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Adaptive rational approximation in Bergman space on bounded symmetric domain

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ABSTRACT

We generalize the pre-orthogonal adaptive Fourier approximation developed by T. Qian et al. [9,7,10] to functions in the Bergman space on the unit disc and the unit ball to the Bergman space $A^2(\mathscr{D})$ on the irreducible bounded symmetric domain $\mathscr{D}$. We show that $A^2(\mathscr{D})$ satisfies the boundary vanishing property, so that the maximum selection principle allows us to give an adaptive expansion of any function $f \in A^2(\mathscr{D})$ in terms of linear combinations of generalized kernel functions in an optimal way.

1. Introduction

Let $\mathscr{D}$ be the subset of all $p \times q$ complex matrices z in $\mathbb{C}^{p\times q}$ such that $I_q - z^* z$ is positive definite, where $p \geq q$, and z^* is the conjugate transpose of z, so $\mathscr{D}$ is a bounded symmetric domain of type I. Let $A^2(\mathscr{D})$ be the Bergman space of all square integrable holomorphic functions on $\mathscr{D}$ with respect to Euclidean volume form dV in $\mathbb{C}^{p\times q}$,

$$A^2(\mathscr{D}) = \left\{ \; f \; \text{ is holomorphic on } \; \mathscr{D} \; \middle| \; \|f\|^2 = \int_{\mathscr{D}} |f|^2 \, dV < \infty \right\}.$$

Bergman space is a Hilbert space with inner product

$$\langle f, g\rangle = \int_{\mathscr{D}} f \, \overline{g} \, dV.$$

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L. K. Hua in [3] constructed the Bergman kernel: for any $z, w \in \mathscr{D}$,

$$K(z,w) = \frac{1}{|\mathscr{D}| \det(I_q - w^* z)^{p+q}} \in A^2(\mathscr{D})$$

and established the reproducing property $f(w) = \langle f, K(\cdot, w)\rangle$ for any $f \in A^2(\mathscr{D})$.

In particular, $A^2(\mathscr{D})$ is a reproducing kernel Hilbert space with positive definite kernel function $k_w(z) = K(z,w)$, and the linear span of all kernel functions $\{ k_w \mid w \in \mathscr{D} \}$ is dense in $A^2(\mathscr{D})$. In the other words, one can use a finite linear combination of k_{w_i} with $w_i \in \mathscr{D}$ to approximate any function in $A^2(\mathscr{D})$.

In the general setting of reproducing kernel Hilbert space (RKHS), the third author proposed POAFD scheme to approximate any element in underlying space with linear combination of kernel functions. We apply this proposal to Bergman space $A^2(\mathscr{D})$ setting in this paper, we outline the idea of POAFD below. More details can be found in our main Theorem 1.

Naively, for any given distinct points $a_1, a_2, \cdots, a_n$ of $\mathscr{D}$, the positive definite property of the reproducing kernel K of $A^2(\mathscr{D})$ implies that $\{k_{a_1}, k_{a_2}, \cdots, k_{a_n}\}$ is linearly independent in $A^2(\mathscr{D})$, one then applies Gram-Schmidt orthonormalization to obtain an orthonormal sequence $\{ \mathscr{B}_1, \ \mathscr{B}_2, \ \cdots, \ \mathscr{B}_n \}$ of $A^2(\mathscr{D})$ such that $\mathscr{B}_1 = k_{a_1}/\|k_{a_1}\|$ and

$$\mathscr{B}_m = \frac{k_{a_m} - \sum_{i=1}^{m-1} \langle k_{a_m}, \mathscr{B}_i\rangle \mathscr{B}_i}{\|k_{a_m} - \sum_{i=1}^{m-1} \langle k_{a_m}, \mathscr{B}_i\rangle \mathscr{B}_i\|} \tag{1}$$

for all $2 \le m \le n$. Then for any $f \in A^2(\mathscr{D})$, let g_n be the image of the orthogonal projection of f onto the $\operatorname{span}\{ \mathscr{B}_1, \cdots, \mathscr{B}_n \} = \operatorname{span}\{ k_{a_i} \mid i = 1, \cdots, n \}$,

$$g_n = \sum_{i=1}^{n} \langle f, \mathscr{B}_i\rangle \mathscr{B}_i \quad \text{and} \quad \|g_n\|^2 = \sum_{i=1}^{n} |\langle f, \mathscr{B}_i\rangle|^2. \tag{2}$$

In [4,8,5], T. Qian proposed the *maximal selection principle* to select an optimal sequence $\{a_1, \cdots, a_n\}$ of points in $\mathscr{D}$ successively such that the modulus $|\langle f, \mathscr{B}_i\rangle|$ of the coefficients in (2) is as large as possible for $i = 1, 2, \cdots, n$.

In particular, when $m = 1$, the first point $a_1 \in \mathscr{D}$ is chosen to satisfy the following

$$|\langle f, \frac{k_{a_1}}{\|k_{a_1}\|}\rangle| = \sup \left\{ |\langle f, \frac{k_b}{\|k_b\|}\rangle| \ \middle| \ b \in \mathscr{D} \right\}. \tag{3}$$

One can establish the existence of a_1 by means of boundary vanishing property (BVP) in Proposition 4 below. For the other points a_m $(2 \le m \le n)$ in $\mathscr{D}$, we establish their existence inductively by means of studying the following objective function g_m defined by

$$g_m(b) = |\langle \ f, \ \mathscr{B}_m^b \ \rangle|, \tag{4}$$

for all $b \in \mathscr{D} \setminus \{a_1, \cdots, a_{m-1}\}$, where $\mathscr{B}_m^b = \frac{k_b - \sum_{i=1}^{m-1} \langle k_b, \mathscr{B}_i\rangle \mathscr{B}_i}{\|k_b - \sum_{i=1}^{m-1} \langle k_b, \mathscr{B}_i\rangle \mathscr{B}_i\|}$.

With the help of BVP again, one can prove that g_m can be extended to a bounded continuous function on $\overline{\mathscr{D}}_m = \overline{\mathscr{D}} \setminus \{a_1, \cdots, a_{m-1}\}$. Then the $\sup\{ g(b) \mid b \in \overline{\mathscr{D}}_m \}$ can be realized by a limit $\lim\limits_{i\to\infty} g_m(b_i)$ where $\{b_i\}_{i\ge 1}$ is a sequence in $\overline{\mathscr{D}}_m$. By passing to its subsequence if necessary, we may assume that the sequence $\{b_i\}_{i\ge 1}$ converges to a point a in $\overline{\mathscr{D}}$. One can use BVP to rule out the case that a lies in the boundary of $\mathscr{D}$, so $a \in \mathscr{D}$. Then we have the following two cases.

In the case of $a \notin \{a_1, \cdots, a_{m-1}\}$, we set $a_m = a$ and $\mathscr{B}_m$ as in (1). However, the other case $a \in \{a_1, \cdots, a_{m-1}\}$ poses a serious problem, one can still set $a_m = a$, which implies that the m points

$a_1, a_2, \cdots, a_m$ are not distinct, so the set $\{k_{a_1}, \cdots, k_{a_m}\}$ is linearly dependent. For this reason, we follow the idea of [8,10] by introducing the following generalized kernels

$$k_{a,\alpha}(z) = \frac{\partial^{|\alpha|}}{\partial \overline{w}^{\alpha}} K(z,w)\Big|_{w=a}$$

for any multi-index $\alpha \in \mathbb{N}^{pq}$.

In this case, we set $a_m = a$. By analyzing the function g_m in more detail, we will prove in section 4 that there exist a positive integer ℓ and a sequence $(c_\alpha)_{|\alpha|=\ell}$ of complex numbers with all $\alpha \in \mathbb{N}^{pq}$ of order $|\alpha| = \ell$, such that

$$E = \sum_{|\alpha|=\ell} c_\alpha k_{a,\alpha} \neq \mathbf{0}, \quad \text{and} \quad \mathscr{B}_m = \frac{E - \sum_{i=1}^{m-1} \langle E, \mathscr{B}_i \rangle \mathscr{B}_i}{\| E - \sum_{i=1}^{m-1} \langle E, \mathscr{B}_i \rangle \mathscr{B}_i \|} \in A^2(\mathscr{D}).$$

In both cases, the following maximal modulus property holds:

$$|\langle f, \mathscr{B}_m \rangle| = \sup \left\{ \left| \left\langle f, \frac{k_b - \sum_{i=1}^{m-1} \langle k_b, \mathscr{B}_i \rangle \mathscr{B}_i}{\| k_b - \sum_{i=1}^{m-1} \langle k_b, \mathscr{B}_i \rangle \mathscr{B}_i \|} \right\rangle \right| \;\middle|\; b \in \mathscr{D} \setminus \{a_1, \cdots, a_{m-1}\} \right\}. \tag{5}$$

We can summarize all of these in the following.

Theorem 1. *(**Maximal selection principle**). Let $\mathscr{D}$ be a bounded symmetric domain of type I. For any $f \in A^2(\mathscr{D})$, there exist*

(i) a sequence $(a_i)_{i \geq 1}$ of points in $\mathscr{D}$,

(ii) a sequence $(k_{a_1,m_1}, k_{a_2,m_2}, \cdots, k_{a_i,m_i}, \cdots)$ of generalized kernel functions in $A^2(\mathscr{D})$, where $m_i \in \mathbb{N}$ and

$$k_{a_i,m_i}(z) = \sum_{|\alpha|=m_i} c_{i,\alpha} \frac{\partial^{|\alpha|} K}{\partial \overline{w}^{\alpha}}(z,w)\Big|_{w=a_i}$$

for some $c_{i,\alpha} \in \mathbb{C}$,

(iii) an orthonormal sequence $\{\mathscr{B}_1, \cdots, \mathscr{B}_i, \cdots\}$ of rational functions in $A^2(\mathscr{D})$ obtained by applying Gram-Schmidt orthonormalization process to the sequence in (ii), such that the following maximal selection principle holds:

$$|\langle f, \mathscr{B}_i \rangle| = \sup \left\{ |\langle f, \mathscr{B}_i^b \rangle| \;\middle|\; b \in \mathscr{D} \setminus \{a_1, a_2, \cdots, a_{i-1}\} \right\} \tag{6}$$

for all $i \geq 1$.

It follows from our main Theorem 1 that for any $f \in A^2(\mathscr{D})$, we can construct an orthonormal sequence $\{\mathscr{B}_i\}_{i \geq 1}$.

Theorem 2. *(**Convergence of POAFD**). With the notations stated above, the sequence $\sum_{i=1}^{n} \langle f, \mathscr{B}_i \rangle \mathscr{B}_i$ in $A^2(\mathscr{D})$ converges to f in $A^2(\mathscr{D})$, i.e.,*

$$f = \sum_{i \geq 1} \langle f, \mathscr{B}_i \rangle \mathscr{B}_i \qquad \text{in} \quad A^2(\mathscr{D}). \tag{7}$$

The right hand side of equation (7) is called a *pre-orthogonal adaptive Fourier decomposition* (POAFD) of f. Similar results could be found in [1], and the monograph [6] written in Chinese is still one of the best source of the recent development of adaptive Fourier decomposition (AFD), including some applications of AFD and its variants in signal processing.

The proof of Theorem 1 is quite long as it occupies most of the paper. The rest of this paper is structured as follows. In section 2, we review some basic facts of the irreducible bounded symmetric domain $\mathscr{D}$ of Type I, the Bergman space $A^2(\mathscr{D})$ and the Bergman kernel K. In section 3, we introduce generalized kernel functions $k_{a,\alpha}$ and proved that they form a linearly independent set. We show that $A^2(\mathscr{D})$ satisfies boundary vanishing property which is related to the maximal selection principle in Theorem 1. In section 4, we establish a maximal selection principle in $A^2(\mathscr{D})$ by finding an explicit limit $\mathscr{B}_i$ in $A^2(\mathscr{D})$ so that the optimal value of $|\langle f, \mathscr{B}_i^b\rangle|$ on $\mathscr{D}$ can be attained. In section 5, we give a proof of Theorem 2.

2. Bergman space $A^2(\mathscr{D})$

2.1. Notations

Let $\mathbb{N}$ be the set of all non-negative integers. Denote by $\mathbb{N}^{pq}$ the set of all multi-indices $\alpha = (\alpha_{1,1}, \ldots, \alpha_{p,q})$ of nonnegative integers. We use $\alpha \geq \alpha'$ to mean $\alpha_{i,j} \geq \alpha'_{i,j}$ for all $i = 1, \ldots, p$ and $j = 1, \ldots, q$. In particular, $\alpha \geq 0$ means all α_i's are nonnegative. We also write $|\alpha| = \alpha_{1,1} + \cdots + \alpha_{p,q}$, $\alpha! = \alpha_{1,1}! \cdots \alpha_{p,q}!$.

For any fixed positive integers $p \geq q$, let $\mathbb{C}^{p\times q}$ be the set of all complex $p \times q$ matrices $(a_{ij})_{1\leq i,j\leq q}$.

A Hermitian matrix $A = (A_{ij})_{1\leq i,j\leq q} \in \mathbb{C}^{q\times q}$ is called *positive semi-definite*, denoted by $A \geq 0$, if

$$\mathbf{z}^* A\mathbf{z} = \sum_{1\leq i,j\leq q} \overline{z_i}\, A_{ij} z_j \geq 0 \tag{8}$$

for any column vector $\mathbf{z} = (z_1, z_2, \cdots, z_q)^T \in \mathbb{C}^q$. Furthermore, A is called *positive definite*, denoted by $A > 0$, when the following condition holds:

$$\mathbf{z}^* A\mathbf{z} = 0 \text{ holds if and only if } \mathbf{z} = \mathbf{0} \text{ in } \mathbb{C}^q.$$

Let I_q be the $q \times q$ identity matrix. Let $\mathscr{D} = \{ z \in \mathbb{C}^{p\times q} \mid I_q - z^* z > 0 \}$ be an irreducible bounded symmetric domain $\mathscr{D}$ of Type I in the Helgason book [2]. This is a non-empty connected open subset in $\mathbb{C}^{p\times q}$. The condition $I_q - z^*z > 0$ implies that $\mathscr{D}$ is bounded in $\mathbb{C}^{p\times q}$, so it has a finite Euclidean volume, denoted by $|\mathscr{D}|$. Our basic reference is Hua's classical treatise [3]. In the following, we will use I to denote I_q. Let $\mathcal{O}(\mathscr{D})$ be the vector space of the holomorphic functions defined on the domain $\mathscr{D}$. Obviously, the coordinate functions $p_{i,j}(z) = z_{i,j}$ are holomorphic on $\mathscr{D}$.

We use $\mathbb{C}^{pq}$ to denote the usual pq-dimensional complex vector space. From now on, we use $\mathbb{C}^{pq}$ instead of $\mathbb{C}^{p\times q}$, and equip $\mathbb{C}^{pq}$ with standard Euclidean norm $\|\cdot\|$. A power series in $\mathbb{C}^{pq}$ with center $a \in \mathbb{C}^{pq}$ is of the form

$$\sum_{\alpha\geq 0} c_\alpha (z-a)^\alpha = \sum_{\alpha_{1,1}\geq 0,\ldots,\alpha_{p,q}\geq 0} c_{\alpha_{1,1}\ldots\alpha_{p,q}} (z_{1,1} - a_{1,1})^{\alpha_{1,1}} \cdots (z_{p,q} - a_{p,q})^{\alpha_{p,q}}, \tag{9}$$

where $c_{\alpha_{1,1}\ldots\alpha_{p,q}}$'s are complex numbers and

$$(z-a)^\alpha = (z_{1,1} - a_{1,1})^{\alpha_{1,1}} \cdots (z_{p,q} - a_{p,q})^{\alpha_{p,q}}.$$

For holomorphic and anti-holomorphic derivatives of functions on domain in $\mathbb{C}^{pq}$, we write

$$\frac{\partial^{|\alpha|}}{\partial z^\alpha} = \frac{\partial^{|\alpha|}}{\partial z_{1,1}^{\alpha_{1,1}} \cdots \partial z_{p,q}^{\alpha_{p,q}}} \quad \text{and} \quad \frac{\partial^{|\beta|}}{\partial \overline{z}^\beta} = \frac{\partial^{|\beta|}}{\partial \overline{z}_{1,1}^{\beta_{1,1}} \cdots \partial \overline{z}_{p,q}^{\beta_{p,q}}},$$

$$f^{(\alpha)}(z) = \frac{\partial^{|\alpha|} f}{\partial z^\alpha}(z) \quad \text{and} \quad f^{(\overline{\beta})}(z) = \frac{\partial^{|\beta|} f}{\partial \overline{z}^\beta}(z),$$

in which we use a bar on top of the order multi-index β for the anti-holomorphic derivatives of order β. Similarly, if $f(z,w)$ is defined on some open subset of $\mathbb{C}^{pq} \times \mathbb{C}^{pq}$, we denote

$$f^{(\alpha,\overline{\beta})}(z,w) = \frac{\partial^{|\alpha|+|\beta|} f}{\partial z^\alpha \partial \overline{w}^\beta}(z,w).$$

2.2. Reproducing kernel of $A^2(\mathscr{D})$

We first recall the definition of reproducing kernel Hilbert space [11].

Definition 1. Let $\mathcal{H}$ be a Hilbert space of complex-valued functions defined on a non-empty set X with an inner product $\langle\ ,\ \rangle$. $\mathcal{H}$ is called a *reproducing kernel Hilbert space* (RKHS) on X, if for any point $x \in X$ the evaluation functional $L_x : \mathcal{H} \to \mathbb{C}$ defined by $L_x(f) = f(x)$ is continuous on $\mathcal{H}$.

The following proposition follows from the Riesz representation theorem.

Proposition 1. *Let $\mathcal{H}$ be a RKHS on X. Then for any $y \in X$, there exists a unique function $k_y \in \mathcal{H}$ such that $f(y) = \langle f, k_y\rangle$ for all $f \in \mathcal{H}$.*

Definition 2. Let $\mathcal{H}$ be a RKHS on X and $y \in X$. The function k_y in Proposition 1 is called the *reproducing kernel function* at the point $y \in X$. In this case, the function $K : X \times X \to \mathbb{C}$ defined by

$$K(x,y) = k_y(x)$$

is called the *reproducing kernel* for $\mathcal{H}$.

Lemma 1. *The linear span of the set $\{\ k_y \ \big|\ y \in X\ \}$ of all reproducing kernel functions k_y is dense in $\mathcal{H}$.*

Proof. Let V be the closure of the linear span of $\{k_y \ \big|\ y \in \mathcal{H}\}$ in $\mathcal{H}$. Suppose contrary that V is a proper closed subspace in $\mathcal{H}$. Then there exists a non-zero f in the orthogonal complement $V^\perp$ of V in $\mathcal{H}$. f is orthogonal to k_y for all $y \in X$ and thus $f(y) = \langle f, k_y\rangle = 0$ for every $y \in X$, i.e., $f = \mathbf{0}$ in $\mathcal{H}$, which is a contradiction. Hence, $V = \mathcal{H}$. □

Theorem 3. *[3] (i) The Bergman space $A^2(\mathscr{D})$ is a reproducing kernel Hilbert space and the reproducing kernel is the Bergman kernel in the form*

$$K(z,w) = |\mathscr{D}|^{-1} \cdot (\det(I - w^* z))^{-(p+q)}. \tag{10}$$

(ii) The reproducing kernel functions $k_w(z) = K(z,w)$ satisfy the following reproducing property: for any $w \in \mathscr{D}$ and for any $f \in A^2(\mathscr{D})$,

$$\langle f, k_w\rangle = f(w).$$

(iii) The linear span of the set $\{k_w \mid w \in \mathscr{D}\}$ is dense in $A^2(\mathscr{D})$, i.e., any function in $A^2(\mathscr{D})$ can be approximated by linear combination of k_w's.

The domain $\mathscr{D}$ in $\mathbb{C}^{pq}$ can be viewed as generalizations of the unit disc $\mathbb{D}$ in $\mathbb{C}$ when $p = q = 1$, and of the unit ball $\mathbb{B}^p$ in $\mathbb{C}^p$ when $p > q = 1$ respectively.

The following example shows that the classical Bergman space $A^2(\mathbb{B}^p)$ on the unit ball $\mathbb{B}^p$ in $\mathbb{C}^p$ is a special case of the space $A^2(\mathscr{D})$. The corresponding topological boundaries and the Bergman kernels are recorded.

Example 1. For $p > q = 1$, we have

$$\mathscr{D} = \left\{ \mathbf{z} = (z_1, \ldots, z_p) \in \mathbb{C}^p \;\middle|\; \mathbf{z}^*\mathbf{z} = \sum_{k=1}^{p} |z_k|^2 < 1 \right\},$$

which is the unit ball $\mathbb{B}^p$ in $\mathbb{C}^p$, and

$$\partial\mathbb{B}^p = \left\{ \mathbf{z} \in \mathbb{C}^p \;\middle|\; \mathbf{z}^*\mathbf{z} = 1 \right\}$$

is its boundary. The Bergman kernel for the Bergman space $A^2(\mathbb{B}^p)$ is given by the following rational function in $\mathbf{z}$ and $\overline{\mathbf{w}}$:

$$K(\mathbf{z}, \mathbf{w}) = \frac{1}{|\mathbb{B}^p|(1 - \mathbf{w}^*\mathbf{z})^{p+1}}. \quad \square$$

The topological boundary $\partial\mathscr{D}$ of $\mathscr{D}$ is the disjoint union of q connected components $\mathcal{B}_1, \cdots, \mathcal{B}_q$, where

$$\mathcal{B}_i = \left\{ z \in \mathbb{C}^{pq} \;\middle|\; I - z^*z \geq 0 \text{ and has rank } q - i \right\}$$

for $1 \leq i \leq q$, and $\mathcal{B}_{i+1} \subset \overline{\mathcal{B}}_i$ for $1 \leq i \leq q - 1$.

The following lemma will be used in the proof of the boundary vanishing property (BVP) of $A^2(\mathscr{D})$ in Proposition 4.

Lemma 2. *Let $A \in \mathscr{D}$ and $B \in \overline{\mathscr{D}} = \mathscr{D} \cup \partial\mathscr{D}$. Then,*

(i) $\det(I - A^*B) \neq 0$;
(ii) if $A_0 \in \partial\mathscr{D}$, then $\lim_{A \to A_0} \det(I - A^*A) = 0$;
(iii) the function $f_A(B) = |\det(I - A^*B)|^{-1}$ extends continuously to $\overline{\mathscr{D}}$ and $f_A(B)$ is bounded for any $B \in \overline{\mathscr{D}}$.

Proof. (i) Suppose contrary that $\det(I - A^*B) = 0$, as $I - A^*B$ is a $q \times q$ matrix, there exists a non-zero $\mathbf{v} \in \mathbb{C}^q$ such that $(I - A^*B)\mathbf{v} = \mathbf{0}$. Then we have $0 = \mathbf{v}^*\mathbf{0} = \mathbf{v}^*(I - A^*B)\mathbf{v} = \mathbf{v}^*\mathbf{v} - \mathbf{v}^*A^*B\mathbf{v} = \|\mathbf{v}\|^2 - (A\mathbf{v})^*(B\mathbf{v})$, which implies that

$$\|\mathbf{v}\|^2 = \langle B\mathbf{v}, A\mathbf{v}\rangle. \tag{11}$$

On the other hand, as $A \in \mathscr{D}$, $B \in \overline{\mathscr{D}}$ and $\mathbf{v} \neq \mathbf{0} \in \mathbb{C}^q$, we have

$$\mathbf{v}^*(I - A^*A)\mathbf{v} > 0 \quad \text{and} \quad \mathbf{v}^*(I - B^*B)\mathbf{v} \geq 0,$$

and hence we have $\|\mathbf{v}\|^2 = \mathbf{v}^*\mathbf{v} = \mathbf{v}^*I\mathbf{v} > \mathbf{v}^*A^*A\mathbf{v} = (A\mathbf{v})^*(A\mathbf{v}) = \|A\mathbf{v}\|^2$, and $\|\mathbf{v}\|^2 \geq \|B\mathbf{v}\|^2$ similarly. Then $\|A\mathbf{v}\|\,\|B\mathbf{v}\| \leq \|A\mathbf{v}\|\,\|\mathbf{v}\| < \|\mathbf{v}\|^2$. It follows from (11) and Cauchy-Schwarz inequality that

$$\|\mathbf{v}\|^2 = \langle B\mathbf{v}, A\mathbf{v}\rangle = |\langle B\mathbf{v}, A\mathbf{v}\rangle| \leq \|B\mathbf{v}\|\|A\mathbf{v}\| < \|\mathbf{v}\|^2,$$

which is a contradiction.
(ii) As $p(A) = \det(I - A^*A)$ is a polynomial in the entries of A and their conjugates, it is continuous on $\overline{\mathscr{D}}$. For any point A_0 in the boundary $\partial\mathscr{D}$, the rank of $I - A_0^*A_0$ is less than q, $\lim_{A\to A_0} p(A) = p(A_0) = 0$.
(iii) For a fixed $A \in \mathscr{D}$, by (i), the function $h(B) = (\det(I - A^*B))^{-1}$ is bounded on $\mathscr{D}$, and hence the result follows. □

3. Generalized kernel functions in $A^2(\mathscr{D})$

In the one-dimensional case [10], the authors constructed generalized kernel functions by differentiating the reproducing kernel functions iteratively. In this section, we generalize their idea to high-dimensional case.

We are going to define the generalized kernel functions $k_{a,\alpha}$ from a given kernel $k_a(z) = K(z,a)$ by taking anti-holomorphic derivatives of order α with respect to second variable, and prove some important and useful properties which will be used in later sections.

3.1. Generalized kernel functions

Definition 3. For any given $a \in \mathscr{D}$, $\alpha \in \mathbb{N}^{pq}$, define *generalized kernel function* $k_{a,\alpha}$ on $\mathscr{D}$ of order α at the point a to be

$$k_{a,\alpha}(z) = \frac{\partial^{|\alpha|}}{\partial \overline{w}^{\alpha}} k_w\Big|_{w=a} = \frac{\partial^{|\alpha|}}{\partial \overline{w}^{\alpha}} K(z,w)\Big|_{w=a}. \tag{12}$$

In particular, $k_{a,\alpha} = k_a$ if $\alpha = (0,\ldots,0) \in \mathbb{N}^{pq}$.

By Theorem 3(iii), any function $f \in A^2(\mathscr{D})$ can be approximated by the linear combination of kernel functions k_a on $\mathscr{D}$. For any given $f \in A^2(\mathscr{D})$, we want to select a sequence $\{a_i\}_{i\geq 1}$ of points in $\mathscr{D}$ successively and to apply Gram-Schmidt orthonormalization process to construct a sequence $\{\mathscr{B}_i\}_{i\geq 1}$ of orthonormal rational functions in $A^2(\mathscr{D})$.

We first study some properties of kernel functions k_a and their anti-holomorphic derivatives. By reproducing property, we have

$$\|k_a\|^2 = \langle k_a, k_a\rangle = k_a(a) = |\mathscr{D}|^{-1} \cdot (\det(I - a^*a))^{-(p+q)}. \tag{13}$$

We define the normalized reproducing kernel functions e_a at $a \in \mathscr{D}$ by

$$e_a(z) = \frac{k_a(z)}{\|k_a\|} = \frac{k_a(z)}{\sqrt{k_a(a)}}. \tag{14}$$

By Theorem 3(iii), the linear span of functions e_a is also dense in $A^2(\mathscr{D})$. Now, by the reproducing property again, we have

$$\langle f, e_a\rangle = \sqrt{|\mathscr{D}|}(\det(I - a^*a))^{\frac{p+q}{2}} f(a). \tag{15}$$

In this section, the main result in Proposition 2 gives a characterization of the generalized kernel functions of higher order. In order to prove Proposition 2, one can prove the following lemma by the Cauchy-Riemann equations and induction.

Lemma 3. *Let $f(z,w)$ be a complex-valued function defined on $\mathbb{C}^{pq} \times \mathbb{C}^{pq}$. Suppose that f satisfies the following 3 conditions:*

(i) $f(w,z) = \overline{f(z,w)}$ *for all* $(z,w) \in \mathbb{C}^{pq} \times \mathbb{C}^{pq}$*;*
(ii) $f(z,w)$ *is holomorphic in the first variable* z*;*
(iii) $f(z,w)$ *is anti-holomorphic in the second variable* w*.*

Then for any $\alpha \in \mathbb{N}^{pq}$*, we have* $f^{(\alpha,\overline{\beta})}(b,a) = \overline{f^{(\beta,\overline{\alpha})}(a,b)}$*.*

Proposition 2. *(i)* $k_{a,\alpha} \in A^2(\mathscr{D})$ *for any* $a \in \mathscr{D}$ *and* $\alpha \in \mathbb{N}^{pq}$*.*
(ii) $\langle F, k_{a,\alpha}\rangle = \frac{\partial^{|\alpha|}F}{\partial z^\alpha}(a)$ *for any* $F \in A^2(\mathscr{D})$*, and*

$$\langle k_{a,\alpha}, k_{b,\beta}\rangle = K^{(\beta,\overline{\alpha})}(b,a). \tag{16}$$

(iii) Let $P = \{\mathbf{a}_1, \cdots, \mathbf{a}_n\}$ *be* n *distinct points in* $\mathscr{D}$ *and* $m \in \mathbb{N}$*. Then*

$$\{k_{\mathbf{a}_i,\alpha_i} \in A^2(\mathscr{D}) \mid \alpha_i \in \mathbb{N}^{pq}, |\alpha_i| \le m, i = 1, \ldots, n\}$$

is linearly independent in $A^2(\mathscr{D})$*.*

Proof. (i) By the definition of $k_w(z) = \frac{1}{|\mathscr{D}|\det(I-w^*z)^{p+q}}$ and $k_{a,\alpha}(z)$ in (12), we have

$$k_{a,\alpha}(z) = \frac{\partial^{|\alpha|}K}{\partial \overline{w}^\alpha}(z,w)\Big|_{w=a} = \frac{p_\alpha(z,w)}{|\mathscr{D}|\left(\det(I-w^*z)\right)^{p+q+|\alpha|}}\Big|_{w=a}, \tag{17}$$

where $p_\alpha(z,w)$ is a polynomial in z and $\overline{w}$. By Lemma 2(i), $\det(I-w^*z)$ is nonzero for any $w \in \mathscr{D}$ and any $z \in \overline{\mathscr{D}}$. By Lemma 2(iii), the function $g(z) = |\det(I-w^*z)|^{-1}$ is bounded for all $w \in \mathscr{D}$ and $z \in \overline{\mathscr{D}}$, and so is $k_{w,\alpha}$. Hence, $k_{w,\alpha} \in A^2(\mathscr{D})$.
(ii) It follows from Theorem 3(iii) that the vector space span$\{k_a \mid a \in \mathscr{D}\}$ is a dense subspace of $A^2(\mathscr{D})$. It suffices to prove (ii) when $F = k_a$ where $a \in \mathscr{D}$. By Lemma 3, we have

$$\langle k_a, k_{b,\alpha}\rangle = \overline{\langle k_{b,\alpha}, k_a\rangle} = \overline{k_{b,\alpha}(a)} = \overline{\frac{\partial^{|\alpha|}K}{\partial \overline{w}^\alpha}(z,w)}\Big|_{(z,w)=(a,b)} = \frac{\partial^{|\alpha|}K}{\partial z^\alpha}(z,w)\Big|_{(z,w)=(b,a)}.$$

This implies that for any $\alpha, \beta \in \mathbb{N}^{pq}$ and $a, b \in \mathscr{D}$, we have

$$\begin{aligned}\langle k_{a,\alpha}, k_{b,\beta}\rangle &= \frac{\partial^{|\beta|}}{\partial z^\beta}(k_{a,\alpha}(z))\Big|_{z=b} = \frac{\partial^{|\beta|}}{\partial z^\beta}\left(\frac{\partial^{|\alpha|}}{\partial \overline{w}^\alpha}(K(z,w)\Big|_{w=a}\right)\Big|_{z=b} \\ &= \left(\frac{\partial^{|\alpha|+|\beta|}}{\partial z^\beta \partial \overline{w}^\alpha}K(z,w)\right)\Big|_{(z,w)=(b,a)} = K^{(\beta,\overline{\alpha})}(b,a).\end{aligned}$$

(iii) Suppose that $\sum_{\mathbf{a}\in P,|\alpha|\le m} c_{\mathbf{a},\alpha}k_{\mathbf{a},\alpha} = 0$, we want to show that $c_{\mathbf{a},\alpha} = 0$ for all $\mathbf{a} \in P$ and multi-indices α of total order at most m.

For any fixed vectors $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_{pq}) \in \mathbb{C}^{pq}$, $\mathbf{a} = (a_1, \cdots, a_{pq}) \in \mathscr{D} \subset \mathbb{C}^{pq}$, and any $t \in \mathbb{R}$, define $f_\lambda \in A^2(\mathscr{D})$ by $f_\lambda(z) = e^{\langle z, t\lambda\rangle} = \exp(t\sum_{i=1}^{pq} z_i\overline{\lambda_i})$. Then $\frac{\partial^{|\alpha|}f_\lambda}{\partial z^\alpha}(\mathbf{a}) = \overline{t\lambda}^\alpha \exp(t\sum_{i=1}^{pq} a_i\overline{\lambda_i}) = t^{|\alpha|}\overline{\lambda}^\alpha \exp(t\langle \mathbf{a}, \lambda\rangle)$, and from (ii),

$$\begin{aligned}0 = \langle f_\lambda, \sum_{\mathbf{a}\in P,|\alpha|\le m} c_{\mathbf{a},\alpha}k_{\mathbf{a},\alpha}\rangle &= \sum_{\mathbf{a}\in P,|\alpha|\le m} \overline{c_{\mathbf{a},\alpha}}\langle f_\lambda, k_{\mathbf{a},\alpha}\rangle \\ &= \sum_{\mathbf{a}\in P,|\alpha|\le m} \overline{c_{\mathbf{a},\alpha}}\frac{\partial^{|\alpha|}f_\lambda}{\partial z^\alpha}(\mathbf{a}) = \sum_{\mathbf{a}\in P,|\alpha|\le m} \overline{c_{\mathbf{a},\alpha}}\,\overline{\lambda}^\alpha t^{|\alpha|}\exp(t\langle \mathbf{a}, \lambda\rangle)\end{aligned}$$

$$= \sum_{\mathbf{a}\in P} \sum_{k=0}^{m} \left(\sum_{|\alpha|=k} \overline{c_{\mathbf{a},\alpha}}\, \overline{\lambda}^{\alpha} \right) t^k \exp(t\langle \mathbf{a}, \lambda\rangle).$$

As $P \subset \mathscr{D} \subset \mathbb{C}^{pq}$ and there are only finitely many vectors in P, there exists a non-empty open subset U in $\mathbb{C}^{pq}$ such that
(U_1) $\lambda^\alpha \neq 0$ for all α of order at most m for all $\lambda \in U$, and
(U_2) for any $\lambda \in U$, the complex numbers $\langle \mathbf{a}, \lambda\rangle$ are distinct for $\mathbf{a} \in P$.

It follows from condition (U_2) that for any fixed $\lambda \in U$, the set of the following functions in t: $\{\ t^k \exp(t\langle \mathbf{a}, \lambda\rangle) \mid \mathbf{a} \in P,\ 0 \leq k \leq m\ \}$ is linearly independent, which implies that the following complex coefficients vanish for any fixed $\lambda \in U$:

$$\overline{\sum_{|\alpha|=k} c_{\mathbf{a},\alpha}\, \lambda^\alpha} = \sum_{|\alpha|=k} \overline{c_{\mathbf{a},\alpha}}\, \overline{\lambda}^{\alpha} = 0.$$

This vanishing condition of the homogeneous polynomial in λ of degree at most m on non-empty open subset U on $\mathbb{C}^{pq}$ and condition (U_1) implies that all the coefficients $c_{\mathbf{a},\alpha}$ in the polynomial above are all zero as well. □

Lemma 4. ***(Cauchy estimate)*** *Let f be a holomorphic function on a n-dimensional polydisc Δ centered at $a \in \mathbb{C}^n$ with positive radius R', and*

$$f(z) = \sum_{\alpha \geq 0} \frac{f^\alpha(a)}{\alpha!} z^\alpha$$

be the Taylor series of f centered at $z = a$. Suppose that $R < R'$ and let

$$M_R(f) = \max\left\{\ |f(z)|\ \middle|\ z = (z_1, \cdots, z_n) \in \Delta,\ |z_i - a_i| \leq R\ \right\}.$$

Then for all $\alpha \in \mathbb{N}^n$, we have $|\frac{f^{(\alpha)}(a)}{\alpha!}| \leq \frac{M_R(f)}{R^{|\alpha|}}$. □

Next we will prove that the Taylor series of the Bergman kernel in the anti-holomorphic variables converges in $A^2(\mathscr{D})$.

Proposition 3. *Let $a \in \mathscr{D} \subset \mathbb{C}^{pq}$, $\mathbf{v} \in \mathbb{C}^{pq}$ and $t \in \mathbb{C}$ with sufficiently small $|t|$ such that $a + t\mathbf{v} \in \mathscr{D}$, then*

(i) the series $\sum_{\alpha\geq 0} k_{a,\alpha} \frac{(\overline{t\mathbf{v}})^\alpha}{\alpha!}$ converges in $A^2(\mathscr{D})$; and
(ii) in the norm topology of $A^2(\mathscr{D})$, we have

$$k_{a+t\mathbf{v}} = \sum_{\alpha\geq 0} \frac{k_{a,\alpha}}{\alpha!} (\overline{t\mathbf{v}})^\alpha.$$

Proof. (i) As $K(z, w) = \frac{1}{|\mathscr{D}| \det(1 - \overline{w}^T z)^{p+q}}$ and it is holomorphic in $z \in \mathbb{C}^{pq}$ and is anti-holomorphic in $w \in \mathbb{C}^{pq}$, or equivalently it is holomorphic in $\overline{w} \in \mathbb{C}^{pq}$. For any multi-indices $\alpha, \beta \in \mathbb{N}^{pq}$, and $a, b \in \mathscr{D}$, it follows from Cauchy estimate in Lemma 4 that

$$\frac{|K^{(\beta,\overline{\alpha})}(b, a)|}{\alpha!\, \beta!} \leq \frac{M_R(K)}{R^{|\alpha|+|\beta|}}, \tag{18}$$

where $M_R(K) = \max\{\, |K(z,w)| \mid |z_{ij} - b_{ij}| \le R, \text{ and } |w_{ij} - a_{ij}| \le R \text{ for all } i,j \,\}$. Moreover, by Proposition 2(ii), $\langle k_{a,\alpha}, k_{b,\beta}\rangle = K^{(\beta,\overline{\alpha})}(b,a)$. Suppose $t \in \mathbb{C}$ with sufficiently small $|t|$, and any $\mathbf{v} \in \mathbb{C}^{pq}$, it follows from (16) that $\|k_{a,\alpha}\|^2 = \langle k_{a,\alpha}, k_{a,\alpha}\rangle = K^{(\alpha,\overline{\alpha})}(a,a)$, and then from (18) and geometric series that

$$\begin{aligned}\sum_{\alpha\ge 0}\Big\|k_{a,\alpha}\frac{\overline{t\mathbf{v}}^{\alpha}}{\alpha!}\Big\| &\le \sum_{\alpha\ge 0}\|k_{a,\alpha}\|\frac{|(t\mathbf{v})^{\alpha}|}{\alpha!} \le \sum_{\alpha\ge 0}\sqrt{\frac{M_R(K)}{R^{2|\alpha|}}}|(t\mathbf{v})^{\alpha}|\\ &\le \sqrt{M_R(K)}\sum_{\alpha\ge 0}\Big|\left(\frac{t\mathbf{v}}{R}\right)^{\alpha}\Big| = \frac{\sqrt{M_R(K)}}{\prod_{ij}(1-\frac{t|v_{ij}|}{R})}. \end{aligned} \tag{19}$$

This implies that the series $\sum_{\alpha\ge 0} k_{a,\alpha}\frac{(\overline{t\mathbf{v}})^{\alpha}}{\alpha!}$ converges in $A^2(\mathscr{D})$.

(ii) By expanding the function $K(a+t\mathbf{v}, b+t\mathbf{w})$ into power series of the holomorphic parameters $t\mathbf{v}$ and anti-holomorphic parameter $\overline{t\mathbf{w}}$, we have

$$\|k_{a+t\mathbf{v}}\|^2 = K(a+t\mathbf{v}, a+t\mathbf{v}) = \sum_{\alpha,\beta\ge 0}\frac{K^{(\alpha,\overline{\beta})}(a,a)}{\alpha!\beta!}(t\mathbf{v})^{\alpha}(\overline{t\mathbf{v}})^{\beta}, \tag{20}$$

in which the series converges uniformly for all $\mathbf{v}$ in the unit ball of $\mathbb{C}^{pq}$ as long as $|t|$ is sufficiently small. Then by Proposition 2(ii) and using the Taylor series expansion in (20), the result follows from the following estimate

$$\begin{aligned}&\Big\|k_{a+t\mathbf{v}} - \sum_{|\beta|\le N}k_{a,\beta}\frac{(\overline{t\mathbf{v}})^{\beta}}{\beta!}\Big\|^2\\ &\le \|k_{a+t\mathbf{v}}\|^2 - 2\sum_{|\beta|\le N}\mathrm{Re}\langle k_{a,\beta}\frac{(\overline{t\mathbf{v}})^{\beta}}{\beta!},\ k_{a+t\mathbf{v}}\,\rangle + \sum_{|\alpha|,|\beta|\le N}\mathrm{Re}\langle k_{a,\beta}\frac{(\overline{t\mathbf{v}})^{\beta}}{\beta!}, k_{a,\alpha}\frac{(\overline{t\mathbf{v}})^{\alpha}}{\alpha!}\rangle\\ &= K(a+t\mathbf{v}, a+t\mathbf{v}) - 2\sum_{|\beta|\le N}\mathrm{Re}\left(K^{(\mathbf{0},\overline{\beta})}(a+t\mathbf{v},a)\frac{(\overline{t\mathbf{v}})^{\beta}}{\beta!}\right)\\ &\quad + \sum_{|\alpha|,|\beta|\le N}\mathrm{Re}\left(\frac{K^{(\alpha,\overline{\beta})}(a,a)}{\alpha!\beta!}(t\mathbf{v})^{\alpha}(\overline{t\mathbf{v}})^{\beta}\right)\\ &= \sum_{\alpha,\beta\ge 0}\frac{K^{(\alpha,\overline{\beta})}(a,a)}{\alpha!\beta!}(t\mathbf{v})^{\alpha}(\overline{t\mathbf{v}})^{\beta} - 2\sum_{\alpha\ge 0,|\beta|\le N}\mathrm{Re}\left(\frac{K^{(\alpha,\overline{\beta})}(a,a)}{\alpha!\beta!}(t\mathbf{v})^{\alpha}(\overline{t\mathbf{v}})^{\beta}\right)\\ &\quad + \sum_{|\alpha|,|\beta|\le N}\mathrm{Re}\left(\frac{K^{(\alpha,\overline{\beta})}(a,a)}{\alpha!\beta!}(t\mathbf{v})^{\alpha}(\overline{t\mathbf{v}})^{\beta}\right)\\ &= \mathrm{Re}\left(\sum_{|\alpha|\ge N,|\beta|\ge N}\frac{K^{(\alpha,\overline{\beta})}(a,a)}{\alpha!\beta!}(t\mathbf{v})^{\alpha}(\overline{t\mathbf{v}})^{\beta}\right).\end{aligned}$$

The last sum is the tail of the Taylor series of $K(a+t\mathbf{v}, a+t\mathbf{v})$ in t which converges uniformly to 0 for all unit vectors $\mathbf{v}$ of $\mathbb{C}^{pq}$ as $N \to +\infty$, provided that $|t|$ is sufficiently small. □

3.2. Boundary vanishing property

In order to prove our main result the maximum selection principle in Theorem 1, it is crucial to check that the reproducing kernel function k_a has the boundary vanishing property in Proposition 4(i). We start with the following easy lemma and proposition.

Lemma 5. *For any $f \in A^2(\mathscr{D})$ and any $a \in \mathscr{D}$, define $H(a) = k_a \in A^2(\mathscr{D})$ and $G(a) = |\langle f, e_a\rangle|$, where $e_a = \frac{k_a}{\|k_a\|}$. Then H and G are continuous on $\mathscr{D}$.*

Proof. As $\|k_a\| > 0$ for all $a \in \mathscr{D}$, it suffices to show that the function $H(a) = k_a$ is continuous from $\mathscr{D}$ to $A^2(\mathscr{D})$. In fact, as $K(z, w)$ is continuous on $\mathscr{D} \times \mathscr{D}$,

$$\|k_b - k_a\|^2 = \|k_b\|^2 + \|k_a\|^2 - 2\mathrm{Re}\langle k_b, k_a\rangle = K(b, b) + K(a, a) - 2\mathrm{Re}K(b, a)$$

converges to 0 as b approaches a. □

Next, we will prove that the function G in Lemma 5 can be extended continuously to $\overline{\mathscr{D}}$.

Proposition 4. *(i) (**Boundary vanishing property**) For any $f \in A^2(\mathscr{D})$ and any $b_0 \in \partial\mathscr{D}$, we have $\lim_{b\to b_0} |\langle f, e_b\rangle| = 0$.*
(ii) For any $f \in A^2(\mathscr{D})$, there exists $a \in \mathscr{D}$ such that $|\langle f, e_a\rangle| = \max\{|\langle f, e_b\rangle| \;\big|\; b \in \mathscr{D}\}$.

Proof. (i) By Lemma 1, the linear span of $\{k_a \;\big|\; a \in \mathscr{D}\}$ is a dense subspace of $A^2(\mathscr{D})$. For any $\epsilon > 0$, there exist a positive integer N, finite sequences $\{a_i\}_{i=1}^N \subset \mathscr{D}$ and $\{c_i\}_{i=1}^N \subset \mathbb{C}$ such that $F = \sum_{i=1}^N c_i k_{a_i}$, and $\|f - F\| < \epsilon$. Since $\|e_b\| = 1$, the triangle inequality and the Cauchy-Schwarz inequality give

$$|\langle f, e_b\rangle| - |\langle F, e_b\rangle| \le |\langle f - F, e_b\rangle| \le \|f - F\| \, \|e_b\| < \epsilon.$$

For any $b \in \mathscr{D}$, by (15), we have

$$\begin{aligned} |\langle f, e_b\rangle| &\le |\langle F, e_b\rangle| + \epsilon = \sqrt{|\mathscr{D}|}\; |\det(I - b^*b)|^{\frac{p+q}{2}}\; |F(b)| + \epsilon \\ &\le |\mathscr{D}|^{-\frac{1}{2}}\; |\det(I - b^*b)|^{\frac{p+q}{2}} \sum_{i=1}^N |c_i|\; |\det(I - a_i^*b)|^{-(p+q)} + \epsilon. \end{aligned} \tag{21}$$

By Lemma 2(ii) and 2(iii), the function $t(b) = |\det(I - a_i^*b)|^{-(p+q)}$ is bounded on $\overline{\mathscr{D}}$ for all $a_i \in \mathscr{D}$. Therefore, the N-sum $\sum_{i=1}^N |c_i|\; |\det(I - a_i^*b)|^{-(p+q)}$ in (21) is bounded. Since $\lim_{b\to b_0} \det(I - b^*b) = 0$, the result follows.
(ii) By Lemma 5, $G(b) = |\langle f, e_b\rangle|$ is continuous on $\mathscr{D}$. By (i), $G(b)$ can be continuously extended to the compact subset $\overline{\mathscr{D}}$ in $\mathbb{C}^{pq}$. Then, consider the following two cases of $M = \max\{G(b) \;\big|\; b \in \overline{\mathscr{D}}\}$.

If $M = 0$, then $0 \le |\langle f, e_b\rangle| \le M = 0$ for all $b \in \mathscr{D}$. By reproducing property of k_b, $f = 0$ on $\mathscr{D}$. In this case, choose a to be any point in $\mathscr{D}$.

If $M > 0$, then it follows from BVP that $G(b) = 0$ for all $b \in \partial\mathscr{D}$, i.e., the maximum M can only be attained at some interior point of $\overline{\mathscr{D}}$. Therefore, there exists $a \in \mathscr{D}$ such that $M = G(a) = |\langle f, e_a\rangle|$. □

4. Maximal selection principle

The goal of this section is to prove Theorem 1 about the existence of pre-orthogonal adaptive Fourier decomposition in $A^2(\mathscr{D})$. We will establish our main result Theorem 1 by a series of lemmas and propositions in the rest of this section. We first construct the sequence $(a_n)_{n\ge1}$ of points in $\mathscr{D}$ needed in Theorem 1(i) inductively on n.

4.1. Initial step for a_1

Initial step. For $n = 1$, define $f_1 = f$. The BVP in Proposition 4 implies that there exists $a_1 \in \mathscr{D}$ such that

$$|\langle f_1, \frac{k_{a_1}}{\|k_{a_1}\|}\rangle| = \sup\left\{ |\langle f_1, \frac{k_b}{\|k_b\|}\rangle| \;\Big|\; b \in \mathscr{D} \right\},$$

which is the equality (6) of Theorem 1. In this initial step, we set

$$m_1 = 0, \quad k_{a_1,m_1} = k_{a_1} \quad \text{and} \quad \mathscr{B}_1 = e_{a_1} = \frac{k_{a_1}}{\|k_{a_1}\|}.$$

Then in the inductive step, let's first assume that we have a sequence $(a_1, \cdots, a_n)$ of not necessarily distinct points in $\mathscr{D}$, and a sequence $(k_{a_1,m_1}, k_{a_2,m_2}, \cdots, k_{a_n,m_n})$ in $A^2(\mathscr{D})$ as stated in Theorem 1. By applying Gram-Schmidt orthonormalization process to $(k_{a_i,m_i})_{1\le i\le n}$, we obtain an orthonormal set $\{\mathscr{B}_1, \cdots, \mathscr{B}_n\}$ in $A^2(\mathscr{D})$. Define $f_j \in A^2(\mathscr{D})$ be the j-th residual function as follows:

$$f_1 = f, \quad \text{and} \quad f_j = f_{j-1} - \langle f_{j-1}, \mathscr{B}_{j-1}\rangle \mathscr{B}_{j-1} \tag{22}$$

for $2 \le j \le n+1$. We can deduce the following.

Proposition 5. *Let $f \in A^2(\mathscr{D})$, $\{\mathscr{B}_1, \cdots, \mathscr{B}_n\}$ be an orthonormal set in $A^2(\mathscr{D})$ and $\{f_j\}_{1\le j\le n+1}$ be defined in (22), then we have*

(i) $\langle f_j, \mathscr{B}_{j-1}\rangle = \langle f_j, \mathscr{B}_{j-2}\rangle = \cdots = \langle f_j, \mathscr{B}_1\rangle = 0$;
(ii) $\langle f_j, \mathscr{B}_j\rangle = \langle f_{j-1}, \mathscr{B}_j\rangle = \cdots = \langle f_2, \mathscr{B}_j\rangle = \langle f_1, \mathscr{B}_j\rangle = \langle f, \mathscr{B}_j\rangle$ *for* $1 \le j \le n$;
(iii) $f_j = f_\ell - \sum_{i=\ell}^{j-1}\langle f, \mathscr{B}_i\rangle \mathscr{B}_i = f - \sum_{i=1}^{j-1}\langle f, \mathscr{B}_i\rangle \mathscr{B}_i$ *for* $1 \le \ell < j \le n+1$;
(iv) $span\{\ \mathscr{B}_1, \cdots, \mathscr{B}_{j-1}, \mathscr{B}_j\ \} = span\{\ k_{a_1,m_1},\ k_{a_2,m_2}, \cdots, k_{a_j,m_j}\ \}$ *for* $1 \le j \le n$;
(v) $\langle f_j, k_{a_i,m_i}\rangle = 0$ *for* $1 \le i < j \le n+1$;
(vi) $k_b - \sum_{j=1}^{n}\langle k_b, \mathscr{B}_j\rangle \mathscr{B}_j = \mathbf{0}$ *if and only if* $b \in \{a_1, \cdots, a_n\}$.

Proof. (i) Using $f_j = f_{j-1} - \langle f_{j-1}, \mathscr{B}_{j-1}\rangle \mathscr{B}_{j-1}$ and $\langle \mathscr{B}_i, \mathscr{B}_j\rangle = \delta_{ij}$, one has

$$\begin{aligned}\langle f_j, \mathscr{B}_{j-1}\rangle &= \langle f_{j-1} - \langle f_{j-1}, \mathscr{B}_{j-1}\rangle \mathscr{B}_{j-1}, \mathscr{B}_{j-1}\rangle \\ &= \langle f_{j-1}, \mathscr{B}_{j-1}\rangle - \langle f_{j-1}, \mathscr{B}_{j-1}\rangle \|\mathscr{B}_{j-1}\|^2 = 0,\end{aligned}$$

the others follow similarly.
(ii) follows from $\langle f_{i+1}, \mathscr{B}_j\rangle = \langle f_i - \langle f_i, \mathscr{B}_i\rangle \mathscr{B}_i\ , \mathscr{B}_j\rangle = \langle f_i, \mathscr{B}_j\rangle$ if $i < j$.
(iii) follows from (ii) and $f_j = f_{j-1} - \langle f_{j-1}, \mathscr{B}_{j-1}\rangle \mathscr{B}_{j-1} = f_{j-1} - \langle f, \mathscr{B}_{j-1}\rangle \mathscr{B}_{j-1}$.
(iv) The result follows from Gram-Schmidt orthonormalization process.
(v) The result follows from (iv).
(vi) If $b \notin \{a_1, \cdots, a_n\}$, then by Proposition 2(iii), the set $\{k_{a_1,m_1}, \cdots, k_{a_n,m_n}\}$ is linearly independent. If $b \in \{a_1, \cdots, a_n\}$, then $k_b \in \text{span}\{k_{a_1,m_1}, \cdots, k_{a_n,m_n}\}$. Hence, by (iv), we have

$$k_b - \sum_{j=1}^{n}\langle k_b, \mathscr{B}_j\rangle \mathscr{B}_j = \mathbf{0} \iff k_b \in \text{span}\{\mathscr{B}_1, \cdots, \mathscr{B}_n\}$$

$$\iff k_b \in \text{span}\{k_{a_1,m_1}, \cdots, k_{a_n,m_n}\} \iff b \in \{a_1, \cdots, a_n\}. \quad \square$$

The following lemma is an application of Theorem 1(iii), and it justifies for the existence in the search of a_{n+1}.

Lemma 6. *Suppose that $f_j \ne 0$ $(1 \le j \le n)$, then $|\langle\ f_j\ ,\ \mathscr{B}_j\ \rangle| \ne 0$.*

Proof. As $f_j \neq 0$, it follows from continuity of f_j that there exist $a \in \mathscr{D}\backslash\{a_1, \cdots, a_{j-1}\}$ and $\delta > 0$ such that $|f_j(w)| > 0$ for all point w in the open ball B centered at a with radius δ. As $\{a_1, \cdots, a_{j-1}\}$ is a finite set in $\mathscr{D}$, one can find a point $z \in B \setminus \{a_1, \cdots, a_{j-1}\} \subseteq \mathscr{D} \setminus \{a_1, \cdots, a_{j-1}\}$.

As $z \neq a_i$ $(1 \leq i \leq j-1)$, Proposition 5(vi) implies that $k_z, \mathscr{B}_1, \cdots, \mathscr{B}_{j-1}$ are linearly independent, and so

$$0 < \|k_z - \sum_{i=1}^{j-1} \langle k_z, \mathscr{B}_i\rangle \mathscr{B}_i\|^2 = \|k_z\|^2 - \sum_{i=1}^{j-1} |\langle k_z, \mathscr{B}_i\rangle|^2 \leq \|k_z\|^2. \tag{23}$$

Applying Gram-Schmidt orthonormalizaton process to k_z with orthonormal set $\{\mathscr{B}_1, \cdots, \mathscr{B}_{j-1}\}$, one can construct

$$\mathscr{B}_j^z = \frac{k_z - \sum_{i=1}^{j-1}\langle k_z, \mathscr{B}_i\rangle \mathscr{B}_i}{\|k_z - \sum_{i=1}^{j-1}\langle k_z, \mathscr{B}_i\rangle \mathscr{B}_i\|} \in A^2(\mathscr{D}).$$

In particular, $\langle \mathscr{B}_j^z, \mathscr{B}_i\rangle = 0$ for $1 \leq i \leq j-1$.

Recall that $f_j = f - \sum_{i=1}^{j-1}\langle f, \mathscr{B}_i\rangle \mathscr{B}_i$ in Proposition 5(iii). Then it follows from the orthonormal set $\{\mathscr{B}_1, \cdots, \mathscr{B}_{j-1}, \mathscr{B}_j^z\}$, the inequality (23), Proposition 5(i), (6) in Theorem 1(iii) and Proposition 5(iii) that

$$\begin{aligned}
0 < \frac{|f_j(z)|}{\|k_z\|} &= \frac{|\langle f_j, k_z\rangle|}{\|k_z\|} = \frac{|\langle f_j, k_z - \sum_{i=1}^{j-1}\langle k_z, \mathscr{B}_i\rangle \mathscr{B}_i\rangle|}{\|k_z\|} \\
&\leq \frac{|\langle f_j, k_z - \sum_{i=1}^{j-1}\langle k_z, \mathscr{B}_i\rangle \mathscr{B}_i\rangle|}{\|k_z - \sum_{i=1}^{j-1}\langle k_z, \mathscr{B}_i\rangle \mathscr{B}_i\|} = |\langle f_j, \mathscr{B}_j^z\rangle| \\
&= |\langle f_j + \sum_{i=1}^{j-1}\langle f, \mathscr{B}_i\rangle \mathscr{B}_i, \mathscr{B}_j^z\rangle| \\
&\leq \sup\left\{ |\langle f, \mathscr{B}_j^b\rangle| \;\middle|\; b \in \mathscr{D} \setminus \{a_1, a_2, \cdots, a_{i-1}\} \right\} \\
&= |\langle f, \mathscr{B}_j\rangle| = |\langle f_j, \mathscr{B}_j\rangle|. \quad \square
\end{aligned}$$

4.2. Inductive step for a_{n+1}

Inductive step. Suppose that $f_{n+1} \neq 0$. We are going to show that there exists a point $a_{n+1} \in \mathscr{D}$ and an associated function $\mathscr{B}_{n+1} \in A^2(\mathscr{D})$ satisfying the following maximal selection principle:

$$|\langle f_{n+1}, \mathscr{B}_{n+1}\rangle| = \sup\left\{ |\langle f_{n+1}, \mathscr{B}_{n+1}^b\rangle| \;\middle|\; b \in \mathscr{D} \setminus \{a_1, \cdots, a_n\} \right\},$$

where the test vector $\mathscr{B}_{n+1}^b$ is the unit vector associated to the non-zero vector $k_b - \sum_{i=1}^n \langle k_b, \mathscr{B}_i\rangle \mathscr{B}_i$ which is orthogonal to the vector subspace span$\{\mathscr{B}_1, \cdots, \mathscr{B}_n\}$ in $A^2(\mathscr{D})$, i.e.,

$$\mathscr{B}_{n+1}^b = \frac{k_b - \sum_{i=1}^n \langle k_b, \mathscr{B}_i\rangle \mathscr{B}_i}{\|k_b - \sum_{i=1}^n \langle k_b, \mathscr{B}_i\rangle \mathscr{B}_i\|} = \frac{e_b - \sum_{i=1}^n \langle e_b, \mathscr{B}_i\rangle \mathscr{B}_i}{\|e_b - \sum_{i=1}^n \langle e_b, \mathscr{B}_i\rangle \mathscr{B}_i\|}, \tag{24}$$

where

$$\|e_b - \sum_{i=1}^n \langle e_b, \mathscr{B}_i\rangle \mathscr{B}_i\|^2 = \|e_b\|^2 - \sum_{i=1}^n |\langle e_b, \mathscr{B}_i\rangle|^2 = 1 - \sum_{i=1}^n |\langle e_b, \mathscr{B}_i\rangle|^2.$$

Definition 4. Let $\mathscr{D}_n = \mathscr{D} \setminus \{a_1, \cdots, a_n\}$ be the punctured domain. Define an objective function g on $\mathscr{D}_n$ as follows: $g(b) = |\langle f_{n+1}, \mathscr{B}^b_{n+1}\rangle|$ for any $b \in \mathscr{D}_n$.

It follows from Proposition 5(i) and Cauchy-Schwarz inequality that $\langle f_{n+1}, \mathscr{B}_j\rangle = 0$,

$$\begin{aligned} g(b) = |\langle f_{n+1}, \mathscr{B}^b_{n+1}\rangle| &= \left|\left\langle f_{n+1}, \frac{k_b - \sum_{j=1}^n \langle k_b, \mathscr{B}_j\rangle \mathscr{B}_j}{\|k_b - \sum_{j=1}^n \langle k_b, \mathscr{B}_j\rangle \mathscr{B}_j\|}\right\rangle\right| \\ &= \frac{|\langle f_{n+1}, k_b\rangle|}{\|k_b - \sum_{j=1}^n \langle k_b, \mathscr{B}_j\rangle \mathscr{B}_j\|} = \frac{|\langle f_{n+1}, e_b\rangle|}{\|e_b - \sum_{j=1}^n \langle e_b, \mathscr{B}_j\rangle \mathscr{B}_j\|} \\ &= \frac{|\langle f_{n+1}, e_b\rangle|}{\sqrt{1 - \sum_{j=1}^n |\langle e_b, \mathscr{B}_j\rangle|^2}} \end{aligned} \tag{25}$$

and

$$g(b) = |\langle f_{n+1}, \mathscr{B}^b_{n+1}\rangle| \le \|f_{n+1}\| \, \|\mathscr{B}^b_{n+1}\| \le \|f_{n+1}\|$$

for all $b \in \mathscr{D}_n$.

For any $b_0 \in \partial\mathscr{D}$, the BVP in Proposition 4(i) implies that $\lim_{b\to b_0} |\langle f_{n+1}, e_b\rangle| = 0$ and $\lim_{b\to b_0} |\langle \mathscr{B}_j, e_b\rangle| = 0$ $(1 \le j \le n)$, hence it follows from (25) that $\lim_{b\to b_0} g(b) = 0$. Therefore, g can be extended to $\mathscr{D}_n \cup \partial\mathscr{D}$ continuously with $g(b) = 0$ for all $b \in \partial\mathscr{D}$.

As g is bounded on $\mathscr{D}_n \cup \partial\mathscr{D}$, the following supremum

$$S = \sup\left\{ \, g(b) \,\middle|\, b \in \mathscr{D}_n \cup \partial\mathscr{D} \,\right\} \tag{26}$$

is finite. Then there exists a sequence $\{b_m\}_{m\ge1}$ of points in $\mathscr{D}$ such that $\lim_{n\to\infty} g(b_m) = S$.

The domain $\mathscr{D}$ is bounded in $\mathbb{C}^{pq}$, and hence its closure $\overline{\mathscr{D}}$ is a compact set. Then the sequence $\{b_m\}_{m\ge1}$ in $\mathscr{D}_n \subset \overline{\mathscr{D}}$ has a convergent subsequence, still denoted by $\{b_m\}_{m\ge1}$, with limit $a = \lim_{m\to\infty} b_m \in \overline{\mathscr{D}}$.

We first prove that $a \notin \partial\mathscr{D}$. Suppose contrary that $a \in \partial\mathscr{D}$. Note that the equality in (25) implies that $g(b) = 0$ for all $b \in \mathscr{D}_n$ if and only if $0 = \langle f_{n+1}, k_b\rangle = f_{n+1}(b)$ for $b \in \mathscr{D}_n$, so $f_{n+1} = 0$ on $\mathscr{D}$ by the continuity of f_{n+1}. Then, by the continuity of g on $\mathscr{D}_n \cup \partial\mathscr{D}$ and BVP, we have

$$S = \lim_{m\to\infty} g(b_m) = g(a) = 0.$$

In particular, $0 \le g(b) \le S = 0$ for all $b \in \mathscr{D}_n$, and hence $g(b) = 0$ on $\mathscr{D}_n$. Then it follows from the note above that $f_{n+1}(b) = 0$ for all $b \in \mathscr{D}$, which violates the assumption $f_{n+1} \ne 0$ stated in the inductive step.

One of the following two cases can happen:

(**A**) $a \in \mathscr{D}_n$, i.e., $a \ne a_i$ for all $1 \le i \le n$; and
(**B**) $a \in \{a_1, \cdots, a_n\}$.

*4.3. The simple case (**A**)*

In case (**A**), we can set $m_{n+1} = 0, a_{n+1} = a$,

$$k_{a_{n+1},m_{n+1}} = k_a, \text{ and } \quad \mathscr{B}_{n+1} = \frac{k_a - \sum_{j=1}^n \langle k_a, \mathscr{B}_j\rangle \mathscr{B}_j}{\|k_a - \sum_{j=1}^n \langle k_a, \mathscr{B}_j\rangle \mathscr{B}_j\|}.$$

In this case, the equality (6) of Theorem 1 obviously holds.

We will consider the case (**B**) in the next subsection.

*4.4. The case (**B**) of repeatedly selected points*

In the following, we assume that the limit point a appeared in the sequence $(a_1, \cdots, a_n)$ of points in $\mathscr{D}$. As $b_m \in \mathscr{D}_n$, so $b_m \neq a$, one can define a sequence $\{\mathbf{v}_m\}_{m\geq 1}$ of unit vectors in $\mathbb{C}^{pq}$ and a sequence $\{t_m\}_{m\geq 1}$ of positive numbers as follows: for any $m \in \mathbb{Z}_+$,

$$t_m = \|b_m - a\| > 0,$$
$$\mathbf{v}_m = \frac{b_m - a}{\|b_m - a\|} = \frac{b_m - a}{t_m} \in \mathbb{C}^{pq}.$$

In particular, $b_m = a + t_m \mathbf{v}_m$. Then $a = \lim_{m\to\infty} b_m$ implies that $\lim_{m\to\infty} t_m = 0$.

As the sequence $\{\mathbf{v}_m\}_{m\geq 1}$ is a sequence in the compact unit sphere in $\mathbb{C}^{pq}$, it has a convergent subsequence, still denoted by $\{\mathbf{v}_m\}_{m\geq 1}$, with limit $\mathbf{v} = (\mathbf{v}(1), \mathbf{v}(2), \cdots, \mathbf{v}(pq))$ in the unit sphere of $\mathbb{C}^{pq}$.

Define $d_m = a + t_m \mathbf{v}$ for all m. Since $\lim_{m\to\infty} t_m = 0$ and a is an interior point of $\mathscr{D}$, we may assume that $d_m \in \mathscr{D}$ for all $m \geq 1$. Moreover, $\lim_{m\to\infty} d_m = a = \lim_{m\to\infty} b_m$. As Recall that the supremum S in (26) is defined by

$$S = \lim_{m\to\infty} g(b_m) = \lim_{m\to\infty} |\langle f_{n+1}, \mathscr{B}_{n+1}^{b_m}\rangle|. \tag{27}$$

We want to replace the sequence $\{\, b_m \,\}_{m\geq 1}$ by the sequence $\{\, d_m \,\}_{m\geq 1}$ such that the limit property (27) also holds for the sequence of $\{d_m\}$ instead of $\{b_m\}$. This is a significant step in the proof of our main Theorem 1.

Proposition 6. *Suppose that $d_m = a + t_m \mathbf{v} \in \mathscr{D}$ for all $m \geq 1$, we have*

(i) the difference sequence $k_{b_m} - k_{d_m}$ converges to $\mathbf{0}$ in the norm of $A^2(\mathscr{D})$;

(ii) the difference sequence

$$\left(k_{b_m} - \sum_{i=1}^{n} \langle\, k_{b_m}, \mathscr{B}_i\rangle \mathscr{B}_i\right) - \left(k_{d_m} - \sum_{i=1}^{n} \langle\, k_{d_m}, \mathscr{B}_i\rangle \mathscr{B}_i\right)$$
$$= (k_{b_m} - k_{d_m}) - \sum_{i=1}^{n} \langle\, k_{b_m} - k_{d_m}, \mathscr{B}_i\rangle \mathscr{B}_i$$

converges to $\mathbf{0}$ in the norm of $A^2(\mathscr{D})$;

(iii) $\lim_{m\to\infty} \left\|k_{b_m} - \sum_{i=1}^{n} \langle\, k_{b_m}, \mathscr{B}_i\rangle \mathscr{B}_i\right\| = \lim_{m\to\infty} \left\|k_{d_m} - \sum_{i=1}^{n} \langle\, k_{d_m}, \mathscr{B}_i\rangle \mathscr{B}_i\right\|$; and

(iv) $\lim_{m\to\infty} g(b_m) = \lim_{m\to\infty} g(d_m)$.

Proof. The proof of (i)–(iv) follow from Lemma 5. □

Remark 1. It follows from Proposition 6(iv) that whenever S is concerned, we may replace the sequence $\{b_m\}_{m\geq 1}$ of points in $\mathscr{D}$ by $\{d_m\}_{m\geq 1}$. From now on, one may assume that $b_m = a + t_m \mathbf{v}$ in $\mathscr{D}$ for all $m \geq 1$ in the following discussion.

As the limit point a appears in the sequence $(a_i)_{1\le i\le n}$ in the case (**B**), choose the smallest index i such that $a = a_i$, then $m_i = 0$. By Proposition 5(i), $\langle f_{n+1}, \mathscr{B}_j\rangle = 0$ for $1 \le j \le n$, and Proposition 5(iv) implies that $0 = \langle f_{n+1}, k_{a_i,m_i}\rangle = \langle f_{n+1}, k_{a_i,0}\rangle = f_{n+1}(a_i) = f_{n+1}(a)$. The facts that $f_{n+1} \ne 0$ and $f_{n+1}(a) = 0$ imply that f is not a constant function. Hence, there exists $\alpha \in \mathbb{N}^{pq}$ such that $f^{(\alpha)}_{n+1}(a) \ne 0$.

Lemma 7. *There exists $\ell \ge 1$ such that*

$$\sum_{|\alpha|=\ell} f^{(\alpha)}_{n+1}(a)\frac{\mathbf{v}^\alpha}{\alpha!} \ne 0.$$

Proof. Suppose contrary that for any $\ell \ge 1$,

$$\sum_{|\alpha|=\ell} f^{(\alpha)}_{n+1}(a)\frac{\mathbf{v}^\alpha}{\alpha!} = 0,$$

which implies that for sufficiently large m,

$$0 = \sum_{\alpha\ge 0} f^{(\alpha)}_{n+1}(a)\frac{\mathbf{v}^\alpha}{\alpha!}t_m^\alpha = f_{n+1}(a + t_m\mathbf{v}) = f_{n+1}(b_m).$$

It follows from the equality in (25) that $g(b_m) = 0$, and hence $S = \lim_{m\to\infty} g(b_m) = 0$, which contradicts to $f_{n+1} \ne 0$. □

Definition 5. Let ℓ be the least positive integer such that

(i) $\displaystyle\sum_{|\alpha|=j} f^{(\alpha)}_{n+1}(a)\frac{\mathbf{v}^\alpha}{\alpha!} = 0$ for all $j = 0, 1, \cdots, \ell - 1$;

(ii) $\displaystyle\sum_{|\alpha|=\ell} f^{(\alpha)}_{n+1}(a)\frac{\mathbf{v}^\alpha}{\alpha!} \ne 0.$

With the help of definition, we define $E = \displaystyle\sum_{|\alpha|=\ell} k_{a,\alpha}\frac{\overline{\mathbf{v}}^\alpha}{\alpha!}$ which is nonzero in $A^2(\mathscr{D})$.

Definition 6. For the case (**B**) of the limit point a stated above, we define

(i) $a_{n+1} = a,\ m_{n+1} = \ell$,

(ii) $k_{a_{n+1},m_{n+1}} = k_{a,\ell} = \displaystyle\sum_{|\alpha|=\ell} k_{a,\alpha}\frac{\overline{\mathbf{v}}^\alpha}{\alpha!}$, and

(iii) $\mathscr{B}_{n+1} = \dfrac{k_{a_{n+1},m_{n+1}} - \sum_{i=1}^n \langle k_{a_{n+1},m_{n+1}}, \mathscr{B}_i\rangle \mathscr{B}_i}{\| k_{a_{n+1},m_{n+1}} - \sum_{i=1}^n \langle k_{a_{n+1},m_{n+1}}, \mathscr{B}_i\rangle \mathscr{B}_i \|}.$

In the following, we will prove that Theorem 1 holds for selection of the triple $(a_{n+1}, m_{n+1}, \mathscr{B}_{n+1})$ stated above.

4.5. Asymptotic analysis

The main result of this subsection is the following Theorem 4. This shows that in the case (**B**), where the limit point a of the sequence $\{b_m = a + t_m\mathbf{v}\}_{m\ge 1}$ appears in the sequence $(a_1, \dots, a_n)$, we can always construct a function $\mathscr{B}_{n+1}$ in $A^2(\mathscr{D})$ associated to a such that $\{\mathscr{B}_1, \dots, \mathscr{B}_n, \mathscr{B}_{n+1}\}$ is an orthonormal set. We finally can state the most important result in this paper, but its proof requires some delicate estimates.

Theorem 4. *The supremum S in (26) is given by*

$$S = \lim_{m\to\infty} g(b_m) = \left|\left\langle f_{n+1}, \frac{E - \sum_{i=1}^n \langle E, \mathscr{B}_i\rangle \mathscr{B}_i}{\|E - \sum_{i=1}^n \langle E, \mathscr{B}_i\rangle \mathscr{B}_i\|}\right\rangle\right|.$$

We divide the proof of the Theorem 4 into a series of lemmas below.

Definition 7. Let $\alpha, \beta \in \mathbb{N}^{pq}$ and $\ell \in \mathbb{N}$. Define

(i) $S_\ell = \{\ \beta \in \mathbb{N}^{pq} \mid \quad |\beta| \geq \ell\ \}$.
(ii) $T_\alpha = \{\ \beta \in \mathbb{N}^{pq} \mid \quad \beta \geq \alpha\ \} = \{\ \alpha + \delta \in \mathbb{N}^{pq} \mid \quad \delta \geq 0\ \}$.

Lemma 8. $S_\ell = \bigcup_{|\alpha|=\ell} T_\alpha$. □

Lemma 9. *In the setup above, we have*

$$\lim_{m\to\infty} \frac{f_{n+1}(b_m)}{t_m^\ell} = \sum_{|\alpha|=\ell} \frac{f_{n+1}^{(\alpha)}(a)}{\alpha!}\mathbf{v}^\alpha.$$

Proof. As stated in Remark 1, we assume that the sequence $\{\, b_m \,\}$ in $\mathscr{D}$ is defined by $b_m = a + t_m\mathbf{v}$ for all $m \geq 1$. Then we write down the Taylor series of f_{n+1} centered at a. It follows from Definition 5 on the least positive integer ℓ and for sufficiently large m that we have

$$f_{n+1}(b_m) = f_{n+1}(a + t_m\mathbf{v}) = t_m^\ell \left(\sum_{|\alpha|=\ell} f_{n+1}^{(\alpha)}(a)\frac{\mathbf{v}^\alpha}{\alpha!} + \sum_{|\alpha|>\ell} f_{n+1}^{(\alpha)}(a)\frac{\mathbf{v}^\alpha}{\alpha!} t_m^{|\alpha|-\ell} \right).$$

Now we want to estimate the second sum, by means of Lemma 8 and Cauchy estimate in Lemma 4. For sufficiently large m, we have

$$\begin{aligned}
&\left|\sum_{|\alpha|>\ell} \frac{f_{n+1}^{(\alpha)}(a)}{\alpha!} t_m^{|\alpha|}\mathbf{v}^\alpha\right| \leq \sum_{|\alpha|>\ell} \left|\frac{f_{n+1}^{(\alpha)}(a)}{\alpha!}\right| t_m^{|\alpha|}|\mathbf{v}^\alpha| \\
&= \sum_{\alpha\in S_{\ell+1}} \left|\frac{f_{n+1}^{(\alpha)}(a)}{\alpha!}\right| t_m^{|\alpha|}|\mathbf{v}^\alpha| \leq \sum_{|\beta|=\ell+1} \quad \sum_{\alpha\in T_\beta} \left|\frac{f_{n+1}^{(\alpha)}(a)}{\alpha!}\right| t_m^{|\alpha|}|\mathbf{v}^\alpha| \\
&= \sum_{|\beta|=\ell+1} \quad \sum_{\delta\geq 0} \left|\frac{f_{n+1}^{(\beta+\delta)}(a)}{(\beta+\delta)!}\right| t_m^{(|\beta|+|\delta|)}|\mathbf{v}^{(\beta+\delta)}| \\
&\leq \sum_{|\beta|=\ell+1} \quad t_m^{\ell+1}|\mathbf{v}^\beta| \sum_{\delta\geq 0} \left|\frac{f_{n+1}^{(\beta+\delta)}(a)}{\delta!}\right| t_m^{|\delta|}|\mathbf{v}^\delta| \\
&\leq t_m^{\ell+1} \sum_{|\beta|=\ell+1} \quad |\mathbf{v}^\beta| \sum_{\delta\geq 0} \frac{M_{2t_m}(f_{n+1}^{(\beta)})}{(2t_m)^{|\delta|}} t_m^{|\delta|}|\mathbf{v}^\delta|
\end{aligned}$$

$$\leq t_m^{\ell+1} \sum_{|\beta|=\ell+1} |\mathbf{v}^\beta| \, M_{2t_m}(f_{n+1}^{(\beta)}) \sum_{\delta\geq 0} | \left(\frac{\mathbf{v}}{2}\right)^\delta |$$

$$= t_m^{\ell+1} \sum_{|\beta|=\ell+1} \frac{|\mathbf{v}^\beta| \, M_{2t_m}(f_{n+1}^{(\beta)})}{\prod_{i=1}^{pq}(1-|\mathbf{v}(i)/2|)}, \tag{28}$$

where $M_{2t_m}(f_{n+1}^{(\beta)}) = \max\left\{ \, |f_{n+1}^{(\beta)}(a+\mathbf{r})| \, \middle| \, \|\mathbf{r}\|_\infty \leq 2t_m \right\}$, and $\mathbf{v} = (\mathbf{v}(1), \mathbf{v}(2), \cdots, \mathbf{v}(pq))$.

The summands in (28) satisfy

$$\lim_{m\to\infty} \frac{|\mathbf{v}^\beta| \, M_{2t_m}(f_{n+1}^{(\beta)})}{\prod_{i=1}^{pq}(1-|\mathbf{v}(i)/2|)} = \frac{|\mathbf{v}^\beta| \, |f_{n+1}^{(\beta)}(a)|}{\prod_{i=1}^{pq}(1-|\mathbf{v}(i)/2|)} < \infty,$$

so we have

$$\lim_{m\to\infty} \frac{1}{t_m^\ell} \left| \sum_{|\alpha|>\ell} \frac{f_{n+1}^{(\alpha)}(a)}{\alpha!} t_m^{|\alpha|} \mathbf{v}^\alpha \right| \leq \lim_{m\to\infty} t_m \sum_{|\beta|=\ell+1} \frac{|\mathbf{v}^\beta| \, M_{2t_m}(f_{n+1}^{(\beta)})}{\prod_{i=1}^{pq}(1-|\mathbf{v}(i)/2|)} = 0,$$

and hence $\lim_{m\to\infty} \dfrac{f_{n+1}(b_m)}{t_m^\ell} = \sum_{|\alpha|=\ell} \dfrac{f_{n+1}^{(\alpha)}(a)}{\alpha!} \mathbf{v}^\alpha$. □

As $S = \lim_{m\to\infty} g(b_m) > 0$ in (26), and

$$S = \lim_{m\to\infty} \left(\frac{\frac{|\langle f_{n+1}, k_{b_m}\rangle|}{t_m^\ell}}{\frac{\|k_{b_m} - \sum_{i=1}^n \langle k_{b_m}, \mathscr{B}_i\rangle \mathscr{B}_i\|}{t_m^\ell}} \right), \tag{29}$$

we have

$$\lim_{m\to\infty} \frac{\|k_{b_m} - \sum_{i=1}^n \langle k_{b_m}, \mathscr{B}_i\rangle \mathscr{B}_i\|}{t_m^\ell} = \frac{\lim_{m\to\infty} \frac{|\langle f_{n+1}, k_{b_m}\rangle|}{t_m^\ell}}{S}, \tag{30}$$

which is finite and non-zero.

Definition 8. In the notations above, we define

$$T_m = \sum_{|\alpha|<\ell} k_{a,\alpha} \frac{\overline{\mathbf{v}}^\alpha}{\alpha!} t_m^{|\alpha|} \quad \text{and} \quad W_m = k_{b_m} - T_m - t_m^\ell E,$$

where $E = \sum_{|\alpha|=\ell} k_{a,\alpha} \frac{\overline{\mathbf{v}}^\alpha}{\alpha!}$ is defined in Theorem 4.

It is obvious that T_m, E, W_m are in $A^2(\mathscr{D})$. And it follows from $b_m = a + t_m\mathbf{v}$ and Proposition 3 that in the norm of $A^2(\mathscr{D})$ we have

$$T_m + t_m^\ell E + W_m = k_{b_m} = k_{a+t_m\mathbf{v}} = \sum_{\alpha\geq 0} k_{a,\alpha} \frac{\overline{\mathbf{v}}^\alpha}{\alpha!} t_m^{|\alpha|}, \tag{31}$$

which implies that the following holds in the norm of $A^2(\mathscr{D})$,

$$W_m = \sum_{|\alpha|>\ell} k_{a,\alpha} \frac{\overline{\mathbf{v}}^\alpha}{\alpha!} t_m^{|\alpha|}. \tag{32}$$

Then it follows from (31) that

$$\begin{aligned} k_{b_m} - \sum_{i=1}^n \langle k_{b_m}, \mathscr{B}_i \rangle \mathscr{B}_i &= \left(T_m - \sum_{i=1}^n \langle T_m, \mathscr{B}_i \rangle \mathscr{B}_i \right) \\ &+ \left(E - \sum_{i=1}^n \langle E, \mathscr{B}_i \rangle \mathscr{B}_i \right) t_m^\ell + \left(W_m - \sum_{i=1}^n \langle W_m, \mathscr{B}_i \rangle \mathscr{B}_i \right). \end{aligned} \tag{33}$$

Lemma 10. $\lim_{m\to\infty} \dfrac{\|W_m - \sum_{i=1}^n \langle W_m, \mathscr{B}_i \rangle \mathscr{B}_i\|}{t_m^\ell} = 0.$

Proof. As $\{\mathscr{B}_1, \dots, \mathscr{B}_n\}$ is an orthonormal set in $A^2(\mathscr{D})$, $\sum_{i=1}^n \langle W_m, \mathscr{B}_i \rangle \mathscr{B}_i$ is the orthogonal projection of W_m onto the subspace spanned by $\{\mathscr{B}_1, \dots, \mathscr{B}_n\}$, so $\|W_m - \sum_{i=1}^n \langle W_m, \mathscr{B}_i \rangle \mathscr{B}_i\| \le \|W_m\|$. It suffices to show the limit $\lim_{m\to\infty} \dfrac{\|W_m\|}{t_m^\ell} = 0$. For any $h \in A^2(\mathscr{D})$ and for sufficiently large m, it follows from (32) that

$$\langle h, W_m \rangle = \langle h, \sum_{|\alpha|>\ell} k_{a,\alpha} \frac{\overline{\mathbf{v}}^\alpha}{\alpha!} t_m^{|\alpha|} \rangle = \sum_{|\alpha|>\ell} \frac{h^{(\alpha)}(a)}{\alpha!} \mathbf{v}^\alpha t_m^{|\alpha|}.$$

Following the same idea of the proof of Lemma 9, we have

$$|\langle h, W_m \rangle| \le \sum_{|\alpha|\ge \ell+1} \left| \frac{h^{(\alpha)}(a)}{\alpha!} \right| |\mathbf{v}|^\alpha t_m^{|\alpha|} \le t_m^{\ell+1} \sum_{|\beta|=\ell+1} |\mathbf{v}^\beta| \frac{M_{2t_m}(h^{(\beta)})}{\prod_{i=1}^{pq} (1 - \frac{|\mathbf{v}(i)|}{2})}. \tag{34}$$

In particular, we can estimate $\dfrac{\|W_m\|^2}{t_m^{2\ell}} = \dfrac{\langle W_m, W_m \rangle}{t_m^{2\ell}}$ by replacing $h = W_m$ in (34).

For this, we need to show the following estimate

$$\begin{aligned} M_{2t_m}(W_m^{(\beta)}) &= \max \left\{ \ |W_m^{(\beta)}(a + 2t_m \mathbf{r})| \ \middle| \ \|\mathbf{r}\|_\infty \le 1 \ \right\} \\ &\le t_m^{\ell+1} \max_{\|\mathbf{r}\|_\infty \le 1} \left\{ \ \sum_{|\gamma|=\ell+1} |\mathbf{v}^\gamma| \frac{M_{2t_m}(K_{a+2t_m \mathbf{r}}^{(\gamma,\overline{\beta})})}{\prod_{i=1}^{pq} \left(1 - \frac{|\mathbf{v}(i)|}{2}\right)} \ \right\}, \end{aligned} \tag{35}$$

where

$$K_{a+2t_m\mathbf{r}}^{(\gamma,\overline{\beta})}(\zeta) = \frac{\partial^{|\beta|+|\gamma|} K}{\partial \overline{w}^\beta \partial z^\gamma}(z, w)\Big|_{(z,w)=(\,\zeta,\, a+2t_m\mathbf{r}\,)}.$$

We proceed to prove inequality (35) in the following.

For any fixed vector $\mathbf{r}$ in $\mathbb{C}^{pq}$ and $\beta \in \mathbb{N}^{pq}$ with $|\beta| = \ell + 1$, it follows from (32) and Lemma 3 that

$$\begin{aligned} &|W_m^{(\beta)}(a + 2t_m \mathbf{r})| = |\langle W_m, k_{a+2t_m\mathbf{r},\beta} \rangle| \\ &= \left| \langle \sum_{|\alpha|>\ell} k_{a,\alpha} \frac{\overline{\mathbf{v}}^\alpha}{\alpha!} t_m^{|\alpha|},\ k_{a+2t_m\mathbf{r},\beta} \rangle \right| = \left| \sum_{|\alpha|>\ell} \langle k_{a,\alpha}^{(\beta)} \frac{\overline{\mathbf{v}}^\alpha}{\alpha!} t_m^{|\alpha|},\ k_{a+2t_m\mathbf{r}} \rangle \right| \\ &= \left| \sum_{|\alpha|>\ell} k_{a,\alpha}^{(\beta)}(a + 2t_m \mathbf{r}) \frac{\overline{\mathbf{v}}^\alpha}{\alpha!} t_m^{|\alpha|} \right| = \left| \sum_{|\alpha|>\ell} \frac{K^{(\beta,\overline{\alpha})}(a + 2t_m \mathbf{r}, a)}{\alpha!} \overline{\mathbf{v}}^\alpha t_m^{|\alpha|} \right| \end{aligned}$$

$$= \left| \sum_{|\alpha|>\ell} \overline{\frac{K^{(\alpha,\overline{\beta})}(a, a+2t_m\mathbf{r})}{\alpha!}} \overline{\mathbf{v}}^\alpha t_m^{|\alpha|} \right| \leq \sum_{|\alpha|>\ell} \left| \frac{K^{(\alpha,\overline{\beta})}(a, a+2t_m\mathbf{r})}{\alpha!} \right| |\mathbf{v}^\alpha| t_m^{|\alpha|}$$

$$\leq t_m^{\ell+1} \sum_{|\gamma|=\ell+1} |\mathbf{v}^\gamma| \frac{M_{2t_m}(K^{(\gamma,\overline{\beta})}_{a+2t_m\mathbf{r}})}{\prod_{i=1}^{pq}\left(1 - \frac{|\mathbf{v}(i)|}{2}\right)}.$$

In the last inequality, we apply the same technique in the proof of the estimate (28) in Lemma 9. Moreover, one can easily use the continuity of the function $K^{(\gamma,\overline{\beta})}(z, a+2t_m\mathbf{r})$ to show that the finite sum of the last expression has a finite limit as m tends to infinity.

The final result follows from the extra factor t_m^2 in (34) and (35) as follows:

$$\frac{\|W_m\|^2}{t_m^{2\ell}} \leq t_m^2 \sum_{|\beta|=|\gamma|=\ell+1} \frac{|\mathbf{v}^\beta|\,|\mathbf{v}^\gamma| \max_{\|\mathbf{r}\|_\infty \leq 1} \left\{ M_{2t_m}(K^{(\gamma,\overline{\beta})}_{a+2t_m\mathbf{r}}) \right\}}{\prod_{i=1}^{pq}\left(1 - \frac{|\mathbf{v}(i)|}{2}\right)^2},$$

in which the last finite double sum converges to finite number as m converges to infinity. □

Lemma 11. *Let* $E_j = \sum_{|\alpha|=j} \frac{\overline{\mathbf{v}}^\alpha}{\alpha!} k_{a,\alpha}$ *and* $F_j = E_j - \sum_{i=1}^{n} \langle E_j, \mathscr{B}_i\rangle \mathscr{B}_i$. *Let* $J = \{\, j \mid 1 \leq j \leq \ell - 1,\ F_j \neq 0 \,\}$. *Then* $\{\, F_j \,\}_{j\in J}$ *is linearly independent.*

Proof. Suppose that $\sum_{j\in J} c_j F_j = 0$, we want to prove that $c_j = 0$ for all $j \in J$ as follows. Then the assumption implies that

$$\sum_{j\in J} c_j E_j = \sum_{j\in J}\sum_{i=1}^{n} c_j \langle E_j, \mathscr{B}_i\rangle \mathscr{B}_i. \tag{36}$$

It follows from Proposition 5(iv) $\mathrm{span}\{\mathscr{B}_1, \cdots, \mathscr{B}_n\} = \mathrm{span}\{k_{a_1,m_1}, \cdots, k_{a_n,m_n}\}$ that there exist $A_1, \cdots, A_n \in \mathbb{C}$ such that

$$\sum_{j\in J} c_j E_j = \sum_{i=1}^{n} A_i k_{a_i,m_i}. \tag{37}$$

Let $I = \{\, i \mid 1 \leq i \leq n, a_i = a \,\}$. By definition, each E_j is a linear combination of generalized kernel functions of same total order j at the same point a, it follows from (37), Proposition 2(iii) and Proposition 5(iv) that for each $j \in J$, there exist $C_1, \cdots, C_n \in \mathbb{C}$ such that

$$c_j E_j = \sum_{i\in I, m_i=j} A_i k_{a,m_i} = \sum_{i=1}^{n} C_i \mathscr{B}_i,$$

$$c_j F_j = c_j E_j - \sum_{i=1}^{n} \langle c_j E_j, \mathscr{B}_i\rangle \mathscr{B}_i = \mathbf{0}.$$

As $j \in J$, so $F_j \neq \mathbf{0}$, and hence $c_j = 0$ for all $j \in J$. □

Finally, Lemma 10 and Lemma 11 imply that the following.

Lemma 12. $T_m - \sum_{i=1}^{n} \langle T_m, \mathscr{B}_i\rangle \mathscr{B}_i = \mathbf{0}$ *in* $A^2(\mathscr{D})$ *for any* $m \geq 1$.

Proof. Apply the triangle inequality to (33), we have

$$\frac{\|k_{b_m} - \sum_{i=1}^{n}\langle k_{b_m}, \mathscr{B}_i\rangle \mathscr{B}_i\|}{t_m^\ell} + \frac{\|W_m - \sum_{i=1}^{n}\langle W_m, \mathscr{B}_i\rangle \mathscr{B}_i\|}{t_m^\ell} + \|E - \sum_{i=1}^{n}\langle E, \mathscr{B}_i\rangle \mathscr{B}_i\| \geq \frac{\|T_m - \sum_{i=1}^{n}\langle T_m, \mathscr{B}_i\rangle \mathscr{B}_i\|}{t_m^\ell}. \tag{38}$$

As m tends to $+\infty$, Lemma 10 implies that the sum of 3 terms in (38) tends to

$$\frac{\lim_{m\to\infty} \frac{|\langle f_{n+1}, k_{b_m}\rangle|}{t_m^\ell}}{S} + 0 + \|E - \sum_{i=1}^{n}\langle E, \mathscr{B}_i\rangle \mathscr{B}_i\| < \infty,$$

so the right hand side in (38) is bounded for all m.

Define $F_j = \sum_{|\alpha|=j} \frac{\overline{\mathbf{v}}^\alpha}{\alpha!}\left(k_{a,\alpha} - \sum_{i=1}^{n}\langle k_{a,\alpha}, \mathscr{B}_i\rangle \mathscr{B}_i\right) \in A^2(\mathscr{D})$, for $0 \leq j \leq \ell - 1$. As $a_{n+1} = a$ appears in the sequence $(a_1, \cdots, a_n)$, so $F_0 = \mathbf{0}$. Recall that $T_m = \sum_{|\alpha|<\ell} k_{a,\alpha}\frac{\overline{\mathbf{v}}^\alpha}{\alpha!} t_m^{|\alpha|}$, so we have

$$\frac{T_m - \sum_{i=1}^{n}\langle T_m, \mathscr{B}_i\rangle \mathscr{B}_i}{t_m^\ell} = \sum_{j=1}^{\ell-1} \frac{1}{t_m^{\ell-j}} F_j \tag{39}$$

which is a polynomial in $\frac{1}{t_m}$, and the coefficient F_j of the term $\frac{1}{t_m^{\ell-j}}$ in (39).

Claim: All F_j $(j = 0, 1, \cdots, \ell - 1)$ are zero in $A^2(\mathscr{D})$.

Suppose contrary that $F_j \neq \mathbf{0}$ for some $j \in \{1, \cdots, \ell - 1\}$. It follows from Lemma 11 that the non-empty set of all non-zero F_j is linearly independent. The estimate in (38) implies that the right hand side of (39) is bounded, so the terms with $\frac{1}{t_m^{\ell-j}}$ of the non-zero F_j are uniformly bounded independent of m. On the other hand, $\lim_{m\to\infty} t_m = 0$, which is a contradiction as $\ell > j$. It follows that all the functions F_j in (39) must be zero, and hence $T_m - \sum_{i=1}^{n}\langle T_m, \mathscr{B}_i\rangle \mathscr{B}_i = \mathbf{0}$. □

By Lemma 10 and Lemma 12, we can determine the following limit.

Lemma 13. *The following limit*

$$\lim_{m\to\infty} \frac{\| \ k_{b_m} - \sum_{i=1}^{n}\langle k_{b_m}, \mathscr{B}_i\rangle \mathscr{B}_i \ \|}{t_m^\ell} = \left\| \sum_{|\alpha|=\ell} \frac{\overline{\mathbf{v}}^\alpha}{\alpha!}\left(k_{a,\alpha} - \sum_{i=1}^{n}\langle k_{a,\alpha}, \mathscr{B}_i\rangle \mathscr{B}_i\right)\right\|$$

and it is non-zero.

Proof. The result follows from Lemma 12, Lemma 10 and triangle inequality that

$$\lim_{m\to\infty} \frac{\|k_{b_m} - \sum_{i=1}^{n}\langle k_{b_m}, \mathscr{B}_i\rangle \mathscr{B}_i\|}{t_m^\ell} = \lim_{m\to\infty} \left\| \left(\frac{T_m}{t_m^\ell} + E + \frac{W_m}{t_m^\ell}\right) - \sum_{k=1}^{n}\left\langle \left(\frac{T_m}{t_m^\ell} + E + \frac{W_m}{t_m^\ell}\right), \mathscr{B}_k\right\rangle \mathscr{B}_k \right\|$$

$$= \lim_{m\to\infty} \left\| \left(E + \frac{W_m}{t_m^\ell}\right) - \sum_{k=1}^{n} \left\langle \left(E + \frac{W_m}{t_m^\ell}\right), \mathscr{B}_k \right\rangle \mathscr{B}_k \right\|$$
$$= \left\| E - \sum_{k=1}^{n} \left\langle E, \mathscr{B}_k \right\rangle \mathscr{B}_k \right\|.$$

Suppose contrary that the limit above is zero, then $E = \sum_{i=1}^{n} \langle E, \mathscr{B}_i \rangle \mathscr{B}_i$ which is non-zero vector in $A^2(\mathscr{D})$. Proposition 5(i) implies that $\langle f_{n+1}, \mathscr{B}_i \rangle = 0$ for all $i = 1, \cdots, n$, and hence

$$\langle f_{n+1}, E \rangle = \sum_{i=1}^{n} \overline{\langle E, \mathscr{B}_i \rangle} \langle f_{n+1}, \mathscr{B}_i \rangle = 0,$$

which violates the choice of E made in Definition 5(ii). □

Proof of Theorem 4. By Lemma 9 and Lemma 13, we have

$$S = \lim_{m\to\infty} g(b_m) = \lim_{m\to\infty} |\langle f_{n+1}, \mathscr{B}_{n+1}^{b_m} \rangle| = \lim_{m\to\infty} \left(\frac{\frac{|\langle f_{n+1}, k_{b_m} \rangle|}{t_m^\ell}}{\frac{\|k_{b_m} - \sum_{i=1}^{n} \langle k_{b_m}, \mathscr{B}_i \rangle \mathscr{B}_i\|}{t_m^\ell}} \right)$$
$$= \frac{|\sum_{|\alpha|=\ell} \frac{f_{n+1}^{(\alpha)}(a)}{\alpha!} \mathbf{v}^{\overline{\alpha}}|}{\left\| \sum_{|\alpha|=\ell} \frac{\overline{\mathbf{v}}^\alpha}{\alpha!} \left(k_{a,\alpha} - \sum_{i=1}^{n} \langle k_{a,\alpha}, \mathscr{B}_i \rangle \mathscr{B}_i\right) \right\|}$$
$$= \left| \left\langle f_{n+1}, \frac{E - \sum_{i=1}^{n} \langle E, \mathscr{B}_i \rangle \mathscr{B}_i}{\|E - \sum_{i=1}^{n} \langle E, \mathscr{B}_i \rangle \mathscr{B}_i\|} \right\rangle \right|, \tag{40}$$

which completes the proof of Theorem 4. □

Note that $\mathscr{B}_{n+1}$ in Definition 6(iii) agrees with (40). This completes the inductive step in constructing the sequences (i) and (ii) in Theorem 1. However, it still remains to check the maximal modulus property (MSP) in Theorem 1(iii) in order to complete the proof of Theorem 1 as follows.

Proof of MSP in Theorem 1. We have defined $\mathscr{B}_1$ in the initial step and $\mathscr{B}_{n+1}$ in the inductive step for cases (**A**) $a \in \mathscr{D}_n$ and (**B**) $a \in \{a_1, \cdots, a_n\}$ respectively, such that the equality (40) implies

$$|\langle f_{n+1}, \mathscr{B}_{n+1} \rangle| = \sup \left\{ |\langle f_{n+1}, \mathscr{B}_{n+1}^b \rangle| \;\middle|\; b \in \mathscr{D} \setminus \{a_1, a_2, \cdots, a_n\} \right\} = S, \tag{41}$$

where $\mathscr{B}_{n+1}$ is given in Definition 6(iii) as in (40). Since $\mathscr{B}_i$ $(i = 1, \cdots, n)$ are mutually orthonormal, by Proposition 5(iii),

$$f_{n+1} = f - \sum_{i=1}^{n} \langle f, \mathscr{B}_i \rangle \mathscr{B}_i. \tag{42}$$

By Gram-Schmidt orthonormalization process, we have $\langle \mathscr{B}_{n+1}^b, \mathscr{B}_i \rangle = 0$ and $\langle \mathscr{B}_{n+1}, \mathscr{B}_i \rangle = 0$ for each $i = 1, \cdots, n$. Then it follows from (42) that

$$\langle f_{n+1}, \mathscr{B}_{n+1}^b \rangle = \langle f, \mathscr{B}_{n+1}^b \rangle \quad \text{and} \quad \langle f_{n+1}, \mathscr{B}_{n+1} \rangle = \langle f, \mathscr{B}_{n+1} \rangle.$$

Hence, the equality (41) is equivalent to

$$|\langle f, \mathscr{B}_{n+1} \rangle| = \sup \left\{ |\langle f, \mathscr{B}_{n+1}^b \rangle| \;\middle|\; b \in \mathscr{D} \setminus \{a_1, a_2, \cdots, a_n\} \right\} = S. \quad \square \tag{43}$$

5. Convergence of POAFD in $A^2(\mathscr{D})$

In the last section, we will prove our second main result in Theorem 2, i.e., to show a POAFD of any function f in Bergman space $A^2(\mathscr{D})$ converges to the same function f.

Definition 9. For any given $f \in A^2(\mathscr{D})$, a sequence $(a_n)_{n\geq 1}$ of points in $\mathscr{D}$ is called a *maximal selection sequence of* f, if there exists an orthonormal sequence $\{\mathscr{B}_j\}_{j\geq 1}$ in $A^2(\mathscr{D})$ associated to a sequence $(a_n)_{n\geq 1}$ constructed as in Theorem 1 satisfying the maximal modulus property (6) in Theorem 1(iii).

For any given $f \in A^2(\mathscr{D})$, we fix a maximal selection sequence $(a_n)_{n\geq 1}$ of f with an orthonormal sequence $\{\mathscr{B}_j\}_{j\geq 1}$ in $A^2(\mathscr{D})$. Then Bessel's inequality implies the following

Lemma 14. *Let $f \in A^2(\mathscr{D})$. Then for any maximal selection sequence $(a_n)_{n\geq 1}$ of f, one has $\sum_{n=1}^{\infty}\langle f, \mathscr{B}_n\rangle\mathscr{B}_n \in A^2(\mathscr{D})$.*

Now we prove the convergence result of POAFD in Theorem 2 for any maximal selection sequence as follows.

Proof of Theorem 2. Suppose contrary that $(a_j)_{j\geq 1}$ is a sequence given by POAFD applied to f, and $\{\mathscr{B}_j\}_{j\geq 1}$ is the corresponding orthonormal sequence in $A^2(\mathscr{D})$ such that the residual

$$h = f - \sum_{j=1}^{\infty}\langle f, \mathscr{B}_j\rangle\mathscr{B}_j \in A^2(\mathscr{D}) \tag{44}$$

is non-zero. We first prove that $\langle h, \mathscr{B}_j\rangle = 0$ for all $j \geq 1$. In fact,

$$\langle h, \mathscr{B}_k\rangle = \langle f - \sum_{j=1}^{\infty}\langle f, \mathscr{B}_j\rangle\mathscr{B}_j, \mathscr{B}_k\rangle = \langle \lim_{N\to\infty}(f - \sum_{j=1}^{N}\langle f, \mathscr{B}_j\rangle\mathscr{B}_j), \mathscr{B}_k\rangle$$
$$= \lim_{N\to\infty}\langle f - \sum_{j=1}^{N}\langle f, \mathscr{B}_j\rangle\mathscr{B}_j, \mathscr{B}_k\rangle = \lim_{N\to\infty}\Big(\langle f, \mathscr{B}_k\rangle - \langle f, \mathscr{B}_k\rangle\langle\mathscr{B}_k, \mathscr{B}_k\rangle\Big) = 0.$$

As $h \neq 0$ and h is holomorphic on $\mathscr{D}$, there exists a closed ball $\overline{B} \subset \mathscr{D}_n$ centered at some point b in $\mathscr{D}$ with positive radius such that $\overline{B}$ is compact and $|h(z)| > 0$ on $\overline{B}$. Then

$$C_0 = \min_{z\in\overline{B}}\frac{|h(z)|}{K(z,z)} > 0.$$

Recall that

$$f_N = f - \sum_{j=1}^{N-1}\langle f, \mathscr{B}_j\rangle\mathscr{B}_j \tag{45}$$

is the N-th residual. We want to estimate $|\langle f_N, \mathscr{B}_N\rangle|$ in two different ways.

Firstly, Lemma 14 implies that $\sum_{j=1}^{\infty}|\langle f, \mathscr{B}_j\rangle|^2 < \infty$. As $C_0 > 0$, there exists $N_0 \in \mathbb{Z}_+$ such that for all $N \geq N_0$, one has

$$\sum_{j=N}^{\infty} |\langle f, \mathscr{B}_j\rangle|^2 < \left(\frac{C_0}{2}\right)^2. \tag{46}$$

By Proposition 5(ii) and inequality (46), we have

$$|\langle f_N, \mathscr{B}_N\rangle| = |\langle f, \mathscr{B}_N\rangle| < \frac{C_0}{2}. \tag{47}$$

Secondly, for any fixed $N \geq N_0$, we select a point $b \in \overline{B}\backslash\{a_1, \ldots, a_N\}$. We consider another sequence $(a_1, \ldots, a_{N-1}, b)$. Let $(\mathscr{B}_1, \ldots, \mathscr{B}_{N-1}, \mathscr{B}_N^b)$ be the Gram-Schmidt orthonormalization of $(\mathscr{B}_1, \ldots, \mathscr{B}_{N-1}, k_b)$, where

$$\mathscr{B}_N^b = \frac{k_b - \sum_{i=1}^{N-1}\langle k_b, \mathscr{B}_i\rangle \mathscr{B}_i}{\|k_b - \sum_{i=1}^{N-1}\langle k_b, \mathscr{B}_i\rangle \mathscr{B}_i\|}.$$

Note that $\mathscr{B}_N$ is selected according to maximum modulus property (6) in Theorem 1:

$$|\langle f, \mathscr{B}_N\rangle| \geq |\langle f_N, \mathscr{B}_N^b\rangle|. \tag{48}$$

Recall that $e_b = \frac{k_b}{\|k_b\|}$. Then $\langle f_N, \mathscr{B}_i\rangle = 0$ for $1 \leq i \leq N-1$ in Proposition 5(i), the inequality

$$\|k_b - \sum_{i=1}^{N-1}\langle k_b, \mathscr{B}_i\rangle \mathscr{B}_i\|^2 = \|k_b\|^2 - \sum_{i=1}^{N-1} |\langle k_b, \mathscr{B}_i\rangle|^2 \leq \|k_b\|^2. \tag{49}$$

The inequalities (47), (48) and (49) imply that

$$\begin{aligned}|\langle f_N, e_b\rangle| = |\langle f_N, \frac{k_b}{\|k_b\|}\rangle| &= \frac{|\langle f_N, k_b - \sum_{i=1}^{N-1}\langle k_b, \mathscr{B}_i\rangle \mathscr{B}_i\rangle|}{\|k_b\|} \\ &\leq \frac{|\langle f_N, k_b - \sum_{i=1}^{N-1}\langle k_b, \mathscr{B}_i\rangle \mathscr{B}_i\rangle|}{\|k_b - \sum_{i=1}^{N-1}\langle k_b, \mathscr{B}_i\rangle \mathscr{B}_i\|} \\ &= |\langle f_N, \mathscr{B}_N^b\rangle| \leq |\langle f_N, \mathscr{B}_N\rangle| < \frac{C_0}{2}.\end{aligned} \tag{50}$$

By (44) and (45),

$$f_N = f - \sum_{i=1}^{N-1}\langle f, \mathscr{B}_i\rangle \mathscr{B}_i = \sum_{j=N}^{\infty}\langle f, \mathscr{B}_j\rangle \mathscr{B}_j + h.$$

It follows from triangle inequality and Cauchy-Schwarz inequality that

$$\begin{aligned}|\langle f_N, e_b\rangle| &= \left|\left\langle h + \sum_{j=N}^{\infty}\langle f, \mathscr{B}_j\rangle \mathscr{B}_j, e_b\right\rangle\right| \geq \left|\frac{h(b)}{K(b,b)}\right| - \left|\left\langle \sum_{j=N}^{\infty}\langle f, \mathscr{B}_j\rangle \mathscr{B}_j, e_b\right\rangle\right| \\ &\geq \min_{z\in\overline{B}}\left|\frac{h(z)}{K(z,z)}\right| - \left\|\sum_{j=N}^{\infty}\langle f, \mathscr{B}_j\rangle \mathscr{B}_j\right\| \|e_b\| \\ &= C_0 - \sqrt{\sum_{j=N}^{\infty} |\langle f, \mathscr{B}_j\rangle|^2} > C_0 - \frac{C_0}{2} = \frac{C_0}{2},\end{aligned}$$

which contradicts to (50). Consequently, $h = 0$ and this completes the proof of Theorem 2. □

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