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Reproducing Kernels of Some Weighted Bergman Spaces

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Abstract
Herein, the theory of Bergman kernel is developed to the weighted case. A general form of weighted Bergman reproducing kernel is obtained, by which we can calculate concrete Bergman kernel functions for specific weights and domains.

Keywords Reproducing kernel · Reproducing kernel Hilbert space · Weighted Bergman spaces

1 Introduction

The theory of Bergman spaces has, in the past several decades, become important in complex analysis of both one and several complex variables, see [7,8]. Recall that, for an arbitrary domain $\Omega \subset \mathbb{C}^n$, the Bergman space $A^p(\Omega)$ is defined as the collection of analytic functions F that satisfy

$$\|F\|_{A^p} = \left\{ \int_\Omega |F(z)|^p dA(z) \right\}^{\frac{1}{p}} < \infty,$$

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where $dA(z) = dxdy$ is the Lebesgue measure on $\mathbb{C}^n$. When $p = 2$, the reproducing kernel plays an important role in the Hilbert space. For the classical Bergman spaces, the reproducing properties and biholomorphic invariance are investigated in [3,5].

Bergman kernels have also been considered in some weighted cases. Since a reproducing kernel delivers certain fundamental information of the corresponding space, it is important to obtain the concrete form of the kernel function. However, we must confess that their weighted Bergman kernel can almost never be calculated explicitly except for some special cases. Among the latter, the Bergman kernels are given for functions defined on the Hermitian ball and polydisc in [7]. Some concrete expressions of the Bergman kernel are also available for some classical homogeneous bounded symmetric domains of Cartan in [6].

Since the set of polynomials is dense in the Bergman spaces of bounded domains, the corresponding reproducing kernels can be obtained from the general representation formula $K(z, w) = \sum_{n=1}^{\infty} \phi_n(z)\overline{\phi_n(w)}$, where $\{\phi_n\}$ is any complete orthonormal basis obtained through orthonormalization of polynomials. However, for unbounded regions, density of polynomials can not be guaranteed. In such case, appropriate weight functions are introduced to make the weighted polynomials dense in certain Bergman spaces of unbounded domains. Then the kernel representations can be similarly obtained. In this paper, we apply the Laplace transform to the case of Bergman spaces on tube domains, which will be an effective and new method to calculate reproducing kernels.

Herein, we develop the theory of the weighted Bergman spaces and obtain a general representation formula of the kernel function for the spaces on tubular domains. As a complementary part to the general study, we calculate the concrete forms of the Bergman kernels for some special weights on the tube domains.

In some previous studies, the concerned reproducing kernels can be computed by using our general representation formula, since the given weight functions satisfy the set conditions in our theorems. For example, taking the weight function $\rho(iy) = \frac{2y^{2q-2}}{\pi\Gamma(2q-1)}$, we can derive that the Bergman–Selberg reproducing kernel on the upper half plane is in the form of $K_q(z, \omega) = \Gamma(2q)\left(\frac{i}{z-\bar{\omega}}\right)^{2q}$ with $q > \frac{1}{2}$, which is introduced in [9]. For the weight function $\rho(iy) = y^{v-1}$, a direct computation gives that the corresponding kernel is of the form $K(z, w) = \frac{2^{v-1}v}{\pi}\left(\frac{z-\bar{w}}{i}\right)^{-v-1}$, which is studied in [2]. For weighted Bergman spaces associated with Lorentz cones, referring to [1], the kernel functions can be also obtained by our formula. The related calculation process is given as an example in the final section. Especially, in the un-weighted case, i.e., letting $\rho(iy) = 1$, it follows that the classical Bergman kernel on the upper half plane is $K(z, w) = \frac{1}{\pi}\left(\frac{i}{z-\bar{w}}\right)^2$.

2 Preliminaries

Let Ω be an arbitrary domain (an open connected set) in the complex n-dimensional Euclidean space $\mathbb{C}^n = \{z = x + iy : x, y \in \mathbb{R}^n\}$. Suppose that $\rho(z)$ is a positive continuous function on Ω that takes the value 0 for $z \notin \Omega$. We consider the weighted volume measure

$$dA_\rho(z) = \rho(z)dA(z),$$

where $dA(z) = dxdy$ is the Lebesgue measure on $\mathbb{C}^n$. For $p > 0$, we denote by L^p_ρ the space of measurable functions on Ω such that

$$\|F\|_{L^p_\rho} = \left(\int_\Omega |F(z)|^p dA_\rho(z)\right)^{\frac{1}{p}} < \infty. \tag{1}$$

The space of such functions is called the weighted Lebesgue space with weight ρ. The quantity $\|F\|_{L^p_\rho}$ is called the norm of the function F, which is a true norm if $p \geq 1$.

We denote by A^p_ρ the collection of functions F that are holomorphic on Ω and satisfy the condition (1). Such a class is called the weighted Bergman space with weight ρ. It is obvious that $A^p_\rho \subset L^p_\rho$.

We first assert that functions in the weighted Bergman space cannot grow too rapidly near the boundary.

Lemma 1 *Point evaluation is a bounded linear functional in each weighted Bergman space A^p_ρ. More specifically, each function $F \in A^p_\rho$ has the property*

$$|F(z)| \leq (\omega_{2n}\varepsilon_z)^{-\frac{1}{p}} (\delta_z)^{-\frac{2n}{p}} \|F\|_{A^p_\rho}. \tag{2}$$

Here, $\omega_{2n} = \frac{\pi^n}{n!}$ is the volume of unit ball $B_{2n}(0, 1)$, $\delta_z = \min\{1, 2^{-1}dist(z, \partial\Omega)\}$ where $dist(z, \partial\Omega)\}$ is the distance from z to the boundary of Ω, and $\varepsilon_z = \min\{\rho(\zeta) : |\zeta - z| \leq \delta_z\}$.

Proof For fixed point $z \in \Omega$, the bounded closed ball $\overline{B_{2n}(z, \delta_z)}$ lies in Ω. Since $\rho(\zeta)$ is a positive continuous function on Ω, then for any $\zeta \in \overline{B_{2n}(z, \delta_z)}$, we have $\varepsilon_z = \min \rho(\zeta) > 0$. Therefore,

$$\begin{aligned} |F(z)|^p &\leq \frac{1}{\omega_{2n}\delta_z^{2n}} \int_{B_{2n}(z,\delta_z)} |F(\zeta)|^p dA(\zeta) \\ &\leq \frac{1}{\omega_{2n}\delta_z^{2n}} \int_{B_{2n}(z,\delta_z)} \frac{|F(\zeta)|^p}{\rho(\zeta)} dA_\rho(\zeta) \\ &\leq \frac{1}{\varepsilon_z\omega_{2n}\delta_z^{2n}} \int_{B_{2n}(z,\delta_z)} |F(\zeta)|^p dA_\rho(\zeta) \\ &\leq \frac{1}{\varepsilon_z\omega_{2n}\delta_z^{2n}} \int_\Omega |F(\zeta)|^p dA_\rho(\zeta) \\ &= \frac{1}{\varepsilon_z\omega_{2n}\delta_z^{2n}} \|F\|^p_{A^p_\rho}, \end{aligned}$$

which is the stated result. □

As a consequence of the above lemma, we conclude that the weighted Bergman space A^p_ρ is a Banach space when $1 \le p < \infty$ and a complete metric space when $0 < p < 1$.

Lemma 2 *Suppose that $\rho(z)$ is a positive continuous function on Ω that takes the value 0 for $z \notin \Omega$. For $0 < p < \infty$, the weighted Bergman space A^p_ρ is closed in L^p_ρ.*

Proof Let $\{F_n\}$ be a sequence in A^p_ρ and assume $F_n \to F$ in L^p_ρ. In particular, $\{F_n\}$ is a Cauchy sequence in L^p_ρ. Applying the previous lemma, we see that $\{F_n\}$ converges uniformly on every compact subset of Ω. Combining with the assumption that $F_n \to F$ in L^p_ρ, we conclude that $F_n \to F$ uniformly on every compact subset of Ω. Therefore, F is analytic in Ω and belongs to A^p_ρ. □

Now let $p = 2$, A^2_ρ is a Hilbert space with inner product

$$\langle F, G\rangle_\rho = \int_\Omega F(z)\overline{G(z)}dA_\rho(z)$$

for $F, G \in A^2_\rho$.

Since each point evaluation functional $\mathcal{T}[F] = F(z)$ of A^2_ρ is bounded, the Riesz representation theorem for Hilbert space guarantees existence of a unique function $K(\zeta, z) = K_z(\zeta) \in A^2_\rho$ such that $F(z) = \langle F, K_z\rangle_\rho$ for every $F \in A^2_\rho$. The function $K(\zeta, z)$ is known as the reproducing kernel with weight ρ, or the weighted Bergman kernel function. It has the reproducing property

$$F(z) = \int_\Omega F(\zeta)\overline{K_z(\zeta)}dA_\rho(\zeta) \tag{3}$$

for each function $F \in A^2_\rho$. Taking $F(z) = K(z, \zeta)$ for some $\zeta \in \Omega$, we see that

$$K(z, \zeta) = \int_\Omega K(\eta, \zeta)\overline{K(\eta, z)}dA_\rho(\zeta) = \overline{K(\zeta, z)}. \tag{4}$$

Thus the kernel function has the symmetry property $K(z, \zeta) = \overline{K(\zeta, z)}$, which also shows that $K(z, \zeta)$ is analytic in z and anti-analytic in ζ. Another consequence is the formula

$$K(z, z) = \int_\Omega |K(z, \zeta)|^2 dA_\rho(\zeta) = \|K(z, \cdot)\|^2_{L^2_\rho} > 0. \tag{5}$$

In view of (5), applying the Schwarz inequality to (3), there holds

$$|F(z)| \le \sqrt{K(z, z)}\|F\|_{A^2_\rho}.$$

Then for each point $\zeta \in \Omega$,

$$\frac{1}{\sqrt{K(\zeta, \zeta)}} \le \|F\|_{A^2_\rho}$$

for all $F \in A^2(\Omega, \rho)$ with $F(\zeta) = 1$. In fact, the lower bound is sharp and uniquely attained by the function $F(z) = \frac{K(z,\zeta)}{K(\zeta,\zeta)}$.

The theory of reproducing kernel Hilbert spaces guarantees that the reproducing kernel $K(\cdot, \cdot)$ is unique.

Recall that a holomorphic mapping $w = \Phi(z)$ from a domain Ω_1 to a domain Ω_2 is said to be biholomorphic if it is one-to-one, onto, and its holomorphic inverse exists.

In fact, the kernel function with weight is biholomorphic invariant in the sense of the following lemma.

Lemma 3 *Suppose that $w = \Phi(z)$ is a biholomorphic mapping of a domain Ω_1 onto a domain Ω_2, $\rho_1(z)$ and $\rho_2(w)$ are two positive continuous functions on domains Ω_1 and Ω_2, respectively, $\rho_1(z) = 0$ for $z \notin \Omega_1$ and $\rho_2(w) = 0$ for $w \notin \Omega_2$, $K_{\rho_1}(z, \zeta)$ and $K_{\rho_2}(w, \varsigma)$ are reproducing kernels of two weighted Bergman spaces $A^2_{\rho_1}$ and $A^2_{\rho_2}$, respectively. If $\rho_1(z) = \rho_2(\Phi(z))$ for all $z \in \Omega_1$, then*

$$K_{\rho_1}(z, \zeta) = (D\Phi)(z) K_{\rho_2}(\Phi(z), \Phi(\zeta)) \overline{(D\Phi)(\zeta)}, \tag{6}$$

where $(D\Phi)(z)$ is the determinant of the holomorphic Jacobian matrix of $w = \Phi(z)$.

Proof Let $F \in A^2_{\rho_1}$, after a change of variables $\varsigma = \Phi(\zeta)$ in the integral,

$$\begin{aligned}
&\int_{\Omega_1} (D\Phi)(z) K_{\rho_2}(\Phi(z), \Phi(\zeta)) \overline{(D\Phi)(\zeta)} F(\zeta) \rho_1(\zeta) dA(\zeta) \\
&= \int_{\Omega_2} (D\Phi)(z) K_{\rho_2}(\Phi(z), \varsigma) \overline{(D\Phi)(\Phi^{-1}(\varsigma))} F(\Phi^{-1}(\varsigma)) \rho_2(\varsigma) (D_R \Phi^{-1})(\varsigma) dA(\varsigma),
\end{aligned}$$

where $D_R \Phi^{-1}$ is the determinant of the real Jacobian matrix of Φ^{-1}. Based on the relationship between the determinant of the real Jacobian matrix and that of the holomorphic Jacobian matrix, i.e., $D_R \Phi^{-1} = |D\Phi^{-1}|^2$ (see [7] Proposition 1.4.10), the above formula simplifies to

$$\int_{\Omega_2} (D\Phi)(z) K_{\rho_2}(\Phi(z), \varsigma) \left\{ \left((D\Phi)(\Phi^{-1}(\varsigma)) \right)^{-1} F(\Phi^{-1}(\varsigma)) \right\} \rho_2(\varsigma) dA(\varsigma).$$

On the other hand, by hypothesis, the expression in braces is an element of $A^2_{\rho_2}$. So the last line equals

$$(D\Phi)(z) \left((D\Phi)(\Phi^{-1}(\Phi(z))) \right)^{-1} F(\Phi^{-1}(\Phi(z))) = F(z).$$

By the uniqueness of the reproducing kernel of the weighted Bergman space $A^2_{\rho_1}$, we see that (6) holds. This completes the proof of Lemma 3. □

3 Main Results

In order to obtain the explicit reproducing kernel of the weighted Bergman kernel corresponding to a specific weights ρ in a concrete domain, we suppose that $\rho(z)$ is a positive continuous function on a tube domain

$$\Omega = T_B = \{z = x + iy : y \in B\}$$

over an open connected subset B of the real n-dimensional Euclidean space $\mathbb{R}^n$. In addition, we assume that $\rho(x + iy) = \rho(iy)$ for all $x \in \mathbb{R}^n$, $y \in B$ and $\rho(z) = 0$ for $z \notin T_B$. In this case, the computation of weighted Bergman kernels on those tube domains greatly benefits from the homogeneity of T_B in the real direction. An important tool is the Laplace transform $F = \mathscr{L}(f)$ of a function f, that is,

$$F = (\mathscr{L} f)(z) = \int_{\mathbb{R}^n} f(t)e^{2\pi i z\cdot t} dt, \tag{7}$$

where $z\cdot t = x\cdot t + iy\cdot t = \sum_{k=1}^{n} z_k \cdot t_k$, and $x\cdot t$, $y\cdot t$ are the Euclidean scalar products for $x, y, t \in \mathbb{R}^n$. The definition will be further justified together with the specific spaces that the test function f belongs to. It is obvious that F is well defined only when f decays sufficiently fast at ∞.

Let

$$I(t) = \int_B \rho(iy)e^{-4\pi y\cdot t} dy, \tag{8}$$

then the set $U_I = \{t : I(t) < \infty\}$ is a convex set, and $\log I(t)$ is a convex function on U_I. The weighted L_I^p space is the set of the measurable function defined on $\mathbb{R}^n$ such that

$$\|f\|_{L_I^p} = \left(\int_{\mathbb{R}^n} |f(t)|^p I(t) dt\right)^{\frac{1}{p}} < \infty.$$

Notice that if $f \in L_I^p$, then $f(t) = 0$ almost everywhere for all $t \notin U_I$, so we can assume that the support of f is contained in the closure of U_I. We also see that A_ρ^2 and L_I^2 are Hilbert spaces with the inner product

$$\langle F, G\rangle_\rho = \int_B \int_{\mathbb{R}^n} F(x + iy)\overline{G(x + iy)}\rho(iy) dx dy$$

for $F, G \in A_\rho^2$ and

$$\langle f, g\rangle_I = \int_{\mathbb{R}^n} f(t)\overline{g(t)} I(t) dt$$

for $f, g \in L_I^2$, respectively.

The main result herein is established as follows.

Theorem 1 *The weighted Bergman kernel $K(z, w)$ of A_ρ^2 is given by*

$$K(z, w) = \int_{\mathbb{R}^n} e^{2\pi i t(z-\overline{w})} I^{-1}(t) dt, \tag{9}$$

where $I(t)$ is defined as (8) and $\rho(z)$ is a positive continuous function on the tube domain T_B and satisfying $\rho(x + iy) = \rho(iy)$ for all $x \in \mathbb{R}^n$, $y \in B$.

To prove Theorem 1, we need the following lemma, which is also an important result by itself.

Lemma 4 *The Laplace transform $\mathscr{L}$ is an isometry from L_I^2 to A_ρ^2 preserving the Hilbert space norms i.e.,*

$$\|\mathscr{L} f\|_{A_\rho^2} = \|f\|_{L_I^2}.$$

Proof First, we prove that if $F(z) \in A_\rho^2$. There exists $f \in L_I^2$ such that $F(z) = (\mathscr{L} f)(z)$, which means that the Laplace transform $\mathscr{L}$ is surjective.

Let $B_0 \subseteq B$ be a bounded connected open set, so there exists a positive constant R_0 such that $B_0 \subseteq D(0, R_0)$. Assume that $l_\varepsilon(z) = (1 + \varepsilon(z_1^2 + \cdots + z_n^2))^N$, where N is an integer and $N > \frac{n}{4}$. Then for $\varepsilon \leq \frac{1}{2R_0^2}$ and $z = x + iy$ with $|y| \leq R_0$,

$$\begin{aligned} |l_\varepsilon(z)| &= |((1 + \varepsilon(z_1^2 + \cdots + z_n^2))^2)^{\frac{N}{2}}| \\ &= \left(\left(1 + \varepsilon(|x|^2 - |y|^2)\right)^2 + 4\varepsilon^2 (x \cdot y)^2\right)^{\frac{N}{2}} \\ &\geq \left(1 + \varepsilon(|x|^2 - |y|^2)\right)^N \geq \left(\frac{1}{2} + \varepsilon|x|^2\right)^N, \end{aligned}$$

i.e., $\left|l_\varepsilon^{-1}(z)\right| \leq \frac{1}{\left(\frac{1}{2}+\varepsilon|x|^2\right)^N}$. Set $F_\varepsilon(z) = F(z) l_\varepsilon^{-1}(z)$, then based on Hölder's inequality,

$$\int_{\mathbb{R}^n} \left|F_{\varepsilon,y}(x)\right| dx \leq \left(\int_{\mathbb{R}^n} \left|F_y(x)\right|^2 dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} \left|l_\varepsilon^{-1}(x + iy)\right|^2 dx\right)^{\frac{1}{2}} < \infty,$$

which implies that $F_{\varepsilon,y}(x) = F_\varepsilon(x + iy) \in L^1(\mathbb{R}^n)$ and

$$\log \int_{\mathbb{R}^n} |F_\varepsilon(x + iy)| \, dx$$

is a convex function on B_0. Therefore, for any compact $K \subseteq B_0$, we have

$$\sup \left\{ \int_{\mathbb{R}^n} |F_\varepsilon(x + iy)| \, dx : y \in K \right\} < \infty. \tag{10}$$

For any $a, b, y \in B_0$, $t \in \mathbb{R}^n$, let

$$\begin{aligned} G_\varepsilon(z) &= F_\varepsilon(z)e^{2\pi i z\cdot t}; \\ J(x', x_n, \tau) &= F_\varepsilon(x + i(a + \tau(b - a))), \quad x = (x', x_n), 0 \le \tau \le 1; \\ N_\varepsilon(y, t) &= \check{F}_{\varepsilon,y}(t)e^{-2\pi y\cdot t}. \end{aligned}$$

In order to show that, for all $a, b \in B_0$, $N_\varepsilon(a, t) = N_\varepsilon(b, t)$ almost everywhere for all $t \in \mathbb{R}^n$, we first assume that $a = (a', a_n)$, $b = (a', b_n)$ and the closed interval $[a, b] = \{a + t(b - a) : 0 \le t \le 1\}$ is contained in B_0. Then (10) implies that the integral

$$\int_0^\infty \int_0^1 \int_{\mathbb{R}^{n-1}} (|J(x', x_n, \tau)| + |J(x', -x_n, \tau)|)dx'd\tau dx_n$$

is finite. This means

$$\varliminf_{R\to\infty} \int_0^1 \int_{\mathbb{R}^{n-1}} (|J(x', R, \tau)| + |J(x', -R, \tau)|)dx'd\tau = 0.$$

Therefore,

$$\begin{aligned} |N_\varepsilon(a, t) - N_\varepsilon(b, t)| &= \left| \int_{\mathbb{R}^n} (G_\varepsilon(x + ib) - G_\varepsilon(x + ia))dx \right| \\ &= \left| \int_{\mathbb{R}^n} \int_0^1 \frac{\partial}{\partial\tau} G_\varepsilon(x + i(a + \tau(b - a))d\tau dx \right| \\ &= \left| \int_{\mathbb{R}^n} \int_0^1 \frac{\partial}{\partial y_n} \left(G_\varepsilon(x + i(a', y_n))\right) \Big|_{y_n = a_n + \tau(b_n - a_n)} (b_n - a_n)d\tau dx \right| \\ &= |b_n - a_n| \left| \int_{\mathbb{R}^n} \int_0^1 i \frac{\partial}{\partial x_n} \left(G_\varepsilon(x + i(a + \tau(b - a)))\right) d\tau dx \right| \\ &\le C_1(t) \varliminf_{R\to\infty} \int_0^1 \int_{\mathbb{R}^{n-1}} (|J(x', R, \tau)| + |J(x', -R, \tau)|)dx'd\tau = 0, \end{aligned}$$

where $C_1(t) = |b_n - a_n|e^{-2\pi|t|(|a|+|b-a|)}$. Remark that B_0 is connected and by an iteration argument, we can show that $g_{\varepsilon,y}(t) = \check{F}_{\varepsilon,y}(t)e^{-2\pi y\cdot t}$ is a function independent of $y \in B_0$. Hence $g_\varepsilon(t) = g_{\varepsilon,y}(t)$ is independent of $y \in B_0$ and $g_\varepsilon(t)e^{2\pi y\cdot t} = \check{F}_{\varepsilon,y}(t) \in L^1(\mathbb{R}^n)$ for all $y \in B_0$.

On the other hand, it is obvious that $F_{\varepsilon,y}(x) \to F_y(x)$ as $\varepsilon \to 0$. Let $N_y(t) = \check{F}_y(t)e^{-2\pi y\cdot t}$, we can also prove that $\check{F}_y(t)e^{-2\pi y\cdot t}$ is independent of $y \in B_0$ and $g(t) = \check{F}_y(t)e^{-2\pi y\cdot t}$ almost everywhere. Indeed, for $a, b \in B_0$ and any compact subset $K \subset \mathbb{R}^n$, let $R_1 = \max\{|t| : t \in K\}$. Then Plancherel's theorem implies that

$$\begin{aligned} \|N_b - N_a\|_{L^2(K)} &\le \|N_b - g_\varepsilon\|_{L^2(K)} + \|g_\varepsilon - N_a\|_{L^2(K)} \\ &\le e^{2\pi R_0 R_1} \left(\|\check{F}_a - \check{F}_{\varepsilon,a}\|_{L^2(K)} + \|\check{F}_{\varepsilon,b} - \check{F}_b\|_{L^2(K)} \right) \end{aligned}$$

$$= e^{2\pi R_0 R_1}\left(\|F_a - F_{\varepsilon,a}\|_{L^2(\mathbb{R}^n)} + \|F_b - F_{\varepsilon,b}\|_{L^2(\mathbb{R}^n)}\right) \to 0$$

as $\varepsilon \to 0$. Hence $\check{F}_a(t)e^{-2\pi a\cdot t} = \check{F}_b(t)e^{-2\pi b\cdot t}$ almost everywhere on $\mathbb{R}^n$.

Next, we show that $g(t)e^{2\pi y\cdot t} \in L^1(\mathbb{R}^n)$. In order to prove this affirmance, decompose $\mathbb{R}^n$ into the union of finite non-overlapping cones $\{\Gamma_k\}_{k=1}^{\infty}$ with common vertex at the origin, i.e., $\mathbb{R}^n = \cup_{k=1}^{N}\Gamma_k$ and let $B_\delta = \overline{D(y_0,\delta)} \subset B_0$. Then for any $y \in D(y_0, \frac{\delta}{4})$ and $y_k \in (y_0 + \Gamma_k)$ satisfying $\frac{3\delta}{4} \le |y_k - y_0| < \delta$, we have $(y_k - y)\cdot t \ge \frac{|y_k - y_0|}{\sqrt{2}}|t| - |y_0 - y||t| \ge (\frac{3}{4\sqrt{2}} - \frac{1}{4})\delta|t| \ge \frac{1}{4}\delta|t|$ for $y_k - y_0, t \in \Gamma_k$. Thus, it follows from Hölder's inequality and Plancherel's theorem that

$$\begin{aligned}\int_{\Gamma_k}|g(t)e^{2\pi y\cdot t}|dt &\le \int_{\Gamma_k}|\check{F}_{y_k}(t)e^{-\pi\frac{\delta_0}{4}|t|}|dt \le \left(\int_{\Gamma_k}|\check{F}_{y_k}(t)|^2dt\right)^{\frac{1}{2}}\left(\int_{\Gamma_k}|e^{-2\pi\frac{\delta_0}{4}|t|}|dt\right)^{\frac{1}{2}}\\ &= \left(\int_{\Gamma_k}|F_{y_k}(t)|^2dt\right)^{\frac{1}{2}}\left(\int_{\Gamma_k}|e^{-2\pi\frac{\delta_0}{4}|t|}|dt\right)^{\frac{1}{2}} < \infty,\end{aligned}$$

which shows that $g(t)e^{2\pi y\cdot t} \in L^1(\Gamma_k)$. Then $g(t)e^{2\pi y\cdot t} \in L^1(\mathbb{R}^n)$. Therefore, we can see that the function $G(z) = \int_{\mathbb{R}^n} g(t)e^{-2\pi i(x+iy)\cdot t}dt$ is well defined and holomorphic on the tube domain T_{B_0}.

Now we can prove that

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^n} g_\varepsilon(t)e^{-2\pi i(x+iy)\cdot t}dt = \int_{\mathbb{R}^n} g(t)e^{-2\pi i(x+iy)\cdot t}dt.$$

For $y \in B_0$,

$$\begin{aligned}&\left|\int_{\mathbb{R}^n}(g_\varepsilon(t) - g(t))e^{-2\pi i(x+iy)\cdot t}dt\right| \le \int_{\mathbb{R}^n}\left|\left(\check{F}_{\varepsilon,y}(t)e^{-2\pi y\cdot t} - \check{F}_y(t)e^{-2\pi y\cdot t}\right)e^{2\pi i z\cdot t}\right|dt\\ &= \sum_{k=1}^{n}\int_{\Gamma_k}\left|\left(\check{F}_{\varepsilon,y_k}(x) - \check{F}_{y_k}(x)\right)e^{-2\pi i(y_k-y)\cdot t}\right|dt\\ &\le \sum_{k=1}^{n}\left(\int_{\Gamma_k}|\check{F}_{\varepsilon,y_k}(x) - \check{F}_{y_k}(x)|^2dt\right)^{\frac{1}{2}}\left(\int_{\Gamma_k}|e^{-2\pi\frac{\delta_0}{4}|t|}|dt\right)^{\frac{1}{2}}\\ &\le \left[\max_{1\le k\le n}\left(\int_{\Gamma_k}|e^{-2\pi\frac{\delta_0}{4}|t|}|dt\right)^{\frac{1}{2}}\right]\sum_{k=1}^{n}\left(\int_{\Gamma_k}|F_{\varepsilon,y_k}(x) - F_{y_k}(x)|^2dt\right)^{\frac{1}{2}} \to 0\end{aligned}$$

as $\varepsilon \to 0$. It follows that $\lim_{\varepsilon\to 0} F_\varepsilon(z) = G(z)$. Combining with $\lim_{\varepsilon\to 0} F_\varepsilon(z) = F(z)$, we get $G(z) = F(z)$ for $z \in T_{B_0}$. Therefore, there exists a measurable function $g(t)$ such that $g(t)e^{2\pi y\cdot t} \in L^1(\mathbb{R}^n)$ for all $y \in B_0$, then

$$F(z) = \int_{\mathbb{R}^n} g(t)e^{-2\pi i z\cdot t}dt$$

holds for $z \in T_{B_0}$. Hence $g(t) = \check{F}_y(t)e^{-2\pi y\cdot t}$ for all $y \in B_0$. Since B is connected, we can choose a sequence of bounded domains $\{B_k\}$ such that $B_0 \subset B_1 \subset B_2 \subset \ldots$ and $B = \cup_{k=0}^{\infty} B_{\delta_k}$. Then $\check{F}_{y_k}(t)e^{-2\pi y_k\cdot t} = \check{F}_y(t)e^{-2\pi y\cdot t}$ for $y \in B_0$ and $y_k \in B_k$, where $k > 0$. These imply that $g(t) = \check{F}(t)e^{-2\pi y\cdot t}$ holds for all $y \in B$. In other words, $F(z) = \int_{\mathbb{R}^n} g(t)e^{-2\pi i z\cdot t}dt$ holds for all $z \in T_B$. By letting $f(t) = g(-t)$, we see that $f \in L_I^2$ and $F(z) = (\mathscr{L}f)(t)$ for any given $F(z) \in A_\rho^2$, which means that the Laplace transform $\mathscr{L}$ is surjective.

Now we prove that (7) is well-defined, injective, and preserves norm, i.e., $\|\mathscr{L}f\|_{A_\rho^2} = \|f\|_{L_I^2}$. Let $F(x+iy) = F_y(x) = (\mathscr{L}f)(x+iy)$ for every fixed $y \in B$. Based on Plancherel's theorem, there holds

$$\int_{\mathbb{R}^n} |F_y(x)|^2 dx = \int_{\mathbb{R}^n} |f(t)|^2 e^{-4\pi y\cdot t}dt.$$

Multiplying by $\rho(iy)$ and performing integral over B on both sides of the above equation, we have

$$\begin{aligned}\|F\|_{A_\rho^2}^2 &= \int_B \left(\int_{\mathbb{R}^n} |F(x+iy)|^2 dx\right)\rho(iy)dy \\ &= \int_B \left(\int_{\mathbb{R}^n} |e^{-2\pi y\cdot t} f(t)|^2 dt\right)\rho(iy)dy \\ &= \int_{\mathbb{R}^n} |f(t)|^2 I(t)dt = \|f\|_{L_I^2}^2.\end{aligned}$$

It then follows that $f(t) = 0$ almost everywhere for all $t \notin U_I$, where $U_I = \{t : I(t) < \infty\}$. Therefore, $f(t)$ is supported in U_I. This completes the proof of Lemma 4. □

Let us now prove Theorem 1.

Proof For $F(z) \in A_\rho^2(T_B)$, there exists $f \in L_I^2$ such that $F(z) = (\mathscr{L}f)(z) = \int_{\mathbb{R}^n} f(t)e^{2\pi i z\cdot t}dt$. And for the kernel $K_{z_0}(z) = K(z, z_0) \in A_\rho^2$, there also exists $f_{z_0} \in L_I^2$ such that $K_{z_0}(z) = (\mathscr{L}f_{z_0})(z) = \int_{\mathbb{R}^n} f_{z_0}(t)e^{2\pi i z\cdot t}dt$ for $z_0, z \in T_B$. On the other hand, Lemma 1 claims that $\mathscr{L}$ is an isometry from L_I^2 to A_ρ^2 preserving the Hilbert space norm. Using the polarization identity of $\|F\|_{A_\rho^2} = \|f\|_{L_I^2}$, it then follows that the inner product is also preserved. Hence we have

$$F(z_0) = \langle F, K_{z_0}\rangle_\rho = \langle f, f_{z_0}\rangle_I = \int_{\mathbb{R}^n} f(t)\overline{f_{z_0}(t)}I(t)dt.$$

Hence

$$\int_{\mathbb{R}^n} f(t)e^{2\pi i z_0\cdot t}dt = \int_{\mathbb{R}^n} f(t)\overline{f_{z_0}(t)}I(t)dt$$

holds for every $f \in L_I^2$, which implies that $e^{2\pi i z_0\cdot t} = \overline{f_{z_0}(t)}I(t)$ almost everywhere on $U_I = \{t \in \mathbb{R}^n : I(t) < \infty\}$. Then, $f_{z_0}(t) = e^{-2\pi i \bar{z}_0\cdot t}I^{-1}(t)$. Here, $I(t)^{-1}$ takes 0 when $I(t) = \infty$ by the definition of $I(t)$. Hence,

$$K_{z_0}(z) = K(z, z_0) = \int_{\mathbb{R}^n} f_{z_0}(t) e^{2\pi i z \cdot t} dt = \int_{\mathbb{R}^n} e^{2\pi i (z - \bar{z}_0) \cdot t} I^{-1}(t) dt.$$

□

Note that the Bergman kernel is uniquely characterized by the following three properties.

(i) $K(z, z_0) = \overline{K(z_0, z)}$ for all $z, z_0 \in T_B$;

(ii) $K(z, z_0)$ reproduces every element in A^2_ρ in the following sense

$$F(z) = \int_B \int_{\mathbb{R}^n} K(z, u + iv) F(u + iv) \rho(iv) du dv$$

for every $F \in A^2_\rho$;

(iii) $K_{z_0} \in A^2_\rho$ for all $z_0 \in T_B$, where $K_{z_0}(z) = K(z, z_0)$.

We shall show that (9) admits these properties. We first prove the symmetric property,

$$\begin{aligned}\overline{K(z_0, z)} &= \overline{\int_{\mathbb{R}^n} e^{2\pi i (z_0 - \bar{z}) \cdot t} I^{-1}(t) dt} = \int_{\mathbb{R}^n} e^{-2\pi i (\bar{z}_0 - z) \cdot t} \overline{I^{-1}(t)} dt \\ &= \int_{\mathbb{R}^n} e^{2\pi i (z - \bar{z}_0) \cdot t} I^{-1}(t) dt = K(z, z_0),\end{aligned}$$

which means (i) holds for the Bergman kernel in the form of (9). We then show $K(z, z_0)$ reproduces every element in A^2_ρ. Indeed, for $F(z), K_{z_0}(z) \in A^2_\rho$,

$$F(z) = \int_{U_I} f(t) e^{2\pi i z \cdot t} dt$$

and

$$K_{z_0}(z) = \int_{\mathbb{R}^n} \left(\frac{e^{-2\pi i \bar{z}_0 \cdot t}}{I(t)} \right) e^{2\pi i t \cdot z} dt.$$

Then the polarization identity implies that

$$\begin{aligned}\langle F, K_{z_0} \rangle_{A^2_\rho} &= \int_B \int_{\mathbb{R}^n} K(z_0, z) F(z) dA_\rho(z) \\ &= \left\langle f(t), \frac{e^{-2\pi i \bar{z}_0 \cdot t}}{I(t)} \right\rangle_{L^2_I} \\ &= \int_{U_I} f(t) \overline{\frac{e^{-2\pi i \bar{z}_0 \cdot t}}{I(t)}} I(t) dt\end{aligned}$$

$$= \int_{U_I} f(t) e^{2\pi i z_0 \cdot t} dt$$
$$= F(z_0).$$

Hence, the second property is proved.

Finally, we prove that $K_{z_0}(z) \in A^2_\rho$. For fixed $z_0 = u + iv \in T_B$, there exists $\delta > 0$ such that $v + P_\delta \subset B$, where $P_\delta = [-\delta, \delta]^n \subset \mathbb{R}^n$. Let $\varepsilon = \min\{\rho(iy) : y \in v + P_\delta\} > 0$, then

$$I(t)e^{4\pi v \cdot t} = \int_B e^{-4\pi(y-v)\cdot t} \rho(y) dy \geq \varepsilon \int_{P_\delta} e^{-4\pi \eta \cdot t} d\eta = \varepsilon \prod_{k=1}^{n} \frac{\sinh(4\pi \delta t_k)}{2\pi \delta t_k}.$$

Therefore, again by the polarization identity,

$$\begin{aligned}
\langle K_{z_0}, K_{z_0} \rangle_{A^2_\rho} &= \int_{T_B} |K_{z_0}(z)|^2 dA_\rho(z) = \langle K_{z_0}, K_{z_0} \rangle_{A^2_\rho} \\
&= \left\langle \frac{e^{-2\pi i \bar{z}_0 \cdot t}}{I(t)}, \frac{e^{-2\pi i \bar{z}_0 \cdot t}}{I(t)} \right\rangle_{L^2_I} = \int_{\mathbb{R}^n} \left| \frac{e^{-2\pi i \bar{z}_0 \cdot t}}{I(t)} \right|^2 I(t) dt \\
&= \int_{\mathbb{R}^n} \frac{e^{-4\pi v \cdot t}}{I(t)} dt \leq \int_{\mathbb{R}^n} \frac{1}{\varepsilon} \left(\prod_{k=1}^{n} \frac{\sinh(4\pi \delta t_k)}{2\pi \delta t_k} \right)^{-1} dt < \infty.
\end{aligned}$$

Therefore, $K_{z_0}(z) \in A^2_\rho$ for $z \in T_B$.

4 Computation of Some Weighted Bergman Kernels

The weighted Bergman kernel has never been computed explicitly. However, the theory would be a bit hollow if we do not compute at least one weighted Bergman kernel. In this section, we calculate some weighted Bergman kernels as examples. The first example is the weighted Bergman kernel for tube over the following cone.

Example 1 Suppose that $B = \{y = (y', y_n) : y_n > |y'|^2\}$ and $\alpha \in \mathbb{R}$. Denote by A^2_α the space of analytic functions on tube domain $\Omega = \{z = x + iy : x \in \mathbb{R}^n, y \in B\}$ such that

$$\|F\|_{A^2_\alpha} = \left(\int_\Omega |F(x+iy)|^2 (y_n - |y'|^2)^\alpha dx dy \right)^{\frac{1}{2}} < \infty. \tag{11}$$

Then the reproducing kernel for the Hilbert space A^2_α is

$$K_\alpha(z, z_0) = C_{1,\alpha}((z' - \bar{z}'_0)^2 - 2i(z_n - \bar{z}_{0,n}))^{-n-\alpha-1}, \tag{12}$$

where $z' = (z_1, \ldots, z_{n-1})$, $z'_0 = (z_{0,1}, \ldots, z_{0,n-1})$ and

$$C_{1,\alpha} = \frac{2^{n+1+2\alpha}\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)\pi^n}. \tag{13}$$

Here, $\Re\{(z' - \bar{z}'_0)^2 - 2i(z_n - \bar{z}_{0,n})\} = 2(y_n + v_n) + (x' - u') \cdot (x' - u') - (y' + v') \cdot (y' + v') > 0$, then $|\arg((z' - \bar{z}'_0)^2 - 2i(z_n - \bar{z}_{0,n}))| < \frac{\pi}{2}$, in which $z = (z', z_n) = (x', x_n) + i(y', y_n)$, $z_0 = (z'_0, z_{0,n}) = (u', u_n) + i(v', v_n) \in T_B$ and $z'^2 = z' \cdot z' = z_1^2 + z_2^2 + \ldots + z_{n-1}^2$.

Proof We first compute $I(t)$,

$$I(t) = \int_B e^{-4\pi y \cdot t}(y_n - |y'|^2)^\alpha dy = \int_0^\infty \int_{|y'|^2 < y_n} (y_n - |y'|^2)^\alpha e^{-4\pi(y_n t_n + y' \cdot t')} dy' dy_n. \tag{14}$$

Performing variable substitution on (14) with $a_n = y_n - |y'|^2$, $a' = y'$, then $y_n = a_n + |a'|^2$. We then obtain

$$\begin{aligned} I(t) &= \int_0^\infty \int_{\mathbb{R}^{n-1}} a_n^\alpha e^{-4\pi(a_n + |a'|^2)t_n} e^{-4\pi a' \cdot t'} da' da_n \\ &= \int_0^\infty a_n^\alpha e^{-4\pi a_n t_n} \left(\prod_{k=1}^{n-1} \int_{\mathbb{R}} e^{-4\pi a_k^2 t_n - 4\pi a_k t_k} da_k \right) da_n. \end{aligned}$$

It is obvious that $I(t) = \infty$ when $t_n \le 0$, For $t_n > 0$,

$$\begin{aligned} I(t) &= \int_0^\infty a_n^\alpha e^{-4\pi a_n t_n} \left(\prod_{k=1}^{n-1} \int_{\mathbb{R}} e^{-4\pi a_k^2 t_n - 4\pi a_k t_k} da_k \right) da_n \\ &= \int_0^\infty a_n^\alpha e^{-4\pi a_n t_n} \left(\prod_{k=1}^{n-1} \int_{\mathbb{R}} e^{-4\pi t_n \left(a_k^2 - \frac{a_k t_k}{t_n} + \left(\frac{t_k}{2t_n}\right)^2 - \left(\frac{t_k}{2t_n}\right)^2 \right)} da_k \right) da_n \\ &= \int_0^\infty a_n^\alpha e^{-4\pi a_n t_n} \left(\prod_{k=1}^{n-1} \int_{\mathbb{R}} e^{-4\pi t_n \left(a_k - \frac{t_k}{2t_n}\right)^2} e^{\frac{\pi t_k^2}{t_n}} da_k \right) da_n. \end{aligned}$$

Let $s = a_k - \frac{t_k}{2t_n}$ and $4\pi t_n s^2 = \eta^2$, then

$$\begin{aligned} I(t) &= \int_0^\infty a_n^\alpha e^{-4\pi a_n t_n} \left(\prod_{k=1}^{n-1} e^{\frac{\pi t_k^2}{t_n}} \int_{\mathbb{R}} e^{-4\pi t_n s^2} ds \right) da_n \\ &= \int_0^\infty a_n^\alpha e^{-4\pi a_n t_n} \left(\prod_{k=1}^{n-1} \frac{e^{\frac{\pi t_k^2}{t_n}}}{2\sqrt{\pi t_n}} \int_{\mathbb{R}} e^{-\eta^2} d\eta \right) da_n \end{aligned}$$

$$= \int_0^\infty a_n^\alpha e^{-4\pi a_n t_n} e^{\frac{\pi |t'|^2}{t_n}} \left(\frac{\sqrt{\pi}}{2\sqrt{\pi t_n}}\right)^{n-1} da_n$$
$$= e^{\frac{\pi |t'|^2}{t_n}} (2\sqrt{t_n})^{1-n} \int_0^\infty a_n^\alpha e^{-4\pi a_n t_n} da_n.$$

Putting $b = -4\pi a_n t_n$, we have

$$I(t) = e^{\frac{\pi |t'|^2}{t_n}} (2\sqrt{t_n})^{1-n} \int_0^\infty \frac{1}{(4\pi t_n)^{\alpha+1}} b^\alpha e^b db$$
$$= \frac{2^{1-n}\Gamma(\alpha+1)}{(4\pi)^{\alpha+1}} e^{\frac{\pi |t'|^2}{t_n}} t_n^{\frac{1}{2}(1-n)-\alpha-1}.$$

Now we can compute the formula of weighted Bergman kernel. According to the representation form of reproducing kernel,

$$K_\alpha(z, z_0) = \int_{\mathbb{R}^n} e^{2\pi i (z-\bar{z}_0)\cdot t} I^{-1}(t) dt = \frac{(4\pi)^{\alpha+1} 2^{n-1}}{\Gamma(\alpha+1)} \int_{\mathbb{R}^n} e^{2\pi i (z-\bar{z}_0)\cdot t - \frac{\pi |t'|^2}{t_n}} t_n^{\frac{1}{2}(n-1)+\alpha+1} dt.$$

Set $w = z - \bar{z}_0$ and $C_\alpha = \frac{(4\pi)^{\alpha+1} 2^{n-1}}{\Gamma(\alpha+1)}$, then

$$K_\alpha(z, z_0) = C_\alpha \int_{\mathbb{R}^n} e^{2\pi i w\cdot t - \frac{\pi |t'|^2}{t_n}} t_n^{\frac{1}{2}(n-1)+\alpha+1} dt$$
$$= C_\alpha \int_0^\infty \int_{\mathbb{R}^{n-1}} e^{2\pi i (w_n t_n + w'\cdot t') - \frac{\pi |t'|^2}{t_n}} t_n^{\frac{1}{2}(n-1)+\alpha+1} dt' dt_n$$
$$= C_\alpha \int_0^\infty e^{2\pi i w_n t_n} t_n^{\frac{1}{2}(n-1)+\alpha+1} \left(\int_{\mathbb{R}^{n-1}} e^{2\pi i w'\cdot t' - \frac{\pi |t'|^2}{t_n}} dt'\right) dt_n$$
$$= C_\alpha \int_0^\infty e^{2\pi i w_n t_n} t_n^{\frac{1}{2}(n-1)+\alpha+1} \left(\prod_{k=1}^{n-1} \int_{\mathbb{R}} e^{2\pi i w_k t_k - \frac{\pi t_k^2}{t_n}} dt_k\right) dt_n,$$

in which

$$\int_{\mathbb{R}} e^{2\pi i w_k t_k - \frac{\pi t_k^2}{t_n}} dt_k = \int_{\mathbb{R}} e^{-\frac{\pi}{t_n}\left(t_k^2 - 2 i w_k t_n t_k + (i w_k t_n)^2 + w_k^2 t_n^2\right)} dt_k$$
$$= e^{-\frac{\pi w_k^2 t_n^2}{t_n}} \int_{\mathbb{R}} e^{-\frac{\pi}{t_n}(t_k - i w_k t_n)^2} dt_k.$$

Let $s = t_k - i w_k t_n$ and $\eta^2 = \frac{\pi}{t_n} s^2$, then

$$\int_{\mathbb{R}} e^{2\pi i w_k t_k - \frac{\pi t_k^2}{t_n}} dt_k = e^{-\frac{\pi w_k^2 t_n^2}{t_n}} \int_{\mathbb{R}} e^{-\frac{\pi}{t_n} s^2} ds = e^{-\frac{\pi w_k^2 t_n^2}{t_n}} \int_{\mathbb{R}} \left(\frac{t_n}{\pi}\right)^{\frac{1}{2}} e^{-\eta^2} d\eta = e^{-\frac{\pi w_k^2 t_n^2}{t_n}} t_n^{\frac{1}{2}}.$$

Therefore,

$$\begin{aligned}K_\alpha(z,z_0) &= C_\alpha\int_0^\infty e^{2\pi i w_n t_n} t_n^{\frac{1}{2}(n-1)+\alpha+1}\prod_{k=1}^{n-1}\left(e^{-\frac{\pi w_k^2 t_n^2}{t_n}} t_n^{\frac{1}{2}}\right)dt_n\\ &= C_\alpha\int_0^\infty e^{2\pi i w_n t_n} t_n^{\frac{1}{2}(n-1)+\alpha+1} e^{-\frac{\pi\sum_{k=1}^{n-1} w_k^2 t_n^2}{t_n}} t_n^{\frac{1}{2}(n-1)} dt_n\\ &= C_\alpha\int_0^\infty e^{\pi t_n(2iw_n - w'\cdot w')} t_n^{n+\alpha} dt_n.\end{aligned}$$

In order to compute $K_\alpha(z,z_0)$, it suffices to show that $\Re(2iw_n - w'\cdot w') < 0$, where $w = z-\bar{z}_0$ for $z, z_0\in T_B$. In fact, since $\Re(iw_n) = -\Im w_n$, $\Re(w'\cdot w') = (\Re w')^2 - (\Im w')^2$, for $w_n = z_n - \bar{z}_{0,n}$,

$$\begin{aligned}\Re(2iw_n - w'\cdot w') &= -2\Im w_n - \sum_{k=1}^{n-1}\left((\Re w_k)^2 - (\Im w_k)^2\right)\\ &= -2(y_n + y_{0,n}) + \sum_{k=1}^{n-1}(y_k + y_{0,k})^2 - \sum_{k=1}^{n-1}(x_k - x_{0,k})^2\\ &\le -2(y_n + y_{0,n}) + \sum_{k=1}^{n-1}(y_k^2 + y_{0,k}^2 + 2y_k y_{0,k})\\ &\le -2(y_n + y_{0,n}) + 2\sum_{k=1}^{n-1}(y_k^2 + y_{0,k}^2)\\ &= 2\left(\left(-y_n + \sum_{k=1}^{n-1} y_k^2\right) + \left(-y_{0,n} + \sum_{k=1}^{n-1} y_{0,k}^2\right)\right) < 0.\end{aligned}$$

Now, we can continue to calculate $K_\alpha(z,z_0)$. Let $a = \pi t_n(w'\cdot w' - 2iw_n)$,

$$\begin{aligned}K_\alpha(z,z_0) &= C_\alpha\int_0^\infty e^{-\pi t_n(w'\cdot w' - 2iw_n)} t_n^{n+\alpha} dt_n\\ &= C_\alpha\int_0^\infty \frac{1}{(\pi(w'\cdot w' - 2iw_n))^{n+\alpha+1}} e^{-a} a^{n+\alpha} da\\ &= \frac{C_\alpha\Gamma(n+\alpha+1)}{(\pi(w'\cdot w' - 2iw_n))^{n+\alpha+1}}\\ &= C_{\alpha,1}(w'\cdot w' - 2iw_n)^{-n-\alpha-1}\\ &= C_{\alpha,1}(z'-\bar{z}_0')^2 - 2i(z_n - \bar{z}_{0,n}))^{-n-\alpha-1},\end{aligned}$$

which shows that (12) holds. □

Example 2 Denote by Ω_n the Siegel domain in $\mathbb{C}^n$, defined as

$$\Omega_n = \{z = x+iy : y_n > |z'|^2\},$$

where $z=(z', z_n)$, $z_n=x_n+iy_n$. Let $\alpha\in\mathbb{R}$ and denote by $A^2_\alpha(\Omega_n)$ the space of analytic functions $F(z)$ in the Siegel domain Ω_n satisfying

$$\|F\|_{A^2_\alpha(\Omega_n)}=\left(\int_{\Omega_n}|F(x+iy)|^2(y_n-|z'|^2)^\alpha dxdy\right)^{\frac{1}{2}}<\infty. \tag{15}$$

Then the reproducing kernel for the Hilbert space $A^2_\alpha(\Omega_n)$ is

$$K_{\alpha,\Omega_n}(z,w)=C_{2,\alpha}(i(\bar{w}_n-z_n)-2z'\cdot\bar{w}')^{-n-\alpha-1}, \tag{16}$$

where $C_{2,\alpha}=2^{-2-\alpha}C_{1,\alpha}$ and $C_{1,\alpha}$ is defined by (13).

Proof Define a transform $\Phi:\Omega_n\to T_B$ as $\zeta=(\zeta',\zeta_n)=\Phi(z)=(2^{\frac{1}{2}}z', z_n-iz'\cdot z')$. Then we observe that the Siegel domain Ω_n is biholomorphically equivalent to the tube domain T_B over $B=\{v\in\mathbb{R}^n: v_n>|v'|^2\}$ via Φ. The inverse of Φ is $z=\Phi^{-1}(\zeta)=(2^{-\frac{1}{2}}\zeta', \zeta_n+\frac{i}{2}\zeta'\cdot\zeta')$, and the determinants of the holomorphic Jacobian matrices of $\zeta=\Phi(z)$ and $z=\Phi^{-1}(\zeta)$ are $(D\Phi)(z)=2^{\frac{1}{2}(n-1)}$ and $(D\Phi^{-1})(\zeta)=2^{-\frac{1}{2}(n-1)}$, respectively. For $\zeta=u+iv\in T_B$ and $z=x+iy\in\Omega_n$,

$$\begin{aligned}
y_n-|z'|^2&=y_n-\sum_{k=1}^{n-1}(x_k^2+y_k^2)\\
&=\Im(\zeta_n+\frac{i}{2}\zeta'\cdot\zeta')-\frac{1}{2}|\zeta'|^2\\
&=v_n+\frac{1}{2}(u'\cdot u'-v'\cdot v')-\frac{1}{2}|\zeta'|^2\\
&=v_n+\frac{1}{2}\sum_{k=1}^{n-1}(u_k^2-v_k^2)-\frac{1}{2}\sum_{k=1}^{n-1}(u_k^2+v_k^2)\\
&=v_n-|v'|^2.
\end{aligned}$$

By Lemma 3, the reproducing kernel K_{α,Ω_n} of the Hibert space $A^2_\alpha(\Omega_n)$ is

$$K_{\alpha,\Omega_n}(z,w)=2^{n-1}K_\alpha(\Phi(z),\Phi(w))=C_{1,\alpha}2^{-2-\alpha}(i(\bar{w}_n-z_n)-2z'\cdot\bar{w}')^{-n-\alpha-1}, \tag{17}$$

where $C_{1,\alpha}$ is defined by (13). □

Example 3 For $\alpha\in\mathbb{R}$, denote by B_n the unit ball $\{z:|z|<1\}$ and $A^2_\alpha(B_n)$ the space of analytic functions $F(z)$ on B_n satisfying

$$\|F\|_{A^2_\alpha(B_n)}=\left(\int_{B_n}|F(z)|^2\frac{(1-|z|^2)^\alpha}{|1+z_n|^{2\alpha}}dxdy\right)^{\frac{1}{2}}<\infty. \tag{18}$$

Then the reproducing kernel for the Hilbert space $A^2_\alpha(B_n)$ is

$$K_{\alpha,B_n}(z,w) = C_{3,\alpha}(1+\bar{w}_n)^\alpha(1+z_n)^\alpha(1-\bar{w}\cdot z)^{-n-\alpha-1}, \tag{19}$$

with $C_{3,\alpha} = 2^{1-3\alpha-n}C_{2,\alpha} = 2^{-4\alpha-n-1}C_{1,\alpha}$, where $C_{1,\alpha}$ is defined by (13).

Proof Note that the Siegel domain Ω_n is an unbounded realization of B_n, i.e., Ω_n is biholomorphically equivalent to B_n. The corresponding biholomorphic automorphism (so-called Cayley transform) $\eta = \Phi(z) : B_n \mapsto \Omega_n$ can be written in explicit form,

$$\eta = \Phi(z) = \left(\frac{2z'}{z_n+1}, \frac{4i(1-z_n)}{1+z_n}\right) \text{ for } z \in B_n.$$

And its inverse form is

$$z = \Phi^{-1}(\eta) = \left(\frac{\eta'}{1-\frac{i}{4}\eta_n}, \frac{1+\frac{i}{4}\eta_n}{1-\frac{i}{4}\eta_n}\right) \text{ for } \eta \in \Omega_n.$$

The determinants of the holomorphic Jacobian matrices of $\eta = \Phi(z)$ and $z = \Phi^{-1}(\eta)$ are

$$(D\Phi)(z) = \frac{-i2^{n+2}}{(1+z_n)^{n+1}} \text{ and } (D\Phi^{-1})(\eta) = \frac{i}{2(1-\frac{i}{4})^{n+1}}$$

, respectively. For $\eta = u + iy \in \Omega_n$ and $z = x + iy \in B_n$, we have

$$\begin{aligned}
\frac{1-|z|^2}{|1+z_n|^2} &= \frac{1-|z'|^2-|z_n|^2}{|1+z_n|^2} \\
&= \frac{1-\left|\frac{\eta'}{1-\frac{i}{4}\eta_n}\right|^2-\left|\frac{1+\frac{i}{4}\eta_n}{1-\frac{i}{4}\eta_n}\right|^2}{\left|1+\frac{1+\frac{i}{4}\eta_n}{1-\frac{i}{4}\eta_n}\right|^2} \\
&= \frac{\left|1+\frac{v_n}{4}-\frac{u_n}{4}i\right|^2-\left|1-\frac{v_n}{4}+\frac{u_n}{4}i\right|^2-\left|\eta'\right|^2}{4} \\
&= \frac{1}{4}\left(v_n-|\eta'|^2\right).
\end{aligned}$$

By Lemma 3, the reproducing kernel K_{α,B_n} of the Hibert space $A^2_\alpha(B_n)$ is

$$\begin{aligned}
K_{\alpha,B_n}(z,w) &= \frac{-i2^{n+2}}{(1+z_n)^{n+1}}K_{\alpha,\Omega_n}(\Phi(z),\Phi(w))\frac{i2^{n+2}}{(1+\bar{w}_n)^{n+1}} \\
&= C_{3,\alpha}(1+\bar{w}_n)^\alpha(1+z_n)^\alpha(1-\bar{w}\cdot z)^{-n-\alpha-1} \qquad (20)
\end{aligned}$$

with $C_{3,\alpha} = 2^{1-3\alpha-n}C_{2,\alpha} = 2^{-4\alpha-n-1}C_{1,\alpha}$, where $C_{1,\alpha}$ is defined by (13). □

Example 4 Suppose that Λ_n is the Lorentz cone (or a forward light cone) defined by $\{y = (y', y_n) : y_n > |y'|\}$. The quadratic function $\Delta(y) = y_n^2 - |y'|^2$ is call the Lorentz form. Let $\alpha \in \mathbb{R}$ and denote by $A^2_\alpha(T_{\Lambda_n})$ the space of analytic functions $F(z)$ in the tube domain T_{Λ_n} over the forward cone Λ_n such that

$$\|F\|_{A^2_\alpha(\Lambda_n)} = \left(\int_{\Lambda_n}\int_{\mathbb{R}^n} |F(x+iy)|^2(\Delta(y))^\alpha dxdy\right)^{\frac{1}{2}} < \infty. \tag{21}$$

Then the reproducing kernel for the Hilbert space $A^2_\alpha(T_{\Lambda_n})$ is

$$K_{\alpha,\Lambda_n}(z,w) = C_{4,\alpha}P(z-\bar{w})^{-\alpha-n}, \tag{22}$$

where $C_{4,\alpha}$ is defined by

$$C_{4,\alpha} = \frac{4^\alpha\Gamma(\alpha+\frac{n}{2}+1)\Gamma(2\alpha+2n)\Gamma(\alpha+\frac{n}{2}+\frac{1}{2})}{\pi^n\Gamma(\alpha+1)\Gamma(2\alpha+n)\Gamma(\alpha+n+\frac{1}{2})} \tag{23}$$

and $P(z) = z_1^2 + \cdots + z_{n-1}^2 - z_n^2$ satisfying $P(z) \subset \mathbb{C}\setminus(-\infty,0]$ for $z \in T_{\Lambda_n}$.

Proof For $t = (t', t_n) \in \mathbb{R}^n$ and Let $a = 4\pi t = (a', a_n)$, then based on the form of kernel in Theorem 1, we have

$$I(t) = \int_B e^{-4\pi y\cdot t}(\Delta(y))^\alpha dy = \int_0^\infty \int_{|y'|<y_n} e^{-a_ny_n - a'\cdot y'}(\Delta(y))^\alpha dy'dy_n.$$

We now choose an orthogonal matrix A such that $Ae_1 = \frac{a'}{|a'|}$, $A^TA = I$ and $\{Ae_1, \ldots, Ae_{n-1}\}$ is also an orthogonal basis in $\mathbb{R}^{n-1}$, where I is the identity matrix and A^T is the transposed matrix of A. Hence $a' \cdot y' = Ae_1|a'| \cdot y' = |a'|y_1$. Write $y = (y', y_n) = (y_1, y'', y_n)$, then

$$\begin{aligned} I(t) &= \int_0^\infty \int_{-y_n}^{y_n} \int_{|y''|<\sqrt{y_n^2-y_1^2}} e^{-a_ny_n-|a'|y_1}(y_n^2 - y_1^2 - |y''|^2)^\alpha dy''dy_1dy_n \\ &= \int_0^\infty \int_0^{y_n} \int_{|y''|<\sqrt{y_n^2-y_1^2}} 2e^{-a_ny_n}\cosh(|a'|y_1)(y_n^2 - y_1^2 - |y''|^2)^\alpha dy''dy_1dy_n. \end{aligned}$$

where $\cosh(s) = \frac{1}{2}(e^s + e^{-s})$. Letting $\sinh(s) = \frac{1}{2}(e^s - e^{-s})$ and $y = \Phi(x) = (x_n \sinh x_1, x'', x_n \cosh x_1)$, together with $0 \le \sinh x_1 < \cosh x_1$, we have $0 < y_1 < y_n$. Performing variables substitution to the above formula, then

$$I(t) = \int_0^\infty \int_0^\infty \int_{|x''|<x_n} 2\cosh(|a'|x_n \sinh x_1)e^{-a_nx_n\cosh x_1}(x_n^2 - |x''|^2)^\alpha (D\Phi)(x)dx''dx_1dx_n,$$

in which $(D\Phi)(x)$ is the determinant of the Jacobian matrix of $\Phi(x)$ and $(D\Phi)(x) = x_n$. Put $t'' = \frac{|x''|}{x_n}$, then

$$\begin{aligned} I(t) &= \int_0^\infty \int_0^\infty \int_{|t''|<1} 2\cosh(|a'|x_n \sinh x_1) e^{-a_n x_n \cosh x_1} (1-|t''|^2)^\alpha x_n^{2\alpha+n-1} dt'' dx_1 dx_n \\ &= 2C_\alpha \int_0^\infty \int_0^\infty \cosh(|a'|x_n \sinh x_1) e^{-a_n x_n \cosh x_1} x_n^{2\alpha+n-1} dx_1 dx_n, \end{aligned}$$

where $C_\alpha = \int_{|t''|<1} (1-|t''|^2)^\alpha dt''$. Now we compute the value of C_α as

$$C_\alpha = \int_0^1 \rho^{n-3} \int_{|\xi|=1} (1-\rho^2)^\alpha d\sigma_{n-2}(\xi) d\rho = S_{n-2} \int_0^1 \rho^{n-3} (1-\rho^2)^\alpha d\rho,$$

where $S_{n-2} = \frac{2\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})}$ is the surface area of the $n-2$ dimensional sphere. Letting $s = \rho^2$,

$$\begin{aligned} C_\alpha &= \frac{S_{n-2}}{2} \int_0^1 s^{\frac{n-3}{2}} (1-s)^\alpha s^{-\frac{1}{2}} ds = \frac{\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})} B\left(\frac{n-2}{2}, \alpha+1\right) \\ &= \frac{\pi^{\frac{n-1}{2}} \Gamma(\frac{n}{2}-1)\Gamma(\alpha+1)}{\Gamma(\frac{n-1}{2})\Gamma(\frac{n}{2}+\alpha)}. \end{aligned}$$

Therefore,

$$\begin{aligned} I(t) &= 2C_\alpha \int_0^\infty \int_0^\infty \cosh(|a'|x_n \sinh x_1) e^{-a_n x_n \cosh x_1} x_n^{2\alpha+n-1} dx_1 dx_n \\ &= C_\alpha \int_0^\infty \int_{-\infty}^\infty e^{-x_n\left(a_n \cosh x_1 - |a'| \sinh x_1\right)} x_n^{2\alpha+n-1} dx_1 dx_n. \end{aligned}$$

Let $s = -x_n \left(a_n \cosh x_1 - |a'| \sinh x_1\right)$, then

$$\begin{aligned} I(t) &= C_\alpha \int_{-\infty}^\infty \frac{1}{(a_n \cosh x_1 - |a'| \sinh x_1)^{2\alpha+n}} \left(\int_0^\infty e^s s^{2\alpha+n-1} ds\right) dx_1 \\ &= C_\alpha \Gamma(2\alpha+n) \int_{-\infty}^\infty \frac{1}{\left(\frac{a_n \cosh x_1 - |a'| \sinh x_1}{\sqrt{a_n^2 - |a'|^2}}\right)^{2\alpha+n} (a_n^2 - |a'|^2)^{\alpha+\frac{n}{2}}} dx_1. \end{aligned}$$

Since $\left(\frac{a_n}{\sqrt{a_n^2-|a'|^2}}\right)^2 - \left(\frac{|a'|^2}{\sqrt{a_n^2-|a'|^2}}\right)^2 = 1$, put $\cosh t_0 = \frac{a_n}{\sqrt{a_n^2-|a'|^2}}$ and $\sinh t_0 = \frac{|a'|^2}{\sqrt{a_n^2-|a'|^2}}$, then

$$\begin{aligned} I(t) &= \frac{C_\alpha \Gamma(2\alpha+n)}{(a_n^2-|a'|^2)^{\alpha+\frac{n}{2}}} \int_{-\infty}^{\infty} \frac{1}{(\cosh t_0 \cosh x_1 - \sinh t_0 \sinh x_1)^{2\alpha+n}} dx_1 \\ &= \frac{C_\alpha \Gamma(2\alpha+n)}{(a_n^2-|a'|^2)^{\alpha+\frac{n}{2}}} \int_{-\infty}^{\infty} \frac{1}{(\cosh(x_1-t_0))^{2\alpha+n}} dx_1 \\ &= \frac{C_\alpha \Gamma(2\alpha+n)}{(a_n^2-|a'|^2)^{\alpha+\frac{n}{2}}} \int_{-\infty}^{\infty} \frac{1}{(\cosh t)^{2\alpha+n}} dt. \end{aligned}$$

Write $J(t) = \int_{-\infty}^{\infty} \frac{1}{(\cosh t)^{2\alpha+n}} dt$ and let $(\cosh t)^2 = s$,

$$J(t) = 2\int_{-\infty}^{\infty} \frac{1}{2s^{\alpha+\frac{n}{2}+\frac{1}{2}}(s-1)^{\frac{1}{2}}} ds = \int_{-\infty}^{\infty} \frac{1}{s^{\alpha+\frac{n}{2}+1}(1-\frac{1}{s})^{\frac{1}{2}}} ds.$$

Set $\frac{1}{s} = k$, then

$$J(t) = \int_0^1 k^{\alpha+\frac{n}{2}+1}(1-k)^{-\frac{1}{2}} k^{-2} dk = B\left(\alpha+\frac{n}{2}, \frac{1}{2}\right) = \frac{\Gamma(\alpha+\frac{n}{2})\Gamma(\frac{1}{2})}{\Gamma(\alpha+\frac{n+1}{2})}.$$

As a result,

$$\begin{aligned} I(t) &= \frac{\pi^{\frac{n-1}{2}} \Gamma(\frac{n}{2}-1)\Gamma(\alpha+1)\Gamma(2\alpha+n)\Gamma(\frac{1}{2})}{\Gamma(\frac{n-1}{2})\Gamma(\alpha+\frac{n+1}{2})} (\Delta(a))^{-\alpha-\frac{n}{2}} \\ &= \frac{\pi^{\frac{n-1}{2}} \Gamma(\frac{n}{2}-1)\Gamma(\alpha+1)\Gamma(2\alpha+n)\Gamma(\frac{1}{2})}{(4\pi)^{2\alpha+n}\Gamma(\frac{n-1}{2})\Gamma(\alpha+\frac{n+1}{2})} (\Delta(t))^{-\alpha-\frac{n}{2}} \\ &= \widetilde{C}_{\alpha,n} (\Delta(t))^{-\alpha-\frac{n}{2}}, \end{aligned}$$

where $\widetilde{C}_{\alpha,n} = \frac{\pi^{\frac{n-1}{2}} \Gamma(\frac{n}{2}-1)\Gamma(\alpha+1)\Gamma(2\alpha+n)\Gamma(\frac{1}{2})}{(4\pi)^{2\alpha+n}\Gamma(\frac{n-1}{2})\Gamma(\alpha+\frac{n+1}{2})}$. In order to obtain the representation form of the reproducing kernel $K_{\alpha,\Lambda_n}(z,w)$, $I(t)$ should be finite, i.e., $t_n^2 > |t'|^2$. Therefore,

$$K(z) = \int_{\mathbb{R}^n} \frac{e^{2\pi i t\cdot z}}{I(t)} dt = \frac{1}{\widetilde{C}_{\alpha,n}} \int_{\mathbb{R}^n} e^{2\pi i t\cdot z} (\Delta(t))^{\alpha+\frac{n}{2}} dt.$$

Let $K_1(z) = \widetilde{C}_{\alpha,n} K(z)$, then performing variables substitution as that in the computation of $I(t)$,

$$\begin{aligned} K_1(iy) &= \int_{\mathbb{R}^n} e^{-2\pi t\cdot y} (\Delta(t))^{\alpha+\frac{n}{2}} dt \\ &= \int_0^\infty \int_{|t'|<t_n} e^{-2\pi y_n t_n - 2\pi t'\cdot y'} (\Delta(t))^{\alpha+\frac{n}{2}} dt' dt_n \\ &= \int_0^\infty \int_{-t_n}^{t_n} \int_{|t''|<\sqrt{t_n^2-t_1^2}} e^{-2\pi y_n t_n - 2\pi |y'| t_1} (t_n^2 - |t_1|^2 - |t''|^2)^{\alpha+\frac{n}{2}} dt'' dt_1 dt_n. \end{aligned}$$

Letting $t = \Phi(u) = (u_n \sinh u_1, u'', u_n \cosh u_1)$, then $(D\Phi)(u) = u_n$. Therefore,

$$K_1(iy) = \int_0^\infty \int_{-\infty}^\infty \int_{|u''|<u_n} e^{-2\pi y_n u_n \cosh u_1 - 2\pi |y'| u_n \sinh u_1} (u_n^2 - |u''|^2)^{\alpha+\frac{n}{2}} u_n du'' du_1 du_n.$$

Put $s'' = \frac{u''}{u_n}$, then

$$K_1(iy) = \int_0^\infty \int_{-\infty}^\infty \int_{|s''|<1} e^{-2\pi y_n u_n \cosh u_1 - 2\pi |y'| u_n \sinh u_1} (1 - |s''|^2)^{\alpha+\frac{n}{2}} u_n^{1+2\alpha+n+n-2} ds'' du_1 du_n,$$

in which

$$\begin{aligned} \int_{|s''|<1} (1-|s''|^2)^{\alpha+\frac{n}{2}} ds'' &= \int_0^1 \rho^{n-2-1} \int_{|\zeta|=1} (1-\rho^2)^{\alpha+\frac{n}{2}} d\sigma_{n-2}(\zeta) d\rho \\ &= S_{D_{n-2}} \int_0^1 \rho^{n-3} (1-\rho^2)^{\alpha+\frac{n}{2}} d\rho \\ &= S_{D_{n-2}} \int_0^1 s^{\frac{n-3}{2}} (1-s)^{\alpha+\frac{n}{2}} \frac{1}{2} s^{-\frac{1}{2}} ds = \frac{S_{D_{n-2}}}{2} B\left(\frac{n-2}{2}, \alpha + \frac{n}{2} + 1\right) \end{aligned}$$

by letting $s = \rho^2$. Here, $S_{D_{n-2}}$ is the surface area of the $n-2$ dimensional unit ball and $B\left(\frac{n-2}{2}, \alpha + \frac{n}{2} + 1\right)$ is the beta function. Let $k_{\alpha,n} = \frac{S_{D_{n-2}}}{2} B\left(\frac{n-2}{2}, \alpha + \frac{n}{2} + 1\right)$. For $u_1 \in \mathbb{R}$, $\sinh u_1 = -\sinh(-u_1)$, we then have

$$K_1(iy) = k_{\alpha,n} \int_0^\infty \int_{-\infty}^\infty e^{-(2\pi y_n u_n \cosh u_1 - 2\pi |y'| u_n \sinh u_1)} u_n^{2\alpha+2n-1} du_1 du_n.$$

Let $2\pi y_n u_n \cosh u_1 - 2\pi |y'| u_n \sinh u_1 = s$, then

$$\begin{aligned} K_1(iy) &= k_{\alpha,n} \int_{-\infty}^\infty \frac{1}{(2\pi y_n \cosh u_1 - 2\pi |y'| \sinh u_1)^{2\alpha+2n}} \left(\int_0^\infty e^{-s} s^{2\alpha+2n-1} ds \right) du_1 \\ &= \frac{k_{\alpha,n} \Gamma(2\alpha+2n)}{(2\pi)^{2\alpha+2n}} \int_{-\infty}^\infty \frac{1}{(y_n \cosh u_1 - |y'| \sinh u_1)^{2\alpha+2n}} du_1 \end{aligned}$$

$$= \frac{k_{\alpha,n}\Gamma(2\alpha+2n)}{(2\pi)^{2\alpha+2n}} \int_{-\infty}^{\infty} \frac{1}{\left(\frac{y_n \cosh u_1 - |y'| \sinh u_1}{\sqrt{y_n^2-|y'|^2}}\right)^{2\alpha+2n} (y_n^2-|y'|^2)^{\alpha+n}} du_1.$$

Put $\frac{y_n}{\sqrt{y_n^2-|y'|^2}} = \cosh t_0$ and $\frac{|y'|}{\sqrt{y_n^2-|y'|^2}} = \sinh t_0$, then

$$\begin{aligned} K_1(iy) &= \frac{k_{\alpha,n}\Gamma(2\alpha+2n)}{(2\pi)^{2\alpha+2n}(y_n^2-|y'|^2)^{\alpha+n}} \int_{-\infty}^{\infty} \frac{1}{(\cosh t_0 \cosh u_1 - \sinh t_0 \sinh u_1)^{2\alpha+2n}} du_1 \\ &= \frac{k_{\alpha,n}\Gamma(2\alpha+2n)}{(2\pi)^{2\alpha+2n}(y_n^2-|y'|^2)^{\alpha+n}} \int_{-\infty}^{\infty} \frac{1}{(\cosh(u_1-t_0))^{2\alpha+2n}} du_1. \end{aligned}$$

Set $u_1 - t_0 = t$, $(\cosh t)^2 = s$ and $\frac{1}{s} = s'$, we have

$$\begin{aligned} K_1(iy) &= \frac{k_{\alpha,n}\Gamma(2\alpha+2n)}{(2\pi)^{2\alpha+2n}(y_n^2-|y'|^2)^{\alpha+n}} \int_{-\infty}^{\infty} \frac{1}{(\cosh t)^{2\alpha+2n}} dt \\ &= \frac{2k_{\alpha,n}\Gamma(2\alpha+2n)}{(2\pi)^{2\alpha+2n}(y_n^2-|y'|^2)^{\alpha+n}} \int_{0}^{\infty} \frac{1}{2s^{\alpha+n}\sqrt{s-1}\sqrt{s}} ds \\ &= \frac{k_{\alpha,n}\Gamma(2\alpha+2n)}{(2\pi)^{2\alpha+2n}(y_n^2-|y'|^2)^{\alpha+n}} \int_{0}^{\infty} \frac{1}{s^{\alpha+n+1}\sqrt{1-\frac{1}{s}}} ds \\ &= \frac{k_{\alpha,n}\Gamma(2\alpha+2n)}{(2\pi)^{2\alpha+2n}(y_n^2-|y'|^2)^{\alpha+n}} \int_{0}^{1} (s')^{\alpha+n+1}(1-s')^{-\frac{1}{2}}(s')^{-2} ds' \\ &= \frac{k_{\alpha,n}\Gamma(2\alpha+2n)}{(2\pi)^{2\alpha+2n}(y_n^2-|y'|^2)^{\alpha+n}} B\left(\alpha+n, \frac{1}{2}\right). \end{aligned}$$

Therefore,

$$K(iy) = \frac{4^\alpha \Gamma(\alpha+\frac{n}{2}+1)\Gamma(2\alpha+2n)\Gamma(\alpha+\frac{n}{2}+\frac{1}{2})}{\pi^n \Gamma(\alpha+1)\Gamma(2\alpha+n)\Gamma(\alpha+n+\frac{1}{2})} (\Delta(y))^{-\alpha-n}.$$

On the other hand, if we let $P(z) = z_1^2 + \cdots + z_{n-1}^2 - z_n^2$, then $P(z) \in \mathbb{C} \setminus (-\infty, 0]$ for $z \in T_{\Lambda_n}$. Indeed,

$$\begin{aligned} P(z) &= z_1^2 + \cdots + z_{n-1}^2 - z_n^2 \\ &= (x_1 + iy_1)^2 + \cdots + (x_{n-1} + iy_{n-1})^2 - z_n^2 \\ &= (x_1^2 - y_1^2) + \cdots + (x_{n-1}^2 - y_{n-1}^2) - (x_n^2 - y_n^2) + 2i(x_1 y_1 + \cdots + x_{n-1} y_{n-1} - x_n y_n) \\ &= \left(\sum_{k=1}^{n-1} x_k^2 - x_n^2\right) + \left(y_n^2 - \sum_{k=1}^{n-1} y_k^2\right) + 2i\left(\sum_{k=1}^{n-1} x_k y_k - x_n y_n\right). \end{aligned}$$

It follows from $\sum_{k=1}^{n-1} x_k y_k = x_n y_n$ that $\sum_{k=1}^{n-1} x_k \frac{y_k}{y_n} = x_n$, then $x_n^2 = (\sum_{k=1}^{n-1} x_k \frac{y_k}{y_n})^2$. Hölder's inequality implies that

$$x_n^2 \leq \left(\sum_{k=1}^{n-1} x_k^2\right)\left(\sum_{k=1}^{n-1} \frac{y_k^2}{y_n^2}\right) < \sum_{k=1}^{n-1} x_k^2.$$

Hence, $P(z) \in \mathbb{C} \setminus (-\infty, 0]$ for $z \in T_B$. Therefore, $(P(z))^{-\alpha-n}$ is well defined for $z \in T_{\Lambda_n}$ and $P(iy) = -y_1^2 - y_2^2 - \cdots - y_{n-1}^2 + y_n^2 = \Delta(y)$ for $y \in \Lambda_n$ and $P(iy)^\alpha = \rho(iy)$ is a weight function on T_{Λ_n}. Then

$$K(iy) = \frac{4^\alpha \Gamma(\alpha + \frac{n}{2} + 1)\Gamma(2\alpha + 2n)\Gamma(\alpha + \frac{n}{2} + \frac{1}{2})}{\pi^n \Gamma(\alpha + 1)\Gamma(2\alpha + n)\Gamma(\alpha + n + \frac{1}{2})} P(iy)^{-\alpha-n}.$$

For $z = x + iy \in T_{\Lambda_n}$, $K(x + iy)$ admits the Taylor expansion formula

$$K(x + iy) = \sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} \frac{K^{(k_1+\cdots+k_n)}(iy)}{k_1! \cdots k_n!} x_1^{k_1} \cdots x_n^{k_n},$$

then

$$K(z) = \frac{4^\alpha \Gamma(\alpha + \frac{n}{2} + 1)\Gamma(2\alpha + 2n)\Gamma(\alpha + \frac{n}{2} + \frac{1}{2})}{\pi^n \Gamma(\alpha + 1)\Gamma(2\alpha + n)\Gamma(\alpha + n + \frac{1}{2})} P(z)^{-\alpha-n}.$$

As a result, for $z, w \in T_{\Lambda_n}$,

$$K_{\alpha,\Lambda_n}(z, w) = K(z - \bar{w}) = C_{4,\alpha} P(z - \bar{w})^{-\alpha-n}.$$

where $C_{4,\alpha}$ is defined by (23). This proves (22). □

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