}) - \int_D f \wedge \Omega_{q-1}(W^{BM}),$$

where $\Omega_{q-1}(W, W^{BM})$ is a rather complicated double differential form that depends explicitly on W and W^{BM}. One then has the representation formula

$$f = \int_{bD} f \wedge \Omega_q(W) + \overline{\partial} T_q^W f \, + T_{q+1}^W \overline{\partial} f \text{ for } f \in C^1_{(0,q)}(\overline{D}).$$

While these formulas were announced by Koppelman in 1967, his untimely death prevented him from publishing proofs. First complete proofs of these results were obtained by I. Lieb [26], who, most importantly—as discussed below—showed how to use these formulas for solving $\overline{\partial}$ on $(0, q)$-forms for general q. Inspired by Lieb's work, shortly thereafter N. Øvrelid obtained somewhat different proofs [33].

By applying this formula with the (holomorphic) HR-generating form on a strictly pseudoconvex domain, one trivially gets $\Omega_q(W^{HR}) = 0$ for $q \geq 1$. Consequently, if f is $\overline{\partial}$-closed, one obtains the rather explicit integral solution operator $T_q^{W^{HR}}$ for $\overline{\partial}$, since then the above representation formula reduces to

$$f = \overline{\partial} T_q^{W^{HR}} f \, .$$

By using these formulas and variations thereof, mathematicians could solve numerous analytic problems on strictly pseudoconvex domains. In particular, it was proved that the operators $T_q^{W^{HR}}$ satisfy *optimal* Hölder estimates such as

$$\left| T_q^{W^{HR}} f \right|_{1/2} \leq C \, |f|_0 \ ,$$

in analogy to the $1/2$ subelliptic estimate proved by Kohn in the L^2-setting [19]. Similar estimates were obtained in numerous other function spaces.

Another important by-product was the precise identification of the principal term of fundamental abstract Hilbert space operators in complex analysis in terms of the singularities of the HR-kernel, the operators $T_q^{W^{HR}}$, and appropriate variations. For example, this was first done for the Szegö kernel by N. Kerzman and E. Stein [18]. A few years later E. Ligocka handled the Bergman kernel [29], and this method allowed a major simplification of the proof of C. Fefferman's famous mapping theorem [10]:

If F is a biholomorphic map between strictly pseudoconvex domains with C^∞ boundaries, then F extends C^∞ to the boundary.

Finally, assuming a special metric adapted to the complex geometry of the boundary (a so-called Levi metric), Lieb and Range handled the canonical solution operator for $\overline{\partial}$ that arises in the L^2 theory of the $\overline{\partial}$-Neumann problem [28].

By using completely different methods based on pseudo-differential operators, shortly thereafter R. Beals, P. Greiner, and N. Stanton were able to handle general metrics [2].

We conclude this section by mentioning that in the early 1980s it was discovered how the general machinery of integral representations, suitably applied to the elementary **local** holomorphic generating form mentioned earlier (the transplant of the Cauchy–Leray kernel for convex domains) could be used to obtain global results directly, avoiding any of the deep results from coherent analytic sheaves and/or PDE that were used by Ramirez and Henkin. (See [36] and [15]).

Just as in the very classical one-dimensional setting three distinct view points—power series, Cauchy–Riemann equations, and Cauchy integral formula—could be used to establish fundamental results, complex analysis on strictly pseudoconvex domains could now be similarly developed by the corresponding distinct higher dimensional versions: coherent sheaf theory, $\overline{\partial}$-methods, and integral representations.

2 Beyond Strictly Pseudoconvex Domains: Many Problems

2.1 L^2 Results and Finite Type

While every smoothly bounded domain in dimension one is strictly pseudoconvex, in higher dimensions new phenomena arise, even within the class of pseudoconvex domains. Such smoothly bounded pseudoconvex domains, where the Levi form is not *positive* definite at every boundary point but just **semi**-definite, are generally called *weakly* pseudoconvex, and they are a genuine higher dimensional phenomenon. In particular, the complex geometry of the boundary can be very complicated, and so far the general case is not well understood. Furthermore, counter examples have been found over the years that demonstrate how certain basic results for strictly pseudoconvex domains do not hold for general weakly pseudoconvex domains. For example, it is not possible to use a holomorphic coordinate change to locally turn the domain into something convex, as in the strictly pseudoconvex case, and in general there are no sup-norm estimates for solutions of $\overline{\partial}$. Of course, one could go even further and consider non-pseudoconvex domains, but this more general case ultimately can be properly investigated only by including envelopes of holomorphy, involving the class of Riemann domains that are complex manifolds spread over $\mathbb{C}^n$.

But weakly pseudoconvex domains already present enough major challenges and open problems, and it is therefore quite reasonable to focus on such domains, at least for the foreseeable future. Historically, the PDE methods based on abstract Hilbert space techniques have been most successful. For example, by introducing appropriate weight functions, in 1973 J.J. Kohn proved global regularity for the Cauchy–Riemann equations on arbitrary weakly pseudoconvex domains, that is, given a form $f \in C^{\infty}_{(0,q+1)}(\overline{D})$ with $\overline{\partial} f = 0$, there exists a solution $u \in C^{\infty}_{(0,q)}(\overline{D})$

of the equation $\overline{\partial} u = f$ [20]. However, counterexamples show that in general there is no such global regularity result for the canonical solution arising from the $\overline{\partial}$-Neumann problem. Instead, it turns out that in order to obtain sharp local regularity estimates one needs to consider the so-called domains of *finite type*. In essence, this condition means that the Levi form vanishes to finite order, or—more precisely—that at every point $P \in bD$ the maximal order of contact of bD with complex analytic varieties (including singularities) is finite. This condition was introduced in dimension 2 by using vector fields by Kohn in 1972, and the appropriate general version was obtained by J. D'Angelo in 1982 [8]. The central role of finite type is evidenced by the key result of Kohn (sufficiency) and D. Catlin (necessity) that finite type at $P \in bD$ is necessary and sufficient for the existence of a (local) subelliptic estimate

$$\|f\|_\epsilon^2 \leq C[\left\|\overline{\partial} f\right\|^2 + \left\|\overline{\partial}^* f\right\|^2 + \|f\|^2],$$

for some $\epsilon > 0$ and all smooth $f \in$dom$(\overline{\partial}^*)$ with compact support in a small neighborhood $U \cap \overline{D}$ of P [4, 21].

This latter result, in particular, implies that Fefferman's mapping theorem (see above) can be extended to the finite type case. However, in spite of numerous efforts, to this author's knowledge it is still unknown whether the corresponding result holds in the arbitrary weakly pseudoconvex case.

2.2 The Obstruction to Holomorphic Kernels

Turning to integral representations, the situation is much worse. One major obstacle is that there is no analogue of the essential **local** ingredient in the construction of the Henkin–Ramirez kernel, that is, the *Levi polynomial* $F^{(r)}$ of a defining function r for the domain D. In order to understand this better, and also in light of the discussion in Section 3, let us review the key concepts. We assume that for a fixed $k \geq 3$ the function r has a bounded C^k norm $|r|_k$ over U. The Taylor expansion of $r(z)$ at the point $\zeta \in U$ up to order 2 can be written as

$$r(z) = r(\zeta) + 2\,\mathrm{Re}\left[-F^{(r)}(\zeta, z)\right] + \mathcal{L}(r, \zeta; \zeta - z) + O(|\zeta - z|^3),$$

where $F^{(r)}(\zeta, z)$ (that is, the Levi polynomial) is defined by

$$F^{(r)}(\zeta, z) = \sum_j \frac{\partial r}{\partial \zeta_j}(\zeta)(\zeta_j - z_j) + \\ (-\frac{1}{2}) \sum_{j,k} \frac{\partial^2 r}{\partial \zeta_j \partial \zeta_k}(\zeta)(\zeta_j - z_j)(\zeta_k - z_k),$$

and $\mathcal{L}(r, \zeta; t)$ is the Levi form of r at ζ applied to the vector $t \in \mathbb{C}^n$.

Assume now that D is *strictly* Levi pseudoconvex; one may then choose the defining function r to be strictly plurisubharmonic on a neighborhood U of bD, i.e., there is $c > 0$ so that

$$\mathcal{L}(r, \zeta; \zeta - z) \geq 2c\,|\zeta - z|^2 .$$

It then follows that there exists $\varepsilon > 0$ so that for $\zeta \in U$ and z with $|\zeta - z| < \varepsilon$ one has

$$2\,\mathrm{Re}\left[F^{(r)}(\zeta, z)\right] \geq r(\zeta) - r(z) + c\,|\zeta - z|^2 \ .$$

In particular, if $\zeta \in bD$, and $z \in \overline{D}$ satisfies $|\zeta - z| < \varepsilon$, one has

$$2\,\mathrm{Re}\left[F^{(r)}(\zeta, z)\right] \geq |r(z)| + c\,|\zeta - z|^2 ,$$

that is, $F^{(r)}(\zeta, z)$ is a holomorphic function in z that has a zero at ζ, but that is non-zero on $\overline{D} \cap B(\zeta, \varepsilon) \setminus \{\zeta\}$. Such a function is also known as a (local) holomorphic support function.

Since trivially $F^{(r)}(\zeta, z) = \sum_j g_j(\zeta, z)(\zeta_j - z_j)$, with g_j holomorphic as well, it follows that

$$W^{(r)}(\zeta, z) = \frac{\sum_j g_j(\zeta, z) d\zeta_j}{F^{(r)}(\zeta, z)}$$

is a (local) *holomorphic* generating form on $bD \times \{z \in D : |\zeta - z| < \varepsilon\}$.

The deep work of Henkin and Ramirez mentioned earlier in Section 1.3 involves passing from this local generating form to a *global* generating form $W^{HR}(\zeta, z)$ that is holomorphic for $z \in D$. The simple local estimates for the Levi polynomial we just stated are critical for all deeper applications of the Henkin–Ramirez kernel.

Unfortunately, for an arbitrary (weakly) pseudoconvex domain, there does not exist any explicit holomorphic analogue of the Levi polynomial. In fact, the 1972 famous example of Kohn and Nirenberg [22] provides a simple pseudoconvex domain D of *finite type* 8, with $0 \in bD$, so that the zero set of any holomorphic h in any neighborhood U of 0 with $h(0) = 0$ will have points both in $D \cap U$ and in $U \setminus \overline{D}$. In particular, this example cannot be locally biholomorphically equivalent to a convex domain! The situation is even worse: shortly thereafter J. E. Fornaess modified and refined the example by showing that this even fails if the biholomorphic coordinate change on $U \cap D$ is only C^1 up to the boundary.

2.3 Partial Results for Convex Domains and in Dimension Two

Given this obstruction, it therefore seemed natural to consider convex domains, where the existence of a holomorphic generating form was known since 1956, thanks to Leray. In order to obtain estimates, some special cases were considered at first. For example, this author considered generalized complex ellipsoids

$$\{z : |z_1|^{m_1} + \ldots + |z_n|^{m_n} < 1\},$$

where the m_j are positive even integers, and proved Hölder estimates for solutions of $\overline{\partial}$ of any order $\epsilon < 1/M$, where $M = \max\{m_j\}$ is the *type* (introduced earlier) of the domain [35]. In 1986 Diederich–Fornaess–Wiegerinck obtained corresponding results on *real* generalized ellipsoids, and in the late 1990s, A. Cumenge, and—independently—Diederich–Fischer–Fornaess proved Hölder estimates for $\overline{\partial}$ on (Euclidean) convex domains of finite type [6, 7, 9]. The latter results made critical use of information about the complex geometry of finite type in the **convex** case obtained by J. McNeal and E. M. Stein [31]. This case is much better understood than the general finite type case, since convexity implies that the type can be identified by just considering the order of contact of the boundary with complex *lines*.

Another case for which positive results are known is in dimension 2. Here, too, finite type is more elementary than in the general case. As originally introduced by Kohn in 1972, finite type in dimension two can be characterized by just considering a non-zero local $(1, 0)$ tangent vector field L and its conjugate $\overline{L}$: bD is of finite type $\leq m$ at $P \in bD$ if and only if L, $\overline{L}$, and their commutators up to order m generate the full (complexified) tangent space at P. In terms of order of contact, this is equivalent to saying that the order of contact at P between bD and (*nonsingular*) holomorphic curves is $\leq m$. Given this much simpler situation, Nagel–Rosay–Stein–Wainger, and—independently—Catlin were able to obtain a rather precise description of the local complex geometry of finite type in dimension two [5, 32].

In order to construct holomorphic kernels, one had to deal with the fundamental obstruction of the Kohn–Nirenberg example. This author used a deep classical result of H. Skoda [44] to construct holomorphic generating forms. Skoda's result is valid on arbitrary pseudoconvex domains (no smoothness assumptions); it uses an ingenious modification of the L^2 techniques of Hörmander [16] to obtain rather precise estimates with weights on holomorphic L^2 functions f and $g_1, \ldots, g_p$ to characterize when f is in the ideal generated by $g_1, \ldots, g_p$.

For $\zeta \in bD$, the constant function $f \equiv 1$ satisfies Skoda's conditions to be in the ideal generated by $g_j = (\zeta_j - z_j)$, $j = 1, \ldots, n$. One therefore obtains precise L^2 estimates with weights for holomorphic solutions (in z) $(w_1(\zeta, z), \ldots, w_n(\zeta, z))$ of the equation

$$\sum_{j=1}^{n} w_j(\zeta, z)(\zeta_j - z_j) = 1.$$

Consequently, $W^{Sk} = \sum_{j=1}^{n} w_j(\zeta, z) d\varsigma_j$ is a holomorphic generating form which, in contrast to the Leray form or the HR-form, is only defined for $z \in D$, and moreover is highly non-explicit. Still, by combining the precise geometric information mentioned above in case the domain is of finite type in dimension 2 with the estimates of Skoda, it was possible to estimate the resulting solution operator $T_1^{W^{Sk}}$ for $\overline{\partial}$ and to prove Hölder estimates for solutions of $\overline{\partial}$ of any order $\epsilon < 1/m$, where m is the maximal type of the domain D [38].

By using completely different methods based on microlocal analysis of pseudo-differential operators on appropriate models of the boundary, Fefferman and Kohn had obtained this same result for solutions of $\overline{\partial}$ a bit earlier [11], even proving the optimal estimate with $\epsilon = 1/m$.

As for dimension ≥ 3, holomorphic integral kernels and pointwise estimates for $\overline{\partial}$ in the general case of weakly pseudoconvex domains of finite type are still unknown, after more than 40 years that the question was first considered. The main problem is that the techniques that have worked in the partial results we just described do not seem to work in general: there is no known model or candidate in higher dimensions for describing the local complex geometry of finite type, such as is known in dimension two or for convex domains.

2.4 Speculation on Some Possible Approaches

One fairly natural approach that has been successful to obtain fundamental results in classical global complex analysis is based on exhausting the given general domain by better domains, for which more detailed information is available. For example, it is known that a pseudoconvex domain D can be exhausted by strictly pseudoconvex domains. In case the boundary is smooth, rather simple strictly plurisubharmonic exhaustion functions, whose sublevel sets are then strictly pseudoconvex, can be constructed from a given defining function. Unfortunately, while this approach has been tried, no useful result has been obtained so far. Still, I believe that there is room for further investigations. In particular, one could explore whether the condition of finite type could be used to obtain improved exhaustions, so that critical estimates can be controlled along the exhaustion.

Another suggestion has its roots in classical work of Kohn and Nirenberg, who proved subelliptic estimates in the strictly pseudoconvex case via *elliptic regularization*, that is, they added a small term $\epsilon G(\varsigma, z)$ to the fundamental operator that made the problem elliptic up to the boundary. This allowed to use more powerful familiar machinery, and the crux was to produce new (**sub**elliptic) estimates that ultimately were independent of ϵ. Letting $\epsilon \to 0$ then preserved these estimates in the original non-elliptic case. So one could try an analogous technique for integral kernels, for example, in the finite type case. The challenge of course is to find an appropriate modification and to prove estimates that remain stable as the correction term goes to zero.

Finally there is the result of Skoda that was used successfully by this author in dimension two. Perhaps the lack of complex geometric information in higher dimensions could be balanced by improving Skoda's theorem by introducing appropriate modifications of his techniques assuming that the domain has smooth boundary of finite type. In the original work, only general L^2 Hilbert space estimates with weights were used, so it would appear that a better version of Skoda's theorem might be possible. Just like numerous variations of Hörmander's L^2 techniques have produced much additional precise global information, say involving L^2 Sobolev norms and appropriate weights, comparable refinements of Skoda's techniques might (should?) produce additional useful information.

Of course, these few suggestions involve very difficult and highly technical speculative methods. But then the problem is really very deep. Even the subelliptic L^2 estimates required major highly non-trivial new methods in order to get from the strictly pseudoconvex case to the finite type case.

3 A New Kernel Approach

Rather than looking for special techniques and new approaches that would allow the local construction of *holomorphic* Cauchy–Fantappié kernels, a few years ago I introduced a modification of the Levi polynomial that leads to a new kernel that is no longer holomorphic, but still retains significant *complex analytic* properties.

The advantages one gains are that this construction works on an *arbitrary pseudoconvex* domain with differentiable boundary, and that it reflects the *analytic/geometric information* contained in the Levi form. Therefore this construction might lead to new applications.

3.1 Motivation: The Basic L^2 A-priori Estimate

The overall plan is motivated by the successful approach in the L^2 theory, especially by Kohn's 1979 work. Most work in this area begins with the fundamental basic L^2 estimate for forms in the domain of the adjoint $\overline{\partial}^*$ that is valid on an arbitrary weakly pseudoconvex domain D. Let us recall this result in case of (0, 1) forms.

We fix a point $P \in bD$, and choose a local orthonormal frame $(\omega_1, \dots, \omega_n)$ for $(1, 0)$ forms on a neighborhood U of $P \in bD$, with ω_n the normal component. Let $\{L_1, \dots, L_n\}$ be the dual frame of $(1, 0)$ vector fields. The basic L^2 estimate then states that there exists a constant C, such that if $f = \sum f_k \overline{\omega_k} \in \mathrm{dom}(\overline{\partial}^*)$ with compact support in U, then

$$\sum_{j,\, k} \left\| \overline{L_j} f_k \right\|^2 + \int_{bD\cap U} \mathcal{L}(r, \zeta; f^{\#}) dS(\zeta) \leq$$

$$\leq C \, [\left\| \overline{\partial} f \right\|^2 + \left\| \overline{\partial}^* f \right\|^2 + \| f \|^2],$$

where $f^{\#} = (f_1, \ldots, f_n)$. The norms here are the standard L^2 norms. Since $f \in \mathrm{dom}(\overline{\partial}^*)$, one has $f_n = 0$ on bD, so that $f^{\#}$ is tangential and hence, by pseudoconvexity, $\mathcal{L}(r, \zeta; f^{\#}) \geq 0$

By analogy, in order to study pointwise estimates such as sup-norm or Hölder estimates, one should start with a pointwise analogue of this basic estimate, as follows. For $q \geq 1$ we define

$$\mathfrak{D}^k_q(D) = C^k_{(0,q)}(\overline{D}) \cap \mathrm{dom}(\overline{\partial}^*),$$

for $k = 1, 2, \ldots$. Fix P, U, and the frames $\omega_1, \omega_2, \ldots, \omega_n$ and $L_1, \ldots, L_n$ as above. Denote by $\mathfrak{D}^k_{qU}$ those forms in $\mathfrak{D}^k_q(D)$ that have compact support in $\overline{D} \cap U$.

For a C^1 form $f = \sum_J f_J \overline{\omega}^J$ of type $(0, q)$ on $\overline{D}$, where the summation is over all strictly increasing q tuples J from $\{1, \ldots, n\}$, we define the norm

$$Q_0(f) = \left|\overline{\partial} f\right|_0 + |\vartheta f|_0 + |f|_0\,,$$

where $|\cdot|_0$ is the sup-norm over D and ϑ is the formal adjoint of $\overline{\partial}$. One would then like to prove something like

$$\left|\overline{L_j} f_J(z)\right| \leq C \; Q_0(f) \text{ for } j = 1, \ldots, n \text{ and each } q\text{-tuple } J$$

for $f \in \mathfrak{D}^k_{qU}$.

Given the well-known subtleties of L^∞ estimates in analysis, such a sharp result is probably not correct. However, a slightly weaker version like

$$\left|\overline{L_j} f_J(z)\right| \cdot dist(z, bD)^\delta \leq C_\delta \cdot Q_0(f) \text{ for all } j, J \text{ and any } \delta > 0$$

would seem quite reasonable and consistent with the basic L^2 estimate.

3.2 A New Kernel

It is readily seen that the basic Bochner–Martinelli–Koppelman representation formula does not yield anything close to the desired a-priori estimate. In particular, it would seem that pseudoconvexity must be used along the way, just as in the L^2 basic estimate.

We assume that D is a bounded pseudoconvex domain in $\mathbb{C}^n$ with C^k boundary bD ($k \geq 3$), and we choose a C^k defining function φ for bD defined on a neighborhood $U = U(bD)$ of bD. In general the level surfaces $M_{-\delta} = \{z : \varphi(z) = -\delta\}$ will **not** be Levi pseudoconvex for $\delta > 0$.

However, one may choose a defining function r of the form

$$r(z) = \varphi(z)\exp(-C\,|z|^2),$$

so that if $C > 0$ is sufficiently large and $U(bD)$ is sufficiently small, its Levi form satisfies for all $\zeta \in D \cap U$

$$\mathcal{L}(r, \zeta; t) = \sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial \zeta_j \partial \overline{\zeta_k}}(\zeta) t_j \overline{t_k} > 0$$

$$\textit{for all } t \in C^n \textit{ with } t \neq 0 \text{ and} \sum_{j=1}^{n} \frac{\partial r}{\partial \zeta_j}(\zeta) t_j = 0.$$

So the level surfaces $M_{r(\zeta)}$ of r are actually strictly pseudoconvex, but the resulting estimates are not uniform in ζ as $r(\zeta) \to 0$, unless bD is strictly pseudoconvex to begin with.

We now fix this particular global defining function r. After shrinking U, we may assume that for a fixed $k \geq 3$ the function r has a bounded C^k norm $|r|_k$ over U. We again consider its Levi polynomial $F^{(r)}(\zeta, z)$ (see Section 2.2). As usual, it follows that

$$2\,\mathrm{Re}\left[F^{(r)}(\zeta, z) - r(\zeta)\right] = -r(\zeta) - r(z) + \\ + \mathcal{L}(r, \zeta; \zeta - z) + O(|\zeta - z|^3).$$

Given a constant $K > 0$ one now defines

$$\Phi_K(\zeta, z) = F^{(r)}(\zeta, z) - r(\zeta) + K\,|\zeta - z|^3\,.$$

One then proves the following key estimate from below: for K suitably large one has

$$|\Phi_K(\zeta, z)| \gtrsim \Big[\left|\mathrm{Im}\,F^{(r)}(\zeta, z)\right| + |r(\zeta)| + |r(z)| + \\ + \mathcal{L}(r, \zeta; \pi_\zeta^t(\zeta - z)) + K\,|\zeta - z|^3\Big],$$

for all $\zeta, z \in \overline{D} \cap U$ with $|\zeta - z| < \varepsilon$, where

$$\pi_\zeta^t : \mathbb{C}^n \to T_\zeta^{1,0}(M_{r(\zeta)}) \subset \mathbb{C}^n$$

is the orthogonal projection. We note that by the property of r, the Levi form term is ≥ 0. As in the familiar strictly pseudoconvex case, $r(\zeta)$ and $\operatorname{Im} F^{(r)}(\zeta, z)$ can be used as coordinates in a C^{k-2} real coordinate system in a neighborhood $B(z, \delta)$ of a fixed point $z \in U$, provided $\delta > 0$ is sufficiently small. The key information in this estimate is that Φ_K has a zero of order one in the complex *normal* direction, while the Levi form completely controls Φ_K from below in the complex *tangential* directions.

Furthermore, if z is fixed, one can introduce special "*z–diagonalizing coordinates for ζ*," so that the Levi form term satisfies

$$\begin{aligned}
&\mathcal{L}(r, \zeta; \pi_\zeta^t(\zeta - z)) + K\,|\zeta - z|^3 \\
&\gtrsim \sum_{j=1}^{n-1} \lambda_j(z)\,\left|\zeta_j - z_j\right|^2 + K/2\,|\zeta - z|^3,
\end{aligned}$$

where the $\lambda_j(z) \geq 0,\ j = 1, \ldots, n-1$, are the eigenvalues of the Levi form at z.

Similarly, one obtains a corresponding version involving the dual frame

$$\begin{aligned}
\partial r \wedge \overline{\partial} r \wedge \partial\overline{\partial} r(\zeta) = \gamma(\zeta)\omega_n \wedge \overline{\omega}_n \wedge \\
\wedge \left[\sum_{j=1}^{n-1} \lambda_j(z) d\zeta_j \wedge \overline{d\zeta_j} + \Omega_1\right],
\end{aligned}$$

where $\gamma(\zeta) > 0$ and Ω_1 is a well-behaved error term that satisfies $|\Omega_1| \leq C\,|\zeta - z|$ for some constant C.

For $\zeta \in bD$, one clearly has a decomposition

$$\Phi_K(\zeta, z) = \sum_{j=1}^{n} \widetilde{g_j}(\zeta, z)(\zeta_j - z_j),$$

where $\widetilde{g_j} = g_j + K\,|\zeta - z|\,(\overline{\zeta_j - z_j})$ (recall g_j from 2.2), so that

$$W^{\mathcal{L}} = \frac{\sum_{j=1}^{n} \widetilde{g}_j d\zeta_j}{\Phi_K}$$

is a local generating form on $bD \times (D \cap U)$ and $|\zeta - z| < \epsilon$. Finally one uses routine techniques to patch this local generating form with the BM generating form, thereby obtaining a global generating form on $bD \times \overline{D}$ that locally agrees with the original one. We still label this global form $W_D^{\mathcal{L}} = W^{\mathcal{L}}$ and call it the *Levi generating form* for the weakly pseudoconvex domain D.

We now consider the Cauchy–Fantappié kernel $\Omega_0(W^{\mathcal{L}}) = (2\pi i)^{-n} W^{\mathcal{L}} \wedge (\overline{\partial}_\zeta W^{\mathcal{L}})^{n-1}$ on $bD \times \overline{D} - \{(\zeta, \zeta) : \zeta \in bD\}$. As usual, for $f \in \mathcal{O}(\mathcal{D}) \cap C\overline{\mathcal{D}})$ one has the representation formula $f(z) = \int_{bD} f\ \Omega_0(W^{\mathcal{L}})(\bullet, z)$. Furthermore we note that for any $f \in L^1(bD)$ the integral transform

$$T^{\mathcal{L}}(f) = \int_{bD} f\ \Omega_0(W^{\mathcal{L}})$$

defines a C^∞ function on D.

While of course $\Omega_0(W^{\mathcal{L}})$ is not holomorphic in the parameter z, this kernel still has some special properties that might make it a useful tool. For example, in contrast to the Bochner–Martinelli kernel, its singularities reflect the complex geometry of the boundary. Furthermore, it is "more holomorphic" than the Bochner–Martinelli kernel, as follows. Given any $0 < \delta < 1/2$ one has an estimate

$$\left|\overline{\partial_z} \int_{bD} f\ \Omega_0(W^{\mathcal{L}})(\bullet, z)\right| \leq C_\delta\, |f|_0 \cdot dist(z, bD)^{\delta-1} \text{ for all } f \in C(bD).$$

For the BM-kernel such an estimate (for any derivative) holds only for $\delta = 0$. The proof of this estimate involves a careful balancing of the estimates from below for $|\Phi_K|$ and the estimates for the numerator that are given above with respect to $z-$ diagonalizing coordinates, while keeping track of numerous error terms. Full details are given in [41].

3.3 Pointwise A-priori Estimates

We shall now discuss how the Koppelman formulas discussed in Section 1.3, applied to the Levi generating form $W^{\mathcal{L}}$, lead to a version of the desired pointwise a-priori estimates. Recall the generating form W^{BM} for the Bochner–Martinelli kernel and the general BMK representation formula

$$f(z) = \int_{bD} f(\zeta) \wedge \Omega_q(W^{BM}) - \overline{\partial_z} \int_D f(\zeta) \wedge \Omega_{q-1}(W^{BM}) -$$

$$- \int_D \overline{\partial_\zeta} f(\zeta) \wedge \Omega_q(W^{BM})\ \ for\ z \in D,$$

for $f \in C^1_{(0,q)}(\overline{D})$ that is valid on an arbitrary smoothly bounded domain.

By utilizing the connection between the BMK-kernels and the fundamental solution ω_q for the complex Laplacian

$$\square = \vartheta\overline{\partial} + \overline{\partial}\vartheta = \frac{1}{4}\Delta$$

on $(0, q)$ forms, where ϑ denotes the *formal* adjoint of $\overline{\partial}$, so that $\vartheta f = \overline{\partial}^* f$ for $f \in \mathfrak{D}^1_q$, one may transform the above BMK formula into

$$f = \int_{bD} f \wedge \Omega_q(W^{BM}) + (\overline{\partial} f, \overline{\partial}\omega_q) + (\overline{\partial}^* f, \vartheta\omega_q)\ for\ f \in \mathfrak{D}^1_q.$$

Here $(\bullet, \bullet)$ denotes the standard L^2 inner product of forms over D, that is

$$(f, g) = \int_D (f, g)(\zeta) dV(\zeta) = \int_D f \wedge *\overline{g}.$$

The fundamental solution ω_q is an isotropic kernel whose regularity properties are well understood. In particular, the operator

$$S^{iso} : f \to S^{iso}(f) = (\overline{\partial} f, \overline{\partial}\omega_q) + (\vartheta f, \vartheta \omega_q)$$

satisfies a Hölder estimate

$$\left| S^{iso}(f) \right|_\delta \leq C_\delta Q_0(f) \, \textit{for all } f \in C^1_{(0,q)}(D) \textit{ and any } \delta < 1.$$

Consequently, the essential information regarding all pointwise estimations is contained in the boundary integral $S^{bD}(f) = \int_{bD} f \wedge \Omega_q(W^{BM})$. Note that the kernel $\Omega_q(W^{BM})$ is isotropic; it treats derivatives in all directions equally, and direct differentiation under the integral in $\int_{bD} f \wedge \Omega_q(W^{BM})$ leads to an expression that will in general blow up like $dist(z, bD)^{-1}$. So this elementary representation of the operator S^{bD} does not provide any useful information.

Note that since $\Omega_n(W^{BM}) \equiv 0$, it trivially follows that $|f|_\delta \leq C_\delta Q_0(f)$ for $f \in \mathfrak{D}^1_n$ and any $\delta < 1$. We shall therefore assume that $q < n$ in what follows.

As we discussed in Section 1.3, by classical results of W. Koppelman one may replace $\Omega_q(W^{BM})$ by any other CF kernel $\Omega_q(W)$ on the boundary bD, as follows.

Given any generating form W on $bD \times D$, one has

$$\int_{bD} f(\zeta) \wedge \Omega_q(W^{BM}) = \int_{bD} f(\zeta) \wedge \Omega_q(W) +$$
$$+ \overline{\partial_z} \int_{bD} f \wedge \Omega_{q-1}(W, W^{BM}) + \int_{bD} \overline{\partial} f \wedge \Omega_q(W, W^{BM}),$$

where the "transition" kernels $\Omega_q(W, W^{BM})$ involve explicit expressions in terms of W and W^{BM}.

Starting with the representation

$$f = S^{bD}(f) + S^{iso}(f) \text{ for } f \in \mathfrak{D}^1_{0,q}(\overline{D}),$$

we shall apply these Koppelman formulas with the *non-holomorphic* Levi generating form $W^{\mathcal{L}}(\zeta, z)$ introduced in the previous section in order to replace the boundary integral $S^{bD}(f) = \int_{bD} f \wedge \Omega_q(W^{BM})$ by

$$S^{bD}_{\mathcal{L}}(f) = \int_{bD} f \wedge \Omega_q(W^{\mathcal{L}}) +$$
$$+ \int_{bD} \overline{\partial} f \wedge \Omega_q(W^{\mathcal{L}}, W^{BM}) + \int_{bD} f \wedge \overline{\partial}_z \Omega_{q-1}(W^{\mathcal{L}}, W^{BM}).$$

By using this representation of $S^{bD}(f)$ by $S^{bD}_{\mathcal{L}}(f)$ we are able to prove the following a-priori estimates on a weakly pseudoconvex domain $D \subset\subset \mathbb{C}^n$.

Theorem *The integral operator*

$$S^{bD}_{\mathcal{L}} : C_{(0,q)}(bD) \to C^{\infty}_{(0,q)}(D)$$

has the following properties. If U is a sufficiently small neighborhood of the point $P \in bD$, there exist constants C_δ depending on $\delta > 0$, so that one has the following uniform estimates for all $f \in \mathfrak{D}^1_{qU}$, $1 \le q < n$, and for $z \in D \cap U$:

1) $\left| f - S^{bD}_{\mathcal{L}}(f) \right|_\delta \le C_\delta \cdot Q_0(f)$ for any $\delta < 1$;
2) $\left| \overline{L_j} S^{bD}_{\mathcal{L}}(f)(z) \right| \le C_\delta \cdot dist(z, bD)^{\delta-1} \cdot Q_0(f)$ for $j = 1, \ldots, n$ and any $\delta < 1/2$;
3) $\left| L_j S^{bD}_{\mathcal{L}}(f)(z) \right| \le C_\delta \cdot dist(z, bD)^{\delta-1} \cdot Q_0(f)$ for $j = 1, \ldots, n-1$ and any $\delta < 1/3$.

Furthermore, if $f_J \overline{\omega}^J$ is a normal component of f with respect to the frame $\overline{\omega}_1, \ldots, \overline{\omega}_n$, one has

$$|f_J|_\delta \le C_\delta Q_0(f) \text{ for any } \delta < 1/2 \text{ if } n \in J.$$

Note that if one had an estimate analogous to 3) also for the normal derivative $L_n S^{bD}_{\mathcal{L}}(f)(z)$ for *some* $\delta > 0$ (with $\delta < 1/3$), standard results would imply the Hölder estimate

$$\left| S^{bD}_{\mathcal{L}}(f) \right|_\delta \le C_\delta Q_0(f).$$

By using 1), one therefore would obtain an estimate

$$| f |_\delta \le C_\delta Q_0(f),$$

i.e., the Hölder analogon of a subelliptic estimate.

It is known that such an estimate does not hold on arbitrary pseudoconvex domains. On the other hand, as outlined in the next section, the theorem provides a starting point in a general setting which, combined with additional suitable properties of the boundary, such as *finite type*, might be useful to obtain appropriate estimates for $L_n S^{bD}_{\mathcal{L}}(f)$ and consequently lead to a Hölder estimate on suitable domains.

The proof of the theorem involves a careful analysis and estimations of the relevant derivatives of the boundary integrals in the representation of $S^{bD}_{\mathcal{L}}(f)$, utilizing, in particular, techniques we mentioned earlier in the context of the kernel $\Omega_0(W^{\mathcal{L}})$, as well as techniques from [28].

One particularly delicate new problem arises with the integral

$$\int_{bD} f \wedge \overline{\partial}_z \Omega_{q-1}(W^{\mathcal{L}}, W^{BM}).$$

Here one needs to shift the differentiation from the kernel in a suitable way onto the form f. This requires an elaborate detour involving Stokes' theorem and the Hodge $*$ operator in order to first transform the boundary integral into standard inner products of forms over D. One then must exploit certain approximate symmetries in the kernels to replace ∂_z with $\overline{\partial}_\varsigma$, so that one can use integration by parts to move the $\overline{\partial}_\varsigma$ differentiation from the kernel onto f in the form $\overline{\partial}^*_\varsigma f$.

The details of the proof are given in [42].

3.4 Outlook: A-priori Hölder Estimates

Given the general integral representation formulas and the "pointwise basic estimates" on arbitrary pseudoconvex domains we have discussed, the main problem before us is to investigate under what additional conditions on the boundary can the estimates be improved to obtain a Hölder estimate

$$|f|_\delta \leq C_\delta Q_0(f) \text{ for some } \delta > 0 \text{ and all } f \in \mathfrak{D}^1_{qU}.$$

Such an estimate would be an analogon in Hölder norm of a subelliptic estimate in the L^2 theory. Just as a subelliptic estimate is the critical ingredient to prove numerous regularity results in Sobolev spaces in the theory of the $\overline{\partial}$-Neumann problem, it is expected that a corresponding Hölder estimate would be a useful tool to prove analogous pointwise estimates for the $\overline{\partial}$-Neumann operator, for solutions of the $\overline{\partial}$-equation, and other related regularity results in Hölder norms.

In case D is strictly pseudoconvex with a Levi metric, the above estimate holds for $\delta = 1/2$, as was shown in 1986 by Lieb and Range [28].

One possible approach on general weakly pseudoconvex domains is based on developing an analogue of the machinery of *subelliptic multipliers* introduced by J.J. Kohn [21]. This would then allow to utilize algebraic/geometric techniques to determine conditions on the boundary that imply the desired estimate.

This work is still mainly in the design stages, and so far there are few definitive results, so we shall just give a brief sketch.

A germ of a C^∞ function μ at P which satisfies

$$|\mu f|_\delta \leq C_\delta Q_0(f) \text{ for all } f \in \mathfrak{D}^1_{qU} \text{ and for some } \delta > 0$$

on a sufficiently small neighborhood U on which μ is defined, is called a *q-Hölder multiplier at P*. We denote the set of such multipliers by $\mathcal{H}^q(P)$.

As in Kohn's work in the L^2 setting, the goal is to set up an algorithm involving, in particular, certain successive differentiations of multipliers, in order to generate more multipliers, and to identify conditions on bD at P that would eventually lead to a non-zero multiplier, that is to prove $1 \in \mathcal{H}^q(P)$, so that one gets the desired Hölder estimate $|f|_\delta \leq C_\delta Q_0(f)$.

It follows readily that $\mathcal{H}^q(P)$ is an ideal in the ring of germs of C^∞ functions at P.

The following two results are easy consequences of the general integral representation formula and the *basic estimates* stated above.

A) *If* $\mu = 0$ *on* bD *near* P*,then* $\mu \in \mathcal{H}^q(P)$ *for any* $q \geq 1$.
B) $1 \in \mathcal{H}^n(P)$, *with the Hölder estimate holding for any* $\delta < 1$.
   Much more delicate is the following result. Recall that r is the suitably chosen defining function for D near the point P.
C) *If* $n = 2$, *the eigenvalue of the Levi form (i.e., the coefficient of the* $(2, 2)$ *form* $\overline{\partial} r \wedge \partial r \wedge \overline{\partial}\partial r$*) is in* $\mathcal{H}^1(P)$.

Attempts to generalize C) to higher dimensions have not been successful so far. In particular, it appears that the restriction to $\delta < 1/2$, resp. $\delta < 1/3$, in the theorem stated above is a major obstacle. It thus seems that one should try to eliminate this restriction, that is, to prove estimate 2) (and perhaps 3)) for any $\delta < 1$. This would be consistent with the known results in the L^2 theory.

3.5 Some Conjectures

If the program to set up an algorithm for Hölder multipliers in the integral representation setting that somehow parallels Kohn's L^2 theory of subelliptic multipliers [21] is successful, it should then be possible to combine these results with Kohn's algorithm and a 1978 theorem of K. Diederich and J. E. Fornaess to obtain proofs of the following conjectures.

Conjecture I *If* bD *is real analytic and pseudoconvex in a neighborhood of* P, *and if there does NOT exist any germ of a complex subvariety* $V \subset bD$ *of dimension* q *with* $P \in V$*, then* $1 \in \mathcal{H}^q(P)$.

Conjecture II *If* D *is pseudoconvex with real analytic boundary, then there exists* $\delta > 0$ *and a solution operator* $S^D_{\overline{\partial}}$ *for* $\overline{\partial}$ *such that* $\left| S^D_{\overline{\partial}}(f) \right|_\delta \leq C_\delta \, |f|_0$.

Furthermore, by using results of Y.T. Siu [43], it might be possible to also prove

Conjecture III *Conjecture II is correct for a smoothly bounded pseudoconvex domain D of finite type.*

To summarize, I believe that an essentially optimal version of the theorem in Section 3.3, that is, the pointwise a-priori estimate on weakly pseudoconvex domains—analogous to the known basic L^2 estimate recalled in Section 3.1—is an important step in order to improve our understanding of the Cauchy–Riemann equations in this general setting. I hope that the discussion in Section 2.4 and in these last two sections may inspire future investigations to learn more about complex geometry and the Cauchy-Riemann equations on weakly pseudoconvex domains.

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On the Riemann Zeta Function and Gaussian Multiplicative Chaos

Eero Saksman and Christian Webb

Abstract We review some aspects of the statistical behavior of the Riemann zeta function on the critical line. Especially, we discuss how its functional statistics is related to Gaussian multiplicative chaos (Saksman and Webb, The Riemann zeta function and Gaussian multiplicative chaos: statistics on the critical line. Preprint arXiv:1609.00027).

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1 Background: Classical Results on Statistics of ζ

The subject of this brief and very partial review is the (functional) statistical behavior of the *Riemann zeta function*

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$$\zeta(s) := \sum_{n=1}^{\infty} n^{-s} \qquad \text{for} \quad \sigma > 1, \qquad s = \sigma + it.$$

We deliberately avoid discussing many technicalities and fine points, and we omit many results since the text below is intended for non-specialists. Basically no proofs are given.

Some vague knowledge of the rudiments of analytic number theory is useful for the reader, but as we give no proofs below, actually very very little will be assumed in order to understand the statements. Basically what one needs to know is that one may continue ζ as a meromorphic function to the whole complex plane, with only one pole. The pole is simple and located at $s = 1$ with residue 1. What comes to growth of $\zeta(s)$ as $s \to \infty$, it behaves in a rather mild manner. In the whole plane $(s-1)\zeta(s)$ is an entire function of order 1, and in the closed half-plane $\{\sigma \geq 1/2\}$ the fastest growth is on the critical line $\{\sigma = 1/2\}$. Almost hundred years old estimates [76, Section 5.18] tell us that

$$\zeta\left(\frac{1}{2} + it\right) = \mathcal{O}(t^{\mu_0}), \quad \text{with some} \quad \mu_0 < 1/6. \tag{1.1}$$

Currently the best known bound is $\mu_0 = 13/84 + \varepsilon$ [17]. The famous Lindelöf hypothesis declares that (1.1) is true for any $\mu_0 > 0$, and would in turn be a consequence of the Riemann hypothesis. Other basic features of ζ include the Euler product formula (see (1.2) below) and the functional equation of ζ. The books [42, 75, 76], among many others, are excellent references for further properties of the Riemann zeta function.

In the remainder of this introductory section, we will formulate explicitly what we mean by statistical behavior and review some classical results concerning this statistical behavior – for further information, we refer the reader to the nice monograph of Laurinčikas [52] devoted to statistical behavior of ζ. After this introduction, we turn to more modern questions regarding the statistical behavior. In particular, we will focus on the connection between statistics of the zeta function, stochastic processes known as log-correlated fields, and the theory of multiplicative chaos. More precisely, in Section 2, we review the connection between the zeta function and log-correlated fields. Section 3 contains a very short introduction to the theory of multiplicative chaos. In Section 4, we discuss the connection between the zeta function and multiplicative chaos. In turn, Section 5 describes how this connection to multiplicative chaos can be used to relate mesoscopic behavior of the zeta function with mesoscopic behavior of certain random matrices. Finally in Section 6, we review some further results and conjectures about the connection between the zeta function and multiplicative chaos.

In what follows we shall denote by $\mathcal{P} = \{2, 3, 5, \ldots\}$ the set of prime numbers.

1.1 Towards Functional Statistics of ζ: The Easy Case of $\sigma > 1$

The statistics in this range are easily described and the result is not surprising, but it provides us with a simple playground to illustrate what one actually means by functional statistics, and it also displays the ubiquitous role of the Euler product formula

$$\zeta(s) := \prod_p (1 - p^{-s})^{-1} \qquad \sigma > 1 \tag{1.2}$$

as well as that of rational independence of $\log p$:s for $p \in \mathcal{P}$. The question of functional statistics of the Riemann zeta function on the vertical line $\sigma = \sigma_0$ where $\sigma_0 \geq 1/2$ roughly amounts to taking a "random" length 1 subinterval I of this line and asking for the statistics of the restrictions $\zeta|_I$. As such the question is perhaps not defined explicitly enough, so in order to make it rigorous we let $\omega \in [0, 1]$ be uniformly distributed, and we consider $[0, 1]$ equipped with the Lebesgue measure as our probability space. Pick a large number $T > 1$ and consider the random function

$$x \mapsto g_T(x) := \zeta(\sigma_0 + ix + iT\omega)$$

defined on $x \in [0, 1]$.[1] One then asks:
BASIC QUESTION: Is there a limiting probability distribution for the random variable $\zeta(\sigma_0 + ix + iT\omega)$ as $T \to \infty$?
One should note that our random variable g_T takes values in the set of (continuous) functions defined on $[0, 1]$. Thus the desired limit distribution is expected to be a probability distribution on suitable class of functions (or perhaps generalized functions...) on the interval $[0, 1]$.

To approach this question, fix $\sigma_0 > 1$ and observe that by the Euler product our g_T takes the form

$$g_T(x) = \prod_p \frac{1}{1 - p^{-\sigma_0 - ix} p^{-i\omega T}}. \tag{1.3}$$

We then make the crucial observation that one may easily describe the limiting statistics of the terms $p^{-i\omega T}$:

[1]One could well let x vary in some larger interval than $[0, 1]$, or even on $\mathbb{R}$, but for simplicity we stick to $x \in [0, 1]$.

Lemma 1.1 *In the limit $T \to \infty$, the random variables $p^{-i\omega T}$, $p \in \mathcal{P}$, tend in the sense of finite dimensional distributions*[2] *to independent random variables that are all uniformly distributed on* $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$.

It is an instructive exercise for the reader to find her/his own proof[3] of this observation. Since $\sigma_0 > 1$, we have uniform convergence in (1.3), and it is then an easy matter to deduce that the statistical limit of g_T is given by the *randomized zeta function* $\zeta_{\mathrm{rand}}(\sigma_0 + ix)_{|[0,1]}$, where

Definition 1.2 $\zeta_{\mathrm{rand}}(s)$ is the random analytic function on $\{\sigma > 1/2\}$ given by the *randomized Euler product*

$$\zeta_{\mathrm{rand}}(s) = \prod_{p \in \mathcal{P}} \left(1 - p^{-is} e^{i\theta_p}\right)^{-1}, \qquad \sigma > 1/2,$$

where θ_p, $p \in \mathcal{P}$, are independent random variables with uniform distribution on $[0, 2\pi]$. $\diamond$

Obviously ζ_{rand} is analytic in $\sigma > 1$, but due to random cancellations arising from the terms $e^{i\theta_p}$ it is relatively easy to check that ζ_{rand} almost surely has an analytic extension to the half-plane $\{\sigma > 1/2\}$. This was first observed by Helson [35], and later on reproved by Bagchi [5] and Hedenmalm, Lindqvist, and Seip [36].

1.2 Bohr, Jessen, and Bagchi: The Case of $\sigma > 1/2$

When one moves from the half-plane $\{\sigma > 1\}$ in to the open right half of the critical strip $\{1/2 < \sigma < 1\}$ nontrivial obstructions arise due to the fact that the Euler product does not converge anymore and hence cannot be directly used to represent ζ. One needs to invoke more serious tools from analytic number theory, and utilize suitable approximative formulas for the Riemann zeta. In this range the statistics of ζ were studied by Bohr and Jessen in the 1930s [12, 13], and they were able to prove the existence of some kind of a (pointwise) statistical limit. Later on, Bagchi [5] identified the limiting functional statistics with $\zeta_{\mathrm{rand}}(s)$ defined above.

Further results related to statistics in the strip $\{1/2 < \sigma < 1\}$ were obtained by Voronin who proved in the 1970s [77] the famous *Voronin universality result*, which states that given a non-vanishing analytic function f on a neighborhood of the ball

[2]To be precise, this means that for any finite collection of primes $p_1, \ldots, p_k \in \mathcal{P}$, the probability distribution of $(p_1^{-i\omega T}, \ldots, p_k^{-i\omega T})$ tends to that of k i.i.d. random variables which are uniformly distributed on the unit circle.

[3]Hint: One may, e.g., easily compute the (mixed) moments of the variables, and the desired statement then boils down to the rational independence of the set $\{\log p : p \in \mathcal{P}\}$, which in turn is a restatement of uniqueness of the prime factor decomposition of natural numbers.

$B(1/2, r)$, where $r < 1/4$, there are translations of ζ to the vertical direction that approximate f with an arbitrary precision. More precisely, for given $\varepsilon > 0$ we may find $T > 1$ so that

$$|f(s) - \zeta(s + iT)| < \varepsilon \qquad \text{for} \quad s \in B(1/2, r).$$

1.3 Selberg: Pointwise Statistics in the Case of $\sigma = 1/2$

The fundamental result concerning the pointwise statistical behavior of $\zeta(1/2 + it)$ is Selberg's central limit theorem on the critical line:

$$(\frac{1}{2}\log\log(T))^{-1/2}\log\left|\zeta(1/2 + iT + i\omega T)\right| \xrightarrow{d} N(0, 1) \tag{1.4}$$

as $T \to \infty$. Originally Selberg [71] proved the same result for the S-function that is essentially the imaginary part of $\log(\zeta(it + 1/2))$. Here one needs to be careful how to define the logarithm unconditionally, see, e.g., [76, Section 9]. Selberg's method was based on estimating moments. Later on he also obtained a similar result for the real part of the logarithm, namely (1.4). These results have been generalized in many ways. For example, it is known that if one drops the absolute values above from the logarithm, and looks at the joint distribution of the real and imaginary parts of the logarithm, the convergence in law is to a multiple of a standard complex Gaussian. Alternatively, one can replace T by $e^{(\log T)^x}$ with $x > 0$ and consider this as a stochastic process in x [39]. We also refer to [15, 16, 56, 69], and the references therein for interesting generalizations of Selberg's result. It is also useful to note that [67] provides a short and self-contained proof of Selberg's central limit theorem.

2 Log-Correlated Fields

2.1 Emergence of Log-Correlated Field: Heuristics and Facts

We shall follow heuristics from Fyodorov's and Keating's work [29]. There one assumes the statistical validity of the Dirichlet series representation of $\log\zeta(s)$ up to the critical line, which leads to suggesting that the statistical behavior of $\log\zeta(1/2+it)$ as a *random function* should resemble that of a *log-correlated field* – formally a stochastic process with a logarithmic singularity on the diagonal of its covariance. We shall discuss the rigorous definition and examples of log-correlated fields in the next subsection, but before that let us try to imitate the heuristics mentioned above. Thus, in what follows, we will simply have fun and calculate in the classical "Eulerian spirit" without concern for validity of convergence of our expressions.

Start from the Euler product and compute

$$\log(\zeta(s)) = \log\left(\prod_p \Big(\frac{1}{1-p^{-s}}\Big)\right) = \sum_p \sum_{k\geq 1} \frac{1}{k} p^{-ks}.$$

Calmly substitute $s = 1/2 + it$ and note that the sum over terms with $k \geq 3$ is uniformly bounded for $\sigma = \mathrm{Re}\, s \geq 1/2$. Also the terms with $k = 2$ "almost converge" (will be "statistically negligible"). So we obtain as the leading order (heuristic) approximation

$$\log(\zeta(1/2+it)) \sim \sum_p \frac{p^{-it}}{\sqrt{p}}, \quad \text{or}$$

$$\log|\zeta(1/2+it)| \sim \mathrm{Re}\Big(\sum_p \frac{p^{-it}}{\sqrt{p}}\Big).$$

Recall from the previous section, that in order to study functional statistics of $\log|\zeta(1/2+it)|$ we choose $T \gg 1$ and pick $\omega \in [0,1]$ at random, and consider the "random shifts" $x \mapsto \log|\zeta(1/2 + i\omega T + ix)|$. When this is substituted to our "leading order approximation" for $\log(\zeta)$ we obtain the quantity

$$\mathrm{Re}\Big(\sum_p \frac{p^{-ix} p^{-i\omega T}}{\sqrt{p}}\Big).$$

According to Lemma 1.1 in the limit $T \to \infty$ we end up with the limiting statistics f, where the random function f is given by

$$f(x) := \mathrm{Re}\left(\sum_p \frac{e^{i\theta_p} p^{-ix}}{\sqrt{p}}\right), \qquad x \in [0,1], \tag{2.1}$$

and again the θ_p's are i.i.d. and uniform on $[0, 2\pi)$. Let us the formally compute the covariance structure of our field f. By denoting $F(x) := \sum_p \frac{e^{i\theta_p} p^{-ix}}{\sqrt{p}}$, a straightforward formal calculation yields

$$\begin{aligned}
\mathbb{E}\, f(x)f(y) &= \frac{1}{4}\mathbb{E}\left(F(x)F(y) + F(x)\overline{F(y)} + \overline{Fx)}F(y) + \overline{F(x)F(y)}\right)\\
&= \frac{1}{2}\mathrm{Re}\left(\sum_p \frac{p^{-i(x-y)}}{p}\right)\\
&= \frac{1}{2}\log|\zeta(1+i(x-y))| \; + \; \text{smooth}\\
&= \frac{1}{2}\log\Big(\frac{1}{|x-y|}\Big) + \; \text{smooth.}
\end{aligned}$$

Thus the covariance function is translation invariant and has a logarithmic singularity at the diagonal! One may check that (2.1) does not define a random function, but it does represent a well-defined random generalized function. Such random generalized functions with a logarithmic singularity on the diagonal have been studied extensively in the setting of Gaussian processes, in which case they are termed Gaussian log-correlated fields, and we shall shortly turn to discussing them.

Another motivation for treating shifts of $\log(\zeta)$ on the critical line statistically as a log-correlated field comes from the Montgomery–Keating–Snaith (see [48, 61]) picture of modeling the zeta function on suitable scales as a characteristic polynomial of a large random matrix which we discuss more in Section 5.1 later on. When this is combined with Lemma 2.3 below, one recovers the principle message of the above heuristics from another point of view.

Finally, it is important to observe that besides the above heuristic line of thinking there is also some real evidence of the log-correlated statistical nature of shifts of $\log(\zeta)$ on the critical line. We have here in mind the results by Bourgade [15], Bourgade and Kuan [16], Rodgers [69], as well as Maples and Rodgers [56]. In [15] Bourgade showed that if one normalizes $\log \zeta$ in a suitable way, then on a certain mesoscopic scale, for some range of the normalized distance $|x - y|$ the statistics behaves like a certain Gaussian field. Another form of this connection was provided in [16, 56, 69], where "mesoscopic linear statistics" of the zeta function zeroes were shown to have a Gaussian limit with a covariance form given by the homogenous $W^{1/2}$-inner product.

2.2 Log-Correlated Gaussian Fields

A centered Gaussian field on a domain $\Omega \subset \mathbb{R}^d$ can be thought of as a random function $X : \Omega \to \mathbb{R}$ such that all the evaluation vectors $(X(z_1), \ldots, X(z_n))$ are multivariate and centered Gaussians. They are (essentially) determined by knowledge of the covariance function $C_X(z, z') := \mathbb{E}\, X(z)X(z')$. As mentioned, we are interested in log-correlated fields, namely the case where this covariance has a logarithmic singularity on the diagonal. In this situation, the rigorous definition is slightly more involved. To illustrate some aspects of the general theory of such objects, let us first focus on an example.

A basic 1-dimensional example is given by the centered Gaussian field X on the torus $\mathbb{T} := \{|z| = 1\}$ (i.e., the complex unit circle).

Definition 2.1 $X_{\mathbf{ci}}$ is the centered Gaussian field on $\mathbb{T}$ defined by the covariance structure

$$C_{X_{\mathbf{ci}}}(z, z') = \text{``}\mathbb{E}\; X_{\mathbf{ci}}(z)X_{\mathbf{ci}}(z')\text{''} = \log\Big(\frac{1}{|z - z'|}\Big) \qquad \text{for} \quad z, z' \in \mathbb{T}.$$

The field $X_{\mathbf{ci}}$ is called *Gaussian Free Field* (GFF), *restricted to* $\mathbb{T}$. As the covariance blows up at the diagonal, the field $X_{\mathbf{ci}}$ cannot be realized as a random function, but it takes values in the space of generalized functions – this means that one needs some care in handling them! $\diamond$

In order to verify the existence of such a field, one simply writes down a random Fourier series:

$$X_{\mathbf{ci}}(e^{i\theta}) := \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}\big(A_n \cos(n\theta) + B_n \sin(n\theta)\big), \tag{2.2}$$

where $A_n \sim N(0,1) \sim B_n \quad (n \geq 1)$ are i.i.d. standard Gaussians. One can readily verify that this series converges almost surely in the Sobolev space of generalized functions $H^{-s}(\mathbb{T})$ for any $s > 0$. That $X_{\mathbf{ci}}$ possesses the right covariance is seen by a (slightly formal) computation: Let $z = e^{it}$ and $z' = e^{it'}$. Then

$$\begin{aligned}
\mathbb{E}\, X_{\mathbf{ci}}(z) X_{\mathbf{ci}}(z') \;&=\; \mathbb{E}\Bigg(\Big(\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}\big(A_k \cos(kt) + B_k \sin(kt)\big)\\
&\qquad\qquad \times \Big(\sum_{m=1}^{\infty} \frac{1}{\sqrt{m}}\big(A_m \cos(mt') + B_m \sin(mt')\big)\Big)\Bigg)
\end{aligned}$$

$$= \sum_{k=1}^{\infty} \frac{1}{k}\Big(\cos(kt)\cos(kt') + \sin(kt)\sin(kt')\Big) = \sum_{k=1}^{\infty} \frac{1}{k}\cos(k(t-t'))$$

$$= \mathrm{Re} \sum_{k=1}^{\infty} \frac{1}{k} e^{ik(t-t')} = \mathrm{Re}\big(\log(1 - z/z')^{-1}\big) \;=\; \log\Big(\frac{1}{|z-z'|}\Big).$$

Another typical example of a log-correlated is the 2-dimensional *Gaussian Free Field* (GFF), which is defined in a domain $\Omega \subset \mathbb{R}^2$, and its covariance is given by $G_\Omega(z,z')$, where G_Ω is the Green's function of Ω (e.g., with zero boundary conditions). Again we have a logarithmic singularity in the covariance. The 2-dimensional free field is perhaps the most important log-correlated field in that it occurs quite often as a scaling limit of various statistical models. However, in view of the fact that applications related to the Riemann zeta function mainly concern one-dimensional log-correlated fields, we have no need here to discuss further the 2-dimensional GFF.

More generally, one considers *log-correlated fields* that are centered Gaussian fields X on a domain $U \subset \mathbb{R}^d$, such that

$$C_X(x,y) := \mathbb{E}\, X(x)X(y) = \log\Big(\frac{1}{|x-y|}\Big) + g(x,y) \qquad \text{for} \quad x, y \in U, \tag{2.3}$$

where g is continuous (or smooth). One can again show that such objects make sense as random elements in suitable Sobolev spaces of generalized functions: see, e.g., [45, Proposition 2.3].

The GFF or, more generally, log-correlated fields almost abound in today's literature on statistical models. For example, they appear as scaling limits of fluctuations of many random models of statistical physics. Before continuing, we take a quick look at one particular instance that is of relevance to us later on.

Example 2.2 We consider the statistical limit behavior of the logarithm of the characteristic polynomial of *random unitary matrices*. Let U_N stand for a $N \times N$ CUE-random matrix. In other words, U_N is the $N \times N$ random unitary matrix whose law is the Haar measure on the unitary group $\mathrm{U}(N)$. We let $\{z_1, \dots z_N\} \subset \mathbb{T}$ stand for the spectrum of U_N and denote by

$$p_n(z) := \prod_{j=1}^{N} (z - z_j)$$

the characteristic polynomial of U_N. We are interested in the behavior of

$$Y_N(\theta) := \log|\det(U_N - e^{i\theta})| = \log|p_n(e^{i\theta})|.$$

Lemma 2.3 (Szegő; Diaconis and Shahshahani; Johansson; Hughes, Keating, and O'Connell) *As $n \to \infty$, Y_n tends to $\frac{1}{\sqrt{2}} X_{\mathbf{ci}}$ in distribution (as a generalized random function).*[4]

Proof A very short sketch: One observes that

$$Y_N(\theta) = \log|\det(U_N - e^{i\theta})| = -\frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k}\Big[e^{-ik\theta}\mathrm{Tr}\, U_N^k + e^{ik\theta}\mathrm{Tr}\, U_N^{-k}\Big].$$

By [24] one has

$$\mathbb{E}\,\mathrm{Tr}\, U_N^j U_N^{-k} = \delta_{k,j}\min(|k|, N)$$

and any finite collection of the variables $\mathrm{Tr}\, U_N^k$ with $k > 0$ converges in law to independent centered Gaussians. This easily yields the convergence as generalized (random) functions. Alternatively, one could as well use the famous strong Szegő theorem [74, Chapter 6] to deduce the result. □

◇

[4] It is not easy to say where this fact was really observed for the first time; however, it is explicitly stated in the paper of [38].

Other examples of statistical models where log-correlated fields arise are, e.g., the dimer model, models for random partitions of integers, and random growth models – see, e.g., [14, 41, 49].

It is not difficult to show (see, e.g., [45, Lemma 2.5]) that log-correlated fields are rather mild generalized functions, since smoothing them just by a little produces immediately fields with Hölder-continuous realizations.

When one deals with log-correlated fields, e.g., when we later on construct multiplicative chaos out of them, it is important to provide good regularizations of them. For that purpose, we call a sequence $(X_n)_{n\geq 1}$ of continuous jointly Gaussian centered fields on U a *standard approximation* of X if it satisfies:

(i) One has

$$\lim_{(m,n)\to\infty} \mathbb{E}\, X_m(x)X_n(y) = C_X(x,y),$$

where convergence is in measure with respect to the Lebesgue measure on $U \times U$.

(ii) There exists a sequence $(c_n)_{n=1}^{\infty}$ such that $c_1 \geq c_2 \geq \ldots > 0$, $\lim_{n\to\infty} c_n = 0$, and for every compact $K \subset U$

$$\sup_{n\geq 1} \sup_{x,y\in K} \left| \mathbb{E}\, X_n(x)X_n(y) - \log \frac{1}{\max(c_n, |x-y|)} \right| < \infty.$$

(iii) We have

$$\sup_{n\geq 1} \sup_{x,y\in U} \left[\mathbb{E}\, X_n(x)X_n(y) - \log \frac{1}{|x-y|} \right] < \infty.$$

We also say that the approximation X_n is "at covariance level" $\log(1/c_n)$, since at the diagonal the covariance of X_n equals $\log(1/c_n) + O(1)$ by (ii).

Example 2.4 For the field $X_{\mathbf{ci}}$ (recall (2.2)) the partial sums

$$X_{\mathbf{ci},N} := \sum_{n=1}^{N} \frac{1}{\sqrt{n}} \big(A_n \cos(n\theta) + B_n \sin(n\theta) \big)$$

yield a standard approximation. This standard approximation additionally has independent increments. ◇

Example 2.5 If X is a log-correlated field and $\varepsilon_n \searrow 0$ as $n \to \infty$, then the usual mollifications

$$X_{\varepsilon_n} := \varphi_{\varepsilon_n} * X$$

provide a standard approximation on the covariance level $\log 1/\varepsilon$. Here $\varphi_\varepsilon = \varepsilon^{-d}\varphi(\cdot/\varepsilon)$, and $\varphi \in C_c^\infty(\mathbb{R}^d)$ with $\int \phi = 1$. $\diamond$

Apart from their natural appearance in various models of probability and mathematical physics, one of the main reasons for studying log-correlated fields is that it is believed that they have many universal features – properties that stem from the logarithmic singularity in the covariance, not from the precise details of the covariance. One such feature is the maximum of the field, which of course only makes sense once the field has been regularized. For example, for a standard approximation at covariance level $\log c_n^{-1}$, one expects that for any compact $K \subset U$ with non-empty interior, $\max_{x\in K} X_n(x) = (1 + o(1))\sqrt{2d}\log c_n^{-1}$. In fact, much stronger results exist – see, e.g., [25, 53].

3 Multiplicative Chaos

3.1 Gaussian Multiplicative Chaos Measures

The foundations of the mathematical theory of *Gaussian multiplicative chaos* were established in the 1980s by Kahane [47]. At that time, the main motivation was the desire to build mathematical models for Kolmogorov's statistical theory of turbulence by providing a continuous counterpart for multiplicative cascades that were originally introduced by Mandelbrot for the same purpose in the early 1970s. During the last 15 years there has been a new wave of interest in multiplicative chaos, partly due to its important connections to Stochastic Loewner Evolution [4, 73], to quantum gravity and scaling limits of random planar maps [10, 23, 26, 50, 57–60], as well as to models in finance and turbulence [68, Section 5]. We will later discuss briefly also the role of multiplicative chaos in random matrix theory.

In order to give a brief and informal description of multiplicative chaos, consider a sequence of a.s. continuous and centered real-valued Gaussian fields X_n, say on an interval $I \subset \mathbb{R}$. The elements of this sequence should be considered as suitable approximations of a (possibly generalized function valued) Gaussian field X. For simplicity, assume that the increments $X_{n+1} - X_n$ are independent. One may then define the random measures λ_n on I by setting

$$\lambda_n(dx) := \exp(X_n(x) - \frac{1}{2}\mathbb{E}\, X_n(x)^2)dx.$$

In this situation, the density is martingale, and basic martingale theory implies that almost surely there exists a (random) limit measure $\lambda = \lim_{n\to\infty} \lambda_n$, where the convergence is understood in the weak*-sense. The measure λ is called the *multiplicative chaos measure* defined by X, often denoted in the physics literature by

$$\lambda = \text{“}\exp(X)\text{”}.$$

Kahane showed that under suitable conditions the limit does not depend on the choice of the approximating sequence (X_n). However, a significant obstacle in defining a meaningful limiting object λ is that it may very well be the zero measure almost surely.

The most important, and in some sense a borderline situation for defining meaningful limiting objects, is when the limit field X is log-correlated, as described in Section 2.2. Then it is natural to try to define "$\exp(\beta X)$," where $\beta > 0$ is a real parameter (the "inverse temperature"). In this case

$$C_{\beta X}(x, y) = \beta^2 \log|x - y| + \beta^2 g(x, y), \qquad x, y \in I,$$

where g is a continuous and bounded function. Kahane's theory implies that the limit measure is almost surely non-zero for $0 < \beta < \sqrt{2}$. For chaos in dimension d the corresponding bound is $\beta < \sqrt{2d}$. The limiting random Borel measure $\lambda = \lambda_\beta$ on the interval I is almost surely singular and its basic properties like multifractal spectrum, tail of the total mass or scaling properties have been investigated – see, e.g., [68] for further information.

For the reader's benefit, let us here sketch the proof the nontriviality of the limit in the so-called L^2-range $\beta \in (0, 1)$, where the proof is particularly simple (similar considerations appeared already in [37]). Assume for simplicity that $(X_n)_{n\geq 1}$ is a standard approximation of X with independent increments – as we saw in Example 2.4, such approximations do exist. We may simply compute $\mathbb{E}\,(\lambda_{\beta,n}(I))^2 \leq C$ for all $n \geq 1$:

$$\begin{aligned}\mathbb{E}\,(\lambda_{\beta,n}(Q_0))^2 \;&=\; \mathbb{E}\left[\Big(\int_I \exp\Big(\beta X_n(x) - \frac{1}{2}\mathbb{E}\,(\beta X_n(x))^2\Big)dx\Big)\right.\\ &\qquad\qquad \left.\times \Big(\int_I \exp\Big(\beta X_n(y) - \frac{1}{2}\mathbb{E}\,(\beta X_n(y))^2\Big)dy\Big)\right]\\ &= \int_{I\times I} \exp\Big(-\frac{1}{2}\mathbb{E}\,(\beta X_n(x))^2 - \frac{1}{2}\mathbb{E}\,(\beta X_n(y))^2\Big)\mathbb{E}\,\exp\Big(\beta X_n(x) + \beta X_n(y)\Big)dxdy\\ &= \int_{I\times I} \exp\Big(-\frac{1}{2}\mathbb{E}\,(\beta X_n(x))^2 - \frac{1}{2}\mathbb{E}\,(\beta X_n(y))^2\Big)\exp\Big(\frac{1}{2}\mathbb{E}\,\big(\beta X_n(x) + \beta X_n(y)\big)^2\Big)dxdy.\end{aligned}$$

The only probability fact we used above is the basic formula

$$\mathbb{E}\,\exp(Y) = \exp(\frac{1}{2}\mathbb{E}\,Y^2), \tag{3.1}$$

when Y is a centered Gaussian random variable. We may thus continue to obtain and

$$\begin{aligned}\mathbb{E}\,(\lambda_{\beta,n}(I))^2 \;&=\; \int_{I\times I} \exp\Big(\beta^2\mathbb{E}\,X_n(x)X_n(y)\Big)dxdy = \int_{I\times I} \exp\Big(\beta^2\mathbb{E}\,C_{X_n}(x, y)\Big)dxdy\\ &\leq \int_{I\times I} \exp\Big(\beta^2\big(\log(1/|x - y|) + O(1)\big)\Big)dxdy.\end{aligned}$$

This entails that the total mass $\lambda_{\beta,n}(I)$ is a positive L^2-bounded martingale with respect to increasing n and naturally defined sigma-fields. Hence the limit $\lambda(I)$ is nontrivial, and non-zero a.s. as one may check by Kolmogorov's 0-1 law.

The topological support of the measure λ_β is the whole interval I. However, almost surely its Hausdorff-dimension as a Borel measure is $1 - \beta^2/2$ (and λ_β is a.s. exact dimensional). The starting point is Kahane's (see [68, Section 4.1]) observation that the chaos measure (if it exists) is concentrated on points $x \in I$ with the property

$$\lim_{n\to\infty} (\mathbb{E}\, X_n(x)^2)^{-1} X_n(x) = \beta, \tag{3.2}$$

i.e., on *β-thick points* of the field X. Especially, the measures λ_β live on disjoint Borel sets for distinct β.

The dependence of the chaos measure on the generating Gaussian field has many delicate features. For example, the universality property (how the law of the limiting object is independent of the precise details of the approximation scheme) is far from trivial for multiplicative chaos [44, 68, 72]. We refer to the nice survey [68] for the basic properties of these measures, to [9] for an elegant proof of the existence of subcritical chaos measures, and to [8, 27, 28] for the existence and basic properties of critical Gaussian chaos (discussed next).

3.2 Critical and Supercritical Chaos

At the threshold $\beta = \beta_c := \sqrt{2d}$ one needs to add a deterministic nontrivial renormalization factor that depends on n in order to obtain the existence of a nontrivial object known as a *critical chaos measure*. Thus, let X_n be a standard approximation of a nice log-correlated field on the covariance level c_n as defined in Section 2.2. Under some mild smoothness conditions for g in the covariance structure (2.3) [28, 44, 46] there exists a nontrivial random measure

$$\lambda_{\sqrt{2d}} := \lim_n (c_n)^{1/2} \exp\Big(\sqrt{2d}\, X_n(x) - d\, \mathbb{E}\, X_n(x)^2\Big) dx, \tag{3.3}$$

where the limit is in, e.g., in the sense of weak convergence of evaluations against test functions. This limit can also be achieved through a random normalization known as the derivative martingale [27, 65].

The supercritical case $\beta > \sqrt{2d}$ has been treated for cascades [55, 78] and for some interesting cases of multiplicative chaos [54]. Here one encounters the phenomenon known as freezing, which in terms of chaos means that in order to obtain convergence to a measure one needs to perform a stronger renormalization than in the critical case, i.e., the extra factor $(c_n)^{1/2}$ in the martingale normalization has to be replaced by the factor

$$c_n^{\frac{3\beta}{2\sqrt{2d}}} e^{c_n\left(\frac{\gamma}{\sqrt{2}} - \sqrt{d}\right)^2}. \tag{3.4}$$

3.3 Complex Chaos

There is a further variant of multiplicative chaos that is important for the connection to the Riemann zeta function which is the concept of *complex multiplicative chaos*, where in the above one allows for complex Gaussian fields. Two basic cases have been studied in the literature. In the first variant one allows the parameter β take complex values, and it turns out that one obtains analyticity in the parameter β for $\beta \in U$, where $U \subset \mathbb{C}$ is an open subset that is the interior of the convex hull of the unit disc and the points $\pm\sqrt{2d}$ (see [6, 7] in the slightly simpler case of multiplicative cascades and [4, 46] in the case of multiplicative chaos). In the second case one assumes that $X = \beta_1 X_1 + i\beta_2 X_2$ with X_1, X_2 independent copies of a log-correlated field and $\beta_1, \beta_2 \in \mathbb{R}$. This case turns out to be more amenable to analysis, due to the independence of the real and imaginary parts, and many aspects of it have been studied thoroughly in [51]. Further study of the case of purely imaginary β is contained in [45]. However, as will be discussed in Section 4.2 the complex chaos we need to study here does not quite fit into either of these models.

4 Riemann Zeta and Multiplicative Chaos

As chaos is the "exponential of a log-correlated field," and one loosely speaking expects random shifts of $\log \zeta(1/2 + it)$ to have log-correlated statistics (in Section 2.1 this was done only to the real part, but we could have as well treated $\log \zeta(1/2 + it)$ itself), this leads us to suggest that perhaps the functional statistics of random shifts of ζ themselves should be given by some kind of multiplicative chaos. This heuristics was made rigorous in [70]. The first main result of [70] states that

Theorem 4.1

(i) *There exists a nontrivial random variable $x \mapsto \zeta_{\mathrm{rand}}(1/2 + ix)$ taking values in $W^{-\alpha,2}(0, 1)$, such that as $T \to \infty$*

$$\zeta(1/2 + ix + iT\omega) \xrightarrow{d} \zeta_{\mathrm{rand}}(1/2 + ix),$$

where the convergence in law is with respect to the strong topology of the Sobolev space $W^{-\alpha,2}(0, 1)$ for any $\alpha > 1/2$.

(ii) *Moreover, the law of the limit ζ_{rand} can be characterized in the following way: as random generalized functions*

$$\zeta_{\mathrm{rand}}(1/2 + ix) = g(x)\nu(x),$$

where ν is a random generalized function known as a Gaussian multiplicative chaos distribution, which can be formally written as

$$\nu(x) = \text{``}\ e^{\mathcal{G}(x)},\ \text{''}$$

where $\mathcal{G}$ is a centered Gaussian field on $(0, 1)$ *with the correlation structure*

$$\mathbb{E}\, \mathcal{G}(x)\mathcal{G}(y) = 0 \quad \text{and} \quad \mathbb{E}\, \mathcal{G}(x)\overline{\mathcal{G}(y)} = \log(\zeta(1+i(x-y))) \quad \text{for } x, y \in (0, 1).$$

The factor g is a random smooth function on $\mathbb{R}$, almost surely has no zeroes, and for which $\mathbb{E}\left(\|g(x)\|^p_{C^\ell(I)} + \|1/g(x)\|^p_{C^\ell(I)}\right)$ is finite for all $p \in \mathbb{R}$, any $\ell \geq 0$, and any finite interval $I \subset \mathbb{R}$.

Above one made use of the standard L^2-based Sobolev spaces: a tempered distribution f on $\mathbb{R}$ belongs to $W^{s,2}(\mathbb{R})$ if its Fourier transform satisfies $(1+|\xi|^2)^{s/2}\widehat{f}(\xi) \in L^2(\mathbb{R})$. A generalized function f on $(0, 1)$ belongs to $W^{s,2}(0, 1)$ if it is a restriction to $(0, 1)$ of some element in $W^{s,2}(\mathbb{R})$. Loosely speaking, this is equivalent to having the s:th derivative of f in $L^2(I)$.

4.1 Some Ingredients of the Proof of Part (i) of Thm 4.1

For simplicity, let us denote by

$$\mu_T(x) := \zeta(1/2 + ix + i\omega T) \quad \text{for} \quad x \in (0, 1)$$

the random function whose limit statistics we would like to understand in the limit $T \to \infty$. It turns out that the truncated Euler products

$$\mu_{T,N}(x) := \prod_{k=1}^{N} (1 - p_k^{-1/2-ix-iT\omega})^{-1},$$

where p_k:s are the primes in an increasing order, yield a good enough approximation to μ_T as $N \to \infty$. The proof uses in a crucial manner the explicit $T \to \infty$ limit of the two-point functions

$$\mathbb{E}\, \mu_T(x)\overline{\mu_T(y)}, \quad \mathbb{E}\, \mu_{T,N}(x)\overline{\mu_T(y)}, \quad \text{and} \quad \mathbb{E}\, \mu_{T,N}(x)\overline{\mu_{T,N}(y)}.$$

These can be controlled by combining simple harmonic analysis with existing techniques for shifted 2-point moments due to Ingham [40] and Bettin [11]. Interestingly enough, the main term in $\mathbb{E}\, \mu_T(x)\overline{\mu_T(y)}$ is given by $(i(x-y))^{-1}$, i.e., the kernel of the Hilbert transform. Using this observation and the well-known L^2-boundedness of the Hilbert transform as a starting point, a careful analysis enables one to deduce suitable uniform estimates, which in turn show that the second moment $\mathbb{E}\, |\mu_T(f)|^2$ converges as soon as $f \in L^2(0, 1)$. Roughly speaking, one establishes the convergence

$$\lim_{N\to\infty}\lim_{T\to\infty}\mathbb{E}\,|\mu_T(f)-\mu_{T,N}(f)|^2=0.$$

The final and rather straightforward piece of information one needs is to note that as $T\to\infty$, the random variable $\mu_{T,N}$ converges in law to the *randomized truncated Euler product* $\zeta_{N,\mathrm{rand}}(1/2+ix)$, where $\zeta_{N,\mathrm{rand}}(s) := \prod_{k=1}^N\left(\frac{1}{1-p_k^{-s}e^{i\theta_k}}\right)$, and the θ_k:s are i.i.d. random variables, each uniformly distributed on $[0,2\pi]$, as in Section 2.

Finally, not too surprisingly any more at this point, $\zeta_{N,\mathrm{rand}}(1/2+ix)$ converges almost surely (in the sense of generalized functions) to the distributional boundary values $\zeta_{\mathrm{rand}}(1/2+ix)$ of the *randomized Riemann zeta function* ζ_{rand} from Definition 1.2. In addition to being a limit of $\zeta_{N,\mathrm{rand}}(1/2+ix)$, $\zeta_{\mathrm{rand}}(1/2+ix)$ should thus be understood as the boundary values (in the sense of generalized functions) of the random analytic function $\zeta_{\mathrm{rand}}(s)$ in the half-plane $\{\sigma>1/2\}$. These boundary values are almost surely honest generalized functions, i.e., they are not given even locally by a measure. In any case, one may conclude that this single random analytic function describes the statistics of random shifts of ζ in the whole closed half-plane $\{\sigma\geq 1/2\}$!

4.2 Some Ingredients of Part (ii) of Thm 4.1

Here one relates the statistical limit ζ_{rand} to a complex Gaussian multiplicative chaos distribution. The theory of complex Gaussian multiplicative chaos is not as well developed as the real case, and as mentioned, our chaos does not fit into the cases studied before. In our situation there is a very special mutual dependence between the real and imaginary parts X_1 and X_2, of the form

$$\mathbb{E}\,X_1(x)X_2(y)=-\frac{\pi}{4}\,\mathrm{sgn}(x-y)\;+\;\text{smooth},$$

where $\mathrm{sgn}(x)$ denotes the sign of x and the covariance is zero when $x=y$. In addition, the 2-point function $\mathbb{E}\,e^{\mathcal{G}(x)+\overline{\mathcal{G}(y)}}$ is not absolutely integrable, which in general indicates that the L^2-theory is not available. Remarkably enough, it is exactly the above peculiar dependence of the real and imaginary part that produces the dominant part $(i(x-y))^{-1}$ to the exponential of the covariance $\mathbb{E}\,\mathcal{G}(x)\overline{\mathcal{G}(y)}$, and hence the basic theory of one-dimensional singular integrals applies to resurrect the L^2-theory. The complex Gaussian chaos ν that appears in Theorem 4.1 has some unique features that arise from the fact that it can be considered as a boundary distribution of a random analytic function. For example, the finiteness of a moment $\mathbb{E}\,|\nu(\phi)|^p$ with $p>4$ can be shown to depend on the smoothness properties of the function ϕ, and thus their properties differ in some respects from the complex chaos considered in [51].

The proof of the second part of the theorem uses the following result (which is [70, Theorem 1.7]) of independent interest, as it provides a direct functional Gaussian approximation in contrast to, e.g., [1, 64].

Theorem 4.2 *For each $N \geq 1$ there exists a decomposition*

$$\log \zeta_{N,\mathrm{rand}}(1/2 + ix) = \mathcal{G}_N(x) + \mathcal{E}_N(x),$$

where $\mathcal{G}_N$ is a Gaussian process on $[-1, 1]$ which can be written in the following way: let $(W_k^{(j)})_{k\in\mathbb{Z}_+, j\in\{0,1\}}$ be i.i.d. standard Gaussians, then

$$\mathcal{G}_N(x) = \sum_{k=1}^{N} \frac{1}{\sqrt{2p_k}} p_k^{-ix} (W_k^{(1)} + iW_k^{(2)}).$$

The function $\mathcal{E}_N$ is smooth and as $N \to \infty$, it a.s. converges uniformly to a random smooth function $\mathcal{E} \in C^\infty[-1, 1]$. Moreover, the maximal error and its derivatives in this decomposition have finite exponential moments:

$$\mathbb{E} \exp\left(\lambda \sup_{N\geq 1} \|\mathcal{E}_N(x)\|_{C^\ell[-1,1]}\right) < \infty \quad \text{for all } \lambda > 0 \quad \textit{and} \quad \ell \geq 0.$$

5 The Mesoscopic Scale: ζ Meets Random Matrices

In random matrix theory (resp. in the study of ζ on the critical line), one typically studies spectral properties of the random matrices on three different scales. For the study of ζ one makes similar definitions by replacing the spectrum by the zeroes of ζ. The *microscopic scale* is where one zooms in and looks at the spectrum around a fixed point on the scale of the distance between the eigenvalues (resp. the zeroes of ζ). The *global, or macroscopic scale* is where on zooms out and looks at all of the eigenvalues simultaneously (or in the setting of the ζ-function, an order one portion of the critical line). Finally one also considers a *mesoscopic scale* where one zooms in, but not to the scale of the distance between the eigenvalues, but to one where one asymptotically sees infinitely many (but still a vanishing fraction) of the eigenvalues.

5.1 The Montgomery(-Dyson) Paradigm

Similarities between the statistics of zeros of the zeta function on the critical line and statistics of eigenvalues of large random matrices, such as GUE random matrices or Haar distributed random unitary matrices, have been of great interest since

Montgomery's seminal pair correlation conjecture. The conjecture roughly states that on a microscopic scale, assuming the Riemann hypothesis and normalizing the zeros suitably, the statistics are indistinguishable from the microscopic statistics of the eigenvalues of the random matrices. Montgomery himself proved a partial result to this direction [61].

The proof of the full Montgomery conjecture seems to be presently out of reach. However, the point of view the conjecture provides has gained additional importance during last 15 years or so. Especially, an important variant (and strengthening) of is due to Keating and Snaith [48]. They suggested that the characteristic polynomial of a Haar distributed unitary matrix should be a good model for the statistical behavior of the zeta function. Even more recently, this has been formulated into very precise conjectures on the microscopic scale by Chhaibi, Najnudel, and Nikeghbali [19]. See also, e.g., [22] and the recent series of papers due to Conrey and Keating [21] and for a brief overview, see the book review of Conrey [20]. In [19] the microscopic statistical limit of the characteristic polynomial of a CUE-random matrix is proven to be a random analytic function whose zeroes behave like a suitable determinantal point process having a sine kernel, which then yields a natural conjecture concerning microscopic limit statistics of the ζ-zeroes.

5.2 *Rigorous Results on the Mesoscopic Scale*

It is then natural to discuss what can be said about the connection between the statistics of the Riemann zeta function and random matrices on the mesoscopic and macroscopic scale. Conjectures about these were made by Fyodorov, Hiary, and Keating, as well as Fyodorov and Keating in [29, 30], especially on the statistical behavior maxima of random shifts over $[0, T]$ in terms of T. The second main result of [70], see Theorem 5.1 below, can be seen as describing rigorously the similarities and differences between the zeta function and random matrix theory on the mesoscopic scale. More precisely, Theorem 5.1 gives a precise description of the statistical functional behavior of the zeta function on the mesoscopic scale *up to a constant multiplicative factor* and observes that the identical result holds on the RMT side. The limiting objects are the same, so this gives a very precise interpretation of the Keating–Snaith conjecture on the mesoscopic scale.

In order to describe the limit statistics for random shifts of ζ under a mesoscopic scaling, let us define a complex Gaussian multiplicative chaos distribution η formally (see [70] for a proper treatment) by

$$\eta(x) = \exp\left[\int_0^1 \frac{e^{-2\pi i x u} - 1}{\sqrt{u}} dB_u^{\mathbb{C}} + \int_1^\infty \frac{e^{-2\pi i x u}}{\sqrt{u}} dB_u^{\mathbb{C}}\right],$$

where $B_u^{\mathbb{C}}$ denotes a complex Brownian motion. Then we have

Theorem 5.1 *There exists a deterministic δ_T tending to zero as $T \to \infty$ such that as $T \to \infty$*

$$\zeta(1/2 + i\delta_T x + i\omega T) \xrightarrow{d} \eta(x)$$

in the topology of $W^{-\alpha,2}_{\mathrm{mult}}(0,1)$ for any $\alpha > 1/2$ (see below for the definition of this space). An analogous result, with the same limiting object, is true for the characteristic polynomial of a CUE-random matrix.

Remark 5.2 Above $W^{-\alpha,2}_{\mathrm{mult}}(0,1)$ stands for the Sobolev space $W^{-\alpha,2}(0,1)$, where each element f is identified with the normalized element $f/\|f\|_{W^{-\alpha,2}(0,1)}$. We omit the precise description here. ◇

6 Results and Conjectures for Statistics of $|\zeta(1/2 + it)|^\beta$

As one expects (recall Section 2.1) that shifts of $\log|\zeta(1/2 + it)|$ converge in some suitable sense to something close to a log-correlated Gaussian field, it is natural to expect that statistical behavior of shifts of $|\zeta(1/2 + it)|^\beta$ should resemble that of a real chaos, at least for small values of β.

6.1 *The Fyodorov–Hiary–Keating Conjecture*

Fyodorov, Hiary, and Keating [30] conjectured that with convergence in probability, one has as $T \to \infty$

$$\int_0^1 |\zeta(1/2 + i\omega T + ix)|^\beta dx = \begin{cases} (\log T)^{\beta^2/4 + o(1)}, & \text{if } 0 < \beta \le 2, \\ (\log T)^{\beta - 1 + o(1)}, & \text{if } \beta > 2. \end{cases} \tag{6.1}$$

This was later on refined and strengthened by Fyodorov and Keating [29] to cover intervals of length $(\log T)^\theta$. In a recent interesting preprint Arguin, Ouimet, and Radziwiłł [3] actually verified an extended form of the conjecture. In addition to such integrals, Fyodorov, Hiary, and Keating also utilized the theory of extrema of log-correlated processes (which we briefly touched on at the end of Section 2) to make conjectures concerning $\max_x \log|\zeta(\frac{1}{2} + i\omega T + ix)|$. While the full extent of these conjectures remain unresolved, there has been significant progress in their proof – see [2, 32, 62].

It is of interest to note that the statistics of the β-moments described by (6.1) matches well what one would get if one would assume real multiplicative chaos limit statistics for the shifts of $|\zeta(1/2 + it)|^\beta$ – this even holds true for the freezing regime as one notes by recalling Section 3.2. We will finish this review by describing conjectures to this direction.

6.2 Multiplicative Chaos as Statistical Limits for Shifts of $|\zeta(1/2+it)|^\beta$?

After what we have described in the previous sections it is natural to expect that multiplicative chaos measures are the correct way to describe the statistical behavior of $|\zeta|^\beta$. More precisely, one would like to know for $0<\beta<2$ ($\beta=\beta_c=2$ being the critical point in this case) the convergence and some basic properties of measures of the form

$$\frac{|\zeta(1/2+ix+i\omega T)|^\beta}{\mathbb{E}\,|\zeta(1/2+ix+i\omega T)|^\beta}dx.$$

One should note that for random matrices there are results on the convergence of $|p_n(\theta)|^\beta$ (with suitable normalization) to the Gaussian chaos measure $\lambda_{\beta/\sqrt{2}}$ defined on the circle $\mathbb{T}$ using the log-correlated field $X_{\mathbf{ci}}$ from Definition 2.1. In [79] this as done in the L^2-range $\beta\in(0,\sqrt{2})$. This was extended to the full subcritical range $\beta\in(0,2)$ in [63].

The convergence of such measures would of course not be too surprising when comparing with Theorem 4.1 and it would fit very well to the Fyodorov–Hiary–Keating conjecture (which would be a consequence of such a result in the subcritical range). However, there are still significant obstacles in proving such a result. First of all, when proving such results one typically needs good asymptotics for the normalizing quantity

$$a(T):=\mathbb{E}\,|\zeta(1/2+ix+i\omega T)|^\beta$$

along with asymptotics for quantities like the two-point function $\mathbb{E}\,|\zeta(1/2+ix+i\omega T)|^\beta|\zeta(1/2+iy+i\omega T)|^\beta$.

Determining the exact asymptotics of $a(T)$ is a long-standing open problem. However, a lower bound of the desired type is known unconditionally [66], and very recently a corresponding unconditional upper bound was established in [34], following conditional ones given in [31]. In addition, some fairly sharp conditional estimates for the shifted moments such as the two-point function are given in [18], but bounds of the correct order in $\log T$ are still unknown.

In any case, it is natural to expect that the correct limiting object can be obtained as a limit of $|\zeta_{N,\mathrm{rand}}|^\beta/\mathbb{E}\,|\zeta_{N,\mathrm{rand}}|^\beta$, for which the limit was identified a s a smooth perturbation of Gaussian (real) chaos in [70]. Hence we are lead to conjectures (put forward in [70]):

Conjecture 6.1 *For $\beta\in(0,2)$ the random densities*

$$(\log T)^{-\frac{1}{4}\beta^2}|\zeta(1/2+ix+i\omega T)|^\beta,\quad x\in[0,1]$$

converge in distribution to a smooth perturbation of a Gaussian chaos measure λ_β.

As a side remark, we note that this entails that one should think of $|\zeta(1/2 + ix + i\omega T)|^{\beta}$ as an approximation to the chaos on covariance level $\log\log T$.

Conjecture 6.2 *The previous conjecture holds for $\beta = \beta_c = 2$ as soon as one adds the normalizing factor* $(\log\log T)^{1/2}$.

Conjecture 6.3 *There are mesoscopic analogues of the above conjectures.*

If one moves a little bit away from the imaginary axis, and considers instead statistics of $|\zeta(1/2+\delta_T+ix+iT\omega)|^{\beta}$, one may get rid of the normalization problem, and the corresponding modified form of Conjecture 6.1 is established in [33] in the L^2-range $\beta \in (0, \sqrt{2})$.

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Some New Aspects in Hypercomplex Analysis

Wolfgang Sprößig

Abstract The aim of this overview article is to inform the reader on some new aspects in the use of hypercomplex methods. We have selected six topics, which are interesting for applications to physics, hydrodynamics, texture analysis, and mathematics itself. In the first part we introduce hyperquaternions and show its usefulness for the algebraical representation of the different kind of physics (P. Girard). The second part is devoted to the analysis on the 3-sphere and its practical applications to problems in texture analysis (H. Schaeben, R. Hielscher). In the third section we describe the fluid flow through porous media with the help of a quaternion operator calculus. For this reason the basics of the calculus ($(D + \alpha)$-holomorphic functions, Bergman–Hodge decomposition of the quaternionic Hilbert space and its projections) has to be introduced (K. Guerlebeck, W. Sproessig). With the help of the Takenaka–Malmquist system a generalized Fourier expansion with the so-called mono-components is shortly described (T. Qian). Furthermore, harmonic conjugates in weighted Bergman spaces of quaternion valued functions are presented (K. Avetisyan). The harmonic conjugates of the Poisson kernel are given explicitly in $\mathbb{R}^3$ and $\mathbb{R}^4$, which are higher dimensional generalizations of the famous Schwarz formula of the classic function theory.

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1 Hyperquaternions

1.1 Introduction

We have the *complex numbers* $\mathbb{C}$ with $e_1 = i$ and $e_1^2 = -1$. *Hamilton's quaternions* $\mathbb{H}$ can be defined by $e_1 = j$, $e_2 = k$, e_1e_2, and $e_\ell^2 = -1$. The algebra of

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biquaternions

$$\mathbb{C} \otimes \mathbb{H} = \mathbb{H} \otimes \mathbb{C} \tag{1.1}$$

is defined by the basis $e_1 = iI$, $e_2 = iJ$, $e_3 = iK$, where $e_\ell^2 = 1$, where $i = i \otimes 1$, $I = 1 \otimes i$, etc. Biquaternions are quaternions with complex coefficients. There is an isomorphy to the *Pauli algebra* generated by *Pauli matrices*

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}; \tag{1.2}$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \quad \sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{1.3}$$

Usually, the unit matrix does not belong to the Pauli matrices. $\mathbb{H} \otimes \mathbb{H}$ are quaternions with quaternionic coefficients also called *tetraquaternions*. Its generators are given by

$$e_0 = j, e_1 = kI, e_2 = kJ, e_3 = Kk, \tag{1.4}$$

with $e_0^2 = -1$ and $e_\ell^2 = 1$ Tetraquaternions with complex coefficients form the so-called *Dirac algebra* $\mathbb{H} \otimes \mathbb{H} \otimes \mathbb{C}$.

Remark 1.1 The name "hyperquaternion" was coined in 1922 by the American mathematician Clarence Lemuel Elisha Moore (1876–1931). He was a mathematics professor in geometry and algebra at the MIT. He had a natural relation to hypercomplex numbers by studies in Bonn (E. Study) and Turin (C. Segre). Nowadays, there are remarkable works of M. Pitkanen and P. Girard in this field.

1.2 Multiplication in $\mathbb{H} \otimes \mathbb{H}$

Let $Q, P \in \mathbb{H} \otimes \mathbb{H}$. Then we define

$$\mathrm{QP} = \begin{pmatrix} q_0p_0 - q_1p_1 - q_2p_2 - q_3p_3 & q_0p_1 + q_1p_0 + q_2p_3 - q_3p_2 \\ q_0p_2 + q_2p_0 - q_3p_1 - q_1p_3 & q_0p_3 + q_3p_0 + q_1p_2 - q_2p_1 \end{pmatrix}.$$

Furthermore, $\mathbb{H} \otimes \mathbb{H} \otimes \mathbb{H}$ are quaternions with tetraquaternionic coefficients, which is a hyperquaternionic structure. It used the rule

$$(u \otimes v)(x \otimes y) = (ux \otimes vy) \quad \text{for} \quad u, v, x, y \in \mathbb{H}. \tag{1.5}$$

The multivector structure is given by the following table:

1	$I = e_3e_2$	$J = e_2e_3$	$K = e_2e_1$,
$i = e_0e_1e_2e_3$	$iI = e_0e_1$	$iJ = e_0e_2$	$iK = e_0e_3$,
$j = e_0$	$jI = e_0e_3e_2$	$jJ = e_0e_1e_3$	$jK = e_0e_2e_1$,
$k = e_1e_2e_3$	$kI = e_1$	$kJ = e_2$	$kK = e_3$.

1.3 Generators

$$m_{i\otimes 1} = \begin{pmatrix} 0 & -i\sigma_2 \\ -i\sigma_2 & 0 \end{pmatrix}; \quad m_{j\otimes 1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}; \quad m_{k\otimes 1} = \begin{pmatrix} -i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{pmatrix};$$

$$m_{1\otimes i} = \begin{pmatrix} 0 & -\sigma_1 \\ \sigma_1 & 0 \end{pmatrix}; \quad m_{1\otimes j} = \begin{pmatrix} -i\sigma_2 & 0 \\ 0 & -i\sigma_2 \end{pmatrix}; \quad m_{1\otimes k} = \begin{pmatrix} 0 & -\sigma_3 \\ \sigma_3 & 0 \end{pmatrix}.$$

These matrices square to -1, anticommute on the same line and commute with the matrices of the other line, then, they constitute distinct quaternionic systems. The products of these matrices generate $m(4, \mathbb{R})$, and thus we get the algebraic isomorphisms

$$\mathbb{H} \otimes \mathbb{H} \cong m(4, \mathbb{R}) \tag{1.6}$$

$$[\mathbb{H} \otimes \mathbb{H}] \otimes \mathbb{C} \cong m(4, \mathbb{C}) \tag{1.7}$$

$$[\mathbb{H} \otimes \mathbb{H}] \otimes \mathbb{H} \cong m(4, \mathbb{H}) \tag{1.8}$$

1.4 Clifford Numbers

W.K. CLIFFORD (1845–1879) was an English mathematician and philosopher. Building on the work of HERMANN GÜNTHER GRASSMANN (1844), he introduced in 1878 the "geometric algebra," a special case of a more general Clifford algebra, which will be introduced. He was the first to propose that gravitation might depend on the underlying geometry. This idea was later used in A. Einstein's relativity theory. The year in that W.K. CLIFFORD died was A. EINSTEIN born. So far Clifford can be seen as forerunner of Einstein's work. See also [1].

Let $n \geq 1$ let $e_0, e_1, \ldots, e_n$ be an orthonormal basis of $\mathbb{R}^{n+1}$, we define a multiplication by

$$e_0e_i = e_ie_0 = e_i, \; i = 0, \ldots, n; \; e_i^2 = -1, \; i = 1, \ldots, n, \tag{1.9}$$

$$e_ie_j = -e_je_i, \; i \neq j, \; i, j = 1, \ldots, n. \tag{1.10}$$

Thus we obtain a basis of the so-called *real Clifford algebra* $C\ell(n)$:

$$e_0; e_1, \ldots, e_n; e_1e_2, \ldots, e_{n-1}e_n; e_1e_2e_3, \ldots; \ldots; e_1e_2\ldots e_n,$$

with e_0 as unit element.

An arbitrary *Clifford number* is given by

$$x = x_0 + \sum_{k=1}^{n} \sum_{0<i_1<\ldots<i_k\leq n} x_{i_1\ldots i_k} e_{i_1\ldots i_k},$$

where we used the abbreviation

$$e_{i_1 i_2\ldots i_k} := e_{i_1} e_{i_2} \ldots e_{i_k}.$$

1.5 Clifford Algebras over the Real Numbers

Every nondegenerate quadratic form on a finite-dimensional real vector space is equivalent to the standard diagonal form:

$$Q(u) = u_1^2 + \cdots + u_p^2 - u_{p+1}^2 - \cdots - u_{p+q}^2, \tag{1.11}$$

where $n = p + q$ is the dimension of the vector space. The pair of integers (p, q) is called the *signature* of the quadratic form. The real vector space with this quadratic form is often denoted $\mathbb{R}^{p,q}$. The *Clifford algebra* on $\mathbb{R}^{p,q}$ is denoted $C\ell_{p,q}(\mathbb{R})(= C\ell_{p,q})$. Obviously, $C\ell(n)$ coincides with $C\ell_{0,n}$.

A standard basis $e_1, \ldots, e_n$ for $\mathbb{R}^{p,q}$ consists of $n = p + q$ mutually orthogonal vectors, p of which square to +1 and q of which square to -1. Of such a basis, the algebra $C\ell_{p,q}$ will therefore have p vectors that square to +1 and q vectors that square to -1. The most important Clifford algebras are those over real and complex vector spaces equipped with nondegenerate quadratic forms. A few low-dimensional cases are:

$C\ell_{0,0}$ is naturally isomorphic to $\mathbb{R}$ since there are no nonzero vectors. $C\ell_{0,1}$ is a two-dimensional algebra generated by e_1 that squares to -1, and is algebra isomorphic to $\mathbb{C}$, the field of complex numbers. $C\ell_{0,2}$ is a four-dimensional algebra spanned by $\{1, e_1, e_2, e_1e_2\}$. The latter three elements all square to -1 and anticommute, and so the algebra is isomorphic to the quaternions $\mathbb{H}$. $C\ell_{0,3}$ is an 8-dimensional algebra isomorphic to the direct $\mathbb{H} \oplus \mathbb{H}$, the *split-biquaternions*. $C\ell_{3,0}$ is an 8-dimensional algebra algebraic-isomorphic to the so-called *Pauli algebra* generated by the Pauli matrices.

1.6 Hyperquaternions for Classification of Physics

We follow the paper [2] of the authors Patrick Girard, Patrick Clarysse, Romaric Pujol, Robert Goutte, Philippe Delachartre. There are used the denotations: with m(n,K) with $K = \mathbb{R}, \mathbb{C}, \mathbb{H}$ we denote the full $n \times n$ matrices with entries from K. As usual $SO(n)$ denotes the special orthogonal group in $\mathbb{R}^n$. $SO(1,3)$ is the special orthogonal group related to the real space $\mathbb{R}^{1,3}$. The notation $SU(n)$ is taken for the special unitary group in $\mathbb{R}^n$. Furthermore, we denote by $USp(n)$ the corresponding unitary symplectic group. nD-physics means physics in $\mathbb{R}^n$. The abbreviations STR and GTR stand for special theory of relativity as well as general theory of relativity. With QM is quantum mechanics abbreviated. The isomorphisms are to see as algebra-isomorphisms. A large part of mathematics in such algebras is concerned with symmetries. The orthogonal group or special orthogonal group describes the linear symmetries in regular quadratic spaces. The special orthogonal group contains as subgroup the spin group which is used to describe actions. The following relations are valid by using tensor product technique:

Algebra	$C\ell_{p,q}$	Symmetry group	Physics
$\mathbb{C}$			1D-physics
$\mathbb{H}$	$C\ell_{0,2}$	SO(2)	2D-physics
$\mathbb{H} \otimes \mathbb{C} \cong m(2,\mathbb{C})$	$C\ell_{3,0}$	SO(3)	3D-physics
$\mathbb{H} \otimes \mathbb{H} \cong m(4,\mathbb{R})$	$C\ell_{3,1}$	SO(1, 3)	STR, GTR
$[\mathbb{H} \otimes \mathbb{H}] \otimes \mathbb{C} \cong m(4,\mathbb{C})$	$C\ell_{2,3}$	SU(4)	Relativistic QM
$[\mathbb{H} \otimes \mathbb{H}] \otimes \mathbb{H} \cong m(4,\mathbb{H})$	$C\ell_{2,4}$	USp(4)	Quaternionic QM
$[\mathbb{H} \otimes \mathbb{H}] \otimes [\mathbb{H} \otimes \mathbb{C}] \cong m(8,\mathbb{C})$	$C\ell_{5,2}$	$SU(8) \supseteq SU(5)$	Standard model

2 Analysis on the 3-Sphere—Some Topics

In crystallographic texture analysis arise the question to have some kind of "harmonic analysis" over the group $SO(3)$, which is easy related to the 3-sphere. One day H. Schaeben (TU Freiberg) an expert for Geology and Geoinformatics asked me the following questions:

How to realize a finite element method on the 3-sphere?
What is the area of a triangle on S^3?

Starting to think on these questions we very quickly come to Poincaré's conjecture.

This conjecture gives a topological characterization of the 3-sphere, which is just a hypersphere that bounds the unit ball in the 4-dimensional Euclidean space. The conjecture formulated by H. POINCARE (1854–1912) in 1900 states:

Every simply connected, closed 3-manifold is homeomorphic to the 3-sphere.

which was proved in 2002–2003 by GRIGORI PERELMAN. This shows that a 3-sphere is from topological point of view very complicated and captivated our imagination.

2.1 Representations of S^3

For practical computations on the sphere one needs more clearness on the representation of S^3. Homeomorphic images are not enough in this case. Therefore we need descriptions of the sphere by suitable coordinates.

Furthermore, we have to use the work of the English geometer W.K. CLIFFORD (1845–1879),the French astronomer and mathematician Y. VILLARCEAU (1813–1883) and the German-Swiss algebraical topologist H. HOPF (1894–1971). The famous physicist and geometer R. PENROSE: wrote ...*Clifford found, that the entire 3-dimensional sphere can be filled up with non-intersecting parallel circles (today called Clifford parallels), each being linked to each of the others.* (see also [3].)

We identify $x = (x_0, x_1, x_2, x_3)$ with $(z_0, z_1) \in \mathbb{C}^2$ with $z_0 = x_0 + ix_1$ and $z_1 = x_2 + ix_3$. S^3 is identified with a subset of $\mathbb{C}^2$ such that $|z_1|^2 + |z_2|^2 = 1$. S^2 is a subset of $\mathbb{C} \times \mathbb{R}$ with the pairs (z, x_3) and $|z|^2 + x_3^2 = 1$. The *Clifford-Hopf mapping h* is given by

$$S^3 \ni (z_1, z_2) \to (2z_0\bar{z}_1, |z_0|^2 - |z_1|^2) \in S^2, \tag{2.1}$$

which is surjective on S^2. For a closer look we refer to the paper by PIERRE ANGLES [4].

Theorem 2.1 *We have $h((z_0, z_1)) = h((w_0, w_1))$ iff $(z_0, z_1) = \lambda(w_0, w_1)$, $\lambda \in \mathbb{C}$ and $|\lambda| = 1$.*

Corollary 2.2 *For any $q \in S^2$ $h^{-1}(q)$ is circle isomorphic to S^1. The mapping h is associated with a "fibration " of S^3. The inverse image of a circle on S^2 is a torus in S^3.*

Moreover one can show that S^3 is the union of two solid tori "glued" together along their common boundary which is called *Clifford torus*. This can be easily seen as follows:

Let $S^3 = \{(w, x, y, z) : w^2 + x^2 + y^2 + z^2 = 1\}$. The common boundary is the Clifford torus $w^2 + x^2 = y^2 + z^2 = 1$. The solid tori can be described by

$$w^2 + x^2 \leq \frac{1}{2}, \quad y^2 + z^2 = 1 - w^2 - x^2, \tag{2.2}$$

$$y^2 + z^2 \leq \frac{1}{2}, \quad w^2 + x^2 = 1 - y^2 - z^2. \tag{2.3}$$

Remark 2.3 From a 4D-to-3D stereographic projection to the Clifford-Hopf fibration one gets the *Clifford torus* composed of interlinked *Villarceau circles*. There exists a nice image in nylander.wordpress.com.

The complicated nature of S^3 makes it not easy to find suitable distortion free notions of distances and volumes. Therefore are used the relations between SO(3) (*special orthogonal group*), $\mathbb{RP}^3$(*real projective space*) and S^3 (*3-sphere*).

We have

$$SO(3) \equiv \mathbb{RP}^3 \equiv S^3/(x \equiv -x). \tag{2.4}$$

$\mathbb{RP}^3$ is locally the Cartesian product of S^1 and S^2, in the same sense like the Möbius band is the Cartesian product of an interval and S^1.

On the sphere S^3 can chosen three coordinates (θ, ϕ, ψ) with $0 < \psi \leq 2\pi$ (parametrization of S^1), $0 < \theta \leq \pi$, $0 < \phi \leq 2\pi$ (spherical coordinates).

$$x_0 = \cos\frac{\theta}{2}\cos\frac{\psi}{2}, \quad x_1 = \cos\frac{\theta}{2}\sin\frac{\psi}{2}, \tag{2.5}$$

$$x_2 = \sin\frac{\theta}{2}\cos\left(\phi + \frac{\psi}{2}\right), \quad x_3 = \sin\frac{\theta}{2}\sin\left(\phi + \frac{\psi}{2}\right). \tag{2.6}$$

Intuitively, we have at first a rotation of angle $\psi \in S^1$ around an axis z, followed by a rotation which places the axis z (versor z) in any position $(\theta, \phi) \in S^2$. The volume element is the product of the surface element in S^2 and the length element of a circle:

$$dV = \sin\theta d\theta d\phi d\psi. \tag{2.7}$$

Such coordinates are called *Clifford-Hopf coordinates*. See it more detailed in the paper [5]

It is well known that the group of all real valued, orthogonal 3×3 matrices with determinant 1 is just $SO(3)$. A rotation $g \in SO(3)$ is defined by a rotational axis $\xi \in S^2$ and a rotational angle $\omega \in [0, \pi)$. The rotation matrix should be now denoted by $R_\xi(\omega)$. The rotational angle $\omega =: \angle g$ of g can be described by the trace of the rotation matrix (g_{ij}).

$$\cos\omega = \frac{g_{11} + g_{22} + g_{33} - 1}{2}. \tag{2.8}$$

With the help of the quaternion multiplication the product $R_{\xi_3}(\omega_3)$ of the two rotations $R_{\xi_1}(\omega_1)$ and $R_{\xi_2}(\omega_2)$ is obtained. The rotation axis and the rotation angle are given by

$$\xi_3 = \sin\frac{\omega_2}{2}\cos\frac{\omega_1}{2}\xi_1 + \sin\frac{\omega_1}{2}\cos\frac{\omega_2}{2}\xi_2 + \cos\frac{\omega 1}{2}\cos\frac{\omega_2}{2}\xi_1 \times \xi_2 \tag{2.9}$$

$$\cos\frac{\omega_3}{2} = \cos\frac{\omega_1}{2}\cos\frac{\omega_2}{2} - \sin\frac{\omega_1}{2}\sin\frac{\omega_2}{2}\xi_1 \cdot \xi_2. \tag{2.10}$$

The rotational angle between two rotations g_1 and g_2 is given by $\angle g_1^{-1}g_2$. This angle can be used to define a metric on $SO(3)$. For the application of the rotation to x one gets the well-known *Rodrigues' formula*

$$R_\xi(\omega)x = \cos\omega x + \sin\omega\xi \times x + (1 - \cos\omega)(\xi \cdot x)\xi. \tag{2.11}$$

One defines the volume element in terms of the rotation axis and the rotation angle

$$dg = d\xi \sin^2\frac{\omega}{2}d\omega \tag{2.12}$$

in order to give sense to the integral $\int\limits_{SO(3)} 1dg$. Therefore

$$\int\limits_{SO(3)} 1dg = 4\int\limits_0^\pi \int\limits_{S^2} 1d\xi \sin^2\frac{\omega}{2}d\omega = 8\pi^2. \tag{2.13}$$

2.2 Tomographic Methods

It is very useful that the subspace of even functions of the space $L_2([-1, 1], \sqrt{1 - t^2})$ and the space of radially symmetric functions with respect to the identity (Id) $L_2(SO(3))$ are isomorphic. A function $u : SO(3) \to \mathbb{R}$ is called *radially symmetric* with respect to the identity if $u(g) = u(g')$ for all $g, g' \in SO(3)$ fulfils the condition $\angle g = \angle g'$, i.e., only the distance of g to Id is relevant. An important example is the *De la Vallée Poussin kernel* which reads as follows:

$$K(t) := \frac{B(\frac{3}{2}, \frac{1}{2})}{B(\frac{3}{2}, \kappa + \frac{1}{2})}t^{2\kappa}, \quad t \in (0, 1], \kappa > 0, \tag{2.14}$$

where B denotes the *Beta function*: $B(x, y) = \int\limits_0^1 t^{x-1}(1 - t)^{y-1}dt$. Now we follow results of *R. Hielscher*:

Theorem 2.4 ([6]) *Let $u \in L_2(SO(3))$ be a radially symmetric function with respect to the identity Id and $U : [-1,1] \ni t \to U(t) = u(g)$ with $g \in SO(3)$ and $|t| := \cos\frac{\angle g}{2}$. Then the mapping $u \to U$ defines an isomorphism. Moreover, it can be proved that any radially symmetric function $u \in L_2(SO(3))$ permits the following series expansion:*

$$u(g) = \sum_{k=0}^{\infty} \hat{U}(2k)\boldsymbol{U}_{2k}(\cos\frac{\angle g}{2}), \quad g \in SO(3), \tag{2.15}$$

where $\hat{U}(2k)$ are the Chebyshev coefficients of even order of the function U and $\mathbf{U}$ *denotes the Chebyshev polynomials of second kind.*

Using the *Peter–Weyl theory* on irreducible representations of compact groups a complete system of the so-called *Wigner-D-functions* $\mathbf{W}_k$ on $SO(3)$ is given [6]. More exactly holds:

Theorem 2.5 *Let be $\mathbf{W}_k : SO(3) \to GL(\mathrm{Harm}_k)(S^2)$ the (left) representations of $SO(3)$ into $\mathrm{Harm}_k(S^2)$. For the matrix entries $\mathbf{W}_k^{m,m'}$ one obtains for $g \in SO(3)$*

$$\mathbf{W}_k^{m,m'}(g) = \int_{S^2} \mathbf{Y}_k^{m'}(g^{-1}\xi)\overline{\mathbf{Y}}_k^{m}(\xi)d\xi, \tag{2.16}$$

where $\mathbf{Y}_k^m$ are harmonic polynomials. For an arbitrary function of $L^2(SO(3))$ one gets a corresponding series expansion in terms of $\mathbf{W}_k^{m,m'}$.

It is well-known that all tomographic methods used in material science and medicine are mathematically based on the so-called *Radon transform*.

$$\mathbf{R}u(x,y) = \int_{\mathbb{R}} u(x+\tau y)d\tau, \quad x, y \in \mathbb{R}^n, \quad \text{(X-ray tomography)} \tag{2.17}$$

which is well-defined for any continuous function u. The inversion of this transformation is a basic ill-posed problem. Its formulation for the *Lie group* SO(3) reads as follows:

$$\mathbf{R} : C(SO(3)) \to C(S^2 \times S^2); \quad (\mathbf{R}u)(h,r) := \frac{1}{2\pi} \int_{G(h,r)} u(g)dg, \tag{2.18}$$

with $G(h,r) = \{g \in SO(3) : gh = r\}, \quad h, r \in S^2$, (see also in [7]).

The inversion of $\mathbf{R}$ is an important problem in *quantitative texture analysis (QTA)*. The Radon transform $(\mathbf{R}u)(h,r)$ can be expressed by the two rotations $R_r(\omega)$ and $g_{h,r}$ a rotation that maps h to r. One has

$$(\mathbf{R}u)(h,r) = \frac{1}{2\pi}\int_0^{2\pi} u(R_r(\omega)g_{h,r})d\omega. \tag{2.19}$$

The Radon transform produces a nice relation between harmonic polynomials of degree k and the Wigner-D-function of degree k. More detailed, it holds

$$\mathbf{RW}_k(h,r) = \frac{2\pi}{k+1/2}\overline{\mathbf{Y}_k(r)}\mathbf{Y}_k(h)^T, \quad h,r \in S^2. \tag{2.20}$$

The proof is based on *Funk-Hecke's theorem*, which is formulated on S^{n-1} as follows: It follows that any $u \in L_2(S^{n-1})$ permits the series expansion

$$u(\omega) = \sum_{k=0}^{\infty} \mathbf{Y}_k(\omega), \tag{2.21}$$

which converges in $L_2(S^{n-1})$. The functions $\mathbf{Y}_k(\omega)$ can be represented by Gegenbauer polynomials C_k^{μ} via an integral formula

$$\mathbf{Y}_k(\omega) = \frac{2k+n-2}{(n-2)\sigma_n}\int_{S^{n-1}} C_k^{\frac{n-1}{2}}((\omega,\zeta))u(\zeta)dS(\zeta), \tag{2.22}$$

where σ_n denotes the area of the unit sphere S^{n-1}. The famous Funk-Hecke formula reads as follows:

$$\int_{S^{n-1}} u(\langle,\omega,\zeta\rangle)\mathbf{Y}_k(\zeta)dS(\zeta) = \sigma_{n-1}\frac{k!(n-3)!}{(k+n-2)!}\int_{-1}^{1} u(t)(1-t^2)^{\frac{n-3}{2}}C_k^{\frac{n-2}{2}}(t)dt\mathbf{Y}_k(\omega).$$

3 Fluid Flow Through Porous Media with the Help of a Quaternionic Operator Calculus

3.1 Some Basic Fluid Flow Equations

A mass of a liquid dq percolating through the surface element $d\sigma$ of a homogeneous porous medium in the direction of the normal n in a time interval dt is given by *Darcy's law*:

$$dq = -k\frac{1}{\mu*}\frac{\partial p}{\partial n}d\sigma dt, \tag{3.1}$$

where p is the so-called *(pore) pressure* and k the *permeability*, a measure of the ability of a material to transmit fluids , $\mu*$ is the so-called *dynamic viscosity* of the percolation material. Further $k/\mu*$ is called *percolation coefficient*.

In his famous paper [8]: H. DARCY described the basic equations of the flow though porous media:

$$\frac{\mu*}{k}u = -\text{grad}p + f \quad \text{(Darcy equation)}, \tag{3.2}$$

$$\text{div } u = 0, \tag{3.3}$$

where u is the *velocity*, p as already mentioned is the *pressure*, and f the body force. Darcy's equations are only true in regions where the velocity is not too large.

Note that the measure of porosity is the fraction of fluid per volume divided by the total volume of the porous body. In the case that this fraction is near to one. H.C. BRINKMAN added in 1949 a further term. The arising system is now called *Brinkman's equations*

$$\frac{\mu*}{k}u = -\text{grad}p + f + \lambda\Delta u, \tag{3.4}$$

$$\text{div } u = 0. \tag{3.5}$$

Here λ is called *effective viscosity*.

Remark 3.1 H.C. BRINKMAN worked at the University of Groningen together with F. ZERNIKE and B. NIJBOER. See also the paper [9].

For situations where the fluid velocity is big enough Darcy's equations are usually replaced by the so-called *Forchheimer equations*. These equations often read as follows:

$$\frac{\mu*}{k}u + au + b|u|u = -\text{grad}p + f, \tag{3.6}$$

$$\text{div } u = 0. \tag{3.7}$$

Remark 3.2 By definition, the *effective viscosity* means the viscosity of Newtonian fluid that gives the same shear stress at the same shear rate.

The analysis of convective fluid flow in a porous medium where the viscosity is considerable varying (with temperature or with salt concentration) it is necessary to use a combination of both the Brinkman and the Forchheimer model. We obtain

$$\frac{\partial u}{\partial t} = \lambda\Delta u - au + b|u|u = -\text{grad}p + f, \quad \text{div } u = 0. \tag{3.8}$$

This last model characterizes a non-slow flow in a saturated porous medium. The solution depends continuously on the *Forchheimer coefficient b* and also from the *Brinkman coefficient* λ. If the effective viscosity tends to zero then the limit model is called *Darcy–Forchheimer equations*. It is known that the energy decays exponentially. A good reference is the paper [10].

3.2 A Quaternion Operator System

First we introduce the notion of a generalized holomorphy. Let D be the operator given by

$$D = \partial_1 e_1 + \partial_2 e_2 + \partial_3 e_3, \tag{3.9}$$

where e_i are the basic quaternions with $-DD = \Delta$.

A componentwise differentiable quaternion valued function Φ_α is called $(D+\alpha)$-*holomorphic* in a domain G if and only if

$$(D + \alpha)\Phi_\alpha = 0 \quad \text{in} \quad G. \tag{3.10}$$

Let

$$K_p(z) = -\frac{\pi i}{2} e^{-\frac{\pi i}{2}p} H_p^{(2)}\left(z e^{-\frac{\pi i}{2}}\right), \quad -\frac{\pi}{2} \leq \arg z < \pi, \tag{3.11}$$

where $H_p^{(2)}$ denotes the Hankel function of second order. The function $K_p(t)$ is also called *MacDonald function* (named after H.M. MacDonald) (1865–1935).

The fundamental solution for the Klein–Gordon operator $-\Delta + \mu^2 = (D + i\mu)(D - i\mu)$ is given by:

$$\mathbf{K}_\alpha(x) := \left(\frac{1}{2\pi}\right)^{n/2} \left(\frac{\alpha}{|x|}\right)^{n/2-1} K_{n/2-1}(\alpha|x|). \tag{3.12}$$

Let us consider the fundamental solution e_α for the operator D_α $(\alpha = i\mu)$. Let $n \geq 3$, we then have

$$e_\alpha(x) := -\left(\frac{\alpha}{2\pi}\right)^{n/2} \frac{1}{|x|^{(n/2-1)}} \left[\frac{x}{|x|} K_{n/2}(\alpha|x|) - K_{n/2-1}(\alpha|x|)\right]. \tag{3.13}$$

$$\left|\int_G e_\alpha(x)dx\right| \leq \frac{1}{\mu} const. \tag{3.14}$$

[11].

Let G be a sufficiently smooth bounded domain in $\mathbb{R}^3$. Introducing the modified *Teodorescu transform* as follows with $\alpha = i\mu$

$$(T_\alpha u)(x) := \int_G e_\alpha(y-x)u(y)dy \quad x \in G. \tag{3.15}$$

Then we have $(D \pm \alpha)T_{\pm\alpha} = I$. In [12] is proved the following Bergman–Hodge decomposition of the Hilbert space

Theorem 3.3 *Let* $\alpha = ia_0$, $a_0 \in \mathbb{R}^+$. *We obtain:*

$$L_2(G) = \ker D_\alpha \cap L_2(G) \oplus D_{-\alpha} \overset{\circ}{W_2^1}(G), \tag{3.16}$$

where $\mathbf{P}_\alpha$ *orthoprojection onto* $(\ker D_\alpha \cap L_2)(G)$ *(Bergman projection)*

$\mathbf{Q}_\alpha$ *orthoprojection onto* $D_{-\alpha} \overset{\circ}{W_2^1}(G)$ $\quad \mathbf{Q}_\alpha = I - \mathbf{P}_\alpha$ *(Pompeiu projection).*

The modified *Cauchy–Fueter operator* is given by

$$(F_{\pm\alpha,\Gamma}u)(x) := \int_\Gamma e_\alpha(x-y)n(y)u(y)d\Gamma_y, \quad x \in G \cup (\mathbb{R}^3 \setminus \overline{G}), \tag{3.17}$$

for $u \in C^1(G) \cap C(\overline{G})$. A corresponding *Borel–Pompeiu formula* is proved. For $u \in C^{0,\beta}(\Gamma)$ $(0 < \beta \leq 1)$ the corresponding *Bitzadse integral operator* is given by

$$(S_{\pm\alpha,\Gamma}u)(x) := 2\int_\Gamma e_\alpha(x-y)n(y)u(y)d\Gamma_y, \quad x \in \Gamma. \tag{3.18}$$

Again we have $S^2_{\pm\alpha,\Gamma} = I$. Thus we have also *Plemelj projections* onto corresponding Hardy spaces of $(D \pm \alpha)$-holomorphic extendable quaternionic functions. We define:

$$P_{\pm\alpha,\Gamma} = \frac{1}{2}(I + S_{\pm\alpha,\Gamma}), \quad Q_{\pm\alpha,\Gamma} = \frac{1}{2}(I - S_{\pm\alpha,\Gamma}). \tag{3.19}$$

Jump formulae of Plemelj–Sokhotzkij type are also proved.

Theorem 3.4 *Let* $f \in W_2^k(G)$, $g \in W_2^{k+3/2}(\Gamma)$. *Then*

$$(-\triangle + \mu^2)u = f \text{ in } G, \tag{3.20}$$

$$u = g \text{ on } \Gamma. \tag{3.21}$$

This problem has the (unique) solution

$$u = F_{\Gamma,-\alpha}g + T_{-\alpha}\mathbf{P}_\alpha D_{-\alpha}h + T_{-\alpha}\mathbf{Q}_\alpha T_\alpha f, \tag{3.22}$$

where h *denotes a* W_2^{k+2}*-extension of* g *into* G

For the proof we again refer to [12].

Theorem 3.5 *The operator*

$$tr_\Gamma T_{-\alpha} F_{\Gamma,\alpha} \ : \ \text{im } P_{\Gamma,\alpha} \cap W_2^{1/2}(\Gamma) \to \ \text{im } Q_{\Gamma,-\alpha} \cap W_2^{3/2}(\Gamma) \tag{3.23}$$

is an isomorphism.

Corollary 3.6 *For the Bergman and Pompeiu projections hold*

$$\mathbf{Q}_\alpha = I - \mathbf{P}_\alpha \quad \textit{and} \quad \mathbf{P}_\alpha = F_{\Gamma,\alpha}(tr_\Gamma T_{-\alpha} F_{\Gamma,\alpha})^{-1} T_{-\alpha}. \tag{3.24}$$

Corollary 3.7 *It holds the following equivalence:*

$$tr_\Gamma T_{-\alpha} u = 0 \Leftrightarrow u \in \text{im}\mathbf{Q}_\alpha. \tag{3.25}$$

Remark 3.8 Let $G \subset \mathbb{R}^{n+1}$ a bounded domain and $\Gamma = \partial G$. For all $x \in \Gamma$ and $0 < r < \text{diam}\Gamma$ has to exist a positive constant c with

$$c^{-1} r^{-1} \leq \mathbf{H}^n(\Gamma) \cap B(x,r) \leq cr^2, \tag{3.26}$$

where $\mathbf{H}^n(F) \quad (F \subset \mathbb{R}^n)$ is the n-dimensional Hausdorff measure. Such a boundary is called *Ahlfors–David boundary*. Surfaces of this type are called *AD-regular*. Liapunov, Lipschitzean, and chord-arc (S. SEMMES) surfaces belong to this class.

In subclasses of hölder-continuous functions operators of Cauchy–Bizadse type are studied. Plemelj–Privalov, Plemelj–Sokhotzky formulae are proved. Riemann boundary value problems are solved. Fractal boundaries also considered more recently. Most progress in these questions was obtained by the Cuban Research Group around J. BORY-REYES (see also [13]). In the Lipschitzean case there are papers by M. MITREA and A. MCINTOSH.

3.3 Representation in Terms of $(D \pm \alpha)$-Holomorphic Functions

Set $\mu := \frac{\mu *}{k\lambda}$ and $\alpha := i\mu$. Applying Teodorescu transforms from the left and consider the validity of the Borel–Pompeiu formula one gets

$$u = T_{-\alpha} T_\alpha D \tilde{p} + T_{-\alpha} T_\alpha \tilde{f} + T_{-\alpha} \Phi_\alpha + \Phi_{-\alpha}, \tag{3.27}$$

with $\tilde{p} = \frac{p}{\lambda}$ and $\tilde{f} = \frac{f}{\lambda}$, where

$$\Phi_{\pm\alpha} \in \ker (D \pm \alpha). \tag{3.28}$$

After using the representation of the orthoprojections

$$\mathbf{P}_\alpha := F_\alpha(\mathrm{tr}_\Gamma T_{-\alpha} F_\alpha)^{-1}\mathrm{tr}_\Gamma T_{-\alpha} \tag{3.29}$$

and

$$\mathbf{Q}_\alpha = I - \mathbf{P}_\alpha \tag{3.30}$$

one obtains

$$u = T_{-\alpha}\mathbf{Q}_\alpha[\tilde{p} + \alpha T_\alpha \tilde{p}] - T_{-\alpha}\mathbf{Q}_\alpha T_\alpha \tilde{f} \tag{3.31}$$

$$0 = \mathrm{Sc}\mathbf{Q}_\alpha[\tilde{p} + \alpha T_\alpha \tilde{p}] - \mathrm{Sc}\mathbf{Q}_\alpha T_\alpha \tilde{f}. \tag{3.32}$$

It holds the equality

$$\|D_{-\alpha}u\|_2^2 + \|\mathbf{Q}_\alpha \tilde{\tilde{p}}\|_2^2 = \|\mathbf{Q}_\alpha T_\alpha \tilde{f}\|_2^2. \tag{3.33}$$

We have already proved that

$$u = T_{-\alpha}\mathbf{Q}_\alpha[\tilde{p} + \alpha T_\alpha \tilde{p}] - T_{-i\alpha}\mathbf{Q}_\alpha T_\alpha \tilde{f}, \tag{3.34}$$

where $\mathbf{Q}_\alpha = I - \mathbf{P}_\alpha$ and

$$\mathbf{P}_\alpha = \mathrm{Vec}\, F_{\Gamma,\alpha}\left(\mathrm{tr}_\Gamma\, T_{-\alpha}\mathrm{Vec}\, F_{\Gamma,\alpha}\right)^{-1}\mathrm{tr}_\Gamma\, T_{-\alpha}, \tag{3.35}$$

where $\mu := \frac{\mu *}{k\lambda}$ (λ, $\mu *$ viscosities and k the permeability and $\alpha = i\mu$. A solution is given by

$$u_s = T_{-\alpha}\mathbf{Q}_{i\alpha}\mathrm{Vec}\, T_\alpha \tilde{f}. \tag{3.36}$$

It remains for the pressure term

$$\mathbf{Q}_\alpha \tilde{\tilde{p}} = \mathbf{Q}_{i\alpha}\mathrm{Sc}T_\alpha \tilde{f}. \tag{3.37}$$

This leads to

$$\tilde{\tilde{p}} = \mathrm{Sc}\, T_\alpha \tilde{f} + \Phi \quad (\Phi \in \ker(D_\alpha)) \tag{3.38}$$

and thus with $H \in \ker(-\Delta + \mu^2)$

$$\tilde{p}_s = \mathrm{Sc}\, T_\alpha \tilde{f} + H. \tag{3.39}$$

4 An Adaptive Fast Fourier Type Decomposition

4.1 Takenaka–Malmquist Systems

Definition 4.1 Let $\alpha \in B_1(0)$. The *Blaschke product* is defined by the factors

$$B_\alpha(z) = \frac{z-\alpha}{1-\overline{\alpha}z}. \tag{4.1}$$

The function $L_k^{(\alpha)} = m_\alpha(z) B_\alpha^k(z)$ with

$$m_\alpha = \frac{\sqrt{1-|\alpha|^2}}{1-\overline{\alpha}z} \quad \text{and} \quad k \in \mathbb{Z} \tag{4.2}$$

is called *discrete Laguerre function*. Moreover, $L_{-n}^1(z) = \overline{L}_n^1(\frac{1}{z})$.

Above representation formulae are used to construct analoga to the well-known *Takenaka–Malmquist* system (TM-system) in quaternions and reduced quaternions. *Reduced quaternions* are quaternions of the form $a_0e_0 + a_1e_1 + a_2e_2$ with a_i are real numbers and e_i basis elements. The TM-system is given by

$$\left\{\begin{aligned} \Phi_1(z) &= \frac{\sqrt{1-|a_1|^2}}{1-\overline{a_1}z}, \\ \Phi_n(z) &= \frac{\sqrt{1-|a_n|^2}}{1-\overline{a_n}z}\prod_{k=1}^{n-1}\frac{z-a_k}{1-\overline{a_k}z}, \ n \geq 2, \end{aligned}\right. \qquad a_k \in \mathbb{D},\ k = 1, 2, \ldots,\ z \in \mathbb{D}. \tag{4.3}$$

See in [14, 15].

Remark 4.2 F. Malmquist was a student of Harold Cramer (Stockholm), who was student of Marcel Riesz, also influenced by G.H. Hardy.

Under some conditions a function f can be represented in a Fourier expansion. There is a proposal for a generalization with the so-called *mono-components* (T. Qian). Such an expansion is given by

$$f(t) = \sum_{i=0}^{\infty} \rho_k(t) \cos\theta_k(t). \tag{4.4}$$

Each of the summands is a mono-component (classical-Fourier-expansion: $\rho_k(t) = \rho_k$ and $\theta_k(t) = k\omega t$). For the convergence is used the famous *Huang algorithm*. This is an algorithm for detecting termination in a distributed system. This greedy algorithm was proposed by Shing-Tsaan Huang in 1989 in the journal of computers. There is a relation to the *Empirical Mode Decomposition (EMD)*.

A similar adaptive orthogonal complete system $\{B_n\}$ of quaternionic valued functions $\mathcal{H}^2(B)$ is obtained. Only as a remark, without construction details we formulate:

Theorem 4.3 ([16, 17]) *By adaptively choosing the parameters $\{a_n\}_{n=1}^{\infty}$ according to the given function a fast decomposition (in terms of energy) is achieved such that*

$$f = \sum_{n=1}^{\infty} B_n \langle f, B_n \rangle. \tag{4.5}$$

The latter decomposition is called *Adaptive fast Fourier type decomposition.*

5 Harmonic Conjugates in Weighted Bergman Spaces

The British mathematicians G.H. Hardy and J.E. Littlewood [18] were the first who considered in 1931 the problem of harmonic conjugation in spaces of holomorphic functions on the unit disk in $\mathbb{C}$. Later E. Stein and G. Weiss generalized this to the half space in $\mathbb{R}^{n+1}$ and characterized Hardy spaces on the upper half space. A. Sudbery in 1979 gave by the help of quaternions the first explicit formula for a harmonic conjugate in $\mathbb{R}^4$. In 1988–1993 D. Constales, F. Brackx, N. van Acker, Z. Xu, J. Chen, and W. Zhang constructed harmonic conjugates to the Poisson kernel. M. Shapiro (1997) and F. Brackx, B. De Knock, H. De Schepper ,D. Eelbode (2006) as well as R. Delanghe and F. Sommen (2002) studied such problems using for the construction singular integral equations.

During the stay of K. Avetisyan in Freiberg and Weimar the following question was studied:

If a given harmonic function belongs to some function space, where are then the harmonic conjugate functions?

Let $f(x) = f(r\zeta)$ a $\mathbb{H}-$valued function in $B(= B_4) = B_1(0) \subset \mathbb{R}^4, 0 \leq r < 1, \quad \zeta \in \partial B = S(= S^3)$. Further let

$$\|f\|_{p,\alpha} := \left(\int_B (1 - |x|)^{\alpha} |f(x)^p dV(x) \right)^{1/p}, 0 < p < \infty, \alpha > -1, \tag{5.1}$$

where dV is the Lebesgue volume measure on B normalized by $V(B) = 1$. We define

$$\mathbf{M}_{\alpha}^{p} := \{f \in \mathbf{M}(B, \mathbb{H}) : \|f\|_{p,\alpha} < \infty\} \tag{5.2}$$

$$h_\alpha^p := \{u \in h(B, \mathbb{R}) : \|u\|_{p,\alpha} < \infty\}. \tag{5.3}$$

$\mathbf{M}_{p,\alpha}$ denotes the *weighted Bergman space of (left) holomorphic functions* in B and h_α^p denotes the *Bergman space of scalar harmonic functions* in B.

Theorem 5.1 *Let $u : B \to \mathbb{R}$ a real-valued harmonic function in the unit ball B. If $u \in h_\alpha^p$ for some $\alpha > -1$ and $0 < p < 1$ or $1 < p < \infty$ then there exists a (left) holomorphic function $f : B \to \mathbb{H}$ such that $f \in \mathbf{M}_\alpha^p$ and $\mathrm{Sc}\, f = u$ in B and*

$$\|f\|_{p,\alpha} \le C(p, \alpha)\|u\|_{p,\alpha}. \tag{5.4}$$

Remark 5.2 The proofs for the two cases $0 < p < 1$ and $1 < p < \infty$ are different. The first case is more complicated.

The starting point for any proof is the following

Theorem 5.3 (A. Sudbery) *Let $v(\mathbf{x})$ be a harmonic function in a domain $G \subset \mathbb{R}^3$, let G be star-shaped with respect to the origin. Here star-shaped means a domain which contains with every $x \in G$ also the line from the origin to x. Then with the Dirac operator D*

$$u(x) = v(\mathbf{x}) - Vec\left(\int_0^1 t^2(Dv)(t\mathbf{x})x\,dt\right) \tag{5.5}$$

is a (left) holomorphic function in $\mathbb{R} \times G \subset \mathbb{H}$.

One can obtain the following inequalities for potential kernels. For the proof we further need estimations of potential kernels:

Let $\beta > \alpha > -1$, $0 \le r < 1$. Then for any $x = r\zeta \in B_n$, $\zeta \in S^n$.

$$\int_{S^n} \frac{d\sigma(\xi)}{|\xi - x|^{\beta+n}} \le C(\beta, n)\frac{1}{(1 - |x|)^{\beta+1}}, \tag{5.6}$$

$$\int_{B_n} \frac{(1 - |y|)^\alpha}{|\zeta - ry|^{\beta+n}} dV_n(y) \le C(\alpha, \beta, n)\frac{1}{(1 - |x|)^{\beta-\alpha}}, \tag{5.7}$$

(see also [19, 20]). Starting from the *classical Hardy inequality* it follows:

Let $f \ge 0$, $1 < p < \infty$

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^p dx \le \left(\frac{p}{p-1}\right)\int_0^\infty [f(x)]^p dx, \tag{5.8}$$

see in [21], one has also the Hardy type inequalities Let $1 \leq p < \infty$, $\beta > -1$ and $g(r) \geq 0$, then

$$\int_0^1 (1-r)^{\beta} \left(\int_0^r g(t)dt \right)^p dr \leq C(\beta, p) \int_0^1 (1-r)^{\beta+p} g^p(r)dr. \tag{5.9}$$

Let $0 < p < 1$ and $g(r)$ is a positive increasing function and $0 \leq r < 1$, then

$$\left(\int_0^1 g(tr)dt \right)^p \leq C(p) \int_0^1 (1-t)^{p-1} g^p(tr)dt. \tag{5.10}$$

On the basis of the famous Hardy-Littlewood-Fefferman-Stein inequality for subharmonic functions $|u|^p$, which reads as follows:

$$|u(x)|^p \leq \frac{C(p,n)}{(1-|x|)^n} \int_{B(x)} |u(y)|^p dV_n(y) \quad \text{(HL-property)} \tag{5.11}$$

with

$$B(x) := \left\{ y \in B_n : |y-x| < \frac{1}{2}(1-|x|) \right\}. \tag{5.12}$$

K. AVETYSIAN obtained for $0 < p < \infty$, $\alpha > -1$ and for all $u \in h(B_n)$

$$\int_{B_n} (1-|x|)^{\alpha} |u(x)|^p dV_n(x) \equiv \int_0^1 (1-r)^{\alpha} M_p^p(u;r)dr, \tag{5.13}$$

with $M_p(u;r) = \|u(r\cdot)\|_{L^p(S,d\sigma)}$, where $0 \leq r < 1$, $0 \leq p < \infty$. $d\sigma$ is the normalized surface measure from S. (see [22]).

6 On Schwarz Type Formulae

6.1 Schwarz Integral Formula in the Complex Plane

Theorem 6.1 (Integral Formula of Schwarz, 1870) *Let z and ζ be variables. B_r is a disk around the origin with radius r. Further let $f = u + iv$ be a function holomorphic in $B_r \subset \mathbb{C}$ and continuous in $\overline{B_r}$. Then for $z \in B_r$ holds*

$$f(z) = \frac{1}{2\pi i} \int\limits_{\partial B_r} \frac{\zeta + z}{\zeta - z} u(\zeta) \frac{d\zeta}{\zeta} + i \operatorname{Im} f(0). \tag{6.1}$$

Analogously a formula with the imaginary part of f holds as well:

$$f(z) = \frac{1}{2\pi} \int\limits_{\partial B_r} \frac{\zeta + z}{\zeta - z} v(\zeta) \frac{d\zeta}{\zeta} + \operatorname{Re} f(0). \tag{6.2}$$

6.2 Schwarz Kernel in $\mathbb{R}^4$

In case of quaternions the following result could be obtained:

Theorem 6.2 *Given a function f, which is defined on $S^3 = \partial B_1(0) = \partial B$, assume $f \in L^2(S^3)$, without loss of generality, we further assume that f is real valued (otherwise we handle with f componentwisely). If a function $F \in \mathcal{H}^2(B)$ satisfies*

$$\lim_{r \to 1^-} Sc(F(r\xi)) = f(\xi), \quad \text{a.e. on } S^3, \tag{6.3}$$

then the adaptive decomposition of F lead to the adaptive decomposition of f. F can be constructed explicitly (not necessarily uniquely) by

$$F(x) = T(f)(x) = \int_{|\omega|=1} S(x,\omega) f(\omega) do(\omega), \quad |x| < 1, \tag{6.4}$$

where $S(x,\omega) = P(x,\omega) + Q(x,\omega)$ is the quaternionic Schwarz kernel with the Poisson kernel

$$P(x,\omega) = \frac{1}{2\pi^2} \frac{1 - |x|^2}{|x - \omega|^4}. \tag{6.5}$$

For the *harmonic conjugate of Poisson's kernel* we have

$$Q(x,\omega) = \operatorname{Vec}\Big(\int_0^1 t^2 (\overline{\partial} P)(tx,\omega) x\, dt\Big) = \Big(\frac{1}{2\pi^2} \int_0^1 \frac{4t^2(1 - t^2|x|^2)}{|tx - \omega|^6} dt\Big) \operatorname{Vec}(\overline{\omega} x)$$

$$= \frac{1}{2\pi^2} \left[\frac{(3+|x|^2)(3-\operatorname{Sc}(\overline{\omega}x))-8}{|x-\omega|^4} - \frac{\arctan \frac{\sqrt{|x|^2-(\operatorname{Sc}(\overline{\omega}x))^2}}{1-\operatorname{Sc}(\overline{\omega}x)}}{\sqrt{|x|^2-(\operatorname{Sc}(\overline{\omega}x))^2}} \right] \frac{\operatorname{Vec}(\overline{\omega}x)}{|x|^2-(\operatorname{Sc}(\overline{\omega}x))^2}$$

is the *Cauchy-type harmonic conjugate* of the Poisson kernel on the unit sphere, [17, 23].

Remark 6.3 One can prove that T is a bounded operator from $L^2(S^3)$ to $\mathcal{H}^2(B)$.

Let $\mathcal{B} = \{x = x_0 + \overrightarrow{x} = x_0 + x_1e_1 + x_2e_2 \in \mathbb{R}^3 : |x| < 1\}$. Denote with $\mathbf{A} := \mathrm{span}_{\mathbb{R}}\{1, e_1, e_2\}$. A function f, which is defined on $\mathcal{B}$ taking values in $\mathbb{H}$, is called *left holomorphic* if it satisfies

$$\mathcal{D}f = e_0\frac{\partial f}{\partial x_0} + e_1\frac{\partial f}{\partial x_1} + e_2\frac{\partial f}{\partial x_2} = 0 \tag{6.6}$$

in its domain. If in addition, f satisfies

$$\|f\|^2 := \frac{1}{4\pi}\sup_{0<r<1}\int_{\eta\in\mathcal{S}}|f(r\eta)|^2 dS < \infty, \tag{6.7}$$

where $\mathcal{S}$ is the boundary of $\mathcal{B}$ and $d\overrightarrow{x} = dx_1dx_2$, then f is said to be an element of the *Hardy space* $\mathcal{H}^2(\mathcal{B})$.

The Cauchy integral formula for this setting is

$$f(x) = \frac{1}{4\pi}\int_{\partial\Omega}\frac{(y-x)^{-1}}{|y-x|}(n(y)dS)f(y), \quad x \in \Omega,\ f \in \mathcal{H}^2(\Omega),\ \Omega = \mathcal{B}. \tag{6.8}$$

For more details about these function classes and their generalizations, see, e.g., [24].

6.3 Schwarz Formula for the Ball in $\mathbb{R}^3$

The following operator:

$$T(f)(x) = F(x) = \int_{\omega\in\mathcal{S}} P(x,\omega)f(\omega)d\omega + \int_{\omega\in\mathcal{S}} Q(x,\omega)f(\omega)d\omega, \quad x \in \mathcal{B},$$

where $P(x,\omega) = \frac{1}{4\pi}\frac{1-|x|^2}{|x-\omega|^3}$ is the *Poisson kernel* and $Q(x,\omega)$ is given by

The so-called *Cauchy-type harmonic conjugate in* $\mathbb{R}^3$ is given by

$$Q(x,\omega) = \mathrm{Vec}\Big(\int_0^1 t(\overline{\mathcal{D}}P)(tx,\omega)x\,dt\Big) \tag{6.9}$$

$$= \Big(\frac{1}{4\pi}\int_0^1 \frac{3t(1-t^2|x|^2)}{|tx-\omega|^5}dt\Big)\mathrm{Vec}(\overline{\omega}x) \tag{6.10}$$

$$= \frac{1}{4\pi}\Big[\frac{(3+|x|^2)(3-\mathrm{Sc}(\overline{\omega}x))-8}{|x-\omega|^3} - 1\Big]\frac{\mathrm{Vec}(\overline{\omega}x)}{|x|^2-(\mathrm{Sc}(\overline{\omega}x))^2}. \tag{6.11}$$

In summing up a real-valued function $f \in L^2(\mathcal{S})$ is mapped to a function $F \in \mathcal{H}^2(\mathcal{B})$, and the real part of the boundary values of F coincides with f.

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Some Connections of Complex Dynamics

Alexandre De Zotti

Abstract We survey some of the connections linking complex dynamics to other fields of mathematics and science. We hope to show that complex dynamics is not just interesting on its own but also has value as an applicable theory.

1 Introduction

Complex dynamics is the study of the iterations of holomorphic maps[1] and the field of *dynamics in one complex variable* is the subfield of complex dynamics concerning the iteration of holomorphic functions defined on Riemann surfaces. Usually this involves an open connected subset U of a Riemann surface such as the complex plane $\mathbb{C}$ or the Riemann sphere $\widehat{\mathbb{C}}$ and a nonconstant holomorphic function f defined on U and whose range intersects U. Hence if one has to explain to nonexperts what complex dynamics is, then one would have to explain many concepts and ideas: complex numbers, holomorphic functions, iteration, and finally, why studying complex dynamics.

The study of the iteration of holomorphic function belongs to both the fields of complex analysis and the theory of dynamical systems. Because of the rigidity of holomorphic functions, the theory of complex dynamics is rich in deep results: a complete combinatorial description of the structure of the Julia set of polynomials and (conjecturally) of the Mandelbrot set [53], application of the thermodynamical formalism to Julia sets (see, for example, the survey [55]) which allows to compute

[1]There is also another unrelated field called *complex dynamics* which can also be described as "nonlinear dynamics" and usually involves the coupling of different systems, hence the use of the word "complex." In our perspective the word "complex" is to be understood as relating to the complex numbers.

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their Hausdorff dimensions, interplay with circle map dynamics and the theory of small divisors (for example, [69]), as examples of realizations of unusual topologies as Julia sets (e.g., [9, 57]) or of pathological dynamical systems [13].

In this article we will attempt to give some examples of what makes complex dynamics an attractive field of research in the point of view of applications. We do not claim to be exhaustive. Section 2 contains some background material on complex dynamics. In the following sections we will focus on three areas: Kleinian groups, root finding algorithms, and the Ising model. For each area we will try to explain some of their relations with the field of complex dynamics and will refer to further references for more in depth exploration. Finally, in Section 6 we will give quick indications about other connections.

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2 Background Material in Complex Dynamics

In this section we cover basic material about complex dynamics. For some references on the topic see, for example, [19, 52, 54] or [3].

The most elementary type of holomorphic functions to study in complex dynamics would be polynomials. But beyond the trivial case of affine maps, the theory of polynomial dynamics is already rich and complex. Let $d \geq 2$ and denote by

$$\mathrm{Poly}(d) \tag{1}$$

the set of polynomials of degree d. Let $P \in \mathrm{Poly}(d)$. Then for each $z = z_0 \in \mathbb{C}$ we define inductively its *orbit* $(z_n)_n$ under P as

$$z_{n+1} := P(z_n) = P^n(z_0), \tag{2}$$

where $P^n = P \circ \ldots \circ P$ denotes the *nth iterate* of P.

Using a direct computation, it is easy to show that if $|z|$ is large enough, then its orbit converges quickly to ∞. This motivates the definition of the *filled Julia set* $K(P)$ of P:

$$K(P) := \{z \in \mathbb{C} : \text{ the orbit of } z \text{ is bounded }\}. \tag{3}$$

Then the *Julia set* $J(P)$ is defined as the boundary of the filled Julia set. The sets $K(P)$ and $J(P)$ are both totally invariant, that is $P(K(P)) = P^{-1}(K(P))$ and $P(J(P)) = P^{-1}(J(P))$. If follows from its definition that the Julia set is characterized by sensible dependence on initial conditions on its neighborhood. We will later see a more general definition for the Julia set.

For example, the Julia set of the map $z \mapsto z^2$ is simply the unit circle $\{z : |z| = 1\}$ and its filled Julia set the closed unit disk. Orbits inside the open disk are attracted by the fixed point at 0 and orbits outside the closed unit disk diverge quickly to ∞. The unit circle is situated at the interface between these two very distinct behaviors.

The Julia set of a polynomial is either connected or consists of uncountably many connected components. And in the latter case when $d = 2$, the Julia set is a Cantor set.

A *critical point* of P is a point where the derivative of P vanishes. One of the main principles of complex dynamics is that the orbits of the critical points determine the global features of the dynamics of the map. This principle is exemplified in the following equivalence: the Julia set of a polynomial is connected if and only all the critical points of P belong to the filled Julia set.

It is easy to see that any polynomial of degree 2 is conjugated via an affine change of variables to a polynomial of the form

$$P_c(z) = z^2 + c \tag{4}$$

where $c \in \mathbb{C}$ is some complex parameter. Moreover, if $c \neq c'$, then P_c and $P_{c'}$ are not affinely conjugated. The family of polynomials P_c is called the *quadratic family*. The quadratic family encompasses the dynamics of all the quadratic polynomials up to affine change of variables. The set of parameters $c \in \mathbb{C}$ is also called the *parameter space* of the quadratic family. This is just the complex plane $\mathbb{C}$ seen as a family of distinct dynamical systems.

In general one studies the properties of a family of holomorphic maps, such as bifurcations, inside the parameter spaces. For example, the *connectedness locus* of a parametrized family is the set of parameter for which the Julia set is connected. The connectedness locus of the quadratic family is more famously known as the *Mandelbrot set*.

As mentioned in the introduction the theory of complex dynamics is concerned with any type of holomorphic functions and the notion of Julia set can be extended to any mapping on a Riemann surface to itself. For that we first need to define the Fatou set.

The *Fatou set* of a holomorphic map f is the set of points z which have a neighborhood on which the family $(f^n)_{n\geq 0}$ of iterates of f forms a normal family. In other words the point z belongs to the Fatou set of f if and only if from any subsequence of $(f^n)_{n\geq 0}$ one can extract a (sub-)subsequence converging on a neighborhood of z for the topology of local uniform convergence. The Fatou set is a totally invariant open set and the *Julia set* is defined as the complement of the Fatou set in the domain of f. This means that the dynamics on the Fatou is stable while the Julia set contains the chaotic part of the dynamics.

When the Riemann surface in question is $\mathbb{C}$ the set of holomorphic functions is the set of entire functions, including the polynomials. On the Riemann sphere $\widehat{\mathbb{C}}$, the holomorphic functions are the *rational maps*. Polynomial maps are also rational maps. A polynomial map is a rational map having a fixed point (identified with $\infty \in \widehat{\mathbb{C}}$) with no other preimage than itself. In particular ∞ is a superattracting fixed point (see below) for any polynomial.

We will also need the following definitions. A *periodic point* for f is a point z such that there exists $p \geq 1$ with $f^p(z) = z$. The minimal value of p such that the above is satisfied is called the *period* of z. When the period is $p = 1$ a periodic point is simply called a *fixed point*. When a point has a finite orbit but is not periodic it is called *preperiodic*.

The derivative of f^p at a periodic point of period p is called the *multiplier* of the periodic point. The multiplier determines the local dynamics of f^p near the periodic point. Let λ be the multiplier of a periodic point z. We have the following classification:

1. If $\lambda = 0$, the periodic point is called *superattracting*.
2. If $|\lambda| < 1$, the periodic point is called *attracting* (superattracting is a special case of attracting).
3. If $|\lambda| = 1$, the periodic point is called *neutral*.
4. If $|\lambda| > 1$, the periodic point is called *repelling*.

In the first two cases the point z belongs to the Fatou set and has a *basin of attraction*. The basin of attraction is an open neighborhood of z consisting of all of the points whose orbit under f^p converges to a point in the (finite) orbit of z.

The repelling periodic points belong to the Julia set and the Julia set is equal to the closure of the set of repelling periodic points of f. The neutral case is the most complicated (and interesting) and the point z might or might not belong to $J(f)$ depending on the map and, more importantly, on the arithmetic properties of λ.

3 Complex Dynamics and Kleinian Groups

The earliest picture of the Mandelbrot set[2] to appear came from the study of discrete subgroups of Möbius transformations. Jørgensen [36] showed that a nonelementary[3] subgroup of $\mathrm{SL}(2, \mathbb{C})$ is discrete if and only if all of its subgroups that are generated by two elements are discrete. This result follows from an inequality that Jørgensen proved in an earlier work [35]. This inequality known as *Jørgensen's inequality* is a necessary condition for a group with two generators to be discrete in $\mathrm{SL}(2, \mathbb{C})$. The proof consists of a rather simple argument by contradiction. Assuming that the group is not discrete one can easily find a pair of elements for which Jørgensen's inequality is not satisfied.

The above results motivated the search for properties of subgroups of $\mathrm{PSL}(2, \mathbb{C})$ generated by two elements that would imply discreteness. Brooks and Matelski [12] generalized Jørgensen's result. This result can be stated as follows. Recall that an

[2]More precisely the conjectured interior of the Mandelbrot set.

[3]A subgroup of $\mathrm{SL}(2, \mathbb{C})$ is *elementary* if any pair of elements of infinite order have a common fixed point. Discreteness of elementary groups can be checked in an easier way than nonelementary groups.

element γ of PSL(2, $\mathbb{C}$) is called *loxodromic* if it is conjugated to $z \mapsto kz$ for some $k \in \mathbb{C}\setminus\{0\}$ with $|k| \neq 1$ or equivalently, if its squared trace $\mathrm{tr}^2\,\gamma$ is a complex number outside the closed interval $[0, 4]$.

Theorem 1 (Brooks and Matelski, 1978) *Let γ_0, γ_1 be the elements of* PSL(2, $\mathbb{C}$) *with γ_0 loxodromic. Then there exists $c = c(\gamma_0, \gamma_1) \in \mathbb{C}$ and $z_0 = z_0(\gamma_0, \gamma_1) \in \mathbb{C}$ such that if the subgroup generated by γ_0 and γ_1 is Kleinian, then the set $\{z_n : n \in \mathbb{Z}_{>0}\}$ is discrete in $\mathbb{C}$, where the sequence z_n is defined by the following induction:*

$$z_{n+1} = z_n^2 + c. \tag{1}$$

The constants c and z_0 can be computed explicitly from γ_0 and γ_1.

More precisely, let $\gamma_0, \gamma_1 \in$ PSL(2, $\mathbb{C}$) with γ_0 loxodromic. Let τ be the *complex translation length* of γ_0. It is defined by the identity $\mathrm{tr}^2\,\gamma_0 = 4\,(\cosh(\tau/2))^2$ with the normalizations $\mathrm{Re}\,\tau \geq 0$ and $\mathrm{Im}\,\tau \in\,]-\pi, \pi]$[4]. Then

$$c = (1 - \cosh(\tau))\cosh(\tau). \tag{2}$$

Now define for $n \in \mathbb{Z}_{\geq 1}$, $\gamma_{n+1} = \gamma_i\gamma_0\gamma_n^{-1}$. Note that for $n \geq 2$, γ_n is loxodromic. Let δ_n be the *complex distance* between the axis of γ_0 and the axis of γ_n. This complex number satisfies $\mathrm{Re}\,\delta_n \geq 0$ and if we denote the fixed points of γ_n by α_n, β_n (in order such that α_n is repelling and β_n is attracting), then $(\cosh(\delta_n/2))^2$ is equal to the cross ratio of $\alpha_0, \alpha_n, \beta_0, \beta_n$. Then

$$z_n = (1 - \cosh(\tau))\cosh\delta_n.^5 \tag{7}$$

If the group generated by γ_0 and γ_1 is discrete, then the set $\left\{\cosh(\delta_n) : n \in \mathbb{Z}_{\geq 2}\right\}$ is discrete in $\mathbb{C}$. The theorem follows from the following inductive relation on the sequence of δ_n [12, p.67]:

$$\cosh(\delta_{n+1}) = (1 - \cosh(\tau))\,(\cosh(\delta_n))^2 + \cosh\tau. \tag{10}$$

[4] Equivalently γ_0 is conjugated to the map $z \mapsto kz$ with $k = e^\tau$ and $\mathrm{Re}\,\tau \geq 0$.

[5] An equivalent formulation is as follows. Let σ be the squared trace of γ_0 and R the value of the cross ratio of $\alpha_0, \alpha_1, \beta_0, \beta_1$. Then

$$c = (2 - \sigma/2)\,(\sigma/2 - 1) \tag{8}$$

and

$$z_1 = (2 - \sigma/2)\,(2R - 1). \tag{9}$$

Then $\frac{z_n}{2-\sigma/2}$ is equal to the image of the complex distance between the axis of γ_n and the axis of γ_0 by the function cosh.

In their article they proceed to draw the filled Julia set of $z^2 + 0.1 + 0.6i$ and the set of $c \in \mathbb{C}$ for which $z \mapsto z^2 + c$ has a stable periodic orbit. It is noteworthy that one of the important features of the field of complex dynamics in the years following the work of Brooks and Matelski is the use of computer graphics in an exploratory way.

Works on the question of the discreteness of groups of Möbius transformations generated by certain generators, and in particular on generalizations of Jorgensen's inequalities, neither started nor ended with the above example (e.g., [11, 18, 35, 37, 38, 42, 43, 67]). One of the important developments appears in the work of Gehring and Martin [29]. Their main theorem is as follows.

Theorem 3.1 (Gehring and Martin, [29]) *Assume that the group generated by $\gamma_0 \in \mathrm{PSL}(2, \mathbb{C})$ and $\gamma_1 \in \mathrm{PSL}(2, \mathbb{C})$ is Kleinian and γ_0 is loxodromic. Define*

$$z_0 = \mathrm{tr}\left[\gamma_0, \gamma_1\right] - 2 \tag{11}$$

and

$$\beta = \mathrm{tr}^2\,\gamma_0 - 4, \tag{12}$$

where $\left[\gamma_0, \gamma_1\right] = \gamma_0\gamma_1\gamma_0^{-1}\gamma_1^{-1}$ is the commutator. Let $K(P_\beta)$ be the filled Julia set of $z \mapsto P_\beta(z) = z^2 - \beta z$.

Then either $z_0 \notin K(P_\beta)$ or z_0 is preperiodic under the iteration of P_β. If z_0 is preperiodic, its orbit never lands on the fixed point 0.

Moreover, if z_0 is preperiodic, there are nontrivial conjugacy relations between γ_0 and γ_1.

These results can be related to other types of link that have been established between the iteration of rational maps and Kleinian groups. These include famously Sullivan's dictionary (compare [50, 51, 65, 66]) but also the study of the coupling of the dynamics of rational maps with Möbius transformations through a procedure called *mating* (see, e.g., [14–17]).

4 Newton's Method and Other Numerical Methods

Since for most polynomials there is no simple formula that expresses the roots in terms of the coefficients one has to use iterative methods to find numerical approximations of their roots. A classic method is Newton's iterative scheme. It is based on the idea that the function whose roots are to be found can be locally replaced by its first order approximation. An approximation of a root is inductively computed using this local approximation of the function. In precise terms, if we want to solve the equation

$$P(z) = 0 \tag{1}$$

we define a sequence of approximations of some root by picking a guess z_0 and then defining the sequence

$$z_{n+1} = N(z_n), \tag{2}$$

where N_P is the *Newton map* and is defined as

$$N_P(z) = z - \frac{P(z)}{P'(z)}. \tag{3}$$

For a large set of choices of z_0 the sequence $(z_n)_n$ will indeed converge to a root of P. Since the invention of Newton's method many other methods for finding the roots of polynomials have been found. Despite its simplicity Newton's method is already quite efficient. This simplicity and its old age has allowed a good understanding of the dynamics of Newton's method. When applied to complex analytic equations, in particular when P is a polynomial, Newton's method can be studied by using complex dynamics.

The study of the dynamics of the Newton map is an old (for example, [20]) and rich topic. Here we are only exploring a very small portion of the theory. In particular we only look at methods for finding roots of a polynomial. The study of Newton's method in complex dynamics is not restricted to this case, see, for example, [4, 31, 32] and more recently [2].

We will focus on two aspects of the theoretical study of Newton's method and related root finding algorithms. Firstly we will ask the question of how big is the set of pairs map-and-initial-guess (f, z_0) for which the method converges. Then we will look for an algorithm to find all the roots of a given polynomial.

4.1 Genericity of Convergence

Given an iterative algorithm it is natural to ask for which initial values this algorithm will converge. One can also ask for which function we are guaranteed to find the roots by using the algorithm. In the best case the algorithm would converge to all or almost all (in the sense of measure) pairs $(P, z) \in \mathrm{Poly}(d) \times \mathbb{C}$ of polynomials and initial guesses. In the context of Newton's method and more general root finding algorithms this idea has been conceptualized by Smale. Smale introduced in [64] the notion of *generally convergent purely iterative algorithm* (GCPIA).

A *purely iterative algorithm* is given by a map $T(P, z) = T_P(z)$ which depends rationally on z and on the coefficients of P. A purely iterative algorithm is *generally convergent* if there exists a dense open set $\Omega \subset \mathrm{Poly}(d) \times \mathbb{C}$ of full measure such that for all $(P, z) \in \Omega$ the sequence $\left(T_P^n(z)\right)_n$ converges to a root of P. A *GPCIA* is a purely iterative algorithm which is generally convergent.

An alternative definition requires only that Ω is open and dense but not necessarily of full measure [49]. To distinguish them from GCPIA we will call such algorithms *GCPIAM*.

Newton's method is not a GCPIA for $\mathrm{Poly}(d)$, $d > 2$. Indeed there are many examples of polynomials for which the Newton map has other attracting basins than the ones of the roots, see, for example, [34]. Using deep results in complex dynamics McMullen was able to show that there is no GCPIA for $\mathrm{Poly}(d)$ with $d \geq 4$ and gave a complete classification of GCPIA for $d = 2, 3$.

Before stating McMullen's result we need the following definition. The *centralizer* $C(T)$ of a rational map T is defined as the subgroup of Möbius transformations which commute with T.

Theorem 4.1 (McMullen, [48], Theorem 1.1)

1. *There is no GCPIAM for* $\mathrm{Poly}(d)$ *with* $d \geq 4$.
2. *Let T be a purely iterative algorithm defined over* $\mathrm{Poly}(3)$. *Then T is a GCPIAM if and only if there exists a rational map $T_0 : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that the following are true.*

   a. *There is $U_0 \subset \mathbb{C}$ open and dense in $\mathbb{C}$ such that for all $z \in U_0$, $T_0^n(z)$ converges to a root of $P_0(z) = z^3 - 1$.*
   b. *The centralizer $C(T)$ contains the group of Möbius transformations permuting the roots of P_0.*
   c. *For all $P \in$* $\mathrm{Poly}(3)$ *with no multiple root, $T_P = M_P \circ T_0 \circ M_P^{-1}$, where M_P is a Möbius transformation mapping the roots of P_0 to the roots of P.*

3. *Let T be a purely iterative algorithm defined over* $\mathrm{Poly}(2)$. *Then T is a GCPIAM if and only if there exists a rational map $T_0 : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ and a rational function $M :$* $\mathrm{Poly}(2) \to \mathrm{PSL}(2, \mathbb{C})/C(T_0)$ *such that the following are true.*

   a. *There is $U_0 \subset \mathbb{C}$ open and dense in $\mathbb{C}$ such that for all $z \in U_0$, $T_0^n(z)$ converges to a root of $P_0(z) = z^2 - 1$.*
   b. *The centralizer of T_0 contains $z \mapsto -z$.*
   c. *If P has no multiple roots, then $M(P)$ maps the roots of P_0 to the roots of P and $T_P = M_P \circ T_0 \circ M_P^{-1}$ for some representant M_P of $M(P)$ in* $\mathrm{PSL}(2, \mathbb{C})$.

In particular the following examples are GCPIA (see [48], Proposition 1.2):

1. Newton's method for quadratic polynomials.
2. The Newton's map of the rational map

$$f(z) = \frac{z^3 + az + b}{3az^2 + 9bz - a^2} \tag{4}$$

is a GCPIA for the cubic polynomials of the form

$$P(z) = z^3 + az + b. \tag{5}$$

The above are also characterized by their fast convergence due to the fact that the roots are superattracting fixed points of the map T_P (compare above reference). Note that finding the roots of a cubic polynomial can easily be replaced by the problem of finding the roots of some P in the form (5). The Newton map for (4) is *expanding* and its only Fatou components are in the basin of some root of P. This implies that it is not just a GCPIAM but also a GCPIA as stated.

When the condition on the complex analycity of the mapping T is relaxed into real analycity (that is by allowing complex conjugate in the formulas), a GCPIA exists for any degree [63].

McMullen later refined their results in [49]. In that article they explain that *braiding* of the roots when going around a polynomial with multiple roots prevents the existence of a mapping T_P which is a GCPIA for the polynomials of degree $d \geq 4$. This is also a very interesting article where complex dynamics is used for studying many aspects of the GCPIAs.

The proof of Theorem 4.1 relies on deep results of complex dynamics. These include the celebrated work of Mañé, Sad, and Sullivan [45] and Thurston's work on the characterization of *postcritically finite rational maps*. A postcritically finite rational map is a rational map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that all of its critical points are either periodic or preperiodic.

If T is a GCPIAM for polynomials of degree $d \geq 2$, then $(T_P)_{P \in \mathrm{Poly}(d)}$ forms a *stable algebraic family*. This means that this is a family of rational maps of fixed degree depending rationally on the coefficients of P (*algebraic family*) and there is a uniform bound on the periods of attracting cycle (*stable*). Indeed the roots are the only attracting periodic points of the family for generic points and there is no bifurcation in the sense of [45].

From a result of Thurston it follows that stable algebraic families either are *trivial* (all the elements in the family are conjugated to each other by a Möbius transformation) or consist of Lattès examples (see [44] or [54], Definition 7.4 for a definition). The latter case is excluded for a GCPIAM since the Julia set of a Lattès example consists of the whole Riemann sphere. It follows from the rigidity of Möbius transformations that a GCPIA cannot exists for $d \geq 4$.

To get around the problem of the nonexistence of GCPIA one can consider instead *towers of algorithms* as defined in [27]. Then it can be shown that the roots of a polynomial of degree d can be computed by a general tower of algorithms if and only if $d \leq 5$ ([27], Corollary 4.3). An explicit algorithm for the quintic in given in the appendix of [27].

Crass [21, 22] has provided methods for solving the quintics and equations of higher degree in a similar manner. These methods involve the iteration of holomorphic maps in higher dimensional complex projective spaces. Subsequent developments also include [23].

4.2 Finding All the Roots

One of the remarkable features of the theory is that it can be used to describe an explicit strategy for finding all the roots of a polynomial with certainty.

Early works on the maximal complexity of Newton's method applied to the search of roots of complex polynomials include Manning's [46]. This work contains the description of an implementable algorithm that ensures the finding of at least one root with complexity bounded a priori by a constant depending only on the degree d (note that their result applies only for $d \geq 10$). This is based on the fact that the Newton map N_P has a repelling fixed point on the Riemann sphere at ∞, explicit bounds on the behavior of N_P and distortions estimates coming from complex analysis.

The question of finding a choice of initial guesses that would guarantee finding all the roots of the polynomial was answered by Hubbard, Schleicher, and Sutherland in [33]. The set they produce depends only on the degree d of P.

Let $\mathscr{P}_d$ be the set of polynomials of degree d with all the roots inside the open unit disk $\mathbb{D}$. Note that there is a simple method to substitute the problem of finding all the roots of an element of $\mathscr{P}_d$ for the problem of finding roots of some arbitrary polynomial.

Theorem 4.2 ([33]) *Let $d \geq 2$. There exists $S_d \subset \mathbb{C}$ finite with at most $1.11d\,(\log d)^2$ elements such that for all $P \in \mathscr{P}_d$ and all root ξ of P there exists $s \in S_d$ such that $N_P^n(s) \to \xi$ as $n \to \infty$.*

Let $\varepsilon > 0$. By compactness there exists $n = n(d, \varepsilon)$ such that for all $P \in \mathscr{P}_d$ and all ξ root of P there exists $s \in S_d$ such that

$$\left|N_P^n(s) - \xi\right| \leq \varepsilon. \tag{6}$$

This ensures that the algorithm effectively finds all the roots in finite time. Schleicher's article [60] provides explicit estimates on $n(d, \varepsilon)$. In theory each guess could require a large number of iterations as the degree becomes large.

The article [33] also contains an explicit construction for the set S_d and finer and better results for when the polynomial is real. The authors use their own algorithm to compute approximations to the invariant measure of Hénon mappings.

In [61] Schleicher and Stoll give a slightly different version of the algorithm mentioned above. This reference also contains many remarks on the implementation and possible improvements. Using some numerical experiments they checked that the theoretically possible large number of iterations (larger than d^2 with $d \approx 10^6$) was not a problem in practice for the specific problems they were looking at. They used it to find the centers of hyperbolic components of the Mandelbrot set and periodic points of iterated polynomials.

The roots are attracting fixed points of the Newton map N_P. The *basin of a root* is the set of points whose orbit converge to the root under the iteration of N_P. This is an open set. The *immediate basin* of a root is the connected component of the basin containing the root. The proof of Theorem 4.2 builds on previous results relating to the shape of the immediate basins of the roots such as [46] and [56].

A summary of the proof is as follows. The only fixed points of the Newton map N_P are the roots of P and ∞. The roots are either superattracting or attracting with multiplier $1 - 1/k$ for some integer $k \geq 2$. The fixed point at ∞ is repelling.

Let $\left(\xi_i\right)_i$ be the roots of P and let U_{ξ_i} be the immediate basin of ξ_i. Define also m_{ξ_i} as the number of critical points of N_P (counted with multiplicity) inside U_{ξ_i}. From [33], Proposition 6, it follows that U_{ξ_i} has m_{ξ_i} accesses to ∞[6]. The idea is to constrain the geometry of these accesses.

Pick a root $\xi = \xi_i$. From [56] (see also [62]) we know that U_ξ is simply connected. Let $\varphi : \mathbb{D} \to U_\xi$ be a conformal isomorphism normalized so that $\varphi(0) = \xi$ and define

$$f := \varphi^{-1} \circ N_P \circ \varphi. \tag{7}$$

The mapping f is proper of degree $m + 1$, where $m = m_\xi$. This mapping can be extended by reflection into a rational map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ of degree $m + 1$.

The rational map f has $m + 2$ fixed points (counted with multiplicity). The point 0 is a (super)attracting fixed point of f. By symmetry this is also the case for the point ∞. The respective multipliers λ_0, λ_{m+1} of 0 and ∞ are either both equal to 0 or to $1 - 1/k$ for some integer $k \geq 2$. It also follows from the symmetry that the other fixed points $\zeta_1, \ldots, \zeta_m$ lie on the unit circle and their respective multipliers are positive real numbers $\lambda_j > 1$.

The holomorphic fixed point formula applied to f (see, e.g., [54, Section 12]) states that

$$\sum_{j=0}^{m+1} \frac{1}{\lambda_j - 1} = -1. \tag{8}$$

Hence

$$\sum_{j=1}^{m} \frac{1}{\lambda_j - 1} \geq 1. \tag{9}$$

It follows that there must be at least one j such that $\lambda_j - 1 \leq m$.

The quotient of the corresponding channel for N_P by the dynamics of N_P is an annulus of modulus $\frac{\pi}{\log \lambda_j}$. Indeed this is the value of the modulus of the annulus obtained by taking the quotient of the upper plane by the action of $z \mapsto \lambda_j z$. Since the degree of f is at most equal to the degree of N_P it follows that

[6] That is the complement of some large disk in U_{ξ_i} has m_{ξ_i} components accumulating to ∞.

$$\frac{\pi}{\log \lambda_j} \geq \frac{\pi}{\log(m+1)} \geq \frac{\pi}{\log d}. \tag{10}$$

Having such a lower bound on the modulus allows to find places where the channel must have a definite extent. This is made precise in [33, Section 5]. Using this, one can pick points independently of P such that at least one of them is in U_ξ.

5 Hierarchical Ising and Potts Models

The Ising and Potts models are mathematical models from solid state physics. The Ising model relates to the ferromagnetic properties of a material. At the base of both models lies a graph whose vertexes represent the locus of a particle/atom and the edges the interaction between these particles. Each vertex is characterized by a state chosen among a finite set of possible values. For the Ising model this set has 2 element while for the Potts model the number of possible states is some positive integer $q \geq 2$.

The Hamiltonian of the system can be computed explicitly for any state. The temperature T, interaction constant J, and the (possibly 0) magnetic field h appear as parameters in the Hamiltonian. From the Hamiltonian one can derive a formula for the partition function.

A *hierarchical lattice* consists of a refining sequence of finite graphs on which the Hamiltonian is computed successively. For example, a *diamond hierarchical lattice* can be defined as follows. The first graph consists of a pair of vertexes joined by a single edge. The refining consists in replacing each edge by two pairs of edges each connecting one of the previous vertexes to a new vertex in the middle. The refining procedure is illustrated in Figure 1. Instead of replacing each edges by 2 branches (pairs of edges), one could also insert $b \geq 2$ branches, see Figure 2. The integer b is the parameter characterizing a diamond hierarchical lattice.

The passage from the partition function of one level of the hierarchy to the next level is performed by a *renormalization group transformation*. This transformation depends on the variable J. For a diamond hierarchical lattice model the renormalization group transformation is identified as a rational map $f : z \mapsto f(z)$, where

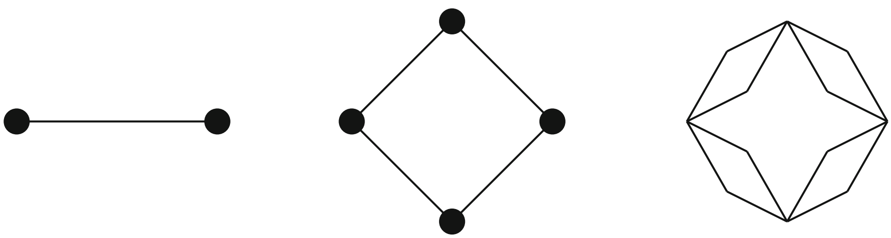

Fig. 1 The first levels of the diamond lattice hierarchy ($b = 2$).

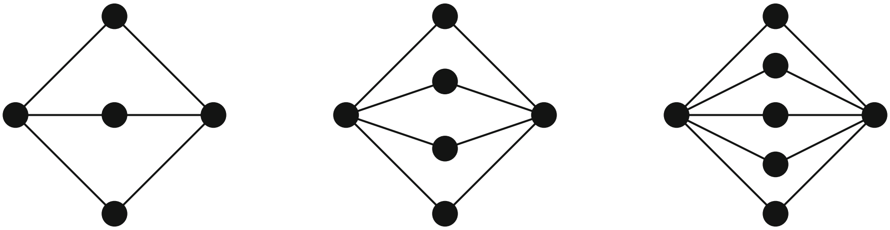

Fig. 2 Examples of diamond lattices with $b = 3, 4, 5$.

$z = z(J, T)$, and its dynamics has a physical relevance. For example, the zeros of the partition functions in the thermodynamic limit (i.e., when the level in the hierarchy tends to ∞) converge to the Julia set of f. The dynamics of this map is the focus of the study of hierarchical Potts/Ising models in complex dynamics.

These models generally do not really represent actual physical systems but are instead used to try to understand what type of properties more complicated and realistic model could have. In general one cannot hope to have an explicit formula for the renormalization transformation of a realistic model.

Hierarchical models are described in [5, 47] and [24]. In the latter, Derrida, De Seze, and Itzykson study the q-state Potts model on a diamond hierarchical lattice with $b = 2$. The renormalization map f for this model can be computed explicitly:

$$f(z) = \left(\frac{z^2 + q - 1}{2z + q - 2}\right)^2. \tag{1}$$

They provide several pictures of the Julia sets corresponding to different values of q in an attempt to get an idea of their fractal structure. This work has been followed by [25] where the geometric properties of the Julia set of f are used to extract information about the model. Those are mainly numerical studies.

Another type of hierarchical model is presented in [10]. The renormalization transformation can also be identified to a rational map. The authors study the structure and Hausdorff dimension of the corresponding Julia set. For another model see also [1, 30].

An important occurrence of the utilization of complex dynamics to study the Ising model is the work of Bleher and Lyubich [6]. They study the Ising model on the diamond hierarchical lattice for arbitrary values of $b \geq 2$ (see also [7]). In that case the renormalization transformation is represented by the rational map

$$f(z) = \frac{4z^b}{\left(1 + z^b\right)^2}. \tag{2}$$

The points 0 and 1 are superattracting fixed points of f. Denote the immediate basin of 0 by Ω_0. The free energy can be expressed as

$$F(z) = \sum_{n=0}^{\infty} \frac{1}{(2b)^n} g \circ f^n(z), \tag{3}$$

where $g(z) = \log(1 + z^b)$. Bleher and Lyubich showed that F is analytic on Ω_0 and that the boundary of Ω_0 is a natural boundary of analyticity for this function (i.e. analytic continuation is not possible along any path that crosses $\partial\Omega_0$). They also derive some physically relevant properties of the model.

The Fatou set of f consists of the respective basins of attractions of 0 and 1. Before proceeding to study the properties of the free energy F, they first showed that Ω_0 is a Jordan domain. Recall that a *quasicircle* is the image of a round circle by a quasiconformal homeomorphism of $\mathbb{C}$. Bleher and Lyubich showed that the boundary of Ω_0 is a quasicircle. Although this proof is rather simple it exemplifies the use of a powerful tool of complex dynamics: the theory of polynomial like maps [26]. This theory explains why copies of the Mandelbrot set seem to appear in every parameter space of holomorphic dynamical systems.

Let $d \geq 2$. A *polynomial like map of degree d* is a triplet $\big(U, U', f\big)$ where U and U' are simply connected open subsets of $\mathbb{C}$ such that U' is compactly contained in U and $f : U' \to U$ is a proper holomorphic mapping of degree d. The *filled Julia set* $K(f)$ of a polynomial like mapping is the set of points whose orbit stays inside the domain U',

$$K(f) := \big\{z \in U' : \forall n,\ f^n(z) \in U'\big\}. \tag{4}$$

The relevance of polynomial like mappings derives from Douady and Hubbard's straightening theorem.

Theorem 5.1 (Douady, Hubbard, [26], Theorem 1) *Let (U, U', f) be a polynomial like mapping of degree $d \geq 2$. Then there exists a quasiconformal map $\psi : \mathbb{C} \to \mathbb{C}$ and a polynomial P of degree d such that*

$$\psi \circ f = g \circ \psi \tag{5}$$

on some neighborhood of $K(f)$ and $\overline{\partial}\psi = 0$ almost everywhere on $K(f)$.

Moreover, if $K(f)$ is connected, then the polynomial P is unique up to conjugation by an affine map.

Thanks to a fine analysis of the map f, Bleher and Lyubich showed that f is polynomial like of degree b on a neighborhood of the closure of Ω_0. Since it has a superattracting fixed point of degree b at 0, the straightening of f is conjugated to the polynomial $z \mapsto z^b$. Since Ω_0 is the basin of 0, it follows that $\partial\Omega_0$ is the image of the circle $\{z : |z| = 1\}$ by a quasiconformal map of the plane, hence it is a quasicircle.

The use of complex dynamics in the field has continued after this work, for example, in [8].

6 Other Connections

There are many other applications of complex dynamics. Eremenko has mentioned other connections in a talk [28] about the interaction between function theory and complex dynamics.

A surprising application is related to gravitational lensing. In [41] Kahvinson and Świątek solved the Sheil-Small and Wilmhurst conjecture. This states that if P is a polynomial of degree $n \geq 2$, then the harmonic polynomial $z - \overline{P(z)}$ has at most $3n - 2$ zeros.

Their proof relies on the classical fact from complex dynamics that any attracting or parabolic periodic point attracts at least one critical point. This can be applied to the holomorphic polynomial $Q(z) = \overline{P\left(\overline{P(z)}\right)}$.

It turns out that this solution has an application in astrophysics exposed in [39]. A similar argument can be used when one replaces the polynomial P by a rational function R. This gives the following theorem.

Theorem 2 (Khavinson, Neumann, [39]) *Let R be a rational function of degree $n \geq 2$. Then the equation $z = \overline{R(z)}$ has at most $5n - 5$ solutions.*

In astrophysics the lensing effect produced by the gravity coming from n point like objects can be modelled via a lens equation (see, for example, [68] and [58]). A corollary ([39], Corollary 1) of the above theorem gives an explicit upper bound on the number of images that such model can produce. See [40] for further developments. For more details about this the reader is advised to consult the excellent [59].

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