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Regularity of Fractional Heat Semigroup Associated with Schrödinger Operators

Pengtao Li [1], Tao Qian [2], Zhiyong Wang [1] and Chao Zhang [3,*]

1 School of Mathematics and Statistics, Qingdao University, Qingdao 266071, China; ptli@qdu.edu.cn (P.L.); 2019020134@qdu.edu.cn (Z.W.)
2 Macau Center for Mathematical Sciences, Macau University of Science and Technology, Macau, China; tqian@must.edu.mo
3 School of Statistics and Mathematics, Zhejiang Gongshang University, Hangzhou 310018, China
* Correspondence: zaoyangzhangchao@163.com

Abstract: Let $L = -\Delta + V$ be a Schrödinger operator, where the potential V belongs to the reverse Hölder class. By the subordinative formula, we introduce the fractional heat semigroup $\{e^{-tL^{\alpha}}\}_{t>0}, 0 < \alpha < 1$, associated with L. By the aid of the fundamental solution of the heat equation: $\partial_t u + Lu = \partial_t u - \Delta u + Vu = 0$, we estimate the gradient and the time-fractional derivatives of the fractional heat kernel $K^L_{\alpha,t}(\cdot,\cdot)$, respectively. This method is independent of the Fourier transform, and can be applied to the second-order differential operators whose heat kernels satisfy the Gaussian upper bounds. As an application, we establish a Carleson measure characterization of the Campanato-type space $BMO^{\gamma}_L(\mathbb{R}^n)$ via the fractional heat semigroup $\{e^{-tL^{\alpha}}\}_{t>0}$.

Keywords: Schrödinger operator; Carleson measure; *BMO*-type space; regularity of semigroup

JEL Classification: 35J10; 42B20; 42B30

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1. Introduction

The aim of this paper is to investigate the fractional heat semigroup of Schrödinger operators

$$L := -\Delta + V(x),$$

where $-\Delta$ denotes the Laplace operator $\Delta = \sum_{i=1}^{n} \partial^2/\partial x_i^2$, and V is a non-negative potential belonging to the reverse Hölder class B_q.

Definition 1. *A non-negative locally L^q integrable function V on $\mathbb{R}^n$ is said to belong to $B_q, 1 < q < \infty$, if there exists $C > 0$ such that the reverse Hölder inequality,*

$$\left(\frac{1}{|B|}\int_B V^q(x)dx\right)^{1/q} \le \frac{C}{|B|}\int_B V(x)dx, \tag{1}$$

holds for every ball B in $\mathbb{R}^n$.

This kind of operator was firstly noted in the famous paper by C. Fefferman [1]. For the special case $V = 0$, $L = -\Delta$, the fractional heat semigroup can be defined via the Fourier transform:

$$\left(e^{-t(-\Delta)^{\alpha}}(f)\right)^{\wedge}(\xi) := e^{-t|\xi|^{2\alpha}}\widehat{f}(\xi),\ \alpha \in (0,1]. \tag{2}$$

In the literature, the fractional heat semigroup $\{e^{-t(-\Delta)^{\alpha}}\}_{t>0}$ has been widely used in the study of partial differential equations, harmonic analysis, potential theory, and modern probability theory. For example, the semigroup $\{e^{-t(-\Delta)^\alpha}\}_{t>0}$ is usually applied to construct the linear part of solutions to fluid equations in mathematical physics, e.g., the generalized Navier–Stokes equation, the quasi-geostrophic equation, and the generalized MHD equations. In the field of probability theory, the researchers use $\{e^{-t(-\Delta)^\alpha}\}_{t>0}$ to describe some kind of Markov process with jumps. For further information and the related applications of fractional heat semigroups $\{e^{-t(-\Delta)^\alpha}\}_{t>0}$, we refer the reader to [2–5].

Denote, by $K_{\alpha,t}(\cdot)$, the integral kernel of $e^{-t(-\Delta)^\alpha}$, i.e.,

$$K_{\alpha,t}(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi - t|\xi|^{2\alpha}} d\xi, \tag{3}$$

and denote, by $K^\theta_{\alpha,t}(\cdot)$, the integral kernel of $(-\Delta)^{\theta/2} e^{-t(-\Delta)^\alpha}$. In [6], by an invariant derivative technique and the Fourier analysis method, Miao–Yuan–Zhang concluded that the kernels $K_{\alpha,t}$ and $K^\theta_{\alpha,t}$ satisfy the following pointwise estimates, respectively (cf., [6], Lemmas 2.1 and 2.2):

$$\begin{cases} K_{\alpha,t}(x) \le \dfrac{Ct}{(t^{1/2\alpha} + |x|)^{n+2\alpha}} & \forall\, (x,t) \in \mathbb{R}^{n+1}_+; \\ K^\theta_{\alpha,t}(x) \le \dfrac{C}{(t^{1/2\alpha} + |x|)^{n+\theta}} & \forall\, (x,t) \in \mathbb{R}^{n+1}_+. \end{cases}$$

Compared with $-\Delta$, for arbitrary Schrödinger operator L with the non-negative potential V, the fractional heat semigroup $\{e^{-tL^\alpha}\}_{t>0}, \alpha \in (0,1)$, can not be defined via (2). In addition, it is obvious that the methods in [6] are invalid for the estimation of the integral kernels of $\{e^{-tL^\alpha}\}_{t>0}$. In this paper, by the functional calculus, we observe that the integral kernel of the Poisson semigroup associated with L can be defined as:

$$P^L_t(x,y) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} K^L_{t^2/4u}(x,y)du = \frac{t}{2\sqrt{\pi}} \int_0^\infty \frac{e^{-t^2/4s}}{s^{3/2}} K^L_s(x,y)ds, \tag{4}$$

where $K^L_s(\cdot,\cdot)$ denotes the integral kernel of e^{-sL}, i.e.,

$$e^{-sL}(f)(x) := \int_{\mathbb{R}^n} K^L_s(x,y) f(y)dy.$$

Recall that $K^L_t(\cdot,\cdot)$ is a positive, symmetric function on $\mathbb{R}^n \times \mathbb{R}^n$, and satisfies $\int_{\mathbb{R}^n} K^L_t(x,y)dy \le 1$. Generally, for $\alpha > 0$, the subordinative formula (cf., [3]) indicates that there exists a continuous function $\eta^\alpha_t(\cdot)$ on $(0,\infty)$, such that:

$$K^L_{\alpha,t}(x,y) = \int_0^\infty \eta^\alpha_t(s) K^L_s(x,y)ds. \tag{5}$$

The identity (5) enables us to estimate $K^L_{\alpha,t}(\cdot,\cdot)$ via the heat kernel $K^L_t(\cdot,\cdot)$. Let $\rho(\cdot)$ be the auxiliary function defined by (12) below. In Propositions 7 and 8, we can obtain the following pointwise estimates of $K^L_{\alpha,t}(\cdot,\cdot)$: for every $N > 0$, there exists a constant C_N, such that:

$$\left|K^L_{\alpha,t}(x,y)\right| \le \frac{C_N t}{(t^{1/2\alpha} + |x-y|)^{n+2\alpha}} \left(1 + \frac{t^{1/2\alpha}}{\rho(x)} + \frac{t^{1/2\alpha}}{\rho(y)}\right)^{-N}, \tag{6}$$

and for every $N > 0$, $0 < \delta' < \min\{1, 2 - n/q\}$, and all $|h| \le t^{1/\alpha}$, there exists a constant C_N, such that:

$$\left|K^L_{\alpha,t}(x+h,y) - K^L_{\alpha,t}(x,y)\right| \le \frac{C_N t(|h|/t^{1/2\alpha})^{\delta'}}{(t^{1/2\alpha} + |x-y|)^{n+2\alpha}} \left(1 + \frac{t^{1/2\alpha}}{\rho(x)} + \frac{t^{1/2\alpha}}{\rho(y)}\right)^{-N}. \tag{7}$$

Based on the estimates (6) and (7), we consider the regularity properties of $K^L_{\alpha,t}(\cdot,\cdot)$. Let $\nabla_{x,t}$ denote the gradient operator on $\mathbb{R}^{n+1}_+$, that is, $\nabla_{x,t} = (\nabla_x, \partial/\partial t)$, where $\nabla_x = (\partial/\partial x_1, \partial/\partial x_2, \ldots, \partial/\partial x_n)$. Generally speaking, for a differential operator L, if the semigroup $\{e^{-tL}\}_{t>0}$ is analytic, then the estimate of the derivative in time of integral kernels can be deduced. However, for the derivatives in spatial variables, it is relatively difficult. Specially, let $H = -\Delta + |x|^2$ be the Hermite operators. The heat kernel related to H, denoted by $K^H_t(\cdot,\cdot)$, can be expressed precisely. Hence, the derivative $\nabla_x K^H_t(\cdot,\cdot)$ can be obtained via a direct computation. (cf., [7,8]). For a general Schrödinger operator, obviously, there does not exist an exact expression of $K^L_t(\cdot,\cdot)$, and the regularity estimates of $K^L_t(\cdot,\cdot)$ cannot be obtained directly as the case of the Hermite operator H. Alternatively, we obtain an energy estimate of the solution to the equation:

$$\partial_t u + Lu = \partial_t u - \Delta u + Vu = 0. \tag{8}$$

By the fundamental solution of $-\Delta$, we prove that, for any $N > 0$, there exists a constant C_N, such that:

$$|\nabla_x K^L_t(x,y)| \le \begin{cases} \dfrac{C_N}{t^{(n+1)/2}} e^{-c|x-y|^2/t}\left(1 + \dfrac{\sqrt{t}}{\rho(x)} + \dfrac{\sqrt{t}}{\rho(y)}\right)^{-N}, & \sqrt{t} \le |x-y|; \\ \dfrac{C_N}{|x-y| t^{n/2}} e^{-c|x-y|^2/t}\left(1 + \dfrac{\sqrt{t}}{\rho(x)} + \dfrac{\sqrt{t}}{\rho(y)}\right)^{-N}, & \sqrt{t} \ge |x-y|; \\ \dfrac{C_N}{t^{(n+1)/2}}\left(1 + \dfrac{\sqrt{t}}{\rho(x)} + \dfrac{\sqrt{t}}{\rho(y)}\right)^{-N}, & \forall\, t, x, y, \end{cases}$$

in Lemma 8. A direct computation, together with the subordinative formula, gives:

$$|\nabla_x K^L_{\alpha,t}(x,y)| \le \frac{C_N t^{1-1/2\alpha}}{(t^{1/2\alpha} + |x-y|)^{n+2\alpha}}\left(1 + \frac{t^{1/2\alpha}}{\rho(x)}\right)^{-N}\left(1 + \frac{t^{1/2\alpha}}{\rho(y)}\right)^{-N};$$

see Proposition 11. By a similar method, we obtain the Hölder regularity of $\nabla_x K^L_{\alpha,t}(\cdot,\cdot)$, i.e., for $|h| < |x-y|/4$ and $\delta' = 1 - n/q$,

$$|\nabla_x K^L_{\alpha,t}(x+h,y) - \nabla_x K^L_{\alpha,t}(x,y)| \le C_N \left(\frac{|h|}{t^{1/2\alpha}}\right)^{\delta'} \frac{1}{t^{1/2\alpha}} \frac{t}{(t^{1/2\alpha} + |x-y|)^{n+2\alpha}};$$

see Proposition 12.

In Section 3.3, we focus on the time-fractional derivatives of $K^L_{\alpha,t}(\cdot,\cdot)$. Recently, there has been an increasing interest in fractional calculus, since time-fractional operators are proven to be very useful for modeling purposes. For example, the following fractional heat equations,

$$\partial^\beta_t u(x,t) = \Delta u(x,t), \tag{9}$$

are used to describe heat propagation in inhomogeneous media. It is known that, as opposed to the classical heat equation, Equation (9) is known to exhibit sub-diffusive behaviour and is related to anomalous diffusions or diffusions in non-homogeneous media, with random fractal structures. Recall that the fractional derivative of $K^L_{\alpha,t}(\cdot,\cdot)$ is defined as:

$$\partial^\beta_t K^L_{\alpha,t}(x,y) := \frac{e^{-i\pi(m-\beta)}}{\Gamma(m-\beta)} \int_0^\infty \partial^m_t K^L_{\alpha,t+u}(x,y) u^{m-\beta} \frac{du}{u},\ \beta > 0 \text{ and } m = [\beta] + 1. \tag{10}$$

For some recent works in the frame of confromable derivatives and Mittag–Leffler kernels, see [9,10]. In Section 3.1, we first obtain the regularity estimates of $t^m \partial^m_t K^L_{\alpha,t}(\cdot,\cdot)$ denoted by $\widetilde{D}^{L,m}_{\alpha,t}(\cdot,\cdot)$; see Proposition 9. Then, the desired estimates of $\partial^\beta_t K^L_{\alpha,t}(\cdot,\cdot)$ can be deduced from (10) and Proposition 9; see Propositions 14–16, respectively.

As an application, in Section 4, we characterize the Camapnato-type spaces associated with L, denoted by $BMO_L^{\gamma}(\mathbb{R}^n)$, via the fractional heat semigroup $\{e^{-tL^{\alpha}}\}_{t>0}$. In the last decades, the characterizations of function spaces associated with Schrödinger operators via semigroups and Carleson measures have attracted the attention of many authors. Let $V \in B_q$, $q > n/2$. Using the family of operators $\{t\partial_t e^{-tL}\}_{t>0}$, the Carleson measure characterization of $BMO_L(\mathbb{R}^n)$ was obtained by Dziubański–Garrigós–Martínez–Torrea–Zienkiewicz [11]. Replacing the potential V by a general Radon measure μ, in [12], Wu–Yan extended the result of [11] to generalized Schrödinger operators. The analogue in the setting of Heisenberg groups was obtained by Lin–Liu [13]. Ma–Stinga–Torrea–Zhang [14] characterized the Campanato-type spaces associated with L via the fractional derivatives of the Poisson semigroup. For further information on this topic, we refer to [15–21] and the references therein. Assume that $L = -\Delta + V$, with $V \in B_q, q > n$. By the regularity estimates obtained in Section 3, we establish the following equivalent characterizations: for $0 < \gamma < \min\{2\alpha,\ 2\alpha\beta\}$,

$$\begin{aligned} f \in BMO_L^{\gamma}(\mathbb{R}^n) \quad &\sim \quad \sup_B \frac{1}{|B|^{1+2\gamma/n}} \int_0^{r_B^{2\alpha}} \int_B |t^{\beta}\partial_t^{\beta} e^{-tL^{\alpha}}(f)(x)|^2 \frac{dxdt}{t} < \infty \\ &\sim \quad \sup_B \frac{1}{|B|^{1+2\gamma/n}} \int_0^{r_B^{2\alpha}} \int_B |t^{1/2\alpha}\nabla_{\alpha} e^{-tL^{\alpha}}(f)(x)|^2 \frac{dxdt}{t} < \infty,, \end{aligned}$$

where $\nabla_{\alpha} := (\nabla_x, \partial_t^{1/2\alpha})$. See Theorems 3 and 4, respectively.

Remark 1.

(i) *The regularity estimates obtained in this paper generalize several results on the regularities of the Schrödinger operators. Letting $\alpha = 1/2$, $K_{1/2,t}^L(\cdot,\cdot) = P_t^L(\cdot,\cdot)$ is the Poisson kernel associated with the Schrödinger operator. For this case, Propositions 11 and 12 come back to ([15], Lemma 3.9). Moreover, the regularities of $\partial_t^{\beta} K_{\alpha,t}^L(\cdot,\cdot)$ obtained in Section 3.3 generalize ([14], Proposition 3.6, (b), (c), and (d)).*

(ii) *The regularity results for $\nabla_x K_t^L(\cdot,\cdot)$ obtained in Section 3.2 all are pointwise estimations, which is stronger than the norm estimates. As a corollary of Lemma 8, by a trivial computation, we can obtain the estimates appearing in ([22], Lemma 2.1) in our suitable setting; see Proposition 10.*

Remark 2. *For the case of $\alpha = 1/2$, the regularities of $\partial_t^{\beta} K_{1/2,t}^L$ have been studied by Ma–Stinga–Torrea–Zhang [14]. We point out that our method is slightly different from that of [14]. In [14], via the Hermite polynomials $H_m(\cdot)$, the authors converted the estimate of $\partial_t^{\beta} P_t^L(\cdot,\cdot)$ to the estimate of $\partial_t K_t^L(\cdot,\cdot)$; see ([14], (3.12)). In Section 3.3, we estimate the time-fractional derivatives of $K_{\alpha,t}^L(\cdot,\cdot)$ via $\partial_t^m K_{\alpha,t}^L(\cdot,\cdot)$, instead of the Hermite polynomials.*

Remark 3.

(i) *In the regularity estimates of $K_{\alpha,t}^L(\cdot,\cdot)$, one of the main tools is the subordinative formula. Due to the analytic property of the heat semigroup $\{e^{-tL}\}_{t>0}$, the estimates of $\partial_t K_t^L(\cdot,\cdot)$ can be deduced from the Cauchy integral formula. Then, we can use the subordinative formula to obtain the related estimates of $\partial_t K_{\alpha,t}^L(\cdot,\cdot)$. However, for the derivatives of $K_t^L(\cdot,\cdot)$ in the spatial variables, i.e., $\nabla_x K_t^L(\cdot,\cdot)$, the method of $\partial_t K_t^L(\cdot,\cdot)$ is invalid and we need a more technical estimate; see Lemmas 8–11 for details.*

(ii) *Following the idea of [11], we can apply the regularities of $K_{\alpha,t}^L(\cdot,\cdot)$ obtained in Section 3 to establish the characterizations of the BMO-type space $BMO_L(\mathbb{R}^n)$. Since the proofs are similar to those in Section 4, we omit the details.*

Some notations:

- $\mathsf{U} \sim \mathsf{V}$ represents that there is a constant $c > 0$, such that $c^{-1}\mathsf{V} \leq \mathsf{U} \leq c\mathsf{V}$, whose right inequality is also written as $\mathsf{U} \lesssim \mathsf{V}$. Similarly, one writes $\mathsf{V} \gtrsim \mathsf{U}$ for $\mathsf{V} \geq c\mathsf{U}$;
- For convenience, the positive constants C may change from one line to another and usually depend on the dimension n, α, β, and other fixed parameters;
- Let B be a ball with the radius r. In the rest of this paper, for $c > 0$, we denote by B_{cr} the ball with the same center and radius cr.

2. Preliminaries

2.1. The Schrödinger Operator

Let $L = -\Delta + V$ be the Schrödinger operator on $\mathbb{R}^n$, $n \geq 3$. Throughout the paper, we will assume that V is a nonzero, non-negative potential, and that it belongs to the reverse Hölder class $B_q, q > n/2,$, which is defined in Definition 1. By Hölder's inequality, we can obtain $B_{q_1} \subset B_{q_2}$ for $q_1 \geq q_2 > 1$. One remarkable feature about the class B_q is that if $V \in B_q$ for some $q > 1$, then there exists $\varepsilon > 0$, which depends only on n and the constant C in (1), such that $V \in B_{q+\varepsilon}$. It is also well known that if $V \in B_q, q > 1$, then $V(x)dx$ is a doubling measure. Namely, for any $r > 0, x \in \mathbb{R}^n$,

$$\int_{B(x,2r)} V(y)dy \leq C_0 \int_{B(x,r)} V(y)dy. \tag{11}$$

The auxiliary function $m(x, V)$ is defined by:

$$\frac{1}{m(x,V)} := \sup\left\{r > 0 : \frac{1}{r^{n-2}} \int_{B(x,r)} V(y)dy \leq 1\right\}. \tag{12}$$

Clearly, $0 < m(x,V) < \infty$ for every $x \in \mathbb{R}^n$, and if $r = 1/m(x,V)$, then $\frac{1}{r^{n-2}} \int_{B(x,r)} V(y)dy = 1$. For simplicity, we sometimes denote $1/m(x,V)$ by $\rho(x)$ in the proofs. We state some properties of $m(x,V)$ which will be used in the proofs of the main results.

Lemma 1. ([23], Lemma 1.2) *There exists a constant $C > 0$, such that for every $0 < r < R < \infty$ and $y \in \mathbb{R}^n$, we have:*

$$\frac{1}{r^{n-2}} \int_{B(y,r)} V(x)dx \leq C\left(\frac{r}{R}\right)^{2-n/q} \frac{1}{R^{n-2}} \int_{B(y,R)} V(x)dx.$$

Lemma 2. ([24], Lemma 3) *For every constant $C_1 > 1$, there exists a constant $C_2 > 1$, such that if*

$$\frac{1}{C_1} \leq \frac{1}{r^{n-2}} \int_{B(x,r)} V(y)dy \leq C_1,$$

then $C_2^{-1} \leq rm(x,V) \leq C_2$.

Lemma 3. ([23], Lemma 1.4) *For every constant $C_1 \geq 1$, there is a constant $C_2 \geq 1$, such that:*

$$\frac{1}{C_2} \leq \frac{m(x,V)}{m(y,V)} \leq C_2$$

for $|x-y| \leq C_1 \frac{1}{m(x,V)}$. Moreover, there exist constants $k_0, C, c > 0$, such that

$$\begin{cases} m(y,V) \leq C(1 + |x-y|m(x,V))^{k_0} m(x,V); \\ m(y,V) \geq cm(x,V)(1 + |x-y|m(x,V))^{-k_0/(1+k_0)}. \end{cases}$$

Lemma 4. ([23], Lemma 1.8) *There exist constants $k_0, C > 0$, such that for $R \geq m(x,V)^{-1}$,*

$$\frac{1}{R^{n-2}} \int_{B(x,R)} V(y)dy \leq C(Rm(x,V))^{k_0}.$$

Lemma 5. ([25], Lemma 1) *Suppose $V \in B_q$, $q > n/2$. Let $m_0 > \log_2 C_0 + 1$, where C_0 is the constant in (11). Then, for any $x_0 \in \mathbb{R}^n$, $R > 0$,*

$$\frac{1}{\{1 + Rm(x_0,V)\}^{m_0}} \int_{B(x_0,R)} V(x)dx \leq CR^{n-2}.$$

As a corollary of ([26], Corollary 4.8), we have:

Lemma 6. *There exist constants C, δ, and l, such that:*

$$\frac{1}{t^{n/2}} \int_{\mathbb{R}^n} e^{-c|x-y|^2/t} V(y)dy \leq \begin{cases} \frac{C}{t}\left(\frac{\sqrt{t}}{\rho(x)}\right)^{\delta}, & \sqrt{t} < m(x,V)^{-1}; \\ \frac{C}{t}\left(\frac{\sqrt{t}}{\rho(x)}\right)^{l}, & \sqrt{t} \geq m(x,V)^{-1}. \end{cases}$$

Since the potential V is non-negative, it follows from the Feynman–Kac formula that the kernels $K_t^L(\cdot,\cdot)$ have a Gaussian upper bound:

$$0 \leq K_t^L(x,y) \leq \frac{1}{(4\pi t)^{n/2}} e^{-|x-y|^2/4t}.$$

Furthermore,

Proposition 1. ([27], Theorem 4.10) *For every $N > 0$, there exist constants C_N and c, such that for all $x, y \in \mathbb{R}^n$,*

$$0 \leq K_t^L(x,y) \leq \frac{C_N}{t^{n/2}} e^{-c|x-y|^2/t} \left(1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)}\right)^{-N}. \tag{13}$$

Proposition 2. ([27], Proposition 4.11) *For every $0 < \delta' < \delta_0 = \min\{1, 2 - n/q\}$ and every $N > 0$, there exist constants $C_N > 0$ and c, such that for $|h| < \sqrt{t}$,*

$$|K_t^L(x+h,y) - K_t^L(x,y)| \leq \frac{C_N}{t^{n/2}} \left(\frac{|h|}{\sqrt{t}}\right)^{\delta'} e^{-c|x-y|^2/t} \left(1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)}\right)^{-N}.$$

Remark 4. *By Proposition 1, it is easy to see that the condition $|h| < \sqrt{t}$ in Proposition 2 can be replaced by $|h| < |x-y|/2$.*

Let $Q_{t,m}^L(x,y) := t^m \partial_t^m K_t^L(x,y)$, $m \in \mathbb{Z}_+$. Then,

Proposition 3. ([28], Proposition 3.3) *Let $m \in \mathbb{Z}_+$.*

(i) *For every $N > 0$, there exist constants $C_N > 0$ and $c > 0$, such that:*

$$|Q_{t,m}^L(x,y)| \leq \frac{C_N}{t^{n/2}} e^{-c|x-y|^2/t} \left(1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)}\right)^{-N}.$$

(ii) *Let $0 < \delta' \leq \delta_0$, where δ_0 appears in Proposition 2. For every $N > 0$, there exist constants $C_N > 0$ and c, such that for $|h| < \sqrt{t}$,*

$$|Q_{t,m}^L(x+h,y) - Q_{t,m}^L(x,y)| \leq \frac{C_N}{t^{n/2}} e^{-c|x-y|^2/t} \left(\frac{|h|}{\sqrt{t}}\right)^{\delta'} \left(1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)}\right)^{-N}.$$

(iii) *For every $N > 0$ and $0 < \delta' \leq \delta_0$, there exists a constant $C_N > 0$, such that:*

$$\Big| \int_{\mathbb{R}^n} Q^L_{t,m}(x,y)dy \Big| \leq C_N \Big(\frac{\sqrt{t}}{\rho(x)}\Big)^{\delta'} \Big(1 + \frac{\sqrt{t}}{\rho(x)}\Big)^{-N}.$$

2.2. Fractional Heat Kernels Associated with L

In this section, we first state some backgrounds on the fractional heat semigroup and the fractional heat kernel associated with L. For the case $V \neq 0$, the fractional heat semigroup associated with L can not be defined using the Fourier multiplier method (2) as the Laplace operator. We strike out on a new path and introduce the fractional heat semigroup via the subordinative formula.

The Schrödinger operator L can be seen as the generator of the semigroup $\{e^{-tL}\}_{t>0}$, i.e.,

$$L(f) := \lim_{t \to 0} \frac{f - e^{-tL} f}{t},$$

where the limit is in $L^2(\mathbb{R}^n)$. L is a self-adjointed, positive operator. The integral kernels of the semigroups $\{e^{-tL}\}_{t>0}$ are denoted by $K^L_t(\cdot,\cdot)$. It is easy to verify that the kernel $K^L_t(\cdot,\cdot)$ satisfies the following:

$$\begin{cases} \text{(i) } K^L_t(x,y) \geq 0, \ x,y \in \mathbb{R}^n; \\ \text{(ii) } K^L_t(x,y) = K^L_t(y,x); \\ \text{(iii) } K^L_{s+t}(x,y) = \displaystyle\int_{\mathbb{R}^n} K^L_s(x,z) K^L_t(z,y) dz; \\ \text{(iv) } \displaystyle\lim_{t \to 0+} \int_{\mathbb{R}^n} K^L_t(x,y) f(y) dy = f(x), \quad \forall f \in L^2(\mathbb{R}^n). \end{cases}$$

For $\alpha \in (0,1)$, the fractional power of L, denoted by L^α, is defined as:

$$L^\alpha(f) = \frac{1}{\Gamma(-\alpha)} \int_0^\infty \Big(e^{-t\sqrt{L}} f(x) - f(x)\Big) \frac{dt}{t^{1+2\alpha}}, \quad \forall f \in L^2(\mathbb{R}^n).$$

Here, $\{e^{-t\sqrt{L}}\}_{t>0}$ denotes the Poisson semigroup related to L, with the kernel $P^L_t(\cdot,\cdot)$ defined as:

$$P^L_t(x,y) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} K^L_{t^2/4u}(x,y) du = \frac{t}{2\sqrt{\pi}} \int_0^\infty \frac{e^{-t^2/4s}}{2s^{3/2}} K^L_s(x,y) ds.$$

By the subordinative formula, we know that there exists a non-negative continuous function $\eta^\alpha_t(\cdot)$ satisfying (cf., [3]):

$$\begin{cases} \eta^\alpha_t(s) = \dfrac{1}{t^{1/\alpha}} \eta^\alpha_1(s/t^{1/\alpha}); \\ \eta^\alpha_t(s) \lesssim \dfrac{t}{s^{1+\alpha}}, \ \forall\, s,t > 0; \\ \displaystyle\int_0^\infty s^{-\gamma} \eta^\alpha_1(s) ds < \infty, \ \gamma > 0; \\ \eta^\alpha_t(s) \simeq \dfrac{t}{s^{1+\alpha}}, \ \forall \ s \geq t^{1/\alpha} > 0, \end{cases} \tag{14}$$

such that $K^L_{\alpha,t}(\cdot,\cdot)$ can be expressed as:

$$K^L_{\alpha,t}(x,y) = \int_0^\infty \eta^\alpha_t(s) K^L_s(x,y) ds; \tag{15}$$

see [3] for some examples of $\eta^\alpha_t(\cdot)$. The function $\eta^\alpha_t(\cdot)$ plays an important role in the estimate of the fractional heat kernel $K^L_{\alpha,t}(\cdot,\cdot)$. Take $\alpha = 1/2$ for example: by (4), we can see that $\eta_t^{1/2}(s) = \frac{t}{2s^{3/2}} e^{-t^2/4s}$, $\forall\, s,t > 0$. It is easy to verify that such $\eta_t^{1/2}(\cdot)$ satisfies conditions in (14). For the special case $L = -\Delta$, a direct computation gives:

$$P_t^{-\Delta}(x,y) = \int_0^\infty \frac{e^{-t^2/4s} t}{2\sqrt{\pi} s^{3/2}} s^{-n/2} e^{-|x-y|^2/s} ds = \frac{c_n t}{(t^2 + |x-y|^2)^{(n+1)/2}},$$

which coincides with the classical Poisson kernel obtained via (3).

2.3. Campanato-Type Spaces Associated with L

The Campanato-type space associated with L is defined as follows:

Definition 2. *The space $BMO_L^\gamma(\mathbb{R}^n)$, $0 < \gamma \leq 1$ is defined as the set of all locally integrable functions f, satisfying that there exists a constant C, such that:*

$$\sup_B \frac{1}{|B|^{1+\gamma/n}} \int_B |f(x) - f(B,V)| dx \leq C, \tag{16}$$

where the supremum is taken over all balls B centered at x_B with radius r_B, and:

$$f(B,V) := \begin{cases} f_B, \ r_B < \rho(x_B); \\ 0, \ r_B \geq \rho(x_B). \end{cases}$$

The norm $\|f\|_{BMO_L^\gamma}$ is defined as the infimum of the constants C, such that (16), above, holds.

Proposition 4. ([14], Proposition 4.3) *Let $B = B(x,r)$ with $r < \rho(x)$. If $f \in BMO_L^\gamma(\mathbb{R}^n), 0 < \gamma \leq 1$, then there exists a constant C_γ, such that $|f_B| \leq C_\gamma (\rho(x))^\gamma \|f\|_{BMO_L^\gamma}$.*

The space $BMO_L^\gamma(\mathbb{R}^n)$ is equivalent to the following Lipschitz-type space related to L:

Definition 3. *For $0 < \gamma \leq 1$, a continuous function f defined on $\mathbb{R}^n$ belongs to the space $C_L^{0,\gamma}(\mathbb{R}^n)$ if*

$$\sup_{x,y \in \mathbb{R}^n} \frac{|f(x) - f(y)|}{|x-y|^\gamma} < \infty \text{ and } \sup_{x \in \mathbb{R}^n} \frac{|f(x)|}{\rho(x)^\gamma} < \infty.$$

Proposition 5. ([14], Proposition 4.6) *If $0 < \gamma \leq 1$, then the spaces $BMO_L^\gamma(\mathbb{R}^n)$ and $C_L^{0,\gamma}(\mathbb{R}^n)$ are equal, and their norms are equivalent.*

It is well known that Hardy spaces $H^p(\mathbb{R}^n), 0 < p \leq 1$, are the predual spaces of Campanato spaces (cf. [29]). In the 2000s, such a dual relationship was extended to function spaces associated with operators; see [11,30–34]. For Schrödinger operator L, the following Hardy-type spaces, $H_L^p(\mathbb{R}^n), 0 < p \leq 1$, were introduced in [26,27]:

Definition 4. *For $0 < p \leq 1$, an integrable function f is an element of the Hardy-type space $H_L^p(\mathbb{R}^n)$ if the maximal function*

$$T^*(f)(x) := \sup_{s>0} |T_s^L(f)(x)|$$

belongs to $L^p(\mathbb{R}^n)$. The quasi-norm in $H_L^p(\mathbb{R}^n)$ is defined by: $\|f\|_{H_L^p} := \|T^*(f)\|_{L^p}$.

Let $\delta_0 = \min\{1, 2 - n/q\}$ and $n/(n+\delta_0) < p \le 1$. An atom of $H_L^p(\mathbb{R}^n)$ associated with a ball $B(x_B, r_B)$ is a function a, such that:

$$\begin{cases} \text{supp } a \subseteq B(x_B, r_B),\ r_B \le \rho(x_B); \\ \|a\|_{L^\infty} \le |B(x_B, r_B)|^{-1/p}; \\ \int_{\mathbb{R}^n} a(x)dx = 0,\ r_B < \rho(x_B)/4. \end{cases}$$

In [27], Dziubański and Zienkiewicz obtained the following atomic characterization of $H_L^p(\mathbb{R}^n)$:

Proposition 6. ([27], Theorem 1.13) *Let $n/(n+\delta_0) < p \le 1$. $f \in H_L^p(\mathbb{R}^n)$ if and only if $f = \sum_j \lambda_j a_j$, where $\{a_j\}$ are H_L^p-atoms and $\sum_j |\lambda_j|^p < \infty$.*

Theorem 1. ([14], Theorem 4.5) *Let $0 \le \gamma < 1$. Then, the dual space of $H_L^{n/(n+\gamma)}(\mathbb{R}^n)$ is $BMO_L^\gamma(\mathbb{R}^n)$. More precisely, any continuous linear functional Φ over $H_L^{n/(n+\gamma)}(\mathbb{R}^n)$ can be represented as*

$$\Phi(a) = \int_{\mathbb{R}^n} f(x)a(x)dx$$

for some function $f \in BMO_L^\gamma(\mathbb{R}^n)$ and all $H_L^{n/(n+\gamma)}$-atoms a. Moreover, the operator norm $\|\Phi\|_{op} \sim \|f\|_{BMO_L^\gamma}$.

Lemma 7. ([14], Lemma 5.4) *Let $q_t(x,y)$ be a function of $x, y \in \mathbb{R}^n$, $t > 0$. Assume that for every $N > 0$, there exists a constant C_N, such that for some $\gamma' \ge \gamma$,*

$$|q_t(x,y)| \le C_N(1 + t/\rho(x) + t/\rho(y))^{-N} t^{-n}(1 + |x-y|/t)^{-(n+\gamma')}.$$

Then, for every $H_L^{n/(n+\gamma)}$-atom g supported on $B(x_0, r)$, there exists $C_{N,x_0,r} > 0$, such that:

$$\sup_{t>0} \left| \int_{\mathbb{R}^n} q_t(x,y)g(y)dy \right| \le C_{N,x_0,r}(1+|x|)^{-(n+\gamma')},\ x \in \mathbb{R}^n.$$

3. Regularities on Fractional Heat Semigroups Associated with L

The aim of this section is to estimate the regularities of the fractional heat kernel $K_{\alpha,t}^L(\cdot,\cdot)$. By the use of (5), we first estimate $\partial_t^m K_{\alpha,t}^L(\cdot,\cdot), m \ge 1$. Then, via the solution to (8), we investigate the spatial gradient of $K_{\alpha,t}^L(\cdot,\cdot)$. At last, we obtain the estimation of the time-fractional derivatives of $K_{\alpha,t}^L(\cdot,\cdot)$.

3.1. Regularities of the Fractional Heat Kernel

We first investigate the regularities of $K_{\alpha,t}^L(\cdot,\cdot)$.

Proposition 7. *Let $\alpha \in (0,1)$ and $V \in B_q, q > n/2$. For every $N > 0$, there exists a constant C_N, such that:*

$$\left|K_{\alpha,t}^L(x,y)\right| \le \frac{C_N t}{(t^{1/2\alpha} + |x-y|)^{n+2\alpha}} \left(1 + \frac{t^{1/2\alpha}}{\rho(x)} + \frac{t^{1/2\alpha}}{\rho(y)}\right)^{-N}.$$

Proof. By Proposition 1, we use (13)–(15) to obtain for any $M, N > 0$, a constant $C_{M,N}$, such that:

$$\begin{aligned} \left|K_{\alpha,t}^L(x,y)\right| &\lesssim \int_0^\infty \frac{t}{s^{1+\alpha}} \left|K_s^L(x,y)\right| ds \\ &\lesssim C_{M,N} \int_0^\infty \frac{te^{-c|x-y|^2/s}}{s^{1+\alpha+n/2}} \left(1 + \frac{\sqrt{s}}{\rho(x)}\right)^{-N} \left(1 + \frac{\sqrt{s}}{\rho(y)}\right)^{-M} ds. \end{aligned}$$

By changing variables, we have:

$$\begin{aligned}\left|K^L_{\alpha,t}(x,y)\right| &\lesssim C_{M,N}\int_0^\infty \frac{t^{1+1/\alpha}}{(t^{1/\alpha}u)^{1+\alpha}}(t^{1/\alpha}u)^{-n/2}e^{-\frac{c|x-y|^2}{t^{1/\alpha}u}}\left(1+\frac{t^{1/2\alpha}\sqrt{u}}{\rho(x)}\right)^{-N}\left(1+\frac{t^{1/2\alpha}\sqrt{u}}{\rho(y)}\right)^{-M}du\\ &\lesssim \frac{C_{M,N}}{t^{n/2\alpha}}\left(\frac{t^{1/2\alpha}}{\rho(x)}\right)^{-N}\left(\frac{t^{1/2\alpha}}{\rho(y)}\right)^{-M}\int_0^\infty \frac{1}{u^{1+\alpha}}u^{-n/2-N/2-M/2}e^{-\frac{c|x-y|^2}{t^{1/\alpha}u}}du.\end{aligned}$$

Let $\frac{|x-y|^2}{t^{1/\alpha}u}=r^2$. Then,

$$\begin{aligned}\left|K^L_{\alpha,t}(x,y)\right| &\le C_{M,N}t^{-n/2\alpha}\left(\frac{t^{1/2\alpha}}{\rho(x)}\right)^{-N}\left(\frac{t^{1/2\alpha}}{\rho(y)}\right)^{-M}\int_0^\infty\left(\frac{|x-y|^2}{t^{1/\alpha}r^2}\right)^{-1-\alpha-n/2-N/2-M/2}e^{-cr^2}\frac{|x-y|^2}{t^{1/\alpha}r^3}dr\\ &\le C_{M,N}\left(\frac{t^{1/2\alpha}}{\rho(x)}\right)^{-N}\left(\frac{t^{1/2\alpha}}{\rho(y)}\right)^{-M}\frac{t^{1+(M+N)/(2\alpha)}}{|x-y|^{2\alpha+n+N+M}}\int_0^\infty r^{2\alpha+n+M+N-1}e^{-cr^2}dr\\ &\le C_{M,N}\left(\frac{t^{1/2\alpha}}{\rho(x)}\right)^{-N}\left(\frac{t^{1/2\alpha}}{\rho(y)}\right)^{-M}\frac{t^{1+(M+N)/(2\alpha)}}{|x-y|^{n+2\alpha+N+M}},\end{aligned}$$

which gives:

$$\left(\frac{t^{1/2\alpha}}{\rho(x)}\right)^{N}\left(\frac{t^{1/2\alpha}}{\rho(y)}\right)^{M}\left|K^L_{\alpha,t}(x,y)\right|\le\frac{C_{M,N}t^{1+(M+N)/(2\alpha)}}{|x-y|^{n+2\alpha+N+M}}. \tag{17}$$

On the other hand, using the change of variables again, we obtain:

$$\begin{aligned}\left|K^L_{\alpha,t}(x,y)\right| &\le C_{M,N}\int_0^\infty s^{-n/2}\eta_1^\alpha\left(\frac{s}{t^{1/\alpha}}\right)\left(1+\frac{\sqrt{s}}{\rho(x)}\right)^{-N}\left(1+\frac{\sqrt{s}}{\rho(y)}\right)^{-M}ds\\ &\le C_{M,N}\int_0^\infty (t^{1/\alpha}\tau)^{-n/2}\frac{1}{t^{1/\alpha}}\eta_1^\alpha(\tau)\left(1+\frac{\sqrt{\tau}t^{1/2\alpha}}{\rho(x)}\right)^{-N}\left(1+\frac{\sqrt{\tau}t^{1/2\alpha}}{\rho(y)}\right)^{-M}t^{1/\alpha}d\tau\\ &\le C_{M,N}t^{-n/2\alpha}\left(\frac{t^{1/2\alpha}}{\rho(x)}\right)^{-N}\left(\frac{t^{1/2\alpha}}{\rho(y)}\right)^{-M}\int_0^\infty \tau^{-n/2-M/2-N/2}\eta_1^\alpha(\tau)d\tau.\end{aligned}$$

The above estimate implies that:

$$\left(\frac{t^{1/2\alpha}}{\rho(x)}\right)^{N}\left(\frac{t^{1/2\alpha}}{\rho(y)}\right)^{M}\left|K^L_{\alpha,t}(x,y)\right|\le\frac{C_{M,N}}{t^{n/2\alpha}}. \tag{18}$$

Now, combining (17) and (18), we have:

$$\left(\frac{t^{1/2\alpha}}{\rho(x)}\right)^{N}\left(\frac{t^{1/2\alpha}}{\rho(y)}\right)^{M}\left|K^L_{\alpha,t}(x,y)\right|\le C_{M,N}\min\left\{\frac{t^{1+(M+N)/(2\alpha)}}{|x-y|^{n+2\alpha+N+M}},\ t^{-n/2\alpha}\right\},$$

which, together with the arbitrariness of M,N, indicates that:

$$\left|K^L_{\alpha,t}(x,y)\right|\le\frac{C_Nt}{(t^{1/2\alpha}+|x-y|)^{n+2\alpha}}\left(1+\frac{t^{1/2\alpha}}{\rho(x)}+\frac{t^{1/2\alpha}}{\rho(y)}\right)^{-N}.$$

This completes the proof of Proposition 7. □

Proposition 8. *Let $\alpha\in(0,1)$ and $V\in B_q, q>n/2$. For any $N>0$, there exists a constant $C_N>0$, such that for every $0<\delta'<\delta_0=\min\{1,2-n/q\}$ and all $|h|\le t^{1/2\alpha}$,*

$$|K^L_{\alpha,t}(x+h,y)-K^L_{\alpha,t}(x,y)| \le \frac{C_N(|h|/t^{1/2\alpha})^{\delta}t}{(t^{1/2\alpha}+|x-y|)^{n+2\alpha}}\left(1+\frac{t^{1/2\alpha}}{\rho(x)}+\frac{t^{1/2\alpha}}{\rho(y)}\right)^{-N}.$$

Proof. The proof is similar to that of Proposition 7. We first assume that $|h|<|x-y|/2$. By the subordinative Formula (15), we can use Proposition 2 to obtain, for any $M,N>0$, a constant $C_{M,N}$, such that:

$$\begin{aligned}\Big|K^L_{\alpha,t}(x+h,y)-K^L_{\alpha,t}(x,y)\Big| &\le C_{M,N}\int_0^\infty \frac{t}{s^{1+\alpha}}s^{-n/2}e^{-c|x-y|^2/s}\Big(|h|/\sqrt{s}\Big)^{\delta'}\Big(1+\frac{\sqrt{s}}{\rho(x)}\Big)^{-N}\Big(1+\frac{\sqrt{s}}{\rho(y)}\Big)^{-M}ds\\ &\le C_{M,N}\Big(\frac{|h|}{t^{1/2\alpha}}\Big)^{\delta'}\Big(\frac{t^{1/2\alpha}}{\rho(x)}\Big)^{-N}\Big(\frac{t^{1/2\alpha}}{\rho(y)}\Big)^{-M}\frac{t^{1+(M+N)/(2\alpha)+\delta'/2\alpha}}{|x-y|^{2\alpha+n+M+N+\delta'}}\\ &\le C_{M,N}\Big(\frac{t^{1/2\alpha}}{\rho(x)}\Big)^{-N}\Big(\frac{t^{1/2\alpha}}{\rho(y)}\Big)^{-M}\frac{t^{1+(M+N)/(2\alpha)}|h|^{\delta'}}{|x-y|^{2\alpha+n+M+N+\delta'}},\end{aligned}$$

which implies:

$$\Big(\frac{t^{1/2\alpha}}{\rho(x)}\Big)^{N}\Big(\frac{t^{1/2\alpha}}{\rho(y)}\Big)^{M}\Big|K^L_{\alpha,t}(x+h,y)-K^L_{\alpha,t}(x,y)\Big| \le \frac{C_{M,N}t^{1+(M+N)/(2\alpha)}|h|^{\delta'}}{|x-y|^{2\alpha+n+M+N+\delta'}}. \tag{19}$$

On the other hand, letting $\tau=s/t^{1/\alpha}$, we have:

$$\begin{aligned}&\Big|K^L_{\alpha,t}(x+h,y)-K^L_{\alpha,t}(x,y)\Big|\\ &\le C_{M,N}\int_0^\infty s^{-n/2}\frac{1}{t^{1/\alpha}}\eta^\alpha_1(s/t^{1/\alpha})\Big(\frac{|h|}{\sqrt{s}}\Big)^{\delta'}\Big(1+\frac{\sqrt{s}}{\rho(x)}\Big)^{-N}\Big(1+\frac{\sqrt{s}}{\rho(y)}\Big)^{-M}ds\\ &\le C_{M,N}\int_0^\infty (t^{1/\alpha}\tau)^{-n/2}\eta^\alpha_1(\tau)\Big(\frac{|h|}{\sqrt{\tau}t^{1/2\alpha}}\Big)^{\delta'}\Big(1+\frac{\sqrt{\tau}t^{1/2\alpha}}{\rho(x)}\Big)^{-N}\Big(1+\frac{\sqrt{\tau}t^{1/2\alpha}}{\rho(y)}\Big)^{-M}d\tau\\ &\le C_{M,N}\Big(\frac{t^{1/2\alpha}}{\rho(x)}\Big)^{-N}\Big(\frac{t^{1/2\alpha}}{\rho(y)}\Big)^{-M}t^{-n/2\alpha-\delta'/2\alpha}|h|^{\delta'}\int_0^\infty \tau^{-n/2-M/2-N/2-\delta'/2}\eta^\alpha_1(\tau)d\tau.\end{aligned}$$

This gives:

$$\Big(\frac{t^{1/2\alpha}}{\rho(x)}\Big)^{N}\Big(\frac{t^{1/2\alpha}}{\rho(y)}\Big)^{M}\Big|K^L_{\alpha,t}(x+h,y)-K^L_{\alpha,t}(x,y)\Big| \le C_{M,N}t^{-n/2\alpha}\Big(\frac{|h|}{t^{1/2\alpha}}\Big)^{\delta'}. \tag{20}$$

The estimates (19) and (20) indicate that:

$$\begin{aligned}&\Big(\frac{t^{1/2\alpha}}{\rho(x)}\Big)^{N}\Big(\frac{t^{1/2\alpha}}{\rho(y)}\Big)^{M}\Big|K^L_{\alpha,t}(x+h,y)-K^L_{\alpha,t}(x,y)\Big|\\ &\quad\le C_N\min\left\{\frac{t^{1+(M+N)/(2\alpha)}|h|^{\delta'}}{|x-y|^{2\alpha+n+M+N+\delta'}},\ t^{-n/2\alpha}\Big(\frac{|h|}{t^{1/2\alpha}}\Big)^{\delta'}\right\}.\end{aligned}$$

Due to the arbitrariness of M, we have:

$$\Big|K^L_{\alpha,t}(x+h,y)-K^L_{\alpha,t}(x,y)\Big| \le \frac{C_Nt}{(t^{1/2\alpha}+|x-y|)^{n+2\alpha}}\Big(\frac{|h|}{t^{1/2\alpha}}\Big)^{\delta'}\Big(1+\frac{t^{1/2\alpha}}{\rho(x)}+\frac{t^{1/2\alpha}}{\rho(y)}\Big)^{-N}.$$

This proves Proposition 8 under the assumption $|h|<|x-y|/2$.

Now, we prove this proposition for the case $|h|<t^{1/2\alpha}$. For $|h|<|x-y|/2<t^{1/2\alpha}$ or $|h|<t^{1/2\alpha}<|x-y|/2$, the desired estimate can be deduced from (19) and (20). The case $|x-y|/2<|h|<t^{1/2\alpha}$ remains to be considered. We split:

$$\Big|K^L_{\alpha,t}(x+h,y)-K^L_{\alpha,t}(x,y)\Big| \le S_1+S_2,$$

where

$$\begin{cases} S_1 := \displaystyle\int_{|h|<\sqrt{s}}\eta^\alpha_t(s)\Big|K^L_{\alpha,s}(x+h,y)-K^L_{\alpha,s}(x,y)\Big|ds;\\ S_2 := \displaystyle\int_{|h|\ge\sqrt{s}}\eta^\alpha_t(s)\Big|K^L_{\alpha,s}(x+h,y)-K^L_{\alpha,s}(x,y)\Big|ds.\end{cases}$$

For S_1, since $|h| < \sqrt{s}$, we can follow the procedure of (20) to deduce that:

$$\begin{aligned} S_1 &\lesssim \int_{|h|<\sqrt{s}} s^{-n/2} \frac{1}{t^{1/\alpha}} \eta_1^\alpha(s/t^{1/\alpha}) \Big(\frac{|h|}{\sqrt{s}}\Big)^{\delta'} \Big(1+\frac{\sqrt{s}}{\rho(x)}\Big)^{-N} \Big(1+\frac{\sqrt{s}}{\rho(y)}\Big)^{-N} ds \\ &\lesssim \int_0^\infty (t^{1/\alpha}\tau)^{-n/2} \eta_1^\alpha(\tau) \Big(\frac{|h|}{\sqrt{\tau} t^{1/2\alpha}}\Big)^{\delta'} \Big(1+\frac{\sqrt{\tau}t^{1/2\alpha}}{\rho(x)}\Big)^{-N} \Big(1+\frac{\sqrt{\tau}t^{1/2\alpha}}{\rho(y)}\Big)^{-N} d\tau \\ &\lesssim \Big(\frac{t^{1/2\alpha}}{\rho(x)}\Big)^{-N} \Big(\frac{t^{1/2\alpha}}{\rho(y)}\Big)^{-N} t^{-n/2\alpha} \Big(\frac{|h|}{t^{1/2\alpha}}\Big)^{\delta'} \int_0^\infty \tau^{-n/2-N-\delta'/2} \eta_1^\alpha(\tau) d\tau \\ &\lesssim \Big(\frac{t^{1/2\alpha}}{\rho(x)}\Big)^{-N} \Big(\frac{t^{1/2\alpha}}{\rho(y)}\Big)^{-N} t^{-n/2\alpha} \Big(\frac{|h|}{t^{1/2\alpha}}\Big)^{\delta'}. \end{aligned}$$

We further divide S_2 into $S_2 = S_{2,1} + S_{2,2}$, where

$$\begin{cases} S_{2,1} := \displaystyle\int_{|h|\geq\sqrt{s}} \eta_t^\alpha(s) \Big|K_{\alpha,s}^L(x,y)\Big| ds; \\ S_{2,2} := \displaystyle\int_{|h|\geq\sqrt{s}} \eta_t^\alpha(s) \Big|K_{\alpha,s}^L(x+h,y)\Big| ds. \end{cases}$$

Noticing $|h| > \sqrt{s}$, for $\delta' > 0$, it follows from Proposition 7 that:

$$\begin{aligned} S_{2,1} &\lesssim \int_{|h|\geq\sqrt{s}} s^{-n/2} \frac{1}{t^{1/\alpha}} \eta_1^\alpha(s/t^{1/\alpha}) \Big(1+\frac{\sqrt{s}}{\rho(x)}\Big)^{-N} \Big(1+\frac{\sqrt{s}}{\rho(y)}\Big)^{-N} ds \\ &\lesssim \int_0^\infty s^{-n/2} \frac{1}{t^{1/\alpha}} \eta_1^\alpha(s/t^{1/\alpha}) \Big(\frac{|h|}{\sqrt{s}}\Big)^{\delta'} \Big(1+\frac{\sqrt{s}}{\rho(x)}\Big)^{-N} \Big(1+\frac{\sqrt{s}}{\rho(y)}\Big)^{-N} ds \\ &\lesssim \int_0^\infty (t^{1/\alpha}\tau)^{-n/2} \eta_1^\alpha(\tau) \Big(\frac{|h|}{\sqrt{\tau} t^{1/2\alpha}}\Big)^{\delta'} \Big(1+\frac{\sqrt{\tau}t^{1/2\alpha}}{\rho(x)}\Big)^{-N} \Big(1+\frac{\sqrt{\tau}t^{1/2\alpha}}{\rho(y)}\Big)^{-N} d\tau \\ &\lesssim \Big(\frac{t^{1/2\alpha}}{\rho(x)}\Big)^{-N} \Big(\frac{t^{1/2\alpha}}{\rho(y)}\Big)^{-N} t^{-n/2\alpha} \Big(\frac{|h|}{t^{1/2\alpha}}\Big)^{\delta'}. \end{aligned}$$

For $S_{2,2}$, similarly, we use Proposition 7, again, to deduce that:

$$\begin{aligned} S_{2,2} &\lesssim \int_{|h|\geq\sqrt{s}} s^{-n/2} \frac{1}{t^{1/\alpha}} \eta_1^\alpha(s/t^{1/\alpha}) \Big(1+\frac{\sqrt{s}}{\rho(x+h)}\Big)^{-2N} \Big(1+\frac{\sqrt{s}}{\rho(y)}\Big)^{-2N} ds \\ &\lesssim \int_0^\infty s^{-n/2} \frac{1}{t^{1/\alpha}} \eta_1^\alpha(s/t^{1/\alpha}) \Big(\frac{|h|}{\sqrt{s}}\Big)^{\delta'} \Big(1+\frac{\sqrt{s}}{\rho(y)}\Big)^{-2N} ds \\ &\lesssim \int_0^\infty (t^{1/\alpha}\tau)^{-n/2} \eta_1^\alpha(\tau) \Big(\frac{|h|}{\sqrt{\tau} t^{1/2\alpha}}\Big)^{\delta'} \Big(1+\frac{\sqrt{\tau}t^{1/2\alpha}}{\rho(y)}\Big)^{-2N} d\tau \\ &\lesssim \Big(\frac{t^{1/2\alpha}}{\rho(y)}\Big)^{-2N} t^{-n/2\alpha} \Big(\frac{|h|}{t^{1/2\alpha}}\Big)^{\delta'}, \end{aligned}$$

which, together with the arbitrariness of N, indicates that:

$$S_{2,2} \lesssim \Big(1+\frac{t^{1/2\alpha}}{\rho(y)}\Big)^{-2N} t^{-n/2\alpha} \Big(\frac{|h|}{t^{1/2\alpha}}\Big)^{\delta'}.$$

Because $|x-y|/2 < t^{1/2\alpha}$, by Lemma 3, it holds that:

$$m(y,V) \geq c\frac{m(x,V)}{(1+|x-y|m(x,V))^{k_0/(1+k_0)}} \gtrsim \frac{m(x,V)}{(1+t^{1/2\alpha}m(x,V))^{k_0/(1+k_0)}},$$

which gives:

$$\begin{aligned}\Big(1+\frac{t^{1/2\alpha}}{\rho(y)}\Big)^{-N} &= \Big(1+t^{1/2\alpha}m(y,V)\Big)^{-N}\\ &\lesssim \frac{(1+t^{1/2\alpha}m(x,V))^{k_0N/(1+k_0)}}{\Big((1+t^{1/2\alpha}m(x,V))^{k_0/(1+k_0)}+t^{1/2\alpha}m(x,V)\Big)^N}\\ &\lesssim \Big(1+\frac{t^{1/2\alpha}}{\rho(y)}\Big)^{-N'}.\end{aligned}$$

The estimates for S_1 and S_2, together with $|x-y|/2 < t^{1/2\alpha}$, imply that:

$$\begin{aligned}\Big|K^L_{\alpha,t}(x+h,y)-K^L_{\alpha,t}(x,y)\Big| &\lesssim t^{-n/2\alpha}\Big(\frac{|h|}{t^{1/2\alpha}}\Big)^{\delta'}\Big(1+\frac{t^{1/2\alpha}}{\rho(x)}\Big)^{-N}\Big(1+\frac{t^{1/2\alpha}}{\rho(y)}\Big)^{-N}\\ &\lesssim \frac{(|h|/t^{1/2\alpha})^{\delta}t}{(t^{1/2\alpha}+|x-y|)^{n+2\alpha}}\Big(1+\frac{t^{1/2\alpha}}{\rho(x)}+\frac{t^{1/2\alpha}}{\rho(y)}\Big)^{-N}.\end{aligned}$$

□

For $m\in\mathbb{Z}^+$ and $t>0$, define: $\widetilde{D}^{L,m}_{\alpha,t}(\cdot,\cdot)=t^m\partial^m_tK^L_{\alpha,t}(\cdot,\cdot)$. We can obtain the following estimates about the kernel: $\widetilde{D}^{L,m}_{\alpha,t}(\cdot,\cdot)$.

Proposition 9. *Let $\alpha\in(0,1)$, $V\in B_q, q>n/2$, $m\in\mathbb{Z}_+$, and $\delta=\min\{2\alpha,\delta_0\}$, where δ_0 appears in Proposition 2.*

(i) *For any $N>0$, there exists a constant $C_N>0$, such that:*

$$|\widetilde{D}^{L,m}_{\alpha,t}(x,y)|\le\frac{C_Nt^m}{(t^{1/2\alpha}+|x-y|)^{n+2\alpha m}}\Big(1+\frac{t^{1/2\alpha}}{\rho(x)}+\frac{t^{1/2\alpha}}{\rho(y)}\Big)^{-N};$$

(ii) *Let $0<\delta'\le\delta$. For any $N>0$, there exists a constant $C_N>0$, such that for all $|h|\le t^{1/2\alpha}$,*

$$|\widetilde{D}^{L,m}_{\alpha,t}(x+h,y)-\widetilde{D}^{L,m}_{\alpha,t}(x,y)|\le C_N\Big(\frac{|h|}{t^{1/2\alpha}}\Big)^{\delta'}\frac{t^m}{(t^{1/2\alpha}+|x-y|)^{n+2\alpha m}}\Big(1+\frac{t^{1/2\alpha}}{\rho(x)}+\frac{t^{1/2\alpha}}{\rho(y)}\Big)^{-N};$$

(iii) *Let $0<\delta'\le\delta$. For any $N>\delta$, there exists a constant $C_N>0$, such that:*

$$\Big|\int_{\mathbb{R}^n}\widetilde{D}^{L,m}_{\alpha,t}(x,y)dy\Big|\le C_N\frac{(t^{1/2\alpha}/\rho(x))^{\delta'}}{(1+t^{1/2\alpha}/\rho(x))^N}.$$

Proof. For (i), since $\eta^\alpha_t(s)=\frac{1}{t^{1/\alpha}}\eta^\alpha_1(s/t^{1/\alpha})$,

$$K^L_{\alpha,t}(x,y) = \int_0^\infty\frac{1}{t^{1/\alpha}}\eta^\alpha_1(s/t^{1/\alpha})K^L_s(x,y)ds=\int_0^\infty\eta^\alpha_1(\tau)K^L_{t^{1/\alpha}\tau}(x,y)d\tau.$$

Hence,

$$\Big|\frac{\partial^m}{\partial t^m}K^L_{\alpha,t}(x,y)\Big| = \Big|\frac{\partial^m}{\partial t^m}\Big(\int_0^\infty\eta^\alpha_1(\tau)K^L_{t^{1/\alpha}\tau}(x,y)d\tau\Big)\Big|=\Big|\int_0^\infty\eta^\alpha_1(\tau)\frac{\partial^m}{\partial t^m}K^L_{t^{1/\alpha}\tau}(x,y)d\tau\Big|.$$

By (i) of Proposition 3 and the higher-order derivative formula of the composite function, we can obtain:

$$\begin{aligned}\Big|\frac{\partial^m}{\partial t^m}K^L_{\alpha,t}(x,y)\Big| &\lesssim \sum_{i=1}^m\Big|\int_0^\infty \eta_1^\alpha(\tau)t^{i/\alpha-m}\tau^i\frac{\partial^i}{\partial s^i}K_s^L(x,y)\Big|_{s=t^{1/\alpha}\tau}d\tau\Big|\\ &\le C_Nt^{-m}\int_0^\infty \eta_1^\alpha(\tau)(t^{1/\alpha}\tau)^{-n/2}e^{-c|x-y|^2/t^{1/\alpha}\tau}\Big(1+\frac{\sqrt{\tau}t^{1/2\alpha}}{\rho(x)}\Big)^{-N}\Big(1+\frac{\sqrt{\tau}t^{1/2\alpha}}{\rho(y)}\Big)^{-N}d\tau.\end{aligned}$$

Notice that $\eta_1^\alpha(\tau)\le C/\tau^{1+\alpha}$. By changing the variables, we obtain:

$$\begin{aligned}\Big|\frac{\partial^m}{\partial t^m}K^L_{\alpha,t}(x,y)\Big| &\le \frac{C_N}{t^{m+n/2\alpha}}\Big(\frac{t^{1/2\alpha}}{\rho(x)}\Big)^{-N}\Big(\frac{t^{1/2\alpha}}{\rho(y)}\Big)^{-N}\int_0^\infty \tau^{-1-\alpha-n/2-N}e^{-c|x-y|^2/t^{1/\alpha}\tau}d\tau\\ &\le \frac{C_Nt^{1+N/\alpha-m}}{|x-y|^{2\alpha+n+2N}}\Big(\frac{t^{1/2\alpha}}{\rho(x)}\Big)^{-N}\Big(\frac{t^{1/2\alpha}}{\rho(y)}\Big)^{-N}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\Big|\frac{\partial^m}{\partial t^m}K^L_{\alpha,t}(x,y)\Big| &= \Big|\int_0^\infty \eta_1^\alpha(\tau)\frac{\partial^m}{\partial t^m}K^L_{t^{1/\alpha}\tau}(x,y)d\tau\Big|\\ &\le C_Nt^{-m}\Big(\frac{t^{1/2\alpha}}{\rho(x)}\Big)^{-N}\Big(\frac{t^{1/2\alpha}}{\rho(y)}\Big)^{-N}\int_0^\infty \eta_1^\alpha(\tau)(\tau t^{1/\alpha})^{-n/2}\tau^{-N}d\tau\\ &\le \frac{C_N}{t^{m+n/2\alpha}}\Big(\frac{t^{1/2\alpha}}{\rho(x)}\Big)^{-N}\Big(\frac{t^{1/2\alpha}}{\rho(y)}\Big)^{-N}.\end{aligned}$$

Finally, we have proved that, for arbitrary $N>0$,

$$\Big(\frac{t^{1/2\alpha}}{\rho(x)}\Big)^{N}\Big(\frac{t^{1/2\alpha}}{\rho(y)}\Big)^{N}\Big|\frac{\partial^m}{\partial t^m}K^L_{\alpha,t}(x,y)\Big|\le C_N\min\Big\{\frac{t^{1+N/\alpha-m}}{|x-y|^{2\alpha+n+2N}},\frac{1}{t^{m+n/2\alpha}}\Big\},$$

which gives:

$$|\widetilde{D}^{L,m}_{\alpha,t}(x,y)|\le\frac{C_Nt^m}{(t^{1/2\alpha}+|x-y|)^{n+2\alpha m}}\Big(1+\frac{t^{1/2\alpha}}{\rho(x)}+\frac{t^{1/2\alpha}}{\rho(y)}\Big)^{-N}.$$

For (ii), via the subordinative Formula (15), we can complete the proof by using (ii) of Proposition 3. We omit the details.

For (iii), it is easy to see that $e^{-sL^\alpha}(f)(x)=\int_0^\infty\eta_s^\alpha(\tau)e^{-\tau L}(f)(x)d\tau$. Hence,

$$t^m\frac{\partial^m}{\partial t^m}K^L_{\alpha,t}(x,y)=t^m\frac{\partial^m}{\partial t^m}\Big(\int_0^\infty\eta_1^\alpha(\tau)K^L_{\tau t^{1/\alpha}}(x,y)d\tau\Big)=C_{m,\alpha}\sum_{i=1}^m\int_0^\infty\eta_1^\alpha(\tau)Q^L_{t^{1/\alpha}\tau,i}(x,y)d\tau.$$

It follows from (iii) of Proposition 3 that:

$$\begin{aligned}\Big|\int_{\mathbb{R}^n}\widetilde{D}^{L,m}_{\alpha,t}(x,y)dy\Big| &\lesssim \int_0^\infty\eta_1^\alpha(\tau)\Big|\int_{\mathbb{R}^n}Q^L_{t^{1/\alpha}\tau,i}(x,y)dy\Big|d\tau\\ &\le C_N\int_0^\infty\eta_1^\alpha(\tau)\frac{(\sqrt{t^{1/\alpha}\tau}/\rho(x))^{\delta'}}{(1+\sqrt{t^{1/\alpha}\tau}/\rho(x))^N}d\tau.\end{aligned}$$

If $t^{1/2\alpha}>\rho(x)$, since $N>\delta$, then:

$$\begin{aligned}\Big|\int_{\mathbb{R}^n}\widetilde{D}^{L,m}_{\alpha,t}(x,y)dy\Big| &\le C_N\rho(x)^{N-\delta'}t^{(\delta'-N)/2\alpha}\int_0^\infty\eta_1^\alpha(\tau)\tau^{(\delta'-N)/2}d\tau\\ &\le \frac{C_N(t^{1/2\alpha}/\rho(x))^{\delta'}}{(1+t^{1/2\alpha}/\rho(x))^N}.\end{aligned}$$

If $t^{1/2\alpha} \leq \rho(x)$, then:

$$\Big|\int_{\mathbb{R}^n} \widetilde{D}^{L,m}_{\alpha,t}(x,y)dy\Big| \leq C_N(t^{1/(2\alpha)}/\rho(x))^{\delta'}\int_0^\infty \eta_1^\alpha(\tau)\tau^{\delta'/2}d\tau.$$

Because the function $\eta_1^\alpha(\cdot)$ is continuous, the integral $\int_0^1 \eta_1^\alpha(\tau)\tau^{\delta'/2}d\tau < \infty$. On the other hand, recalling that $\eta_1^\alpha(\tau) \leq 1/\tau^{1+\alpha}$, we obtain:

$$\int_1^\infty \eta_1^\alpha(\tau)\tau^{\delta'/2}d\tau \lesssim \int_1^\infty \frac{1}{\tau^{1+\alpha}}\tau^{\delta'/2}d\tau < \infty,$$

which implies that:

$$\Big|\int_{\mathbb{R}^n} \widetilde{D}^{L,m}_{\alpha,t}(x,y)dy\Big| \leq \frac{C_N(t^{1/2\alpha}/\rho(x))^{\delta'}}{(1+t^{1/(2\alpha)}/\rho(x))^N}.$$

□

3.2. Estimation on the Spatial Gradient

In this section, we investigate the spatial gradient of $K^L_{\alpha,t}(\cdot,\cdot)$, $\alpha > 0$. For the special case $\alpha = 1/2$, i.e., the Poisson kernel, the regularity estimates have been obtained in ([15], Lemma 3.9).

Lemma 8. *Suppose that $V \in B_q$ for some $q > n$. For every $N > 0$, there exist constants $C_N > 0$ and $c > 0$, such that for all $x, y \in \mathbb{R}^n$ and $t > 0$, the kernels $K^L_t(\cdot,\cdot)$ satisfy the following estimates:*

$$|\nabla_x K^L_t(x,y)| \leq \begin{cases} \dfrac{C_N}{t^{(n+1)/2}}e^{-c|x-y|^2/t}\Big(1+\dfrac{\sqrt{t}}{\rho(x)}+\dfrac{\sqrt{t}}{\rho(y)}\Big)^{-N}, & \sqrt{t} \leq |x-y|; \\ \dfrac{C_N}{|x-y|t^{n/2}}e^{-c|x-y|^2/t}\Big(1+\dfrac{\sqrt{t}}{\rho(x)}+\dfrac{\sqrt{t}}{\rho(y)}\Big)^{-N}, & \sqrt{t} > |x-y|. \end{cases}$$

Proof. Let $\Gamma_0(\cdot,\cdot)$ be the fundamental solution of $-\Delta$ in $\mathbb{R}^n$, i.e.,

$$\Gamma_0(x,y) = -\frac{1}{n(n-2)\omega(n)}\frac{1}{|x-y|^{n-2}},\ n \geq 3,$$

where $\omega(n)$ denotes the area of the unit sphere in $\mathbb{R}^n$. Fix $t > 0$ and $x_0, y_0 \in \mathbb{R}^n$. Assume that $u(\cdot,\cdot)$ is a weak solution to the equation:

$$\partial_t u + Lu = \partial_t u + (-\Delta)u + Vu = 0.$$

Let $\eta \in C_0^\infty(B(x_0,2R))$, with some $R > 0$, such that $\eta = 1$ on $B(x_0,3R/2)$, $|\nabla\eta| \leq C/R$, and $|\nabla^2\eta| \leq C/R^2$. Noticing that $\partial_t u + Lu = 0$, we can obtain:

$$\begin{aligned} -\Delta(u\eta) &= -\sum_{i=1}^n\Big(\frac{\partial^2 u}{\partial x_i^2}\cdot\eta + 2\frac{\partial u}{\partial x_i}\frac{\partial\eta}{\partial x_i} + u\cdot\frac{\partial^2\eta}{\partial x_i^2}\Big) \\ &= -\Delta u\cdot\eta - 2\nabla u\cdot\nabla\eta - u\cdot\Delta\eta \\ &= -(\partial_t u)\eta - Vu\eta - 2\nabla u\nabla\eta - u\nabla\eta, \end{aligned}$$

which, together with integration by parts, gives:

$$
\begin{aligned}
-\int_{\mathbb{R}^n}\Gamma_0(x,y)\nabla u(y,t)\cdot\nabla\eta(y)dy &= -\sum_{i=1}^{n}\int_{\mathbb{R}^n}\Gamma_0(x,y)\frac{\partial u(y,t)}{\partial y_i}\frac{\partial\eta(y)}{\partial y_i}dy\\
&=\sum_{i=1}^{n}\int_{\mathbb{R}^n}\Big(\frac{\partial}{\partial y_i}\Gamma_0(x,y)\Big)\frac{\partial\eta(y)}{\partial x_i}u(y,t)dy+\sum_{i=1}^{n}\int_{\mathbb{R}^n}\Gamma_0(x,y)\Big(\frac{\partial^2\eta(y)}{\partial x_i^2}\Big)u(y,t)dy\\
&=\int_{\mathbb{R}^n}\nabla_y\Gamma_0(x,y)\cdot\nabla\eta(y)u(y,t)dy+\int_{\mathbb{R}^n}\Gamma_0(x,y)\cdot\Delta\eta(y)u(y,t)dy.
\end{aligned}
$$

Then we can obtain:

$$
\begin{aligned}
u(x,t)\eta(x) &= \int_{\mathbb{R}^n}\Gamma_0(x,y)\Big\{-V(y)u(y,t)\eta(y)-\eta(y)\partial_t u(y,t)-2\nabla u(y,t)\cdot\nabla\eta(y)-u(y,t)\Delta\eta(y)\Big\}dy\\
&= \int_{\mathbb{R}^n}\Gamma_0(x,y)\Big\{-V(y)u(y,t)\eta(y)-\eta(y)\partial_t u(y,t)+\Delta\eta(y)\cdot u(y,t)\Big\}dy\\
&\quad +2\int_{\mathbb{R}^n}\nabla_y\Gamma_0(x,y)\nabla\eta(y)u(y,t)dy.
\end{aligned}
$$

Notice that it follows from Lemma 5 that (cf., [23], (1.7)):

$$
\int_{B(x_0,2R)}\frac{V(y)}{|x-y|^{n-1}}dy\le\frac{C}{R^{n-1}}\int_{B(x_0,2R)}V(y)dy\le\frac{C}{R}\Big(1+\frac{R}{\rho(x_0)}\Big)^{m_0},\ m_0>1.
$$

Thus, for $x\in B(x_0,R)$, it holds that:

$$
\begin{aligned}
|\nabla_x u(x,t)| &= |\nabla_x(u(x,t)\eta(x))|\\
&\le \int_{B(x_0,2R)}\frac{V(y)|u(y,t)||\eta(y)|}{|x-y|^{n-1}}dy+\int_{B(x_0,2R)}\frac{|\partial_t u(y,t)||\eta(y)|}{|x-y|^{n-1}}dy+\frac{C}{R^{n+1}}\int_{B(x_0,2R)}|u(y,t)|dy\\
&\le \frac{C}{R}\sup_{B(x_0,2R)}|u(y,t)|\Big\{\Big(1+\frac{R}{\rho(x_0)}\Big)^{m_0}+1\Big\}+CR\sup_{B(x_0,R)}|\partial_t u(y,t)|.
\end{aligned}
$$

Take $u(x,t)=K_t^L(x,y_0)$ and $R<\min\{|x_0-y_0|/8,\ \rho(x_0)\}$. We obtain:

$$
|\nabla_x K_t^L(x_0,y_0)|\le\frac{C}{R}\sup_{B(x_0,2R)}|K_t^L(x,y_0)|\Big\{\Big(1+\frac{R}{\rho(x_0)}\Big)^{m_0}+1\Big\}+R\sup_{B(x_0,2R)}|\partial_t K_t^L(x,y_0)|.
$$

If $x\in B(x_0,2R)$, then $|x-y_0|\sim|x_0-y_0|$. Additionally, $\rho(x)\sim\rho(x_0)$ for $|x-x_0|<2R<2\rho(x_0)$. It follows, from Propositions 1 and 3, that for any $N>0$ there exists a constant C_N, such that:

$$
\begin{cases}
\sup\limits_{x\in B(x_0,2R)}|K_t^L(x,y_0)|\le\dfrac{C_N}{t^{n/2}}e^{-c|x_0-y_0|^2/t}\Big(1+\dfrac{\sqrt{t}}{\rho(x_0)}+\dfrac{\sqrt{t}}{\rho(y_0)}\Big)^{-N};\\
\sup\limits_{x\in B(x_0,2R)}|t\partial_t K_t^L(x,y_0)|\le\dfrac{C_N}{t^{n/2}}e^{-c|x_0-y_0|^2/t}\Big(1+\dfrac{\sqrt{t}}{\rho(x_0)}+\dfrac{\sqrt{t}}{\rho(y_0)}\Big)^{-N}.
\end{cases}
\tag{21}
$$

Finally, it can be deduced from (21) that:

$$
|\nabla_x K_t^L(x_0,y_0)| \le \frac{C_N}{R}t^{-n/2}e^{-c|x_0-y_0|^2/t}\Big(1+\frac{\sqrt{t}}{\rho(x_0)}+\frac{\sqrt{t}}{\rho(y_0)}\Big)^{-N}\Big\{1+\frac{R^2}{t}\Big\}.
$$

The rest of the proof is divided into three cases:
Case 1: $R>\sqrt{t}$. For this case, $\sqrt{t}<R<\min\{|x_0-y_0|/8,\ \rho(x_0)\}$. We split

$$
|\nabla_x K_t^L(x_0,y_0)|\le C_N(M_1+M_2),
$$

where
$$\begin{cases} M_1 := \dfrac{\sqrt{t}}{R}\dfrac{1}{t^{(n+1)/2}}e^{-c|x_0-y_0|^2/t}\Big(1+\dfrac{\sqrt{t}}{\rho(x_0)}+\dfrac{\sqrt{t}}{\rho(y_0)}\Big)^{-N}; \\ M_2 := \dfrac{\sqrt{t}}{R}\dfrac{1}{t^{(n+1)/2}}e^{-c|x_0-y_0|^2/t}\Big(1+\dfrac{\sqrt{t}}{\rho(x_0)}+\dfrac{\sqrt{t}}{\rho(y_0)}\Big)^{-N}\dfrac{R^2}{t}. \end{cases}$$

It is obvious that:
$$M_1 \lesssim \frac{1}{t^{(n+1)/2}}e^{-c|x_0-y_0|^2/t}\Big(1+\frac{\sqrt{t}}{\rho(x_0)}+\frac{\sqrt{t}}{\rho(y_0)}\Big)^{-N}.$$

Similarly, for the term M_2, we can also obtain:
$$\begin{aligned} M_2 &\lesssim \frac{1}{t^{(n+1)/2}}e^{-c|x_0-y_0|^2/t}\Big(1+\frac{\sqrt{t}}{\rho(x_0)}+\frac{\sqrt{t}}{\rho(y_0)}\Big)^{-N}\frac{R^2}{t} \\ &\lesssim \frac{1}{t^{(n+1)/2}}e^{-c|x_0-y_0|^2/t}\Big(1+\frac{\sqrt{t}}{\rho(x_0)}+\frac{\sqrt{t}}{\rho(y_0)}\Big)^{-N}\frac{|x_0-y_0|^2}{t} \\ &\lesssim \frac{1}{t^{(n+1)/2}}e^{-c|x_0-y_0|^2/t}\Big(1+\frac{\sqrt{t}}{\rho(x_0)}+\frac{\sqrt{t}}{\rho(y_0)}\Big)^{-N}. \end{aligned}$$

Case 2: $0 < R \le \sqrt{t} < \min\{|x_0-y_0|/8,\ \rho(x_0)\}$. We write:
$$|\nabla_x K_t^L(x_0,y_0)| \le \frac{C_N}{t^{(n+1)/2}}e^{-c|x_0-y_0|^2/t}\Big(1+\frac{\sqrt{t}}{\rho(x_0)}+\frac{\sqrt{t}}{\rho(y_0)}\Big)^{-N}\Big\{\frac{\sqrt{t}}{R}+\frac{R}{\sqrt{t}}\Big\}.$$

Because $R < \sqrt{t} < \min\{|x_0-y_0|/8,\ \rho(x_0)\}$, taking the infimum for R yields:
$$|\nabla_x K_t^L(x_0,y_0)| \le \frac{C_N}{t^{(n+1)/2}}e^{-c|x_0-y_0|^2/t}\Big(1+\frac{\sqrt{t}}{\rho(x_0)}+\frac{\sqrt{t}}{\rho(y_0)}\Big)^{-N}.$$

Case 3: $0 < R < \min\{|x_0-y_0|/8,\ \rho(x_0)\} < \sqrt{t}$. Similarly, we can see that:
$$|\nabla_x K_t^L(x_0,y_0)| \le \frac{C_N}{t^{(n+1)/2}}e^{-c|x_0-y_0|^2/t}\Big(1+\frac{\sqrt{t}}{\rho(x_0)}+\frac{\sqrt{t}}{\rho(y_0)}\Big)^{-N}\Big(\frac{\sqrt{t}}{R}+\frac{R}{\sqrt{t}}\Big).$$

Since $0 < R < \min\{|x_0-y_0|/8,\ \rho(x_0)\} < \sqrt{t}$, the function $\sqrt{t}/R + R/\sqrt{t}$ is decreasing and with the infimum at $R = \min\{|x_0-y_0|/8,\ \rho(x_0)\}$. Then,
$$\begin{aligned} |\nabla_x K_t^L(x_0,y_0)| &\le \frac{C_N}{t^{(n+1)/2}}e^{-c|x_0-y_0|^2/t}\Big(1+\frac{\sqrt{t}}{\rho(x_0)}+\frac{\sqrt{t}}{\rho(y_0)}\Big)^{-N} \\ &\quad\times\Big\{\frac{\sqrt{t}}{\min\{|x_0-y_0|/8,\ \rho(x_0)\}}+\frac{\min\{|x_0-y_0|/8,\ \rho(x_0)\}}{\sqrt{t}}\Big\}. \end{aligned} \tag{22}$$

Case 3.1: $\rho(x_0) \le |x_0-y_0|/8$. Since N is arbitrary, we can deduce from (22) that:
$$\begin{aligned} |\nabla_x K_t^L(x_0,y_0)| &\le \frac{C_N}{t^{(n+1)/2}}e^{-c|x_0-y_0|^2/t}\Big(1+\frac{\sqrt{t}}{\rho(x_0)}+\frac{\sqrt{t}}{\rho(y_0)}\Big)^{-N}\Big(\frac{\sqrt{t}}{\rho(x_0)}+\frac{\rho(x_0)}{\sqrt{t}}\Big) \\ &\le \frac{C_N}{t^{(n+1)/2}}e^{-c|x_0-y_0|^2/t}\Big(1+\frac{\sqrt{t}}{\rho(x_0)}+\frac{\sqrt{t}}{\rho(y_0)}\Big)^{-N}. \end{aligned}$$

Case 3.2: $\rho(x_0) > |x_0 - y_0|/8$. For this case, by (22) again, it holds that:

$$\begin{aligned}|\nabla_x K_t^L(x_0,y_0)| &\leq \frac{C_N}{t^{(n+1)/2}}e^{-c|x_0-y_0|^2/t}\Big(1+\frac{\sqrt{t}}{\rho(x_0)}+\frac{\sqrt{t}}{\rho(y_0)}\Big)^{-N}\Big(\frac{\sqrt{t}}{|x_0-y_0|}+\frac{|x_0-y_0|}{\sqrt{t}}\Big)\\ &\leq \frac{C_N}{t^{(n+1)/2}}e^{-c|x_0-y_0|^2/t}\Big(1+\frac{\sqrt{t}}{\rho(x_0)}+\frac{\sqrt{t}}{\rho(y_0)}\Big)^{-N}\\ &\quad+\frac{C_N}{|x_0-y_0|t^{n/2}}e^{-c|x_0-y_0|^2/t}\Big(1+\frac{\sqrt{t}}{\rho(x_0)}+\frac{\sqrt{t}}{\rho(y_0)}\Big)^{-N}.\end{aligned}$$

Finally, we obtain the following estimates:

$$\begin{aligned}&|\nabla_x K_t^L(x_0,y_0)|\\ &\leq \begin{cases}\frac{C_N}{t^{(n+1)/2}}e^{-c|x_0-y_0|^2/t}\Big(1+\frac{\sqrt{t}}{\rho(x_0)}+\frac{\sqrt{t}}{\rho(y_0)}\Big)^{-N}, & \sqrt{t}\leq \min\Big\{|x_0-y_0|/8,\ \rho(x_0)\Big\};\\ \frac{C_N}{t^{n/2}}e^{-c|x_0-y_0|^2/t}\big(\frac{1}{\sqrt{t}}+\frac{1}{|x_0-y_0|}\big)\Big(1+\frac{\sqrt{t}}{\rho(x_0)}+\frac{\sqrt{t}}{\rho(y_0)}\Big)^{-N}, & \sqrt{t}> \min\Big\{|x_0-y_0|/8,\ \rho(x_0)\Big\}.\end{cases}\end{aligned}$$

Then, if $\sqrt{t}\geq |x_0-y_0|/8$,

$$|\nabla_x K_t^L(x_0,y_0)|\leq \frac{C_N}{|x_0-y_0|t^{n/2}}e^{-c|x_0-y_0|^2/t}\Big(1+\frac{\sqrt{t}}{\rho(x_0)}+\frac{\sqrt{t}}{\rho(y_0)}\Big)^{-N}.$$

This proves Lemma 8. □

Our spatial gradient estimates in this paper all are pointwise estimations, which is stronger than the norm estimates. From the spatial gradient estimates in Lemma 8, we can obtain the estimates appearing in ([22], Lemma 2.1) in the following.

Proposition 10. *Suppose that $V\in B_q$ for some $q>n$, $n\geq 3$. For $\alpha>0$, the spatial derivative of $K^L_{\alpha,t}(\cdot,\cdot)$ satisfies the following L^1-estimate and L^2-estimate, respectively.*

$$\begin{cases}\|\nabla_x K^L_{\alpha,t}(\cdot,y)\|_{L^1(e^{\alpha|x-y|/\sqrt{t}}dx)}\lesssim t^{-1/2};\\ \|\nabla_x K^L_{\alpha,t}(\cdot,y)\|_{L^2(e^{\alpha|x-y|/\sqrt{t}}dx)}\lesssim t^{-\frac{n+2}{4}}.\end{cases}$$

Proof. We only give the details for the L^1-estimate, and the estimate for $\|\cdot\|_{L^2}$ can be dealt with similarly. By Lemma 8, we obtain:

$$\|\nabla_x K^L_{\alpha,t}(\cdot,y)\|_{L^1(e^{\alpha|x-y|/\sqrt{t}}dx)}\lesssim M_1+M_2,$$

where

$$\begin{cases}M_1:=\displaystyle\int_{|x-y|\geq\sqrt{t}}\frac{1}{t^{(n+1)/2}}e^{-c|x-y|^2/t}e^{\alpha|x-y|/\sqrt{t}}dx;\\ M_2:=\displaystyle\int_{|x-y|<\sqrt{t}}\frac{1}{t^{n/2}|x-y|}e^{-c|x-y|^2/t}e^{\alpha|x-y|/\sqrt{t}}dx.\end{cases}$$

By the change of variables, we can obtain:

$$M_2\ \lesssim\ \frac{1}{t^{1/2}}\int_0^1 e^{-cu^2}e^{\alpha u}u^{n-2}du\lesssim t^{-1/2}.$$

For M_1, applying the change of variables again,

$$\begin{aligned} M_1 &\lesssim \frac{1}{t^{1/2}} \int_1^\infty e^{-\alpha u^2} e^{\alpha u} u^{n-1} du \\ &\lesssim \frac{1}{t^{1/2}} \int_{1-\alpha/2c}^\infty e^{-v^2} (v + \alpha/2c)^{n-1} dv. \end{aligned}$$

Case 1: $\alpha \le 2c$. For this case, it is obvious that $M_1 \lesssim t^{-1/2}$.
Case 2: $\alpha > 2c$. Then, we spilt $M_1 \le M_{1,1} + M_{1,2}$, where:

$$\begin{cases} M_{1,1} := \dfrac{1}{t^{1/2}} \displaystyle\int_{\alpha/2c-1}^\infty e^{-v^2} (v + \alpha/2c)^{n-1} dv; \\ M_{1,2} := \dfrac{1}{t^{1/2}} \displaystyle\int_{1-\alpha/2c}^{\alpha/2c-1} e^{-v^2} (v + \alpha/2c)^{n-1} dv. \end{cases}$$

Obviously, $M_{1,1} \lesssim t^{-1/2}$. For $M_{1,2}$, we have:

$$M_{1,2} = \sum_{k=0}^{n-1} C_{n-1}^k (\alpha/2c)^{n-1-k} M_{1,2,k},$$

where

$$M_{1,2,k} = \frac{1}{t^{1/2}} \int_{1-\alpha/2c}^{\alpha/2c-1} e^{-v^2} v^k dv.$$

Noting that

$$M_{1,2,k} = \begin{cases} 0, & k \text{ is odd;} \\ \dfrac{2}{t^{1/2}} \displaystyle\int_0^{\alpha/2c-1} e^{-v^2} v^k dv, & k \text{ is even,} \end{cases}$$

we can obtain $M_1 \lesssim t^{-1/2}$. □

Lemma 9. *Suppose that $V \in B_q$ for some $q > n$. For every $N > 0$, there exists a constant $C_N > 0$, such that for all $x, y \in \mathbb{R}^n$ and $t > 0$, the semigroup kernels $K_t^L(\cdot,\cdot)$ satisfy the following estimate:*

$$|\nabla_x K_t^L(x,y)| \le \frac{C_N}{t^{(n+1)/2}} \left(1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(x)}\right)^{-N}.$$

Proof. Assume that $u(\cdot,\cdot)$ is a weak solution of the equation

$$\partial_t u + Lu = \partial_t u + (-\Delta)u + Vu = 0.$$

Similar to Lemma 8, we can prove that for all $x \in B(x_0, R)$,

$$|\nabla_x u(x,t)| \le \frac{C}{R} \sup_{B(x_0,2R)} |u(y,t)| \left\{ \left(1 + \frac{R}{\rho(x_0)}\right)^{m_0} + 1 \right\} + CR \sup_{B(x_0,R)} |\partial_t u(y,t)|.$$

Take $u(x,t) = K_t^L(x,y_0)$ for fixed y_0, and let $R \in (0, \min\{\rho(x_0), \sqrt{t}\})$. It can be deduced from Propositions 1 and 3 that:

$$\sup_{x \in B(x_0,2R)} \left\{ |K_t^L(x,y_0)| + |t\partial_t K_t^L(x,y_0)| \right\} \le \frac{C_N}{t^{n/2}} \left(1 + \frac{\sqrt{t}}{\rho(x_0)} + \frac{\sqrt{t}}{\rho(y_0)}\right)^{-N}.$$

This, together with $R < \rho(x_0)$, implies that:

$$
\begin{aligned}
|\nabla_x K_t^L(x_0,y_0)| &\leq \frac{C_N}{R}\frac{1}{t^{n/2}}\Big(1+\frac{\sqrt{t}}{\rho(x_0)}+\frac{\sqrt{t}}{\rho(y_0)}\Big)^{-N}\Big\{1+\frac{R^2}{t}\Big\} \\
&\leq \frac{C_N}{t^{(n+1)/2}}\Big(1+\frac{\sqrt{t}}{\rho(x_0)}+\frac{\sqrt{t}}{\rho(y_0)}\Big)^{-N}\Big(\frac{\sqrt{t}}{R}+\frac{R}{\sqrt{t}}\Big).
\end{aligned}
\tag{23}
$$

If $\sqrt{t} \leq \rho(x_0)$, taking the infimum for R on both sides of (23) reaches:

$$
|\nabla_x K_t^L(x_0,y_0)| \leq \frac{C_N}{t^{(n+1)/2}}\Big(1+\frac{\sqrt{t}}{\rho(x_0)}+\frac{\sqrt{t}}{\rho(y_0)}\Big)^{-N}.
$$

If $\sqrt{t} > \rho(x_0)$, note that the function $h(t) := R/\sqrt{t}+\sqrt{t}/R$ is decreasing on $R \in (0,\rho(x_0))$. Taking the infimum again, we obtain:

$$
\begin{aligned}
|\nabla_x K_t^L(x_0,y_0)| &\leq \frac{C_N}{t^{(n+1)/2}}\Big(1+\frac{\sqrt{t}}{\rho(x_0)}+\frac{\sqrt{t}}{\rho(y_0)}\Big)^{-N}\Big\{\frac{\sqrt{t}}{\rho(x_0)}+\frac{\rho(x_0)}{\sqrt{t}}\Big\} \\
&\leq \frac{C_N}{t^{(n+1)/2}}\Big(1+\frac{\sqrt{t}}{\rho(x_0)}+\frac{\sqrt{t}}{\rho(y_0)}\Big)^{-N}.
\end{aligned}
$$

This completes the proof of Lemma 9. □

Now, we give the gradient estimate of $K_{\alpha,t}^L(\cdot,\cdot)$.

Proposition 11. *Suppose $\alpha > 0$ and $V \in B_q$ for some $q > n$. For every $N > 0$, there exists a constant $C_N > 0$, such that for all $x, y \in \mathbb{R}^n$ and $t > 0$,*

$$
|t^{1/2\alpha}\nabla_x K_{\alpha,t}^L(x,y)| \leq \frac{C_N t}{(t^{1/2\alpha}+|x-y|)^{n+2\alpha}}\Big(1+\frac{t^{1/2\alpha}}{\rho(x)}+\frac{t^{1/2\alpha}}{\rho(y)}\Big)^{-N}.
$$

Proof. The subordinate formula gives:

$$
\nabla_x K_{\alpha,t}^L(x,y) = \int_0^\infty \eta_t^\alpha(s)\nabla_x K_s^L(x,y)ds,
$$

which, together with Lemma 8, implies that $\big|\nabla_x K_{\alpha,t}^L(x,y)\big| \leq C_N(L_1+L_2)$, where:

$$
\begin{cases}
L_1 := \displaystyle\int_0^{|x-y|^2} \eta_t^\alpha(s)\frac{1}{s^{(n+1)/2}}e^{-c|x-y|^2/s}\Big(1+\frac{\sqrt{s}}{\rho(x)}+\frac{\sqrt{s}}{\rho(y)}\Big)^{-N}ds; \\
L_2 := \displaystyle\int_{|x-y|^2}^\infty \eta_t^\alpha(s)\frac{1}{s^{n/2}|x-y|}e^{-c|x-y|^2/s}\Big(1+\frac{\sqrt{s}}{\rho(x)}+\frac{\sqrt{s}}{\rho(y)}\Big)^{-N}ds.
\end{cases}
$$

For L_1, letting $s = t^{1/\alpha}u$, we can obtain:

$$
\begin{aligned}
L_1 &\leq \int_0^\infty \frac{t}{(t^{1/\alpha}u)^{1+\alpha}}(t^{1/\alpha}u)^{-(n+1)/2}e^{-\frac{c|x-y|^2}{t^{1/\alpha}u}}\Big(1+\frac{t^{1/2\alpha}\sqrt{u}}{\rho(x)}\Big)^{-N}\Big(1+\frac{t^{1/2\alpha}\sqrt{u}}{\rho(y)}\Big)^{-N}t^{1/\alpha}du \\
&\leq \Big(\frac{t^{1/2\alpha}}{\rho(x)}\Big)^{-N}\Big(\frac{t^{1/2\alpha}}{\rho(y)}\Big)^{-N}t^{-(n+1)/2\alpha}\int_0^\infty u^{-(n+1)/2-(1+\alpha+N)}e^{-\frac{c|x-y|^2}{t^{1/\alpha}u}}du \\
&\leq \Big(\frac{t^{1/2\alpha}}{\rho(x)}\Big)^{-N}\Big(\frac{t^{1/2\alpha}}{\rho(y)}\Big)^{-N}\frac{t^{1+N/\alpha}}{|x-y|^{2\alpha+2N+n+1}}.
\end{aligned}
$$

Similarly, for the term L_2, a change of variables yields:

$$
\begin{aligned}
L_2 &\le \frac{1}{|x-y|}\int_0^\infty \frac{t}{(t^{1/\alpha}u)^{1+\alpha}}(t^{1/\alpha}u)^{-n/2}e^{-\frac{c|x-y|^2}{t^{1/\alpha}u}}\Big(1+\frac{t^{1/2\alpha}\sqrt{u}}{\rho(x)}\Big)^{-N}\Big(1+\frac{t^{1/2\alpha}\sqrt{u}}{\rho(y)}\Big)^{-N}t^{1/\alpha}du \\
&\le \frac{1}{t^{n/2\alpha}|x-y|}\Big(\frac{t^{1/2\alpha}}{\rho(x)}\Big)^{-N}\Big(\frac{t^{1/2\alpha}}{\rho(y)}\Big)^{-N}\int_0^\infty u^{-n/2-(1+\alpha+N)}e^{-\frac{c|x-y|^2}{t^{1/\alpha}u}}du \\
&\le \frac{1}{|x-y|t^{n/2\alpha}}\Big(\frac{t^{1/2\alpha}}{\rho(x)}\Big)^{-N}\Big(\frac{t^{1/2\alpha}}{\rho(y)}\Big)^{-N}\int_0^\infty \Big(\frac{t^{1/\alpha}r^2}{|x-y|^2}\Big)^{1+\alpha+N+n/2}e^{-cr^2}\frac{|x-y|^2}{t^{1/\alpha}r^3}dr \\
&\le \Big(\frac{t^{1/2\alpha}}{\rho(x)}\Big)^{-N}\Big(\frac{t^{1/2\alpha}}{\rho(y)}\Big)^{-N}\frac{t^{1+N/\alpha}}{|x-y|^{2\alpha+2N+n+1}}.
\end{aligned}
$$

The estimates for L_1 and L_2 indicate that:

$$\Big(\frac{t^{1/2\alpha}}{\rho(x)}\Big)^{N}\Big(\frac{t^{1/2\alpha}}{\rho(y)}\Big)^{N}|\nabla_x K^L_{\alpha,t}(x,y)| \le \frac{C_N t^{1+N/\alpha}}{|x-y|^{2\alpha+2N+n+1}}.$$

On the other hand, by Lemma 9 and changing variables $\tau = s/t^{1/\alpha}$, we obtain:

$$
\begin{aligned}
|\nabla_x K^L_{\alpha,t}(x,y)| &\le C_N\int_0^\infty s^{-(n+1)/2}\frac{1}{t^{1/\alpha}}\eta_1^\alpha\big(\frac{s}{t^{1/\alpha}}\big)\Big(1+\frac{\sqrt{s}}{\rho(x)}\Big)^{-N}\Big(1+\frac{\sqrt{s}}{\rho(y)}\Big)^{-N}ds \\
&\le C_N\int_0^\infty (t^{1/\alpha}\tau)^{-(n+1)/2}\eta_1^\alpha(\tau)\Big(1+\frac{t^{1/2\alpha}\sqrt{\tau}}{\rho(x)}\Big)^{-N}\Big(1+\frac{t^{1/2\alpha}\sqrt{\tau}}{\rho(y)}\Big)^{-N}d\tau \\
&\le \frac{C_N}{t^{(n+1)/2\alpha}}\Big(\frac{t^{1/2\alpha}}{\rho(x)}\Big)^{-N}\Big(\frac{t^{1/2\alpha}}{\rho(y)}\Big)^{-N}.
\end{aligned}
$$

Finally, we obtain:

$$\Big(\frac{t^{1/2\alpha}}{\rho(x)}\Big)^{N}\Big(\frac{t^{1/2\alpha}}{\rho(y)}\Big)^{N}|\nabla_x K^L_{\alpha,t}(x,y)| \le C_N\min\Big\{t^{-(n+1)/2\alpha},\ \frac{t^{1+N/\alpha}}{|x-y|^{n+1+2N+2\alpha}}\Big\}.$$

The arbitrariness of N indicates that:

$$|\nabla_x K^L_{\alpha,t}(x,y)| \le \frac{C_N}{t^{1/2\alpha}}\frac{t}{(t^{1/2\alpha}+|x-y|)^{n+2\alpha}}\Big(1+\frac{t^{1/2\alpha}}{\rho(x)}\Big)^{-N}\Big(1+\frac{t^{1/2\alpha}}{\rho(y)}\Big)^{-N}.$$

□

Below, we estimate the Lipschitz continuity of $|\nabla_x K^L_t(\cdot,\cdot)|$.

Lemma 10. *Suppose that $\alpha > 0$ and $V \in B_q$ for some $q > n$. Let $\delta' = 1 - n/q$. For every $N > 0$, there exist constants $C_N > 0$ and $c > 0$, such that for all $x, y \in \mathbb{R}^n$, $t > 0$ and $|h| < |x-y|/4$,*

$$
\begin{aligned}
&|\nabla_x K^L_t(x+h,y) - \nabla_x K^L_t(x,y)| \\
&\quad\le \begin{cases} \dfrac{C_N}{t^{(n+1)/2}}\Big(\dfrac{|h|}{\sqrt{t}}\Big)^{\delta'}e^{-c|x-y|^2/t}\Big(1+\dfrac{\sqrt{t}}{\rho(x)}+\dfrac{\sqrt{t}}{\rho(y)}\Big)^{-N}, & \sqrt{t}\le|x-y|; \\ \dfrac{C_N}{t^{n/2}|x-y|}\Big(\dfrac{|h|}{|x-y|}\Big)^{\delta'}e^{-c|x-y|^2/t}\Big(1+\dfrac{\sqrt{t}}{\rho(x)}+\dfrac{\sqrt{t}}{\rho(y)}\Big)^{-N}, & \sqrt{t}\ge|x-y|. \end{cases}
\end{aligned}
$$

Proof. The proof is similar to that of Lemma 8. Let $\Gamma_0(\cdot,\cdot)$ be the fundamental solution of $-\Delta$ in $\mathbb{R}^n$. Assume that $\partial_t u + (-\Delta)u + Vu = 0$. Let $\eta \in C_0^\infty(B(x_0,2R))$, such that $\eta = 1$ on $B(x_0,3R/2)$, $|\nabla\eta| \le C/R$, and $|\nabla^2\eta| \le C/R^2$. It is easy to see that:

$$(-\Delta)(u\eta) = (-\Delta u)\eta - 2\nabla u\cdot\nabla\eta + u\cdot(-\Delta\eta).$$

Similar to the proof of Lemma 8, an integration by parts implies that:

$$-\int_{\mathbb{R}^n}\Gamma_0(x,y)\nabla u(y,t)\cdot\nabla\eta(y)dy=\int_{\mathbb{R}^n}\nabla_y\Gamma_0(x,y)\nabla\eta(y)u(y,t)dy+\int_{\mathbb{R}^n}\Gamma_0(x,y)\Delta\eta(y)u(y,t)dy,$$

which yields:

$$\begin{aligned} u(x,t)\eta(x) &= \int_{\mathbb{R}^n}\Gamma_0(x,y)\Big\{(-\partial_t u(y,t))\eta(y)-V(y)u(y,t)\eta(y)+u(y,t)\Delta\eta(y)\Big\}dy \\ &\quad +2\int_{\mathbb{R}^n}\nabla_y\Gamma_0(x,y)u(y,t)\cdot\nabla\eta(y)dy. \end{aligned}$$

Then, for $x\in B(x_0,R)$, $u(x,t)=u(x,t)\eta(x)$, and

$$\begin{aligned} \nabla_x^2 u(x,t) &= \nabla_x^2\Big\{\int_{\mathbb{R}^n}\Gamma_0(x,y)\Big\{(-\partial_t u(y,t))\eta(y)-V(y)u(y,t)\eta(y)+u(y,t)\Delta\eta(y)\Big\}dy \\ &\quad +2\int_{\mathbb{R}^n}\nabla_y\Gamma_0(x,y)u(y,t)\cdot\nabla\eta(y)dy\Big\}, \end{aligned}$$

which gives $\|\nabla_x^2 u(\cdot,t)\|_q\le\sum_{i=1}^4\|S_i(\cdot,t)\|_q$, where:

$$\begin{cases} S_1(x,t):=\int_{\mathbb{R}^n}|\nabla_x^2\Gamma_0(x,y)|\cdot|\eta(y)|\cdot|\partial_t u(y,t)|dy; \\ S_2(x,t):=\int_{\mathbb{R}^n}|\nabla_x^2\Gamma_0(x,y)|\cdot|\Delta\eta(y)|\cdot|u(y,t)|dy; \\ S_3(x,t):=2\int_{\mathbb{R}^n}|\nabla_x^2\nabla_y\Gamma_0(x,y)|\cdot|\nabla\eta(y)|\cdot|u(y,t)|dy; \\ S_4(x,t):=\int_{\mathbb{R}^n}|\nabla_x^2\Gamma_0(x,y)|\cdot|\eta(y)|V(y)|\cdot|u(y,t)|dy. \end{cases}$$

Now, we estimate the terms $\|S_i(\cdot,t)\|_q, i=1,2,3,4$ separately. For the term $\|S_1(\cdot,t)\|_q$, because it is well known that $\nabla_x^2\Gamma_0(\cdot,\cdot)$ is a Calderón–Zygmund kernel (see [35]), we have:

$$\begin{aligned} \|S_1(\cdot,t)\|_q &= \Big\|\int_{\mathbb{R}^n}|\eta(y)||\partial_t u(y,t)||\nabla_x^2\Gamma_0(\cdot,y)|dy\Big\|_q \\ &\lesssim \Big\{\sup_{y\in B(x_0,2R)}|\partial_t u(y,t)|\Big\}\Big\|\int_{\mathbb{R}^n}|\eta(y)||\nabla_x^2\Gamma_0(\cdot,y)|dy\Big\|_q \\ &\lesssim \|\eta\|_q\Big\{\sup_{y\in B(x_0,2R)}|\partial_t u(y,t)|\Big\}. \end{aligned}$$

The estimate of S_2 is similar to that of S_1. Noting that $\eta=1$ on $B(x_0,3R/2)$, we can obtain:

$$S_2(x,t)\lesssim\int_{B(x_0,2R)\backslash B(x_0,3R/2)}\frac{|u(y,t)||\Delta\eta(y)|}{|x-y|^n}dy\lesssim\Big\{\sup_{y\in B(x_0,2R)}|u(y,t)|\Big\}\int_{B(x_0,2R)\backslash B(x_0,3R/2)}\frac{|\Delta\eta(y)|}{|x-y|^n}dy.$$

For $3R/2<|y-x_0|<2R$ and $x\in B(x_0,R)$, a direct computation gives $|x-y|\sim R$ and

$$\begin{aligned} S_2(x,t) &\lesssim \frac{\sup_{y\in B(x_0,2R)}|u(y,t)|}{R^n}\int_{B(x_0,2R)\backslash B(x_0,3R/2)}|\Delta\eta(y)|dy \\ &\lesssim \frac{\sup_{y\in B(x_0,2R)}|u(y,t)|}{R^{n+2}}\int_{B(x_0,2R)}dy\lesssim\frac{\sup_{B(x_0,2R)}|u(y,t)|}{R^2}. \end{aligned}$$

Additionally,

$$\begin{aligned}
\|S_2(\cdot,t)\|_q &\lesssim \Big\{\sup_{y\in B(x_0,2R)}|u(y,t)|\Big\}\|\Delta\eta\|_q\\
&\lesssim \Big\{\sup_{y\in B(x_0,2R)}|u(y,t)|\Big\}\Big(\int_{B(x_0,2R)\backslash B(x_0,3R/2)}|\Delta\eta(y)|^q dy\Big)^{1/q}\\
&\lesssim R^{n/q-2}\Big\{\sup_{y\in B(x_0,2R)}|u(y,t)|\Big\}.
\end{aligned}$$

Following the same procedure, we apply the Young inequality to obtain:

$$\begin{aligned}
\|S_3(\cdot,t)\|_q &\lesssim \Big\{\sup_{y\in B(x_0,2R)}|u(y,t)|\Big\}\Big\|\int_{B(x_0,2R)\backslash B(x_0,3R/2)}|\nabla_x^2\nabla_y\Gamma_0(\cdot,y)|\cdot|\nabla\eta(y)|dy\Big\|_q\\
&\lesssim \Big\{\sup_{B(x_0,2R)}|u(y,t)|\Big\}\|\nabla\eta\|_q\cdot\Big\|\nabla_x^2\nabla_y\Gamma_0(x,\cdot)\chi_{B(x_0,2R)\backslash B(x_0,3R/2)}\Big\|_1\\
&\lesssim \Big\{\sup_{y\in B(x_0,2R)}|u(y,t)|\Big\}R^{n/q-1}\Big(\int_{B(x_0,2R)\backslash B(x_0,3R/2)}\frac{dy}{|x-y|^{n+1}}\Big)\\
&\lesssim R^{n/q-2}\Big\{\sup_{y\in B(x_0,2R)}|u(y,t)|\Big\}.
\end{aligned}$$

At last, for the term S_4, by Lemma 5 and the condition $V\in B_q$, we can obtain, via the L^p-boundedness of the operator with the kernel $\nabla_x^2\Gamma_0(\cdot,\cdot)$, the following:

$$\begin{aligned}
\|S_4(\cdot,t)\|_q &\lesssim \Big\|V(\cdot)|u(\cdot,t)||\eta(\cdot)|\Big\|_q\\
&\lesssim \Big\{\sup_{y\in B(x_0,2R)}|u(y,t)|\Big\}\Big(\int_{B(x_0,2R)}V^q(y)dy\Big)^{1/q}\\
&\lesssim \Big\{\sup_{y\in B(x_0,2R)}|u(y,t)|\Big\}R^{n/q}\Big(\frac{1}{|B(x_0,2R)|}\int_{B(x_0,2R)}V^q(y)dy\Big)^{1/q}\\
&\lesssim \Big\{\sup_{y\in B(x_0,2R)}|u(y,t)|\Big\}R^{n/q-2}\Big(\frac{1}{R^{n-2}}\int_{B(x_0,2R)}V(y)dy\Big)\\
&\lesssim \Big\{\sup_{y\in B(x_0,2R)}|u(y,t)|\Big\}R^{n/q-2}\Big(1+\frac{R}{\rho(x_0)}\Big)^{m_0}.
\end{aligned}$$

The estimates for $\|S_i(\cdot,t)\|_q, i=1,2,3,4$ indicate that:

$$\|\nabla^2_{\cdot}u(\cdot,t)\|_{L^q(B(x_0,R))}\lesssim R^{n/q-2}\Big\{\Big(1+\frac{R}{\rho(x_0)}\Big)^{m_0}+1\Big\}\Big\{\sup_{y\in B(x_0,2R)}|u(y,t)|\Big\}+R^{n/q}\Big\{\sup_{y\in B(x_0,2R)}|\partial_t u(y,t)|\Big\}.$$

Let $u(x_0,t)=K_t^L(x_0,y_0)$. Then,

$$\begin{aligned}
&|\nabla_x K_t^L(x_0+h,y_0)-\nabla_x K_t^L(x_0,y_0)|\\
&\lesssim |h|^{1-n/q}\Big(\int_{B(x_0,R)}|\nabla_x^2 K_t^L(x,y_0)|^q dx\Big)^{1/q}\\
&\lesssim \Big(\frac{|h|}{\sqrt{t}}\Big)^{1-n/q}\Big(\frac{\sqrt{t}}{R}\Big)^{1-n/q}\frac{1}{R}\Big\{\Big(1+\frac{R}{\rho(x_0)}\Big)^{m_0}+1\Big\}\Big\{\sup_{x\in B(x_0,2R)}|K_t^L(x,y_0)|+\frac{R^2}{t}\sup_{x\in B(x_0,2R)}|t\partial_t K_t^L(x,y_0)|\Big\}\\
&\le \frac{C_N}{R}\Big(\frac{|h|}{\sqrt{t}}\Big)^{1-n/q}\Big(\frac{\sqrt{t}}{R}\Big)^{1-n/q}\Big\{\Big(1+\frac{R}{\rho(x_0)}\Big)^{m_0}+1\Big\}\Big\{\sup_{x\in B(x_0,2R)}\frac{1}{t^{n/2}}e^{-c|x-y_0|^2/t}\Big(1+\frac{\sqrt{t}}{\rho(x)}\Big)^{-N}\Big(1+\frac{\sqrt{t}}{\rho(y_0)}\Big)^{-N}\Big\}\\
&\quad +C_N\frac{R^2}{t}\Big\{\sup_{x\in B(x_0,2R)}\frac{1}{t^{n/2}}e^{-c|x-y_0|^2/t}\Big(1+\frac{\sqrt{t}}{\rho(x)}\Big)^{-N}\Big(1+\frac{\sqrt{t}}{\rho(y_0)}\Big)^{-N}\Big\}.
\end{aligned}$$

Take $0 < R < \min\{\rho(x_0), |x_0 - y_0|/8\}$. If $x \in B(x_0, 2R)$, then $|x - x_0| < 2\rho(x_0)$, that is, $\rho(x) \sim \rho(x_0)$. Moreover, if $x \in B(x_0, 2R)$, $|x - x_0| < 2R < |x_0 - y_0|/4$, which means that $|x - y_0| \sim |x_0 - y_0|$. We can obtain:

$$\begin{aligned} &|\nabla_x K_t^L(x_0 + h, y_0) - \nabla_x K_t^L(x_0, y_0)| \\ &\le \frac{C_N}{t^{(n+1)/2}}\Big(\frac{|h|}{\sqrt{t}}\Big)^{1-n/q} e^{-c|x_0-y_0|^2/t}\Big(1+\frac{\sqrt{t}}{\rho(x_0)}\Big)^{-N}\Big(1+\frac{\sqrt{t}}{\rho(y_0)}\Big)^{-N}\Big(\frac{\sqrt{t}}{R}\Big)^{1-n/q}\Big(\frac{\sqrt{t}}{R}+\frac{R}{\sqrt{t}}\Big), \end{aligned} \tag{24}$$

Define a function $F(x) = x^{1-n/q}(x + 1/x)$, $x > 0$. Then, we can see that for $x > \sqrt{n/(2q-n)}$, $F'(x) > 0$, i.e., F is increasing, which means that the function $f := F(\sqrt{t}/R)$ is decreasing for $R \in (0, \sqrt{(2q-n)t/n})$. Below, we divide the rest of the proof into two cases:

Case 1: $0 < R < \min\{\rho(x_0), |x_0 - y_0|\} < \sqrt{t}$. We further divide the discussion into two subcases:

Case 1.1: $0 < R < \rho(x_0) < \sqrt{t} < \sqrt{(2q-n)t/n}$, i.e., $\rho(x_0) \le |x_0 - y_0|$. For this case, the function $f(R)$ has the infimum $f(\rho(x_0))$ for $R \in (0, \rho(x_0))$. Then, taking the infimum for R on both sides of (24), we can use the fact that $\rho(x_0) < \sqrt{t}$ to obtain:

$$\begin{aligned} &|\nabla_x K_t^L(x_0 + h, y_0) - \nabla_x K_t^L(x_0, y_0)| \\ &\le \frac{C_N}{t^{(n+1)/2}}\Big(\frac{|h|}{\sqrt{t}}\Big)^{1-n/q} e^{-c|x_0-y_0|^2/t}\Big(1+\frac{\sqrt{t}}{\rho(x_0)}\Big)^{-N}\Big(1+\frac{\sqrt{t}}{\rho(y_0)}\Big)^{-N}\Big\{\Big(\frac{\sqrt{t}}{\rho(x_0)}\Big)^{1-n/q}\Big(\frac{\sqrt{t}}{\rho(x_0)}+1\Big)\Big\} \\ &\le \frac{C_N}{t^{(n+1)/2}}\Big(\frac{|h|}{\sqrt{t}}\Big)^{1-n/q} e^{-c|x_0-y_0|^2/t}\Big(1+\frac{\sqrt{t}}{\rho(x_0)}\Big)^{-N}\Big(1+\frac{\sqrt{t}}{\rho(y_0)}\Big)^{-N}. \end{aligned}$$

Case 1.2: $0 < R < |x_0 - y_0| < \sqrt{t}$, i.e., $\rho(x_0) > |x_0 - y_0|$. Similar to Case 1.1, taking the infimum for $f(R)$ gives:

$$\begin{aligned} \Big(\frac{\sqrt{t}}{R}\Big)^{1-n/q}\Big(\frac{\sqrt{t}}{R}+\frac{R}{\sqrt{t}}\Big) &\lesssim \Big(\frac{\sqrt{t}}{|x_0-y_0|}\Big)^{1-n/q}\Big(\frac{\sqrt{t}}{|x_0-y_0|}+\frac{|x_0-y_0|}{\sqrt{t}}\Big) \\ &\lesssim \Big(\frac{\sqrt{t}}{|x_0-y_0|}\Big)^{1-n/q}\Big(\frac{\sqrt{t}}{|x_0-y_0|}+1\Big). \end{aligned}$$

Then, we obtain:

$$\begin{aligned} &|\nabla_x K_t^L(x_0 + h, y_0) - \nabla_x K_t^L(x_0, y_0)| \\ &\le \frac{C_N}{t^{(n+1)/2}}\Big(\frac{|h|}{\sqrt{t}}\Big)^{1-n/q}\Big(\frac{\sqrt{t}}{|x_0-y_0|}\Big)^{1-n/q} e^{-c|x_0-y_0|^2/t}\Big(1+\frac{\sqrt{t}}{\rho(x_0)}\Big)^{-N}\Big(1+\frac{\sqrt{t}}{\rho(y_0)}\Big)^{-N}\Big(\frac{\sqrt{t}}{|x_0-y_0|}+1\Big). \end{aligned}$$

It is easy to see that:

$$\begin{aligned} &|\nabla_x K_t^L(x_0 + h, y_0) - \nabla_x K_t^L(x_0, y_0)| \\ &\lesssim \Big(\frac{|h|}{|x_0-y_0|}\Big)^{1-n/q}\frac{1}{t^{(n+1)/2}} e^{-c|x_0-y_0|^2/t}\Big(1+\frac{\sqrt{t}}{\rho(x_0)}\Big)^{-N}\Big(1+\frac{\sqrt{t}}{\rho(y_0)}\Big)^{-N}\Big(\frac{\sqrt{t}}{|x_0-y_0|}+1\Big) \\ &\lesssim \Big(\frac{|h|}{|x_0-y_0|}\Big)^{1-n/q}\frac{1}{t^{(n+1)/2}} e^{-c|x_0-y_0|^2/t}\Big(1+\frac{\sqrt{t}}{\rho(x_0)}\Big)^{-N}\Big(1+\frac{\sqrt{t}}{\rho(y_0)}\Big)^{-N}\frac{\sqrt{t}}{|x_0-y_0|} \\ &\quad+\Big(\frac{|h|}{|x_0-y_0|}\Big)^{1-n/q}\frac{1}{t^{(n+1)/2}} e^{-c|x_0-y_0|^2/t}(1+\frac{\sqrt{t}}{\rho(x_0)})^{-N}\Big(1+\frac{\sqrt{t}}{\rho(y_0)}\Big)^{-N} \\ &\lesssim \Big(\frac{|h|}{|x_0-y_0|}\Big)^{1-n/q}\Big(\frac{1}{t^{(n+1)/2}}+\frac{1}{t^{n/2}|x_0-y_0|}\Big) e^{-c|x_0-y_0|^2/t}\Big(1+\frac{\sqrt{t}}{\rho(x_0)}\Big)^{-N}\Big(1+\frac{\sqrt{t}}{\rho(y_0)}\Big)^{-N} \\ &\lesssim \Big(\frac{|h|}{|x_0-y_0|}\Big)^{1-n/q}\frac{1}{t^{n/2}|x_0-y_0|} e^{-c|x_0-y_0|^2/t}\Big(1+\frac{\sqrt{t}}{\rho(x_0)}\Big)^{-N}\Big(1+\frac{\sqrt{t}}{\rho(y_0)}\Big)^{-N}. \end{aligned}$$

Case 2: $\sqrt{t} < \min\{\rho(x_0), |x_0 - y_0|/8\}$. Similar to Case 1, we divide the discussion into two subcases again:

Case 2.1: $0 < R < \sqrt{t} < \sqrt{(2q-n)t/n} < \min\{\rho(x_0), |x_0 - y_0|/8\}$. It follows from (24) that:

$$\begin{aligned} &|\nabla_x K_t^L(x_0+h, y_0) - \nabla_x K_t^L(x_0, y_0)| \\ &\leq C_N \Big(\frac{|h|}{\sqrt{t}}\Big)^{1-n/q} \frac{e^{-c|x_0-y_0|^2/t}}{t^{(n+1)/2}} \Big(1+\frac{\sqrt{t}}{\rho(x_0)}\Big)^{-N} \Big(1+\frac{\sqrt{t}}{\rho(y_0)}\Big)^{-N} \Big(\frac{\sqrt{t}}{R}\Big)^{1-n/q} \Big(\frac{\sqrt{t}}{R}+\frac{R}{\sqrt{t}}\Big). \end{aligned} \tag{25}$$

Taking the infimum on both sides (25) reaches:

$$|\nabla_x K_t^L(x_0+h, y_0) - \nabla_x K_t^L(x_0, y_0)| \leq \frac{C_N}{t^{(n+1)/2}} \Big(\frac{|h|}{\sqrt{t}}\Big)^{1-n/q} e^{-c|x_0-y_0|^2/t} \Big(1+\frac{\sqrt{t}}{\rho(x_0)}\Big)^{-N} \Big(1+\frac{\sqrt{t}}{\rho(y_0)}\Big)^{-N}.$$

Case 2.2: $0 < R < \sqrt{t} < \min\{\rho(x_0), |x_0 - y_0|/8\} < \sqrt{(2q-n)t/n}$. Similarly, taking the infimum on both sides of (25), we obtain:

$$\begin{aligned} &|\nabla_x K_t^L(x_0+h, y_0) - \nabla_x K_t^L(x_0, y_0)| \\ &\leq C_N \Big(\frac{|h|}{\sqrt{t}}\Big)^{1-n/q} \frac{1}{t^{(n+1)/2}} e^{-c|x_0-y_0|^2/t} \Big(1+\frac{\sqrt{t}}{\rho(x_0)}\Big)^{-N} \Big(1+\frac{\sqrt{t}}{\rho(y_0)}\Big)^{-N} \\ &\times \bigg\{ \Big(\frac{\sqrt{t}}{\min\{\rho(x_0), |x_0-y_0|/8\}}\Big)^{1-n/q} \Big(\frac{\sqrt{t}}{\min\{\rho(x_0), |x_0-y_0|/8\}} + \frac{\min\{\rho(x_0), |x_0-y_0|/8\}}{\sqrt{t}}\Big) \bigg\}. \end{aligned}$$

If $\rho(x_0) < |x_0 - y_0|/8$, then:

$$\begin{aligned} &|\nabla_x K_t^L(x_0+h, y_0) - \nabla_x K_t^L(x_0, y_0)| \\ &\leq C_N \Big(\frac{|h|}{\sqrt{t}}\Big)^{1-n/q} \frac{1}{t^{(n+1)/2}} e^{-c|x_0-y_0|^2/t} \Big(1+\frac{\sqrt{t}}{\rho(x_0)}\Big)^{-N} \Big(1+\frac{\sqrt{t}}{\rho(y_0)}\Big)^{-N} \\ &\quad \times \Big\{ \Big(\frac{\sqrt{t}}{\rho(x_0)}\Big)^{1-n/q} \Big(\frac{\sqrt{t}}{\rho(x_0)} + \frac{\rho(x_0)}{\sqrt{t}}\Big) \Big\} \\ &\leq C_N \Big(\frac{|h|}{\sqrt{t}}\Big)^{1-n/q} \frac{1}{t^{(n+1)/2}} e^{-c|x_0-y_0|^2/t} \Big(1+\frac{\sqrt{t}}{\rho(x_0)}\Big)^{-N} \Big(1+\frac{\sqrt{t}}{\rho(y_0)}\Big)^{-N}. \end{aligned}$$

If $\rho(x_0) \geq |x_0 - y_0|/8$, we have:

$$\begin{aligned} &|\nabla_x K_t^L(x_0+h, y_0) - \nabla_x K_t^L(x_0, y_0)| \\ &\leq \frac{C_N}{t^{(n+1)/2}} \Big(\frac{|h|}{\sqrt{t}}\Big)^{1-n/q} e^{-c|x_0-y_0|^2/t} \Big(1+\frac{\sqrt{t}}{\rho(x_0)}\Big)^{-N} \Big(1+\frac{\sqrt{t}}{\rho(y_0)}\Big)^{-N} \\ &\quad \times \Big\{ \Big(\frac{\sqrt{t}}{|x_0-y_0|}\Big)^{1-n/q} \Big(\frac{\sqrt{t}}{|x_0-y_0|} + \frac{|x_0-y_0|}{\sqrt{t}}\Big) \Big\} \\ &\leq \frac{C_N}{t^{(n+1)/2}} \Big(\frac{|h|}{|x_0-y_0|}\Big)^{1-n/q} e^{-c|x_0-y_0|^2/t} \Big(1+\frac{\sqrt{t}}{\rho(x_0)}\Big)^{-N} \Big(1+\frac{\sqrt{t}}{\rho(y_0)}\Big)^{-N} \Big(\frac{\sqrt{t}}{|x_0-y_0|} + \frac{|x_0-y_0|}{\sqrt{t}}\Big) \\ &\leq \frac{C_N}{t^{n/2}|x_0-y_0|} \Big(\frac{|h|}{|x_0-y_0|}\Big)^{1-n/q} e^{-c|x_0-y_0|^2/t} \Big(1+\frac{\sqrt{t}}{\rho(x_0)}\Big)^{-N} \Big(1+\frac{\sqrt{t}}{\rho(y_0)}\Big)^{-N} \\ &\quad + \frac{C_N}{t^{(n+1)/2}} \Big(\frac{|h|}{|x_0-y_0|}\Big)^{1-n/q} e^{-c|x_0-y_0|^2/t} \Big(1+\frac{\sqrt{t}}{\rho(x_0)}\Big)^{-N} \Big(1+\frac{\sqrt{t}}{\rho(y_0)}\Big)^{-N}. \end{aligned}$$

If $\sqrt{t} < |x_0 - y_0|$, then:

$$|\nabla_x K_t^L(x_0+h, y_0) - \nabla_x K_t^L(x_0, y_0)| \leq C_N \Big(\frac{|h|}{\sqrt{t}}\Big)^{1-n/q} \frac{e^{-c|x_0-y_0|^2/t}}{t^{(n+1)/2}} \Big(1+\frac{\sqrt{t}}{\rho(x_0)}+\frac{\sqrt{t}}{\rho(y_0)}\Big)^{-N}.$$

If $\sqrt{t} \geq |x_0 - y_0|$, then:

$$|\nabla_x K_t^L(x_0+h, y_0) - \nabla_x K_t^L(x_0, y_0)| \leq C_N \Big(\frac{|h|}{|x_0-y_0|}\Big)^{1-n/q} \frac{e^{-c|x_0-y_0|^2/t}}{t^{n/2}|x_0-y_0|} \Big(1 + \frac{\sqrt{t}}{\rho(x_0)} + \frac{\sqrt{t}}{\rho(y_0)}\Big)^{-N}.$$

□

Lemma 11. *Suppose that $V \in B_q$ for some $q > n$. Let $\delta' = 1 - n/q$. For every $N > 0$, there exists a constant $C_N > 0$, such that for all $x, y \in \mathbb{R}^n$ and $t > 0$, the semigroup kernels $K_t^L(\cdot,\cdot)$ satisfy the following estimate: for $|h| < |x-y|/4$,*

$$|\nabla_x K_t^L(x+h, y) - \nabla_x K_t^L(x, y)| \leq \frac{C_N}{t^{(n+1)/2}} \Big(\frac{|h|}{\sqrt{t}}\Big)^{\delta'} \Big(1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)}\Big)^{-N}.$$

Proof. Similar to Lemma 10, we take $R \in (0, \min\{\rho(x_0), \sqrt{t}\})$ and obtain:

$$\begin{aligned}
&|\nabla_x K_t^L(x_0+h, y_0) - \nabla_x K_t^L(x_0, y_0)| \\
&\leq \frac{C_N}{t^{(n+1)/2}} \Big(\frac{|h|}{\sqrt{t}}\Big)^{1-n/q} e^{-c|x_0-y_0|^2/t} \Big(1 + \frac{\sqrt{t}}{\rho(x_0)}\Big)^{-N} \Big(1 + \frac{\sqrt{t}}{\rho(y_0)}\Big)^{-N} \Big\{\Big(\frac{\sqrt{t}}{R}\Big)^{1-n/q} \Big(\frac{\sqrt{t}}{R} + \frac{R}{\sqrt{t}}\Big)\Big\} \\
&\leq \frac{C_N}{t^{(n+1)/2}} \Big(\frac{|h|}{\sqrt{t}}\Big)^{1-n/q} \Big(1 + \frac{\sqrt{t}}{\rho(x_0)}\Big)^{-N} \Big(1 + \frac{\sqrt{t}}{\rho(y_0)}\Big)^{-N} \Big\{\Big(\frac{\sqrt{t}}{R}\Big)^{1-n/q} \Big(\frac{\sqrt{t}}{R} + \frac{R}{\sqrt{t}}\Big)\Big\}.
\end{aligned} \tag{26}$$

Case 1: $\rho(x_0) \leq \sqrt{t}$. This implies $0 < R < \rho(x_0) < \sqrt{t} < \sqrt{(2q-n)t/n}$. We can obtain:

$$\begin{aligned}
&|\nabla_x K_t^L(x_0+h, y_0) - \nabla_x K_t^L(x_0, y_0)| \\
&\leq \frac{C_N}{t^{(n+1)/2}} \Big(\frac{|h|}{\sqrt{t}}\Big)^{1-n/q} \Big(1 + \frac{\sqrt{t}}{\rho(x_0)}\Big)^{-N} \Big(1 + \frac{\sqrt{t}}{\rho(y_0)}\Big)^{-N} \Big\{\Big(\frac{\sqrt{t}}{\rho(x_0)}\Big)^{1-n/q} \Big(\frac{\sqrt{t}}{\rho(x_0)} + \frac{\rho(x_0)}{\sqrt{t}}\Big)\Big\} \\
&\leq \frac{C_N}{t^{(n+1)/2}} \Big(\frac{|h|}{\sqrt{t}}\Big)^{1-n/q} \Big(1 + \frac{\sqrt{t}}{\rho(x_0)}\Big)^{-N} \Big(1 + \frac{\sqrt{t}}{\rho(y_0)}\Big)^{-N}.
\end{aligned}$$

Case 2: $\rho(x_0) > \sqrt{t}$. For this case, $0 < R < \sqrt{t}$. Then, the following two cases are considered:

$$\begin{cases} \textit{Case } 2.1: & 0 < R < \sqrt{t} < \rho(x_0) < \sqrt{(2q-n)t/n}; \\ \textit{Case } 2.2: & 0 < R < \sqrt{t} < \sqrt{(2q-n)t/n} < \rho(x_0). \end{cases}$$

It is obvious that Case 2.1 comes back to Case 1. For Case 2.2, letting $R \to \sqrt{t}$ on the right-hand side of (26), we have:

$$|\nabla_x K_t^L(x_0+h, y_0) - \nabla_x K_t^L(x_0, y_0)| \leq \frac{C_N}{t^{(n+1)/2}} \Big(\frac{|h|}{\sqrt{t}}\Big)^{1-n/q} \Big(1 + \frac{\sqrt{t}}{\rho(x_0)}\Big)^{-N} \Big(1 + \frac{\sqrt{t}}{\rho(y_0)}\Big)^{-N}.$$

□

Proposition 12. *Suppose that $\alpha > 0$ and $V \in B_q$ for some $q > n$. Let $\delta' = 1 - n/q$. For every $N > 0$, there exists a constant $C_N > 0$, such that for all $x, y \in \mathbb{R}^n$ and $t > 0$, the fractional heat kernels $K_{\alpha,t}^L(\cdot,\cdot)$ satisfy the following estimate: for $|h| < |x-y|/4$,*

$$|\nabla_x K_{\alpha,t}^L(x+h, y) - \nabla_x K_{\alpha,t}^L(x, y)| \leq C_N \Big(\frac{|h|}{t^{1/2\alpha}}\Big)^{\delta'} \frac{1}{t^{1/2\alpha}} \frac{t}{(t^{1/2\alpha} + |x-y|)^{n+2\alpha}}.$$

Proof. By the subordinative formula and Lemma 10, we can obtain:

$$|\nabla_x K_{\alpha,t}^L(x+h, y) - \nabla_x K_{\alpha,t}^L(x, y)| \leq C_N \int_0^\infty \frac{t}{s^{1+\alpha}} \Big|\nabla_x K_t^L(x+h, y) - \nabla_x K_t^L(x, y)\Big| ds \leq C_N(M_6 + M_7),$$

where

$$\begin{cases} M_6 := \int_0^{|x-y|} \frac{t}{s^{1+\alpha}} \Big(\frac{|h|}{\sqrt{s}}\Big)^{\delta'} \frac{1}{s^{(n+1)/2}} e^{-c|x-y|^2/s} \Big(1+\frac{\sqrt{s}}{\rho(x)}\Big)^{-N} \Big(1+\frac{\sqrt{s}}{\rho(y)}\Big)^{-N} ds; \\ M_7 := \int_{|x-y|}^{\infty} \frac{t}{s^{1+\alpha}} \Big(\frac{|h|}{|x-y|}\Big)^{\delta'} \frac{1}{s^{n/2}|x-y|} e^{-c|x-y|^2/s} \Big(1+\frac{\sqrt{s}}{\rho(x)}\Big)^{-N} \Big(1+\frac{\sqrt{s}}{\rho(y)}\Big)^{-N} ds. \end{cases}$$

We first estimate M_6 and apply a change of variables to obtain:

$$\begin{aligned} M_6 &\lesssim \int_0^{\infty} \frac{t}{s^{1+\alpha}} \Big(\frac{|h|}{\sqrt{s}}\Big)^{\delta'} \frac{1}{s^{(n+1)/2}} e^{-c|x-y|^2/s} \Big(1+\frac{\sqrt{s}}{\rho(x)}\Big)^{-N} \Big(1+\frac{\sqrt{s}}{\rho(y)}\Big)^{-N} ds \\ &\lesssim t^{-(n+1)/2\alpha} \Big(\frac{t^{1/2\alpha}}{\rho(x)}\Big)^{-N} \Big(\frac{t^{1/2\alpha}}{\rho(y)}\Big)^{-N} \Big(\frac{|h|}{t^{1/2\alpha}}\Big)^{\delta'} \int_0^{\infty} u^{-N-1-\alpha-(n+1)/2-\delta'/2} e^{-c|x-y|^2/t^{1/\alpha}u} du \\ &\lesssim t^{-(n+1)/2\alpha} \Big(\frac{t^{1/2\alpha}}{\rho(x)}\Big)^{-N} \Big(\frac{t^{1/2\alpha}}{\rho(y)}\Big)^{-N} \Big(\frac{|h|}{t^{1/2\alpha}}\Big)^{\delta'} \int_0^{\infty} \Big(\frac{r^2 t^{1/\alpha}}{|x-y|^2}\Big)^{N+1+\alpha+(n+1)/2+\delta'/2} e^{-cr^2} \frac{|x-y|^2}{t^{1/\alpha} r^3} dr \\ &\lesssim \Big(\frac{t^{1/2\alpha}}{\rho(x)}\Big)^{-N} \Big(\frac{t^{1/2\alpha}}{\rho(y)}\Big)^{-N} \Big(\frac{|h|}{|x-y|}\Big)^{\delta'} \frac{t^{1+N/\alpha}}{|x-y|^{2\alpha+2N+n+1}}. \end{aligned}$$

Similarly, for M_7, we have:

$$\begin{aligned} M_7 &\lesssim \frac{1}{|x-y|} \int_0^{\infty} \frac{t}{s^{1+\alpha}} \Big(\frac{|h|}{\sqrt{s}}\Big)^{\delta'} \frac{1}{s^{n/2}} e^{-c|x-y|^2/s} \Big(1+\frac{\sqrt{s}}{\rho(x)}\Big)^{-N} \Big(1+\frac{\sqrt{s}}{\rho(y)}\Big)^{-N} ds \\ &\lesssim \Big(\frac{t^{1/2\alpha}}{\rho(x)}\Big)^{-N} \Big(\frac{t^{1/2\alpha}}{\rho(y)}\Big)^{-N} \Big(\frac{|h|}{|x-y|}\Big)^{\delta'} \frac{t^{1+N/\alpha}}{|x-y|^{2\alpha+2N+n+1}}, \end{aligned}$$

which gives:

$$|\nabla_x K^L_{\alpha,t}(x+h,y) - \nabla_x K^L_{\alpha,t}(x,y)| \le \frac{C_N t^{1+N/\alpha}}{|x-y|^{2\alpha+2N+n+1}} \Big(\frac{|h|}{|x-y|}\Big)^{\delta'} \Big(\frac{t^{1/2\alpha}}{\rho(x)}\Big)^{-N} \Big(\frac{t^{1/2\alpha}}{\rho(y)}\Big)^{-N}.$$

On the other hand, we can deduce from Lemma 11 that:

$$\begin{aligned} &|\nabla_x K^L_{\alpha,t}(x+h,y) - \nabla_x K^L_{\alpha,t}(x,y)| \\ &\le C_N \int_0^{\infty} s^{-(n+1)/2} \frac{1}{t^{1/\alpha}} \eta_1^{\alpha}\Big(\frac{s}{t^{1/\alpha}}\Big) \Big(\frac{|h|}{\sqrt{s}}\Big)^{\delta'} \Big(1+\frac{\sqrt{s}}{\rho(x)}\Big)^{-N} \Big(\frac{\sqrt{s}}{\rho(y)}\Big)^{-N} ds \\ &\le C_N \Big(\frac{t^{1/2\alpha}}{\rho(x)}\Big)^{-N} \Big(\frac{t^{1/2\alpha}}{\rho(y)}\Big)^{-N} \Big(\frac{|h|}{t^{1/2\alpha}}\Big)^{\delta'} \frac{1}{t^{(n+1)/2\alpha}}. \end{aligned}$$

Finally, the arbitrariness of N indicates that:

$$\begin{aligned} &\Big(1+\frac{t^{1/2\alpha}}{\rho(x)}\Big)^{N} \Big(1+\frac{t^{1/2\alpha}}{\rho(y)}\Big)^{N} |\nabla_x K^L_{\alpha,t}(x+h,y) - \nabla_x K^L_{\alpha,t}(x,y)| \\ &\quad \le C_N \min\left\{ \Big(\frac{|h|}{|x-y|}\Big)^{\delta'} \frac{t^{1+N/\alpha}}{|x-y|^{2\alpha+2N+n+1}},\ \Big(\frac{|h|}{t^{1/2\alpha}}\Big)^{\delta'} \frac{1}{t^{(n+1)/2\alpha}} \right\}, \end{aligned}$$

which proves Proposition 12. □

Proposition 13. *Assume that $V \in B_q$ for some $q > n$. Let $\alpha \in (0, 1/2 - n/2q)$. For every $N > 0$,*

$$\Big| t^{1/2\alpha} \nabla_x e^{-tL^{\alpha}}(1)(x) \Big| \lesssim \min\left\{ \Big(\frac{t^{1/2\alpha}}{\rho(x)}\Big)^{1+2\alpha},\ \Big(\frac{t^{1/2\alpha}}{\rho(x)}\Big)^{-N} \right\}.$$

Proof. We divide the proof into two cases:

Case 1: $t^{1/2\alpha} > \rho(x)$. By Proposition 11, we use a direct computation to obtain:

$$\begin{aligned} |t^{1/2\alpha}\nabla_x e^{-tL^\alpha}(1)(x)| &\lesssim \int_{\mathbb{R}^n} \frac{t}{(t^{1/2\alpha}+|x-y|)^{n+2\alpha}}\Big(1+\frac{t^{1/2\alpha}}{\rho(x)}\Big)^{-N}\Big(1+\frac{t^{1/2\alpha}}{\rho(y)}\Big)^{-N}dy \\ &\lesssim \Big(\frac{t^{1/2\alpha}}{\rho(x)}\Big)^{-N}\int_{\mathbb{R}^n} \frac{t}{(t^{1/2\alpha}+|x-y|)^{n+2\alpha}}dy \\ &\lesssim \Big(\frac{t^{1/2\alpha}}{\rho(x)}\Big)^{-N}. \end{aligned}$$

Because $t^{1/2\alpha} > \rho(x)$, then

$$|t^{1/2\alpha}\nabla_x e^{-tL^\alpha}(1)(x)| \lesssim \min\Big\{\Big(\frac{t^{1/2\alpha}}{\rho(x)}\Big)^{\delta}, \Big(\frac{t^{1/2\alpha}}{\rho(x)}\Big)^{-N}\Big\}.$$

Case 2: $t^{1/2\alpha} \le \rho(x)$. It follows from (5) that:

$$t^{1/2\alpha}\nabla_x e^{-tL^\alpha}(1)(x) = t^{1/2\alpha}\nabla_x\int_{\mathbb{R}^n} K^L_{\alpha,t}(x,y)dy = I_1+I_2,$$

where:

$$\begin{cases} I_1 := t^{1/2\alpha}\displaystyle\int_{\rho^2(x)}^{\infty}\eta^\alpha_t(s)\Big(\int_{\mathbb{R}^n}\nabla_x K^L_s(x,y)dy\Big)ds; \\ I_2 := t^{1/2\alpha}\displaystyle\int_0^{\rho^2(x)}\eta^\alpha_t(s)\Big(\int_{\mathbb{R}^n}\nabla_x K^L_s(x,y)dy\Big)ds. \end{cases}$$

We claim that:

$$L_s(x) := \int_{\mathbb{R}^n}\nabla_x K^L_s(x,y)dy \lesssim \frac{1}{\sqrt{s}}. \tag{27}$$

In fact, by Lemma 8, we have $L_s(x) \lesssim L_{s,1}(x)+L_{s,2}(x)$, where:

$$\begin{cases} L_{s,1}(x) := \displaystyle\int_{\{y:\sqrt{s}\le|x-y|\}}\frac{1}{s^{(n+1)/2}}e^{-c|x-y|^2/s}\Big(1+\frac{\sqrt{s}}{\rho(x)}+\frac{\sqrt{s}}{\rho(y)}\Big)^{-N}dy; \\ L_{s,2}(x) := \displaystyle\int_{\{y:|y-x|<\sqrt{s}\}}\frac{1}{s^{n/2}|x-y|}e^{-c|x-y|^2/s}\Big(1+\frac{\sqrt{s}}{\rho(x)}+\frac{\sqrt{s}}{\rho(y)}\Big)^{-N}dy. \end{cases}$$

Taking N as large enough, it is easy to see that:

$$L_{s,1}(x) \lesssim \int_{\mathbb{R}^n}\frac{1}{s^{(n+1)/2}}e^{-c|x-y|^2/s}dy \lesssim \frac{1}{\sqrt{s}}.$$

Similarly, a direct calculus gives, together with changing variables: $|x-y|/\sqrt{s}=u$,

$$L_{s,2}(x) \lesssim \frac{1}{\sqrt{s}}\int_{\mathbb{R}^n}\frac{1}{s^{n/2}|x-y|/\sqrt{s}}e^{-c|x-y|^2/s}dy \lesssim \frac{1}{\sqrt{s}}\int_0^\infty u^{n-2}e^{-cu^2}du \lesssim \frac{1}{\sqrt{s}}.$$

Then, we can deduce from (27) that:

$$I_1 \lesssim t^{1/2\alpha}\int_{\rho^2(x)}^\infty \eta^\alpha_t(s)\frac{ds}{\sqrt{s}} \lesssim t^{1+1/2\alpha}\int_{\rho^2(x)}^\infty\frac{1}{s^{\alpha+3/2}}ds \lesssim \Big(\frac{t^{1/2\alpha}}{\rho(x)}\Big)^{1+2\alpha}. \tag{28}$$

For I_2, it follows from the perturbation formula (see [36], p. 497, (2.3), also [11], (5.25)), that:

$$h_u(x-y)-K^L_u(x,y) = \int_0^u\int_{\mathbb{R}^n} h_s(x-z)V(z)K_{u-s}(z,y)dzds$$

and that for $\delta = 2 - n/q > 1$,

$$|\sqrt{u}\nabla_x e^{-uL}(1)(x)| \lesssim \int_0^u \sqrt{\frac{u}{s}}\Big(\frac{\sqrt{s}}{\rho(x)}\Big)^{\delta}\frac{ds}{s} \lesssim \Big(\frac{\sqrt{u}}{\rho(x)}\Big)^{\delta}.$$

Therefore, noting that $0 < 2\alpha < 1 - n/q$, we can use the change of variables to obtain:

$$\begin{aligned} I_2 &\lesssim t^{1/2\alpha}\int_0^{\rho^2(x)} \eta_t^{\alpha}(s)\frac{1}{\sqrt{s}}\Big(\sqrt{s}\int_{\mathbb{R}^n}\nabla_x K_s^L(x,y)dy\Big)ds \\ &\lesssim t^{1/2\alpha}\int_0^{\rho^2(x)} \frac{1}{t^{1/\alpha}}\eta_1^{\alpha}(s/t^{1/\alpha})\frac{1}{\sqrt{s}}\Big(\frac{\sqrt{s}}{\rho(x)}\Big)^{\delta}ds \\ &\lesssim t^{1/2\alpha-1/\alpha}\int_0^{\rho(x)^2/t^{1/\alpha}} \eta_1^{\alpha}(\tau)\frac{1}{\sqrt{t^{1/\alpha}\tau}}\Big(\frac{\sqrt{t^{1/\alpha}\tau}}{\rho(x)}\Big)^{\delta}t^{1/\alpha}d\tau \\ &\lesssim \Big(\frac{t^{1/2\alpha}}{\rho(x)}\Big)^{1+2\alpha}. \end{aligned} \tag{29}$$

The above estimates, (29) and (28), imply that:

$$|t^{1/2\alpha}\nabla_x e^{-tL^{\alpha}}(1)(x)| \lesssim \Big(\frac{t^{1/2\alpha}}{\rho(x)}\Big)^{1+2\alpha}.$$

□

3.3. Estimation on Time-Fractional Derivatives

In this section, we give some gradient estimates for the fractional heat kernel associated with the variable t. Define an operator:

$$D_{\alpha,t}^{L,\beta}(f) = t^{\beta}\partial_t^{\beta}e^{-tL^{\alpha}}f, \ \ \alpha \in (0,1), \text{ and } \beta > 0.$$

Denote, by $D_{\alpha,t}^{L,\beta}(\cdot,\cdot)$, the integral kernel of $D_{\alpha,t}^{L,\beta}$. Then, we can obtain the following proposition:

Proposition 14. *Let $\alpha \in (0,1)$, $V \in B_q, q > n/2$ and $\beta > 0$. For every $N > \alpha\beta$, there exists a constant $C_N > 0$, such that:*

$$|D_{\alpha,t}^{L,\beta}(x,y)| \le \frac{C_N t^{\beta}}{(t^{1/2\alpha}+|x-y|)^{n+2\alpha\beta}}\Big(1+\frac{t^{1/2\alpha}}{\rho(x)}+\frac{t^{1/2\alpha}}{\rho(y)}\Big)^{-N}. \tag{30}$$

Proof. The following two cases are considered:

Case 1: $\beta \in (0,1)$. It is easy to see that:

$$\begin{aligned} t^{\beta}\partial_t^{\beta}e^{-tL^{\alpha}} &= c_{\beta}t^{\beta}\int_0^{\infty}\partial_t e^{-(t+s)L^{\alpha}}\frac{ds}{s^{\beta}} = c_{\beta}t^{\beta}\int_0^{\infty}(-L)^{\alpha}e^{-(t+s)L^{\alpha}}\frac{ds}{s^{\beta}} \\ &= c_{\beta}t^{\beta}\int_0^{\infty}(t+s)(-L)^{\alpha}e^{-(t+s)L^{\alpha}}\frac{ds}{(t+s)s^{\beta}}, \end{aligned}$$

which, together with Proposition 9, gives:

$$
\begin{aligned}
\left|D_{\alpha,t}^{L,\beta}(x,y)\right| &\leq C_N t^{\beta}\int_0^{\infty}\frac{1}{((t+s)^{1/2\alpha}+|x-y|)^{n+2\alpha}}\Big(1+\frac{(t+s)^{1/2\alpha}}{\rho(x)}\Big)^{-N}\Big(1+\frac{(t+s)^{1/2\alpha}}{\rho(y)}\Big)^{-N}\frac{ds}{s^{\beta}} \\
&\leq C_N t^{\beta}\int_0^{\infty}\frac{1}{(t+s)^{n/2\alpha+1}}\Big(\frac{(t+s)^{1/2\alpha}}{\rho(x)}\Big)^{-N}\Big(\frac{(t+s)^{1/2\alpha}}{\rho(y)}\Big)^{-N}\frac{ds}{s^{\beta}} \\
&\leq C_N t^{\beta}\rho(x)^N\rho(y)^N\int_0^{\infty}(t+s)^{-n/2\alpha-N/\alpha-1}s^{-\beta}ds \\
&\leq \frac{C_N}{t^{n/2\alpha}}\Big(\frac{t^{1/2\alpha}}{\rho(x)}\Big)^{-N}\Big(\frac{t^{1/2\alpha}}{\rho(y)}\Big)^{-N}.
\end{aligned}
$$

On the other hand, since $e^{-tL^{\alpha}}=\int_0^{\infty}\eta_1^{\alpha}(\tau)e^{-t^{1/\alpha}\tau L}d\tau$,

$$
\begin{aligned}
t^{\beta}\partial_t^{\beta}e^{-tL^{\alpha}} &= C_{\beta}t^{\beta}\int_0^{\infty}\partial_r\Big(\int_0^{\infty}\eta_1^{\alpha}(\tau)e^{-(t+r)^{1/\alpha}\tau L}d\tau\Big)\frac{dr}{r^{\beta}} \\
&= C_{\beta}t^{\beta}\int_0^{\infty}\Big(\int_0^{\infty}\eta_1^{\alpha}(\tau)(t+r)^{1/\alpha-1}\tau Le^{-(t+r)^{1/\alpha}\tau L}d\tau\Big)\frac{dr}{r^{\beta}} \\
&= C_{\beta}t^{\beta}\int_0^{\infty}\Big(\int_0^{\infty}Q^L_{(t+r)^{1/\alpha}\tau,1}\frac{dr}{(t+r)r^{\beta}}\Big)\eta_1^{\alpha}(\tau)d\tau.
\end{aligned}
$$

By Proposition 3, we can obtain:

$$
\begin{aligned}
\left|D_{\alpha,t}^{L,\beta}(x,y)\right| &\leq C_N t^{\beta}\int_0^{\infty}\eta_1^{\alpha}(\tau)\bigg\{\int_0^{\infty}((t+r)^{1/\alpha}\tau)^{-n/2}e^{-c|x-y|^2/(t+r)^{1/\alpha}\tau} \\
&\qquad\Big(1+\frac{(t+r)^{1/2\alpha}\sqrt{\tau}}{\rho(x)}\Big)^{-N}\Big(1+\frac{(t+r)^{1/2\alpha}\sqrt{\tau}}{\rho(y)}\Big)^{-N}\frac{dr}{r^{\beta}(t+r)}\bigg\}d\tau \\
&\leq C_N\frac{t^{\beta}\rho(x)^N\rho(y)^N}{|x-y|^{n+2\alpha\beta}}\int_0^{\infty}\eta_1^{\alpha}(\tau)\tau^{\alpha\beta-N}\Big(\int_0^{\infty}(t+r)^{\beta-N/\alpha-1}r^{-\beta}dr\Big)d\tau \\
&\lesssim \frac{C_N t^{\beta}}{|x-y|^{n+2\alpha\beta}}\Big(\frac{t^{1/2\alpha}}{\rho(x)}\Big)^{-N}\Big(\frac{t^{1/2\alpha}}{\rho(y)}\Big)^{-N}.
\end{aligned}
$$

By the arbitrariness of N, we obtain:

$$
\begin{aligned}
|D_{\alpha,t}^{L,\beta}(x,y)| &\leq C_N\min\Big\{\frac{1}{t^{n/2\alpha}},\ \frac{t^{\beta}}{|x-y|^{n+2\alpha\beta}}\Big\}\Big(1+\frac{t^{1/2\alpha}}{\rho(x)}\Big)^{-N}\Big(1+\frac{t^{1/2\alpha}}{\rho(y)}\Big)^{-N} \\
&\leq \frac{C_N t^{\beta}}{(t^{1/2\alpha}+|x-y|)^{n+2\beta\alpha}}\Big(1+\frac{t^{1/2\alpha}}{\rho(x)}+\frac{t^{1/2\alpha}}{\rho(y)}\Big)^{-N}.
\end{aligned}
$$

Case 2: $\beta\geq 1$. Let $m=[\beta]+1$. We can obtain:

$$
\begin{aligned}
t^{\beta}\partial_t^{\beta}e^{-tL^{\alpha}} &= c_{\beta}t^{\beta}\int_0^{\infty}\partial_t^m e^{-(t+s)L^{\alpha}}\frac{ds}{s^{\beta+1-m}} \\
&= c_{\beta}t^{\beta}\int_0^{\infty}(-L)^{m\alpha}e^{-(t+s)L^{\alpha}}\frac{ds}{s^{1+\beta-m}} \\
&= c_{\beta}t^{\beta}\int_0^{\infty}(t+s)^m(-L)^{m\alpha}e^{-(t+s)L^{\alpha}}\frac{ds}{(t+s)^m s^{1+\beta-m}}.
\end{aligned}
$$

It follows from Proposition 9 that:

$$\begin{aligned}\left|D^{L,\beta}_{\alpha,t}(x,y)\right| &\leq C_N t^{\beta}\int_0^{\infty}\frac{(t+s)^m}{(t+s)^{n/2\alpha+m}}\Big(\frac{(t+s)^{1/2\alpha}}{\rho(x)}\Big)^{-N}\Big(\frac{(t+s)^{1/2\alpha}}{\rho(y)}\Big)^{-N}\frac{ds}{(t+s)^m s^{1+\beta-m}}\\ &\leq C_N t^{\beta}\rho(x)^N\rho(y)^N\int_0^{\infty}(t+s)^{-n/2\alpha-N/\alpha-m}\frac{ds}{s^{1+\beta-m}}\\ &\lesssim \frac{C_N}{t^{n/2\alpha}}\Big(1+\frac{t^{1/2\alpha}}{\rho(x)}\Big)^{-N}\Big(1+\frac{t^{1/2\alpha}}{\rho(y)}\Big)^{-N}.\end{aligned}$$

On the other hand, we obtain:

$$\begin{aligned}\left|D^{L,\beta}_{\alpha,t}(x,y)\right| &\leq t^{\beta}\int_0^{\infty}\Big\{\int_0^{\infty}\Big|\partial_r^m K^L_{(t+r)^{1/\alpha}\tau}(x,y)\Big|\frac{dr}{r^{\beta+1-m}}\Big\}\eta_1^{\alpha}(\tau)d\tau\\ &\leq C_N t^{\beta}\int_0^{\infty}\Big\{\int_0^{\infty}(t+r)^{-m}((t+r)^{1/\alpha}\tau)^{-n/2}e^{-c|x-y|^2/(t+r)^{1/\alpha}\tau}\\ &\quad\Big(\frac{\sqrt{\tau}(t+r)^{1/2\alpha}}{\rho(x)}\Big)^{-N}\Big(\frac{\sqrt{\tau}(t+r)^{1/2\alpha}}{\rho(y)}\Big)^{-N}\frac{dr}{r^{\beta+1-m}}\Big\}\eta_1^{\alpha}(\tau)d\tau\\ &\leq C_N\frac{t^{\beta}\rho(x)^N\rho(y)^N}{|x-y|^{n+2\alpha\beta}}t^{-N/\alpha}\int_0^{\infty}\Big(\int_0^{\infty}(1+u)^{-m+\beta-N/\beta}u^{m-\beta-1}du\Big)\eta_1^{\alpha}(\tau)\tau^{\alpha\beta-N}d\tau\\ &\lesssim \frac{C_N t^{\beta}}{|x-y|^{n+2\alpha\beta}}\Big(1+\frac{t^{1/2\alpha}}{\rho(x)}\Big)^{-N}\Big(1+\frac{t^{1/2\alpha}}{\rho(y)}\Big)^{-N},\end{aligned}$$

which indicates that (30) holds. □

In the next proposition, we give the Lipschitz continuity of $D^L_{\alpha,t}(\cdot,\cdot)$.

Proposition 15. *Let $\alpha\in(0,1)$, $V\in B_q, q>n/2$, and $\beta>0$. Let $0<\delta'\leq\delta=\min\{2\alpha,\delta_0\}$. For every $N>\alpha\beta$, there exists a constant $C_N>0$, such that for all $|h|\leq t^{1/2\alpha}$,*

$$|D^{L,\beta}_{\alpha,t}(x+h,y)-D^{L,\beta}_{\alpha,t}(x,y)|\leq C_N\Big(\frac{|h|}{t^{1/2\alpha}}\Big)^{\delta'}\frac{t^{\beta}}{(t^{1/2\alpha}+|x-y|)^{n+2\alpha\beta}}\Big(1+\frac{t^{1/2\alpha}}{\rho(x)}+\frac{t^{1/2\alpha}}{\rho(y)}\Big)^{-N}.$$

Proof. It is equivalent to verify:

$$|D^{L,\beta}_{\alpha,t}(x+h,y)-D^{L,\beta}_{\alpha,t}(x,y)|\leq C_N\Big(\frac{|h|}{t^{1/2\alpha}}\Big)^{\delta'}\min\Big\{\frac{1}{t^{n/2\alpha}},\ \frac{t^{\beta}}{|x-y|^{n+2\alpha\beta}}\Big\}\Big(1+\frac{t^{1/2\alpha}}{\rho(x)}+\frac{t^{1/2\alpha}}{\rho(y)}\Big)^{-N}. \tag{31}$$

Without loss of generality, for $m=[\beta]+1$, it holds that:

$$t^{\beta}\partial_t^{\beta}e^{-tL^{\alpha}}=c_{\beta}t^{\beta}\int_0^{\infty}(t+s)^m(-L)^{m\alpha}e^{-(t+s)L^{\alpha}}\frac{ds}{(t+s)^m s^{1+\beta-m}}.$$

By Proposition 9, we can obtain:

$$\begin{aligned}&\left|D^{L,\beta}_{\alpha,t}(x+h,y)-D^{L,\beta}_{\alpha,t}(x,y)\right|\\ &\leq C_N t^{\beta}\int_0^{\infty}\frac{(t+s)^m(|h|/(t+s)^{1/2\alpha})^{\delta'}}{((t+s)^{1/2\alpha}+|x-y|)^{n+2\alpha m}}\Big(1+\frac{(t+s)^{1/2\alpha}}{\rho(x)}\Big)^{-N}\Big(1+\frac{(t+s)^{1/2\alpha}}{\rho(y)}\Big)^{-N}\frac{ds}{(t+s)^m s^{1+\beta-m}}\\ &\leq C_N t^{\beta}|h|^{\delta'}\rho(x)^N\rho(y)^N\int_0^{\infty}(t+s)^{-(n+\delta')/2\alpha-N/\alpha-m}\frac{ds}{s^{1+\beta-m}}\\ &\lesssim C_N t^{-n/2\alpha}\Big(\frac{t^{1/2\alpha}}{\rho(x)}\Big)^{-N}\Big(\frac{t^{1/2\alpha}}{\rho(y)}\Big)^{-N}\Big(\frac{|h|}{t^{1/2\alpha}}\Big)^{\delta'}.\end{aligned}$$

On the other hand, we obtain:

$$
\begin{aligned}
&\left|D^{L,\beta}_{\alpha,t}(x+h,y)-D^{L,\beta}_{\alpha,t}(x,y)\right| \\
&\le C_N t^{\beta}\int_0^\infty\Bigg\{\int_0^\infty (t+r)^{-m-n/2\alpha}\tau^{-n/2}\Big(\frac{|h|}{\sqrt{(t+r)^{1/\alpha}\tau}}\Big)^{\delta'} \\
&\quad\times e^{-c|x-y|^2/(t+r)^{1/\alpha}\tau}\Big(\frac{\sqrt{\tau}(t+r)^{1/2\alpha}}{\rho(x)}\Big)^{-N}\Big(\frac{\sqrt{\tau}(t+r)^{1/2\alpha}}{\rho(y)}\Big)^{-N}\frac{dr}{r^{\beta+1-m}}\Bigg\}\eta^{\alpha}_1(\tau)d\tau \\
&\le C_N\frac{t^{\beta}|h|^{\delta'}\rho(x)^N\rho(y)^N}{|x-y|^{n+2\alpha\beta}}\int_0^\infty\Bigg\{\int_0^\infty (t+r)^{-m+\beta-N/\alpha-\delta'/2\alpha}r^{m-\beta-1}dr\Bigg\}\eta^{\alpha}_1(\tau)\tau^{\alpha\beta-N-\delta'/2}d\tau \\
&\lesssim \frac{C_N t^{\beta}}{|x-y|^{n+2\alpha\beta}}\Big(\frac{|h|}{t^{1/2\alpha}}\Big)^{\delta'}\Big(1+\frac{t^{1/2\alpha}}{\rho(x)}\Big)^{-N}\Big(1+\frac{t^{1/2\alpha}}{\rho(y)}\Big)^{-N},
\end{aligned}
$$

which implies (31). □

Proposition 16. *Let $\alpha\in(0,1)$, $\beta>0$, $V\in B_q, q>n/2$, and $0<\delta'\le\min\{2\alpha,2\alpha\beta,\delta_0\}$. For every $N>0$, there exists a constant $C_N>0$, such that:*

$$
\Big|\int_{\mathbb{R}^n}D^{L,\beta}_{\alpha,t}(x,y)dy\Big|\le C_N\frac{(t^{1/2\alpha}/\rho(x))^{\delta'}}{(1+t^{1/2\alpha}/\rho(x))^N},\quad x\in\mathbb{R}^n.
$$

Proof. Let $m=[\beta]+1$. By (iii) of Proposition 9, we change the order of integrations to obtain:

$$
\begin{aligned}
\Big|\int_{\mathbb{R}^n}D^{L,\beta}_{\alpha,t}(x,y)dy\Big| &= \Big|\int_{\mathbb{R}^n}\Big\{t^{\beta}\int_0^\infty\partial^m_t D^{L,\beta}_{\alpha,t+s}(x,y)\frac{ds}{s^{1+\beta-m}}\Big\}dy\Big| \\
&\le t^{\beta}\int_0^\infty\int_{\mathbb{R}^n}\left|\partial^m_t D^{L,\beta}_{\alpha,t+s}(x,y)\right|\frac{dyds}{s^{1+\beta-m}} \\
&\le C_N t^{\beta}\int_0^\infty\frac{((t+s)^{1/2\alpha}/\rho(x))^{\delta'}}{(1+(t+s)^{1/2\alpha}/\rho(x))^N}\frac{ds}{s^{1+\beta-m}(t+s)^m}.
\end{aligned}
$$

If $t^{1/2\alpha}>\rho(x)$, then:

$$
\begin{aligned}
\Big|\int_{\mathbb{R}^n}D^{L,\beta}_{\alpha,t}(x,y)dy\Big| &\le C_N t^{\beta}\rho(x)^{N-\delta'}\int_0^\infty (t+s)^{\delta'/2\alpha-N/2\alpha-m}s^{m-\beta-1}ds \\
&\le C_N\frac{(t^{1/2\alpha}/\rho(x))^{\delta'}}{(1+t^{1/2\alpha}/\rho(x))^N}.
\end{aligned}
$$

If $t^{1/2\alpha}\le\rho(x)$, then:

$$
\begin{aligned}
\Big|\int_{\mathbb{R}^n}D^{L,\beta}_{\alpha,t}(x,y)dy\Big| &\le C_N t^{\beta}\int_0^\infty\Big((t+s)^{1/2\alpha}/\rho(x)\Big)^{\delta'}\frac{ds}{s^{1+\beta-m}(t+s)^m} \\
&\le C_N t^{\beta}\rho(x)^{\delta'}\int_0^\infty (t+s)^{\delta'/2\alpha-m}s^{m-1-\beta}ds \\
&\lesssim C_N\Big(t^{1/2\alpha}/\rho(x)\Big)^{\delta'}\le C_N\frac{(t^{1/2\alpha}/\rho(x))^{\delta'}}{(1+t^{1/2\alpha}/\rho(x))^N},
\end{aligned}
$$

which completes the proof of Proposition 16. □

4. Characterization of Campanato–Morrey Spaces Associated with L

Firstly, we deduce a reproducing formula:

Lemma 12. *Let $\alpha \in (0,1)$ and $\beta > 0$. The operator $t^\beta \partial_t^\beta e^{-tL^\alpha}$ defines an isometry from $L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^{n+1}_+, dxdt/t)$. Moreover, in the sense of $L^2(\mathbb{R}^n)$, it holds that:*

$$f(x) = c_{\alpha,\beta} \lim_{N\to\infty} \lim_{\epsilon\to 0} \int_\epsilon^N (t^\beta \partial_t^\beta e^{-tL^\alpha})^2(f)(x) \frac{dt}{t}.$$

Proof. Note that for $dE(\lambda)$, the spectral resolution of the operator L, it follows from

$$e^{-tL^\alpha} = \int_0^\infty e^{-t\lambda^\alpha} dE(\lambda)$$

that:

$$\begin{aligned} t^\beta \partial_t^\beta e^{-tL^\alpha} &= t^\beta \int_0^\infty \partial_t^m \Big(\int_0^\infty e^{-(t+s)\lambda^\alpha} dE(\lambda) \Big) \frac{ds}{s^{1+\beta-m}} \\ &= t^\beta \int_0^\infty \Big(\int_0^\infty (-1)^m \lambda^{\alpha m} e^{-(t+s)\lambda^\alpha} dE(\lambda) \Big) \frac{ds}{s^{1+\beta-m}} \\ &= C t^\beta \int_0^\infty (-1)^m \lambda^{m\alpha} e^{-t\lambda^\alpha} \lambda^{\alpha(\beta-m)} dE(\lambda) \\ &= \int_0^\infty (t\lambda^\alpha)^\beta e^{-t\lambda^\alpha} dE(\lambda). \end{aligned}$$

Then, for $f \in L^2(\mathbb{R}^n)$, we have:

$$\begin{aligned} \|t^\beta \partial_t^\beta e^{-tL^\alpha}(f)(x)\|^2_{L^2(\mathbb{R}^{n+1}_+, dxdt/t)} &= \int_0^\infty \Big(\int_{\mathbb{R}^n} |t^\beta \partial_t^\beta e^{-tL^\alpha}(f)(x)|^2 dx \Big) \frac{dt}{t} \\ &= \int_0^\infty \Big\langle t^\beta \partial_t^\beta e^{-tL^\alpha}(f),\, t^\beta \partial_t^\beta e^{-tL^\alpha}(f) \Big\rangle \frac{dt}{t} \\ &= \int_0^\infty \Big\langle (t^\beta \partial_t^\beta e^{-tL^\alpha})^2(f),\, f \Big\rangle \frac{dt}{t} \\ &= \int_0^\infty \int_0^\infty t^{2\beta} \lambda^{2\alpha\beta} e^{-2t\lambda^\alpha} \frac{dt}{t} dE_{f,f}(\lambda) \\ &= \frac{\Gamma(\beta)}{2^\beta} \|f\|_2^2. \end{aligned}$$

Below, we only prove that for every pair of sequences $n_k \uparrow \infty$ and $\epsilon_k \downarrow 0$ as $k \to \infty$,

$$\lim_{k\to\infty} \int_{n_k}^{n_{k+m}} (t^\beta \partial_t^\beta e^{-tL^\alpha})^2 f(x) \frac{dt}{t} = \lim_{k\to\infty} \int_{\epsilon_k}^{\epsilon_{k+m}} (t^\beta \partial_t^\beta e^{-tL^\alpha})^2 f(x) \frac{dt}{t} = 0. \tag{32}$$

If (32) holds, there exists a function $h \in L^2(\mathbb{R}^n)$, such that:

$$\lim_{k\to\infty} \int_{\epsilon_k}^{n_k} (t^\beta \partial_t^\beta e^{-tL^\alpha})^2 f(x) \frac{dt}{t} = h(x),$$

which implies that for all $g \in L^2(\mathbb{R}^n)$,

$$\begin{aligned} \langle h,\, g \rangle &= \Big\langle \lim_{k\to\infty} \int_{\epsilon_k}^{n_k} (t^\beta \partial_t^\beta e^{-tL^\alpha})^2 f \frac{dt}{t},\, g \Big\rangle \\ &= \lim_{k\to\infty} \int_{\epsilon_k}^{n_k} \Big\langle (t^\beta \partial_t^\beta e^{-tL^\alpha})^2 f,\, g \Big\rangle \frac{dt}{t} \\ &= \lim_{k\to\infty} \int_{\epsilon_k}^{n_k} \Big\langle t^\beta \partial_t^\beta e^{-tL^\alpha} f,\, t^\beta \partial_t^\beta e^{-tL^\alpha} g \Big\rangle \frac{dt}{t} \\ &= C_{\alpha,\beta} \langle f,\, g \rangle. \end{aligned}$$

This means $h = C_{\alpha,\beta} f$. Now, we verify (32). As $k \to \infty$, we can apply the functional calculus to deduce that:

$$\begin{aligned} \Big\| \int_{n_k}^{n_{k+m}} (t^\beta \partial_t^\beta e^{-tL^\alpha})^2 f(x) \frac{dt}{t} \Big\|_{L^2}^2 &= \Big\| \int_{n_k}^{n_{k+m}} \Big(\int_0^\infty t^{2\beta} \lambda^{2\alpha\beta} e^{-2t\lambda^\alpha} dE_f(\lambda) \Big) \frac{dt}{t} \Big\|^2 \\ &= \Big\| \int_0^\infty \Big(\int_{n_k}^{n_{k+m}} t^{2\beta} \lambda^{2\alpha\beta} e^{-2t\lambda^\alpha} \frac{dt}{t} \Big) dE_f(\lambda) \Big\|^2 \\ &\leq \int_0^\infty \Big(\int_{n_k}^{n_{k+m}} t^{2\beta} \lambda^{2\alpha\beta} e^{-2t\lambda^\alpha} \frac{dt}{t} \Big)^2 dE_{f,f}(\lambda) \to 0, \end{aligned}$$

since

$$\lim_{k\to\infty} \Big| \int_{n_k}^{n_{k+m}} t^{2\beta} \lambda^{2\alpha\beta} e^{-2t\lambda^\alpha} \frac{dt}{t} \Big| = 0.$$

The integral

$$\lim_{k\to\infty} \int_{\epsilon_k}^{\epsilon_{k+m}} (t^\beta \partial_t^\beta e^{-tL^\alpha})^2 f(x) \frac{dt}{t}$$

can be dealt with similarly. □

The following inequality was established by Harboure–Salinas–Viviani [37]:

Lemma 13. ([37], (5.3)) *Let $0 < \gamma \leq 1$. For any pair of measurable functions F and G on $\mathbb{R}^{n+1}_+$, we have:*

$$\begin{aligned} &\iint_{\mathbb{R}^{n+1}_+} |F(x,t)| \cdot |G(x,t)| \frac{dxdt}{t} \\ &\leq C \sup_B \left\{ \frac{1}{|B|^{1+2\gamma/n}} \iint_{\widehat{B}} |F(x,t)|^2 \frac{dxdt}{t} \right\}^{1/2} \left\{ \int_{\mathbb{R}^n} \left(\int_0^\infty \int_{|x-y|<t} |G(y,t)|^2 \frac{dydt}{t^{n+1}} \right)^{n/2(n+\gamma)} dx \right\}^{1+\gamma/n}. \end{aligned}$$

In Lemma 13, letting

$$\begin{cases} F(x,t) := t^{2\alpha\beta} \partial_s^\beta e^{-sL^\alpha} \mid_{s=t^{2\alpha}} (f)(x) = Q^{L,\beta}_{\alpha,t}(f)(x); \\ G(x,t) := t^{2\alpha\beta} \partial_s^\beta e^{-sL^\alpha} \mid_{s=t^{2\alpha}} (g)(x) = Q^{L,\beta}_{\alpha,t}(g)(x), \end{cases}$$

we have:

$$\begin{aligned} &\iint_{\mathbb{R}^{n+1}_+} |Q^{L,\beta}_{\alpha,t}(f)(x)| \cdot |Q^{L,\beta}_{\alpha,t}(g)(x)| \frac{dxdt}{t} \\ &\leq C \sup_B \left(\frac{1}{|B|^{1+2\gamma/n}} \iint_{\widehat{B}} |Q^{L,\beta}_{\alpha,t}(f)(x)|^2 \frac{dxdt}{t} \right)^{1/2} \\ &\times \left\{ \int_{\mathbb{R}^n} \left(\int_0^\infty \int_{|x-y|<t} |Q^{L,\beta}_{\alpha,t}(g)(x)|^2 \frac{dydt}{t^{n+1}} \right)^{n/2(n+\gamma)} dx \right\}^{1+\gamma/n}. \end{aligned} \tag{33}$$

On the left-hand side of (33), since

$$\begin{cases} Q^{L,\beta}_{\alpha,t}(f)(x) = t^{2\alpha\beta} L^{\alpha\beta} e^{-t^{2\alpha} L^\alpha}(f); \\ Q^{L,\beta}_{\alpha,t}(g)(x) = t^{2\alpha\beta} L^{\alpha\beta} e^{-t^{2\alpha} L^\alpha}(g), \end{cases}$$

we can obtain, via the change of variables,

$$
\begin{aligned}
&\iint_{\mathbb{R}^{n+1}_+} |Q^{L,\beta}_{\alpha,t}(f)(x)| \cdot |Q^{L,\beta}_{\alpha,t}(g)(x)| \frac{dxdt}{t} \\
&= \iint_{\mathbb{R}^{n+1}_+} |t^{2\alpha\beta} L^{\alpha\beta} e^{-t^{2\alpha}L^\alpha}(f)(x)| \cdot |t^{2\alpha\beta} L^{\alpha\beta} e^{-t^{2\alpha}L^\alpha}(g)(x)| \frac{dxdt}{t} \\
&= \iint_{\mathbb{R}^{n+1}_+} |s^{\beta} L^{\alpha\beta} e^{-sL^\alpha}(f)(x)| \cdot |s^{\beta} L^{\alpha\beta} e^{-sL^\alpha}(g)(x)| \frac{dxds}{s} \\
&= \iint_{\mathbb{R}^{n+1}_+} |s^{\beta} \partial_s^\beta e^{-sL^\alpha}(f)(x)| \cdot |s^{\beta} \partial_s^\beta e^{-sL^\alpha}(g)(x)| \frac{dxds}{s}.
\end{aligned}
$$

On the right-hand side of (33), using change of variables again, we obtain:

$$
\begin{aligned}
&\sup_B \left(\frac{1}{|B|^{1+2\gamma/n}} \int_0^{r_B} \int_B |t^{2\alpha\beta} L^{\alpha\beta} e^{-t^{2\alpha}L^\alpha}(f)(x)|^2 \frac{dxdt}{t} \right)^{1/2} \\
&\lesssim \sup_B \left(\frac{1}{|B|^{1+2\gamma/n}} \int_0^{r_B^{2\alpha}} \int_B |s^{\beta} L^{\alpha\beta} e^{-sL^\alpha}(f)(x)|^2 \frac{dxds}{s} \right)^{1/2} \\
&= \sup_B \left(\frac{1}{|B|^{1+2\gamma/n}} \int_0^{r_B^{2\alpha}} \int_B |s^{\beta} \partial_s^\beta e^{-sL^\alpha}(f)(x)|^2 \frac{dxds}{s} \right)^{1/2};
\end{aligned}
$$

meanwhile,

$$
\begin{aligned}
&\left\{ \int_{\mathbb{R}^n} \left(\int_0^\infty \int_{|x-y|<t} |t^{2\alpha\beta} L^{\alpha\beta} e^{-t^{2\alpha}L^\alpha}(g)(y)|^2 \frac{dydt}{t^{n+1}} \right)^{n/2(n+\gamma)} dx \right\}^{1+\gamma/n} \\
&\lesssim \left\{ \int_{\mathbb{R}^n} \left(\int_0^\infty \int_{|x-y|<s^{1/2\alpha}} |s^{\beta} L^{\alpha\beta} e^{-sL^\alpha}(g)(y)|^2 \frac{s^{1/2\alpha-1}dyds}{s^{(n+1)/2\alpha}} \right)^{n/2(n+\gamma)} dx \right\}^{1+\gamma/n} \\
&\lesssim \left\{ \int_{\mathbb{R}^n} \left(\int_0^\infty \int_{|x-y|<s^{1/2\alpha}} |s^{\beta} \partial_s^\beta e^{-sL^\alpha}(g)(y)|^2 \frac{s^{1/2\alpha-1}dyds}{s^{(n+1)/2\alpha}} \right)^{n/2(n+\gamma)} dx \right\}^{1+\gamma/n}.
\end{aligned}
$$

Finally, we have:

$$
\begin{aligned}
&\iint_{\mathbb{R}^{n+1}_+} |s^{\beta} \partial_s^\beta e^{-sL^\alpha}(f)(x)| \cdot |s^{\beta} \partial_s^\beta e^{-sL^\alpha}(g)(x)| \frac{dxds}{s} \\
&\lesssim \sup_B \left(\frac{1}{|B|^{1+2\gamma/n}} \int_0^{r_B^{2\alpha}} \int_B |s^{\beta} \partial_s^\beta e^{-sL^\alpha}(f)(x)|^2 \frac{dxds}{s} \right)^{1/2} \\
&\quad \times \left\{ \int_{\mathbb{R}^n} \left(\iint_{|x-y|<s^{1/2\alpha}} |s^{\beta} \partial_s^\beta e^{-sL^\alpha}(g)(y)|^2 \frac{s^{1/2\alpha-1}dyds}{s^{(n+1)/2\alpha}} \right)^{n/2(n+\gamma)} dx \right\}^{1+\gamma/n}.
\end{aligned} \tag{34}
$$

For $\alpha \in (0,1)$ and $\beta > 0$, define an area function $S^L_{\alpha,\beta}$ as follows:

$$
S^L_{\alpha,\beta}(h)(x) := \left(\iint_{\Gamma_\alpha(x)} |t^\beta \partial_t^\beta e^{-tL^\alpha}(h)(y)|^2 \frac{dydt}{t^{n/2\alpha+1}} \right)^{1/2},
$$

where $\Gamma_\alpha(x)$ denotes the cone $\{(y,t) : |x-y| < t^{1/2\alpha}\}$.

Lemma 14. *Let $\alpha \in (0,1)$ and $\beta > 0$. The area function $S^L_{\alpha,\beta}$ is bounded on $L^2(\mathbb{R}^n)$.*

Proof. Let

$$
g^L_{\alpha,\beta}(h)(x) := \left(\int_0^\infty |t^\beta \partial_t^\beta e^{-tL^\alpha} h(x)|^2 \frac{dt}{t} \right)^{1/2}.
$$

We can obtain:

$$\begin{aligned}\|g^L_{\alpha,\beta}(h)\|_2^2 &= \int_{\mathbb{R}^n}\Big(\int_0^\infty |t^\beta\partial_t^\beta e^{-tL^\alpha}h(x)|^2\frac{dt}{t}\Big)dx \\ &= \int_0^\infty\Big(\int_{\mathbb{R}^n}|t^\beta\partial_t^\beta e^{-tL^\alpha}h(x)|^2dx\Big)\frac{dt}{t} \\ &= \int_0^\infty\Big\langle t^\beta\partial_t^\beta e^{-tL^\alpha}h,\ t^\beta\partial_t^\beta e^{-tL^\alpha}h\Big\rangle\frac{dt}{t} \\ &= \int_0^\infty\Big\langle (t^\beta\partial_t^\beta e^{-tL^\alpha})^2h,\ h\Big\rangle\frac{dt}{t} \\ &= \int_0^\infty\int_0^\infty t^{2\beta}\lambda^{2\alpha\beta}e^{-t\lambda^\alpha}dE_{h,h}(\lambda)\frac{dt}{t} \\ &\lesssim \|h\|_2^2.\end{aligned} \tag{35}$$

Hence, it follows from (35) that:

$$\begin{aligned}\|S^L_{\alpha,\beta}h\|_2^2 &= \int_{\mathbb{R}^n}\Big(\iint_{\Gamma_\alpha(x)}|t^\beta\partial_t^\beta e^{-tL^\alpha}(h)(y)|^2\frac{dydt}{t^{\frac{n}{2\alpha}+1}}\Big)dx \\ &= \int_{\mathbb{R}^n}\Big(\int_0^\infty\int_{\mathbb{R}^n}|t^\beta\partial_t^\beta e^{-tL^\alpha}(h)(y)|^2\chi_{\Gamma_\alpha(x)}(y)\frac{dydt}{t^{\frac{n}{2\alpha}+1}}\Big)dx \\ &\lesssim \int_0^\infty\int_{\mathbb{R}^n}|t^\beta\partial_t^\beta e^{-tL^\alpha}(h)(y)|^2\Big(\int_{\mathbb{R}^n}\chi_{\Gamma_\alpha(y)}(x)dx\Big)\frac{dydt}{t^{\frac{n}{2\alpha}+1}} \\ &\lesssim \int_0^\infty\int_{\mathbb{R}^n}|t^\beta\partial_t^\beta e^{-tL^\alpha}(h)(y)|^2\frac{dydt}{t}\lesssim\|h\|_2^2.\end{aligned}$$

□

Theorem 2. *Assume that $\alpha \in (0,1)$, $\beta > 0$ and $0 < \gamma < \min\{2\alpha, 2\alpha\beta\}$. Let f be a linear combination of $H_L^{n/(n+\gamma)}$-atoms. There exists a constant C, such that:*

$$\|S^L_{\alpha,\beta}(f)\|_{L^{n/(n+\gamma)}} \le C\|f\|_{H_L^{n/(n+\gamma)}}.$$

Proof. Let a be an $H_L^{n/(n+\gamma)}$-atom associated with a ball $B = B(x_0, r)$. Then, we write:

$$\|S^L_{\alpha,\beta}(a)\|^{n/(n+\gamma)}_{L^{n/(n+\gamma)}} \le I + II,$$

where

$$\begin{cases} I := \int_{8B}|S^L_{\alpha,\beta}(a)(x)|^{n/(n+\gamma)}dx; \\ II := \int_{(8B)^c}|S^L_{\alpha,\beta}(a)(x)|^{n/(n+\gamma)}dx. \end{cases}$$

We use Lemma 14 and Hölder's inequality to obtain:

$$I \lesssim \Big(\int_{8B}|S^L_{\alpha,\beta}a(x)|^2dx\Big)^{n/2(n+\gamma)}|B|^{(n+2\gamma)/2(n+\gamma)} \lesssim \|a\|_2^{n/(n+\gamma)}|B|^{(n+2\gamma)/2(n+\gamma)} \lesssim 1.$$

Now, we deal with II in the following two cases:

Case 1: $r < \rho(x_0)/4$. For this case, $\int_{\mathbb{R}^n} a(x)dx = 0$. We write $(S^L_{\alpha,\beta}(a)(x))^2 \lesssim I_1(x) + I_2(x)$, where:

$$\begin{cases} I_1(x) := \int_0^{(|x-x_0|/2)^{2\alpha}} \int_{|x-y|<t^{1/2\alpha}} \left\{ \int_{\mathbb{R}^n} (t^\beta \partial_t^\beta e^{-tL^\alpha}(y,x') - t^\beta \partial_t^\beta e^{-tL^\alpha}(y,x_0))a(x')dx' \right\}^2 \frac{dydt}{t^{n/2\alpha+1}}; \\ I_2(x) := \int_{(|x-x_0|/2)^{2\alpha}}^\infty \int_{|x-y|<t^{1/2\alpha}} \left\{ \int_{\mathbb{R}^n} (t^\beta \partial_t^\beta e^{-tL^\alpha}(y,x') - t^\beta \partial_t^\beta e^{-tL^\alpha}(y,x_0))a(x')dx' \right\}^2 \frac{dydt}{t^{n/2\alpha+1}}. \end{cases}$$

We first estimate I_1. Since $|x-y| \le |x-x_0|/2$, then $|y-x'| \sim |y-x_0|$ for $x' \in B(x_0, r)$ and $x \notin B(x_0, 8r)$. We can use Propositions 14 and 15 to deduce that there exists $\delta' > \gamma$, such that:

$$\begin{aligned} |I_1(x)| &\lesssim \int_0^{(|x-x_0|/2)^{2\alpha}} \int_{|x-y|<t^{1/2\alpha}} \left\{ \int_B \left(\frac{|x'-x_0|}{t^{1/2\alpha}} \right)^{\delta'} \frac{t^\beta}{(t^{1/2\alpha}+|y-x_0|)^{n+2\alpha\beta}} \frac{dx'}{|B|^{1+\gamma/n}} \right\}^2 \frac{dydt}{t^{n/2\alpha+1}} \\ &\lesssim \int_0^{(|x-x_0|/2)^{2\alpha}} \int_{|x-y|<t^{1/2\alpha}} \left\{ \int_B \left(\frac{r}{t^{1/2\alpha}} \right)^{\delta'} \frac{t^\beta}{t^{\beta+n/2\alpha}(1+|y-x_0|/t^{1/2\alpha})^{n+2\alpha\beta}} \frac{dx'}{|B|^{1+\gamma/n}} \right\}^2 \frac{dydt}{t^{n/2\alpha+1}} \\ &\lesssim \int_0^{(|x-x_0|/2)^{2\alpha}} \int_{|x-y|<t^{1/2\alpha}} \left(\frac{r}{t^{1/2\alpha}} \right)^{2\delta'} \frac{1}{t^{n/\alpha}(1+|y-x_0|/t^{1/2\alpha})^{2n+4\alpha\beta}} \frac{1}{|B|^{2\gamma/n}} \frac{dydt}{t^{n/2\alpha+1}}. \end{aligned}$$

Because $0 < t < |x-x_0|^{2\alpha}/2^{2\alpha}$ and $|x-y| < t^{1/2\alpha}$, then $|x-y| < |x-x_0|/2$. This implies $|y-x_0| \gtrsim |x-x_0|/2$. We have:

$$\begin{aligned} |I_1(x)| &\lesssim \int_0^{(|x-x_0|/2)^{2\alpha}} \int_{|x-y|<t^{1/2\alpha}} \left(\frac{r}{t^{1/2\alpha}} \right)^{2\delta'} \frac{1}{t^{n/\alpha}(1+|x-x_0|/t^{1/2\alpha})^{2n+4\alpha\beta}} \frac{1}{|B|^{2\gamma/n}} \frac{dydt}{t^{n/2\alpha+1}} \\ &\lesssim \int_0^{(|x-x_0|/2)^{2\alpha}} \left(\frac{r}{t^{1/2\alpha}} \right)^{2\delta'} \frac{1}{t^{n/\alpha}(1+|x-x_0|/t^{1/2\alpha})^{2n+4\alpha\beta}} \frac{1}{|B|^{2\gamma/n}} \frac{dt}{t} \\ &\lesssim \frac{r^{2(\delta'-\gamma)}}{|x-x_0|^{2(n+\delta')}}, \end{aligned}$$

which, via a direct computation, gives:

$$\int_{(8B)^c} |I_1(x)|^{n/2(n+\gamma)} dx \lesssim \int_{(8B)^c} \left(\frac{r^{\delta'-\gamma}}{|x-x_0|^{n+\delta'}} \right)^{n/(n+\gamma)} dx \le C.$$

Let us continue with I_2. Similarly, it follows from Proposition 15 that:

$$\begin{aligned} |I_2(x)| &\lesssim \int_{|x-x_0|^{2\alpha}/2^{2\alpha}}^\infty \int_{|x-y|<t^{1/2\alpha}} \left\{ \int_B \left(\frac{|x'-x_0|}{t^{1/2\alpha}} \right)^{\delta'} \frac{1}{t^{n/2\alpha}} \frac{dx'}{|B|^{1+\gamma/n}} \right\}^2 \frac{dydt}{t^{n/2\alpha+1}} \\ &\lesssim \int_{|x-x_0|^{2\alpha}/2^{2\alpha}}^\infty \int_{|x-y|<t^{1/2\alpha}} \left(\frac{r}{t^{1/2\alpha}} \right)^{2\delta'} \frac{1}{t^{n/\alpha}} \frac{1}{|B|^{2\gamma/n}} \frac{dydt}{t^{n/2\alpha+1}} \\ &\lesssim \frac{r^{2(\delta'-\gamma)}}{|x-x_0|^{2n+2\delta'}}. \end{aligned}$$

Hence, we still have $\int_{(8B)^c} |I_2(x)|^{n/2(n+\gamma)} dx \lesssim 1$.

Case 2: $\rho(x_0)/4 < r < \rho(x_0)$. For this case, the atom a has no canceling condition. We have $(S^L_{\alpha,\beta}(a)(x))^2 \lesssim I_3(x) + I_4(x) + I_5(x)$, where:

$$\begin{cases} I_3(x) := \int_0^{r^{2\alpha}/4^\alpha} \int_{|x-y|<t^{1/2\alpha}} \left| \int_{\mathbb{R}^n} t^\beta \partial_t^\beta e^{-tL^\alpha}(y,x')a(x')dx' \right|^2 \frac{dydt}{t^{n/2\alpha+1}}; \\ I_4(x) := \int_{r^{2\alpha}/4^\alpha}^{|x-x_0|^{2\alpha}/4^{2\alpha}} \int_{|x-y|<t^{1/2\alpha}} \left| \int_{\mathbb{R}^n} t^\beta \partial_t^\beta e^{-tL^\alpha}(y,x')a(x')dx' \right|^2 \frac{dydt}{t^{n/2\alpha+1}}; \\ I_5(x) := \int_{|x-x_0|^{2\alpha}/4^{2\alpha}}^{\infty} \int_{|x-y|<t^{1/2\alpha}} \left| \int_{\mathbb{R}^n} t^\beta \partial_t^\beta e^{-tL^\alpha}(y,x')a(x')dx' \right|^2 \frac{dydt}{t^{n/2\alpha+1}}. \end{cases}$$

Because $x \in (8B)^c$ and $x' \in B$, then $|x' - x_0| < |x - x_0|/8$. On the other hand, for $t \in (0, r^{2\alpha}/4^\alpha)$, $|y - x| < t^{1/2\alpha} \le r/2 < |x - x_0|/8$. This means that $|y - x'| \ge c|x - x_0|$. We can obtain:

$$\begin{aligned} |I_3(x)| &\lesssim \int_0^{r^{2\alpha}/4^\alpha} \int_{|x-y|<t^{1/2\alpha}} \left(\int_B \frac{t^\beta}{(t^{1/2\alpha} + |x - x_0|)^{n+2\alpha\beta}} \frac{dx'}{|B|^{1+\gamma/n}} \right)^2 \frac{dydt}{t^{n/2\alpha+1}} \\ &\lesssim \frac{1}{|B|^{2\gamma/n}} \int_0^{r^{2\alpha}/4^\alpha} \int_{|x-y|<t^{1/2\alpha}} \frac{t^{2\beta}}{(t^{1/2\alpha} + |x - x_0|)^{2n+4\alpha\beta}} \frac{dydt}{t^{n/2\alpha+1}} \\ &\lesssim \frac{1}{|B|^{2\gamma/n}} \int_0^{r^{2\alpha}/4^\alpha} \frac{t^{2\beta}}{(t^{1/2\alpha} + |x - x_0|)^{2n+4\alpha\beta}} \frac{dt}{t} \\ &\lesssim \frac{r^{4\alpha\beta-2\gamma}}{|x - x_0|^{2n+4\alpha\beta}}, \end{aligned}$$

which indicates that:

$$\int_{(8B)^c} |I_3(x)|^{n/2(n+\gamma)} dx \lesssim \int_{(8B)^c} \left(\frac{r^{2\alpha\beta-\gamma}}{|x - x_0|^{n+2\alpha\beta}} \right)^{n/(n+\gamma)} dx \le C.$$

Similarly,

$$\begin{aligned} |I_4(x)| &\lesssim \int_{r^{2\alpha}/4^\alpha}^{|x-x_0|^{2\alpha}/4^{2\alpha}} \int_{|x-y|<t^{1/2\alpha}} \left\{ \int_B \frac{t^\beta}{(t^{1/2\alpha} + |y - x'|)^{n+2\alpha\beta}} \frac{1}{|B|^{1+\gamma/n}} \left(\frac{\rho(x')}{t^{1/2\alpha}} \right)^N dx' \right\}^2 \frac{dydt}{t^{n/2\alpha+1}} \\ &\lesssim \int_{r^{2\alpha}/4^\alpha}^{|x-x_0|^{2\alpha}/4^{2\alpha}} \int_{|x-y|<t^{1/2\alpha}} \left\{ \int_B \frac{t^\beta}{(t^{1/2\alpha} + |y - x'|)^{n+2\alpha\beta}} \frac{1}{|B|^{1+\gamma/n}} \left(\frac{\rho(x_0)}{t^{1/2\alpha}} \right)^N dx' \right\}^2 \frac{dydt}{t^{n/2\alpha+1}}. \end{aligned}$$

Notice that $r/2 \le t^{1/2\alpha} \lesssim |x - x_0|/4$ for $t \in (r^{2\alpha}/4^\alpha, |x - x_0|^{2\alpha}/4^{2\alpha})$. It can be deduced from the triangle inequality that $|y - x'| \sim |x - x_0|$. Then,

$$\begin{aligned} |I_4(x)| &\lesssim \int_{r^{2\alpha}/4^\alpha}^{|x-x_0|^{2\alpha}/4^{2\alpha}} \int_{|x-y|<t^{1/2\alpha}} \frac{1}{t^{n/\alpha}(1 + |x - x_0|/t^{1/2\alpha})^{2n+4\alpha\beta}} \frac{1}{|B|^{2\gamma/n}} \left(\frac{\rho(x_0)}{t^{1/2\alpha}} \right)^{2N} \frac{dydt}{t^{n/2\alpha+1}} \\ &\lesssim \int_{r^{2\alpha}/4^\alpha}^{|x-x_0|^{2\alpha}/4^{2\alpha}} \left(\frac{t^{n/2\alpha+\beta}}{t^{n/2\alpha}|x - x_0|^{n+2\alpha\beta}} \frac{r^N}{t^{N/2\alpha} r^\gamma} \right)^2 \frac{dt}{t} \\ &\lesssim \frac{r^{4\alpha\beta-2\gamma}}{|x - x_0|^{2n+4\alpha\beta}}. \end{aligned}$$

The estimate for I_5 is similar to that of I_4. In fact, due to $r \sim \rho(x_0)$,

$$
\begin{aligned}
|I_5(x)| &\lesssim \int_{|x-x_0|^{2\alpha}/4^{2\alpha}}^{\infty} \int_{|x-y|<t^{1/2\alpha}} \left\{ \int_B \frac{t^{\beta}}{(t^{1/2\alpha}+|y-x'|)^{n+2\alpha\beta}} \frac{dx'}{r^{\gamma+n}} \left(\frac{\rho(x_0)}{t^{1/2\alpha}}\right)^N \right\}^2 \frac{dydt}{t^{n/2\alpha+1}} \\
&\lesssim \int_{|x-x_0|^{2\alpha}/4^{2\alpha}}^{\infty} \int_{|x-y|<t^{1/2\alpha}} \left(\frac{r}{t^{1/2\alpha}}\right)^{2N} \frac{1}{r^{2\gamma}} \frac{dydt}{t^{3n/2\alpha+1}} \\
&\lesssim \frac{r^{2N-2\gamma}}{|x-x_0|^{2(n+N)}}.
\end{aligned}
$$

The estimates for I_4 and I_5 indicate that:

$$\int_{(8B)^c} |I_4(x)+I_5(x)|^{n/2(n+\gamma)} dx \lesssim 1.$$

□

Lemma 15. *Let $\alpha \in (0,1)$, $q_t(\cdot,\cdot)$ be a function of $x, y \in \mathbb{R}^n$ and $t > 0$. Assume that for each $N > 0$, there exists a constant C_N, such that for $\theta > \gamma$,*

$$|q_t(x,y)| \le C_N \Big(1 + \frac{t^{1/2\alpha}}{\rho(x)} + \frac{t^{1/2\alpha}}{\rho(y)}\Big)^{-N} t^{-n/2\alpha} \Big(1 + \frac{|x-y|}{t^{1/2\alpha}}\Big)^{-(n+\theta)}. \tag{36}$$

Then, for any $H_L^{n/(n+\gamma)}$-atom a supported on $B(x_0,r)$, there exists a constant $C_{x_0,r}$, such that:

$$\sup_{t>0} \Big| \int_{\mathbb{R}^n} q_t(x,y)a(y)dy \Big| \le C_{N,x_0,r}(1+|x|)^{-n-\theta}, x \in \mathbb{R}^n.$$

Proof. If $x \in B(x_0,2r)$, then $1+|x| \le 1+|x-x_0|+|x_0| \le 1+2r+|x_0|$. It follows from the condition $\|a\|_\infty \le |B(x_0,r)|^{-1-\gamma/n}$ that:

$$
\begin{aligned}
\Big| \int_{\mathbb{R}^n} q_t(x,y)a(y)dy \Big| &\lesssim \int_{B(x_0,r)} |q_t(x,y)||a(y)|dy \\
&\lesssim \int_{B(x_0,r)} t^{-n/2\alpha} \Big(1+\frac{|x-y|}{t^{1/2\alpha}}\Big)^{-n-\theta} r^{-n-\gamma} dy \\
&\lesssim r^{-n-\gamma} \frac{(1+2r+|x_0|)^{n+\theta}}{(1+2r+|x_0|)^{n+\theta}} \\
&\lesssim C_{N,x_0,r}(1+|x|)^{-n-\theta}.
\end{aligned}
$$

If $x \notin B(x_0,2r)$, then for any $y \in B(x_0,r)$, $|x-y| \sim |x-x_0|$. On the other hand, $\rho(y) \sim \rho(x_0)$, since $r < \rho(x_0)$ and $|y-x_0| < r$. By (36), we have:

$$
\begin{aligned}
|q_t(x,y)| &\lesssim t^{-n/2\alpha}\Big(1+\frac{|x-y|}{t^{1/2\alpha}}\Big)^{-(n+\theta)} \Big(1+\frac{t^{1/2\alpha}}{\rho(x_0)}\Big)^{-N} \\
&\lesssim t^{-n/2\alpha}\Big(1+\frac{|x-y|}{t^{1/2\alpha}}\Big)^{-(n+\theta)} \Big(\frac{t^{1/2\alpha}}{\rho(x_0)}\Big)^{-\theta},
\end{aligned}
$$

which implies that:

$$
\begin{aligned}
\Big| \int_{\mathbb{R}^n} q_t(x,y)a(y)dy \Big| &\lesssim \Big(\frac{t^{1/2\alpha}}{\rho(x_0)}\Big)^{-\theta} t^{-n/2\alpha} \Big(\frac{|x-x_0|}{t^{1/2\alpha}}\Big)^{-(n+\theta)} \int_{\mathbb{R}^n} |a(y)|dy \\
&\lesssim (\rho(x_0))^{\theta} r^{-\gamma} |x-x_0|^{-(n+\theta)} := C_{\gamma,x_0,r}|x-x_0|^{-(n+\theta)}.
\end{aligned}
$$

Because $x \notin B(x_0, 2r)$, set $x = x_0 + 2rz$, where $|z| \geq 1$. Then, $1 + |x| \leq 1 + |x_0| + 2r|z|$ and

$$\frac{1 + |x_0| + 2r}{2r}|x - x_0| = (1 + |x_0| + 2r)|z| \geq 1 + |x_0| + 2r|z|,$$

which implies that $|x_0 - x| \geq (1 + |x|)/C_{x_0,r}$. This completes the proof of Lemma 15. □

Lemma 16. *Given $\alpha \in (0,1)$, $\beta > 0$ and $0 < \gamma < \min\{2\alpha, 2\alpha\beta\}$. Let $f \in L^1(\mathbb{R}^n, (1 + |x|)^{-(n+\gamma+\epsilon)}dx)$ for any $\epsilon > 0$, and let a be an $H_L^{n/(n+\gamma)}$-atom. Then, for*

$$\begin{cases} F(x,t) := t^\beta \partial_t^\beta e^{-tL^\alpha}(f)(x); \\ G(x,t) := t^\beta \partial_t^\beta e^{-tL^\alpha}(a)(x), \end{cases}$$

there exists a constant $C_{\alpha,\beta}$, such that:

$$C_{\alpha,\beta} \int_{\mathbb{R}^n} f(x)\overline{a(x)}dx = \iint_{\mathbb{R}^{n+1}_+} F(x,t)\overline{G(x,t)}\frac{dxdt}{t}.$$

Proof. Assume that a is an $H_L^{n/(n+\gamma)}$-atom associated with a ball $B(x_0, r)$. By Lemma 13 and Theorem 2, we obtain:

$$\begin{aligned} I &= \iint_{\mathbb{R}^{n+1}_+} F(x,t)\overline{G(x,t)}\frac{dxdt}{t} \\ &= \lim_{\epsilon\to 0}\int_\epsilon^{1/\epsilon}\int_{\mathbb{R}^n} t^\beta\partial_t^\beta e^{-tL^\alpha}(f)(x)\overline{t^\beta\partial_t^\beta e^{-tL^\alpha}(a)(x)}\frac{dxdt}{t} \\ &= \lim_{\epsilon\to 0}\int_\epsilon^{1/\epsilon}\int_{\mathbb{R}^n} D_{\alpha,t}^{L,\beta}(f)(x)\overline{D_{\alpha,t}^{L,\beta}(a)(x)}\frac{dxdt}{t}. \end{aligned}$$

The inner integration satisfies the following:

$$\begin{aligned} \Big|\int_{\mathbb{R}^n} D_{\alpha,t}^{L,\beta}(f)(x)\overline{D_{\alpha,t}^{L,\beta}(a)(x)}dx\Big| &\leq \int_{\mathbb{R}^n}\Big|D_{\alpha,t}^{L,\beta}(f)(x)\Big|\cdot\Big|\overline{D_{\alpha,t}^{L,\beta}(a)(x)}\Big|dx \\ &\leq \int_{\mathbb{R}^n}\Big|D_{\alpha,t}^{L,\beta}(f)(x)\Big|\bigg\{\sup_{t>0}\Big|\overline{D_{\alpha,t}^{L,\beta}(a)(x)}\Big|\bigg\}dx. \end{aligned}$$

By Proposition 14, we can see that:

$$\begin{aligned} |D_{\alpha,t}^{L,\beta}(x,y)| &\lesssim \frac{t^\beta}{(t^{1/2\alpha} + |x-y|)^{n+2\alpha\beta}}\Big(1 + \frac{t^{1/2\alpha}}{\rho(x)} + \frac{t^{1/2\alpha}}{\rho(y)}\Big)^{-N} \\ &\lesssim t^{-n/2\alpha}\frac{1}{(1 + |x-y|/t^{1/2\alpha})^{n+2\alpha\beta}}\Big(1 + \frac{t^{1/2\alpha}}{\rho(x)} + \frac{t^{1/2\alpha}}{\rho(y)}\Big)^{-N}. \end{aligned}$$

If $x \in B(x_0, 2r)$, then $1 + |x| \leq 1 + |x - x_0| + |x_0| \leq 1 + 2r + |x_0|$. It follows from the condition $\|a\|_\infty \leq |B(x_0, r)|^{-1-\gamma/n}$ that:

$$\begin{aligned} \Big|\int_{\mathbb{R}^n} D_{\alpha,t}^{L,\beta}(x,y)a(y)dy\Big| &\lesssim \int_{B(x_0,r)}|D_{\alpha,t}^{L,\beta}(x,y)||a(y)|dy \\ &\lesssim \int_{B(x_0,r)} t^{-n/2\alpha}\Big(1 + \frac{|x-y|}{t^{1/2\alpha}}\Big)^{-n-2\alpha\beta}r^{-n-\gamma}dy \\ &\lesssim r^{-n-\gamma}\frac{(1 + 2r + |x_0|)^{n+2\alpha\beta}}{(1 + 2r + |x_0|)^{n+2\alpha\beta}} \\ &\lesssim C_{N,x_0,\gamma}(1 + |x|)^{-n-2\alpha\beta}. \end{aligned} \tag{37}$$

If $x \notin B(x_0, 2r)$, then for any $y \in B(x_0, r)$, $|x-y| \sim |x-x_0|$. On the other hand, $\rho(y) \sim \rho(x_0)$, since $r < \rho(x_0)$ and $|y-x_0| < r$. By Proposition 14, we have:

$$\begin{aligned} |D^{L,\beta}_{\alpha,t}(x,y)| &\lesssim \frac{t^\beta}{(t^{1/2\alpha}+|x-y|)^{n+2\alpha\beta}}\Big(1+\frac{t^{1/2\alpha}}{\rho(x)}+\frac{t^{1/2\alpha}}{\rho(y)}\Big)^{-N} \\ &\lesssim t^{-n/2\alpha}\Big(1+\frac{|x-y|}{t^{1/2\alpha}}\Big)^{-(n+2\alpha\beta)}\Big(1+\frac{t^{1/2\alpha}}{\rho(x_0)}\Big)^{-N} \\ &\lesssim t^{-n/2\alpha}\Big(1+\frac{|x-y|}{t^{1/2\alpha}}\Big)^{-(n+2\alpha\beta)}\Big(\frac{t^{1/2\alpha}}{\rho(x_0)}\Big)^{-2\alpha\beta}, \end{aligned}$$

which implies that:

$$\begin{aligned} \Big|\int_{\mathbb{R}^n} D^{L,\beta}_{\alpha,t}(x,y)a(y)dy\Big| &\lesssim \Big(\frac{t^{1/2\alpha}}{\rho(x_0)}\Big)^{-2\alpha\beta} t^{-n/2\alpha}\Big(\frac{|x-x_0|}{t^{1/2\alpha}}\Big)^{-(n+2\alpha\beta)}\int_{\mathbb{R}^n}|a(y)|dy \\ &\lesssim (\rho(x_0))^{2\alpha\beta} r^{-\gamma}|x-x_0|^{-(n+2\alpha\beta)} := C_{\gamma,x_0,r}|x-x_0|^{-(n+2\alpha\beta)}. \end{aligned}$$

Because $x \notin B(x_0, 2r)$, set $x = x_0 + 2rz$, where $|z| \geq 1$. Then, $1+|x| \leq 1+|x_0|+2r|z|$ and

$$\frac{1+|x_0|+2r}{2r}|x-x_0| = (1+|x_0|+2r)|z| \geq 1+|x_0|+2r|z|,$$

which implies that $|x_0 - x| \geq (1+|x|)/C_{x_0,r}$, and

$$\Big|\int_{\mathbb{R}^n} D^{L,\beta}_{\alpha,t}(x,y)a(y)dy\Big| \leq C_{\gamma,x_0,r}(1+|x|)^{-n-2\alpha\beta}. \tag{38}$$

The above estimate indicates that $D^{L,\beta}_{\alpha,t}(\cdot,\cdot)$ satisfies (36) with $\theta = 2\alpha\beta$. On the other hand, it can be deduced from (37) and (38) that:

$$\begin{aligned} \int_{\mathbb{R}^n}|D^{L,\beta}_{\alpha,t}(f)(x)||D^{L,\beta}_{\alpha,t}(a)(x)|dx &\lesssim \int_{\mathbb{R}^n}\Big(\int_{\mathbb{R}^n}\frac{|f(y)|t^\beta}{(|x-y|+t^{1/2\alpha})^{n+2\alpha\beta}}dy\Big)\frac{dx}{(1+|x|)^{n+2\alpha\beta}} \\ &\lesssim I_1 + I_2, \end{aligned}$$

where

$$\begin{cases} I_1 := \displaystyle\iint_{|x-y|>|y|/2}\frac{|f(y)|t^\beta}{(|x-y|+t^{1/2\alpha})^{n+2\alpha\beta}}\frac{dxdy}{(1+|x|)^{n+2\alpha\beta}}; \\ I_2 := \displaystyle\iint_{|x-y|\leq|y|/2}\frac{|f(y)|t^\beta}{(|x-y|+t^{1/2\alpha})^{n+2\alpha\beta}}\frac{dxdy}{(1+|x|)^{n+2\alpha\beta}}. \end{cases}$$

If $|x-y| < |y|/2$, then $|y| \leq |x-y|+|x| \leq |y|/2+|x|$, i.e., $|y| \leq 2|x|$.

$$I_2 \lesssim \int_{\mathbb{R}^n}\Big(\int_{\mathbb{R}^n}\frac{t^\beta}{(|x-y|+t^{1/2\alpha})^{n+2\alpha\beta}}dx\Big)\frac{|f(y)|}{(1+|y|)^{n+2\alpha\beta}}dy < \infty.$$

For I_1, since $|x-y| > |y|/2$, we have:

$$\int_{\mathbb{R}^n}\frac{|f(y)|t^\beta}{(|x-y|+t^{1/2\alpha})^{n+2\alpha\beta}}dy \lesssim \frac{1}{t^{n/(2\alpha)}}\int_{\mathbb{R}^n}\frac{|f(y)|}{(1+|y|/t^{1/(2\alpha)})^{n+2\alpha\beta}}dy.$$

If $0 < t < 1$, then:

$$\frac{1}{t^{n/(2\alpha)}}\int_{\mathbb{R}^n}\frac{|f(y)|}{(1+|y|/t^{1/(2\alpha)})^{n+2\alpha\beta}}dy \lesssim \frac{1}{t^{n/(2\alpha)}}\int_{\mathbb{R}^n}\frac{|f(y)|}{(1+|y|)^{n+2\alpha\beta}}dy.$$

If $t > 1$, then:

$$\begin{aligned}\frac{1}{t^{n/(2\alpha)}}\int_{\mathbb{R}^n}\frac{|f(y)|}{(1+|y|/t^{1/(2\alpha)})^{n+2\alpha\beta}}dy &\lesssim \frac{1}{t^{n/(2\alpha)}}\int_{\mathbb{R}^n}\frac{|f(y)|}{(1/t^{1/(2\alpha)}+|y|/t^{1/(2\alpha)})^{n+2\alpha\beta}}dy\\ &\lesssim t^{\beta}\int_{\mathbb{R}^n}\frac{|f(y)|}{(1+|y|)^{n+2\alpha\beta}}dy.\end{aligned}$$

Hence, there exists a constant C_t, such that:

$$\begin{aligned} I_1 &\lesssim \iint_{\mathbb{R}^n\times\mathbb{R}^n}\frac{|f(y)|t^{\beta}}{(|x-y|+t^{1/2\alpha})^{n+2\alpha\beta}}\frac{dxdy}{(1+|y|)^{n+2\alpha\beta}}\\ &\lesssim C_t\int_{\mathbb{R}^n}\Big(\int_{\mathbb{R}^n}\frac{|f(y)|}{(|y|+1)^{n+2\alpha\beta}}dy\Big)\frac{dx}{(1+|x|)^{n+2\alpha\beta}}<\infty.\end{aligned}$$

Notice that

$$\begin{aligned}\int_{\mathbb{R}^n}D^{L,\beta}_{\alpha,t}(f)(x)\overline{D^{L,\beta}_{\alpha,t}(a)(x)}dx &= \int_{\mathbb{R}^n}f(x)\overline{(D^{L,\beta}_{\alpha,t})^2(a)(x)}dx\\ &= \int_{\mathbb{R}^n}f(x)\overline{t^{2\beta}\partial_t^{2\beta}e^{-2tL^{\alpha}}(a)(x)}dx,\end{aligned}$$

which, together with the Fubini theorem, indicates that:

$$\begin{aligned} I &= \lim_{\epsilon\to0}\int_{\epsilon}^{1/\epsilon}\Big\{\int_{\mathbb{R}^n}f(y)\overline{t^{2\beta}\partial_t^{2\beta}e^{-2tL^{\alpha}}(a)(y)}dy\Big\}\frac{dt}{t}\\ &= \lim_{\epsilon\to0}\int_{\mathbb{R}^n}f(y)\Big\{\int_{\epsilon}^{1/\epsilon}\overline{t^{2\beta}\partial_t^{2\beta}e^{-2tL^{\alpha}}(a)(y)}\frac{dt}{t}\Big\}dy.\end{aligned}$$

For the term

$$\int_{\epsilon}^{1/\epsilon}\overline{t^{2\beta}\partial_t^{2\beta}e^{-2tL^{\alpha}}(a)(y)}\frac{dt}{t},$$

we can see that

$$\begin{aligned}\Big|\int_{\epsilon}^{1/\epsilon}t^{2\beta}\partial_t^{2\beta}e^{-2tL^{\alpha}}(a)(y)\frac{dt}{t}\Big| &\le \Big|\int_{\epsilon}^{\infty}t^{2\beta}\partial_t^{2\beta}e^{-2tL^{\alpha}}(a)(y)\frac{dt}{t}\Big|+\Big|\int_{1/\epsilon}^{\infty}t^{2\beta}\partial_t^{2\beta}e^{-2tL^{\alpha}}(a)(y)\frac{dt}{t}\Big|\\ &= \Big|\int_{\mathbb{R}^n}\int_{\epsilon}^{\infty}t^{2\beta}\partial_t^{2\beta}e^{-2tL^{\alpha}}(x,y)\frac{dt}{t}a(x)dx\Big|\\ &\quad+\Big|\int_{\mathbb{R}^n}\int_{1/\epsilon}^{\infty}t^{2\beta}\partial_t^{2\beta}e^{-2tL^{\alpha}}(x,y)\frac{dt}{t}a(x)dx\Big|.\end{aligned}$$

By the change of variables, we obtain:

$$\begin{aligned}\Big|\int_{\epsilon}^{\infty}t^{2\beta}\partial_t^{2\beta}e^{-2tL^{\alpha}}(x,y)\frac{dt}{t}\Big| &= \Big|C_\beta\int_{\epsilon}^{\infty}t^{2\beta}\int_0^{\infty}\partial_t^{m}e^{-(2t+s)L^{\alpha}}(x,y)\frac{ds}{s^{1+2\beta-m}}\frac{dt}{t}\Big|\\ &= \Big|C_\beta\int_{\epsilon}^{\infty}t^{2\beta}\int_0^{\infty}L^{\alpha m}e^{-(2t+s)L^{\alpha}}(x,y)\frac{ds}{s^{1+2\beta-m}}\frac{dt}{t}\Big|\\ &\simeq \Big|C_\beta\int_{2\epsilon}^{\infty}t^{2\beta}\int_0^{\infty}\partial_t^{m}e^{-(t+s)L^{\alpha}}(x,y)\frac{ds}{s^{1+2\beta-m}}\frac{dt}{t}\Big|\\ &\simeq \Big|\int_{2\epsilon}^{\infty}t^{2\beta}\partial_t^{2\beta}e^{-tL^{\alpha}}(x,y)\frac{dt}{t}\Big|.\end{aligned}$$

The rest of the proof is divided into three cases:

Case 1: $2\beta < 1$. For this case, $[2\beta]+1=1$. Then, a change of variables reaches:

$$\begin{aligned}\int_{2\epsilon}^{\infty} t^{2\beta}\partial_t^{2\beta} e^{-tL^\alpha}(x,y)\frac{dt}{t} &= \int_{2\epsilon}^{\infty} t^{2\beta}\int_0^\infty \partial_t e^{-(t+s)L^\alpha}(x,y)\frac{ds}{s^{2\beta}}\frac{dt}{t}\\ &= \int_{2\epsilon}^{\infty} t^{2\beta}\int_t^\infty \partial_u e^{-uL^\alpha}(x,y)\frac{du}{(u-t)^{2\beta}}\frac{dt}{t}\\ &= \int_{2\epsilon}^{\infty}\partial_u e^{-uL^\alpha}(x,y)\Big(\int_{2\epsilon}^{u}\Big(\frac{t}{u-t}\Big)^{2\beta}\frac{dt}{t}\Big)du\\ &= \int_{2\epsilon}^{\infty}\partial_u e^{-uL^\alpha}(x,y)\Big(\int_{2\epsilon/u}^{1}\Big(\frac{w}{1-w}\Big)^{2\beta}\frac{dw}{w}\Big)du.\end{aligned}$$

Notice that

$$\lim_{u\to(2\epsilon)^+} e^{-2\epsilon L^\alpha}(x,y)\int_{2\epsilon/u}^{1}\Big(\frac{w}{1-w}\Big)^{2\beta}\frac{dw}{w}=0,$$

and, as $u\to\infty$,

$$|e^{-uL^\alpha}(x,y)|\le \frac{u}{(u^{1/2\alpha}+|x-y|)^{n+2\alpha}}\le\frac{1}{u^{n/2\alpha}}\to 0.$$

An application of integration by parts gives:

$$\begin{aligned}\int_{2\epsilon}^{\infty} t^{2\beta}\partial_t^{2\beta} e^{-tL^\alpha}(x,y)\frac{dt}{t} &= -\int_{2\epsilon}^{\infty} e^{-uL^\alpha}(x,y)\frac{\partial}{\partial u}\Big(\int_{2\epsilon/u}^{1}\Big(\frac{w}{1-w}\Big)^{2\beta}\frac{dw}{w}\Big)du\\ &= -\int_{2\epsilon}^{\infty} e^{-uL^\alpha}(x,y)\Big(\frac{2\epsilon}{u-2\epsilon}\Big)^{2\beta}\frac{du}{u}\\ &= I+II,\end{aligned}$$

where

$$\begin{cases} I:=-\int_{2\epsilon}^{\infty} e^{-uL^\alpha}(x,y)\Big(\frac{2\epsilon}{u-2\epsilon}\Big)^{2\beta}\chi_A(u)\frac{du}{u};\\ II:=-\int_{2\epsilon}^{\infty} e^{-uL^\alpha}(x,y)\Big(\frac{2\epsilon}{u-2\epsilon}\Big)^{2\beta}\chi_{A^c}(u)\frac{du}{u},\end{cases}$$

and where $A:=\{u:\ u-2\epsilon\le\epsilon+|x-y|^{2\alpha}\}$. By Proposition 7,

$$\begin{aligned}|I| &\lesssim \int_{2\epsilon}^{\infty}\frac{u}{(u^{1/2\alpha}+|x-y|)^{n+2\alpha}}\Big(1+\frac{u^{1/2\alpha}}{\rho(x)}+\frac{u^{1/2\alpha}}{\rho(y)}\Big)^{-N}\Big(\frac{2\epsilon}{u-2\epsilon}\Big)^{2\beta}\chi_A(u)\frac{du}{u}\\ &\lesssim \frac{\epsilon^{2\beta}}{(\epsilon^{1/2\alpha}+|x-y|)^{n+2\alpha}}\Big(1+\frac{\epsilon^{1/2\alpha}}{\rho(x)}+\frac{\epsilon^{1/2\alpha}}{\rho(y)}\Big)^{-N}\int_{2\epsilon}^{3\epsilon+|x-y|^{2\alpha}}(u-2\epsilon)^{-2\beta}du\\ &\lesssim \frac{1}{\epsilon^{n/2\alpha}}\frac{1}{(1+|x-y|/\epsilon^{1/2\alpha})^{n+4\alpha\beta}}\Big(1+\frac{\epsilon^{1/2\alpha}}{\rho(x)}+\frac{\epsilon^{1/2\alpha}}{\rho(y)}\Big)^{-N}.\end{aligned}$$

For II,

$$\begin{aligned}|II| &\lesssim \int_{2\epsilon}^{\infty}\frac{u}{(u^{1/2\alpha}+|x-y|)^{n+2\alpha}}\Big(1+\frac{\epsilon^{1/2\alpha}}{\rho(x)}+\frac{\epsilon^{1/2\alpha}}{\rho(y)}\Big)^{-N}\Big(\frac{2\epsilon}{u-2\epsilon}\Big)^{2\beta}\chi_{A^c}(u)\frac{du}{u}\\ &\lesssim \int_{3\epsilon+|x-y|^{2\alpha}}^{\infty}\frac{u}{(u^{1/2\alpha}+|x-y|)^{n+2\alpha}}\Big(1+\frac{\epsilon^{1/2\alpha}}{\rho(x)}+\frac{\epsilon^{1/2\alpha}}{\rho(y)}\Big)^{-N}\Big(\frac{2\epsilon}{\epsilon+|x-y|^{2\alpha}}\Big)^{2\beta}\frac{du}{u}\\ &\lesssim \Big(\frac{2\epsilon}{\epsilon+|x-y|^{2\alpha}}\Big)^{2\beta}\Big(1+\frac{\epsilon^{1/2\alpha}}{\rho(x)}+\frac{\epsilon^{1/2\alpha}}{\rho(y)}\Big)^{-N}\int_{3\epsilon+|x-y|^{2\alpha}}^{\infty}(u^{1/2\alpha}+|x-y|)^{-n-2\alpha}du\\ &\lesssim \frac{1}{\epsilon^{n/2\alpha}}\frac{1}{(1+|x-y|/\epsilon^{1/2\alpha})^{n+4\alpha\beta}}\Big(1+\frac{\epsilon^{1/2\alpha}}{\rho(x)}+\frac{\epsilon^{1/2\alpha}}{\rho(y)}\Big)^{-N}.\end{aligned}$$

Case 2: $2\beta = 1$. A direct computation gives:

$$\begin{aligned} \Big| \int_{2\epsilon}^{\infty} t\partial_t e^{-tL^\alpha}(x,y)\frac{dt}{t}\Big| &\lesssim |e^{-2\epsilon L^\alpha}(x,y)| \\ &\lesssim \frac{1}{\epsilon^{n/2\alpha}}\frac{1}{(1+|x-y|/\epsilon^{1/2\alpha})^{n+2\alpha}}\Big(1+\frac{\epsilon^{1/2\alpha}}{\rho(x)}+\frac{\epsilon^{1/2\alpha}}{\rho(y)}\Big)^{-N}. \end{aligned}$$

Case 3: $2\beta > 1$. Let $k \in \mathbb{Z}_+$, such that $k-1 < 2\beta \leq k$, $k \geq 2$. We obtain:

$$\begin{aligned} M &= \int_{2\epsilon}^{\infty} t^{2\beta}\partial_t^{2\beta} e^{-tL^\alpha}(x,y)\frac{dt}{t} \\ &= \int_{2\epsilon}^{\infty} t^{2\beta}\Big(\int_0^{\infty}\partial_t^k e^{-(t+s)L^\alpha}(x,y)\frac{ds}{s^{1+2\beta-k}}\Big)\frac{dt}{t} \\ &= \int_{2\epsilon}^{\infty} t^{2\beta}L^{\alpha k}\Big(\int_0^{\infty} e^{-(t+s)L^\alpha}(x,y)\frac{ds}{s^{1+2\beta-k}}\Big)\frac{dt}{t} \\ &= \int_{2\epsilon}^{\infty} t^{2\beta}\Big(\int_t^{\infty} L^{\alpha k}e^{-uL^\alpha}(x,y)\frac{du}{(u-t)^{1+2\beta-k}}\Big)\frac{dt}{t} \\ &= \int_{2\epsilon}^{\infty} t^{2\beta}\Big(\int_t^{\infty} \partial_u^k e^{-uL^\alpha}(x,y)\frac{du}{(u-t)^{1+2\beta-k}}\Big)\frac{dt}{t} \\ &= \int_{2\epsilon}^{\infty} \partial_u^k e^{-uL^\alpha}(x,y)\Big(\int_{2\epsilon}^{u} t^{2\beta}(u-t)^{k-2\beta-1}\frac{dt}{t}\Big)du \\ &= \int_{2\epsilon}^{\infty} u^{k-1}\partial_u^k e^{-uL^\alpha}(x,y)\Big(\int_{2\epsilon/u}^{1} w^{2\beta}(1-w)^{k-2\beta-1}\frac{dw}{w}\Big)du, \end{aligned}$$

where, in the last step, we have used the change of variables: $w = t/u$. Notice that

$$\begin{aligned} u^{k-1}\partial_u^k e^{-uL^\alpha}(x,y) &= \partial_u(u^{k-1}\partial_u^{k-1}e^{-uL^\alpha}(x,y)) - (k-1)\partial_u(u^{k-2}\partial_u^{k-2}e^{-uL^\alpha}(x,y)) \\ &\quad + \cdots + (-1)^{k-1}(k-1)!\partial_u e^{-uL^\alpha}(x,y). \end{aligned}$$

Then, the integration by parts yields $M = \sum_{m=1}^{k-1} C_m I_m$, where $C_m = (-1)^{m-1}(k-1)!/(k-m+1)!$ and

$$I_m := \int_{2\epsilon}^{\infty} u^m\partial_u^m e^{-uL^\alpha}(x,y)\frac{(2\epsilon)^{2\beta}u^{1-k}}{(u-2\epsilon)^{1+2\beta-k}}\frac{du}{u}.$$

We obtain:

$$|I_m| \lesssim \int_{2\epsilon}^{\infty}\frac{u^m}{(u^{1/2\alpha}+|x-y|)^{n+2\alpha m}}\Big(1+\frac{u^{1/2\alpha}}{\rho(x)}+\frac{u^{1/2\alpha}}{\rho(y)}\Big)^{-N}\frac{(2\epsilon)^{2\beta}u^{1-k}}{(u-2\epsilon)^{1+2\beta-k}}\frac{du}{u} \lesssim I_m^{(1)} + I_m^{(2)},$$

where

$$\begin{cases} I_m^{(1)} := \dfrac{\epsilon^{2\beta}}{(\epsilon^{1/2\alpha}+|x-y|)^{n+2\alpha m}}\Big(1+\dfrac{\epsilon^{1/2\alpha}}{\rho(x)}+\dfrac{\epsilon^{1/2\alpha}}{\rho(y)}\Big)^{-N}\displaystyle\int_{2\epsilon}^{3\epsilon}\frac{1}{(u-2\epsilon)^{1+2\beta-k}}\frac{du}{u^{k-m}}; \\ I_m^{(2)} := \dfrac{\epsilon^{2\beta}}{(\epsilon^{1/2\alpha}+|x-y|)^{n+2\alpha m}}\Big(1+\dfrac{\epsilon^{1/2\alpha}}{\rho(x)}+\dfrac{\epsilon^{1/2\alpha}}{\rho(y)}\Big)^{-N}\displaystyle\int_{3\epsilon}^{\infty}\frac{1}{(u-2\epsilon)^{1+2\beta-k}}\frac{du}{u^{k-m}}. \end{cases}$$

For $I_m^{(1)}$, since $2\epsilon < u < 3\epsilon$, we obtain:

$$\begin{aligned}
I_m^{(1)} &\lesssim \frac{\epsilon^{2\beta}}{(\epsilon^{1/2\alpha}+|x-y|)^{n+2\alpha m}}\Big(1+\frac{\epsilon^{1/2\alpha}}{\rho(x)}+\frac{\epsilon^{1/2\alpha}}{\rho(y)}\Big)^{-N}\frac{1}{\epsilon^{k-m}}\int_{2\epsilon}^{3\epsilon}\frac{du}{(u-2\epsilon)^{1+2\beta-k}}\\
&\lesssim \frac{\epsilon^{2\beta}}{(\epsilon^{1/2\alpha}+|x-y|)^{n+2\alpha m}}\Big(1+\frac{\epsilon^{1/2\alpha}}{\rho(x)}+\frac{\epsilon^{1/2\alpha}}{\rho(y)}\Big)^{-N}\frac{1}{\epsilon^{k-m}}\epsilon^{k-2\beta}\\
&\lesssim \frac{1}{\epsilon^{n/2\alpha}}\frac{1}{(1+|x-y|/\epsilon^{1/2\alpha})^{n+2\alpha m}}\Big(1+\frac{\epsilon^{1/2\alpha}}{\rho(x)}+\frac{\epsilon^{1/2\alpha}}{\rho(y)}\Big)^{-N}.
\end{aligned}$$

Similarly, for $I_m^{(2)}$, because $u \in (3\epsilon, \infty)$, then $1/u \lesssim 1/(u-2\epsilon)$. Noticing that $m < 2\beta$, we obtain:

$$\begin{aligned}
I_m^{(2)} &\lesssim \frac{\epsilon^{2\beta}}{(\epsilon^{1/2\alpha}+|x-y|)^{n+2\alpha m}}\Big(1+\frac{\epsilon^{1/2\alpha}}{\rho(x)}+\frac{\epsilon^{1/2\alpha}}{\rho(y)}\Big)^{-N}\int_{3\epsilon}^{\infty}\frac{du}{(u-2\epsilon)^{1+2\beta-m}}\\
&\lesssim \frac{\epsilon^{2\beta}}{(\epsilon^{1/2\alpha}+|x-y|)^{n+2\alpha m}}\Big(1+\frac{\epsilon^{1/2\alpha}}{\rho(x)}+\frac{\epsilon^{1/2\alpha}}{\rho(y)}\Big)^{-N}\epsilon^{m-2\beta}\\
&\lesssim \frac{1}{\epsilon^{n/2\alpha}}\frac{1}{(1+|x-y|/\epsilon^{1/2\alpha})^{n+2\alpha m}}\Big(1+\frac{\epsilon^{1/2\alpha}}{\rho(x)}+\frac{\epsilon^{1/2\alpha}}{\rho(y)}\Big)^{-N}.
\end{aligned}$$

By Lemma 15, the above estimates in Cases 1–3 indciate that:

$$\sup_{\epsilon>0}\Big|\int_{\epsilon}^{1/\epsilon} t^{2\beta}\partial_t^{2\beta}e^{-2tL^{\alpha}}(a)(y)\frac{dt}{t}\Big| \lesssim (1+|y|)^{-(n+\gamma+\epsilon)},$$

where

$$\epsilon = \begin{cases} 4\alpha\beta-\gamma, & 2\beta<1;\\ 2\alpha-\gamma, & 2\beta=1;\\ 2\alpha m-\gamma, & 2\beta>1. \end{cases}$$

Therefore, we can use Lemma 12 to complete the proof. □

Finally, we can obtain the following characterization of $BMO_L^{\gamma}(\mathbb{R}^n)$ corresponding to the time-fractional derivative:

Theorem 3. *Let $V \in B_q, q > n/2$. Assume that $\alpha \in (0,1), \beta > 0$, and $0 < \gamma < \min\{1, 2\alpha, 2\beta\alpha\}$. Let f be a function, such that:*

$$\int_{\mathbb{R}^n}\frac{|f(x)|}{(1+|x|)^{n+\gamma+\epsilon}}dx < \infty \tag{39}$$

for some $\epsilon > 0$. The following statements are equivalent:

(i) $f \in BMO_L^{\gamma}(\mathbb{R}^n)$;

(ii) *There exists $C_{\alpha,\beta}$, such that $\|D_{\alpha,t}^{L,\beta}(f)\|_{\infty} \le C_{\alpha,\beta}t^{\gamma/2\alpha}$;*

(iii) *For all $B = B(x_B, r_B) \subset \mathbb{R}^n$,*

$$\Big(\frac{1}{|B|}\int_0^{r_B^{2\alpha}}\int_B |D_{\alpha,t}^{L,\beta}(f)(x)|^2\frac{dxdt}{t}\Big)^{1/2} \lesssim |B|^{\gamma/n}. \tag{40}$$

Proof. (i)$\Longrightarrow$(ii). If $f \in BMO_L^{\gamma}(\mathbb{R}^n)$, then $|t^{\beta}\partial_t^{\beta}e^{-tL^{\alpha}}f(x)| \lesssim I + II$, where:

$$\begin{cases} I := \Big|\int_{\mathbb{R}^n} D_{\alpha,t}^{L,\beta}(x,y)(f(y)-f(x))dy\Big|;\\ II := \Big|f(x)\int_{\mathbb{R}^n} D_{\alpha,t}^{L,\beta}(x,y)dy\Big|. \end{cases}$$

For I, we have:

$$I \lesssim \|f\|_{BMO_L^\gamma} \int_{\mathbb{R}^n} \frac{t^\beta |x-y|^\gamma}{(t^{1/2\alpha}+|x-y|)^{n+2\alpha\beta}} dy \lesssim t^{\gamma/2\alpha} \|f\|_{BMO_L^\gamma}.$$

We further divide the estimation of II into the following two cases:
Case 1: $\rho(x) \le t^{1/2\alpha}$. By Proposition 14,

$$\begin{aligned} II &\lesssim \|f\|_{BMO_L^\gamma} \rho(x)^\gamma \Big| \int_{\mathbb{R}^n} D_{\alpha,t}^{L,\beta}(x,y) dy \Big| \\ &\lesssim \|f\|_{BMO_L^\gamma} t^{\gamma/2\alpha} \int_{\mathbb{R}^n} \frac{t^\beta}{(t^{1/2\alpha}+|x-y|)^{n+2\alpha\beta}} dy \\ &\lesssim \|f\|_{BMO_L^\gamma} t^{\gamma/2\alpha}. \end{aligned}$$

Case 2: $\rho(x) > t^{1/2\alpha}$. We use Proposition 16 to obtain that there exists $\delta' > \gamma$, such that:

$$\begin{aligned} II &\lesssim \|f\|_{BMO_L^\gamma} \rho(x)^\gamma \Big| \int_{\mathbb{R}^n} D_{\alpha,t}^{L,\beta}(x,y) dy \Big| \\ &\lesssim \|f\|_{BMO_L^\gamma} \rho(x)^\gamma \frac{(t^{1/2\alpha}/\rho(x))^{\delta'}}{(1+t^{1/2\alpha}/\rho(x))^N} \\ &\lesssim \|f\|_{BMO_L^\gamma} t^{\gamma/2\alpha} \frac{(t^{1/2\alpha}/\rho(x))^{\delta'-\gamma}}{(1+t^{1/2\alpha}/\rho(x))^N} \\ &\lesssim \|f\|_{BMO_L^\gamma} t^{\gamma/2\alpha}. \end{aligned}$$

(ii)$\Longrightarrow$(iii). Assume that (ii) holds. Then,

$$\Big(\frac{1}{|B|} \int_0^{r_B^{2\alpha}} \int_B |D_{\alpha,t}^{L,\beta}(f)(x)|^2 \frac{dxdt}{t} \Big)^{1/2} \lesssim \Big(\frac{1}{|B|} \int_0^{r_B^{2\alpha}} \int_B t^{\gamma/\alpha} \frac{dxdt}{t} \Big)^{1/2} \lesssim |B|^{\gamma/n}.$$

(iii)$\Longrightarrow$(i). Assume that (40) holds. Let a be an $H_L^{n/(n+\gamma)}$-atom associated with $B = B(x_B, r_B)$. Then, by Lemma 16,

$$\int_{\mathbb{R}^n} f(x)\overline{a(x)}dx \simeq \iint_{\mathbb{R}^{n+1}_+} t^\beta \partial_t^\beta e^{-tL^\alpha}(f)(x) \overline{t^\beta \partial_t^\beta e^{-tL^\alpha}(a)(x)} \frac{dt}{t},$$

which, together with (34) and Theorem 2, gives:

$$\begin{aligned} \Big| \int_{\mathbb{R}^n} f(x)\overline{a(x)}dx \Big| &\lesssim \sup_B \Big(\frac{1}{|B|^{1+2\gamma/n}} \int_0^{r_B^{2\alpha}} \int_B |t^\beta \partial_t^\beta e^{-tL^\alpha}(f)(x)|^2 \frac{dxdt}{t} \Big)^{1/2} \\ &\quad \times \Big\{ \int_{\mathbb{R}^n} \Big(\iint_{|x-y|<s^{1/2\alpha}} |t^\beta \partial^\beta e^{-tL^\alpha}(a)(y)|^2 \frac{t^{1/2\alpha-1}dyds}{t^{(n+1)/2\alpha}} \Big)^{n/2(n+\gamma)} dx \Big\}^{1+\gamma/n} \\ &\lesssim \|S_{\alpha,\beta}^L(a)\|_{L^{n/(n+\gamma)}} \sup_B \Big(\frac{1}{|B|^{1+2\gamma/n}} \int_0^{r_B^{2\alpha}} \int_B |t^\beta \partial_t^\beta e^{-tL^\alpha}(f)(x)|^2 \frac{dxdt}{t} \Big)^{1/2} \\ &\lesssim \|a\|_{H_L^{n/(n+\gamma)}}. \end{aligned}$$

Hence,

$$T(g) := \int_{\mathbb{R}^n} f(x)\overline{g(x)}dx,\ g \in H_L^{n/(n+\gamma)}(\mathbb{R}^n),$$

is a bounded linear functional on $H_L^{n/(n+\gamma)}(\mathbb{R}^n)$; equivalently, $f \in \big(H_L^{n/(n+\gamma)}(\mathbb{R}^n)\big)^* = BMO_L^\gamma(\mathbb{R}^n)$. $\square$

Below, we consider the characterization of $BMO_L^\gamma(\mathbb{R}^n)$ via the the spatial gradient. Define a general gradient as $\nabla_\alpha := (\nabla_x, \partial_t^{1/2\alpha})$.

Theorem 4. *Let $V \in B_q, q > n$. Assume that $\alpha \in (0, 1/2 - n/2q)$, $\beta > 0$, and $0 < \gamma < \min\{1, 2\alpha, 2\alpha\beta\}$. Let f be a function satisfying (39). The following statements are equivalent:*

(i) $f \in BMO_L^\gamma(\mathbb{R}^n)$;

(ii) *There exists a constant $C > 0$, such that:*

$$\|t^{1/2\alpha}\nabla_\alpha e^{-tL^\alpha} f\|_\infty \le C t^{\gamma/2\alpha};$$

(iii) *$u(x,t) = e^{-tL^\alpha} f(x)$ satisfies that, for any balls $B = B(x_B, r_B)$,*

$$\frac{1}{|B|^{1+2\gamma/n}} \int_0^{r_B^{2\alpha}} \int_B |t^{1/2\alpha}\nabla_\alpha e^{-tL^\alpha}(f)(x)|^2 \frac{dxdt}{t} \le C. \tag{41}$$

Proof. (i) $\Longrightarrow$ (ii). Let $f \in BMO_L^\gamma(\mathbb{R}^n)$. By Theorem 3, $\|t^{1/2\alpha}\partial_t^{1/2\alpha} e^{-tL^\alpha}(f)\|_\infty \le C_{\alpha,\beta} t^{\gamma/2\alpha}$. One writes:

$$\begin{aligned} t^{1/2\alpha}\nabla_x e^{-tL^\alpha} f(x) &= \int_{\mathbb{R}^n} t^{1/2\alpha}\nabla_x K_{\alpha,t}^L(x,z)\Big(f(z) - f(x)\Big)dz + f(x) t^{1/2\alpha}\nabla_x e^{-tL^\alpha}(1)(x) \\ &:= I(x) + II(x). \end{aligned}$$

We first estimate the term I. Because $f \in BMO_L^\gamma(\mathbb{R}^n)$, then $|f(x) - f(z)| \le \|f\|_{BMO_L^\gamma}|x - z|^\gamma$. Since

$$|t^{1/2\alpha}\nabla_x K_{\alpha,t}^L(x,z)| \lesssim \frac{t}{(t^{1/2\alpha} + |x-z|)^{n+2\alpha}},$$

and a direct computation gives:

$$\begin{aligned} |I(x)| &\lesssim \|f\|_{BMO_L^\gamma} \int_{\mathbb{R}^n} |t^{1/2\alpha}\nabla_x K_{\alpha,t}^L(x,z)| \cdot |x-z|^\gamma dz \\ &\lesssim \|f\|_{BMO_L^\gamma} \int_{\mathbb{R}^n} \frac{t|x-z|^\gamma}{(t^{1/2\alpha} + |x-z|)^{n+2\alpha}} dz \\ &\lesssim t^{\gamma/2\alpha}\|f\|_{BMO_L^\gamma}. \end{aligned}$$

By Proposition 13, we have:

$$|t^{1/2\alpha}\nabla_x e^{-tL^\alpha}(1)| \lesssim \min\Big\{(t^{1/2\alpha}/\rho(x))^{1+2\alpha},\ (t^{1/2\alpha}/\rho(x))^{-N}\Big\}.$$

The estimate of II is divided into two cases:
Case 1: $\rho(x) \le t^{1/2\alpha}$. $f \in BMO_L^\gamma(\mathbb{R}^n)$ implies that $|f(x)| \lesssim \rho^\gamma(x)\|f\|_{BMO_L^\gamma}$. Then,

$$\begin{aligned} II(x) &\lesssim \|f\|_{BMO_L^\gamma}\rho^\gamma(x)|t^{1/2\alpha}\nabla_x e^{-tL^\alpha}(1)(x)| \\ &\lesssim \|f\|_{BMO_L^\gamma}\rho^\gamma(x)\Big(\frac{t^{1/2\alpha}}{\rho(x)}\Big)^{-N} \\ &\lesssim \|f\|_{BMO_L^\gamma} t^{\gamma/2\alpha}\Big(\frac{t^{1/2\alpha}}{\rho(x)}\Big)^{-N-\gamma} \\ &\lesssim t^{\gamma/2\alpha}\|f\|_{BMO_L^\gamma}. \end{aligned}$$

Case 2: $\rho(x) > t^{1/2\alpha}$. We can obtain:

$$\begin{aligned} II(x) &\lesssim \rho^{\gamma}(x)\|f\|_{BMO_L^{\gamma}}|t^{1/2\alpha}\nabla_x e^{-tL^{\alpha}}(1)(x)| \\ &\lesssim \rho^{\gamma}(x)\|f\|_{BMO_L^{\gamma}}\Big(\frac{t^{1/2\alpha}}{\rho(x)}\Big)^{1+2\alpha} \\ &\lesssim t^{\gamma/2\alpha}\|f\|_{BMO_L^{\gamma}}\Big(\frac{t^{1/2\alpha}}{\rho(x)}\Big)^{1+2\alpha-\gamma} \\ &\lesssim t^{\gamma/2\alpha}\|f\|_{BMO_L^{\gamma}}. \end{aligned}$$

(ii)$\Longrightarrow$(iii). For every ball $B = B(x_B, r_B)$,

$$\int_0^{r_B^{2\alpha}}\int_B\Big|t^{1/2\alpha}\nabla_\alpha e^{-tL^{\alpha}}f(x)\Big|^2\frac{dxdt}{t} \lesssim \int_0^{r_B^{2\alpha}}\int_B t^{\gamma/\alpha}\frac{dxdt}{t} \lesssim r_B^{n+2\gamma},$$

which implies that (41) holds.

(iii)$\Longrightarrow$(i). Assume that (41) holds. For any ball $B = B(x_B, r_B)$, it holds that:

$$\sup_B r_B^{-(n+2\gamma)}\int_0^{r_B^{2\alpha}}\int_{B(x_B,r_B)}|t^{1/2\alpha}\partial_t^{1/2\alpha}e^{-tL^{\alpha}}(f)(x)|^2\frac{dxdt}{t} < \infty.$$

It is a corollary of Theorem 3 that $f \in BMO_L^{\gamma}(\mathbb{R}^n)$ with

$$\|f\|_{BMO_L^{\gamma}} \lesssim \sup_B r_B^{-(n+2\gamma)}\int_0^{r_B^{2\alpha}}\int_{B(x_B,r_B)}|t^{1/2\alpha}\partial_t^{1/2\alpha}e^{-tL^{\alpha}}(f)(x)|^2\frac{dxdt}{t} < \infty.$$

□

A positive measure ν on $\mathbb{R}_+^{n+1}$ is called a κ-Carleson measure if

$$\|\nu\|_{\mathcal{C}} := \sup_{x\in\mathbb{R}^n, r>0}\frac{\nu(B(x,r)\times(0,r))}{|B(x,r)|^{\kappa}} < \infty.$$

The following result can be deduced from Theorem 4 immediately:

Theorem 5. *Let $V \in B_q, q > n$. Assume that $\alpha \in (0, 1/2 - n/2q)$, $\beta > 0$, and $0 < \gamma \le 1$, with*

$$0 < \gamma < \min\{2\alpha, 2\alpha\beta\}.$$

Let $d\nu_\alpha$ be a measure defined by:

$$d\nu_\alpha(x,t) := |t\nabla e^{-t^{2\alpha}L^{\alpha}}(f)(x)|^2 dxdt/t,\ (x,t) \in \mathbb{R}_+^{n+1}.$$

(i) *If $f \in BMO_L^{\gamma}(\mathbb{R}^n)$, then $d\nu_\alpha$ is a $(1 + 2\gamma/n)$-Carleson measure;*
(ii) *Conversely, if $f \in L^1((1 + |x|)^{-n-1}dx)$ and $d\nu_\alpha$ is a $(1 + 2\gamma/n)$-Carleson measure, then $f \in BMO_L^{\gamma}(\mathbb{R}^n)$.*

Moreover, in any case, there exists a constant $C > 0$, such that:

$$C^{-1}\|f\|^2_{BMO_L^{\gamma}} \le \|d\nu_\alpha\|_{\mathcal{C}} \le C\|f\|^2_{BMO_L^{\gamma}}.$$

Proof. (i). In Theorem 3, letting $\beta = 1$, we obtain for $f \in BMO_L^{\gamma}(\mathbb{R}^n)$,

$$\frac{1}{|B|^{1+2\gamma/n}}\int_0^{r_B^{2\alpha}}\int_B|t\partial_t e^{-tL^{\alpha}}(f)(x)|^2\frac{dxdt}{t} \lesssim \|f\|^2_{BMO_L^{\gamma}},$$

which, together with a change of variable, gives:

$$\begin{aligned}
\frac{1}{|B|^{1+2\gamma/n}}\int_0^{r_B}\int_B |t\partial_t e^{-t^{2\alpha}L^\alpha}(f)(x)|^2\frac{dxdt}{t} &= \frac{1}{|B|^{1+2\gamma/n}}\int_0^{r_B}\int_B |t^{2\alpha}L^\alpha e^{-t^{2\alpha}L^\alpha}(f)(x)|^2\frac{dxdt}{t} \\
&= \frac{1}{|B|^{1+2\gamma/n}}\int_0^{r_B^{2\alpha}}\int_B |t\partial_t e^{-tL^\alpha}(f)(x)|^2\frac{dxdt}{t} \\
&\lesssim \|f\|^2_{BMO_L^\gamma}.
\end{aligned}$$

The estimation

$$\frac{1}{|B|^{1+2\gamma/n}}\int_0^{r_B}\int_B |t\nabla_x e^{-t^{2\alpha}L^\alpha}f(x)|^2\frac{dxdt}{t} \lesssim \|f\|^2_{BMO_L^\gamma}$$

can be obtained in the manner of Theorem 3.

(ii). Assume that $d\nu_\alpha$ is a $(1+2\gamma/n)$-Carleson measure, i.e.,

$$\sup_B \frac{1}{|B|^{1+2\gamma/n}}\int_0^{r_B}\int_B |t\nabla e^{-t^{2\alpha}L^\alpha}(f)(x)|^2\frac{dxdt}{t} < \infty.$$

Subsequently,

$$\sup_B \frac{1}{|B|^{1+2\gamma/n}}\int_0^{r_B}\int_B |t\partial_t e^{-t^{2\alpha}L^\alpha}(f)(x)|^2\frac{dxdt}{t} < \infty.$$

It can be deduced from Theorem 4 that $f \in BMO_L^\gamma(\mathbb{R}^n)$. □

5. Conclusions

In this paper, with the aid of the fundamental solution of the heat equation associated with the Schrödinger operators, we estimate the gradient and the time-fractional derivatives of the fractional heat kernel $K^L_{\alpha,t}(\cdot,\cdot)$, respectively. Finally, as an application, we establish a Carleson measure characterization of the Campanato-type space $BMO_L^\gamma(\mathbb{R}^n)$ via the fractional heat semigroup $\{e^{-tL^\alpha}\}_{t>0}$.

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