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COMPLEX VARIABLES AND ELLIPTIC EQUATIONS
https://doi.org/10.1080/17476933.2021.2007377

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ABSTRACT

We study a weak greedy type algorithm called the weak pre-orthogonal adaptive Fourier decomposition (WPOAFD) for the Bergman space $A^2(\Omega)$ on a bounded pseudoconvex domain Ω with smooth boundary. We show that any function $f \in A^2(\Omega)$ can be approximated by linear combinations of kernel functions in weak greedy sense.

ARTICLE HISTORY
Received 1 May 2021
Accepted 9 November 2021

COMMUNICATED BY
S. Berhanu

KEYWORDS
Bergman space; Bergman kernel; pseudoconvex domain; weak pre-orthogonal adaptive Fourier decomposition; boundary vanishing property; weak maximal selection principle

AMS SUBJECT CLASSIFICATIONS
32A25; 32A36; 32T15; 46E20; 46E22

1. Introduction

Let Ω be a bounded pseudoconvex domain with smooth boundary in $\mathbb{C}^n$. The Bergman space $A^2(\Omega)$ on Ω is the space of square-integrable holomorphic functions on Ω with the inner product

$$\langle f, g\rangle = \int_\Omega f\,\bar{g}\,\mathrm{d}V,$$

where $\mathrm{d}V$ is the Lebesgue measure on $\mathbb{C}^n$. Let $K(z,w)$ be the Bergman kernel of Ω. Then $A^2(\Omega)$ with $K(z,w)$ is a reproducing kernel Hilbert space (RKHS) [1]. For any $f \in A^2(\Omega)$ and any $a \in \Omega$, the reproducing property holds:

$$f(a) = \langle f(\cdot), K(\cdot, a)\rangle = \int_\Omega f(z)\overline{K(z,a)}\,\mathrm{d}V(z).$$

It is well known that $K(z,w) = \sum_{i=1}^{\infty} \phi_i(z)\overline{\phi_i(w)}$, where $\{\phi_i\}_{i\geq 1}$ is any orthonormal basis of $A^2(\Omega)$ [1].

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For any fixed $a \in \Omega$, define $k_a(z) = K(z, a)$ to be the kernel function associated to a. Note that

$$\|k_a\|^2 = K(a, a) = \sum_{i \geq 1} |\phi_i(a)|^2 \geq \frac{1}{|\Omega|}. \tag{1}$$

In fact, inequality (1) holds as one can choose ϕ_1 to be the constant function

$$\phi_1(z) = \frac{1}{\sqrt{|\Omega|}}$$

for all $z \in \Omega$.

By the reproducing property, the span of the set $\{k_a \mid a \in \Omega\}$ of kernel functions is dense in $A^2(\Omega)$. Hence, for any countable dense set $\{a_i\}_{i \geq 1}$ in Ω, the corresponding sequence $\{k_{a_i}\}_{i \geq 1}$ forms a complete system in $A^2(\Omega)$. Furthermore, we have the following simple observation.

Lemma 1.1: *The set $\{k_a\}_{a \in \Omega}$ of all Bergman kernel functions is a linearly independent set if Ω is bounded.*

Proof: Suppose to the contrary that there exist finitely many distinct points $a_1, \ldots, a_m \in \Omega \subset \mathbb{C}^n$ such that $\{k_{a_1}, \ldots, k_{a_m}\}$ in $A^2(\Omega)$ is linearly dependent. So there exists a non-zero vector $(c_1, c_2, \ldots, c_m) \in \mathbb{C}^m$ such that in $A^2(\Omega)$, we have

$$\sum_{j=1}^{m} c_j k_{a_j} = 0. \tag{2}$$

For any fixed vectors $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{C}^n$, define $f_\lambda \in A^2(\Omega)$ by $f_\lambda(z) = e^{t\langle z, \lambda \rangle} = \exp(t \sum_{i=1}^{n} z_i \overline{\lambda_i})$. As the sequence $(a_1, \ldots, a_m)$ consists of finitely many distinct points in $\mathbb{C}^n$, there exists a non-empty open subset U in $\mathbb{C}^n$ such that for any $\lambda \in U$, the complex numbers $\langle a_i, \lambda \rangle$ are distinct for $1 \leq i \leq m$. It follows from (2) and the reproducing property of the kernel functions k_{a_i} that

$$0 = \langle f_\lambda, 0 \rangle = \left\langle f_\lambda, \sum_{j=1}^{m} \overline{c_j} k_{a_j} \right\rangle = \sum_{j=1}^{m} c_j f_\lambda(a_j) = \sum_{j=1}^{m} c_j \exp(t \langle a_j, \lambda \rangle).$$

For any fixed $\lambda \in U$, the set $\{\exp(t\langle a_i, \lambda \rangle) \mid 1 \leq i \leq m\}$ of functions in t is linearly independent. This implies that all the coefficients c_j are zero, which contradicts to the non-zero assumption of the vector $(c_i)_{1 \leq i \leq m}$, and so the result follows. ■

By Lemma 1.1 and inequality (1), we can apply the Gram–Schmidt orthonormalization process to $\{k_{a_i}\}_{i \geq 1}$ associated to a dense sequence of distinct points in Ω, to obtain a complete orthonormal system $\{\mathscr{B}_i\}_{i \geq 1}$, where $\mathscr{B}_1 = \frac{k_{a_1}}{\|k_{a_1}\|}$ and

$$\mathscr{B}_i = \frac{k_{a_i} - \sum_{j=1}^{i-1} \langle k_{a_i}, \mathscr{B}_j \rangle \mathscr{B}_j}{\|k_{a_i} - \sum_{j=1}^{i-1} \langle k_{a_i}, \mathscr{B}_j \rangle \mathscr{B}_j\|} \tag{3}$$

for all $i \geq 2$. Then, for any $f \in A^2(\Omega)$, we have

$$\lim_{N\to\infty} \left\| \sum_{1\leq i\leq N} \langle f, \mathscr{B}_i \rangle \mathscr{B}_i - f \right\| = 0. \tag{4}$$

In general, if the given fixed sequence $\{a_i\}_{i\geq 1}$ of points in Ω is not dense, the corresponding set $\{k_{a_i}\}_{i\geq 1}$ may not form a complete system in Ω, i.e. $\overline{\text{span}\{k_{a_i}\}_{i\geq 1}} \neq A^2(\Omega)$. Nevertheless, for any given $f \in A^2(\Omega)$, instead of considering a fixed set of parameter points, due to the flexibility of selecting parameter points a_i's of k_{a_i}'s with respect to the function f, one can try to select a_i successively to produce an orthonormal sequence $(\mathscr{B}_i)_{i\geq 1}$ as above so that the values $|\langle f, \mathscr{B}_i \rangle|$ are as large as possible out of the sum in (4). This is a starting point of an adaptive Fourier decomposition [2–6].

In this paper, we apply a recently developed greedy type algorithm called the weak pre-orthogonal adaptive Fourier decomposition (WPOAFD) to any given function $f \in A^2(\Omega)$. Our main result is as follows.

Theorem 1.1 (Weak Maximal Selection Principle): *For any $f \in A^2(\Omega)$ and any sequence $\{\rho_i \,|\, 0 < \rho_0 \leq \rho_i < 1,\ i = 1, 2, \dots\}$, there exists a sequence $(a_i)_{i\geq 1}$ of distinct points in Ω such that the orthonormal sequence $(\mathscr{B}_i)_{i\geq 1}$ of functions in $A^2(\Omega)$ obtained by applying Gram–Schmidt orthonormalization process to the sequence $(k_{a_i})_{i\geq 1}$ of kernel functions satisfies the inequality for all $i \geq 1$:*

$$|\langle f, \mathscr{B}_i \rangle| \geq \rho_i \sup\{|\langle f, \mathscr{B}_i^b \rangle| \,|\, b \in \Omega \setminus \{a_1, a_2, \dots, a_{i-1}\}\}, \tag{5}$$

where $\mathscr{B}_i^b = \frac{k_b - \sum_{j=1}^{i-1} \langle k_b, \mathscr{B}_j \rangle \mathscr{B}_j}{\|k_b - \sum_{j=1}^{i-1} \langle k_b, \mathscr{B}_j \rangle \mathscr{B}_j\|}$.

It follows from our main Theorem 1.1 that for any $f \in A^2(\Omega)$, we can construct an orthonormal sequence $(\mathscr{B}_i)_{i\geq 1}$.

Theorem 1.2 (Convergence of WPOAFD): *With the notations stated above, the sequence $\sum_{i=1}^{n} \langle f, \mathscr{B}_i \rangle \mathscr{B}_i$ converges to f in $A^2(\Omega)$, i.e.*

$$f = \sum_{i\geq 1} \langle f, \mathscr{B}_i \rangle \mathscr{B}_i \quad \text{in } A^2(\Omega). \tag{6}$$

The right hand side of Equation (6) is called a *weak pre-orthogonal adaptive Fourier decomposition* (WPOAFD) of f.

Remark 1.1: In the case of the Bergman spaces of classical bounded symmetric domains, our result of WPOAFD can be strengthened to POAFD one in which the inequality in (5) is replaced by equality with all the $\rho_i = 1$ [7]. In fact, this is one of the motivations of this work. In the case of POAFD [3,4,8–11], we need to introduce the class of generalized kernel functions by passing to limit functions when two consecutive points a_n and a_{n+1} coincide.

The rest of this paper is structured as follows. In Section 2, we show that $A^2(\Omega)$ satisfies an important property called boundary vanishing property (BVP) so that WPOAFD can be implemented in $A^2(\Omega)$. In Section 3, we give a proof of Theorem 1.1. In Section 4, we prove the convergence of WPOAFD in Theorem 1.2.

2. Boundary vanishing property (BVP) of $A^2(\Omega)$

We first recall the definition of bounded pseudoconvex domains with smooth boundary [12]:

Definition 2.1: Let $\Omega = \{z \in \mathbb{C}^n \mid \psi(z) < 0\}$ be a bounded domain in $\mathbb{C}^n$, where ψ is a real-valued function on $\mathbb{C}^n$. The boundary $\partial\Omega = \{z \in \mathbb{C}^n \mid \psi(z) = 0\}$ of Ω is called *smooth* if ψ is a real-valued smooth function on $\mathbb{C}^n$ with $(\frac{\partial\psi}{\partial z_1}, \ldots, \frac{\partial\psi}{\partial z_n}) \neq 0$ on $\partial\Omega$. A bounded domain Ω with smooth boundary $\partial\Omega$ is called *pseudoconvex* if for any $p \in \partial\Omega$ and any $w = (w_1, \ldots, w_n) \in \mathbb{C}^n$ with $\sum_{j=1}^n \frac{\partial\psi}{\partial z_j}(p)\, w_j = 0$, we have

$$\sum_{j,k=1}^{n} \frac{\partial^2\psi}{\partial z_j \partial \bar{z}_k}(p) w_j \bar{w}_k \geq 0.$$

Let $A^\infty(\Omega)$ be the set of all holomorphic functions on Ω which are smooth up to the boundary of Ω, and $A^2(\Omega)$ be the Bergman space on Ω. In our study for BVP of $A^2(\Omega)$, we need a growth estimate of $K(z,z)$ for z near a boundary point.

Theorem 2.1 ([13]): *Let K be the Bergman kernel of $A^2(\Omega)$ and $b \in \partial\Omega$. Then there exists a constant $C > 0$ and a neighborhood U of b in $\mathbb{C}^n$ such that*

$$K(a,a) \geq \frac{C}{d^2(a)} \quad \text{for all } a \in U \cap \Omega,$$

where $d(a)$ is the distance from a to $\partial\Omega$.

Lemma 2.1: *For any $f \in A^2(\Omega)$ and any $a \in \Omega$, define*

$$e_a = \frac{k_a}{\|k_a\|}, \quad H(a) = k_a \in A^2(\Omega), \quad \text{and} \quad G(a) = |\langle f, e_a\rangle|.$$

Then H and G are continuous on Ω.

Proof: We first show that the function $H(a) = k_a$ is continuous from Ω to the Hilbert space $A^2(\Omega)$ with the norm $\|\cdot\|$. In fact, it follows from the continuity of K on $\Omega \times \Omega$ [1, Proposition 1.1.7.] that

$$\begin{aligned}\|k_b - k_a\|^2 &= \|k_b\|^2 + \|k_a\|^2 - \langle k_b, k_a\rangle - \langle k_a, k_b\rangle \\ &= K(b,b) + K(a,a) - K(a,b) - K(b,a)\end{aligned}$$

converges to 0 as b tends to $a \in \Omega$. This also implies that $\lim_{b\to a} \|k_b\| = \|k_a\|$ and hence $\lim_{b\to a} e_b = \lim_{b\to a} \frac{k_b}{\|k_b\|} = \frac{k_a}{\|k_a\|} = e_a$. Then the continuity of G follows from $\lim_{b\to a} e_b = e_a$ and $|G(b) - G(a)| \leq |\langle f, e_b - e_a\rangle| \leq \|f\|\, \|e_b - e_a\|$. ■

Proposition 2.1: (i) *(Boundary Vanishing Property) For any $f \in A^2(\Omega)$ and any $b_0 \in \partial\Omega$, we have* $\lim_{b\to b_0} |\langle f, e_b\rangle| = 0$.

(ii) *(Maximal Modulus Coefficient Property) For any $f \in A^2(\Omega)$, there exists $a \in \Omega$ such that* $|\langle f, e_a\rangle| = \max\{|\langle f, e_b\rangle| \mid b \in \Omega\}$.

Proof: (i) For any $\epsilon > 0$ and any $f \in A^2(\Omega)$, if follows from [14, Lemma 3.1.4.] that there exists $g \in A^\infty(\Omega)$ such that

$$\|f - g\| < \frac{\epsilon}{2}. \tag{7}$$

Since $g \in A^\infty(\Omega)$, there exists $M > 0$ such that

$$|g(b)| \leq M \quad \text{for all } b \in \bar{\Omega}. \tag{8}$$

By Theorem 2.1, there exists a neighborhood U of b_0 such that

$$\frac{1}{\sqrt{K(b,b)}} \leq \frac{\epsilon}{2M} \quad \text{for all } b \in U \cap \Omega. \tag{9}$$

By reproducing property of k_b, the normalized kernel function e_b at $b \in \Omega$ is given by

$$e_b = \frac{k_b}{\|k_b\|} = \frac{k_b}{\sqrt{\langle k_b, k_b\rangle}} = \frac{k_b}{\sqrt{k_b(b)}} = \frac{k_b}{\sqrt{K(b,b)}}.$$

Since $\|e_b\| = 1$, the triangle inequality, the Cauchy–Schwarz inequality, the reproducing property of k_b and the inequalities (7), (8), (9) give

$$\begin{aligned} |\langle f, e_b\rangle| &\leq |\langle f - g, e_b\rangle| + |\langle g, e_b\rangle| \leq \|f - g\| + \frac{|g(b)|}{\sqrt{K(b,b)}} \\ &\leq \frac{\epsilon}{2} + M \cdot \frac{\epsilon}{2M} < \epsilon. \end{aligned}$$

(ii) By Lemma 2.1, $G(b) = |\langle f, e_b\rangle|$ is continuous on Ω. By (i), $G(b)$ can be continuously extended to the bounded and closed subset $\bar{\Omega}$ in $\mathbb{C}^n$ and $G(b) = 0$ for all $b \in \partial\Omega$. Then $L = \max\{G(b) \mid b \in \bar{\Omega}\}$ is finite.

If $L = 0$, then $0 \leq |\langle f, e_b\rangle| \leq L = 0$ for all $b \in \Omega$. By reproducing property of k_b, $f = 0$ on Ω. In this case, choose a to be any point in Ω.

If $L > 0$, then it follows from (i) that the maximum L can only be attained at some interior point of Ω. Therefore, there exists $a \in \Omega$ such that $L = G(a) = |\langle f, e_a\rangle|$. ■

Define $f_j \in A^2(\Omega)$ be the jth residual function of f as follows:

$$f_1 = f, \quad \text{and} \quad f_j = f_{j-1} - \langle f_{j-1}, \mathscr{B}_{j-1}\rangle \mathscr{B}_{j-1} \tag{10}$$

for $2 \leq j \leq n + 1$. We can deduce the following.

Proposition 2.2: *Let $f \in A^2(\Omega)$, $\{\mathscr{B}_1, \ldots, \mathscr{B}_n\}$ be an orthonormal set in $A^2(\Omega)$ and $(f_j)_{1\leq j\leq n+1}$ be defined in (10). Then we have*

(i) $\langle f_j, \mathscr{B}_{j-1}\rangle = \langle f_j, \mathscr{B}_{j-2}\rangle = \cdots = \langle f_j, \mathscr{B}_1\rangle = 0$;
(ii) $\langle f_j, \mathscr{B}_j\rangle = \langle f_{j-1}, \mathscr{B}_j\rangle = \cdots = \langle f_2, \mathscr{B}_j\rangle = \langle f_1, \mathscr{B}_j\rangle = \langle f, \mathscr{B}_j\rangle$ *for* $1 \leq j \leq n$;
(iii) $f_j = f_\ell - \sum_{i=\ell}^{j-1}\langle f, \mathscr{B}_i\rangle \mathscr{B}_i = f - \sum_{i=1}^{j-1}\langle f, \mathscr{B}_i\rangle \mathscr{B}_i$ *for* $1 \leq \ell < j \leq n + 1$.

Proof: (i) Using $f_j = f_{j-1} - \langle f_{j-1}, \mathscr{B}_{j-1}\rangle \mathscr{B}_{j-1}$ and $\langle \mathscr{B}_i, \mathscr{B}_j\rangle = \delta_{ij}$, where δ_{ij} is the Kronecker delta, we have

$$\begin{aligned}\langle f_j, \mathscr{B}_{j-1}\rangle &= \langle f_{j-1} - \langle f_{j-1}, \mathscr{B}_{j-1}\rangle \mathscr{B}_{j-1}, \mathscr{B}_{j-1}\rangle \\ &= \langle f_{j-1}, \mathscr{B}_{j-1}\rangle - \langle f_{j-1}, \mathscr{B}_{j-1}\rangle \|\mathscr{B}_{j-1}\|^2 = 0,\end{aligned}$$

the others follow similarly.

(ii) The result follows from $\langle f_{i+1}, \mathscr{B}_j\rangle = \langle f_i - \langle f_i, \mathscr{B}_i\rangle \mathscr{B}_i, \mathscr{B}_j\rangle = \langle f_i, \mathscr{B}_j\rangle$ if $i < j$.

(iii) The result follows from (ii) and $f_j = f_{j-1} - \langle f_{j-1}, \mathscr{B}_{j-1}\rangle \mathscr{B}_{j-1} = f_{j-1} - \langle f, \mathscr{B}_{j-1}\rangle \mathscr{B}_{j-1}$. ■

Definition 2.2: Let $\Omega_n = \Omega \setminus \{a_1, \dots, a_n\}$ be the punctured domain. Define an objective function g_{n+1} on Ω_n as follows: $g_{n+1}(b) = |\langle f_{n+1}, \mathscr{B}^b_{n+1}\rangle|$ for any $b \in \Omega_n$, where $\mathscr{B}^b_{n+1} = \frac{k_b - \sum_{j=1}^n \langle k_b, \mathscr{B}_j\rangle \mathscr{B}_j}{\|k_b - \sum_{j=1}^n \langle k_b, \mathscr{B}_j\rangle \mathscr{B}_j\|}$.

It follows from Proposition 2.2(i) and Cauchy–Schwarz inequality that $\langle f_{n+1}, \mathscr{B}_j\rangle = 0$,

$$\begin{aligned}g_{n+1}(b) = |\langle f_{n+1}, \mathscr{B}^b_{n+1}\rangle| &= \left|\left\langle f_{n+1}, \frac{k_b - \sum_{j=1}^n \langle k_b, \mathscr{B}_j\rangle \mathscr{B}_j}{\|k_b - \sum_{j=1}^n \langle k_b, \mathscr{B}_j\rangle \mathscr{B}_j\|}\right\rangle\right| \\ &= \frac{|\langle f_{n+1}, k_b\rangle|}{\|k_b - \sum_{j=1}^n \langle k_b, \mathscr{B}_j\rangle \mathscr{B}_j\|} = \frac{|\langle f_{n+1}, e_b\rangle|}{\|e_b - \sum_{j=1}^n \langle e_b, \mathscr{B}_j\rangle \mathscr{B}_j\|} \\ &= \frac{|\langle f_{n+1}, e_b\rangle|}{\sqrt{1 - \sum_{j=1}^n |\langle e_b, \mathscr{B}_j\rangle|^2}} \end{aligned} \tag{11}$$

and

$$g_{n+1}(b) = |\langle f_{n+1}, \mathscr{B}^b_{n+1}\rangle| \le \|f_{n+1}\| \, \|\mathscr{B}^b_{n+1}\| \le \|f_{n+1}\|$$

for all $b \in \Omega_n$.

For any $b_0 \in \partial\Omega$, the BVP in Proposition 2.1(i) implies that $\lim_{b\to b_0} |\langle f_{n+1}, e_b\rangle| = 0$ and $\lim_{b\to b_0} |\langle \mathscr{B}_j, e_b\rangle| = 0$ $(1 \le j \le n)$, hence $\lim_{b\to b_0} g_{n+1}(b) = 0$. Therefore, g_{n+1} can be extended to $\Omega_n \cup \partial\Omega$ continuously with $g_{n+1}(b) = 0$ for all $b \in \partial\Omega$.

As g_{n+1} is bounded on $\Omega_n \cup \partial\Omega$, the following supremum

$$S = \sup\{g_{n+1}(b) \in \mathbb{R} \mid b \in \Omega_n \cup \partial\Omega\} \tag{12}$$

is finite.

3. Weak maximal selection principle in $A^2(\Omega)$

We now present proof of Theorem 1.1.

Proof of Theorem 1.1.: The goal is to select a sequence $\{a_1, \dots, a_n\}$ of distinct points in Ω successively such that each modulus $|\langle f, \mathscr{B}_i\rangle|$ is large enough for $i = 1, 2, \dots, n$. More precisely, when $i = 1$, by Proposition 2.1, the first point $a_1 \in \Omega$ can always be chosen such that $\mathscr{B}_1 = \frac{k_{a_1}}{\|k_{a_1}\|}$ and

$$|\langle f, \mathscr{B}_1\rangle| = \left|\left\langle f, \frac{k_{a_1}}{\|k_{a_1}\|}\right\rangle\right| = \sup\left\{\left|\left\langle f, \frac{k_b}{\|k_b\|}\right\rangle\right| \,|\, b \in \Omega\right\}. \tag{13}$$

If the supremum in (13) is zero, then the decomposition terminates.

For the other points a_i $(i \geq 2)$ in Ω, we establish their existence inductively as follows. Suppose that the previous n distinct points $a_1, \ldots, a_n$ are chosen and $\mathscr{B}_i = \mathscr{B}_i^{a_i}$ such that

$$|\langle f_i, \mathscr{B}_i \rangle| \geq \rho_i \sup\{|\langle f_i, \mathscr{B}_i^b \rangle| \mid b \in \Omega \setminus \{a_1, a_2, \ldots, a_{i-1}\}\}, \quad (14)$$

where f_i is the ith residual function defined in (10), $0 < \rho_0 < \rho_i < 1$ and $i = 1, \ldots, n$. If any one of the residual functions f_i is zero, then the decomposition terminates.

Otherwise, for any ρ_{n+1} with $0 < \rho_0 < \rho_{n+1} < 1$, by the definition of supremum S in (12), there exists $a_{n+1} \in \Omega \setminus \{a_1, \ldots, a_n\}$ such that

$$|\langle f_{n+1}, \mathscr{B}_{n+1} \rangle| \geq \rho_{n+1} \sup\{|\langle f_{n+1}, \mathscr{B}_{n+1}^b \rangle| \mid b \in \Omega \setminus \{a_1, \ldots, a_n\}\}, \quad (15)$$

where $\mathscr{B}_{n+1} = \frac{k_{a_{n+1}} - \sum_{j=1}^n \langle k_{a_{n+1}}, \mathscr{B}_j \rangle \mathscr{B}_j}{\|k_{a_{n+1}} - \sum_{j=1}^n \langle k_{a_{n+1}}, \mathscr{B}_j \rangle \mathscr{B}_j\|}$ and $\mathscr{B}_{n+1}^b = \frac{k_b - \sum_{j=1}^n \langle k_b, \mathscr{B}_j \rangle \mathscr{B}_j}{\|k_b - \sum_{j=1}^n \langle k_b, \mathscr{B}_j \rangle \mathscr{B}_j\|}$.

By constructions, we have $\langle \mathscr{B}_{n+1}, \mathscr{B}_i \rangle = 0$ and $\langle \mathscr{B}_{n+1}^b, \mathscr{B}_i \rangle = 0$ for each $i = 1, \ldots, n$. Then it follows from Proposition 2.2(ii) that $\langle f_{n+1}, \mathscr{B}_{n+1} \rangle = \langle f, \mathscr{B}_{n+1} \rangle$ and $\langle f_{n+1}, \mathscr{B}_{n+1}^b \rangle = \langle f, \mathscr{B}_{n+1}^b \rangle$. Hence, (14) and (15) are, respectively, equivalent to

$$|\langle f, \mathscr{B}_i \rangle| \geq \rho_i \sup\{|\langle f, \mathscr{B}_i^b \rangle| \mid b \in \Omega \setminus \{a_1, a_2, \ldots, a_{i-1}\}\}$$

and

$$|\langle f, \mathscr{B}_{n+1} \rangle| \geq \rho_{n+1} \sup\{|\langle f, \mathscr{B}_{n+1}^b \rangle| \mid b \in \Omega \setminus \{a_1, \ldots, a_n\}\}. \qquad \blacksquare$$

4. Convergence of WPOAFD in $A^2(\Omega)$

In the last section, we shall prove our second main result in Theorem 1.2, i.e. to show a WPOAFD of any function f in Bergman space $A^2(\Omega)$ converges to the same function f.

Definition 4.1: For any given $f \in A^2(\Omega)$ and any sequence $\{\rho_i\}_{i \geq 0}$ with $0 < \rho_0 \leq \rho_i < 1$ for all $i \geq 1$, a sequence $(a_i)_{i \geq 1}$ of distinct points in Ω is called a *weak maximal selection sequence of* f if there exists an orthonormal sequence $(\mathscr{B}_i)_{i \geq 1}$ in $A^2(\Omega)$ associated to a sequence $(a_i)_{i \geq 1}$ constructed as in Theorem 1.1.

For any given $f \in A^2(\Omega)$, we fix a weak maximal selection sequence $(a_i)_{i \geq 1}$ of f with an orthonormal sequence $(\mathscr{B}_i)_{i \geq 1}$ in $A^2(\Omega)$. Then one can extend the given $(\mathscr{B}_i)_{i \geq 1}$ to an orthonormal basis of $A^2(\Omega)$, so Bessel's inequality implies the following lemma.

Lemma 4.1: *Let $f \in A^2(\Omega)$. Then for any weak maximal selection sequence $(a_i)_{i \geq 1}$ of f, we have $\sum_{i \geq 1} \langle f, \mathscr{B}_i \rangle \mathscr{B}_i \in A^2(\Omega)$.*

Now, we prove the convergence result of WPOAFD in Theorem 1.2 for any weak maximal selection sequence as follows.

Proof of Theorem 1.2.: Suppose to the contrary that $(a_i)_{i \geq 1}$ is a sequence given by WPOAFD applied to f, and $(\mathscr{B}_i)_{i \geq 1}$ is the corresponding orthonormal sequence in $A^2(\Omega)$ such that the residual

$$h = f - \sum_{i \geq 1} \langle f, \mathscr{B}_i \rangle \mathscr{B}_i \in A^2(\Omega) \quad (16)$$

is non-zero. Then the sequence $(a_i)_{i\geq 1}$ is not finite in this case. We first prove that $\langle h, \mathscr{B}_i\rangle = 0$ for all $i \geq 1$. In fact,

$$\langle h, \mathscr{B}_k\rangle = \left\langle f - \sum_{i\geq 1}\langle f, \mathscr{B}_i\rangle\mathscr{B}_i, \mathscr{B}_k\right\rangle = \left\langle \lim_{N\to\infty}\left(f - \sum_{1\leq i\leq N}\langle f, \mathscr{B}_i\rangle\mathscr{B}_i\right), \mathscr{B}_k\right\rangle$$

$$= \lim_{N\to\infty}\left\langle f - \sum_{1\leq i\leq N}\langle f, \mathscr{B}_i\rangle\mathscr{B}_i, \mathscr{B}_k\right\rangle = \lim_{N\to\infty}(\langle f, \mathscr{B}_k\rangle - \langle f, \mathscr{B}_k\rangle\langle \mathscr{B}_k, \mathscr{B}_k\rangle) = 0.$$

As $h \neq 0$ and h is holomorphic on Ω, there exists a closed ball $\bar{B} \subset \Omega$ centered at some point b in Ω with positive radius such that $\bar{B}$ is compact and $|h(z)| > 0$ on $\bar{B}$. Then set

$$C_0 = \min_{x\in\bar{B}} \frac{|h(x)|}{K(x,x)} > 0.$$

Recall that f_N is the Nth residual of f in (10). We estimate $|\langle f_N, \mathscr{B}_N\rangle|$ in two different ways.

Firstly, Lemma 4.1 implies that $\sum_{i=1}^{\infty} |\langle f, \mathscr{B}_i\rangle|^2 < \infty$. As $C_0 > 0$, there exists a positive integer N_0 such that for all $N \geq N_0$, one has

$$\sum_{i=N}^{\infty} |\langle f, \mathscr{B}_i\rangle|^2 < \left(\frac{\rho_0 C_0}{2}\right)^2. \tag{17}$$

By Proposition 2.2(ii) and inequality (17), we have

$$|\langle f_N, \mathscr{B}_N\rangle| = |\langle f, \mathscr{B}_N\rangle| < \frac{\rho_0 C_0}{2}. \tag{18}$$

Secondly, for any fixed $N \geq N_0$, we select a point $b \in \bar{B}\backslash\{a_1, \ldots, a_N\}$. We consider another sequence $(a_1, \ldots, a_{N-1}, b) \in \Omega^N$. Let $(\mathscr{B}_1, \ldots, \mathscr{B}_{N-1}, \mathscr{B}_N^b)$ be the Gram–Schmidt orthonormalization of $(\mathscr{B}_1, \ldots, \mathscr{B}_{N-1}, k_b)$, where

$$\mathscr{B}_N^b = \frac{k_b - \sum_{i=1}^{N-1}\langle k_b, \mathscr{B}_i\rangle\mathscr{B}_i}{\|k_b - \sum_{i=1}^{N-1}\langle k_b, \mathscr{B}_i\rangle\mathscr{B}_i\|}. \tag{19}$$

Since $\mathscr{B}_N$ is selected according to the weak maximal modulus property (5) in Theorem 1.1, we have

$$\begin{aligned}|\langle f_N, \mathscr{B}_N\rangle| &\geq \rho_N \sup\{|\langle f_N, \mathscr{B}_N^z\rangle| \mid z \in \Omega \setminus \{a_1, a_2, \ldots, a_{N-1}\}\}\\ &\geq \rho_N|\langle f_N, \mathscr{B}_N^b\rangle| \geq \rho_0\,|\langle f_N, \mathscr{B}_N^b\rangle|.\end{aligned} \tag{20}$$

In order to arrive at a contradiction, consider $e_b = \frac{k_b}{\|k_b\|}$ for $b \in \bar{B}\backslash\{a_1, \ldots, a_N\}$ and note that

$$\|k_b - \sum_{i=1}^{N-1}\langle k_b, \mathscr{B}_i\rangle\mathscr{B}_i\|^2 = \|k_b\|^2 - \sum_{i=1}^{N-1}|\langle k_b, \mathscr{B}_i\rangle|^2 \leq \|k_b\|^2. \tag{21}$$

On the one hand, Proposition 2.2(i), (21), (19), (20) and (18) imply that

$$|\langle f_N, e_b\rangle| = \left|\left\langle f_N, \frac{k_b}{\|k_b\|}\right\rangle\right| = \frac{|\langle f_N, k_b - \sum_{i=1}^{N-1}\langle k_b, \mathscr{B}_i\rangle\mathscr{B}_i\rangle|}{\|k_b\|}$$

$$\leq \frac{|\langle f_N, k_b - \sum_{i=1}^{N-1}\langle k_b, \mathscr{B}_i\rangle \mathscr{B}_i\rangle|}{\|k_b - \sum_{i=1}^{N-1}\langle k_b, \mathscr{B}_i\rangle \mathscr{B}_i\|}$$
$$= |\langle f_N, \mathscr{B}_N^b\rangle| \leq \frac{1}{\rho_0}|\langle f_N, \mathscr{B}_N\rangle| < \frac{1}{\rho_0}\frac{\rho_0 C_0}{2} = \frac{C_0}{2}. \tag{22}$$

On the other hand, by (10) and (16),

$$f_N = f - \sum_{i=1}^{N-1}\langle f, \mathscr{B}_i\rangle \mathscr{B}_i = \sum_{i=N}^{\infty}\langle f, \mathscr{B}_i\rangle \mathscr{B}_i + h. \tag{23}$$

Then it follows from (23), triangle inequality, Cauchy–Schwarz inequality and (17) that

$$|\langle f_N, e_b\rangle| = \left|\left\langle h + \sum_{i=N}^{\infty}\langle f, \mathscr{B}_i\rangle \mathscr{B}_i, e_b\right\rangle\right| \geq \left|\frac{h(b)}{K(b,b)}\right| - \left|\left\langle \sum_{i=N}^{\infty}\langle f, \mathscr{B}_i\rangle \mathscr{B}_i, e_b\right\rangle\right|$$
$$\geq \min_{z\in\bar{B}}\left|\frac{h(z)}{K(z,z)}\right| - \left\|\sum_{i=N}^{\infty}\langle f, \mathscr{B}_i\rangle \mathscr{B}_i\right\| \|e_b\|$$
$$= C_0 - \sqrt{\sum_{i=N}^{\infty}|\langle f, \mathscr{B}_i\rangle|^2} > C_0 - \frac{\rho_0 C_0}{2} > \frac{C_0}{2},$$

which contradicts to (22).

Consequently, $h = 0$ and this completes the proof of Theorem 1.2. ∎

Acknowledgments

The authors would be grateful to the anonymous referees for their suggestions on studying pseudo-convex domains with smooth boundaries.

Disclosure statement

No potential conflict of interest was reported by the author(s).

Funding

This work was supported by University of Macau Multi-Year Research Grant Ref. No. MYRG2018-00168-FST, the Fundo para o Desenvolvimento das Ciências e da Tecnologia (Science and Technology Development Fund, Macau SAR) (File nos. FDCT/031/2016/A1, 0123/2018/A3).

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References

[1] Krantz SG. Geometric analysis of the Bergman kernel and metric. New York (NY): Springer; 2013. (Graduate texts in mathematics; 268).

[2] Alpay D, Colombo F, Qian T, Sabadini I. Adaptative decomposition: the case of the Drury–Arveson space. J Fourier Anal Appl. 2017;23(6):1426–1444.
[3] Qian T, Wang YB. Adaptive Fourier series – a variation of greedy algorithm. Adv Comput Math. 2011;34(3):279–293.
[4] Qian T. 一种新的基于非线性相位的 Fourier 理论及其应用 [A novel Fourier theory on non-linear phases and applications]. Adv Math (China). 2018;47(3):321–347 (Chinese).
[5] Baratchart L, Mai WX, Qian T. Greedy algorithms and rational approximation in one and several variables. In: Bernstein S, Kaehler U, Sabadini I, Sommen F, editors. Modern trends in hypercomplex analysis. New York (NY): Springer; 2016. p. 19–33. (Trends in mathematics).
[6] Qian T. Two-dimensional adaptive Fourier decomposition. Math Methods Appl Sci. 2016;39(10):2431–2448.
[7] Wu HT, Leong IT, Qian T. Adaptive rational approximation in Bergman space on bounded symmetric domain. J Math Anal Appl. 2021;506:125591. doi:10.1016/j.jmaa.2021.125591
[8] Qian T, Wang YB. Remarks on adaptive Fourier decomposition. Int J Wavelets Multiresolution Inf Process. 2013;11(1):1–14.
[9] Qian T. 自适应 Fourier 变换 [Adaptive Fourier transform]. Beijing: Science Press; 2015 (Chinese).
[10] Qian T. A novel Fourier theory on non-linear phases and applications. Preprint; 2018. Available from: arXiv:1805.06101.
[11] Qu W, Dang P. Rational approximation in a class of weighted Hardy spaces. Complex Anal Oper Theory. 2019;13:1827–1852.
[12] Hörmander L. An introduction to complex analysis in several variables. Amsterdam: North-Holland Publishing Co.; 1990. (North-Holland mathematical library; 7).
[13] Fu S. A sharp estimate on the Bergman kernel of a pseudoconvex domain. Proc Am Math Soc. 1994;121(3):979–980.
[14] Catlin D. Boundary behavior of holomorphic functions on pseudoconvex domains. J Differ Geom. 1980;15(4):605–625.