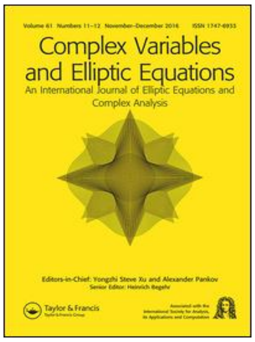

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Time–frequency transform involving nonlinear modulation and frequency-varying dilation

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ABSTRACT
This paper designs a general type time–frequency transform whose kernel function involves a nonlinear modulation and a frequency-varying dilation. The corresponding inversion formula is established.

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1. Introduction

In modern time–frequency analysis there are three basic operators which play important roles in time–frequency representations, namely *modulation* $\mathcal{M}$, *translation* $\mathcal{T}$ *and dilation* $\mathcal{D}$. They are defined as

$$\mathcal{M}_\omega f(\cdot) = e^{i\omega\cdot} f(\cdot), \quad \mathcal{T}_b f(\cdot) = f(\cdot - b), \quad \mathcal{D}_a f(\cdot) = a^{-\frac{1}{2}} f\left(\frac{\cdot}{a}\right),$$

for $b, \omega \in \mathbb{R}$, $a \in \mathbb{R}_+$, respectively.

For modulation and translation there hold the following non-commutative relations:

$$\mathcal{T}_b \mathcal{M}_\omega = e^{-ib\omega} \mathcal{M}_\omega \mathcal{T}_b = \mathcal{R}_{-b\omega} \mathcal{M}_\omega \mathcal{T}_b.$$

Here the rotation transform is defined by $\mathcal{R}_d : f \to e^{id} f$.

The time–frequency shift $\mathcal{R}_{d+\frac{b\omega}{2}} \mathcal{T}_b \mathcal{M}_\omega$ satisfies the composition rule

$$(\mathcal{R}_{d+\frac{b\omega}{2}} \mathcal{T}_b \mathcal{M}_\omega)(\mathcal{R}_{\tilde{d}+\frac{\tilde{b}\tilde{\omega}}{2}} \mathcal{T}_{\tilde{b}} \mathcal{M}_{\tilde{\omega}}) = \mathcal{R}_{\left[\left(d+\tilde{d}+\frac{1}{2}(\tilde{b}\omega - b\tilde{\omega})\right)+\frac{1}{2}(b+\tilde{b})(\omega+\tilde{\omega})\right]} \mathcal{T}_{b+\tilde{b}} \mathcal{M}_{\omega+\tilde{\omega}}$$

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that suggests the full Heisenberg group $\mathbb{H} := (\mathbb{R}^3, *)$ with the multiplication

$$(b, \omega, d) * (\tilde{b}, \tilde{\omega}, \tilde{d}) = \left(b + \tilde{b}, \omega + \tilde{\omega}, d + \tilde{d} + \frac{1}{2}(\tilde{b}\omega - b\tilde{\omega})\right).$$

The unitary operator $\mathcal{R}_{d+\frac{b\omega}{2}}\mathcal{T}_b\mathcal{M}_\omega$ is called the Schrödinger representation of the full Heisenberg group whose representation coefficient is given by

$$\begin{aligned}\langle f, \mathcal{R}_{d+\frac{b\omega}{2}}\mathcal{T}_b\mathcal{M}_\omega\phi\rangle &= \mathcal{R}_{-d-\frac{b\omega}{2}}\langle f, \mathcal{T}_b\mathcal{M}_\omega\phi\rangle \\ &= \mathcal{R}_{-d-\frac{b\omega}{2}}\langle f, \mathcal{R}_{-b\omega}\mathcal{M}_\omega\mathcal{T}_b\phi\rangle = \mathcal{R}_{-d+\frac{b\omega}{2}}\langle f, \mathcal{M}_\omega\mathcal{T}_b\phi\rangle\end{aligned}$$

for $f \in L^2(\mathbb{R})$ and a fixed function $\phi \in L^2(\mathbb{R})$. Up to the phase factor $e^{i(-d+\frac{b\omega}{2})}$, the coefficient of the Schrödinger representation coincides with the windowed Fourier transform [1]

$$\mathcal{V}_\phi f(b, \omega) := \langle f, \mathcal{M}_\omega\mathcal{T}_b\phi\rangle = \int_{\mathbb{R}} f(x)\overline{\phi(x-b)}e^{-i\omega x}\mathrm{d}x. \tag{1}$$

The corresponding inversion formula is

$$f(x) = (2\pi\|\phi\|)^{-1}\iint_{\mathbb{R}^2} \mathcal{V}_\phi f(b, \omega)\mathcal{M}_\omega\mathcal{T}_b\phi(x)\mathrm{d}b\mathrm{d}\omega, \quad \text{a.e. } x \in \mathbb{R}. \tag{2}$$

We remark that when $d = 0$ the unitary operator $\mathcal{R}_{d+\frac{b\omega}{2}}\mathcal{T}_b\mathcal{M}_\omega$ has the symmetric form $\mathcal{M}_{\frac{\omega}{2}}\mathcal{T}_b\mathcal{M}_{\frac{\omega}{2}}$ as *time–frequency shift*, namely,

$$\mathcal{M}_{\frac{\omega}{2}}\mathcal{T}_b\mathcal{M}_{\frac{\omega}{2}} = \mathcal{R}_{\frac{-b\omega}{2}}\mathcal{M}_\omega\mathcal{T}_b = \mathcal{R}_{\frac{b\omega}{2}}\mathcal{T}_b\mathcal{M}_\omega.$$

The dilation and translation satisfy the non-commutative relation

$$\mathcal{D}_a\mathcal{T}_b = \mathcal{T}_{ab}\mathcal{D}_a \quad (\text{or } \mathcal{T}_b\mathcal{D}_a = \mathcal{D}_a\mathcal{T}_{\frac{b}{a}}).$$

The unitary operator $\mathcal{T}_b\mathcal{D}_a$ compliances the composition rule

$$(\mathcal{T}_b\mathcal{D}_a)(\mathcal{T}_{\tilde{b}}\mathcal{D}_{\tilde{a}}) = \mathcal{T}_{b+a\tilde{b}}\mathcal{D}_{a\tilde{a}}$$

for $(a, b), (\tilde{a}, \tilde{b}) \in (0, +\infty) \times \mathbb{R}$ that motivates the affine group $\mathbb{A} := ((0, +\infty) \times \mathbb{R}, *)$ with the multiplication

$$(a, b) * (\tilde{a}, \tilde{b}) = (a\tilde{a}, b + a\tilde{b}).$$

The affine group [2] has the representation coefficient $\langle f, \mathcal{T}_b\mathcal{D}_a\psi\rangle$ for $f \in L^2(\mathbb{R})$ and a fixed wavelet function $\psi \in L^2(\mathbb{R})$. The latter leads to the wavelet transform

$$\mathcal{W}_\psi f(a, b) := \langle f, \mathcal{T}_b\mathcal{D}_a\psi\rangle = \int_{\mathbb{R}} f(x)a^{-\frac{1}{2}}\overline{\psi}\left(\frac{x-b}{a}\right)\mathrm{d}x. \tag{3}$$

The inversion formula reads [3]

$$f(x) = C_\psi^{-1}\int_0^\infty \frac{\mathrm{d}a}{a^2}\int_{\mathbb{R}} \mathcal{W}_\psi f(a, b)\mathcal{T}_b\mathcal{D}_a\psi(x)\mathrm{d}b, \quad \text{a.e. } x \in \mathbb{R}. \tag{4}$$

In [4], the group $\mathbb{H}$ is extended to the affine Weyl–Heisenberg group

$$\mathbb{H}_1 := \big((0,\infty)\times\mathbb{R}^3, *\big)$$

with the multiplication law

$$(a,b,\omega,d)*(\tilde{a},\tilde{b},\tilde{\omega},\tilde{d}) = \left(a\tilde{a}, b+a\tilde{b}, \omega+\frac{1}{a}\tilde{\omega}, d+\tilde{d}+\frac{1}{2}(a\langle\omega,\tilde{b}\rangle-\frac{1}{a}\langle b,\tilde{\omega}\rangle)\right).$$

The affine Weyl–Heisenberg group $\mathbb{H}_1$ has the group representation

$$\rho_{\mathbb{H}_1}(a,b,\omega,d) := \mathcal{R}_d\mathcal{M}_{\frac{1}{2}\omega}\mathcal{T}_b\mathcal{M}_{\frac{1}{2}\omega}\mathcal{D}_a,$$

which has the representation coefficient $\langle f, \mathcal{M}_{\frac{1}{2}\omega}\mathcal{T}_b\mathcal{M}_{\frac{1}{2}\omega}\mathcal{D}_a\psi\rangle$. The affine Weyl–Heisenberg group leads to the more general transformation

$$\mathcal{T}_\psi f(b,\omega) = \int_{\mathbb{R}} f(x)a^{-\frac{1}{2}}\overline{\psi}\left(\frac{x-b}{a}\right)e^{-i\omega x}\mathrm{d}x. \tag{5}$$

In this note, we will extend the transform (5) to the setting with nonlinear modulation and frequency-varying dilation, and investigate the reconstruction formula.

2. Nonlinear modulation and time–frequency transform

For a fixed function $\mu(\omega):\omega\in\mathbb{R}$ (some conditions that have to be met by this function will be set later), define the operator of nonlinear modulation by

$$\mathcal{M}_{\mu(\omega)}: f(\cdot)\to e^{i\mu(\omega)\cdot}f(\cdot). \tag{6}$$

For a real-variable and non-negative function $\lambda(\omega):\omega\in\mathbb{R}$, define the dilation operator with varying frequency by

$$\mathcal{D}_{\lambda(\omega)}: f(\cdot)\to \lambda(\omega)^{-\frac{1}{2}}f\left(\frac{\cdot}{\lambda(\omega)}\right). \tag{7}$$

Being acted by the operator $\mathcal{M}_{\mu(\omega)}\mathcal{T}_t\mathcal{D}_{\lambda(\omega)}$, a given basic atom ϕ gives rise to a class of atoms of the type $|\lambda(\omega)|^{-\frac{1}{2}}\phi(\frac{\cdot-t}{\lambda(\omega)})e^{i\mu(\omega)\cdot}$. For convenience of the discussion of the new type time–frequency transform we modify it to the following form:

$$\phi^{t,\omega}(x) := \gamma(\omega)|\lambda(\omega)|^{\frac{1}{2}}M_{\mu(\omega)}\mathcal{T}_tD_{\lambda(\omega)}\phi(x) = \gamma(\omega)\phi\left(\frac{x-t}{\lambda(\omega)}\right)e^{i\mu(\omega)x},\quad x\in\mathbb{R}, \tag{8}$$

and consider the time–frequency transforms

$$\mathcal{T}f(t,\omega) = \langle f,\phi^{t,\omega}\rangle = \overline{\gamma}(\omega)\int_{\mathbb{R}} f(x)\overline{\phi}\left(\frac{x-t}{\lambda(\omega)}\right)e^{-i\mu(\omega)x}\mathrm{d}x,\quad (t,\omega)\in\mathbb{R}^2. \tag{9}$$

Formally, the transform (9) looks like a generalization of the windowed Fourier transform (1), wavelet transform (3) and the transform (5). But it is essentially different from them due to the fact that both the dilation and the modulation depend on the frequency variable ω. We think that this idea is natural because the essence of dilation and modulation is to characterize the 'frequency' (vibration) of signals. Compared with the classical cases, the price of the nonlinearity of the modulation and the dilation in the kernel function here is that it lacks a group structure.

It is known that the windowed Fourier transform $\mathcal{V}_\phi$ is an isometry from $L^2(\mathbb{R})$ into $L^2(\mathbb{R}^2)$. The wavelet transform $\mathcal{W}_\psi$ maps $L^2(\mathbb{R})$ into $L^2(\mathbb{R}^2, C_\psi^{-1}a^{-2}\mathrm{d}a\mathrm{d}b)$, the space of all complex valued functions F on $\mathbb{R}^2$ equipped with the norm

$$\|F\| = C_\psi^{-1}\int_{\mathbb{R}}\frac{1}{a^2}\mathrm{d}a\int_{\mathbb{R}}|F(a,b)|^2\mathrm{d}b.$$

Here $C_\psi = 2\pi\int_{\mathbb{R}}|\omega|^{-1}|\hat{\psi}(\omega)|^2$ and the Fourier transform $\hat{f}$ for $f \in L^2(\mathbb{R})$ is defined by [5]

$$\hat{f}(\omega) = \mathcal{F}f(\omega) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(t)e^{-i\omega t}\mathrm{d}t, \quad \omega \in \mathbb{R}.$$

Specifically, for wavelet transform the image space $\mathcal{W}_\psi(L^2(\mathbb{R}))$ is a *reproducing kernel Hilbert subspaces* (RKHS) of the Hilbert space $L^2(\mathbb{R}^2, C_\psi^{-1}a^{-2}\mathrm{d}a\mathrm{d}b)$. The kernel function is

$$K(a,b;\tilde{a},\tilde{b}) = \overline{\left(\mathcal{W}_\psi\mathcal{T}_b\mathcal{D}_a\psi\right)(\tilde{a},\tilde{b})} = \langle\mathcal{T}_{\tilde{b}}\mathcal{D}_{\tilde{a}}\psi, \mathcal{T}_b\mathcal{D}_a\psi\rangle.$$

The windowed Fourier transform has a parallel theory: The image $\mathcal{V}_\phi(L^2(\mathbb{R}))$ is a subspace of the Hilbert space $L^2(\mathbb{R}^2)$ and also a RKHS with the kernel function [6]

$$K(\omega,b;\tilde{\omega},\tilde{b}) = \langle\mathcal{M}_{\tilde{\omega}}\mathcal{T}_{\tilde{b}}\phi, \mathcal{M}_\omega\mathcal{T}_b\phi\rangle.$$

Our first purpose is to understand the image space $\mathcal{T}(L^2(\mathbb{R}))$. We hope to choose a suitable univariate function r of the frequency variable ω such that the image space $\mathcal{T}(L^2(\mathbb{R}))$ is just a *reproducing kernel Hilbert space* (RKHS) of $L^2(\mathbb{R}^2, \frac{\mathrm{d}t\mathrm{d}\omega}{r(\omega)})$ equipped with the norm

$$\|F\|^2 = \iint_{\mathbb{R}^2}|F(t,\omega)|^2\frac{\mathrm{d}t\mathrm{d}\omega}{r(\omega)} \tag{10}$$

being given by the inner product

$$\langle F,G\rangle_{t,\omega} = \iint_{\mathbb{R}^2}F(t,\omega)\overline{G(t,\omega)}\,\frac{\mathrm{d}t\mathrm{d}\omega}{r(\omega)}. \tag{11}$$

Note that the functions γ, λ, μ and r are univariate functions dependent on the frequency variable ω, which we need to construct. We remark that it is difficult to extend them to bivariate functions of time and frequency variables. The reason can be seen from the proof of the first lemma of the next section.

We need two useful transformations. The first is the Hilbert transform which is defined by

$$\mathcal{H}f(t) = \frac{1}{\pi}\text{p.v.}\int_{\mathbb{R}} \frac{f(x)}{t-x}\text{d}x, \quad t \in \mathbb{R},$$

where the improper integral must converge in the sense of Cauchy principle. The second is the integral transform $\mathcal{I}$ by

$$\mathcal{I}f(t) = \int_t^\infty f(x)\text{d}x,$$

from which we know that $\mathcal{I}f$ is essentially an antiderivative of $-f$.

3. Technical lemmas

We will establish some lemmas which are crucial for the proof of our main results.

Lemma 3.1: *Suppose that both f and $\mathcal{I}f$ are in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Then for any real numbers A, B, there holds the following identity:*

$$\mathcal{F}^{-1}\left(\frac{\hat{f}(\cdot)\text{sgn}(\cdot - A)}{\cdot - B}\right)(t) = \mathcal{M}_A\mathcal{H}\left(\mathcal{M}_{B-A}\mathcal{I}\left(\mathcal{M}_{-B}f\right)\right)(t)$$
$$= e^{iAt}\mathcal{H}\left(e^{i(B-A)t}\mathcal{I}\left(e^{-iBt}f(t)\right)\right). \tag{12}$$

Proof: Denote by $g = \mathcal{F}^{-1}\left(\frac{\hat{f}(\cdot)\text{sgn}(\cdot-A)}{\cdot-B}\right)$, that is, $\hat{g} = \frac{\hat{f}(\cdot)\text{sgn}(\cdot-A)}{\cdot-B}$. Then we have

$$\mathcal{T}_{-A}\hat{g}(\omega) = \frac{\hat{f}(\omega+A)}{-i(\omega+A-B)}(-i\text{sgn}(\omega)).$$

By using the identity $\mathcal{F}\mathcal{M}_t = \mathcal{T}_t\mathcal{F}$ and the property of $\frac{\pi}{2}$-phase shift

$$\mathcal{F}(\mathcal{H}f)(\omega) = -i\text{sgn}(\omega)\hat{f}(\omega)$$

of the Hilbert transform, it gives

$$\left(\mathcal{M}_{-A}g\right)^{\wedge}(\omega) = \left(\mathcal{H}g_1\right)^{\wedge}(\omega) \tag{13}$$

with

$$\hat{g}_1(\omega) = \frac{\hat{f}(\omega+A)}{-i(\omega+A-B)}. \tag{14}$$

To view Equation (13) in the time domain, we get

$$g(t) = e^{iAt}\mathcal{H}g_1(t). \tag{15}$$

We now compute g_1 in the time domain according to its definition in (14). Applying the translation operator $\mathcal{T}_{A-B}$ to both sides of (14), we have

$$\mathcal{T}_{A-B}\hat{g}_1(\omega) = \frac{\hat{f}(\omega+B)}{-i\omega} = \frac{\mathcal{T}_{-B}\hat{f}(\omega)}{-i\omega} = \left(\mathcal{I}\left(\mathcal{M}_{-B}f\right)\right)^{\wedge}(\omega). \tag{16}$$

In the last step of the above computation, we used the relations $(\mathcal{F}f')(\omega) = i\omega\hat{f}(\omega)$ and $(\mathcal{I}f)' = -f$.

Applying the inverse Fourier transform to both sides of (16), we have

$$\mathcal{M}_{A-B}g_1(t) = \mathcal{I}\left(\mathcal{M}_{-B}f\right)(t).$$

Consequently,

$$g_1(t) = \mathcal{M}_{B-A}\mathcal{I}\left(\mathcal{M}_{-B}f\right)(t).$$

Finally, substituting this equation into (15), we conclude (12). This completes the proof of the lemma. ■

The next lemma concerns about the Fourier transform of $\phi^{t,\omega}$ in (8).

Lemma 3.2: *Suppose that $\phi^{t,\omega}$ is defined in (8). Then the Fourier transform of $\phi^{t,\omega}$ is*

$$\begin{aligned}\mathcal{F}\left(\phi^{t,\omega}\right)(\xi) &= \gamma(\omega)|\lambda(\omega)|^{\frac{1}{2}}T_{\mu(\omega)}M_{-t}D_{\frac{1}{\lambda(\omega)}}\hat{\phi}(\xi)\\ &= \gamma(\omega)|\lambda(\omega)|e^{-it(\xi-\mu(\omega))}\hat{\phi}\left(\lambda(\omega)(\xi-\mu(\omega))\right).\end{aligned}$$

Proof: Applying the formula $\mathcal{F}\mathcal{M}_t = \mathcal{T}_t\mathcal{F}$, $\mathcal{F}\mathcal{T}_t = \mathcal{M}_{-t}\mathcal{F}$ and $\mathcal{F}\mathcal{D}_t = \mathcal{D}_{\frac{1}{t}}\mathcal{F}$, we have

$$\begin{aligned}\mathcal{F}\left(\phi^{t,\omega}\right)(\xi) &= \mathcal{F}\left(\gamma(\omega)|\lambda(\omega)|^{\frac{1}{2}}M_{\mu(\omega)}T_tD_{\lambda(\omega)}\phi\right)(\xi)\\ &= \gamma(\omega)|\lambda(\omega)|^{\frac{1}{2}}T_{\mu(\omega)}M_{-t}D_{\frac{1}{\lambda(\omega)}}\hat{\phi}(\xi)\\ &= \gamma(\omega)|\lambda(\omega)|e^{-it(\xi-\mu(\omega))}\hat{\phi}\left(\lambda(\omega)(\xi-\mu(\omega))\right).\end{aligned}$$ ■

The next lemma offers an alternative form of $\mathcal{T}f$.

Lemma 3.3: *For any $f \in L^2(\mathbb{R})$, the following identity holds:*

$$\mathcal{T}f(t,\omega) = \langle f, \phi^{t,\omega}\rangle = e^{-it\mu(\omega)}\left\langle \hat{f}\gamma(\omega)|\lambda(\omega)|^{\frac{1}{2}}\mathcal{T}_{\mu(\omega)}\mathcal{D}_{\frac{1}{\lambda(\omega)}}\overline{\hat{\phi}}, \mathcal{M}_{-t}1\right\rangle. \tag{17}$$

Proof: Applying Lemma (3.2) and the unitary property of the Fourier transform, we have

$$\begin{aligned}\mathcal{T}f(t,\omega) &= \langle f, \phi^{t,\omega}\rangle = \left\langle \hat{f}, \mathcal{F}\phi^{t,\omega}\right\rangle\\ &= \left\langle \hat{f}, \gamma(\omega)|\lambda(\omega)|^{\frac{1}{2}}\mathcal{T}_{\mu(\omega)}\mathcal{M}_{-t}\mathcal{D}_{\frac{1}{\lambda(\omega)}}\hat{\phi}\right\rangle.\end{aligned}$$

Using the canonical commutation relation of the modulation and the translation

$$\mathcal{T}_x\mathcal{M}_\omega = e^{-ix\omega}\mathcal{M}_\omega\mathcal{T}_x,$$

we get

$$\begin{aligned}\mathcal{T}f(t,\omega) &= \left\langle \hat{f}, \gamma(\omega)|\lambda(\omega)|^{\frac{1}{2}} e^{it\mu(\omega)}\mathcal{M}_{-t}\mathcal{T}_{\mu(\omega)}\mathcal{D}_{\frac{1}{\lambda(\omega)}}\hat{\phi}\right\rangle \\ &= e^{-it\mu(\omega)}\left\langle \hat{f}, \gamma(\omega)|\lambda(\omega)|^{\frac{1}{2}}\mathcal{M}_{-t}\mathcal{T}_{\mu(\omega)}\mathcal{D}_{\frac{1}{\lambda(\omega)}}\hat{\phi}\right\rangle \\ &= e^{-it\mu(\omega)}\left\langle \hat{f}\gamma(\omega)|\lambda(\omega)|^{\frac{1}{2}}\mathcal{T}_{\mu(\omega)}\mathcal{D}_{\frac{1}{\lambda(\omega)}}\overline{\hat{\phi}}, \mathcal{M}_{-t}1\right\rangle.\end{aligned}$$

The proof of this lemma is completed. ■

The next lemma is crucial for reproducibility of the time–frequency transform defined in (9).

Lemma 3.4: *Suppose that both f and g are in $L^2(\mathbb{R})$. Then the following identity holds:*

$$\begin{aligned}&\iint_{\mathbb{R}^2}\langle f,\phi^{t,\omega}\rangle\overline{\langle g,\phi^{t,\omega}\rangle}\,\frac{\mathrm{d}t\mathrm{d}\omega}{r(\omega)} \\ &= 2\pi\iint_{\mathbb{R}^2}\gamma^2(\omega)\lambda^2(\omega)\left|\hat{\phi}\left[\lambda(\omega)\left(t-\mu(\omega)\right)\right]\right|^2\hat{f}(t)\overline{\hat{g}(t)}\,\frac{\mathrm{d}t\mathrm{d}\omega}{r(\omega)}. \qquad (18)\end{aligned}$$

Proof: Note that the left-hand side of (18) is essentially the inner product $\langle\mathcal{T}f,\mathcal{T}g\rangle_{t,\omega}$ of $\mathcal{T}f$ and $\mathcal{T}g$ in $L^2(\mathbb{R}^2, \frac{\mathrm{d}t\mathrm{d}\omega}{r(\omega)})$. Applying Lemma (3.3), it follows that

$$\begin{aligned}&\left\langle\mathcal{T}f,\mathcal{T}g\right\rangle_{t,\omega} \\ &= \Big\langle e^{-it\mu(\omega)}\langle\gamma(\omega)|\lambda(\omega)|^{\frac{1}{2}}\hat{f}\mathcal{T}_{\mu(\omega)}\mathcal{D}_{\frac{1}{\lambda(\omega)}}\overline{\hat{\phi}},\mathcal{M}_{-t}1\rangle, \\ &\quad e^{-it\mu(\omega)}\langle\gamma(\omega)|\lambda(\omega)|^{\frac{1}{2}}\hat{g}\mathcal{T}_{\mu(\omega)}\mathcal{D}_{\frac{1}{\lambda(\omega)}}\overline{\hat{\phi}},\mathcal{M}_{-t}1\rangle\Big\rangle_{t,\omega} \\ &= \Big\langle \langle\gamma(\omega)|\lambda(\omega)|^{\frac{1}{2}}\hat{f}T_{\mu(\omega)}\mathcal{D}_{\frac{1}{\lambda(\omega)}}\overline{\hat{\phi}},\mathcal{M}_{-t}1\rangle,\ \langle\gamma(\omega)|\lambda(\omega)|^{\frac{1}{2}}\hat{g}\mathcal{T}_{\mu(\omega)}\mathcal{D}_{\frac{1}{\lambda(\omega)}}\overline{\hat{\phi}},\mathcal{M}_{-t}1\rangle\Big\rangle_{t,\omega} \\ &= 2\pi\left\langle\mathcal{F}\left(\gamma(\omega)|\lambda(\omega)|^{\frac{1}{2}}\hat{f}\mathcal{T}_{\mu(\omega)}\mathcal{D}_{\frac{1}{\lambda(\omega)}}\overline{\hat{\phi}}\right)(-t),\right. \\ &\quad\left.\mathcal{F}\left(\gamma(\omega)|\lambda(\omega)|^{\frac{1}{2}}\hat{g}\mathcal{T}_{\mu(\omega)}\mathcal{D}_{\frac{1}{\lambda(\omega)}}\overline{\hat{\phi}}\right)(-t)\right\rangle_{t,\omega} \\ &= 2\pi\int_{\mathbb{R}}\left\langle\mathcal{F}\left(\gamma(\omega)|\lambda(\omega)|^{\frac{1}{2}}\hat{f}\mathcal{T}_{\mu(\omega)}\mathcal{D}_{\frac{1}{\lambda(\omega)}}\overline{\hat{\phi}}\right)(-t),\right. \\ &\quad\left.\mathcal{F}\left(\gamma(\omega)|\lambda(\omega)|^{\frac{1}{2}}\hat{g}\mathcal{T}_{\mu(\omega)}\mathcal{D}_{\frac{1}{\lambda(\omega)}}\overline{\hat{\phi}}\right)(-t)\right\rangle_t\frac{\mathrm{d}\omega}{r(\omega)},\end{aligned}$$

where $\langle\cdot,\cdot\rangle_t$ in the last step of the above equation is the inner product of the space $L^2(\mathbb{R})$. By using the unitarity of the Fourier transform again, we obtain that

$$\begin{aligned}
&\langle \mathcal{T}f,\mathcal{T}g\rangle_{t,\omega}\\
&= 2\pi\int_{\mathbb{R}}\left\langle \gamma(\omega)|\lambda(\omega)|^{\frac{1}{2}}\hat{f}T_{\mu(\omega)}D_{\frac{1}{\lambda(\omega)}}\overline{\hat{\phi}}(t),\ \gamma(\omega)|\lambda(\omega)|^{\frac{1}{2}}\hat{g}T_{\mu(\omega)}D_{\frac{1}{\lambda(\omega)}}\overline{\hat{\phi}}(t)\right\rangle_t\frac{\mathrm{d}\omega}{r(\omega)}\\
&= 2\pi\int_{\mathbb{R}}\left\langle \gamma(\omega)|\lambda(\omega)|\hat{f}\mathcal{T}_{\mu(\omega)}\overline{\hat{\phi}}(\lambda(\omega)t),\ \gamma(\omega)|\lambda(\omega)|\hat{g}\mathcal{T}_{\mu(\omega)}\overline{\hat{\phi}}(\lambda(\omega)t)\right\rangle_t\frac{\mathrm{d}\omega}{r(\omega)}\\
&= 2\pi\int_{\mathbb{R}}\left\langle \gamma(\omega)|\lambda(\omega)|\hat{f}\overline{\hat{\phi}}(\lambda(\omega)(t-\mu(\omega))),\ \gamma(\omega)|\lambda(\omega)|\hat{g}\overline{\hat{\phi}}(\lambda(\omega)(t-\mu(\omega)))\right\rangle_t\frac{\mathrm{d}\omega}{r(\omega)}.
\end{aligned}$$

Writing the inner product in the integral form we then conclude (18). The proof of this lemma is complete. ■

4. Reproducibility

Equation (18) is the starting point of our discussion. We hope that the integral of the right-hand side of Equation (18) is separable, that is,

$$\begin{aligned}
&2\pi\iint_{\mathbb{R}^2}\gamma^2(\omega)\lambda^2(\omega)\left|\hat{\phi}\left[\lambda(\omega)\left(t-\mu(\omega)\right)\right]\right|^2\hat{f}(t)\overline{\hat{g}(t)}\,\frac{\mathrm{d}t\mathrm{d}\omega}{r(\omega)}\\
&\quad=\int_{\mathbb{R}}|\phi(\omega)|^2\mathrm{d}m_1(\omega)\int_{\mathbb{R}}f(t)\bar{g}(t)\mathrm{d}m_2(t)
\end{aligned}$$

for some measures $\mathrm{d}m_1$ and $\mathrm{d}m_2$, and then the inversion formula holds

$$f(x)=\frac{1}{C_\phi}\iint_{\mathbb{R}^2}\langle f,\phi^{t,\omega}\rangle\tilde{\phi}_{t,\omega}(x)\frac{\mathrm{d}t\mathrm{d}\omega}{r(\omega)},$$

and correspondingly

$$C_\phi^{-1}\iint_{\mathbb{R}^2}\langle\cdot,\phi^{t,\omega}\rangle\tilde{\phi}_{t,\omega}\frac{\mathrm{d}t\mathrm{d}\omega}{r(\omega)}=Id,$$

which means that f can be reconstructed from $\mathcal{T}f$. Here, the constant C_ϕ is dependent on ϕ and $\tilde{\phi}_{t,\omega}(\cdot)$ is some univariate function. Note that $\tilde{\phi}_{t,\omega}$ is a synthesis atom. We cannot ensure that $\tilde{\phi}_{t,m}(\cdot)$ has the same structure as $\phi^{t,m}$. Without doubt, selections of functions r,γ,λ,μ are crucial.

We need to change the form of the right-hand side integral of (18):

$$\begin{aligned}
&\int_{\mathbb{R}^2}\gamma^2(\omega)\lambda^2(\omega)\left|\hat{\phi}\left[\lambda(\omega)\left(t-\mu(\omega)\right)\right]\right|^2\hat{f}(t)\overline{\hat{g}(t)}\,\frac{\mathrm{d}t\mathrm{d}\omega}{r(\omega)}\\
&=\int_{\mathbb{R}}\left\{\int_{\mathbb{R}}\frac{\gamma^2(\omega)\lambda^2(\omega)}{r(\omega)}\left|\hat{\phi}\left[\lambda(\omega)\left(t-\mu(\omega)\right)\right]\right|^2\mathrm{d}\omega\right\}\hat{f}(t)\overline{\hat{g}(t)}\mathrm{d}t\\
&=\int_{\mathbb{R}}\left\{\int_{\mathbb{R}}\frac{\gamma^2(\omega)\lambda^2(\omega)}{r(\omega)}\left|\hat{\phi}\left[\lambda(\omega)\left(t-\mu(\omega)\right)\right]\right|^2\frac{\mathrm{d}[\lambda(\omega)\left(t-\mu(\omega)\right)]}{t\lambda'(\omega)-(\lambda(\omega)\mu(\omega))'}\right\}\hat{f}(t)\overline{\hat{g}(t)}\mathrm{d}t.
\end{aligned}$$

The candidates of new time–frequency transform come from the two cases: $\lambda'(\omega)=C_1(\lambda(\omega)\mu(\omega))'$ with $C_1\in\mathbb{R}\setminus\{0\}$ or $(\lambda(\omega)\mu(\omega))'=0$. We will investigate both these cases separately below. From now on, we assume $\lambda\prime(\omega)>0,\omega\in\mathbb{R}$.

4.1. (i) The case $\lambda'(\omega) = C_1(\lambda(\omega)\mu(\omega))'$

In this case, $\lambda(\omega) = C_1\lambda(\omega)\mu(\omega) + C_2$ for some nonzero real constant C_2. Thus $\mu(\omega) = \frac{1}{C_1} - \frac{C_2}{C_1}\frac{1}{\lambda(\omega)}$ or equivalently $\lambda(\omega) = \frac{C_2}{1-C_1\mu(\omega)}$ and then

$$\lambda(\omega)(t-\mu(\omega)) = \lambda(\omega)\left(t-\frac{1}{C_1}\right) + \frac{C_2}{C_1}.$$

The integral of the right-hand side of (18) becomes

$$\int_{\mathbb{R}}\left\{\int_{\mathbb{R}}\frac{\gamma^2(\omega)\lambda^2(\omega)}{r(\omega)}\left|\hat{\phi}\left[\lambda(\omega)\left(t-\frac{1}{C_1}\right)+\frac{C_2}{C_1}\right]\right|^2\frac{d[\lambda(\omega)\left(t-\frac{1}{C_1}\right)+\frac{C_2}{C_1}]}{\lambda'(\omega)(t-C_1)}\right\}\hat{f}(t)\overline{\hat{g}(t)}\mathrm{d}t.$$

By imposing the condition

$$\frac{\gamma^2(\omega)\lambda^2(\omega)}{r(\omega)\lambda'(\omega)} = \frac{1}{2\pi}$$

and applying change of variable $y = \lambda(\omega)\left(t-\frac{1}{C_1}\right)+\frac{C_2}{C_1}$ and using (12), Equation (18) becomes

$$\int_{\mathbb{R}^2}\langle f,\phi^{t,\omega}\rangle\overline{\langle g,\phi^{t,\omega}\rangle}\frac{\mathrm{d}t\mathrm{d}\omega}{r(\omega)} = \|\phi\|_2^2\int_{\mathbb{R}}\frac{\hat{f}(t)sgn\left(t-\frac{1}{C_1}\right)}{t-C_1}\overline{\hat{g}}(t)\mathrm{d}t.$$

Define the function $\tilde{f}$ by

$$\mathcal{F}\tilde{f}(\cdot) := \frac{\hat{f}(\cdot)sgn\left(\cdot-\frac{1}{C_1}\right)}{\cdot-C_1}.$$

Recalling Lemma 3.1, we know that

$$\tilde{f}(x) = e^{i\frac{x}{C_1}}\mathcal{H}\left(e^{i\left(C_1-\frac{1}{C_1}\right)x}\mathcal{I}\left(e^{-iC_1x}f(x)\right)\right) = \mathcal{M}_{\frac{1}{C_1}}\mathcal{H}\mathcal{M}_{C_1-\frac{1}{C_1}}\mathcal{I}\mathcal{M}_{-C_1}f(x) \tag{19}$$

and obtain the formula

$$\iint_{\mathbb{R}^2}\langle f,\phi^{t,\omega}\rangle\overline{\langle g,\phi^{t,\omega}\rangle}\frac{\mathrm{d}t\mathrm{d}\omega}{r(\omega)} = \|\phi\|_2^2\langle\tilde{f},g\rangle. \tag{20}$$

The above equation (20) may be written as

$$\left\langle\iint_{\mathbb{R}^2}\langle f,\phi^{t,\omega}\right\rangle\phi^{t,\omega}\frac{\mathrm{d}t\mathrm{d}\omega}{r(\omega)},g\rangle = \|\phi\|_2^2\langle\tilde{f},g\rangle$$

that suggests a representation formula in the weak sense for $\tilde{f}$:

$$\tilde{f}(x) = e^{i\frac{x}{C_1}}\mathcal{H}\left(e^{i\left(C_1-\frac{1}{C_1}\right)x}\mathcal{I}\left(e^{-iC_1x}f(x)\right)\right) = \frac{1}{\|\phi|_2^2}\iint_{\mathbb{R}^2}\langle f,\phi^{t,\omega}\rangle\phi^{t,\omega}(x)\frac{\mathrm{d}t\mathrm{d}\omega}{r(\omega)}.$$

The latter implies the desired inversion formula for f:

$$f(x) = \frac{1}{\|\phi\|_2^2} \iint_{\mathbb{R}^2} \langle f, \phi^{t,\omega} \rangle \tilde{\phi}_{t,\omega}(x) \, \frac{\mathrm{d}t\mathrm{d}\omega}{r(\omega)}. \tag{21}$$

Precisely, the above relation is in the weak sense:

$$\iint_{\mathbb{R}^2} \langle f, \phi^{t,\omega} \rangle \langle \tilde{\phi}_{t,\omega}, g \rangle \frac{\mathrm{d}t\mathrm{d}\omega}{r(\omega)} = \|\phi\|_2^2 \langle f, g \rangle \tag{22}$$

for any $f, g \in L^2(\mathbb{R})$. Here, the synthesis atom is of the form

$$\tilde{\phi}_{t,\omega}(x) = -e^{iC_1 x} \frac{\mathrm{d}}{\mathrm{d}x} \left[e^{i(\frac{1}{C_1} - C_1)x} \mathcal{H} \left(e^{-i\frac{x}{C_1}} \phi^{t,\omega}(x) \right) \right], \tag{23}$$

or equivalently, of an alternative form in terms of the basic operators

$$\tilde{\phi}_{t,\omega}(x) = -\mathcal{M}_{C_1} \frac{\mathrm{d}}{\mathrm{d}x} \left[\mathcal{M}_{\frac{1}{C_1} - C_1} \mathcal{H} \left(\mathcal{M}_{-\frac{1}{C_1}} \phi^{t,\omega}(x) \right) \right]. \tag{24}$$

Indeed, there hides a gap between the formulae (20) and (22). The following lemma fills in it.

Lemma 4.1: *Equation (20) is sufficient for Equation (22) to hold for any $f, g \in L^2(\mathbb{R})$.*

Proof: Let h be any function in the Schwartz class $\mathbb{S}(\mathbb{R})$ consisting of all infinitely differentiable and infinitely decaying functions. Set $g = \mathcal{M}_{\frac{1}{C_1}} \mathcal{H} \mathcal{M}_{C_1 - \frac{1}{C_1}} \frac{\mathrm{d}}{\mathrm{d}x} \mathcal{M}_{-C_1} h$ in Equation (20) and obtain that

$$\begin{aligned} &\iint_{\mathbb{R}^2} \langle f, \phi^{t,\omega} \rangle \overline{\left\langle \mathcal{M}_{\frac{1}{C_1}} \mathcal{H} \mathcal{M}_{C_1 - \frac{1}{C_1}} \frac{\mathrm{d}}{\mathrm{d}x} \mathcal{M}_{-C_1} h, \phi^{t,\omega} \right\rangle} \frac{\mathrm{d}t\mathrm{d}\omega}{r(\omega)} \\ &\quad = \|\phi\|_2^2 \left\langle \tilde{f}, \mathcal{M}_{\frac{1}{C_1}} \mathcal{H} \mathcal{M}_{C_1 - \frac{1}{C_1}} \frac{\mathrm{d}}{\mathrm{d}x} \mathcal{M}_{-C_1} h \right\rangle, \end{aligned} \tag{25}$$

where $\tilde{f}$ is defined in (19).

One one hand, noting that the adjoint operators of modulation and the Hilbert transform satisfy $\mathcal{M}_\omega^* = \mathcal{M}_{-\omega}$, $\mathcal{H}^* = -\mathcal{H}$, using decaying property of h and utilizing the formula of integration by parts, we get

$$\begin{aligned} &\left\langle \mathcal{M}_{\frac{1}{C_1}} \mathcal{H} \mathcal{M}_{C_1 - \frac{1}{C_1}} \frac{\mathrm{d}}{\mathrm{d}x} \mathcal{M}_{-C_1} h, \ \phi^{t,\omega} \right\rangle = \left\langle \mathcal{H} \mathcal{M}_{C_1 - \frac{1}{C_1}} \frac{\mathrm{d}}{\mathrm{d}x} \mathcal{M}_{-C_1} h, \ \mathcal{M}_{-\frac{1}{C_1}} \phi^{t,\omega} \right\rangle \\ &= \left\langle \mathcal{M}_{C_1 - \frac{1}{C_1}} \frac{\mathrm{d}}{\mathrm{d}x} \mathcal{M}_{-C_1} h, \ -\mathcal{H} \mathcal{M}_{-\frac{1}{C_1}} \phi^{t,\omega} \right\rangle \end{aligned}$$

$$
\begin{aligned}
&= \left\langle \frac{\mathrm{d}}{\mathrm{d}x}\mathcal{M}_{-C_1}h,\ -\mathcal{M}_{-C_1+\frac{1}{C_1}}\mathcal{H}\mathcal{M}_{-\frac{1}{C_1}}\phi^{t,\omega}\right\rangle \\
&= -\mathcal{M}_{-C_1}h(x)\frac{\mathrm{d}}{\mathrm{d}x}\mathcal{M}_{-C_1+\frac{1}{C_1}}\mathcal{H}\mathcal{M}_{-\frac{1}{C_1}}\phi^{t,\omega}(x)|_{x=-\infty}^{\infty} \\
&\quad + \left\langle \mathcal{M}_{-C_1}h,\ \frac{\mathrm{d}}{\mathrm{d}x}\mathcal{M}_{-C_1+\frac{1}{C_1}}\mathcal{H}\mathcal{M}_{-\frac{1}{C_1}}\phi^{t,\omega}\right\rangle \\
&= \left\langle \mathcal{M}_{-C_1}h,\ \frac{\mathrm{d}}{\mathrm{d}x}\mathcal{M}_{-C_1+\frac{1}{C_1}}\mathcal{H}\mathcal{M}_{-\frac{1}{C_1}}\phi^{t,\omega}\right\rangle \\
&= \left\langle h,\ \mathcal{M}_{C_1}\frac{\mathrm{d}}{\mathrm{d}x}\mathcal{M}_{-C_1+\frac{1}{C_1}}\mathcal{H}\mathcal{M}_{-\frac{1}{C_1}}\phi^{t,\omega}\right\rangle = -\left\langle h, \tilde{\phi}_{t,\omega}\right\rangle.
\end{aligned}
$$

On the other hand, a similar argument leads to

$$
\begin{aligned}
&\left\langle \tilde{f}, \mathcal{M}_{\frac{1}{C_1}}\mathcal{H}\mathcal{M}_{C_1-\frac{1}{C_1}}\frac{\mathrm{d}}{\mathrm{d}x}\mathcal{M}_{-C_1}h\right\rangle \\
&\quad = \left\langle \mathcal{M}_{\frac{1}{C_1}}\mathcal{H}\mathcal{M}_{C_1-\frac{1}{C_1}}\mathcal{I}\mathcal{M}_{-C_1}f, \mathcal{M}_{\frac{1}{C_1}}\mathcal{H}\mathcal{M}_{C_1-\frac{1}{C_1}}\frac{\mathrm{d}}{\mathrm{d}x}\mathcal{M}_{-C_1}h\right\rangle \\
&\quad = \left\langle \mathcal{H}\mathcal{M}_{C_1-\frac{1}{C_1}}\mathcal{I}\mathcal{M}_{-C_1}f, \mathcal{H}\mathcal{M}_{C_1-\frac{1}{C_1}}\frac{\mathrm{d}}{\mathrm{d}x}\mathcal{M}_{-C_1}h\right\rangle \\
&\quad = -\left\langle \mathcal{M}_{C_1-\frac{1}{C_1}}\mathcal{I}\mathcal{M}_{-C_1}f, \mathcal{M}_{C_1-\frac{1}{C_1}}\frac{\mathrm{d}}{\mathrm{d}x}\mathcal{M}_{-C_1}h\right\rangle \\
&\quad = -\left\langle \mathcal{I}\mathcal{M}_{-C_1}f, \frac{\mathrm{d}}{\mathrm{d}x}\mathcal{M}_{-C_1}h\right\rangle \\
&\quad = -\mathcal{I}\mathcal{M}_{-C_1}f(x)\mathcal{M}_{-C_1}h(x)|_{x=-\infty}^{\infty} - \left\langle \mathcal{M}_{-C_1}f, \mathcal{M}_{-C_1}h\right\rangle \\
&\quad = -\left\langle \mathcal{M}_{-C_1}f, \mathcal{M}_{-C_1}h\right\rangle = -\left\langle f, h\right\rangle.
\end{aligned}
$$

Then by (25), we obtain that

$$
\iint_{\mathbb{R}^2} \left\langle f, \phi^{t,\omega}\right\rangle \overline{\left\langle h, \tilde{\phi}_{t,\omega}\right\rangle}\,\frac{\mathrm{d}t\mathrm{d}\omega}{r(\omega)} = \|\phi\|_2^2 \left\langle f, h\right\rangle
$$

holds for any $h \in \mathbb{S}(\mathbb{R})$. Finally, by a density argument, we conclude (22). ■

The above discussion leads to the following theorem.

Theorem 4.2: *Suppose that* λ, r *and* γ *are real-variable and real-valued functions. Assume that* $\lambda'(\omega) > 0$, r *and* γ *satisfy that* $\frac{\gamma^2(\omega)\lambda^2(\omega)}{r(\omega)\lambda'(\omega)} = \frac{1}{2\pi}$. *Define the decomposition atom by*

$$
\phi^{t,\omega}(x) = \gamma(\omega)\phi\left(\frac{x-t}{\lambda(\omega)}\right)e^{i\left(\frac{1}{C_1}-\frac{C_2}{C_1}\frac{1}{\lambda(\omega)}\right)x} \tag{26}
$$

for any function $\phi \in L^2(\mathbb{R})$ *with* $\|\phi\|_2 = 1$ *and any fixed nonzero real numbers* C_1 *and* C_2, *and the synthesis atom by*

$$
\tilde{\phi}_{t,\omega}(x) = -e^{iC_1x}\frac{\mathrm{d}}{\mathrm{d}x}\left[e^{i(\frac{1}{C_1}-C_1)x}\mathcal{H}\left(e^{-i\frac{x}{C_1}}\phi^{t,\omega}(x)\right)\right]. \tag{27}
$$

Then the inversion formula of (9) is

$$f(x) = \int_{\mathbb{R}^2} \langle f, \phi^{t,\omega} \rangle \tilde{\phi}_{t,\omega}(x) \, \frac{\mathrm{d}t\mathrm{d}\omega}{r(\omega)},$$

where the identical relation holds in the weak sense for any function $f \in L^2(\mathbb{R})$.

Now we investigate the image space $\mathcal{T}(L^2(\mathbb{R}))$ of the time–frequency transform $\mathcal{T}$ defined in (9) with the special atom (26). We will show that $\mathcal{T}(L^2(\mathbb{R}))$ is a RKHS of $L^2(\mathbb{R}^2, \frac{\mathrm{d}t\mathrm{d}\omega}{r(\omega)})$. For any $F \in \mathcal{T}(L^2(\mathbb{R}))$, there is a function $f \in L^2(\mathbb{R})$ such that

$$F(t', \omega') = \mathcal{T}f(t', \omega') = \left\langle f, \phi^{t',\omega'} \right\rangle, \quad (t', \omega') \in \mathbb{R}^2.$$

Recalling the formula (22) and setting $g = \phi^{t',\omega'}$ there, it follows

$$\iint_{\mathbb{R}^2} \langle f, \phi^{t,\omega} \rangle \overline{\langle \phi^{t',\omega'}, \tilde{\phi}_{t,\omega} \rangle} \, \frac{\mathrm{d}t\mathrm{d}\omega}{r(\omega)} = \|\phi\|_2^2 \langle f, \phi^{t',\omega'} \rangle.$$

Then there holds

$$\begin{aligned} F(t', \omega') &= \frac{1}{\|\phi\|_2^2} \iint_{\mathbb{R}^2} \langle f, \phi^{t,\omega} \rangle \overline{\langle \phi^{t',\omega'}, \tilde{\phi}_{t,\omega} \rangle} \, \frac{\mathrm{d}t\mathrm{d}\omega}{r(\omega)} \\ &= \frac{1}{\|\phi\|_2^2} \iint_{\mathbb{R}^2} F(t,\omega) \overline{\langle \phi^{t',\omega'}, \tilde{\phi}_{t,\omega} \rangle} \, \frac{\mathrm{d}t\mathrm{d}\omega}{r(\omega)} \\ &= \frac{1}{\|\phi\|_2^2} \iint_{\mathbb{R}^2} F(t,\omega) K(t,\omega; t', \omega') \, \frac{\mathrm{d}t\mathrm{d}\omega}{r(\omega)} \end{aligned}$$

with the kernel

$$K(t,\omega; t', \omega') = \overline{\langle \phi^{t',\omega'}, \tilde{\phi}_{t,\omega} \rangle} = \langle \tilde{\phi}_{t,\omega}, \phi^{t',\omega'} \rangle.$$

4.2. *(ii) The case* $(\lambda(\omega)\mu(\omega))' = 0$

In this case, $\mu(\omega) = \frac{C}{\lambda(\omega)}$ for any nonzero real number C. By noting that

$$\lambda(\omega)(t - \mu(\omega)) = \lambda(\omega)t - C,$$

the integral of the right-hand side of (18) becomes

$$\int_{\mathbb{R}} \left\{ \int_{\mathbb{R}} \frac{\gamma^2(\omega)\lambda^2(\omega)}{r(\omega)} \left| \hat{\phi}\left(\lambda(\omega)t - C\right) \right|^2 \frac{\mathrm{d}\left(\lambda(\omega)t - C\right)}{\lambda'(\omega)t} \right\} \hat{f}(t)\overline{\hat{g}(t)}\mathrm{d}t.$$

By imposing the condition

$$\frac{\gamma^2(\omega)\lambda^2(\omega)}{r(\omega)\lambda'(\omega)} = \frac{1}{2\pi}$$

and applying change of variable $y = \lambda(\omega)t - C$ and using (12), Equation (18) becomes

$$\iint_{\mathbb{R}^2} \langle f, \phi^{t,\omega} \rangle \overline{\langle g, \phi^{t,\omega} \rangle} \, \frac{\mathrm{d}t\mathrm{d}\omega}{r(\omega)} = \|\phi\|_2^2 \int_{\mathbb{R}} \frac{\hat{f}(t)\mathrm{sgn}(t)}{t} \overline{\hat{g}}(t)\mathrm{d}t. \tag{28}$$

Set

$$\tilde{f}(x) = \mathcal{H}\left(\mathcal{I}\left(f(x)\right)\right).$$

We get the equivalent form of (28)

$$\iint_{\mathbb{R}^2} \langle f, \phi^{t,\omega}\rangle \overline{\langle g, \phi^{t,\omega}\rangle} \frac{\mathrm{d}t\mathrm{d}\omega}{r(\omega)} = \|\phi\|_2^2 \langle \tilde{f}, g\rangle. \tag{29}$$

Equation (29) formally leads to

$$\mathcal{H}\left(\mathcal{I}\left(f(x)\right)\right) = \frac{1}{\|\phi|_2^2} \iint_{\mathbb{R}^2} \langle f, \phi^{t,\omega}\rangle \phi^{t,\omega}(x) \frac{\mathrm{d}t\mathrm{d}\omega}{r(\omega)}$$

and then the inversion formula

$$f(x) = \frac{1}{\|\phi\|_2^2} \iint_{\mathbb{R}^2} \langle f, \phi^{t,\omega}\rangle \tilde{\phi}_{t,\omega}(x) \frac{\mathrm{d}t\mathrm{d}\omega}{r(\omega)} \tag{30}$$

with

$$\tilde{\phi}_{t,\omega}(x) = -\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathcal{H}\phi^{t,\omega}(x)\right).$$

The inversion formula (30) is in the weak sense

$$\iint_{\mathbb{R}^2} \langle f, \phi^{t,\omega}\rangle \overline{\langle g, \tilde{\phi}_{t,\omega}\rangle} \frac{\mathrm{d}t\mathrm{d}\omega}{r(\omega)} = \|\phi\|_2^2 \langle f, g\rangle. \tag{31}$$

The above discussion leads to the following theorem.

Theorem 4.3: *Suppose that λ is a real variable function with $\lambda'(\omega) > 0$. Assume that λ,r and γ satisfy the condition $\frac{\gamma^2(\omega)\lambda^2(\omega)}{r(\omega)\lambda'(\omega)} = \frac{1}{2\pi}$. Define the decomposition atom by*

$$\phi^{t,\omega}(x) = \gamma(\omega)\phi\left(\frac{x-t}{\lambda(\omega)}\right) e^{i\frac{C}{\lambda(\omega)}x} \tag{32}$$

for any real function $\phi \in L^2(\mathbb{R})$ with $\|\phi\|_2 = 1$ and any fixed nonzero real number C, and the synthesis atom by

$$\tilde{\phi}_{t,\omega}(x) = -\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathcal{H}\phi^{t,\omega}(x)\right). \tag{33}$$

Then the inversion formula of (9) is

$$f(x) = \iint_{\mathbb{R}^2} \langle f, \phi^{t,\omega}\rangle \tilde{\phi}_{t,\omega}(x) \frac{\mathrm{d}t\mathrm{d}\omega}{r(\omega)},$$

where the convergence of the integral is in the weak sense for any function $f \in L^2(\mathbb{R})$.

Denote by $\mathcal{T}(L^2(\mathbb{R}))$ the image space of the time–frequency transform $\mathcal{T}$ defined in (9) with the special atom (32). For $\phi^{t,\omega}$ defined in (32), setting $g = \phi^{t',\omega'}$ in the formula (31), a similar argument concludes that, for any $F \in \mathcal{T}(L^2(\mathbb{R}))$, there exists a function $f \in L^2(\mathbb{R})$ such that

$$F(t', \omega') = \frac{1}{\|\phi\|_2^2} \iint_{\mathbb{R}^2} F(t, \omega) K(t, \omega; t', \omega') \, \frac{\mathrm{d}t\mathrm{d}\omega}{r(\omega)}$$

with the kernel

$$K(t, \omega; t', \omega') = \langle \tilde{\phi}_{t,\omega}, \phi^{t',\omega'} \rangle$$

and $\tilde{\phi}_{t,\omega}$ defined in (33). It indicates that $\mathcal{T}(L^2(\mathbb{R}))$ in the case (ii) is also a RKHS of $L^2(\mathbb{R}^2, \frac{\mathrm{d}t\mathrm{d}\omega}{r(\omega)})$.

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