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Riemann Boundary Value Problems for Monogenic Functions on the Hyperplane

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Abstract. In this paper we systematically study the Riemann boundary value problems on the hyperplane for monogenic functions in Clifford analysis. The concept of the principal part of a sectionally regular function with the hyperplane as its jump surface is first introduced. Based on this concept the general forms of the Riemann boundary value problems on the hyperplane for monogenic functions are formulated. Then, the explicit expressions and explicit solvable conditions for solutions with any finite integer order at infinity are obtained.

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1. Introduction

It is well known that the boundary value problems (BVPs) for analytic functions in the classical complex analysis are an important branch of mathematical analysis, and have been actively studied for a long time, due to its theoretical elegance and ample applications in physics and other areas such as elastic theory, hydromechanics, fracture mechanics, etc. The theory of boundary value problems for analytic functions has been investigated systematically in the literature [11,19,24], and some applications have been studied in the monographs [20,23].

It has been proved that the function theory over Clifford algebra is an appropriate setting to generalize many aspects of the function theory of the one complex variable to higher dimensions [2,6,14,15,29]. It is a natural expectation to develop the boundary value problems for analytic functions in

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the classical complex analysis to those for regular functions in the hypercomplex analysis. In fact, some results in the former have already been extended to the latter with the Clifford algebra setting (see [7,16,17] and references therein). However, as far as the authors know, there are many obstacles for BVPs for monogenic functions in hypercomplex analysis. Here the difficulty mainly lies in the fact that the basic theory on BVPs for monogenic function in Clifford analysis is not fully established, and some commonly used technical tools also have not been systematically formed. For Riemann boundary value problems for closed smooth surfaces Γ ("closed" means that Γ is the boundary of a bounded region), [22] gives the latest results. There are more difficulties in the discussion for Riemann boundary value problems on hyperplanes. In fact, even in the classical complex analysis some boundary value problems on the real axis have not been fully discussed in outstanding monographs [19,24]. To develop a theory of boundary value problems on hyperplanes in Clifford analysis we are facing more obstacles. In [30], Xu and Zhou tried to solve the Hilbert boundary value problems on the hyperplane under the condition that the solutions vanish or are bounded at the infinity. In [13], Gong and Du continued to discuss the Riemann boundary value problems on the hyperplane under the condition that the solutions have a finite non-negative order at the infinity. These studies, however, contain some mistakes and are lack of sufficient theoretical basis as being pointed out in detail below.

In the present paper, we will systematically study the Riemann boundary value problems on the hyperplane for monogenic functions in Clifford analysis. The particular contribution of this study is the initiation of the negative orders at the infinity and the solutions of the corresponding boundary value problems that overcome some primitive difficulties. The results for the negative order category are also new for the Riemann type boundary value problems in classical one complex variable case [19,24]. For Riemann boundary value problems with nonnegative order singularity at the infinity, we also construct the necessary supporting theory in detail, which updates and corrects the results in [13,30]. In order to obtain these results, we must develop some theory of hypercomplex boundary value problems and innovate some tools for the boundary behavior of monogenic functions. The paper is organized as follows. In §2 we provide necessary preliminary knowledge in Clifford analysis. In §3, we introduce the Cauchy type integrals on the hyperplane and discuss their regularity. This kind of regularity discussion, as necessity in solving the BVPs in this paper, is more difficult but interesting than those Cauchy type integrals on finite smooth surfaces. Many authors ignored this point, and the present study, to the authors' knowledge, is the first time to seriously look at this issue. In order to a thorough study to the Riemann boundary value problems with negative orders at the infinity, we also need to discuss the Cauchy type integrals on some unbounded subdomain of the hyperplane. In §4, we further discuss the boundary behavior of the Cauchy type integrals on the hyperplane, which plays an important role for solving the BVPs. The results presented there improve the classical results, that are the so-called Plemelj-Sochocki formula and Privalov–Muskhelishvili theorem (the 2P Theorems for short) [11,19,24]. The authors wonder why, since the 2P Theorems are the cornerstones of solving BVPs, some authors refer to it directly without proof, especially in the case of hyperplanes, which is quite different from the case of real axis in the classical complex analysis. We also discuss the boundary behavior of the Cauchy type integrals at the infinity, which is the premise to obtain the general solution of Riemann boundary value problems by using the Liouville type theorem. It is very different from the corresponding results in classical complex analysis and cannot be simply simulated. We use the twisted inversion to get the boundedness of the Cauchy type integrals at the infinity. Many authors used the twisted inversion without providing or referring anywhere about the calculation of its Jacobian determinant here. In §5, the definitions of the principal part and the order at the infinity for a monogenic function cut along a hyperplane are established, which are rather difficult since the infinity is not as usual an isolated singular point for such functions. Before this, people failed to discuss the boundary value problem with arbitrary order at the infinity, for they would be difficult to find a reasonable description of the principal part. This work even improves the Liouville type theorem. In §6, the Riemann boundary value problem for the sectionally regular functions with the hypercomlex plane as its jump surface is suitably formulated and successfully solved. For all the allowed positive (including zero) and negative orders of the solutions at the infinity the solution formulas are obtained and the conditions of the solvability are specified.

The results of this paper can be used to solve the Hilbert boundary value problems on the hypercomplex plane, which will be discussed in a separate and forthcoming article.

2. Hypercomplex Functions

We begin by providing the necessary preliminary knowledge in Clifford algebra and Clifford analysis which are used throughout this paper [2,6].

2.1. Clifford Analysis

Let $C(V_n)$ be a 2^n-dimensional real linear space. To expediently introduce the product on it, we write its basis by $\{e_A, A=(h_1,\ldots,h_r)\in\mathcal{P}N, 1\le h_1<\ldots<h_r\le n\}$, where N stands for the set $\{1,\ldots,n\}$ and $\mathcal{P}N$ is to denote the family of all order-preserving subsets of N in the above fixed way. Sometimes, $e_\emptyset$ is written as e_0 and e_A as $e_{h_1\ldots h_r}$ for $A=\{h_1,\ldots,h_r\}\in\mathcal{P}N$. The product on $C(V_n)$ is defined by

$$\begin{cases} e_Ae_B=(-1)^{\#(A\cap B)}(-1)^{P(A,B)}e_{A\boldsymbol{\Delta}B}, & \text{if } A,B\in\mathcal{P}N,\\ \lambda\mu=\sum\limits_{A\in\mathcal{P}N}\sum\limits_{B\in\mathcal{P}N}\lambda_A\mu_Be_Ae_B, & \text{if } \lambda=\sum\limits_{A\in\mathcal{P}N}\lambda_Ae_A, \mu=\sum\limits_{A\in\mathcal{P}N}\mu_Ae_A, \end{cases} \tag{2.1}$$

where the notation $\#(A)$ denotes the number of the elements in A, $P(A,B)=\sum_{j\in B}P(A,j)$ with $P(A,j)=\#\{i: i\in A, i>j\}$, the symmetric difference set $A\boldsymbol{\Delta}B$ is also the order-preserving one in the above way, as well as $\lambda_A\in\mathbb{R}$ (real number set) is the coefficient of the e_A–component of the Clifford number λ. It follows at once from the multiplication rule (2.1) that e_0 is the identity element written now as 1 and in particular,

$$\begin{cases} e_i^2 = -1, & \text{if} \quad i = 1, \ldots, n, \\ e_i e_j = -e_j e_i, & \text{if} \quad 1 \le i < j \le n, \\ e_{h_1} e_{h_2} \ldots e_{h_r} = e_{h_1 h_2 \ldots h_r}, & \text{if} \quad 1 \le h_1 < h_2 < \ldots < h_r \le n. \end{cases} \tag{2.2}$$

It is clear that $C(V_n)$ is a real linear, associative and non-commutative algebra by algebraically spanning the linear subspace $V_n = \text{span}\{e_1, e_2, \ldots, e_n\}$. It is called the Clifford algebra over V_n. The elements $\lambda = \lambda_0 + \lambda_1 e_1 + \ldots + \lambda_n e_n$ for $\lambda_0, \ldots, \lambda_n \in \mathbb{R}$ are called paravectors.

We constantly use the following involution. It is defined by

$$\begin{cases} \overline{e_A} = (-1)^{\frac{\#(A)(\#(A)+1)}{2}} e_A, & \text{if} \quad A \in \mathcal{P}N, \\ \overline{\lambda} = \sum\limits_{A \in \mathcal{P}N} \lambda_A \overline{e_A}, & \text{if} \quad \lambda = \sum\limits_{A \in \mathcal{P}N} \lambda_A e_A. \end{cases} \tag{2.3}$$

In the sequel, λ_A is also written as $[\lambda]_A$, in particular, the coefficient $\lambda_\emptyset$ is denoted by λ_0 or $[\lambda]_0$, which is called the scalar part of the Clifford number λ. An inner product $(\cdot, \cdot)$ on $C(V_n)$ is defined by putting for any λ and μ in $C(V_n)$

$$(\lambda, \mu) = \left[\lambda \overline{\mu}\right]_0 = \sum\nolimits_A \lambda_A \, \mu_A, \tag{2.4}$$

where $\lambda = \sum_A \lambda_A e_A$, $\mu = \sum_A \mu_A e_A$ and the symbol $\sum_A$ is abbreviated from $\sum\limits_{A \in \mathcal{P}N}$. Thus, the corresponding norm on $C(V_n)$ reads,

$$\left|\lambda\right| = \sqrt{(\lambda, \lambda)} = \left[\sum_A \lambda_A^2\right]^{\frac{1}{2}}. \tag{2.5}$$

In this way, $C(V_n)$ is a real Hilbert space and at the same time it is a Banach algebra with the equivalent norm

$$\left|\lambda\right|_0 = 2^{\frac{n}{2}} \left|\lambda\right|, \tag{2.6}$$

that is

$$\left|\lambda\mu\right|_0 \le \left|\lambda\right|_0 \left|\mu\right|_0, \quad \left|\lambda\mu\right| \le 2^{\frac{n+1}{2}} \left|\lambda\right|\left|\mu\right|. \tag{2.7}$$

In particular, if λ is a paravector and $\mu \in C(V_n)$, then [2]

$$\left|\lambda\mu\right| = \left|\mu\lambda\right| = \left|\lambda\right|\left|\mu\right|. \tag{2.8}$$

Let Ω be a non-empty subset of $\mathbb{R}^{n+1}$. Hypercomplex functions f defined in Ω and with values in $C(V_n)$ will be considered, *i.e.*, $f : \Omega \longrightarrow C(V_n)$. They are of the form

$$f(x) = \sum\nolimits_A f_A(x) e_A, \quad x = (x_0, x_1, \ldots, x_n) \in \Omega \subset \mathbb{R}^{n+1}, \tag{2.9}$$

where $f_A(x)$ is the e_A–component function of $f(x)$. Obviously, f_A' are real–valued functions in Ω. Whenever a property such as differentiability and continuity is ascribed to f, it is clear that in fact all the component functions f_A possess the cited property. So $f \in C^r(\Omega, C(V_n))$ is very clear. The conjugate of the function f is the function $\overline{f}$ given by

$$\overline{f}(x) = \sum_A f_A(x) \overline{e_A}, \quad x \in \Omega. \tag{2.10}$$

Clearly,

$$f \in C^r(\Omega, C(V_n)) \Longleftrightarrow \overline{f} \in C^r(\Omega, C(V_n)). \tag{2.11}$$

Obviously, $C(V_{n-1})$ is the subalgebra of $C(V_n)$ where $V_{n-1} = \text{span}\{e_1, e_2, \ldots, e_{n-1}\}$. Then, $\lambda \in C(V_n)$ has the unique decomposition [13, 30]

$$\lambda = \lambda_1 + \lambda_2 e_n \quad \text{where} \quad \lambda_1, \lambda_2 \in C(V_{n-1}), \tag{2.12}$$

i.e.,

$$C(V_n) = C(V_{n-1}) \oplus C(V_{n-1}) e_n. \tag{2.13}$$

We define

$$\text{Re}(\lambda) = \lambda_1, \qquad \text{Im}(\lambda) = \lambda_2. \tag{2.14}$$

It is clear that the decomposition (2.12) is the generalization of the representation of the classical complex number. In other words, (2.14) is the generalization of operators Re and Im acting on the complex $\mathbb{C}$.

For a hypercomplex function f given by (2.9), if

$$\big(\text{Im} f\big)(x) \triangleq \text{Im}\big(f(x)\big) \equiv 0, \;\; x = (x_0, x_1, \ldots, x_n) \in \Omega \subset \mathbb{R}^{n+1}, \tag{2.15}$$

i.e., $f : \Omega \longrightarrow C(V_{n-1})$, then we say that it is a $C(V_{n-1})$-valued function, briefly, a para real-valued function which mimics the case of the real-valued function in the classical complex analysis.

When $x = (x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1}$, we introduce the mapping

$$\text{capital} : x \longmapsto X = \sum_{i=0}^{n} x_i e_i, \tag{2.16}$$

which is one proper isomorphism between $\mathbb{R}^{n+1}$ and the linear subspace $\text{span}\{e_0, e_1, \ldots, e_n\}$ of $C(V_n)$. In the sequel, we simply treat X as x. This is Vahlen's choice [27]. Thus,

$$\text{Im}(x) = x_n \;\; \text{while} \;\; x \in \mathbb{R}^{n+1}, \tag{2.17}$$

and

$$\text{Re}(x) = x_0 + x_1 e_1 + \ldots + x_{n-1} e_{n-1}. \tag{2.18}$$

Define

$$\mathbb{R}^{n+1}_{+} = \{x, \text{Im}(x) > 0\}, \; \mathbb{R}^{n+1}_{-} = \{x, \text{Im}(x) < 0\}, \; \mathbb{R}^{n+1}_{0} = \{x, \text{Im}(x) = 0\}. \tag{2.19}$$

$\mathbb{R}^{n+1}_{+}$ and $\mathbb{R}^{n+1}_{-}$ are called, respectively, the Poincaré upper halfspace and the Poincaré lower halfspace, while the hyperplane $\mathbb{R}^{n+1}_{0}$ is called the para real plane in $\mathbb{R}^{n+1}$. It is a significant advantage that the paravectors given in (2.18) play a treble role as elements of $\mathbb{R}^n$ and $\mathbb{R}^{n+1}_0$ as well as $C(V_{n-1}) \subset C(V_n)$.

2.2. $\widehat{H}$ Class of Functions

We need to introduce some classes of hypercomplex functions used frequently in this paper. Below we will use $\frac{1}{x}$ to represent the inverse x^{-1} of $x \in \mathbb{R}^{n+1}\backslash\{0\}$.

Definition 2.1. Assume f is defined on $\Omega \subseteq \mathbb{R}^{n+1}$. If

$$\left|f(t) - f(s)\right| \leq M\left|t - s\right|^{\mu} \quad (0 < \mu \leq 1) \tag{2.20}$$

for arbitrary points t, s on Ω, where M and μ are finite constants, then f is said to satisfy Hölder condition of order μ, denoted by $f \in H^{\mu}(\Omega)$. And μ and M are called, repectively, the Hölder index and a Hölder coefficient of f. If the order μ is not emphasized, it may be denoted briefly by $f \in H(\Omega)$.

Definition 2.2. Assume f is defined on $\Omega \subseteq \mathbb{R}^{n+1}$. If

$$\left|f(\xi) - f(\zeta)\right| \leq M\left|\frac{1}{\xi} - \frac{1}{\zeta}\right|^{\mu} \quad (0 < \mu \leq 1) \tag{2.21}$$

for arbitrary points ξ, ζ on $\Omega\setminus\{0\}$, where M and μ are finite constants, then f is said to satisfy †-Hölder condition of order μ, denoted by $f \in H^{\mu}_{\dagger}(\Omega)$. And μ and M are called, repectively, the †-Hölder index and a †-Hölder coefficient of f. If the order μ is not emphasized, it may be denoted briefly by $f \in H_{\dagger}(\Omega)$.

Definition 2.3. If $f \in H^{\mu}(\Omega) \cap H^{\mu}_{\dagger}(\Omega)$, then f is said to satisfy $\widehat{H}$ condition of order μ on Ω, denoted by $f \in \widehat{H}^{\mu}(\Omega)$ or briefly $f \in \widehat{H}(\Omega)$.

The conditions (2.20) and (2.21) are, respectively, called the Hölder condition and †-Hölder condition of the $\widehat{H}(\Omega)$ class function f, which may be simplified. To do this, we introduce the following notations.

$$B_{n+1}(w, R) = \left\{\zeta \in \mathbb{R}^{n+1},\ |\zeta - w| < R\right\},\ B_n(t, R) = \left\{\zeta \in \mathbb{R}^{n+1}_0,\ |\zeta - t| < R\right\} \tag{2.22}$$

are, respectively, the ball in $\mathbb{R}^{n+1}$ with radius R at center $w \in \mathbb{R}^{n+1}$ and the ball in $\mathbb{R}^{n+1}_0$ with radius R at center $t \in \mathbb{R}^{n+1}_0$.

$$B_{n+1}(R, \infty) = \left\{\zeta \in \mathbb{R}^{n+1},\ |\zeta| > R\right\},\ B_n(R, \infty) = \left\{\zeta \in \mathbb{R}^{n+1}_0,\ |\zeta| > R\right\} \tag{2.23}$$

are, respectively, the ball in $\mathbb{R}^{n+1}$ with radius R at center $\infty \in \mathbb{R}^{n+1}$ and the ball in $\mathbb{R}^{n+1}_0$ with radius R at center $\infty \in \mathbb{R}^{n+1}_0$. The notation $\overline{A}$ denotes the closure of the set A, such as

$$\overline{B}_n(t, R) = \left\{\zeta \in \mathbb{R}^{n+1}_0,\ |\zeta - t| \leq R\right\},\ \overline{B}_n(R, \infty) = \left\{\zeta \in \mathbb{R}^{n+1}_0,\ |\zeta| \geq R\right\}. \tag{2.24}$$

are, respectively, the closures of $B_n(t, R)$ and $B_n(R, \infty)$.

Example 2.1. [see[13]] $f \in \widehat{H}^{\mu}\left(\mathbb{R}^{n+1}_0\right)$ if and only if

$$\left|f(\xi) - f(\zeta)\right| \leq M\left|\xi - \zeta\right|^{\mu}, \quad \xi, \zeta \in \overline{B}_n(0, R), \tag{2.25}$$

and

$$\Big|f(\xi)-f(\zeta)\Big| \le M\left|\frac{1}{\xi}-\frac{1}{\zeta}\right|^{\mu}, \quad \xi,\zeta\in \overline{B}_n(R,\infty), \tag{2.26}$$

where $M>0$ and $R>0$ are some constants. In fact, (2.25) and (2.26) result in, respectively,

$$\Big|f(\xi)-f(\zeta)\Big| \le MR^{2\mu}\left|\frac{1}{\xi}-\frac{1}{\zeta}\right|^{\mu} \text{ while } \quad \xi,\zeta\in \overline{B}_n(0,R)\setminus\{0\}, \tag{2.27}$$

and

$$\Big|f(\xi)-f(\zeta)\Big| \le \frac{M}{R^{2\mu}}\Big|\xi-\zeta\Big|^{\mu} \text{ while } \quad \xi,\zeta\in \overline{B}_n(R,\infty). \tag{2.28}$$

In general, we may prove the following result, which makes it easier to identify $f\in\widehat{H}^{\mu}(\Omega)$.

Remark 2.1. $f\in\widehat{H}^{\mu}(\Omega)$ if and only if

$$\begin{cases} \Big|f(\xi)-f(\zeta)\Big| \le M\Big|\xi-\zeta\Big|^{\mu} \text{ while } \quad \xi,\zeta\in\Omega\cap\overline{B}_{n+1}(0,R), \\ \Big|f(\xi)-f(\zeta)\Big| \le M\left|\frac{1}{\xi}-\frac{1}{\zeta}\right|^{\mu} \text{ while } \quad \xi,\zeta\in\Omega\setminus B_{n+1}(0,R), \end{cases} \tag{2.29}$$

where $M>0$ and $R>0$ are some constants. In other words,

$$f\in\widehat{H}^{\mu}(\Omega) \Longleftrightarrow f\in H^{\mu}\Big(\Omega\cap\overline{B}_{n+1}(0,R)\Big)\bigcap H^{\mu}_{\dagger}\Big(\Omega\setminus B_{n+1}(0,R)\Big). \tag{2.30}$$

Lemma 2.1. [[13]] *If $f\in H^{\mu}_{\dagger}\big(B_n(R,\infty)\big)$, which is called that f satisfies $H_{\dagger}$ condition near ∞, then,*

$$f(\infty)=\lim_{x\in B_n(R,\infty),\, x\to\infty} f(x) \tag{2.31}$$

exists and

$$\Big|f(x)-f(\infty)\Big|\le\frac{M}{|x|^{\mu}},\ x\in B_n(R,\infty) \quad \Big(\textit{the Hölder condition at} z{=}\infty\Big), \tag{2.32}$$

where M is a constant.

Proof. From (2.21) and the Cauchy's criterion, we easily get this lemma. □

By using (2.31) and (2.32), we introduce a new class of hypercomplex functions. Such kind of functions only have the weak pointwise Hölder condition at infinity.

Definition 2.4. Let f be a function defined on $B_n(R,\infty)$. If (2.31) exists and it satisfies the condition (2.32), then we say that it satisfies the pointwise Hölder condition at the infinity, denoted by $f\in H^{\mu}_{\dagger}(\infty)$ or briefly $f\in H_{\dagger}(\infty)$.

We also use the more general definition below.

Definition 2.5. Let f be a function defined on $\Omega\subseteq\mathbb{R}^{n+1}$ with ∞ as its cluster point. If

$$f(\infty)=\lim_{w\in\Omega,\, w\to\infty} f(w) \tag{2.33}$$

exists and

$$\left|f(w)-f(\infty)\right| \leq \frac{M}{|w|^{\mu}} \left(0<\mu\leq 1\right),\ w \in \Omega\backslash\{0\} \left(\textit{the Hölder condition at } z{=}\infty\right) \tag{2.34}$$

where M is a constant, then we say that f satisfies the pointwise Hölder condition at the infinity in Ω, denoted by $f \in H^{\mu}_{\dagger}(\infty)$, or also briefly $f \in H_{\dagger}(\infty)$.

Remark 2.2. Moreover, $\widehat{H}^{\mu}\left(\mathbb{R}_0^{n+1}\right)$ class function f is a continuous bounded function on the whole $\mathbb{R}_0^{n+1}$ and there exists the Chebyshev norm

$$\left\|f\right\|_{\infty} = \left\|f\right\|_{\mathbb{R}_0^{n+1}} = \max\left\{\left|f(x)\right|,\ x \in \mathbb{R}_0^{n+1}\right\}. \tag{2.35}$$

Analogously, if $f \in H_{\dagger}\left(\overline{B}_n(R,\infty)\right)$, then it is also a continuous bounded function and there exists the Chebyshev norm on $\overline{B}_n(R,\infty)$

$$\left\|f\right\|_{\overline{B}_n(R,\infty)} = \max\left\{\left|f(x)\right|,\ x \in \overline{B}_n(R,\infty)\right\}. \tag{2.36}$$

Lemma 2.2. *If* $0 < \nu < \mu \leq 1$, *then* $\widehat{H}^{\nu}(\Omega) \subseteq \widehat{H}^{\mu}(\Omega)$.

Proof. Obviously, (2.20) and (2.21) result in, respectively,

$$\left|f(\xi) - f(\zeta)\right| \leq 2M\left|\xi - \zeta\right|^{\nu} \quad \text{while} \quad \xi, \zeta \in \overline{B}_{n+1}(0,1) \cap \Omega, \tag{2.37}$$

and

$$\left|f(\xi) - f(\zeta)\right| \leq 2M\left|\frac{1}{\xi} - \frac{1}{\zeta}\right|^{\nu} \quad \text{while} \quad \xi, \zeta \in \Omega \setminus B_{n+1}(0,1). \tag{2.38}$$

By Remark 2.1, the proof is completed. □

To find out the relationship between the class H and $H_{\dagger}$, we shall introduce the twisted inversion [1,27]

$$\xi = \dagger(x) = x^{\dagger} = -\frac{1}{x}, \qquad x \in \widetilde{\mathbb{R}}^{n+1} = \mathbb{R}^{n+1} \cup \{\infty\}, \tag{2.39}$$

where we agree

$$0^{\dagger} = \infty \ \textit{and}\ \infty^{\dagger} = 0. \tag{2.40}$$

$x^{\dagger}$ is called the twisted inversion point of x. For $\Omega \subseteq \mathbb{R}^{n+1}$,

$$\Omega^{\dagger} = \left\{x^{\dagger},\ x \in \Omega\right\} \tag{2.41}$$

is called the twisted inversion set of Ω. For example,

$$\left(\mathbb{R}_0^{n+1}\backslash\{0\}\right)^{\dagger} = \mathbb{R}_0^{n+1}\backslash\{0\},\ \ \left(\mathbb{R}_{\pm}^{n+1}\right)^{\dagger} = \mathbb{R}_{\pm}^{n+1}. \tag{2.42}$$

Obviously

$$\left(x^{\dagger}\right)^{\dagger} = x,\ \ \left(\Omega^{\dagger}\right)^{\dagger} = \Omega. \tag{2.43}$$

Then, if f is defined in Ω we introduce its associated twisted inversion function

$$f^{\dagger}(\xi) \triangleq f\left(\xi^{\dagger}\right) = f\left(-\frac{1}{\xi}\right),\quad \xi \in \Omega^{\dagger}. \tag{2.44}$$

It is easily seen, by (2.41), that

$$\left(f^{\dagger}\right)^{\dagger}=f. \tag{2.45}$$

Remark 2.3. (2.26) now may be improved as

$$\left|f(\xi)-f(\zeta)\right|\leq M\left|\xi^{\dagger}-\zeta^{\dagger}\right|^{\mu}\quad\text{while}\quad \xi,\zeta\in\overline{B}_n(R,\infty). \tag{2.46}$$

Lemma 2.3. *(1) $f\in H^{\mu}_{\dagger}\left(\mathbb{R}^{n+1}_0\right)$ if and only if $f^{\dagger}\in H^{\mu}\left(\mathbb{R}^{n+1}_0\right)$; (2) $f\in H^{\mu}\left(\mathbb{R}^{n+1}_0\right)$ if and only if $f^{\dagger}\in H^{\mu}_{\dagger}\left(\mathbb{R}^{n+1}_0\right)$; (3) $f\in\widehat{H}^{\mu}\left(\mathbb{R}^{n+1}_0\right)$ if and only if $f^{\dagger}\in\widehat{H}^{\mu}\left(\mathbb{R}^{n+1}_0\right)$.*

Proof. By using (2.45), we only need show (1). Necessity of (1):

$$\left|f^{\dagger}(\xi)-f^{\dagger}(\zeta)\right|=\left|f\left(\xi^{\dagger}\right)-f\left(\zeta^{\dagger}\right)\right|\leq M\left|\frac{1}{\xi^{\dagger}}-\frac{1}{\zeta^{\dagger}}\right|^{\mu}=M\left|\xi-\zeta\right|^{\mu},\ \xi,\zeta\in\mathbb{R}^{n+1}_0. \tag{2.47}$$

Sufficiency of (1):

$$\left|f\left(\xi^{\dagger}\right)-f\left(\zeta^{\dagger}\right)\right|=\left|f^{\dagger}(\xi)-f^{\dagger}(\zeta)\right|\leq M\left|\xi-\zeta\right|^{\mu}=M\left|\frac{1}{\xi^{\dagger}}-\frac{1}{\zeta^{\dagger}}\right|^{\mu},\ \xi,\zeta\in\mathbb{R}^{n+1}_0\setminus\{0\}. \tag{2.48}$$

(2.47) and (2.48) show that the assert (1) holds. □

In general, we have the following result.

Lemma 2.4. *(1) $f\in H^{\mu}_{\dagger}(\Omega)$ if and only if $f^{\dagger}\in H^{\mu}\left(\Omega^{\dagger}\right)$; (2) $f\in H^{\mu}(\Omega)$ if and only if $f^{\dagger}\in H^{\mu}_{\dagger}\left(\Omega^{\dagger}\right)$; (3) $f\in\widehat{H}^{\mu}(\Omega)$ if and only if $f^{\dagger}\in\widehat{H}^{\mu}\left(\Omega^{\dagger}\right)$.*

Let $f_{\mathbb{m}}(x)=x^m f(x)$. When $f_{\mathbb{m}}\in H^{\mu}_{\dagger}(\mathbb{R}^{n+1}_0)$, then we write $f\in H^{\mu}_{m,\dagger}(\mathbb{R}^{n+1}_0)$, or briefly $f\in H_{m,\dagger}(\mathbb{R}^{n+1}_0)$. Similarly, if $f_{\mathbb{m}}\in H^{\mu}_{\dagger}\left(\mathbb{R}^{n+1}_0,\infty\right)$, then we write $f\in H^{\mu}_{m,\dagger}\left(\mathbb{R}^{n+1}_0,\infty\right)$ or briefly $f\in H_{m,\dagger}\left(\infty\right)$. In this way, $f\in H^{\mu}_{m,\dagger}\left(\overline{B}_n(R,\infty)\right)$, $f\in H_{m,\dagger}\left(\overline{B}_{n+1}(R,\infty)\right)$ and $f\in H^{\mu}_{m,\dagger}\left(\mathbb{R}^{n+1}_{\pm},\infty\right)$ etc. are clear. The following two lemmas will be used in the discussion on Riemann boundary value problems below.

Lemma 2.5. *If $f\in H_{m,\dagger}(\Omega,\infty)$ $(m>0)$, then $f(\infty)=0$, more precisely, $\left|f(w)\right|=O\left(|w|^{-m}\right)$ near $w=\infty$ on Ω, which is denoted by $f\in O^{-m}(\infty)$.*

Proof. The proof is easy by $f(w)=w^{-m}f_{\mathbb{m}}(w)$ $\left(w^{-m}=\left[w^{-1}\right]^m\right)$ and Definition 2.5. □

Lemma 2.6. *Let $0\leq m<k$ and $R>0$. Then, we have*

(1) $H^{\mu}_{k,\dagger}\left(\overline{B}_n(R,\infty)\right)\subseteq H^{\nu}_{m,\dagger}\left(\overline{B}_n(R,\infty)\right)$, where $\nu=\min\left\{k-m,\mu\right\}$,

(2) $H_{k,\dagger}\left(\infty\right)\subseteq\left\{f,\ f_{\mathbb{m}}(x)=O\left(x^{-\nu}\right)\ \text{near}\ \infty\right\}\subseteq H_{m,\dagger}\left(\infty\right)$ where $\nu>0$.

Proof. If $f\in H^{\mu}_{k,\dagger}\left(\overline{B}_n(R,\infty),\infty\right)$, noting Remark 2.2 we have, for $|\xi|,|\zeta|\geq R$,

$$\left|f_{\mathbb{m}}(\xi)-f_{\mathbb{m}}(\zeta)\right|\leq\frac{2}{R^{k-m}}\left|f_{\mathbb{k}}(\xi)-f_{\mathbb{k}}(\zeta)\right|+\left\|f_{\mathbb{k}}\right\|_{\overline{B}_n(R,\infty)}\left|\xi^{m-k}-\zeta^{m-k}\right|. \tag{2.49}$$

So, there exists constant M such that

$$\Big|f_{\mathbb{m}}(\xi) - f_{\mathbb{m}}(\zeta)\Big| \le M\Big|\xi^{\dagger} - \zeta^{\dagger}\Big|^{\nu} \quad \Big(\nu = \min\big\{k - m, \mu\big\}\Big), \tag{2.50}$$

which results in (1).

The proof of (2) is simpler. In fact,

$$f_{\mathbb{m}}(\infty) = \lim_{x\to\infty}\big[x^{m-k} f_{\mathbb{k}}(x)\big] = 0 \ \text{ near } \ \infty, \tag{2.51}$$

and

$$\Big|f_{\mathbb{m}}(x) - f_{\mathbb{m}}(\infty)\Big| = \Big|f_{\mathbb{m}}(x)\Big| \le \frac{|f_{\mathbb{k}}(\infty)| + 1}{|x|^{k-m}} \ \text{ near } \ \infty, \tag{2.52}$$

which results in (2). □

Example 2.2. Let

$$g(x) = \begin{cases} 1, & |x| \le 1, \\ \dfrac{1}{|x|^{s+1}}, & |x| > 1, \end{cases} \tag{2.53}$$

where $s \ge 0$. Then $g \in \widehat{H}_{s,0}\Big(\mathbb{R}_0^{n+1}\Big)$ but $g \notin \widehat{H}_{\ell}\Big(\mathbb{R}_0^{n+1}\Big)$ when $\ell > s+1$, where

$$\widehat{H}_{s,0}\Big(\mathbb{R}_0^{n+1}\Big) = H\Big(\mathbb{R}_0^{n+1}\Big)\bigcap \widehat{H}_{s,\dagger}\Big(\mathbb{R}_0^{n+1}\Big)\bigcap\Big\{g, g_{\mathbb{s}}(\infty) = 0\Big\}. \tag{2.54}$$

2.3. Monogenic Functions

Let Ω be a domain of $\mathbb{R}^{n+1}$. Introduce the following Dirac operator

$$D = \sum_{k=0}^{n} e_k \frac{\partial}{\partial x_k} : \ C^{(r)}\left(\Omega, C\left(V_n\right)\right) \longrightarrow C^{(r-1)}\big(\Omega, C\big(V_n\big)\big), \tag{2.55}$$

its action on functions from the left and from the right being governed by the rules

$$D[f] = \sum_{k=0}^{n}\sum_{A} e_k e_A \frac{\partial f_A}{\partial x_k}, \qquad [f]D = \sum_{k=0}^{n}\sum_{A} e_A e_k \frac{\partial f_A}{\partial x_k}. \tag{2.56}$$

Definition 2.6. We say a function $f \in C^{(r)}\left(\Omega, C\left(V_n\right)\right)$ $(r \ge 1)$ to be left (right) regular in Ω if $D[f] = 0$ ($[f]D = 0$) in Ω. Sometimes we also say that f is left (right) regular in Ω. f is said to be biregular in Ω if and only if it is both left and right regular.

Remark 2.4. Generally speaking, if

$$f(x) = \sum_{j=1}^{m} f_j(x)\lambda_j \quad \Big(f_j \in C(\Omega, \mathbb{R}) \text{ and } \lambda_j \in C(V_n)\Big), \tag{2.57}$$

then

$$D[f] = \sum_{k=0}^{n}\sum_{j=1}^{m} e_k\lambda_j \frac{\partial f_j}{\partial x_k}, \quad [f]D = \sum_{k=0}^{n}\sum_{j=1}^{m} \lambda_j e_k \frac{\partial f_j}{\partial x_k}. \tag{2.58}$$

Example 2.3. Let

$$E(x)=\frac{\overline{x}}{|x|^{n+1}}=\frac{x^{-1}}{|x|^{n-1}}=\frac{1}{|x|^{n-1}x}\,,\quad x\in\mathbb{R}^{n+1}\setminus\{0\}. \tag{2.59}$$

Then E is biregular [2,9], which is called the Cauchy kernel function.

Example 2.4. The hypercomplex variables

$$z_j=z_j(x)=x_je_0-x_0e_j\ \ \big(j=1,\ldots,n\big) \tag{2.60}$$

are biregular [2,4,5,31,32].

Example 2.5. All derivatives of E, i.e.,

$$W_{\{\ell_1,\ell_2,\ldots\ell_k\}}(x)=(-1)^k\frac{\partial^k E}{\partial x_{\ell_1}\partial x_{\ell_2}\ldots\partial x_{\ell_k}}(x),\quad x\in\mathbb{R}^{n+1}\setminus\{0\}, \tag{2.61}$$

are biregular [2,4,5,31,32], and

$$W_{\{\ell_1,\ell_2,\ldots\ell_k\}}(x)=O\Big(|x|^{-(n+k)}\Big)\ \text{near}\ \ \infty, \tag{2.62}$$

where ℓ_j's are k elements out of the set $N=\big\{1,\ldots,n\big\}$ repetition being allowed, in other words, $\big\{\ell_1,\ldots,\ell_k\big\}\in N^k$.

3. Cauchy Type Integrals and Singular Integrals

In this section, we first introduce the integrals of hypercomplex functions, specially the Cauchy type integrals and singular integrals on the hyperplane $\mathbb{R}_0^{n+1}$ which are the important tools for solving the boundary value problems.

3.1. Integrals of Hypercomplex Functions

To discuss integrals of functions with values in $C(V_n)$, we first introduce the differential space with basis $\{\mathrm{d}x_0,\ldots,\mathrm{d}x_n\}$ denoted by T_{n+1}. Let $G\,(T_{n+1})$ be the Grassmann algebra over T_{n+1} with basis $\{\mathrm{d}x_A, A=(h_1,\ldots,\ h_r)\in\mathcal{P}N_0, 0\le h_1<h_2<\ldots<h_r\le n\}$, where N_0 is the set $\{0,1,\ldots,n\}$ and $\mathcal{P}N_0$ the family of all order-preserving subsets of N_0 in the above fixed way. The exterior product on $G\,(T_{n+1})$ is defined by

$$\begin{cases}\mathrm{d}x_A\wedge\mathrm{d}x_B=(-1)^{P(A,B)}\mathrm{d}x_{A\cup B}, & \text{if}\ A,B\in\mathcal{P}N_0, A\cap B=\emptyset,\\ \mathrm{d}x_A\wedge\mathrm{d}x_B=0, & \text{if}\ A,B\in\mathcal{P}N_0, A\cap B\neq\emptyset,\\ \eta\wedge\upsilon=\sum_A{}'\sum_B{}'\eta_A\upsilon_B\mathrm{d}x_A\wedge\mathrm{d}x_B, \ \text{if}\ \eta=\sum_A{}'\eta_A\mathrm{d}x_A,\ \upsilon=\sum_A{}'\upsilon_A\mathrm{d}x_A,\end{cases} \tag{3.1}$$

where η_A and υ_A are real, $\sum{}'_A$ is the sum for all $A\in\mathcal{P}N_0$ and $P(A,B)$ is as before. Obviously, as a rule,

$$\begin{cases}\mathrm{d}x_\emptyset=1,\\ \mathrm{d}x_{h_1}\wedge\mathrm{d}x_{h_2}\ldots\wedge\mathrm{d}x_{h_r}=\mathrm{d}x_{h_1h_2\ldots h_r}, & \text{if}\ \ 0\le h_1<h_2\ldots<h_r\le n,\\ \mathrm{d}x_A\wedge\mathrm{d}x_B=(-1)^{\#(A)\#(B)}\mathrm{d}x_B\wedge\mathrm{d}x_A, & \text{if}\ \ A,B\in\mathcal{P}N_0.\end{cases} \tag{3.2}$$

We construct the direct product algebra $\mathcal{W} = \big(C(V_n), G(T_{n+1})\big)$, then consider the function $\digamma : \mathbb{R}_0^{n+1} \longrightarrow \mathcal{W}$ of the form

$$\digamma(x) = \sum_A \sum_{\#(B)=p}' \digamma_{A,B}(x) e_A \mathrm{d}x_B, \tag{3.3}$$

where all $\digamma_{A,B}$ are of the class $C^{(r)}$ $(r \geq 1)$ on $\mathbb{R}_0^{n+1}$, p is fixed and $0 < p \leq n+1$. $\digamma$ is called a $C(V_n)$-valued p-differential form. Let furthermore Γ be a p-chain on $\mathbb{R}^{n+1}$, then we define

$$\int_\Gamma \digamma(x) = \sum_A \sum_{\#(B)=p}' e_A \int_\Gamma \digamma_{A,B}(x) \mathrm{d}x_B. \tag{3.4}$$

In the sequel, we shall use the following $C(V_n)$-valued n–differential form, which is exact, so written as

$$\mathrm{d}\sigma = \sum_{k=0}^{n} (-1)^k e_k \mathrm{d}\widehat{x}_k \equiv \sum_{k=0}^{n} (-1)^k e_k \mathrm{d}x_0 \wedge \ldots \wedge \mathrm{d}x_{k-1} \wedge \mathrm{d}x_{k+1} \ldots \wedge \mathrm{d}x_n. \tag{3.5}$$

Remark 3.1. Sometimes we write it as $\mathrm{d}\sigma_n(x)$ in detail, then $\mathrm{d}\sigma_{n-1}(\varsigma)$ is the $C(V_{n-1})$-valued $(n-1)$-differential form for $\varsigma \in \mathbb{R}^n$, which will used below.

3.2. Cauchy Principal Value Integrals at the Infinity

In this paper, we will devote to discussing the Cauchy principal value integrals at the infinity on the hyperplane $\mathbb{R}_0^{n+1}$.

Definition 3.1. Suppose that $f, g \in C\left(\mathbb{R}_0^{n+1}, C(V_n)\right)$. If

$$\int_{\mathbb{R}_0^{n+1}} g(x)\, \mathrm{d}\sigma f(x) = \lim_{R \to +\infty} \int_{B_n(0,R)} g(x)\, \mathrm{d}\sigma\, f(x) \tag{3.6}$$

exists, then we say that the left side integral is convergent and call it the principal value integral on $\mathbb{R}_0^{n+1}$ at ∞.

Remark 3.2. Obviously,

$$\int_{\mathbb{R}_0^{n+1}} g(x)\, \mathrm{d}\sigma f(x) = \int_{\mathbb{R}_0^{n+1}} g(x)\, (-e_n)\, f(x)\, \mathrm{d}S, \tag{3.7}$$

where

$$\mathrm{d}S = \mathrm{d}x_0 \mathrm{d}x_1 \ldots \mathrm{d}x_{n-1} = e_n \mathrm{d}\sigma \tag{3.8}$$

is the elementary surface measure on the hyperplane $\mathbb{R}_0^{n+1}$.

Example 3.1. (see[13]) Let E be the Cauchy kernel function given in (2.59), then

$$\frac{1}{\mathsf{V}_{n+1}} \int_{\mathbb{R}_0^{n+1}} E(x-w)\, \mathrm{d}\sigma = \begin{cases} \dfrac{1}{2}, & w \in \mathbb{R}_+^{n+1}, \\ -\dfrac{1}{2}, & w \in \mathbb{R}_-^{n+1}, \end{cases} \tag{3.9}$$

and

$$\frac{1}{\mathsf{V}_{n+1}} \int_{\mathbb{R}_0^{n+1}} \mathrm{d}\sigma\, E(x-w) = \begin{cases} \dfrac{1}{2}, & w \in \mathbb{R}_+^{n+1}, \\ -\dfrac{1}{2}, & w \in \mathbb{R}_-^{n+1}, \end{cases} \tag{3.10}$$

where

$$\bigvee_{n+1} = \frac{2\pi^{(n+1)/2}}{\Gamma((n+1)/2)} \tag{3.11}$$

is the area of the unit sphere

$$S^n = \Big\{x,\ |x| = 1,\ x \in \mathbb{R}^{n+1}\Big\}. \tag{3.12}$$

To show this example, we first prove some lemmas in [13] for easy reference and promotion.

Lemma 3.1. [[13]] *Assume x is a fixed point in $\mathbb{R}_0^{n+1}$, then for a large enough R, we have*

$$\operatorname{mes}\bigg(\Big[B_n(x,R)\Big]\boldsymbol{\Delta}\Big[B_n(0,R)\Big]\bigg) \le 2\mathcal{V}_n\left[R^n - \left(R - \frac{|x|}{2}\right)^n\right] \ \ for\ |x| \le 2R, \tag{3.13}$$

where $A\boldsymbol{\Delta}B$ represents the symmetric difference of set A and set B, $\operatorname{mes}(A)$ *denotes the measure of A, $\mathcal{V}_n$ is the volume of unit ball in $\mathbb{R}_0^{n+1}$.*

Proof. From

$$\left\{y, \left|y - \frac{x}{2}\right| < R - \frac{|x|}{2},\ y \in \mathbb{R}_0^{n+1}\right\} \subseteq \Big[B_n(x,R)\Big] \bigcap \Big[B_n(0,R)\Big],\ \ |x| \le R, \tag{3.14}$$

(3.13) is obvious. □

Lemma 3.2. ([13]) *Let w be a fixed point on $\mathbb{R}_\pm^{n+1}$, $f \in C\left(\mathbb{R}_0^{n+1}, C\left(V_n\right)\right)$ and $\big\|f\big\|_\infty < M$. Then*

$$\lim_{R\to+\infty}\left[\int_{B_n(0,R)} E(x-w)\mathrm{d}\sigma f(x) - \int_{B_n(\mathrm{Re}(w),R)} E(x-w)\mathrm{d}\sigma f(x)\right] = 0, \tag{3.15}$$

where $w \in \mathbb{R}_\pm^{n+1}$.

Proof. It is easily seen that, for $R > \big|\mathrm{Re}(w)\big|$,

$$\begin{aligned}
&\overset{\Delta}{=} \left|\int_{B_n(0,R)} E(x-w)\mathrm{d}\sigma f(x) - \int_{B_n(\mathrm{Re}(w),R)} E(x-w)\mathrm{d}\sigma f(x)\right| \\
&\le M\int_{\left[B_n(0,R)\right]\boldsymbol{\Delta}\left[B_n(\mathrm{Re}(w),R)\right]} \Big|E(x-w)\Big|\mathrm{d}S \\
&\le M\int_{\left[B_n(0,R)\right]\boldsymbol{\Delta}\left[B_n(\mathrm{Re}(w),R)\right]} \frac{1}{\Big[|\mathrm{Re}(x-w)|^2 + |\mathrm{Im}(w)|^2\Big]^{\frac{n}{2}}}\mathrm{d}S \\
&\le \frac{2M\mathcal{V}_n\left[R^n - \left(R - \frac{1}{2}|\mathrm{Re}(w)|\right)^n\right]}{\Big[\left(R - |\mathrm{Re}(w)|\right)^2 + |\mathrm{Im}(w)|^2\Big]^{\frac{n}{2}}} \qquad \Big(\text{by Lemma 3.1}\Big),
\end{aligned} \tag{3.16}$$

in which we used the following relationships

$$\big|\mathrm{Re}(x-w)\big| = \big|x - \mathrm{Re}(w)\big| \ge R \ge R - |\mathrm{Re}(w)| \ \ \text{while}\, x \notin B_n\big(\mathrm{Re}(w), R\big) \tag{3.17}$$

and

$$\big|\mathrm{Re}(x-w)\big| = \big|x - \mathrm{Re}(w)\big| \geq R - |\mathrm{Re}(w)| \quad \text{while} \quad x \notin B_n\big(0, R\big). \tag{3.18}$$

Obviously, (3.16) results in (3.15). □

Proof of (3.9). Let

$$\mathcal{G} = \frac{1}{\mathsf{V}_{n+1}} \int_{B_n(\mathrm{Re}(w),R)} E(x-w)\mathrm{d}\sigma. \tag{3.19}$$

By Lemma 3.2, we just need prove

$$\lim_{R\to+\infty} \mathcal{G} \triangleq \lim_{R\to+\infty} \frac{1}{\mathsf{V}_{n+1}} \int_{B_n(\mathrm{Re}(w),R)} E(x-w)\mathrm{d}\sigma = \begin{cases} \frac{1}{2}, & w \in \mathbb{R}^{n+1}_{+}, \\ -\frac{1}{2}, & w \in \mathbb{R}^{n+1}_{-}. \end{cases} \tag{3.20}$$

Let $w = (w_0, w_1, \ldots, w_n) \in \mathbb{R}^{n+1}_{\pm}$. We divide $\mathcal{G}$ into two parts.

$$\begin{aligned} \mathcal{G} &= \frac{1}{\mathsf{V}_{n+1}} \int_{B_n(\mathrm{Re}(w),R)} \frac{w_n e_n}{[|\mathrm{Re}(x-w)|^2 + w_n^2]^{\frac{n+1}{2}}} \mathrm{d}\sigma \\ &\quad + \frac{1}{\mathsf{V}_{n+1}} \int_{B_n(\mathrm{Re}(w),R)} \frac{\overline{\mathrm{Re}(x-w)}}{[|\mathrm{Re}(x-w)|^2 + w_n^2]^{\frac{n+1}{2}}} \mathrm{d}\sigma \\ &\triangleq \mathcal{G}_1 + \mathcal{G}_2. \end{aligned} \tag{3.21}$$

Obviously, $\lim\limits_{R\to+\infty} \mathcal{G}_1$ is a Poisson integral in the n-dimensional Euclidean space [26], *i.e.*,

$$\lim_{R\to+\infty} \mathcal{G}_1 = \frac{1}{\mathsf{V}_{n+1}} \int_{\mathrm{R}^n} \frac{w_n}{[|\mathrm{Re}(x-w)|^2 + w_n^2]^{\frac{n+1}{2}}} \mathrm{d}x_0 \ldots \mathrm{d}x_{n-1} = \pm\frac{1}{2}, \tag{3.22}$$

where $w \in \mathbb{R}^{n+1}_{\pm}$.

By the coordinate transformation $r = |\mathrm{Re}(x-w)|$, $\xi = \overline{\mathrm{Re}(x-w)}/r$ in R^{n+1}_0, we easily get

$$\begin{aligned} &\mathcal{G}_2 \\ &= \frac{1}{\mathsf{V}_{n+1}} \int_{B_n(\mathrm{Re}(w),R)} \frac{\overline{\mathrm{Re}(x-w)}}{[|\mathrm{Re}(x-w)|^2 + w_n^2]^{\frac{n+1}{2}}} \mathrm{d}\sigma \\ &= \frac{-e_n}{\mathsf{V}_{n+1}} \int_0^R \frac{r^n}{(r^2+w_n^2)^{(n+1)/2}} \left[\int_{|\xi|=1} \xi \mathrm{d}\xi_0 \ldots \mathrm{d}\xi_{n-2} \right] \mathrm{d}r \ \left(\xi \in \mathbb{R}^{n+1}_0\right) \\ &= \frac{-e_n}{\mathsf{V}_{n+1}} \int_0^R \frac{r^n}{(r^2+w_n^2)^{(n+1)/2}} \left[\int_{|\varsigma|=1, \varsigma\in\mathbb{R}^n} \mathrm{d}\sigma_{n-1}(\varsigma) \right] \mathrm{d}r \ \left(\text{by Remark 3.1}\right) \\ &= 0, \end{aligned} \tag{3.23}$$

in which we use the fact that, by the Cauchy's Theorem [2,9], the above inner integral vanishes, *i.e.*,

$$\int_{|\varsigma|=1} \mathrm{d}\sigma_{n-1}(\varsigma) = 0 \ \left(\varsigma \in \mathbb{R}^n\right). \tag{3.24}$$

(3.21), (3.22) and (3.23) result in (3.20). □

Example 3.2. If $f, g \in \widehat{H}^{\mu}\left(\mathbb{R}_0^{n+1}\right)$, then

$$\Big(\mathcal{U}\,[f]\Big)(w) = \frac{1}{\mathsf{V}_{n+1}} \int_{\mathbb{R}_0^{n+1}} E(x-w)\,\mathrm{d}\sigma\,\big[f(x)-f(\infty)\big], \quad w \in \mathbb{R}_{\pm}^{n+1} \tag{3.25}$$

and

$$\Big([g]\,\mathcal{U}\Big)(w) = \frac{1}{\mathsf{V}_{n+1}} \int_{\mathbb{R}_0^{n+1}} \big[g(x)-g(\infty)\big]\,\mathrm{d}\sigma\,E(x-w), \quad w \in \mathbb{R}_{\pm}^{n+1} \tag{3.26}$$

exist.

In fact, by (2.8) and (2.32), we have, for example, when $|x| > 2|w|$

$$\Big|E(x-w)(-e_n)\big[f(x)-f(\infty)\big]\Big| = \Big|E(x-w)\Big|\,\Big|(-e_n)\Big|\,\Big|\big[f(x)-f(\infty)\big]\Big| \leq \frac{2M}{|x|^{n+\mu}}. \tag{3.27}$$

So, the integral (3.25) exists (absolutely converges) and it is an ordinary (improper) integral. Similarly, the integral (3.26) exists as an ordinary (improper) integral.

Theorem 3.1. *If* $f \in \widehat{H}^{\mu}\left(\mathbb{R}_0^{n+1}\right)$, *then*

$$\big(\mathcal{S}[f]\big)(w) = \frac{1}{\mathsf{V}_{n+1}} \int_{\mathbb{R}_0^{n+1}} E(x-w)\,\mathrm{d}\sigma\,f(x), \quad w \in \mathbb{R}_{\pm}^{n+1} \tag{3.28}$$

and

$$\big([f]\mathcal{S}\big)(w) = \frac{1}{\mathsf{V}_{n+1}} \int_{\mathbb{R}_0^{n+1}} f(x)\,\mathrm{d}\sigma\,E(x-w), \quad w \in \mathbb{R}_{\pm}^{n+1} \tag{3.29}$$

exist.

(3.28) and (3.29) are called, respectively, the left Cauchy type integral and the right Cauchy type integral with the density f on the hyperplane $\mathbb{R}_0^{n+1}$.

Proof. Now, we see that, by Example 3.1 and Example 3.2,

$$\begin{aligned}
&\big(\mathcal{S}[f]\big)(w)\\
&= \frac{1}{\mathsf{V}_{n+1}} \int_{\mathbb{R}_0^{n+1}} E(x-w)\mathrm{d}\sigma f(\infty) + \frac{1}{\mathsf{V}_{n+1}} \int_{\mathbb{R}_0^{n+1}} E(x-w)\mathrm{d}\sigma[f(x)-f(\infty)]\\
&= \pm\frac{1}{2} f(\infty) + \Big(\mathcal{U}[f]\Big)(w), \quad w \in \mathbb{R}_{\pm}^{n+1}.
\end{aligned} \tag{3.30}$$

Similarly, (3.29) is also convergent. □

Remark 3.3. In fact, if

$$\begin{cases} f \in C\Big(\mathbb{R}_0^{n+1}, C\left(V_n\right)\Big),\\ \Big|f(x)\Big| \leq \dfrac{M}{|x|^{\mu}} \text{ near } x=\infty \ \Big(M \mathit{is\,a\,constant}\Big), \end{cases} \tag{3.31}$$

denoted as

$$f \in C\Big(\mathbb{R}_0^{n+1}\Big) \bigcap O^{-\mu}(\infty), \tag{3.32}$$

then both $\big(\mathcal{S}[f]\big)(w)$ and $\big([f]\mathcal{S}\big)(w)$ given, respectively, in (3.28) and (3.29) exist and they are just ordinary (improper) integrals.

3.3. Cauchy Type Integrals on General Unbounded Domains

We also need the Cauchy type integral on general unbounded domains in the discussion for Riemann boundary value problems hereinbelow.

Definition 3.2. Suppose that Ω is a unbounded subdomain of $\mathbb{R}_0^{n+1}$, $f, g \in C\big(\Omega, C\left(V_n\right)\big)$. If

$$\int_\Omega g(x)\,\mathrm{d}\sigma f(x) = \lim_{R\to+\infty}\int_{B_n(0,R)\cap\Omega} g(x)\,\mathrm{d}\sigma\, f(x), \tag{3.33}$$

exists, then we say that it is convergent and the principal value integral on Ω at ∞.

For example, the Cauchy type integral on outside of a ball is an usual one, *i.e.*, for fixed $t \in \mathbb{R}_0^{n+1}$ and $r \geq 0$,

$$\Big(\mathcal{S}_{|x-t|\geq r}[f]\Big)(w) = \frac{1}{V_{n+1}}\int_{\mathbb{R}_0^{n+1}\setminus B_n(t,r)} E(x-w)\,\mathrm{d}\sigma\, f(x),\ \ w\in\mathbb{R}_\pm^{n+1}\cup B_n(t,r) \tag{3.34}$$

defined by

$$\Big(\mathcal{S}_{|x-t|\geq r}[f]\Big)(w) = \lim_{R\to+\infty}\frac{1}{V_{n+1}}\int_{\overline{B}_n(0,R)\setminus B_n(t,r)} E(x-w)\mathrm{d}\sigma f(x), \quad w\in\mathbb{R}_\pm^{n+1}\cup B_n(t,r). \tag{3.35}$$

In exactly the same way that we prove Lemma 3.2, we have the following more general result.

Lemma 3.3. *Let Ω be a unbounded subdomain of $\mathbb{R}_0^{n+1}$, $f \in C\left(\Omega, C\left(V_n\right)\right)$ and $\big|f(x)\big| \leq M$ for $x\in\Omega$. Then*

$$\lim_{R\to+\infty}\left[\int_{B_n(0,R)\cap\Omega} E(x-w)\mathrm{d}\sigma f(x) - \int_{B_n(\mathrm{Re}(w),R)\cap\Omega} E(x-w)\mathrm{d}\sigma f(x)\right] = 0,\quad w\in\mathbb{R}_\pm^{n+1}. \tag{3.36}$$

Example 3.3. Let $\Omega = \mathbb{R}_0^{n+1}\setminus B_n(\mathrm{Re}(w), r)$. As before, we may prove that, if $f \in C\big(\Omega, C(V_{n+1})\big)$ and f satisfies the †-Hölder condition at ∞ then the Cauchy type integral $\mathcal{S}_\Omega[f]$ on Ω exists. In particular,

$$\Big(\mathcal{S}_{|x-\mathrm{Re}(w)|\geq r}[1]\Big)(w) = \frac{1}{\mathsf{V}_{n+1}}\int_{\mathbb{R}_0^{n+1}\setminus B_n(\mathrm{Re}(w),r)} E(x-w)\mathrm{d}\sigma,\ w\in\mathbb{R}_\pm^{n+1} \tag{3.37}$$

exists and

$$\left|\Big(\mathcal{S}_{|x-\mathrm{Re}(w)|\geq r}[1]\Big)(w)\right| \leq \frac{1}{2}\ \Big(r\geq 0\Big),\quad w\in\mathbb{R}_\pm^{n+1}. \tag{3.38}$$

In fact, in this case we still have, similar to (3.21),

$$\mathcal{G}(R) = \frac{1}{\mathsf{V}_{n+1}}\int_{r\leq|x-\mathrm{Re}(w)|\leq R} E(x-w)\mathrm{d}\sigma \triangleq \mathcal{G}_1(R) + \mathcal{G}_2(R), \tag{3.39}$$

where

$$\mathcal{G}_2(R) = \frac{1}{\mathsf{V}_{n+1}} \int_{r\leq|x-\mathrm{Re}(w)|\leq R} \frac{\overline{\mathrm{Re}(x-w)}}{\left[|\mathrm{Re}(x-w)|^2+w_n^2\right]^{\frac{n+1}{2}}} \mathrm{d}\sigma = 0, \tag{3.40}$$

and since $\mathcal{G}_1(R)$ is monotonous for R there are the limits

$$\lim_{R\to+\infty} \Big[\pm \mathcal{G}_1(R)\Big] = \frac{1}{\mathsf{V}_{n+1}} \int_{r\leq|x-\mathrm{Re}(w)|} \frac{\pm w_n}{\left[|\mathrm{Re}(x-w)|^2+w_n^2\right]^{\frac{n+1}{2}}} \mathrm{d}x_0 \ldots \mathrm{d}x_{n-1} \leq \frac{1}{2} \tag{3.41}$$

by (3.22).

Thus, by Lemma 3.3 we get (3.38).

In (3.15) we improve R as a function $R = R(w)$ and rewrite Δ in (3.16) as

$$\Delta(w) = \int_{B_n(0,R(w))} E(x-w)\mathrm{d}\sigma f(x) - \int_{B_n(\mathrm{Re}(w),R(w))} E(x-w)\mathrm{d}\sigma f(x), w \in \mathbb{R}^{n+1}_{\pm}. \tag{3.42}$$

Then another variant of Lemma 3.1 is more interesting.

Lemma 3.4. *Let $\Delta(w)$ be given in (3.42) with $R(w) \geq 2|w|$ and $f \in C\left(\mathbb{R}^{n+1}_0,\right.$ $\left.C\left(V_n\right)\right)$ with $\left\|f\right\|_{\mathbb{R}^{n+1}_0} < M$. Then*

$$\lim_{w\in\mathbb{R}^{n+1}_{\pm},\, w\to+\infty} \Delta(w) = 0 \tag{3.43}$$

and $\Delta(w)$ is bounded.

Proof. When $R = R(w) \geq 2|w| > 2|\mathrm{Re}(w)|$, (3.13) and (3.16) still hold. So, the proof of (3.43) is exactly the same as the proof of (3.15). And from (3.16) with $R = R(w)$ we get

$$\left|\Delta(w)\right| \leq 2M\mathcal{V}_n \left[\frac{R(w)}{R(w)-|\mathrm{Re}(w)|}\right]^n \leq 2^{n+1}M\mathcal{V}_n \triangleq C, \tag{3.44}$$

where C is a constant independent of w. □

Example 3.4. Both the integrals

$$\left(\mathcal{S}_{|x|\geq\frac{1}{2}|w|}[1]\right)(w) = \frac{1}{\mathsf{V}_{n+1}} \int_{|x|\geq\frac{1}{2}|w|} E(x-w)\mathrm{d}\,\sigma, \quad w \in \mathbb{R}^{n+1}_{\pm} \tag{3.45}$$

and

$$\left(\mathcal{S}_{|x|\leq\frac{1}{2}|w|}[1]\right)(w) = \frac{1}{\mathsf{V}_{n+1}} \int_{|x|\leq\frac{1}{2}|w|} E(x-w)\mathrm{d}\,\sigma, \quad w \in \mathbb{R}^{n+1}_{\pm} \tag{3.46}$$

are bounded.

(3.46) is easy to obtain. In fact,

$$\begin{aligned}
&\left|\left(\mathcal{S}_{|x|\le\frac{1}{2}|w|}[1]\right)(w)\right| \quad \left(w\in\mathbb{R}^{n+1}_{\pm}\right)\\
&\le \frac{1}{\mathsf{V}_{n+1}}\int_{|x|\le\frac{1}{2}|w|}\frac{1}{|x-w|^n}\mathrm{d}x_0\ldots\mathrm{d}x_{n-1}\\
&\le \frac{1}{\mathsf{V}_{n+1}}\int_{|x|\le\frac{1}{2}|w|}\left[\frac{|w|}{2}\right]^{-n}\mathrm{d}x_0\ldots\mathrm{d}x_{n-1}\left(\text{by }|x-w|\ge|w|-|x|\ge\frac{1}{2}|w|\right)\\
&\le \frac{\mathcal{V}_n}{\mathsf{V}_{n+1}}\quad\left(\mathcal{V}_n\text{is the volume of unit ball in}\mathbb{R}^{n+1}_0\right).
\end{aligned} \tag{3.47}$$

(3.47) and Example 3.1 result in

$$\left|\left(\mathcal{S}_{|x|\ge\frac{1}{2}|w|}[1]\right)(w)\right|\le\left|\left(\mathcal{S}_{|x|\le\frac{1}{2}|w|}[1]\right)(w)\right|+\left|\left(\mathcal{S}[1]\right)(w)\right|\le\frac{\mathcal{V}_n}{\mathsf{V}_{n+1}}+\frac{1}{2}. \tag{3.48}$$

3.4. Regularity of the Cauchy Type Integrals

Sometimes, (3.28) and (3.29) are treated as Cauchy principal value integrals at ∞ with a parameter variable w. For this kind of integrals we need to introduce the concept of uniform convergence below.

Definition 3.3. Let Σ be a domain in $\mathbb{R}^{n+1}$, $f,g\in C\left(\mathbb{R}^{n+1}_0\times\Sigma, C(V_n)\right)$ and $w\in\Sigma$. If

$$\lim_{R\to+\infty}\int_{|x|\le R}g(x,\varsigma)\mathrm{d}\sigma f(x,\varsigma)=\int_{\mathbb{R}^{n+1}_0}g(x,\varsigma)\mathrm{d}\sigma f(x,\varsigma),\ \ \varsigma\in B_{n+1}(w,r)\subset\Sigma \tag{3.49}$$

is uniform for some $r>0$, then we say this Cauchy type integral with the parameter variable ς to be locally uniformly convergent at w. If it is locally uniformly convergent at each point in Σ then we say that it is locally uniformly convergent on Σ.

Remark 3.4. In other words, (3.49) is locally uniformly convergent at w if and only if for any $\epsilon>0$ there is $R>0$ such that, for some $r>0$ and any $\varsigma\in B_{n+1}(w,r)$,

$$\begin{aligned}
\left|\int_{|x|\ge R}g(x,\varsigma)\mathrm{d}\sigma f(x,\varsigma)\right| &= \left|\int_{\mathbb{R}^{n+1}_0}g(x,\varsigma)\mathrm{d}\sigma f(x,\varsigma)-\int_{|x|\le R}g(x,\varsigma)\mathrm{d}\sigma f(x,\varsigma)\right|\\
&<\epsilon.
\end{aligned} \tag{3.50}$$

Remark 3.5. $\int_{\mathbb{R}^{n+1}_0}\mathcal{F}(x,w)\mathrm{d}\sigma$ is convergent (locally uniformly convergent) at w if and only if all integrals $\int_{\mathbb{R}^{n+1}_0}\mathcal{F}_A(x,w)\mathrm{d}\sigma$ are convergent (locally uniformly convergent) at w, where $\mathcal{F}_A$ are the e_A-component of $\mathcal{F}$.

Now we consider the regularity of the Cauchy type integral. As we know, it has not been seriously treated, but many authors often give tacit consent to it in their articles due to its fundamental nature. To prove the regularity, we need the Leibniz rule for the Cauchy type integral at ∞. Firstly, we start to consider the Leibniz rule for the simplest case of $\mathbb{R}$-valued hypercomplex functions.

Let Σ be an open set in $\mathbb{R}^{n+1}$, Ω be its subdomain and $\overline{\Omega} \subset \Sigma$. $\phi(x, w)$ is an $\mathbb{R}$-valued hypercomplex function defined on $\mathbb{R}_0^{n+1} \times \Sigma$.

Lemma 3.5. *(Leibniz rule for integrals of $\mathbb{R}$-valued functions) Assume that*

(1) $\phi, \dfrac{\partial \phi}{\partial w_\ell} \Big(\ell = 0, 1, \ldots, n\Big) \in C\Big(\mathbb{R}_0^{n+1} \times \Omega, \mathbb{R}\Big)$;

(2) $\mathbb{E}(w) = \displaystyle\int_{\mathbb{R}_0^{n+1}} \phi(x, w)\, \mathrm{d}S$ *converges*;

(3) $\displaystyle\int_{\mathbb{R}_0^{n+1}} \frac{\partial \phi}{\partial w_\ell}(x, w)\mathrm{d}S \; \Big(\ell = 0, 1, \ldots, n\Big)$ *are locally uniformly convergent on* Ω. *Then*

$$\frac{\partial}{\partial w_\ell} \int_{\mathbb{R}_0^{n+1}} \phi(x, w)\mathrm{d}S = \int_{\mathbb{R}_0^{n+1}} \frac{\partial \phi}{\partial w_\ell}(x, w)\mathrm{d}S \in C\Big(\Omega, \mathbb{R}\Big) \; \Big(\ell = 0, 1, \ldots, n\Big). \tag{3.51}$$

Proof. Without any loss of generality, we suppose $\Omega = B_{n+1}(w^*, r)$. Firstly, for any $\epsilon > 0$ there is $R > 0$ such that, by the uniform integrability in Condition (3) and Remark 3.4,

$$\left| \int_{|x| \geq R} \frac{\partial \phi}{\partial w_\ell}(x, \varsigma)\mathrm{d}S \right| < \epsilon \quad \text{for} \quad \text{any} \quad \varsigma \in \Omega. \tag{3.52}$$

Noting that if $0 < |\lambda| < r$ then

$$\frac{\mathbb{E}(w) - \mathbb{E}(w + \lambda \varepsilon_\ell)}{\lambda} = \int_{\mathbb{R}_0^{n+1}} \frac{\partial \phi}{\partial w_\ell}\Big(x, w + \theta \lambda \varepsilon_\ell\Big)\mathrm{d}S \quad \Big(0 < \theta < 1\Big), \tag{3.53}$$

where $\varepsilon_\ell = (\delta_{0,\ell}, \delta_{1,\ell}, \ldots, \delta_{k,\ell}, \ldots, \delta_{n,\ell})$ with the Kronecker symbol $\delta_{k,\ell}$, thus

$$\begin{aligned} \boldsymbol{\Delta} &= \left| \frac{\mathbb{E}(w) - \mathbb{E}(w + \lambda \varepsilon_\ell)}{\lambda} - \int_{\mathbb{R}_0^{n+1}} \frac{\partial \phi}{\partial w_\ell}(x, w)\mathrm{d}S \right| \\ &\leq \left| \int_{\mathbb{R}_0^{n+1}} \frac{\partial \phi}{\partial w_\ell}\Big(x, w + \theta \lambda \varepsilon_\ell\Big)\mathrm{d}S - \int_{\mathbb{R}_0^{n+1}} \frac{\partial \phi}{\partial w_\ell}(x, w)\mathrm{d}S \right| \\ &\leq 2\epsilon + \left| \int_{|x| \leq R} \frac{\partial \phi}{\partial w_\ell}\Big(x, w + \theta \lambda \varepsilon_\ell\Big)\mathrm{d}S - \int_{|x| \leq R} \frac{\partial \phi}{\partial w_\ell}(x, w)\mathrm{d}S \right| \\ &\leq 2\epsilon + M\omega\left(\frac{\partial \phi}{\partial w_\ell}, \big|\lambda\big|\right), \end{aligned} \tag{3.54}$$

where M is a constant and $\omega\left(\dfrac{\partial \phi}{\partial w_\ell}, \big|\lambda\big|\right)$ is the modulus of continuity of $\dfrac{\partial \phi}{\partial w_\ell}$ on $\overline{B_n(0, R)} \times \overline{\Omega}$, *i.e.*,

$$\begin{aligned} &\omega\left(\frac{\partial \phi}{\partial w_\ell}, h\right) \\ &\quad = \sup\left\{ \left| \frac{\partial \phi}{\partial w_\ell}(x, \varsigma) - \frac{\partial \phi}{\partial w_\ell}(t, \xi) \right|, |x - t|, |\varsigma - \xi| \leq h, x, t \in \overline{B_n(0, R)}, \varsigma, \xi \in \overline{\Omega} \right\}. \end{aligned} \tag{3.55}$$

By using $\dfrac{\partial\phi}{\partial w_\ell} \in C\left(\overline{B_n(0,R)} \times \overline{\Omega}, \mathbb{R}\right)$ and (3.54), we get

$$\limsup_{\lambda\to 0} \mathbf{\Delta} \le 2\epsilon \tag{3.56}$$

which results in, by the arbitrariness of ϵ,

$$\lim_{\lambda\to 0} \mathbf{\Delta} = 0, \tag{3.57}$$

i.e.,

$$\frac{\partial\mathbb{E}}{\partial w_\ell}(w) = \int_{\mathbb{R}_0^{n+1}} \frac{\partial\phi}{\partial w_\ell}(x,w)\mathrm{d}S, \quad \ell = 0,1,\ldots,n. \tag{3.58}$$

By the same way, we have

$$\begin{aligned}\delta &= \left|\frac{\partial\mathbb{E}}{\partial w_\ell}(\varsigma) - \frac{\partial\mathbb{E}}{\partial w_\ell}(w)\right| \quad \left(\varsigma, w \in \Omega\right)\\ &= \left|\int_{\mathbb{R}_0^{n+1}} \frac{\partial\phi}{\partial w_\ell}(x,\varsigma)\mathrm{d}S - \int_{\mathbb{R}_0^{n+1}} \frac{\partial\phi}{\partial w_\ell}(x,w)\mathrm{d}S\right|\\ &\le 2\epsilon + \left|\int_{|x|\le R} \frac{\partial\phi}{\partial w_\ell}(x,\varsigma)\mathrm{d}S - \int_{|x|\le R} \frac{\partial\phi}{\partial w_\ell}(x,w)\mathrm{d}S\right|\end{aligned} \tag{3.59}$$

which implies

$$\limsup_{\varsigma\to w} \delta \le 2\epsilon, \quad i.e., \quad \lim_{\varsigma\to w} \delta = 0, \tag{3.60}$$

so,

$$\frac{\partial\mathbb{E}}{\partial w_\ell} \in C(\Omega,\mathbb{R}), \ \ell = 0,1,\ldots,n. \tag{3.61}$$

(3.58) and (3.61) result in (3.51). □

Theorem 3.2. (Leibniz's rule for the Cauchy principal integrals) *Let Σ be an open set in $\mathbb{R}^{n+1}$, Ω be its subdomain and $\overline{\Omega} \subset \Sigma$. If*

(1) $f \in C\left(\mathbb{R}_0^{n+1}, C(V_n)\right)$, ϕ, $\dfrac{\partial\phi}{\partial w_\ell}\left(\ell = 0,1,\ldots,n\right) \in C\left(\mathbb{R}_0^{n+1} \times \Omega, C(V_n)\right)$,

(2) $\Phi^B(w) = \int_{\mathbb{R}_0^{n+1}} \phi(x,w)\,\mathrm{d}\sigma f_B(x)\ \left(B \in \mathcal{PN}\right)$ *exist, where f_B is the e_B-component of f,*

(3) $\Upsilon^B(w) = \int_{\mathbb{R}_0^{n+1}} \dfrac{\partial\phi}{\partial w_\ell}(x,w)\,\mathrm{d}\sigma f_B(x)\ \left(B \in \mathcal{PN}\right)$ *are locally uniformly convergent on Ω. Then*

$$\left(D\left[\int_{\mathbb{R}_0^{n+1}} \phi(x,w)\mathrm{d}\sigma f(x)\right]\right)(w) = \int_{\mathbb{R}_0^{n+1}} \left(D_w[\phi]\right)(x,w)\mathrm{d}\sigma f(x),\ w \in \Omega, \tag{3.62}$$

and

$$\left(\left[\int_{\mathbb{R}_0^{n+1}} f(x)\mathrm{d}\sigma\phi(x,w)\right]D\right)(w) = \int_{\mathbb{R}_0^{n+1}} f(x)\mathrm{d}\sigma\left([\phi]D_w\right)(x,w),\ w \in \Omega, \tag{3.63}$$

where D_w is the Dirac operator acting to the second variable w of the function $\phi(x,w)$.

Remark 3.6. Obviously, (1) is equivalent to that

$$f_B \in C\Big(\mathbb{R}_0^{n+1}, \mathbb{R}\Big), \quad \phi_A, \frac{\partial \phi_A}{\partial w_\ell}\Big(\ell = 0, 1, \ldots, n\Big) \in C\Big(\mathbb{R}_0^{n+1} \times \Omega, \mathbb{R}\Big). \tag{3.64}$$

Moreover, it is easy to see, by (2.1),

$$\left[\sum_A e_A \lambda_A\right] e_B = \sum_A e_C (-1)^{\#(A\cap B)} (-1)^{P(A,B)} \lambda_A. \tag{3.65}$$

where $C = A\boldsymbol{\Delta}B$. By Remark 3.5, we know that the condition (2) is equivalent to that

$$\int_{\mathbb{R}_0^{n+1}} \phi_A(x,w) f_B(x)\, \mathrm{d}S \ \big(A, B \in \mathcal{P}N\big) \text{ are convergent.} \tag{3.66}$$

Finally, the condition (3) is equivalent to that

$$\int_{\mathbb{R}_0^{n+1}} \frac{\partial \phi_A}{\partial w_\ell}(x,w) f_B(x) \mathrm{d}S \ \big(A, B \in \mathcal{P}N\big) \tag{3.67}$$

are locally uniformly convergent on Ω.

Proof of Theorem 3.2. We only prove (3.62). (3.63) may be similarly proved. Without loss of generality, taking $B_{n+1}(w,r) = \{\varsigma, |\varsigma - w| < r\} \subset \Omega$. we only consider $\Omega = B_{n+1}(w,r)$ for (3.62). Denote

$$\Phi(w) = \int_{\mathbb{R}_0^{n+1}} \phi(x,w) \mathrm{d}\sigma f(x), \ \ w \in \Omega, \tag{3.68}$$

$$\Upsilon(w) = \int_{\mathbb{R}_0^{n+1}} \Big(D_w[\phi]\Big)(x,w) \mathrm{d}\sigma f(x), \ w \in \Omega. \tag{3.69}$$

We have, by Remark 2.4,

$$\Upsilon(w) = \sum_{\ell=0}^{n} \sum_A \sum_B \Big[e_j\, e_A\, (-e_n)\, e_B\Big] \left[\int_{\mathbb{R}_0^{n+1}} \frac{\partial}{\partial w_\ell} \phi_A(x,w) f_B(x) \mathrm{d}S\right]. \tag{3.70}$$

By Remark 3.6 and Lemma 3.5, we know

$$\frac{\partial}{\partial w_\ell}\left[\int_{\mathbb{R}_0^{n+1}} \phi_A(x,w) f_B(x) \mathrm{d}S\right] = \int_{\mathbb{R}_0^{n+1}} \frac{\partial \phi_A}{\partial w_\ell}(x,w) f_B(x) \mathrm{d}S. \tag{3.71}$$

Thus, by Remark 2.4,

$$\big(D[\Phi]\big)(w) = \sum_{\ell=0}^{n} \sum_A \sum_B \Big[e_j\, e_A (-e_n)\, e_B\Big] \frac{\partial}{\partial w_\ell}\left[\int_{\mathbb{R}_0^{n+1}} \phi_A(x,w) f_B(x) \mathrm{d}S\right] = \Upsilon(w), \tag{3.72}$$

i.e, (3.62) holds. □

Remark 3.7. The Leibniz rule for integrals on finite smooth surfaces is first proved in [8,10]. The proof for the Leibniz rule of the Cauchy principal integral on hyperplane $\mathbb{R}_0^{n+1}$ here is more difficult. With its help, we now state the following main result, which has been used in many articles, for example, in [13,30], but has not been proved.

Theorem 3.3. (Regularity of the Cauchy type integral) *If* $f \in \widehat{H}^{\mu}\left(\mathbb{R}_0^{n+1}\right)$, *then* $\mathcal{S}[f]$ *given in (3.28) is left regular and* $[f]\mathcal{S}$ *given in (3.29) is right regular.*

Proof. Obviously,

$$f \in C\Big(\mathbb{R}_0^{n+1}, C(V_n)\Big), E, \frac{\partial E}{\partial w_\ell}\Big(\ell = 0, 1, \ldots, n\Big) \in C\Big(\mathbb{R}_0^{n+1} \times \mathbb{R}_{\pm}^{n+1}, C(V_n)\Big), \tag{3.73}$$

where

$$\frac{\partial E}{\partial w_\ell}(x, w) = \frac{(n+1)\,(\overline{x-w})\,(x_\ell - w_\ell)}{|x-w|^{n+3}} + \frac{(-1)^{\delta_{0,\ell}} e_\ell}{|x-w|^{n+1}},\ \ \ell = 0, 1, \ldots, n. \tag{3.74}$$

Noting $f_B \in \widehat{H}^{\mu}\left(\mathbb{R}_0^{n+1}\right)$ since $f \in \widehat{H}^{\mu}\left(\mathbb{R}_0^{n+1}\right)$, so

$$\mathcal{S}[f_B]\ (B \in \mathcal{P}N)\ \text{areconvergent.} \tag{3.75}$$

By (2.8) and (3.74), we know

$$\begin{aligned}&\left|\frac{\partial E}{\partial w_\ell}(x, w)(-e_n) f_B(x)\right| \\ &\quad = \left|\frac{\partial E}{\partial w_\ell}(x, w)\right|\,\Big|f_B\Big| \le \frac{(n+2)\|f\|_\infty}{|x-w|^{n+1}} \le \frac{(n+2)\|f\|_\infty}{|x|^{n+1}} \frac{1}{1 - |x^{-1}||w|},\end{aligned} \tag{3.76}$$

which results in

$$\left|\frac{\partial E}{\partial w_\ell}(x, w)(-e_n) f_B\right| \le \frac{2(n+2)\|f\|_\infty}{|x|^{n+1}}\ \text{ while }\ |w| < R < \frac{1}{2}|x|, \tag{3.77}$$

so,

$$\int_{\mathbb{R}_0^{n+1}} \frac{\partial E}{\partial w_\ell}(x, w) \mathrm{d}\sigma f_B(x)\ (B \in \mathcal{P}N)\ \text{are locally uniformly convergent on}\ \mathbb{R}_{\pm}^{n+1}. \tag{3.78}$$

By (3.71), (3.75), (3.78), Theorem 3.2 and Example 2.3, we get Theorem 3.3. □

3.5. Singular Integrals

To discuss the boundary behavior of the Cauchy type integral, we need introduce another kind of Cauchy principal value integrals which has also the unaided significance.

We consider the integrals

$$\begin{aligned}\Big(\mathcal{S}[f]\Big)(t) &= \frac{1}{\mathsf{V}_{n+1}} \int_{\mathbb{R}_0^{n+1}} E(x, t)\, \mathrm{d}\sigma\, f(x) \\ &= \frac{1}{\mathsf{V}_{n+1}} \int_{\mathbb{R}_0^{n+1}} \frac{\overline{x-t}}{|x-t|^{n+1}}\, \mathrm{d}\sigma\, f(x),\quad t \in \mathbb{R}_0^{n+1}\end{aligned} \tag{3.79}$$

and

$$\begin{aligned}\Big([f]\mathcal{S}\Big)(t) &= \frac{1}{\mathsf{V}_{n+1}} \int_{\mathbb{R}_0^{n+1}} f(x)\ \mathrm{d}\sigma\, E(x, t) \\ &= \frac{1}{\mathsf{V}_{n+1}} \int_{\mathbb{R}_0^{n+1}} f(x)\ \mathrm{d}\sigma\, \frac{\overline{x-t}}{|x-t|^{n+1}},\quad t \in \mathbb{R}_0^{n+1}.\end{aligned} \tag{3.80}$$

It is evident that such integrals are divergent in general.

Nevertheless,

$$\lim_{R\to+\infty,\,\delta\to 0^+}\Big(\mathcal{S}_{\delta,R}[f]\Big)(t) \triangleq \lim_{R\to+\infty,\,\delta\to 0^+}\frac{1}{\bigvee_{n+1}}\int_{\delta\le|x-t|,\,|x|\le R} E(x-t)\mathrm{d}\sigma\, f(x) \tag{3.81}$$

and

$$\lim_{R\to+\infty,\delta\to 0^+}\Big([f]\mathcal{S}_{\delta,R}\Big)(t) \triangleq \lim_{R\to+\infty,\delta\to 0^+}\frac{1}{\bigvee_{n+1}}\int_{\delta\le|x-t|,|x|\le R} f(x)\mathrm{d}\sigma\, E(x-t) \tag{3.82}$$

maybe exist.

Definition 3.4. If (3.81) and (3.82) exist, we call, respectively, (3.79) and (3.80) the left Cauchy principal value integral and the right Cauchy principal value integral at points both ∞ and t, or simply singular integrals.

Example 3.5. If $f(x)\equiv 1$, then $\big(\mathcal{S}[1]\big)(t)$ given in (3.79) and $\big([1]\mathcal{S}\big)(t)$ given in (3.80) exist. Specifically,

$$\big(\mathcal{S}[1]\big)(t) = \frac{1}{\bigvee_{n+1}}\int_{\mathbb{R}_0^{n+1}} E(x-t)\,\mathrm{d}\sigma = 0, \quad t\in\mathbb{R}_0^{n+1}, \tag{3.83}$$

and

$$\big([1]\mathcal{S}\big)(t) = \frac{1}{\bigvee_{n+1}}\int_{\mathbb{R}_0^{n+1}} \mathrm{d}\sigma\, E(x-t) = 0, \quad t\in\mathbb{R}_0^{n+1}. \tag{3.84}$$

To show this example, we need the following lemma, which and its proof are quite similar to Lemma 3.2.

Lemma 3.6. ([13]) *Let* $\big|f(x)\big| \le M$ *for* $x\in B_n(0,R)\bigcup B_n(t,R)$. *Assume* $t\in\mathbb{R}_0^{n+1}$ *and the positive real* R *is large enough, then*

$$\lim_{R\to+\infty}\left[\int_{\delta\le|x-t|,|x|\le R} E(x-t)\,\mathrm{d}\sigma f(x) - \int_{\delta\le|x-t|\le R} E(x-t)\,\mathrm{d}\sigma f(x)\right] = 0, t\in\mathbb{R}_0^{n+1}. \tag{3.85}$$

Proof. We see

$$\begin{aligned}
&\left|\int_{\delta\le|x-t|,\,|x|\le R} E(x-t)\mathrm{d}\sigma f(x) - \int_{\delta\le|x-t|\le R} E(x-t)\mathrm{d}\sigma f(x)\right| \\
&\quad\le M\int_{\left[B_n(0,R)\right]\boldsymbol{\Delta}\left[B_n(t,R)\right]} \Big|E(x-t)\Big|\mathrm{d}S \\
&\quad\le M\int_{\left[B_n(0,R)\right]\boldsymbol{\Delta}\left[B_n(t,R)\right]} \frac{1}{|x-t|^n}\mathrm{d}S \\
&\quad\le \frac{2MV_n\left[R^n - \left(R-\frac{1}{2}|t|\right)^n\right]}{\left[R-|t|\right]^n}\ \Big(\text{ by Lemma 3.1, (3.17) and (3.18)}\Big),
\end{aligned} \tag{3.86}$$

which results in (3.85). ∎

Proof of Example 3.5. By (3.85) and the spherical transformation $x-t=rw$, $r=|x-t|$, we have

$$\begin{aligned}&\big(\mathcal{S}[1]\big)(t)\\&=-\frac{1}{\mathrm{V}_{n+1}}\lim_{\delta\to0^+,\,R\to+\infty}\int_\delta^R\int_{|w|=1}\frac{w}{r}\mathrm{d}w\mathrm{d}r\quad\Big(w\in\mathbb{R}_0^{n+1}\Big)\\&=-\frac{1}{\mathrm{V}_{n+1}}\lim_{\delta\to0^+,\,R\to+\infty}\int_\delta^R\frac{1}{r}\left[\int_{|\varsigma|=1}\mathrm{d}\sigma_{n-1}(\varsigma)\right]\mathrm{d}r\quad\Big(\varsigma\in\mathbb{R}^n\Big)\\&=0\quad\Big(\text{by (3.24)}\Big).\end{aligned}\tag{3.87}$$

Similarly, we can obtain (3.84). □

Remark 3.8. In fact, we have proved in (3.87)

$$\int_{\delta\le|x-t|\le R}E(x-t)\mathrm{d}\sigma=0,\quad t\in\mathbb{R}_0^{n+1}.\tag{3.88}$$

So,

$$\int_{|x-t|\le R}E(x-t)\mathrm{d}\sigma\triangleq\lim_{\delta\to0^+}\int_{\delta\le|x-t|\le R}E(x-t)\mathrm{d}\sigma=0,\ \ t\in\mathbb{R}_0^{n+1},\tag{3.89}$$

and

$$\int_{|x-t|\ge\delta}E(x-t)\mathrm{d}\sigma\triangleq\lim_{R\to+\infty}\int_{\delta\le|x-t|,\,|x|\le R}E(x-t)\mathrm{d}\sigma=0,\ t\in\mathbb{R}_0^{n+1}.\tag{3.90}$$

Theorem 3.4. *If $f\in\widehat{H}\left(\mathbb{R}_0^{n+1}\right)$, then the integrals $\big(\mathcal{S}[f]\big)(w)$ and $\big([f]\mathcal{S}\big)(w)$ exist for $w\in\mathbb{R}^{n+1}$.*

Proof. The case of $w\in\mathbb{R}_\pm^{n+1}$ is proved in Theorem 3.1. Now, we only consider the case of $w=t\in\mathbb{R}_0^{n+1}$ for $\big(\mathcal{S}[f]\big)(w)$. Let

$$\begin{aligned}&\big(\mathfrak{U}[f]\big)\\&=\lim_{R\to+\infty,\delta\to0^+}\frac{1}{\mathrm{V}_{n+1}}\int_{\delta\le|x-t|,\,|x|\le R}E(x-t)\,\mathrm{d}\sigma\,[f(x)-f(\infty)],\quad t\in\mathbb{R}_0^{n+1}.\end{aligned}\tag{3.91}$$

Taking r such that $R>r>\delta$, we have

$$\begin{aligned}&\lim_{R\to+\infty,\delta\to0^+}\frac{1}{\mathrm{V}_{n+1}}\int_{\delta\le|x-t|,\,|x|\le R}E(x-t)\,\mathrm{d}\sigma\,[f(x)-f(\infty)]\\&=\lim_{R\to+\infty}\frac{1}{\mathrm{V}_{n+1}}\int_{r\le|x-t|,\,|x|\le R}E(x-t)\,\mathrm{d}\sigma\,[f(x)-f(\infty)]\\&\quad+\lim_{\delta\to0^+}\frac{1}{\mathrm{V}_{n+1}}\int_{\delta\le|x-t|\le r}E(x-t)\,\mathrm{d}\sigma\,[f(x)-f(t)]\quad\Big(\text{by (3.88)}\Big)\\&\triangleq I_1+I_2.\end{aligned}\tag{3.92}$$

Since $f\in\widehat{H}\left(\mathbb{R}_0^{n+1}\right)$, when $t\in\mathbb{R}_0^{n+1}$ we have

$$\Big|E(x-t)\big[f(x)-f(\infty)\big]\Big|=O\Big(|x|^{-n-\mu}\Big)\ \ \textit{as}\ \ |x|\to\infty,\tag{3.93}$$

and

$$\Big|E(x-t)\big[f(x)-f(t)\big]\Big| = O\Big(|x-t|^{-n+\mu}\Big) \ \ as \ \ |x-t| \to 0. \tag{3.94}$$

So, both

$$I_1 = \frac{1}{\bigvee_{n+1}} \int_{r \le |x-t|} E(x-t)\, \mathrm{d}\sigma\, [f(x) - f(\infty)] \tag{3.95}$$

and

$$I_2 = \frac{1}{\bigvee_{n+1}} \int_{|x-t| \le r} E(x-t)\, \mathrm{d}\sigma\, [f(x) - f(t)] \tag{3.96}$$

are the ordinary improper integrals by (3.93) and (3.94). (3.92), (3.95) and (3.96) show $\big(\mathcal{U}[f]\big)(t)$ is meaningful for $t \in \mathbb{R}_0^{n+1}$. Thus, by Example 3.5,

$$\big(\mathcal{S}[f]\big)(t) = \big(\mathfrak{U}[f]\big)(t), \quad t \in \mathbb{R}_0^{n+1} \tag{3.97}$$

exists. □

In the below, we also write directly

$$\Big(\mathfrak{U}[f]\Big)(t) = \frac{1}{\bigvee_{n+1}} \int_{\mathbb{R}_0^{n+1}} E(x-t)\, \mathrm{d}\sigma\, [f(x) - f(\infty)], \quad t \in \mathbb{R}_0^{n+1}. \tag{3.98}$$

The following remark will be used.

Remark 3.9. We point out that, if $f \in H^\mu\left(\mathbb{R}_0^{n+1}\right) \cap f \in O^{-\mu}(\infty)$ then $\big(\mathcal{S}[f]\big)(t)$ and $\big([f]\mathcal{S}\big)(t)$, respectively, given in (3.79) and (3.80) exist. In fact, by (3.89)

$$\begin{aligned} &\big(\mathcal{S}[f]\big)(t) \\ &= \frac{1}{\bigvee_{n+1}} \int_{|x| \ge R} E(x-t)\, \mathrm{d}\sigma\, f(x) + \frac{1}{\bigvee_{n+1}} \int_{|x-t| \le R} E(x-t)\, \mathrm{d}\sigma\, [f(x) - f(t)] \\ &= \frac{1}{\bigvee_{n+1}} \int_{\Omega} E(x-t)\, \mathrm{d}\sigma\, f(x) + \frac{1}{\bigvee_{n+1}} \int_{\mathbb{R}_0^{n+1} \setminus \Omega} E(x-t)\, \mathrm{d}\sigma\, f(x), \end{aligned} \tag{3.99}$$

where all integrals are ordinary (improper) integrals at ∞ or t except the last one is a Cauchy principle value integral at t. For the bounded closed domain $\Omega \subseteq \mathbb{R}_0^{n+1}$, we define the Cauchy principle value integral

$$\frac{1}{\bigvee_{n+1}} \int_{\Omega} E(x-t) \mathrm{d}\sigma f(x) = \lim_{\delta \to 0^+} \frac{1}{\bigvee_{n+1}} \int_{\Omega \setminus \{x,\, |x-t| \le \delta\}} E(x-t)\, \mathrm{d}\sigma f(x),\ t \in \Omega^\circ \tag{3.100}$$

if the right side limits exists, where Ω° is the interior of Ω.

4. Boundary Behavior of Cauchy Type Integrals

In this section, we will discuss the boundary behavior of Cauchy type integrals for both the hyperplane $\mathbb{R}_0^{n+1}$ and the infinity.

4.1. Boundary Behavior of Cauchy Type Integrals for $\mathbb{R}_0^{n+1}$

We consider the integrals

$$\big(\mathfrak{D}[f]\big)(w) = \frac{1}{\mathsf{V}_{n+1}} \int_{\mathbb{R}_0^{n+1}} E(x-w)\,\mathrm{d}\sigma \left[f(x)-f\big(\mathrm{Re}(w)\big)\right], \quad w \in \mathbb{R}^{n+1} \tag{4.1}$$

and

$$\big([f]\mathfrak{D}\big)(w) = \frac{1}{\mathsf{V}_{n+1}} \int_{\mathbb{R}_0^{n+1}} \left[f(x)-f\big(\mathrm{Re}(w)\big)\right] \mathrm{d}\sigma\, E(x-w), \quad w \in \mathbb{R}^{n+1}. \tag{4.2}$$

Then we have the important result stated as follows.

Theorem 4.1. (Hölder continuity of $\mathfrak{D}[f]$ and $[f]\mathfrak{D}$) *If* $f \in \widehat{H}^{\mu}\left(\mathbb{R}_0^{n+1}\right)$ *then*

$$[f]\mathfrak{D},\, \mathfrak{D}[f] \in \begin{cases} H^{\mu}\left(\mathbb{R}^{n+1}\right), \text{ if } \ 0<\mu<1, \\ H^{\nu}\left(\mathbb{R}^{n+1}\right), \text{ if } \ \mu=1 \ (\text{where } \ 0<\nu<1). \end{cases} \tag{4.3}$$

Some technical lemma are needed to prove Theorem 4.1. We first introduce a symbol. Let $\Sigma \subseteq \mathbb{R}^{n+1}$, for an arbitrary point $w \in \mathbb{R}^{n+1}$, we will denote the point on Σ nearest to w by w_Σ (if such points are more than one in number, w_Σ may be any one of them).

Obviously, we have

$$\left|x-w_\Sigma\right| \le 2\left|x-w\right|, \quad x \in \Sigma \subseteq \mathbb{R}^{n+1},\ w \in \mathbb{R}^{n+1}. \tag{4.4}$$

If $f \in H^{\mu}(\Sigma)$, then we also have

$$\left|f(x)-f(w_\Sigma)\right| \le 2^{\mu} M(f)\left|x-w\right|^{\mu}, \quad x \in \Sigma \subseteq \mathbb{R}^{n+1},\ w \in \mathbb{R}^{n+1}, \tag{4.5}$$

where $M(f)$ is the Hölder coefficient of f. In particular, when $\Sigma = \mathbb{R}_0^{n+1}$ then

$$w_{\mathbb{R}_0^{n+1}} = \mathrm{Re}(w), \tag{4.6}$$

$$\left|f\left(v_{\mathbb{R}_0^{n+1}}\right) - f\left(w_{\mathbb{R}_0^{n+1}}\right)\right| \le M(f)\left|v-w\right|^{\mu}, \quad v,w \in \mathbb{R}^{n+1}. \tag{4.7}$$

In the sequel, notations C and M will be used for some constants which may vary from one occurrence to the next.

Lemma 4.1. *Let* $f \in H^{\mu}\left(\mathbb{R}_0^{n+1}\right)$ *and* t *be a fixed point on* $\mathbb{R}_0^{n+1}$. *Then*

$$\left|\int_{\overline{B}_n(t,r)} E(x-w)\,\mathrm{d}\sigma\left[f(x)-f\big(\mathrm{Re}(w)\big)\right]\right| \le M\, r^{\mu}, \quad w \in \mathbb{R}^{n+1}, \tag{4.8}$$

and

$$\left|\int_{\overline{B}_n(t,r)} \left[f(x)-f\big(\mathrm{Re}(w)\big)\right]\mathrm{d}\sigma\, E(x-w)\right| \le M\, r^{\mu}, \quad w \in \mathbb{R}^{n+1}, \tag{4.9}$$

where M *is a constant independent of* t *and* w.

Proof. We may prove the following stronger inequality.

$$\int_{\overline{B}_n(t,r)} \left|E(x-w)\right|\left|\mathrm{d}\sigma\right|\left|f(x)-f\big(\mathrm{Re}(w)\big)\right| \le M\, r^{\mu}, \quad w \in \mathbb{R}^{n+1}. \tag{4.10}$$

By (4.5) and (4.6) we get

$$\int_{\overline{B}_n(t,r)} \Big| E(x-w)\Big|\Big|\mathrm{d}\sigma\Big|\Big|\Big[f(x)-f\big(\mathrm{Re}(w)\big)\Big]\Big| \le M_1\mathfrak{P}(w), \tag{4.11}$$

where

$$\mathfrak{P}(w)=\int_{\overline{B}_n(t,r)} \frac{1}{\big|x-w\big|^{n-\mu}}\,\mathrm{d}S. \tag{4.12}$$

So, we just have to prove the following inequality

$$\mathfrak{P}(w)\le M_2 r^{\mu},\ \ w\in\mathbb{R}^{n+1}. \tag{4.13}$$

By (4.4) with $\Sigma=\overline{B}_n(t,r)$, it is easy to see (only for the case of $n>1$ and $n=1$ is simple) that

$$\begin{aligned}
&\mathfrak{P}(w)\\
&\quad\le \int_{\overline{B}_n(t,r)} \frac{2^{n-\mu}}{\big|x-w_{\Sigma}\big|^{n-\mu}}\,\mathrm{d}S\\
&\quad\le \int_0^{2r}\frac{2^{n-\mu}}{\rho^{1-\mu}}\,\mathrm{d}\rho\int_0^{\pi}\sin^{n-2}\varphi_1\mathrm{d}\varphi_1\ldots\int_0^{\pi}\sin\varphi_{n-2}\mathrm{d}\varphi_{n-2}\int_0^{2\pi}\mathrm{d}\varphi_{n-1}\\
&\quad\le \frac{2^n\pi^{\frac{n}{2}}}{\mu\Gamma(\frac{n}{2})}\,r^{\mu}.
\end{aligned} \tag{4.14}$$

(4.11) and (4.14) imply (4.8). □

Lemma 4.2. *If* $\Sigma=\big\{x,\, x\in\mathbb{R}_0^{n+1},\, |x-t|\ge r\big\}=\mathbb{R}_0^{n+1}\backslash B_n(t,r)$*, then*

$$\mathfrak{Y}(w)=\int_{\mathbb{R}_0^{n+1}\backslash B_n(t,r)}\frac{1}{|x-w|^{n+k}}\,\mathrm{d}S\le\frac{M}{|w-w_{\Sigma}|^k}\ \ (k>0),\quad w\in\mathbb{R}^{n+1}\setminus\Sigma, \tag{4.15}$$

where M is some constant independent of Σ.

Proof. Let

$$\mathfrak{R}(a,b,k)=\int_{\mathbb{R}_0^{n+1}}\frac{1}{[|x-a|+b]^{n+k}}\,\mathrm{d}S,\quad a\in\mathbb{R}_0^{n+1},\ b,\ k>0. \tag{4.16}$$

It is easy to see that (only for $n>1$, the case of $n=1$ is simpler)

$$\begin{aligned}
&\mathfrak{R}(a,b,k)\\
&\le\int_0^{\infty}\frac{1}{[\rho+b]^{1+k}}\mathrm{d}\rho\int_0^{\pi}\sin^{n-2}\varphi_1\mathrm{d}\varphi_1\ldots\int_0^{\pi}\sin\varphi_{n-2}\mathrm{d}\varphi_{n-2}\int_0^{2\pi}\mathrm{d}\varphi_{n-1}\\
&\le\frac{\pi^{\frac{n}{2}}}{k\Gamma(\frac{n}{2})b^k}.
\end{aligned} \tag{4.17}$$

Thus, we have

$$\begin{aligned}
&\mathfrak{Y}(w)\\
&\quad \le \int_\Sigma \frac{3^{n+k}}{|3(x-w)|^{n+k}}\,\mathrm{d}S\\
&\quad \le \int_{\mathbb{R}_0^{n+1}} \frac{3^{n+k}}{\left[\left|x-w_\Sigma\right|+\left|w-w_\Sigma\right|\right]^{n+k}}\,\mathrm{d}S \quad \left(\text{by (4.4) since} w_\Sigma \text{exists}\right)\\
&\quad \le 3^{n+k}\,\mathfrak{R}\left(w_\Sigma, \left|w-w_\Sigma\right|, k\right).
\end{aligned} \tag{4.18}$$

(4.15) follows from (4.17) and (4.18). □

Proof of Theorem 4.1. We just prove the case of $\mathfrak{D}[f]$. The proof will be divided into two steps. We may assume $\mu < 1$ by Lemma 2.2.

Step 1: Let $t \in \mathbb{R}_0^{n+1}$, $w \in \mathbb{R}^{n+1}$ and $\eta = \left|w-t\right|$. Set $\Sigma = B_n(t, 2\eta)$, we have

$$\begin{aligned}
&\left|\left(\mathfrak{D}[f]\right)(w) - \left(\mathfrak{D}[f]\right)(t)\right|\\
&\quad \le \frac{1}{\mathsf{V}_{n+1}}\left|\int_\Sigma E(x-w)\mathrm{d}\sigma\left[f(x)-f\left(w_{\mathbb{R}_0^{n+1}}\right)\right]\right| + \frac{1}{\mathsf{V}_{n+1}}\left|\int_\Sigma E(x-t)\mathrm{d}\sigma\left[f(x)-f(t)\right]\right|\\
&\qquad + \frac{1}{\mathsf{V}_{n+1}}\left|\int_{\mathbb{R}_0^{n+1}\setminus\Sigma} E(x-w)-E(x-t)\Big]\mathrm{d}\sigma\left[f(x)-f\left(w_{\mathbb{R}_0^{n+1}}\right)\right]\right| \quad \left(\text{by (4.6)}\right)\\
&\quad \triangleq \delta_1+\delta_2+\delta_3.
\end{aligned} \tag{4.19}$$

By Lemma 4.1,

$$\delta_1, \delta_2 \le C\left|w-t\right|^\mu. \tag{4.20}$$

Invoking Hile's Lemma [12]

$$\left|E(x-w)-E(x-t)\right| \le \frac{|w-t|}{|x-w|^{n+1}}\sum_{j=0}^{n-1}\frac{|x-w|^{j+1}}{|x-t|^{j+1}} \tag{4.21}$$

and

$$\frac{1}{2} \le \frac{|x-t|-|t-w|}{|x-t|} \le \frac{|x-w|}{|x-t|} \le \frac{|x-t|+|t-w|}{|x-t|} \le 2 \text{ while } 2|t-w| \le |x-t|, \tag{4.22}$$

therefore, by Lemma 4.2,

$$\delta_3 \le M_2|w-t|\int_{\mathbb{R}_0^{n+1}\setminus\Sigma}\frac{1}{|x-w|^{n+1-\mu}}\,\mathrm{d}S \le C\left|w-t\right|^\mu. \tag{4.23}$$

Combining (4.19), (4.20) and (4.23) we obtain

$$\left|\left(\mathfrak{D}[f]\right)(w) - \left(\mathfrak{D}[f]\right)(t)\right| \le C\left|w-t\right|^\mu, \quad t \in \mathbb{R}_0^{n+1},\ w \in \mathbb{R}^{n+1}, \tag{4.24}$$

which is so called Privalov theorem.

Step 2: Let $v, w \in \mathbb{R}^{n+1}$. Denote the distance from the segment $[v, w]$ to $\mathbb{R}_0^{n+1}$ by d, then there is $\mathcal{K} \in [v, w]$ such that $d = \left|\mathcal{K} - \mathcal{K}_{\mathbb{R}_0^{n+1}}\right| = \left|\mathcal{K} - \mathrm{Re}(\mathcal{K})\right|$. In fact, $\mathcal{K} \in [v, w]$ such that $t = \mathrm{Re}(\mathcal{K}) = \min\{|\mathrm{Re}(\xi)|,\ \xi \in [v, w]\}$, so $d = |\mathcal{K} - t|$.

Case 1: $\big|v-w\big| \geq \big|\mathcal{K}-t\big|$. Under this case, by the Privalov theorem in **Step 1**, we get

$$\begin{aligned}
&\Big|\Big(\mathfrak{D}[f]\Big)(v) - \Big(\mathfrak{D}[f](w)\Big| \\
&\quad \leq \Big|\Big(\mathfrak{D}[f]\Big)(v) - \Big(\mathfrak{D}[f]\Big)(t)\Big| + \Big|\Big(\mathfrak{D}[f]\Big)(t) - \Big(\mathfrak{D}[f]\Big)(w)\Big| \\
&\quad \leq C\Big[|v-t|^{\mu} + |w-t|^{\mu}\Big] \quad \Big(\text{by } (4.24)\Big) \\
&\quad \leq 4C\big|v-w\big|^{\mu} \quad \Big(\text{by } |v-t| \leq |v-\mathcal{K}| + |\mathcal{K}-t| \leq 2|v-w|\Big).
\end{aligned} \tag{4.25}$$

Case 2: $\big|v-w\big| < \big|\mathcal{K}-t\big|$. Under this case, both v and w are simultaneously in $\mathbb{R}^{n+1}_{+}$, or simultaneously in $\mathbb{R}^{n+1}_{-}$, and

$$\big|x-v\big| \leq \big|x-w\big| + \big|v-w\big| \leq |x-w| + |t-\mathcal{K}| \leq 2\big|x-w\big|, \quad x \in \mathbb{R}^{n+1}_{0}. \tag{4.26}$$

By the same way, we have,

$$\big|x-w\big| \leq 2\big|x-v\big|, \quad x \in \mathbb{R}^{n+1}_{0}. \tag{4.27}$$

Hence

$$\frac{1}{2} \leq \frac{|x-v|}{|x-w|} \leq 2, \quad x \in \mathbb{R}^{n+1}_{0}. \tag{4.28}$$

Also,

$$\big|x-t\big| \leq \big|x-v\big| + \big|v-\mathcal{K}\big| + \big|\mathcal{K}-t\big| < \big|x-v\big| + 2\big|\mathcal{K}-t\big| \leq 3\big|x-v\big|, \quad x \in \mathbb{R}^{n+1}_{0}. \tag{4.29}$$

In the same way,

$$\big|x-t\big| < 3\big|x-w\big|, \quad x \in \mathbb{R}^{n+1}_{0}. \tag{4.30}$$

Thus, we have, by Example 3.1,

$$\begin{aligned}
&\Big(\mathfrak{D}[f]\Big)(v) - \Big(\mathfrak{D}[f]\Big)(w) \quad \Big(\text{say, } v, w \in \mathbb{R}^{n+1}_{+}\Big) \\
&\quad = \frac{1}{\bigvee_{n+1}}\left\{\int_{\mathbb{R}^{n+1}_{0}} E(x-v)\mathrm{d}\sigma\Big[f(x) - f(t)\Big]\right. \\
&\qquad \left. + \int_{\mathbb{R}^{n+1}_{0}} E(x-w)\mathrm{d}\sigma\Big[f(t) - f(x)\Big]\right\} + \frac{1}{2}\Big[f(\mathrm{Re}(w)) - f(\mathrm{Re}(v))\Big] \\
&\quad \triangleq \bigwedge\nolimits_1 + \bigwedge\nolimits_2 .
\end{aligned} \tag{4.31}$$

From (4.7) we immediately have

$$\big|\bigwedge\nolimits_2\big| \leq M(f)\big|v-w\big|^{\mu}. \tag{4.32}$$

Note

$$\begin{aligned}
&\big|\wedge_1\big| \\
&\le \frac{1}{\mathsf{V}_{n+1}}\left|\int_{\overline{B}_n(t,|v-w|)} E(x-v)\mathrm{d}\sigma\Big[f(x)-f(t)\Big]\right| \\
&\quad+\frac{1}{\mathsf{V}_{n+1}}\left|\int_{\overline{B}_n(t,|v-w|)} E(x-w)\mathrm{d}\sigma\Big[f(x)-f(t)\Big]\right| \\
&\quad+\frac{1}{\mathsf{V}_{n+1}}\left|\int_{\mathbb{R}_0^{n+1}\setminus B_n(t,|v-w|)} \Big[E(x-v)-E(x-w)\Big]\mathrm{d}\sigma\Big[f(x)-f(t)\Big]\right| \\
&\triangleq \sigma_1+\sigma_2+\sigma_3.
\end{aligned} \tag{4.33}$$

We have, by Lemma 4.1,

$$\sigma_1, \sigma_2 \le C|v-w|^\mu. \tag{4.34}$$

By Hile's inequality (4.21) and (4.20), in a manner similar to (4.23) we get

$$\sigma_3 \le C|v-w|^\mu. \tag{4.35}$$

Then (4.33), (4.34) and (4.35) together imply

$$\big|\bigwedge\nolimits_1\big| \le C\big|v-w\big|^\mu. \tag{4.36}$$

The relations (4.31), (4.32) and (4.36) imply

$$\left|\Big(\mathfrak{D}[f]\Big)(v)-\Big(\mathfrak{D}[f]\Big)(w)\right| \le C\Big|v-w\Big|^\mu, \quad v, w \in \mathbb{R}^{n+1}, \tag{4.37}$$

which is so called Muskhelisvili theorem.

Now, the proof of $\mathfrak{D}[f] \in H^\mu\Big(\mathbb{R}^{n+1}\Big)$ for $0 < \mu < 1$ is complete. The proof of the result for $[f]\mathfrak{D}$ is similar. □

Introduce the following singular integral operators

$$\Big(\mathcal{S}^{\pm}[f]\Big)(w) = \begin{cases} \Big(\mathcal{S}[f]\Big)(w), & \text{if } w \in \mathbb{R}_{\pm}^{n+1}, \\ \pm\frac{1}{2}f(t) + \Big(\mathcal{S}[f]\Big)(t), & \text{if } w = t \in \mathbb{R}_0^{n+1}, \end{cases} \tag{4.38}$$

$$\Big([f]\mathcal{S}^{\pm}\Big)(w) = \begin{cases} \Big([f]\mathcal{S}\Big)(w), & \text{if } w \in \mathbb{R}_{\pm}^{n+1}, \\ \pm\frac{1}{2}f(t) + \Big([f]\mathcal{S}\Big)(t), & \text{if } w = t \in \mathbb{R}_0^{n+1}. \end{cases} \tag{4.39}$$

Now, we immediately get the following important theorem from Theorem 4.1, Example 3.1 and Example 3.3.

Theorem 4.2. [Hölder continuity of Cauchy type integrals] *If* $f \in \widehat{H}\left(\mathbb{R}_0^{n+1}\right)$, *then*

$$\mathcal{S}^+(f),\ [f]\mathcal{S}^+ \in H^\mu\left(\overline{\mathbb{R}_+^{n+1}}\right), \quad \mathcal{S}^-[f],\ [f]\mathcal{S}^- \in H^\mu\left(\overline{\mathbb{R}_-^{n+1}}\right). \tag{4.40}$$

As usual, if the limits

$$\Big(\mathcal{S}[f]\Big)^+(t) = \lim_{v\to t,\, v\in\mathbb{R}_+^{n+1}} \Big(\mathcal{S}[f]\Big)(v), \quad \Big(\mathcal{S}[f]\Big)^-(t) = \lim_{v\to t,\, v\in\mathbb{R}_-^{n+1}} \Big(\mathcal{S}[f]\Big)(v) \tag{4.41}$$

exists, they are called the positive and negative boundary values of $\mathcal{S}[f]$, respectively. Similarly, we call

$$\Big([f]\mathcal{S}\Big)^{+}(t)=\lim_{v\to t,\,v\in\mathbb{R}_{+}^{n+1}}\Big([f]\mathcal{S}\Big)(v),\ \ \Big([f]\mathcal{S}\Big)^{-}(t)=\lim_{v\to t,\,v\in\mathbb{R}_{-}^{n+1}}\Big([f]\mathcal{S}\Big)(v) \tag{4.42}$$

the positive and negative boundary values of $[f]\mathcal{S}$, respectively.

We note that the Plemelj–Sochocki formulae for the boundary values of the Cauchy type integrals $\big(S[f]\big)(z)$ and $\big([f]S\big)(z)$ become immediate corollaries of Theorem 4.2.

Corollary 4.1. *(Plemelj–Sochocki formulae) If* $f\in\widehat{H}\left(\mathbb{R}_0^{n+1}\right)$*, then*

$$\begin{cases}\Big(\mathcal{S}[f]\Big)^{+}(t)=\Big(\mathcal{S}^{+}[f]\Big)(t)=\dfrac{1}{2}f(t)+\Big(\mathcal{S}[f]\Big)(t),\\ \Big(\mathcal{S}[f]\Big)^{-}(t)=\Big(\mathcal{S}^{-}[f]\Big)(t)=-\dfrac{1}{2}f(t)+\Big(\mathcal{S}[f]\Big)(t),\end{cases}\quad t\in\mathbb{R}_0^{n+1}, \tag{4.43}$$

$$\begin{cases}\Big([f]\mathcal{S}\Big)^{+}(t)=\Big([f]\mathcal{S}^{+}\Big)(t)=\dfrac{1}{2}f(t)+\Big([f]\mathcal{S}\Big)(t),\\ \Big([f]\mathcal{S}\Big)^{-}(t)=\Big([f]\mathcal{S}^{-}\Big)(t)=-\dfrac{1}{2}f(t)+\Big([f]\mathcal{S}\Big)(t),\end{cases}\quad t\in\mathbb{R}_0^{n+1}, \tag{4.44}$$

hold.

Remark 4.1. Theorem 4.2 is called the Privalov-Muskhelishvili theorem, which is a corollary of Theorem 4.1, and the Plemelj–Sochocki formula is a corollary of it. Therefore, Theorem 4.1 unifies and improves the 2P Theorems. The Plemelj–Sochocki formula plays an important role in the solution of boundary value problems. For Cauchy type integrals on closed smooth surfaces, [8,10] give a strict and simple proof for the Plemelj–Sochocki formula. We note that although the proof in the paper is exactly the same as that in [8,10], we get a better conclusion that $\mathfrak{D}[f]$ is a function of H^{μ} in the whole plane. The introduction of operator $\mathfrak{D}$ is the key technique. So, here we first give a rigorous proof for the Plemelj-Sochocki formulae of Cauchy type integrals over R_0^{n+1}, instead of simulating from the corresponding result of the classical complex analysis. The proof here is also valid for the classical case of $n=1$. In the classical case, (4.43) is based on the Plemelj-Sochocki formulae for an open smooth curve [11,19,24], but there have many obstacles to overcome in the case of $n>1$.

4.2. Boundary Behavior of Cauchy Type Integrals at Infinity

To discuss the boundary behavior of the Cauchy singular integral operator at the infinity, we shall consider the following twisted inversion functions of (3.25) and (3.26).

$$\Big(\mathfrak{U}[f]\Big)^{\dagger}(v)=\frac{1}{\mathsf{V}_{n+1}}\int_{\mathbb{R}_0^{n+1}}E\left(x-v^{\dagger}\right)\mathrm{d}\sigma\Big[f(x)-f(\infty)\Big],\ v\in\mathbb{R}_{+}^{n+1}\cup\mathbb{R}_{-}^{n+1}, \tag{4.45}$$

and

$$\Big([f]\mathfrak{U}\Big)^{\dagger}(v) = \frac{1}{\mathsf{V}_{n+1}} \int_{\mathbb{R}_0^{n+1}} \Big[f(x) - f(\infty)\Big] \mathrm{d}\sigma E\left(x - v^{\dagger}\right),\ v \in \mathbb{R}_+^{n+1} \cup \mathbb{R}_-^{n+1}. \tag{4.46}$$

Lemma 4.3. *If* $f \in \widehat{H}^{\mu}\left(\mathbb{R}_0^{n+1}\right)$ $\big(0 < \mu < 1\big)$ *then*

$$\Big(\mathfrak{U}[f]\Big)^{\dagger}(v) = \frac{1}{\mathsf{V}_{n+1}} \int_{\mathbb{R}_0^{n+1}} |v|^{n-1} v E(\xi - v) |\xi|^{-n-1} \xi \mathrm{d}\sigma(\xi) \Big[f^{\dagger}(\xi) - f^{\dagger}(0)\Big], v \in \mathbb{R}_{\pm}^{n+1}, \tag{4.47}$$

and

$$\Big([f]\mathfrak{U}\Big)^{\dagger}(v) = \frac{1}{\mathsf{V}_{n+1}} \int_{\mathbb{R}_0^{n+1}} \Big[f^{\dagger}(\xi) - f^{\dagger}(0)\Big] \mathrm{d}\sigma(\xi) |v|^{n-1} v\, E(\xi - v) |\xi|^{-n-1} \xi, v \in \mathbb{R}_{\pm}^{n+1}. \tag{4.48}$$

Proof. Write, for $R > \delta > 0$,

$$\Big(\mathfrak{U}_{\delta,R}[f]\Big)(w) = \frac{1}{\mathsf{V}_{n+1}} \int_{\delta \leq |x| \leq R} E\left(x - w\right) \mathrm{d}\sigma \Big[f(x) - f(\infty)\Big], w \in \mathbb{R}_+^{n+1} \cup \mathbb{R}_-^{n+1}. \tag{4.49}$$

Then

$$\Big(\mathfrak{U}_{\delta,R}[f]\Big)^{\dagger}(v) = \frac{1}{\mathsf{V}_{n+1}} \int_{\delta \leq |x| \leq R} E\left(x - v^{\dagger}\right) \mathrm{d}\sigma \Big[f(x) - f^{\dagger}(0)\Big], v \in \mathbb{R}_+^{n+1} \cup \mathbb{R}_-^{n+1}, \tag{4.50}$$

and

$$\begin{aligned} &\lim_{R \to +\infty, \delta \to 0^+} \Big(\mathfrak{U}_{\delta,R}[f]\Big)^{\dagger}(v) \\ &\quad = \lim_{R \to +\infty} \frac{1}{\mathsf{V}_{n+1}} \int_{|x| \leq R} E\left(x - v^{\dagger}\right) \mathrm{d}\sigma \left[f(x) - f(\infty)\right] \\ &\qquad + \lim_{\delta \to 0^+} \frac{1}{\mathsf{V}_{n+1}} \int_{\delta \leq |x|} E\left(x - v^{\dagger}\right) \mathrm{d}\sigma \left[f(x) - f(\infty)\right] \\ &\quad = \Big(\mathfrak{U}[f]\Big)^{\dagger}(v), \quad v \in \mathbb{R}_+^{n+1} \cup \mathbb{R}_-^{n+1} \quad \Big((\text{by } (4.45))\Big). \end{aligned} \tag{4.51}$$

Let

$$x = \xi^{\dagger}, \quad v = w^{\dagger}, \tag{4.52}$$

we get the conformal invariance of the Cauchy kernel [18]

$$E(x - w) = |v|^{n-1} v\, E(\xi - v)\, |\xi|^{n-1} \xi = |\xi|^{n-1} \xi\, E(\xi - v)\, |v|^{n-1} v. \tag{4.53}$$

In (4.49), using the variable substitution (2.39) on $\mathbb{R}_0^{n+1}$, we obtain

$$\begin{aligned} &\Big(\mathfrak{U}_{\delta,R}[f]\Big)^{\dagger}(v) \\ &\quad = \frac{1}{\mathsf{V}_{n+1}} \int_{\frac{1}{R} \leq |\xi| \leq \frac{1}{\delta}} |v|^{n-1} v\, E(\xi - v)\, |\xi|^{n-1} \xi \mathrm{d}\sigma(\xi) \Big[f^{\dagger}(\xi) - f^{\dagger}(0)\Big] \det J_{\dagger_0}(\xi), \end{aligned} \tag{4.54}$$

where $J_{\dagger_0}(\xi)$ is the Jacobian matrix of the restriction $\dagger_0$ on $\mathbb{R}_0^{n+1}$ of the twisted inversion $\dagger$, i.e., $\dagger_0 = \dagger\big|_{\mathbb{R}_0^{n+1}}$. We may find

$$J_{\dagger_0}(\xi) = \Big\{c_{j,\ell}\Big\}_{j,\ell=0}^{n-1} \text{ with } c_{j,\ell} = \begin{cases} \dfrac{1}{|\xi|^4}\Big[2\xi_0\xi_\ell - \delta_{j,\ell}|\xi|^2\Big],\ j=0, \\ -\dfrac{1}{|\xi|^4}\Big[2\xi_j\xi_\ell - \delta_{j,\ell}|\xi|^2\Big],\ j\neq 0. \end{cases} \tag{4.55}$$

It is proved in Remark 4.2 below that

$$\det\Big[J_{\dagger_0}(\xi)\Big] = \frac{1}{|\xi|^{2n}}. \tag{4.56}$$

Now (4.54) becomes

$$\Big(\mathfrak{U}_{\delta,R}[f]\Big)^\dagger(v) = \frac{1}{\mathsf{V}_{n+1}}\int_{\frac{1}{R}\le|\xi|\le\frac{1}{\delta}} \mathfrak{M}(v,\xi)\mathrm{d}\sigma(\xi)\Big[f^\dagger(\xi)-f^\dagger(0)\Big],\ v\in\mathbb{R}_+^{n+1}\cup\mathbb{R}_-^{n+1}, \tag{4.57}$$

where

$$\mathfrak{M}(v,\xi) = |v|^{n-1}v\,E(\xi-v)\,|\xi|^{-n-1}\xi, \quad v\in\mathbb{R}_+^{n+1}\cup\mathbb{R}_-^{n+1},\ \ \xi\in\mathbb{R}_0^{n+1}. \tag{4.58}$$

Thus, letting $\delta\to 0^+$ and $R\to+\infty$ on the both sides of (4.57), noting (4.51), we obtain

$$\Big(\mathfrak{U}[f]\Big)^\dagger(v) = \frac{1}{\mathsf{V}_{n+1}}\int_{\mathbb{R}_0^{n+1}} |v|^{n-1}v\,E(\xi-v)\,|\xi|^{-n-1}\xi\,\mathrm{d}\sigma(\xi)\,\Big[f^\dagger(\xi)-f^\dagger(0)\Big],\ \ v\in\mathbb{R}_\pm^{n+1}, \tag{4.59}$$

in which the right side integral is an ordinary (improper) integral, since

$$\left|\mathfrak{M}(v,\xi)\Big[f^\dagger(\xi)-f^\dagger(0)\Big]\right| = \begin{cases} O\big(|\xi|^{-2n}\big), & \text{as } \xi\to\infty, \\ O\big(|\xi|^{-n-\mu}\big), & \text{as } \xi\to 0. \end{cases} \tag{4.60}$$

In the same way, we may prove (4.48). □

Remark 4.2. This result (4.56) was used in many articles [13, 18, 25, 34] without a proof or reference. The author gives a proof for case $n = 2$ by the specific calculation in [33], while in [34] he does not give any proof for general case. For convenience, a simple proof will be given in the following, which is based on discussion between Zhongxiang Zhang and Jinyuan Du. In fact, by using the inductive method, we can prove

$$\begin{aligned} I_n(c) &= \begin{vmatrix} 2\xi_0^2-c^2 & 2\xi_0\xi_1 & \dots & 2\xi_0\xi_{n-1} \\ 2\xi_0\xi_1 & 2\xi_1-c^2 & \dots & 2\xi_1\xi_{n-1} \\ \dots & \dots & \dots & \dots \\ 2\xi_0\xi_{n-1} & 2\xi_1\xi_{n-1} & \dots & 2\xi_{n-1}^2-c^2 \end{vmatrix} \\ &= (-1)^{n-1}c^{2(n-1)}\left[2\sum_{j=0}^{n-1}\xi_j^2 - c^2\right], \end{aligned} \tag{4.61}$$

where c is any constant. Taking $c = |\xi|$ $\left(\xi \in \mathbb{R}_0^{n+1}\right)$ in the above equality, we get

$$\det\left[J_{\dagger_0}(\xi)\right] = \frac{(-1)^{n-1}}{|\xi|^{4n}} I_n\left(|\xi|\right). \tag{4.62}$$

(4.56) then follows from (4.61) and (4.62).

Lemma 4.4. *When $f \equiv 1$, we have*

$$\left(\mathcal{S}[1]\right)^{\dagger}(v) = \frac{1}{\mathsf{V}_{n+1}} \int_{\mathbb{R}_0^{n+1}} |v|^{n-1} v E(\xi - v) |\xi|^{-n-1} \xi \mathrm{d}\sigma(\xi) = \pm\frac{1}{2},\ v \in \mathbb{R}_{\pm}^{n+1}, \tag{4.63}$$

$$\left([1]\mathcal{S}\right)^{\dagger}(v) = \frac{1}{\mathsf{V}_{n+1}} \int_{\mathbb{R}_0^{n+1}} \mathrm{d}\sigma(\xi) |v|^{n-1} v E(\xi - v) |\xi|^{-n-1} \xi = \pm\frac{1}{2},\ v \in \mathbb{R}_{\pm}^{n+1}. \tag{4.64}$$

Proof. By (3.9) and noting

$$v = w^{\dagger} \in \mathbb{R}_{\pm}^{n+1} \quad \text{if and only if} \quad w \in \mathbb{R}_{\pm}^{n+1}, \tag{4.65}$$

we have

$$\left(\mathcal{S}[1]\right)^{\dagger}(v) = \pm\frac{1}{2},\ \ v \in \mathbb{R}_{\pm}^{n+1}. \tag{4.66}$$

Also, using the variable substitution $\dagger_0$, we obtain

$$\frac{1}{\mathsf{V}_{n+1}} \int_{\delta \le |x| \le R} E\left(x - v^{\dagger}\right) \mathrm{d}\sigma = \frac{1}{\mathsf{V}_{n+1}} \int_{\frac{1}{R} \le |\xi| \le \frac{1}{\delta}} |v|^{n-1} v E(\xi - v) |\xi|^{-n-1} \xi \mathrm{d}\sigma(\xi). \tag{4.67}$$

So, letting $R \to +\infty$, $\delta \to 0^+$ in (4.67), and by (4.66), we have

$$\frac{1}{\mathsf{V}_{n+1}} \int_{\mathbb{R}_0^{n+1}} |v|^{n-1} v\, E(\xi - v)\, |\xi|^{-n-1} \xi \mathrm{d}\sigma(\xi) = \left(\mathcal{S}[1]\right)^{\dagger}(v) = \pm\frac{1}{2}. \tag{4.68}$$

The proof of (4.64) is similar. □

Remark 4.3. The integral in the left band side of (4.68) is an ordinary improper integral at ∞ and a principal value integral at 0.

Lemma 4.3 and Lemma 4.4 result in the following result.

Lemma 4.5. *[Variable substitution formula for Cauchy type integrals] If $f \in \widehat{H}^{\mu}\left(\mathbb{R}_0^{n+1}\right)$ $(0 < \mu < 1)$, then*

$$\left(\mathcal{S}[f]\right)^{\dagger}(v) = \frac{1}{\mathsf{V}_{n+1}} \int_{\mathbb{R}_0^{n+1}} |v|^{n-1} v E(\xi - v)\, |\xi|^{-n-1} \xi\, \mathrm{d}\sigma(\xi)\, f^{\dagger}(\xi),\ v \in \mathbb{R}_{\pm}^{n+1}, \tag{4.69}$$

and

$$\left([f]\mathcal{S}\right)^{\dagger}(v) = \frac{1}{\mathsf{V}_{n+1}} \int_{\mathbb{R}_0^{n+1}} f^{\dagger}(\xi) \mathrm{d}\sigma(\xi) |v|^{n-1} v E(\xi - v) |\xi|^{-n-1} \xi\ v \in \mathbb{R}_{\pm}^{n+1}, \tag{4.70}$$

in which the right hand side integrals are the Cauchy principle value integrals at both 0 and ∞.

Example 4.1. In the special case $n = 1$, we take $e_1 = i$ and write $\mathbb{R}^2_\pm = \mathbb{Z}_\pm$, i.e., the upper half and the lower half complex planes. It is easy to see that, from (4.69) and (4.70),

$$\Big(\mathcal{S}[f]\Big)^\dagger(v) = \Big(\mathcal{S}\Big[f^\dagger\Big]\Big)(v) - \Big(\mathcal{S}\Big[f^\dagger\Big]\Big)(0) = \Big([f]\mathcal{S}\Big)^\dagger(v), \quad v \in \mathbb{Z}_\pm, \tag{4.71}$$

Remark 4.4. By Lemma 2.3 and Theorem 4.2, we know $\mathcal{S}^\pm\big[f^\dagger\big] \in H^\mu\big(\overline{\mathbb{Z}_\pm}\big)$. Also, noting $\Big(\mathcal{S}^\pm[f]\Big)^\dagger \in H^\mu\Big(\overline{\mathbb{Z}_\pm}\Big)$. Thus, $\mathcal{S}[f] \in H^\mu\Big(\overline{\mathbb{Z}_\pm}\Big)$. And we end up with

$$\mathcal{S}^\pm[f] \in \widehat{H}^\mu\big(\overline{\mathbb{Z}_\pm}\big) \quad \text{while} \quad n = 1. \tag{4.72}$$

We guess that the result corresponding to (4.72) for $n > 1$ still holds. We tried to prove it for sometime without success.

Conjecture. If $f \in \widehat{H}^\mu\Big(\mathbb{R}^{n+1}_0\Big)$, then $\mathcal{S}^\pm[f] \in \widehat{H}^\mu\Big(\overline{\mathbb{R}^{n+1}_\pm}\Big)$ for $n > 1$.

We will prove the following weaker result for further solving the Riemann boundary value problems in §6.

Theorem 4.3. *If* $f \in \widehat{H}^\mu\Big(\mathbb{R}^{n+1}_0\Big)$, *then* $\mathcal{S}^\pm[f] \in H^\mu\Big(\overline{\mathbb{R}^{n+1}_\pm}\Big) \bigcap H^\nu_\dagger(\infty)$ *for any* $\nu \in (0, \mu)$.

Proof. The conclusion $\mathcal{S}^\pm(f) \in H^\mu\Big(\overline{\mathbb{R}^{n+1}_\pm}\Big)$ is just Theorem 4.2. We are going to leave the proof for the conclusion $\mathcal{S}^\pm[f] \in H^\nu_\dagger(\infty)$ in a more general theorem in §6 (see Remark 6.8). □

Now we prove a weaker result as follows for the time begin, which is a fundament of solving Riemann boundary value problems.

Lemma 4.6. [Vanishing of $\mathfrak{U}(f)$ at the infinity] *If* $f \in \widehat{H}^\mu\big(\mathbb{R}^{n+1}_0\big)$, *then*

$$\Big(\mathfrak{U}[f]\Big)(\infty) = \lim_{|v|\to\infty,\, v\in\mathbb{R}^{n+1}} \Big(\mathfrak{U}[f]\Big)(v) = 0, \tag{4.73}$$

and

$$\Big([f]\mathfrak{U}\Big)(\infty) = \lim_{|v|\to\infty,\, v\in\mathbb{R}^{n+1}} \Big([f]\mathfrak{U}\Big)(v) = 0. \tag{4.74}$$

[19,24] proved the above lemma for the classical case $n = 1$ very simply by a fractional linear transformations. However, the proof in [19,24] involves a number of concrete calculations that are difficult to implement in Clifford algebra when $n \geq 2$. In the following we will give an analytical proof, that works for both $n = 1$ and $n \geq 2$. In order to work for $n = 1$, we assume that $0 < \mu < 1$, which is always possible according to Lemma 2.2.

Proof of Lemma 4.6. We divide the proof into two steps and only show (4.73).

Step 1. We first estimate the result when $\big|\mathrm{Im}(v)\big| \geq c$ $(c > 0)$. Obviously,

$$\Big|\Big(\mathfrak{U}[f]\Big)(v)\Big| = \left|\int_{\mathbb{R}^{n+1}_0} E(x-v)\mathrm{d}\sigma\big[f(x) - f(\infty)\big]\right| \leq M\mathfrak{B}(v),\ v \in \mathbb{R}^{n+1}_\pm, \tag{4.75}$$

where

$$\mathfrak{B}(v) = \int_{\mathbb{R}_0^{n+1}} \frac{1}{|x-v|^n |x|^\mu} \,\mathrm{d}S, \quad v \in \mathbb{R}_\pm^{n+1}. \tag{4.76}$$

To estimate it, we split it into two pieces.

$$\delta_1(v) = \int_{|x|\le \frac{1}{2}|v|} \frac{1}{|x-v|^n |x|^\mu} \,\mathrm{d}S, \quad v \in \mathbb{R}_\pm^{n+1}, \tag{4.77}$$

and

$$\delta_2(v) = \int_{|x|\ge \frac{1}{2}|v|} \frac{1}{|x-v|^n |x|^\mu} \,\mathrm{d}S, \quad v \in \mathbb{R}_\pm^{n+1}. \tag{4.78}$$

Let us first point out an obvious fact. In order to compare the classical case, which is $n=1$, we can assume $0<\mu<1$ according to Lemma 2.2, then the following result is still true for the case $n=1$. Therefore, the subsequent conclusions, such as the following Theorem 4.4, are also valid for $n=1$, so, a new proof of the classical result for the boundary behavior at infinity of the Cauchy type integral on the real axis in [11, 19, 24] is given here.

For $\delta_1(v)$, We have,

$$\begin{aligned}
&\delta_1(v)\\
&\quad = \int_{|x|\le \frac{1}{2}|v|} \frac{1}{|x-v|^n |x|^\mu} \,\mathrm{d}S\\
&\quad\ \le \frac{2^n}{|v|^n} \int_{|x|\le \frac{1}{2}|v|} \frac{1}{|x|^\mu} \,\mathrm{d}S \ \left(|x-v| \ge |v|-|x| \ge |v| - \frac{1}{2}|v| = \frac{1}{2}|v|\right)\\
&\le \frac{2^n}{|v|^n} \int_0^{\frac{1}{2}|v|} \rho^{n-1-\mu}\,\mathrm{d}\rho \int_0^\pi \sin^{n-2}\varphi_1 \mathrm{d}\varphi_1 \ldots \int_0^\pi \sin\varphi_{n-2}\mathrm{d}\varphi_{n-2} \int_0^{2\pi} \mathrm{d}\varphi_{n-1}\\
&\le \frac{C}{|v|^\mu} \ \Big(\text{just here, taking} 0<\mu<1 \text{when} n=1\Big),
\end{aligned} \tag{4.79}$$

where C is a constant.

For $\delta_2(v)$, we also have

$$\begin{aligned}
&\delta_2(v)\\
&\quad = \int_{|x|\ge \frac{1}{2}|v|} \frac{1}{|x-v|^n |x|^\mu} \,\mathrm{d}S\\
&\quad \le \frac{2^{\mu-\epsilon}}{|v|^{\mu-\epsilon}} \int_{|x|\ge \frac{1}{2}|v|} \frac{1}{|x-v|^n |x|^\epsilon} \,\mathrm{d}S \ \Big(0<\epsilon\le\mu\Big)\\
&\quad \le \frac{2^{\mu-\epsilon}3^\epsilon}{|v|^{\mu-\epsilon}} \int_{|x|\ge \frac{1}{2}|v|} \frac{1}{|x-v|^n (3|x|)^\epsilon} \,\mathrm{d}S\\
&\quad \le \frac{2^{\mu-\epsilon}3^\epsilon}{|v|^{\mu-\epsilon}} \int_{|x|\ge \frac{1}{2}|v|} \frac{1}{|x-v|^{n+\epsilon}} \,\mathrm{d}S \ \Big(\text{by } |3x| = |2x| + |x| \ge |v| + |x| \ge |x-v|\Big)\\
&\quad = \frac{2^{\mu-\epsilon}3^\epsilon}{|v|^{\mu-\epsilon}} \mathfrak{Y}(v) \ \left((4.15) \text{ with} \, k=\epsilon, t=0 \,\text{and}\, r = \frac{|v|}{2}\right)\\
&\quad \le \frac{M}{|v|^{\mu-\epsilon}|\mathrm{Im}(v)|^\epsilon} \ \Big(\text{ by Lemma 4.2}\Big),
\end{aligned} \tag{4.80}$$

where M is a constant. In particular,

$$\delta_2 \le \frac{M}{c^\epsilon |v|^{\mu-\epsilon}} \ \Big(0<\epsilon<\mu\Big), \quad |\mathrm{Im}(v)| \ge c. \tag{4.81}$$

(4.79) and (4.81) result in that there exists the banding limit

$$\lim_{v\to\infty,\,|\mathrm{Im}(v)|\geq c>0,\,v\in\mathbb{R}^{n+1}_{\pm}}\Big(\mathfrak{U}[f]\Big)(v)=0. \tag{4.82}$$

Step 2. The banding limit for $\big|\mathrm{Im}(v)\big|\leq 1$. For concreteness, we take v in the Poincaré upper halfspace, then, for any but fixed $c>0$

$$\begin{aligned}&\Big|\Big(\mathfrak{U}[f]\Big)(v)\Big| \;\; \Big(|\mathrm{Im}(v)|\leq 1\Big) \;\; \Big(0\leq \mathrm{Im}(v)\leq 1\Big)\\ &\quad\leq\Big|\Big(\mathfrak{U}[f]\Big)(v)-\Big(\mathfrak{U}[f]\Big)(v+c\,e_n)\Big|+\Big|\Big(\mathfrak{U}[f]\Big)(v+c\,e_n)\Big| \;\; \Big(c>0\Big)\\ &\quad\leq Mc^{\mu}+\Big|\Big(\mathfrak{U}[f]\Big)(v+c\,e_n)\Big| \qquad \Big(\text{by Theorem 4.1}\Big).\end{aligned} \tag{4.83}$$

By (4.82) and (4.83), noting $c\leq\big|\mathrm{Im}(v+ce_n)\big|$,

$$\limsup_{v\to\infty,\;0<|\mathrm{Im}(v)|\leq 1}\Big|\Big(\mathfrak{U}[f]\Big)(v)\Big|\leq Mc^{\mu}, \tag{4.84}$$

which results in

$$\lim_{v\to\infty,\;0<|\mathrm{Im}(v)|\leq 1}\Big(\mathfrak{U}[f]\Big)(v)=0, \tag{4.85}$$

by arbitrariness of c.

(4.82) and (4.84) imply (4.73). □

Remark 4.5. It is not hard to see that, if $f\in H^{\mu}\left(\mathbb{R}^{n+1}_0\right)\bigcap H^{\mu}_{\dagger}(\infty)$ then (4.73) still holds.

Remark 4.6. Let

$$\theta(v)=\left\langle v,\mathbb{R}^{n+1}_0\right\rangle\in\left(0,\frac{\pi}{2}\right] \tag{4.86}$$

be the included angle between v and $\mathbb{R}^{n+1}_0$. Take $\epsilon=\mu$. Then, by (4.79) and (4.80),

$$\mathfrak{B}(v)\leq\frac{M}{\sin^{\mu}\theta(v)}\frac{1}{|v|^{\mu}},\quad v\in\mathbb{R}^{n+1}_{\pm}. \tag{4.87}$$

Thus, by (4.79), (4.80) and Lemma 4.6, letting $\mathbb{R}^{n+1}_0(\theta)$ denote the angular domain $\{v,\,\theta(v)\leq\theta\}$ of $\mathbb{R}^{n+1}$, then

$$\Big|\Big(\mathfrak{U}[f]\Big)(v)-\Big(\mathfrak{U}[f]\Big)(\infty)\Big|\leq\frac{M}{\sin^{\mu}\theta}\frac{1}{|v|^{\mu}},\quad v\in\mathbb{R}^{n+1}_0(\theta), \tag{4.88}$$

which is to say that $\mathfrak{U}[f]$ satisfies the non tangential $H_{\dagger}$ condition at ∞.

Theorem 4.4. (Boundary behavior of the Cauchy type integrals at ∞) *If* $f\in\widehat{H}^{\mu}\big(\mathbb{R}^{n+1}_0\big)$, *then*

$$\Big(\mathcal{S}^{\pm}[f]\Big)(\infty)=\lim_{v\to\infty,\,v\in\mathbb{R}^{n+1}_{\pm}}\Big(\mathcal{S}^{\pm}[f]\Big)(v)=\pm\frac{1}{2}f(\infty), \tag{4.89}$$

$$\Big([f]\mathcal{S}^{\pm}\Big)(\infty)=\lim_{v\to\infty,\,v\in\mathbb{R}^{n+1}_{\pm}}\Big([f]\mathcal{S}^{\pm}\Big)(v)=\pm\frac{1}{2}f(\infty). \tag{4.90}$$

Proof. By Lemma 4.6 and Example 3.1, we immediately get (4.89) and (4.90). □

Corollary 4.2. [Boundary behavior of singular integrals at infinity] *If* $f \in \widehat{H}^{\mu}\left(\mathbb{R}_0^{n+1}\right)$, *then*

$$\left(\mathcal{S}[f]\right)\Big|_{\mathbb{R}_0^{n+1}}(\infty) = \lim_{t\to\infty,\, t\in\mathbb{R}_0^{n+1}} \left(\mathcal{S}[f]\right)(t) = 0, \tag{4.91}$$

and

$$\left([f]\mathcal{S}\right)\Big|_{\mathbb{R}_0^{n+1}}(\infty) = \lim_{t\to\infty,\, t\in\mathbb{R}_0^{n+1}} \left([f]\mathcal{S}\right)(t) = 0. \tag{4.92}$$

Proof. By Lemma 4.6 and Example 3.5, we also immediately get (4.91) and (4.92). □

Remark 4.7. By Remark 4.4, we know that if $f \in C\left(\mathbb{R}_0^{n+1}\right)$ and $f \in O^{-\mu}(\infty)$ then Theorem 4.2, Theorem 4.3 and Corollary 4.2 are also true.

Theorem 4.5. *If* $f \in H^{\mu}\left(\mathbb{R}_0^{n+1}\right) \bigcap O^{-\mu}(\infty)$, *then*

$$\left(\mathcal{S}[f]\right)(\infty) = \left(\mathfrak{U}[f]\right)(\infty) = 0, \quad \left([f]\mathcal{S}\right)(\infty) = \left([f]\mathfrak{U}\right)(\infty) = 0, \tag{4.93}$$

where

$$\begin{aligned} \left(\mathcal{S}[f]\right)(\infty) &= \lim_{w\to\infty,\, w\in\mathbb{R}^{n+1}} \left(\mathcal{S}[f]\right)(w), \\ \left([f]\mathcal{S}\right)(\infty) &= \lim_{w\to\infty,\, w\in\mathbb{R}^{n+1}} \left([f]\mathcal{S}\right)(w). \end{aligned} \tag{4.94}$$

Remark 4.8. For clarity, we sometimes write ∞ in (4.73), (4.74), (4.89), (4.90), (4.91) and (4.92) by $\infty\left(\infty\big|_{\mathbb{R}^{n+1}}\right)$, $\infty\big|_{\mathbb{R}_\pm^{n+1}}$, $\infty\big|_{\mathbb{R}_0^{n+1}}$, respectively.

5. Sectionally Regular fFunctions with $\mathbb{R}_0^{n+1}$ as Jump Surface

To suitably present the interested Riemann boundary value problems, we introduce sectionally regular (holomorphic) functions with $\mathbb{R}_0^{n+1}$ as their jump surface, and discuss their principal parts as well as their orders at the infinity.

5.1. Sectionally Regular Functions

Firstly, we introduce sectionally regular functions with the hyperplane $\mathbb{R}_0^{n+1}$ as the jump surface.

Definition 5.1. A function F is said to be sectionally left (right) regular with the hyperplane $\mathbb{R}_0^{n+1}$ as its jump surface, if it is left (right) monogenic in $\mathbb{R}_+^{n+1}$ and $\mathbb{R}_-^{n+1}$, and has the positive boundary values $F^+(t)$ and the negative boundary values $F^-(t)$ in $\mathbb{R}_0^{n+1}$, where

$$F^+(t) = \lim_{w\to t,\, w\in\mathbb{R}_+^{n+1},\, t\in\mathbb{R}_0^{n+1}} F(w), \quad F^-(t) = \lim_{w\to t,\, w\in\mathbb{R}_-^{n+1},\, t\in\mathbb{R}_0^{n+1}} F(w). \tag{5.1}$$

In the following, we only consider the sectionally left regular function and simply call it the sectionally regular function. If F is a sectionally regular function with the hyperplane $\mathbb{R}_0^{n+1}$ as its jump surface, we write

$$\begin{aligned} F^+(w) &= \begin{cases} F(w), & w \in \mathbb{R}_+^{n+1}, \\ F^+(t), & w = t \in \mathbb{R}_0^{n+1}, \end{cases} \\ F^-(w) &= \begin{cases} F(w), & w \in \mathbb{R}_-^{n+1}, \\ F^-(t), & w = t \in \mathbb{R}_0^{n+1}. \end{cases} \end{aligned} \tag{5.2}$$

Remark 5.1. It is easy to prove that $F^+ \in C\left(\overline{\mathbb{R}_+^{n+1}}, C(V_n)\right)$ and $F^- \in C\left(\overline{\mathbb{R}_-^{n+1}}, C(V_n)\right)$. This fact will be used in the sequel.

Example 5.1. By using the regularity and the Plemelj-Sochocki formulae for the Cauchy type integrals established in, respectively, Theorem 3.3 and Corollary 4.1, we immediately obtain that if $f \in \widehat{H}^\mu\left(\mathbb{R}_0^{n+1}\right)$, then the Cauchy type integral $\mathcal{S}[f]$ is a sectionally regular with the hyperplane $\mathbb{R}_0^{n+1}$ as its jump surface.

5.2. Principal Part and Order at the Infinity

The infinity is not necessarily the isolated singularity of a sectionally regular functions with the hyperplane $\mathbb{R}_0^{n+1}$ as the jump surfaces. We must generalize the definition of their principal part at the infinity. More generally, for a function F monogenic in the region where $\mathbb{R}^{n+1}$ is cut off the hyperplane $\mathbb{R}_0^{n+1}$, denoted as $F \in \mathcal{M}\big(\mathbb{R}^{n+1} \setminus \mathbb{R}_0^{n+1}\big)$, we introduce its principal part at ∞ as following.

Definition 5.2. Let $F \in \mathcal{M}\big(\mathbb{R}^{n+1} \setminus \mathbb{R}_0^{n+1}\big)$. If there exists an entire function $\mathcal{E}$ such that

$$\lim_{w \to \infty,\, w \in \mathbb{R}_\pm^{n+1}} \big[F(w) - \mathcal{E}(w)\big] = 0, \tag{5.3}$$

then we call $\mathcal{E}$ the (generalized) principal part of F at $w = \infty$, denoted by $G.P[F, \infty](w)$.

Some symbols in [32] will be used below. Let $Z(x) = (z_1(x), \ldots, z_n(x))$ and $\alpha = [\alpha_1, \ldots, \alpha_n]$ where z_j's are the hypercomplex variables given in Example 2.4 and α_j's are nonnegative integers, then the symmetry power Z^α is a biregular function in $\mathbb{R}^{n+1}$ defined as the sum of all possible z_i products each of which contains z_i factor exactly α_i times. For example, for $n = 2$,

$$(z_1, z_2)^{[0,0]} = 1, \;\; (z_1, z_2)^{[1,1]} = z_1 z_2 + z_2 z_1, \;\; (z_1, z_2)^{[2,0]} = z_1^2. \tag{5.4}$$

For $\alpha = [\alpha_1, \alpha_2, \ldots, \alpha_n] \in N_0^n$, we denote

$$\alpha! = \alpha_1! \alpha_2! \ldots \alpha_n!\,, \qquad \big|\alpha\big| = \sum_{j=1}^n \alpha_j, \tag{5.5}$$

and

$$\partial^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}. \tag{5.6}$$

Introduce the mapping

$$\boldsymbol{\alpha} : \big(\ell_1, \ell_2, \dots, \ell_k\big) \mapsto \alpha = \big[\alpha_1, \dots \alpha_n\big] \tag{5.7}$$

where α_j is the number of times j appearing in $\big(l_1, l_2, \dots, l_k\big) \in N^k$. Now we may rewrite $W_{\{\ell_1,\ell_2,\dots,\ell_k\}}$ in Example 2.5 by

$$\begin{aligned} W_{\{\ell_1,\ell_2,\dots,\ell_k\}}(w) &= (-1)^k \Big(\partial^{\boldsymbol{\alpha}(\ell_1,\ell_2,\dots,\ell_k)}[E]\Big)(w) \\ &= (-1)^k \Big(\partial^{\alpha}[E]\Big)(w), \quad x \in \mathbb{R}^{n+1}\backslash\{0\}. \end{aligned} \tag{5.8}$$

So, $x^{n+|\alpha|}\big(\partial^{\alpha}[E]\big)(x)$ is bounded by Example 2.5.

Remark 5.2. If F has the isolated singular point $w = \infty$, then it has a Laurent series expansion near the infinity [4]

$$F(w) = \sum_{|\alpha|=0}^{+\infty} Z^{\alpha}(w)\lambda_{\alpha} + \sum_{|\alpha|=0}^{+\infty} \Big(\partial^{\alpha}[E]\Big)(w)\mu_{\alpha} \quad \text{near} \quad w = \infty, \tag{5.9}$$

where E is the Cauchy kernel. We denote its principal part by

$$P.P[F,\infty](w) = \sum_{|\alpha|=0}^{+\infty} Z^{\alpha}(w)\lambda_{\alpha}, \quad w \in \mathbb{R}^{n+1}, \tag{5.10}$$

In this case, we may prove the following result

$$G.P[F,\infty] = P.P\big[F,\infty\big]. \tag{5.11}$$

In fact, by (5.3), (5.9) and (5.10) we have

$$\lim_{w\in\mathbb{R}^{n+1}_{\pm},\, w\to\infty} \Big[P.P[F,\infty](w) - G.P[F,\infty](w)\Big] = 0. \tag{5.12}$$

Noting that both $P.P[F,\infty]$ and $G.P[F,\infty]$ are entire, we know that (5.12) is equivalent to

$$\lim_{w\in\mathbb{R}^{n+1},\, w\to\infty} \Big[P.P[F,\infty](w) - G.P[F,\infty](w)\Big] = 0, \tag{5.13}$$

and consequently we get (5.11) by Liouville's theorem [2].

In general, $w = \infty$ may not be an isolated singular point of F, and in the case, $P.P[F,\infty]$ is not defined. For example, in the classical complex analysis the following example is easily given [28]. Let

$$F(w) = \frac{\ln(-w)}{w^m} \quad (m = 1, 2, \dots) \tag{5.14}$$

where the logarithm function $\ln w$ is the principal branch in the complex plane cut along $(-\infty, 0]$, *i.e.*,

$$\begin{aligned} \ln w = \ln|w| + i\arg(w)\ (-\pi < \arg(w) < \pi), \\ w \in \mathbb{C} \setminus (-\infty, +\infty) \subset \mathbb{C} \setminus (-\infty, 0], \end{aligned} \tag{5.15}$$

and we can see that $F(w)$ has no $P.P(F,\infty)$, but

$$G.P[F,\infty](w) = 0. \tag{5.16}$$

So, the concept of $G.P$ is more extensive than the concept of $P.P$ in the classical sense.

Example 5.2. If $f \in \widehat{H}^{\mu}\left(\mathbb{R}_0^{n+1}\right)$, by Lemma 4.6 and Theorem 4.4, then $G.P(\mathcal{U}[f],\infty)=0$ and $G.P\big(w^{-1}\mathcal{S}[f],\infty\big)=0$. Moreover, by Theorem 4.4, $G.P(\mathcal{U}[f],\infty)=G.P(\mathcal{S}[f],\infty)=0$ when $f\in H^{\mu}\left(\mathbb{R}_0^{n+1}\right)\cap O^{-\mu}(\infty)$.

Remark 5.3. The principal part $G.P[F,\infty]$ is unique. For example, if both $\mathcal{E}_1$ and $\mathcal{E}_2$ are the principal parts $G.P[F,\infty]$, we then have $P.P\left[\mathcal{E}_1-\mathcal{E}_2,\infty\right]=G.P\big[\mathcal{E}_1-\mathcal{E}_2,\infty\big]=0$ by Remark 5.2. This results in $\mathcal{E}_1=\mathcal{E}_2$ by Liouville's theorem.

For F defined on $\mathbb{R}^{n+1}\setminus\mathbb{R}_0^{n+1}$, we sometimes need the concept of the order of F at the infinity.

Definition 5.3. Let $F\in\mathcal{M}\big(\mathbb{R}^{n+1}\setminus\mathbb{R}_0^{n+1}\big)$. If

$$0<\beta=\limsup_{w\in\mathbb{R}_{\pm}^{n+1},\, w\to\infty}\left|w^{-m}F(w)\right|<+\infty, \tag{5.17}$$

we say F to be of order m at $w=\infty$, denoted as $\operatorname{Ord}(F,\infty)=m$.

Definition 5.4. Let $m\geq 0$. We call

$$f(w)=\sum_{|\alpha|=0}^{m} Z^{\alpha}(w)\,c_{\alpha}\ \text{with}\ \sum_{|\alpha|=m}|c_{\alpha}|\neq 0 \tag{5.18}$$

a hypercomplex symmetric polynomial of degree m, denoted by $\operatorname{Deg}(f)=m$.

Lemma 5.1. (see [22]) *Let f be a hypercomplex symmetric polynomial. Then* $\operatorname{Deg}(f)=m$ *if and only if* $\operatorname{Ord}(f,\infty)=m$.

For the boundary behavior of function $\Phi\in\mathcal{M}\left(\mathbb{R}^{n+1}\setminus\mathbb{R}_0^{n+1}\right)$ at the infinity, there are three kinds of common statements for growth.

$$(A)\ G.P(w^{-(m+1)}\Phi,\infty)=0,\ \text{namely}\ \Phi(w)=o\left(w^{m+1}\right)\ \text{near}\ w=\infty, \tag{5.19}$$

$$(B)\ \limsup_{w\in\mathbb{R}_{\pm}^{n+1},\, w\to\infty}|w|^{-m}|\Phi(w)|=\beta,\ \text{namely}\ |\Phi(w)|=O\left(|w|^{m}\right)\ \text{near}\ w=\infty, \tag{5.20}$$

$$(C)\quad \operatorname{Ord}(\Phi,\infty)\leq m,\quad \text{namely}\ \operatorname{Ord}(\Phi,\infty)=k\ \text{with}\ k\leq m. \tag{5.21}$$

Remark 5.4. Obviously, (C) implies (B), while (B) implies (A). So the condition (A) is the weakest. We use the condition (A) in the Riemann boundary value problems below, which is an innovation of [13].

In [22], we get the following Liouville type theorem.

Theorem 5.1. (Liouville type theorem [22]) *If f is left entire with the growth condition $|f(w)|=O\left(|w|^{m}\right)$ near $w=\infty$, then it is a hypercomplex symmetric polynomial of degree not exceeding m, that is,*

$$f(w)=\begin{cases}\displaystyle\sum_{|\alpha|=0}^{m}\frac{1}{|\alpha|!}\,Z^{\alpha}(w)\,c_{\alpha}, & \textit{when } m\geq 0,\\ 0, & \textit{when } m<0,\end{cases} \tag{5.22}$$

where c_{α} are some hypercomplex constants.

Remark 5.5. In [22], we assumed the condition (C). It is seen that the condition (B) is used only in the proof of the Liouville type theorem in [12].

Sometimes it is more convenient that we write $|F(w)| \overset{sup}{\approx} |G(w)|$ when $w \to \infty$, *i.e.*, there are positive constants M and m ($M > m > 0$) such that

$$m \leq \limsup_{w\to\infty} \frac{|F(w)|}{|G(w)|} \leq M. \tag{5.23}$$

The following is an obvious fact.

Lemma 5.2. $|F(w)| \overset{sup}{\approx} |w^k|$ *near* $w = \infty$ *if and only if* $\mathrm{Ord}\,(F, \infty) = k$.

Example 5.3. $Z^\alpha(w) \overset{sup}{\approx} w^{|\alpha|}$ near $w = \infty$, so $\mathrm{Ord}\,\big(Z^\alpha, \infty\big) = |\alpha|$. In fact, it is not difficult to see that

$$\frac{z_j}{|w|} = \frac{1}{\sqrt{n}} \text{ when } w_0 \neq 0,\ w_j = R\,\big(j = 1, \ldots, n\big) \text{ imply } \frac{|Z^\alpha|}{|w|^{|\alpha|}} = \frac{|\alpha|!}{n^{\frac{1}{2}|\alpha|}\ \alpha!} \tag{5.24}$$

and

$$\frac{|z_j|}{|w|} \leq 1\,(j = 1, \ldots, n) \ \text{ imply } \ \frac{|Z^\alpha|}{|w|^{|\alpha|}} \leq \frac{|\alpha|!}{\alpha!}. \tag{5.25}$$

(5.24) and (5.25) result in

$$\frac{|\alpha|!}{n^{\frac{1}{2}|\alpha|}\ \alpha!} \leq \limsup_{w\to\infty} \frac{|Z^\alpha|}{|w|^{|\alpha|}} \leq \frac{|\alpha|!}{\alpha!}. \tag{5.26}$$

Let

$$Q_m(w) = \sum_{|\alpha|=m} \big[\partial^\alpha E\big](w)\lambda_\alpha, \quad w \in \mathbb{R}^{n+1}\backslash\{0\}, \tag{5.27}$$

where λ_α's are some constants in $C(V_n)$. We call it a hypercomplex Laurent polynomial.

In [22], we also gave the following result.

Lemma 5.3. [see [22]] $\mathrm{Ord}\big(Q_m, \infty\big) = -n - m$, *or* $Q_m(w) \overset{sup}{\approx} w^{-n-m}$ *near* $w = \infty$, *if and only if*

$$\sum_{|\alpha|=m} |\lambda_\alpha| \neq 0. \tag{5.28}$$

Lemma 5.4. *Let* $Q(w) = w^{n+m}Q_m(w)$ *where* Q_m *is a hypercomplex Laurent polynomial given in (5.27). Then* $G.P(Q, \infty) = 0$ *if and only if all* $\lambda_\alpha = 0$.

Proof. Sufficiency. It is obvious. In fact, $Q(w) \equiv 0$ in this case. Necessity. By Lemma 5.3 we know that (5.28) implies $\limsup_{w\to\infty} Q(w) > 0$, which is contradictory with $G.P(Q, \infty) = 0$. □

Remark 5.6. Similarly, let

$$Q^*(w) = Z^{n+m}(w)Q_m(w), \quad w \in \mathbb{R}^{n+1}\backslash\{0\}, \tag{5.29}$$

by Example 5.3,

$$P.P(Q^*, \infty) = 0 \ \text{ if and only if } \ \lambda_\alpha = 0. \tag{5.30}$$

6. Riemann Boundary Value Problems

In this section the Riemann boundary value problems for the sectionally regular functions with the hypercomplex plane as its jump surface will be discussed. After formulation of the problems, explicit representations of the solutions and the conditions of the solvability are given in detail.

6.1. Painlevé Problem

The simplest Riemann boundary value problem is so-called the Painlevé problem. Let us start our discussion from the Painlevé problem.

Painlevé problem Find a sectionally regular function Φ, with $\mathbb{R}_0^{n+1}$ as its jump surface, satisfying the boundary value condition

$$\Phi^+(x) = \Phi^-(x), \quad x \in \mathbb{R}_0^{n+1}. \tag{6.1}$$

Obviously, an entire function is a solution of the Painlevé problem. Conversely, whether a solution of the Painlevé problem is an entire function motivates the so-called Painlevé theorem. In [22], the Painlevé theorem with the smooth jump surface has been proved. Here the jump surface is a hypercomplex plane. For ease of reference, we would rather give a direct proof as follows.

Theorem 6.1. (Painlevé theorem) *If f is left (right) regular in $\mathbb{R}_\pm^{n+1}$ and $f \in C\big(\mathbb{R}^{n+1}, C(V_n)\big)$, then f is left (right) entire in $\mathbb{R}^{n+1}$.*

We only need to prove the following local theorem.

Lemma 6.1. *Let $\Omega = (a_0, b_0) \times (a_1, b_1) \times \ldots \times (a_n, b_n) \subset \mathbb{R}^{n+1}$ with $a_n < 0 < b_n$ be a rectangular parallelepiped. If f is left (right) monogenic in $\Omega \backslash \mathbb{R}_0^{n+1}$ and $f \in C\big(\Omega, C(V_n)\big)$, then f is left (right) monogenic in Ω.*

Proof. Take c_j, d_j $(j = 0, 1, \ldots, n)$ such that $a_j < c_j < d_j < b_j$ with $c_n < 0 < d_n$, where

$$\Box = (c_0, d_0) \times (c_1, d_1) \times \ldots \times (c_n, d_n). \tag{6.2}$$

We split it into two pieces

$$\Box_+ = (c_0, d_0) \times (c_1, d_1) \times \ldots \times (0, d_n), \quad \Box_- = (c_0, d_0) \times (c_1, d_1) \times \ldots \times (c_n, 0). \tag{6.3}$$

Then

$$\begin{aligned} &\big(\mathcal{S}_+[f]\big)(w) \quad \Big(w \in \mathbb{R}_\pm^{n+1}\Big) \\ &= \frac{1}{\mathsf{V}_{n+1}} \int_{\partial(\Box_+)} E(x - w)\, \mathrm{d}\sigma\, f(x) \\ &= \frac{1}{\mathsf{V}_{n+1}} \int_{\partial(\Box_{\epsilon^-})} E(x - w) \mathrm{d}\sigma\, f(x) + \frac{1}{\mathsf{V}_{n+1}} \int_{\partial(\Box_{\epsilon^+})} E(x - w) \mathrm{d}\sigma\, f(x) \\ &\triangleq \rho_1 + \rho_2, \end{aligned} \tag{6.4}$$

where

$$\Box_{\epsilon^-} = (c_0, d_0) \times (c_1, d_1) \times \ldots \times (0, \epsilon), \; \Box_{\epsilon^+} = (c_0, d_0) \times (c_1, d_1) \times \ldots \times (\epsilon, d_n) \tag{6.5}$$

with the induced orientation by the exterior normal.

Then, by the Cauchy theorem [2],

$$\rho_2 = \begin{cases} f(w), & w \in \square_{\epsilon^+}, \\ 0, & w \in \square_-, \end{cases} \tag{6.6}$$

and

$$\Big|\rho_1\Big| \leq C\Big[\epsilon\big\|f\big\|_{\overline{\square}} + \omega_{\overline{\square}}(f,\epsilon)\Big] \ \Big(C \text{is some constant}\Big), \tag{6.7}$$

where $\overline{\square}$ is the closure of $\square$, $\|f\|_{\overline{\square}}$ and $\omega_{\overline{\square}}$ are respectively the Chebyshev norm and the modulus of continuity of f on the closure $\overline{\square}$, which implies

$$\lim_{\epsilon \to 0^+} \rho_1 = 0. \tag{6.8}$$

(6.6) and (6.8) result in

$$\Big(\mathcal{S}_+[f]\Big)(w) = \begin{cases} f(w), & w \in \square_+, \\ 0, & w \in \square_-. \end{cases} \tag{6.9}$$

In exactly the same way, we may get

$$\begin{aligned} \Big(\mathcal{S}_-[f]\Big)(w) &= \frac{1}{\mathrm{V}_{n+1}} \int_{\partial(\square_-)} E(x-w)\mathrm{d}\sigma f(x) \\ &= \begin{cases} 0, & w \in \square_+, \\ f(w), & w \in \square_-. \end{cases} \end{aligned} \tag{6.10}$$

From (6.9) and (6.10) we have

$$\Big(\mathcal{S}_{\partial\square}[f]\Big)(w) = \frac{1}{\mathrm{V}_{n+1}} \int_{\partial\square} E(x-w)\,\mathrm{d}\sigma\, f(x) = f(w), \quad w \in \square_{\pm}. \tag{6.11}$$

Noting the continuity of $\mathcal{S}_{\partial\square}[f]$ on $\square$, we know that it has the representation of the Cauchy type integral

$$f(w) = \Big(\mathcal{S}_{\partial\square}[f]\Big)(w) = \frac{1}{\mathrm{V}_{n+1}} \int_{\partial\square} E(x-w)\,\mathrm{d}\sigma\, f(x), \quad w \in \square, \tag{6.12}$$

and thus it is regular on $\square$ by the Cauchy type integral theorem in [8,10]. □

Conclusion 6.1. (Painlevé problem) The solutions of Painlevé problem (6.1) are all entire functions.

Remark 6.1. The Painlevé problem is one of the foundations for solving the Riemann boundary value problems. In some of the earlier work [13,30], it was directly cited, without seriously proving it or pointing out the literature from which it came.

6.2. Liouville Problem

In the Painlevé problem, if the growth at $w = \infty$ is restricted, then it becomes the Liouville problem.

Liouville problem Find a sectionally regular function Φ, with $\mathbb{R}_0^{n+1}$ as its jump plane, such that

$$\begin{cases} \Phi^+(x) = \Phi^-(x), \quad x \in \mathbb{R}_0^{n+1}, \\ G.P\left[w^{-(m+1)}\Phi, \infty\right] = 0 \ \left(m \text{ is an integer}\right). \end{cases} \tag{6.13}$$

To solve the above Liouville problem, we need to generalize the Liouville type Theorem 5.1.

Theorem 6.2. (Generalized Liouville type theorem) *If f is a left (right) entire function with the growth condition $G.P\left(w^{-(m+1)}f, \infty\right) = 0$ at the infinity, then it is a hypercomplex symmetric polynomial of degree not exceeding m, namely,*

$$f(w) = \begin{cases} \sum_{|\alpha|=0}^{m} \frac{1}{|\alpha|!} Z^{\alpha}(w)\, c_\alpha, \ \textit{when } m \geq 0, \\ 0, \qquad\qquad \textit{when } m < 0, \end{cases} \tag{6.14}$$

where c_α are C_{n+m}^m hypercomplex constants.

Proof. By the Liouville type theorem (Theorem 5.1), we obtain

$$f(w) = \begin{cases} \sum_{|\alpha|=0}^{m+1} \frac{1}{|\alpha|!} Z^{\alpha}(w)\, c_\alpha, \ \text{when } m \geq 0, \\ 0, \qquad\qquad \text{when } m < 0. \end{cases} \tag{6.15}$$

If $\sum_{|\alpha|=m+1} \left|c_\alpha\right| \neq 0$ then $\mathrm{Deg}(f) = m+1$. Thus, $\mathrm{Ord}(f) = m+1$ by Remark 5.3, which contradicts with the growth condition (A) in (5.19). □

Remark 6.2. This theorem generalizes the classical Liouville type theorem [22]. Moreover, it shows that the conditions (A) in (5.19), (B) in (5.20) and (C) in (5.21) are equivalent with each other when Φ is an entire function.

Conclusion 6.2. (Liouville problem) The solution of the Liouville problem (6.13) is just arbitrary hypercomplex symmetric polynomial P_m of degree not exceeding m when $m \geq 0$ and $\Phi = 0$ when $m < 0$, i.e.,

$$\Phi(w) = P_m(w) = \sum_{|\alpha|=0}^{m} \frac{1}{|\alpha|!} Z^{\alpha}(w)\, c_\alpha \tag{6.16}$$

with the agreement $P_m = 0$ while $m < 0$, where c_α's are C_{n+m}^m arbitrary hypercomplex constants.

Proof. The desired result (6.16) follows from the solution of the Painlevé problem and the generalized Liouville Theorem 6.2. □

Remark 6.3. We see that the Liouville problem (6.13) and the corresponding Liouville problem with a closed smooth surface as its jump discussed in [22] are similar in the form, but they are distinct essentially by Remark 5.2. The tool for the generalized principal part needs to be referred here and the condition for growth at the infinity is less restrictive here.

6.3. Jump Problem R_m

In this section, we discuss the following jump problem R_m.

Jump problem R_m. Find a sectionally regular function $\Phi(w)$, with $\mathbb{R}_0^{n+1}$ as its jump plane, such that

$$\begin{cases} \Phi^+(x) - \Phi^-(x) = g(x), \ x \in \mathbb{R}_0^{n+1} \ \big(\text{boundary value condition}\big), \\ G.P\left[w^{-(m+1)}\Phi, \infty\right] = 0 \ \big(\text{growth condition at}\infty\big), \end{cases} \tag{6.17}$$

where

$$g \in \widehat{H}^{\mu}_{m_0}\left(\mathbb{R}_0^{n+1}\right) \ \text{ with } \ m_0 = \max\left\{0, -(m+1)\right\}. \tag{6.18}$$

When $m \geq 0$ we call it the jump problem with non-negative order. This case is discussed in [13, 30], but the theoretical basis of the results obtained is insufficient (see Remark 6.4 below). When $m < 0$ we call it the jump problem with negative order (in fact, its solution has zero point at ∞). The discussion of this situation is rather technical and will be postponed to the later part of this section.

We firstly discuss the jump problem with non-negative order, in this case, $m_0 = 0$, *i.e.*, we assume $g \in \widehat{H}\left(\mathbb{R}_0^{n+1}\right)$.

Conclusion 6.3. (Non-negative order) When $m \geq 0$ and $g \in \widehat{H}\left(\mathbb{R}_0^{n+1}\right)$, the general solution of the jump problem (6.17) with the non-negative order is

$$\begin{aligned} \Phi(w) &= \Big(\mathcal{S}[g]\Big)(w) + P_m(w) \\ &= \frac{1}{\bigvee_{n+1}} \int_{\mathbb{R}_0^{n+1}} E(x-w)\mathrm{d}\sigma g(x) + \sum_{|\alpha|=0}^{m} \frac{1}{|\alpha|!} Z^{\alpha}(w)\, c_{\alpha}, \ w \in \mathbb{R}_{\pm}^{n+1}, \end{aligned} \tag{6.19}$$

where P_m is arbitrary hypercomplex symmetric polynomial of degree not exceeding m.

Proof. Firstly, $\big(\mathcal{S}[g]\big)(w)$ is a special solution of R_0 problem by Example 5.1, Example 5.2 and the Plemelj-Sochocki formulae in Corollary 4.1. Certainly, it is also a solution of the jump problem with the non-negative orders.

Secondly, Φ is the solution of R_m problem (6.17), if and only if $\Delta = \Phi - \mathcal{S}[g]$ is the solution of the Liouville problem

$$\begin{cases} \Delta^+(x) = \Delta^-(x), \quad x \in \mathbb{R}_0^{n+1}, \\ G.P\left[w^{-(m+1)}\Delta, \infty\right] = 0. \end{cases} \tag{6.20}$$

Hence, $\Delta = P_m$ by Conclusion 6.2, which results in (6.19). ☐

Remark 6.4. It should be pointed out that both the two steps in the above proof have to be based on Theorem 4.4. The articles [14] , however, is based on the insufficient Corollary 4.2 which is an oversight.

Due to extensive complications, there has been no study aware devoting to this case, even for the classical context $n = 1$. This case will be discussed in detail in the following part of this article. The generalized principal part tool must be used here, even in the case $n = 1$, which should be one reason why in the well known monograph books [19, 24] R_m are not thoroughly studied but restricted to the bounded solutions, *i.e.*, $m = 0$. The results of the negative order R_m problem in this paper are also essentially generalized to the study for the case of classical $n = 1$ in [19,24]. More interestingly, subsequent work has found that these results have important applications in the asymptotic analysis of polynomials [3,21].

An important jump problem with negative order is R_{-1}, which is the fundamental one of R_m jump problem with $m < 0$.

Jump problem R_{-1}. Find a sectionally regular function $\Phi(w)$, with $\mathbb{R}_0^{n+1}$ as its jump plane, such that

$$\begin{cases} \Phi^+(x) - \Phi^-(x) = g(x), \ x \in \mathbb{R}_0^{n+1} \quad \text{(boundary value condition)}, \\ G.P\,[\Phi, \infty] = 0 \quad \text{(growth condition at the infinity)}, \end{cases} \tag{6.21}$$

where $g \in \widehat{H}^\mu\left(\mathbb{R}_0^{n+1}\right)$.

Conclusion 6.4. (R_{-1} problem) When $g \in \widehat{H}\left(\mathbb{R}_0^{n+1}\right)$ the jump problem R_{-1} (6.21) has the unique solution

$$\Phi(w) = \Big(\mathcal{S}(g)\Big)(w) = \frac{1}{\bigvee_{n+1}} \int_{\mathbb{R}_0^{n+1}} E(x-w)\,\mathrm{d}\sigma\, g(x), \quad w \in \mathbb{R}_\pm^{n+1}, \tag{6.22}$$

if and only if

$$g(\infty) = \lim_{x \in \mathbb{R}_0^{n+1},\ x \to \infty} g(x) = 0. \tag{6.23}$$

Proof. **Condition of solvability.** If the jump problem R_{-1} has a solution Φ, then, by the growth condition at $w = \infty$ in (6.21),

$$\Phi(\infty) = \lim_{w \in \mathbb{R}_\pm^{n+1},\ w \to \infty} \Phi(w) = 0. \tag{6.24}$$

This results in

$$\Phi^\pm(\infty) = \lim_{x \in \mathbb{R}_0^{n+1},\ x \to \infty} \Phi^\pm(x) = 0. \tag{6.25}$$

So, taking the limit under the boundary value condition of (6.21), we get, by the Plemelj-Sochocki formulae,

$$g(\infty) = \lim_{x \in \mathbb{R}_0^{n+1},\ x \to \infty} g(x) = \lim_{x \in \mathbb{R}_0^{n+1},\ x \to \infty} \Big[\Phi^+(x) - \Phi^-(x)\Big] = 0. \tag{6.26}$$

Solvability. In fact, by Example 5.1, Corollary 4.1 and Example 5.2, it is obvious that if the condition of solvability (6.23) is fulfilled, then $\mathcal{S}[g]$ is a solution of (6.21).

Uniqueness of the solution. Similarly, Φ is the solution of R_{-1} problem (6.21), if and only if $\Delta = \Phi - \mathcal{S}[g]$ is the solution of the Liouville problem (6.20) with $m = -1$. Hence, $\Delta = 0$ by Conclusion 6.2, that is to say that $\mathcal{S}[g]$ is the unique solution of the jump problem R_{-1}. □

Remark 6.5. (6.25) and (6.26) show that the boundary value condition in the R_{-1} problem also holds for the extended hypercomplex plane

$$\tilde{\mathbb{R}}_0^{n+1} = \mathbb{R}_0^{n+1} \cup \left\{\infty\big|_{\mathbb{R}_0^{n+1}}\right\}, \tag{6.27}$$

where $\infty\big|_{\mathbb{R}_0^{n+1}}$ given in Remark 4.8 represents the infinity at $\mathbb{R}_0^{n+1}$. Likewise, if Φ is the solution of the jump problem R_0, then

$$\Phi^+\Big(\infty\big|_{\mathbb{R}_0^{n+1}}\Big) - \Phi^-\Big(\infty\big|_{\mathbb{R}_0^{n+1}}\Big) = g\Big(\infty\big|_{\mathbb{R}_0^{n+1}}\Big) \tag{6.28}$$

holds.

Now we come to discuss the jump problem (6.17) with $m < -1$. Similar to (6.26), we have, from the growth condition at the infinity in (6.17),

$$g_{\mathrm{r}}(\infty) = \lim_{x\in\mathbb{R}_0^{n+1},\ x\to\infty} x^r g(x) = 0 \quad \text{where} \quad r = -(m+1). \tag{6.29}$$

Obviously, we have, for $s \geq 0$,

$$\Big|g_{\mathrm{s}}(x)\Big| \leq \frac{M}{|x|^{\mu}} \text{ or} \Big|g(x)\Big| \leq \frac{M}{|x|^{s+\mu}} \Big(x \in \mathbb{R}_0^{n+1}\backslash\{0\}\Big) \text{ when } g \in \widehat{H}_{s,0}\Big(\mathbb{R}_0^{n+1}\Big). \tag{6.30}$$

So, in discussing for the jump problem (6.17) with $m < -1$ we will assume directly

$$g \in \widehat{H}^{\mu}_{r,0}\Big(\mathbb{R}_0^{n+1}\Big) \quad \Big(r = -(m+1)\Big). \tag{6.31}$$

Conclusion 6.5. (Case $-n < m < -1$) Let $-n < m < -1$ and $r = -(m+1)$. If $g \in \widehat{H}_{r,0}\Big(\mathbb{R}_0^{n+1}\Big)$, then the jump problem (6.17) has the unique solution

$$\Phi(w) = \Big(\mathcal{S}[g]\Big)(w) = \frac{1}{\mathsf{V}_{n+1}} \int_{\mathbb{R}_0^{n+1}} E(x-w)\,\mathrm{d}\sigma\, g(x), \quad w \in \mathbb{R}_{\pm}^{n+1}. \tag{6.32}$$

Proof. Obviously, the solution Φ of the R_m $(m < -1)$ problem is also the solution of the R_{-1} problem. Thus, we only need to consider the growth condition of Φ at ∞, that is, prove

$$G.P\Big[w^{-(m+1)}\Big(\mathcal{S}[g]\Big)(w), \infty\Big] = 0, \tag{6.33}$$

or

$$\lim_{w\to\infty, w\in\mathbb{R}^{\pm}} w^{-(m+1)}\Big(\mathcal{S}[g]\Big)(w) = 0. \tag{6.34}$$

To show (6.34), we need to establish a series of lemmas. So, we shall give its proof in Remark 6.9 later. □

We rewrite the following functions and explore them further.

$$\Big(\mathcal{S}_1(g)\Big)(w) = \frac{1}{\mathsf{V}_{n+1}} \int_{|x|\leq\frac{1}{2}|w|} E(x,w)\,\mathrm{d}\sigma\, g\,(x)\,, \quad w \in \mathbb{R}_{\pm}^{n+1}, \tag{6.35}$$

and

$$\Big(\mathcal{S}_2(g)\Big)(w) = \frac{1}{\mathsf{V}_{n+1}} \int_{|x|\geq\frac{1}{2}|w|} E(x,w)\,\mathrm{d}\sigma\, g\,(x)\,, \quad w \in \mathbb{R}_{\pm}^{n+1}. \tag{6.36}$$

For, $g \in C\Big(\mathbb{R}_0^{n+1}\Big)$ and $\rho > 0$, let

$$\Big(\mathcal{S}^{\rho}(g)\Big)(w) = \frac{1}{\mathsf{V}_{n+1}} \int_{\overline{B}_n(0,\rho)} E(x,w)\,\mathrm{d}\sigma\, g\,(x)\,, \quad w \in \mathbb{R}^{n+1}\backslash\overline{B}_n(0,\rho). \tag{6.37}$$

Lemma 6.2. *The Cauchy type integral $S^\rho(g)$ has the Laurent series expansion near the infinity*

$$\Big(S^\rho[g]\Big)(w) = -\sum_{k=0}^{\infty} Q_k(w), \quad \big|w\big| > \rho, \tag{6.38}$$

where

$$Q_k(w) = \sum_{|\alpha|=k} \frac{(-1)^{|\alpha|}\partial^\alpha E(w)}{|\alpha|!} \frac{1}{\bigvee_{n+1}} \int_{B_n(0,\rho)} Z^\alpha(x)\,\mathrm{d}\sigma\, g(x). \tag{6.39}$$

Moreover,

$$\Big|Q_k(w)\Big| \leq M\, C_{k+n-1}^{k+1} \Big[1+k^2\Big] \frac{\rho^k}{\big|w\big|^{n+k}} \int_{B_n(0,\rho)} \big|g(x)\big|\mathrm{d}S, \quad |w| > \rho, \tag{6.40}$$

where the constant $M > 0$ only depends upon the dimension n.

Proof. When $\overline{B}_n(0,\rho)$ in (6.39) is replaced by $\Gamma = \partial B_{n+1}(0,\rho)$, this lemma will reduce to Lemma 12.1.1 in [2]. We can prove this lemma directly by the method in [2]. □

Lemma 6.3. *If $g \in \widehat{H}_{s,0}^{\mu}\left(\mathbb{R}_0^{n+1}\right)$ and $0 \leq s < n$, then $G.P(w^s \mathcal{S}_1[g], \infty) = 0$, more precisely,*

$$\big|w\big|^s \Big|\mathcal{S}_1[g](w)\Big| \leq \frac{M}{|w|^\mu} \quad \textit{near} \quad \infty. \tag{6.41}$$

Proof. Taking $\rho = \dfrac{1}{2}|w|$ in (6.39) and rewriting Q_k in this case by $\mathcal{Q}_k$, *i.e.*,

$$\mathcal{Q}_k(w) = \sum_{|\alpha|=k} \frac{(-1)^{|\alpha|}\partial^\alpha E(w)}{|\alpha|!} \frac{1}{\bigvee_{n+1}} \int_{B_n\left(0,\frac{1}{2}|w|\right)} Z^\alpha(x)\mathrm{d}\sigma g(x), \tag{6.42}$$

we get

$$\begin{aligned} \Big|\mathcal{Q}_k(w)\Big| &\leq M_1\, C_{k+n-1}^{k+1} \Big(1+k^2\Big) \frac{1}{|w|^n} \left[\frac{1}{2}\right]^k \int_{B_n\left(0,\frac{|w|}{2}\right)} \big|g(x)\big|\mathrm{d}S \\ &\triangleq M_1\, C_{k+n-1}^{k+1} \Big(1+k^2\Big) \frac{1}{|w|^n} \left[\frac{1}{2}\right]^k \Big[\P_1 + \P_2\Big], \end{aligned} \tag{6.43}$$

where

$$\P_1 = \int_{|x|\leq 1} \big|g(x)\big|\mathrm{d}x_0 \ldots \mathrm{d}x_{n-1} \leq M_2 \Big\|g\Big\|_\infty, \tag{6.44}$$

$$\begin{aligned} \P_2 &= \int_{1\leq|x|\leq\frac{1}{2}|w|} \big|g(x)\big|\mathrm{d}x_0 \ldots \mathrm{d}x_{n-1} \quad \Big(|w| > 1\Big) \\ &\leq M_3 \int_{1\leq|x|\leq\frac{1}{2}|w|} \frac{1}{|x|^{s+\mu}}\mathrm{d}x_0 \ldots \mathrm{d}x_{n-1} \quad \Big(\text{by } (6.31)\Big) \\ &\leq M_4 \int_1^{|w|} \rho^{n-1-s-\mu}\,\mathrm{d}\rho \\ &\leq M_5 \big|w\big|^{n-s-\mu} \quad \Big(\text{taking} 0 < \mu < 1\Big). \end{aligned} \tag{6.45}$$

Thus,

$$\left|\mathcal{Q}_k(w)\right| \leq M_6\, C_{k+n-1}^{k+1}\left(1+k^2\right)\left[\frac{1}{2}\right]^k\left[\frac{1}{|w|^n}+\frac{1}{|w|^{s+\mu}}\right]. \tag{6.46}$$

We get, for $|w|>1$ and $s<n$,

$$\begin{aligned}\left|w\right|^{s+\mu}\left|\mathcal{S}_1[g](w)\right| &\leq M_7\left|w\right|^{s+\mu}\sum_{k=1}^{\infty}\left|\mathcal{Q}_k(w)\right| \\ &\leq M_8\sum_{k=0}^{\infty}C_{k+n-1}^{k+1}\left(1+k^2\right)\left[\frac{1}{2}\right]^k \triangleq M,\end{aligned} \tag{6.47}$$

which is the result required. □

Remark 6.6. In fact, we have proved that, if $\ell=\operatorname{Ord}\big(\mathcal{S}_1[g],\infty\big)$ exists then $\ell\leq -s$.

Lemma 6.4. *If $g\in\widehat{H}^{\mu}_{s,0}\left(\mathbb{R}^{n+1}_0\right)$ $(s\geq 0)$, then $G.P\big(w^s\mathcal{S}_2[g],\infty\big)=0$, more precisely,*

$$\left|w\right|^s\left|\mathcal{S}_2[g](w)\right| \leq \frac{M}{|w|^{\nu}} \quad \textit{for } \left|\operatorname{Im}(w)\right|\geq c>0 \textit{ and } 0<\nu<\mu, \tag{6.48}$$

where M is a constant.

Proof. Firstly, by (4.78) and (4.80),

$$\begin{aligned}\left|\mathcal{S}_2[g](w)\right| &\leq \int_{|x|\geq\frac{1}{2}|w|}\frac{|g_{\mathbf{s}}(x)|}{|x-w|^n|x|^{s+\mu}}\,\mathrm{d}S \\ &\leq \frac{2^s}{|w|^s}\delta_2(w) \leq \frac{M_1}{c^{\nu}|w|^{s+\nu}} \quad \left(0<\nu<\mu\right),\end{aligned} \tag{6.49}$$

where M_1 is a constant. □

Divide $\mathfrak{D}[f]$ into three pieces

$$\left(\mathfrak{D}_1(f)\right)(w)=\frac{1}{\mathsf{V}_{n+1}}\int_{|x|\leq\frac{1}{2}|w|}E(x-w)\mathrm{d}\sigma\Big[f(x)-f\big(\operatorname{Re}(w)\big)\Big],\ w\in\mathbb{R}^{n+1}, \tag{6.50}$$

$$\left(\mathfrak{D}_2[f]\right)(w)=\frac{1}{\mathsf{V}_{n+1}}\int_{|x-\operatorname{Re}(w)|\leq\frac{1}{3}|w|}E(x-w)\mathrm{d}\sigma\Big[f(x)-f\big(\operatorname{Re}(w)\big)\Big],\ w\in\mathbb{R}^{n+1}, \tag{6.51}$$

$$\left(\mathfrak{D}_3[f]\right)(w)=\frac{1}{\mathsf{V}_{n+1}}\int_{|x|\geq\frac{1}{2}|w|,|x-\operatorname{Re}(w)|\geq\frac{1}{3}|w|}E(x-w)\mathrm{d}\sigma\Big[f(x)-f\big(\operatorname{Re}(w)\big)\Big],\ w\in\mathbb{R}^{n+1}. \tag{6.52}$$

Lemma 6.5. *If $g\in\widehat{H}^{\mu}_{s,0}\left(\mathbb{R}^{n+1}_0\right)$ and $0\leq s<n$, then $G.P\big(w^s\mathfrak{D}_1[g],\infty\big)=0$, more precisely,*

$$\left|w\right|^s\left|\mathfrak{D}_1[g](w)\right| \leq \frac{M}{|w|^{\mu}} \textit{ near } w=\infty \textit{ and } \left|\operatorname{Im}(w)\right|<c \ \left(w\in\mathbb{R}^{n+1}_{\pm}\right). \tag{6.53}$$

Proof. Noting

$$\big(\mathfrak{D}_1(g)\big)(w)=\big(\mathcal{S}_1[g]\big)(w)+g\big(\operatorname{Re}(w)\big)\big(\mathcal{S}_1[1]\big)(w), \tag{6.54}$$

and

$$
\begin{aligned}
&\Big|g\big(\mathrm{Re}(w)\big)\Big| \\
&\quad\leq \frac{\big|g_{\mathbf{s}}\big(\mathrm{Re}(w)\big)\big|}{\big|\big(\mathrm{Re}(w)\big)\big|^{s}} \\
&\quad\leq \frac{M}{\big|\big(\mathrm{Re}(w)\big)\big|^{s+\mu}} \leq \frac{2M}{|w|^{s+\mu}} \quad \text{near } \ w=\infty \ \Big(\big|\mathrm{Im}(w)\big|<c\Big),
\end{aligned}
\tag{6.55}
$$

by Lemma 6.3 and (3.47) in Example 3.4 we immediately get (6.53). □

Remark 6.7. It is easy to see that, if we substitute the condition $\mathrm{Ord}\big(\mathcal{S}_1[g],\infty\big) < -s$ for the condition $0 \leq s < n$, then (6.41) automatically holds. Thus, under this case, both Lemma 6.3 and Lemma 6.5 still hold.

Lemma 6.6. *If $g \in \widehat{H}_{s,0}\left(\mathbb{R}_0^{n+1}\right)$ $(s \geq 0)$, then $G.P\big(w^s\mathfrak{D}_2[g],\infty\big) = 0$, more precisely,*

$$
\big|w\big|^{s}\Big|\Big(\mathfrak{D}_2[g]\Big)(w)\Big| \leq \frac{M}{|w|^{\mu}} \, \textit{near} \, w=\infty \, \textit{and} \, \big|\mathrm{Im}(w)\big|<c\,\Big(w\in\mathbb{R}^{n+1}\Big), \tag{6.56}
$$

where M is a constant.

Proof. It is easily seen that, if $\big|x-\mathrm{Re}(w)\big| < \dfrac{1}{3}|w|$ and $\mathrm{Im}(w) < c$, then

$$
\Big|x\Big| \geq \Big|w\Big| - \Big|x-w\Big| \geq \Big|w\Big| - \left[\frac{|w|}{3} + \big|\mathrm{Im}(w)\big|\right] > \frac{|w|}{3} \quad \text{near } w=\infty, \tag{6.57}
$$

and

$$
\Big|\mathrm{Re}(w)\Big| \geq \Big|w\Big| - \Big|Im(w)\Big| > \frac{|w|}{2} \quad \text{near } w=\infty. \tag{6.58}
$$

Divide $\mathfrak{D}_2[g]$ into two parts

$$
\begin{aligned}
&\big(\mathfrak{D}_2[g]\big)(w) \\
&\quad= \frac{1}{\mathsf{V}_{n+1}} \int_{|x-\mathrm{Re}(w)|\leq\frac{1}{3}|w|} E(x-w)\,\mathrm{d}\sigma\, x^{-s}\Big[g_{\mathbf{s}}(x) - g_{\mathbf{s}}\big(\mathrm{Re}(w)\big)\Big] \\
&\qquad+ \frac{1}{\mathsf{V}_{n+1}} \int_{|x-\mathrm{Re}(w)|\leq\frac{1}{3}|w|} E(x-w)\,\mathrm{d}\sigma \left[x^{-s} - \big(\mathrm{Re}(w)\big)^{-s}\right]\Big\{g_{\mathbf{s}}\big(\mathrm{Re}(w)\big)\Big\} \\
&\quad\triangleq \mathcal{P}_1 + \mathcal{P}_2\Big\{g_{\mathbf{s}}\big(\mathrm{Re}(w)\big)\Big\}.
\end{aligned}
\tag{6.59}
$$

Noting that by (2.8)

$$
\begin{aligned}
\Big|x^{-1} - \big(\mathrm{Re}(w)\big)^{-1}\Big| &= \Big|x^{-1}\Big[\mathrm{Re}(w) - x\Big]\big(\mathrm{Re}(w)\big)^{-1}\Big| \\
&= \Big|x\Big|^{-1}\Big|x - \big(\mathrm{Re}(w)\big)\Big|\Big|\big(\mathrm{Re}(w)\big)\Big|^{-1},
\end{aligned}
\tag{6.60}
$$

we have

$$\begin{aligned} &\left|\mathcal{P}_1\right| \\ &\quad\le \frac{M_1}{|w|^s}\int_{|x-\mathrm{Re}(w)|\le\frac{1}{3}|w|}\left|E(x-w)\right|\left|x^{-1}-\left(\mathrm{Re}(w)\right)^{-1}\right|^{\mu}\mathrm{d}S\ \Big(\text{by (6.57)}\Big) \\ &\quad\le \frac{M_2}{|w|^{s+2\mu}}\int_{|x-\mathrm{Re}(w)|\le\frac{1}{3}|w|} \\ &\qquad\times\left|E(x-w)\right|\left|\mathrm{Re}(w)-x\right|^{\mu}\mathrm{d}S\ \Big(\text{by (6.57), (6.58), (6.60)}\Big) \\ &\quad\le \frac{M_3}{|w|^{s+\mu}}\ \Big(\text{by (4.10)}\Big). \end{aligned} \tag{6.61}$$

By induction, it is easy to prove that

$$\left|x^k-y^k\right|\le k\Big[\max\left\{|x|,|y|\right\}\Big]^{k-1}\left|x-y\right| \text{ for any} x,y\in\mathbb{R}^{n+1}\text{and} k\in N_0. \tag{6.62}$$

In fact, by using the inductive hypothesis, there follows

$$\begin{aligned} &\left|x^{k+1}-y^{k+1}\right| \\ &\quad\le \left|x^{k+1}-x^k y\right|+\left|x^k-y^k\right|\left|y\right| \\ &\quad\le |x|^k\left|x-y\right|+k\Big[\max\left\{|x|,|y|\right\}\Big]^{k-1}\left|x-y\right|\left|y\right| \\ &\quad\le (k+1)\Big[\max\left\{|x|,|y|\right\}\Big]^{k}\left|x-y\right|. \end{aligned} \tag{6.63}$$

So,

$$\begin{aligned} &\left|\mathcal{P}_2\right| \\ &\quad\le \frac{1}{\mathsf{V}_{n+1}}\int_{|x-\mathrm{Re}(w)|\le\frac{1}{3}|w|}\left|E(x-w)\right|\left|\mathrm{d}\sigma\right|\left|x^{-s}-\left(\mathrm{Re}(w)\right)^{-s}\right|\ \Big(\text{by (6.59)}\Big) \\ &\quad\le \frac{sM_1}{|w|^{s-1}}\int_{|x-\mathrm{Re}(w)|\le\frac{1}{3}|w|}\left|E(x-w)\right|\left|x^{-1}-\left(\mathrm{Re}(w)\right)^{-1}\right|\mathrm{d}S \\ &\qquad\times\Big(\text{by (6.57), (6.58), (6.62)}\Big) \\ &\quad\le \frac{M_2}{|w|^{s+1}}\int_{|x-\mathrm{Re}(w)|\le\frac{1}{3}|w|}\left|E(x-w)\right|\left|x-\left(\mathrm{Re}(w)\right)\right|\mathrm{d}S\ \Big(\text{by (6.60)}\Big) \\ &\quad\le \frac{M_3}{|w|^{s+1}}\int_{|x-\mathrm{Re}(w)|\le\frac{1}{3}|w|}\frac{1}{|x-w|^{n-1}}\mathrm{d}x_0\ldots\mathrm{d}x_{n-1} \\ &\quad\le \frac{M_4}{|w|^{s}}\quad\Big(\text{by (4.13)}\Big). \end{aligned} \tag{6.64}$$

(6.59), (6.61), (6.64) and (6.55) imply (6.56). ◻

Lemma 6.7. *If* $g\in\widehat{H}_{s,0}\left(\mathbb{R}_0^{n+1}\right)$ $(s\ge 0)$, *then* $G.P\big(w^s\mathfrak{D}_3[g],\infty\big)=0$, *more precisely,*

$$|w|^s\left|\Big(\mathfrak{D}_3[g]\Big)(w)\right|\le\frac{M}{|w|^{\mu}}\quad\text{near}\quad w=\infty\quad\text{and}\quad\left|\mathrm{Im}(w)\right|<c\Big(w\in\mathbb{R}^{n+1}\Big), \tag{6.65}$$

where M *is a constant.*

Proof. Since the ball $\overline{B}_n\big(\mathrm{Re}(w),\frac{1}{2}|w|\big)$ is disjoint from the ball $\overline{B}_n\big(0,\frac{1}{3}|w|\big)$ when $\left|\mathrm{Im}(w)\right|<c$ and w is sufficiently large such that $|\mathrm{Re}(w)|>[\frac{1}{2}+\frac{1}{3}]|w|$, we have

$$\begin{aligned}
&\big(\mathfrak{D}_3[g]\big)(w)\\
&= \frac{1}{\mathsf{V}_{n+1}} \int_{|x|\geq \frac{1}{2}|w|,\, |x-\mathrm{Re}\,(w)|\geq \frac{1}{3}|w|} E(x-w)\,\mathrm{d}\sigma\, g(x) - \Bigg[\Big(\mathcal{S}[1]\Big)(w) \\
&\quad -\Big(\mathcal{S}_{|x|\leq \frac{1}{2}|w|}[1]\Big)(w) - \Big(\mathcal{S}_{|x-\mathrm{Re}\,(w)|\leq \frac{1}{3}|w|}[1]\Big)(w)\Bigg] g\big(\mathrm{Re}(w)\big) \\
&= \mathbb{p}_1 + \mathbb{p}_2 + \mathbb{p}_3 + \mathbb{p}_4.
\end{aligned} \tag{6.66}$$

We first have

$$\begin{aligned}
&\Big|\mathbb{p}_1\Big|\\
&\leq \frac{1}{\mathsf{V}_{n+1}} \int_{|x|\geq \frac{1}{2}|w|,\, |x-\mathrm{Re}\,(w)|\geq \frac{1}{3}|w|} \big|E(x-w)\big|\,\big|\mathrm{d}\sigma\big|\,\big|g(x)\big| \\
&\leq M_1 \int_{|x|\geq \frac{1}{2}|w|,\, |x-\mathrm{Re}\,(w)|\geq \frac{1}{3}|w|} \Big|E(x-w)\Big| \frac{1}{|x|^{s+\mu}}\,\mathrm{d}S \Big(\text{by } (6.55)\Big) \\
&\leq M_2 \int_{|x|\geq \frac{1}{2}|w|,\, |x-\mathrm{Re}(w)|\geq \frac{1}{3}|w|} \frac{1}{|x|^{n+s+\mu}}\,\mathrm{d}x_0\mathrm{d}x_1\ldots\mathrm{d}x_{n-1} \\
&\Big(\text{by} 4|x-w| \geq |x-w| + 3|x-\mathrm{Re}(w)| \geq |x-w| + |w| \geq |x|\Big) \\
&\leq M_2 \int_{|x|\geq \frac{1}{2}|w|} \frac{1}{|x|^{n+s+\mu}}\,\mathrm{d}x_0\mathrm{d}x_1\ldots\mathrm{d}x_{n-1} \\
&\leq \frac{M_3}{|w|^{s+\mu}}.
\end{aligned} \tag{6.67}$$

By (6.55), (3.9) in Example 3.1, (3.47) in Example 3.4 and (3.38) in Example 3.3, we have

$$\Big|\mathbb{p}_2\Big|, \Big|\mathbb{p}_3\Big|, \Big|\mathbb{p}_4\Big| \leq \frac{M_4}{|w|^{s+\mu}}. \tag{6.68}$$

(6.66), (6.67) and (6.68) imply (6.65). □

Auxiliary Theorem 6.1. *If* $0 \leq s < n$ *and* $g \in \widehat{H}_{s,0}\left(\mathbb{R}_0^{n+1}\right)$*, then G.P* $\left(w^s\mathcal{S}[g], \infty\right) = 0$*, more precisely,*

$$\Big|\Big(\mathcal{S}[g]\Big)(w)\Big| \leq \frac{M}{|w|^{s+\nu}} \quad \textit{for} \;\; 0 < \nu < \mu \;\; \Big(w \in \overline{\mathbb{R}_{\pm}^{n+1}}\Big), \tag{6.69}$$

Proof. By the Plemelj-Sochocki formulae, it is sufficient to prove (6.69) when $w \in \mathbb{R}_{\pm}^{n+1}$. To do so, we treat two cases separately.

Case 1. $\mathrm{Im}(w) \geq 1$. In this case, we have

$$\big(\mathcal{S}[g]\big)(w) = \big(\mathcal{S}_1[g]\big)(w) + \big(\mathcal{S}_2[g]\big)(w), \quad w \in \mathbb{R}_{\pm}^{n+1}. \tag{6.70}$$

By Lemma 6.3 and Lemma 6.4 we get (6.69) when $w \in \mathbb{R}_{\pm}^{n+1}$.

Case 2. $\mathrm{Im}(w) \leq 1$. In this case, we have

$$\begin{aligned}\big(\mathcal{S}[g]\big)(w) = &\pm \frac{1}{2}\, g\big(\mathrm{Re}(w)\big) + \big(\mathfrak{D}_1[g]\big)(w) + \big(\mathfrak{D}_2[g]\big)(w) \\ &+ \big(\mathfrak{D}_3[g]\big)(w), \quad w \in \mathbb{R}^{n+1}_{\pm}.\end{aligned} \tag{6.71}$$

By (6.55), Lemma 6.5, Lemma 6.6 and Lemma 6.7, we also get (6.69) when $w \in \mathbb{R}^{n+1}_{\pm}$. □

Theorem 6.3. (Boundary behavior of the Cauchy type integrals–First Version) *Let* $0 \leq s < n$ *and* $f \in \widehat{H}_s\left(\mathbb{R}^{n+1}_0\right)$, *then*

$$\left|\Big(\mathcal{S}[f]\Big)(w) - \frac{1}{2} f(\infty)\right| \leq \frac{M}{|w|^{s+\nu}} \quad for \;\; 0 < \nu < \mu \;\; \Big(w \in \mathbb{R}^{n+1}_{+}\Big) \tag{6.72}$$

and

$$\left|\Big(\mathcal{S}[f]\Big)(w) + \frac{1}{2} f(\infty)\right| \leq \frac{M}{|w|^{s+\nu}} \quad for \;\; 0 < \nu < \mu \;\; \Big(w \in \mathbb{R}^{n+1}_{-}\Big). \tag{6.73}$$

Proof. Let

$$g(t) = f(t) - f(\infty), \;\; t \in \mathbb{R}^{n+1}_0, \tag{6.74}$$

by Auxiliary Theorem 6.1, the desired inequalities hold. □

Remark 6.8. Taking $s = 0$, the conclusion about the boundary behavior of the Cauchy type integrals $\mathcal{S}^{\pm}[f]$ in Theorem 4.3, *i.e.*, $\mathcal{S}^{\pm}[f] \in H_{\dagger}(\infty)$, holds.

Remark 6.9. (**The proof of the growth condition** (6.34) **in Conclusion** 6.5) By using Auxiliary Theorem 6.1, the growth condition (6.34) holds.

Remark 6.10. [22] points out that, the solution Φ of the Riemann boundary value problems R_m $(m < 0)$ on the closed smooth surfaces must satisfy $\mathrm{Ord}(\Phi, \infty) \leq -n$, even there is no regular function F near the infinity that satisfies $\mathrm{Ord}(F, \infty) = m$ with $-n < m < 0$ when $n > 1$. In other words, the solution of R_{-1} is also the solution of the solution of R_m $(-n < m < -1)$. In the setting of this paper, this phenomenon does not occur, *i.e.*, there exists Φ such that it is the solution of the R_m problem (6.17) $(-n < m < -1)$ but not the solution of the R_{m+1} problem. For example, we take the input function g given by (2.53) in Example 2.2. Then, $\mathcal{S}[g]$ is the solution of R_{-s} but not the solution of R_{-s+1}, otherwise $\mathrm{Ord}(g, \infty) \leq -(s+1)$ by (6.29).

To discuss the R_m problems with $m < -n$ we need to establish the improved version of Auxiliary Theorem 6.1. The Remark 6.5 suggests that we can improve Auxiliary Theorem 6.1 into a stronger version.

Auxiliary Theorem 6.2. *If* $s \geq 0$ *and* $f \in \widehat{H}_{s,0}\left(\mathbb{R}^{n+1}_0\right)$, *then we have the following claims.*

(1) $G.P(|w|^s \mathcal{S}_1[f], \infty) = 0$ *is equivalent to* $G.P(|w|^s \mathcal{S}[f], \infty) = 0$,

(2) $\mathcal{S}_1[f] = O\big(|w|^{-(s+\mu)}\big)$ *near* ∞ *is equivalent to* $\mathcal{S}[f] = O\left(|w|^{-(s+\mu)}\right)$ *near* ∞.

Proof. In exactly the same way to Auxiliary Theorem 6.1, through (6.70), (6.71), (6.54), (6.55), Lemma 6.4, Lemma 6.5, Lemma 6.6 and Lemma 6.7, we get the two claims. □

When Auxiliary Theorem 6.2 is applied to discern the growth condition in the R_m problem (6.17), we must give a measurement for the behavior $\mathcal{S}_1[g]$ at the infinity which is similar to Lemma 6.3. To do so, we first introduce a result in [22].

Let

$$Q_k(x) = \sum_{|\alpha|=k} \big[\partial^\alpha E\big](x)\,\lambda_\alpha \tag{6.75}$$

be a hypercomplex Laurent polynomial where λ_α's are some hypercomplex constants.

Lemma 6.8. (see [22]) *Let Q_k be a hypercomplex Laurent polynomial given in (6.75). Then* $\operatorname{Ord}\big(Q_k,\infty\big) = -n-k$ *if and only if*

$$\sum_{|\alpha|=k} |\lambda_\alpha| \neq 0, \tag{6.76}$$

in other words, there is α such that $|\alpha| = k$ but $\lambda_\alpha \neq 0$.

Obviously, if $g \in \widehat{H}_{r,0}\Big(\mathbb{R}_0^{n+1}\Big)$, then $g = O\big(x^{-r-\mu}\big)$ near $x=\infty$. Thus, by Example 5.3 we have $Z^\alpha(x)g(x) = O\big(|x|^{|\alpha|-r-\mu}\big)$, so, all integrals

$$\mathfrak{Z}_\alpha = \int_{\mathbb{R}_0^{n+1}} Z^\alpha(x)\mathrm{d}\sigma g(x),\ \ \big|\alpha\big| = 0,1,\dots,r-n\ \ \big(r \geq n\big) \tag{6.77}$$

are well defined. Let

$$\Xi = \Big\{\big|\alpha\big|,\ \mathfrak{Z}_\alpha \neq 0,\ \big|\alpha\big| = 0,1,\dots,r-n\Big\}. \tag{6.78}$$

Lemma 6.9. *If $g \in \widehat{H}^{\mu}_{r,0}\Big(\mathbb{R}_0^{n+1}\Big)$ and $r \geq n$, then*

(1) $\operatorname{Ord}\big(\mathcal{S}_1,\infty\big) = -n-N$, *where* $N = \min\big\{\ell,\ \ell \in \Xi\big\}$ *when Ξ is not empty,*
(2) $\big(\mathcal{S}_1[g]\big)(w) = O\big(|w|^{-r-\mu}\big)$ *near $w=\infty$ when Ξ is empty.*

Proof. We need to estimate (6.46) more finely. In fact [2,6],

$$\begin{aligned}
&\Big|\mathcal{Q}_k(w)\Big| \quad \Big(\text{if } \ |w|>2 \ \text{ and } \ k \geq r-n+1\Big)\\
&\quad\leq M_1 C^{k+1}_{k+n-1}\Big(1+k^2\Big)\frac{1}{|w|^{n+k}}\int_{B_n\left(0,\frac12|w|\right)} \big|x^k g(x)\big|\mathrm{d}s\\
&\quad\leq M_1 C^{k+1}_{k+n-1}\Big(1+k^2\Big)\frac{1}{|w|^{n+k}}\left[\int_{B_n(0,1)} \big|x^k g(x)\big|\mathrm{d}s + \int_{B_n\left(1,\frac12|w|\right)} \big|x^k g(x)\big|\mathrm{d}s\right]\\
&\quad\leq C^{k+1}_{k+n-1}\Big(1+k^2\Big)\frac{M_2}{\big|w\big|^{n+k}}\bigg[\Big\|g\Big\|_\infty\\
&\qquad + \int_{B_n\left(1,\frac12|w|\right)} \frac{1}{|x|^{r+\mu-k}}\mathrm{d}x_0\dots\mathrm{d}x_{n-1}\bigg]\ \Big(\text{by } (6.55)\Big)\\
&\quad\leq M_3 C^{k+1}_{k+n-1}\Big(1+k^2\Big)\frac{1}{|w|^{n+k}}\left[1+\left[\frac{|w|}{2}\right]^{-r-\mu+k+n}\right]\\
&\quad\leq M_4 C^{k+1}_{k+n-1}\Big(1+k^2\Big)\left[\frac{1}{\big|w\big|^{n+k}} + \frac{1}{2^k|w|^{r+\mu}}\right]
\end{aligned}$$

$$\leq M_4 C_{k+n-1}^{k+1}\Big(1+k^2\Big)\left[\frac{1}{2^{n+k-r-\mu}}+\frac{1}{2^k}\right]\frac{1}{|w|^{r+\mu}}\ \Big(|w|>2,\, k\geq r-n+1\Big)$$
$$\leq M_5 C_{k+n-1}^{k+1}\Big(1+k^2\Big)\left[\frac{1}{2^k}\right]\frac{1}{|w|^{r+\mu}}. \tag{6.79}$$

Let

$$\Big(\mathcal{S}_{1,2}[g]\Big)(w)=\sum_{k=r-n+1}^{\infty}\mathcal{Q}_k(w),\quad |w|>2, \tag{6.80}$$

then we have

$$\Big|w\Big|^{r+\mu}\Big|\Big(\mathcal{S}_{1,2}[g]\Big)(w)\Big|\leq M\sum_{k=r-n+1}^{+\infty}\Big[1+k^2\Big]C_{k+n-1}^{k+1}\left[\frac{1}{2}\right]^k<+\infty, \tag{6.81}$$

where the constant M is independent on w.

Rewrite

$$\Big(\mathcal{S}_{1,1}[g]\Big)(w)=\sum_{|\alpha|=0}^{r-n}\Big[\partial^\alpha E\Big](w)\mathfrak{Z}_\alpha-\sum_{|\alpha|=0}^{r-n}\Big[\partial^\alpha E\Big](w)\mathcal{Z}_\alpha(w)\triangleq\triangle(w)-\nabla(w) \tag{6.82}$$

where $\mathfrak{Z}_\alpha$'s are given in (6.77) and

$$\mathcal{Z}_\alpha(w)=\int_{\frac{1}{2}|w|\leq x\leq+\infty}Z^\alpha(x)\mathrm{d}\sigma g(x). \tag{6.83}$$

When $\Xi\neq\emptyset$, by Lemma 6.8, we have

$$\lim_{w\in\mathbb{R}_\pm^{n+1},w\to\infty}\Big|w\Big|^{n+N}\left|\sum_{|\alpha|=k}\Big[\partial^\alpha E\Big](w)\mathfrak{Z}_\alpha\right|=0\,\text{for}\,0\leq k\leq r-n\,\text{and}\,k\neq N, \tag{6.84}$$

since $\mathcal{Q}_k=0$ when $k<N$ by the definition for N and $w^{n+k}\mathcal{Q}_k(w)$ is bounded when $N<k<r-n$ by Example 2.5 and (5.8). Thus, by Lemma 6.8,

$$\limsup_{w\in\mathbb{R}_\pm^{n+1},w\to\infty}\Big|w\Big|^{n+N}\Big|\triangle(w)\Big|=\limsup_{w\in\mathbb{R}_\pm^{n+1},w\to\infty}\Big|w\Big|^{n+N}\left|\sum_{|\alpha|=N}\Big[\partial^\alpha E\Big](w)\mathfrak{Z}_\alpha\right|=a, \tag{6.85}$$

where $0<a<\infty$. Thus,

$$\mathrm{Ord}\Big(\triangle,\infty\Big)=-n-N. \tag{6.86}$$

From (6.83), Example 5.3 and (6.55), we also have

$$\begin{aligned}\Big|\mathcal{Z}_\alpha(w)\Big|&\leq M_1\int_{\frac{1}{2}|w|\leq x\leq+\infty}\Big|x\Big|^{|\alpha|-r-\mu}\mathrm{d}x_0\ldots\mathrm{d}x_{n-1}\\&\leq\frac{M_2}{|w|^{-|\alpha|+r+\mu-n}}\ \ \Big(|\alpha|\leq r-n\Big),\end{aligned} \tag{6.87}$$

which results in, by (2.62) in Example 2.5,

$$\Big|\Big[\partial^\alpha E\Big](w)\mathcal{Z}_\alpha(w)\Big|\leq\frac{M_3}{|w|^{r+\mu}}. \tag{6.88}$$

This implies

$$\begin{cases} \Big|\nabla(w)\Big| \leq \dfrac{M}{|w|^{r+\mu}} \ \text{ near } \infty \ \Big(w \in \mathbb{R}^{n+1}_{\pm},\ M\text{is a constant}\Big), \\ \lim\limits_{w\in\mathbb{R}^{n+1}_{\pm},\, w\to\infty} \Big|w\Big|^{n+N}\Big|\nabla(w)\Big| = 0 \ \text{ when } \ \Xi \neq \emptyset \ \Big(\text{by } r \geq n\Big). \end{cases} \tag{6.89}$$

Now, noting

$$\Big(\mathcal{S}_1[g]\Big)(w) = \begin{cases} \Big(\mathcal{S}_{1,1}[g]\Big)(w) + \Big(\mathcal{S}_{1,2}[g]\Big)(w), \text{ when } \ \Xi \neq \emptyset, \\ \nabla(w) + \Big(\mathcal{S}_{1,2}[g]\Big)(w), \qquad\qquad \text{when } \ \Xi = \emptyset, \end{cases} \tag{6.90}$$

by (6.81), (6.82), (6.86) and (6.89) we finally get the assertions (1) and (2) in this lemma. □

Theorem 6.4. *If $s \geq n$ and $g \in \widehat{H}^{\mu}_{s,0}\left(\mathbb{R}^{n+1}_0\right)$, then $G.P\big(w^s\mathcal{S}[g], \infty\big) = 0$, more precisely,*

$$\Big|\Big(\mathcal{S}(g)\Big)(w)\Big| \leq \frac{M}{|w|^{s+\nu}} \quad for \quad 0 < \nu < \mu \ \ \Big(w \in \overline{\mathbb{R}^{n+1}_{\pm}}\Big). \tag{6.91}$$

Combining Theorem 6.4 and Theorem 6.3 we get the following theorem.

Theorem 6.5. (Boundary behavior of the Cauchy type integrals-Second Version) *If $s \geq 0$ and $f \in \widehat{H}_s\left(\mathbb{R}^{n+1}_0\right)$, then*

$$\Big|\Big(\mathcal{S}[f]\Big)(w) - \frac{1}{2}f(\infty)\Big| \leq \frac{M}{|w|^{s+\nu}} \quad for \ \ 0 < \nu < \mu \ \ \Big(w \in \mathbb{R}^{n+1}_{+}\Big) \tag{6.92}$$

and

$$\Big|\Big(\mathcal{S}[f]\Big)(w) + \frac{1}{2}f(\infty)\Big| \leq \frac{M}{|w|^{s+\nu}} \quad for \ \ 0 < \nu < \mu \ \ \Big(w \in \mathbb{R}^{n+1}_{-}\Big). \tag{6.93}$$

Conclusion 6.6. (Order $m \leq -n$) Let $m \leq -n$ and $g \in \widehat{H}_{-(m+1),0}\Big(\mathbb{R}^{n+1}_0\Big)$, then the jump problem (6.17) has unique solution (6.32) when the C^n_{-m-1} conditions of solvability

$$\int_{\mathbb{R}^{n+1}_0} Z^{\alpha}(x)\mathrm{d}\sigma g(x) = 0, \ \ |\alpha| = 0, 1, \ldots, -(n+1+m) \tag{6.94}$$

are fulfilled.

Proof. Uniqueness of the solution. The solution of the R_m ($m \leq -n$) problem must be the solution of the R_{-1} problem, so it is (6.32).

Sufficiency of condition of solvability. The growth condition holds by Theorem 6.4.

Necessity of condition of solvability. By Auxiliary Theorem 6.2 and Lemma 6.9, the growth condition holds only when $\Xi = \emptyset$. □

Summing up the above Conclusion 6.3, Conclusion 6.4, Conclusion 6.5 and Conclusion 6.6, we get the main result below.

Theorem 6.6. *For the Riemann boundary value problem* R_m *(6.17), four cases will happen.*

(1) *Let* $m \geq 0$*,* $g \in \widehat{H}^{\mu}\left(\mathbb{R}_0^{n+1}\right)$*, then its general solution is (6.19) with* C_{n+m}^{m} *free hypercomplex constants.*

(2) *Let* $m = -1$*. If* $g \in \widehat{H}\left(\mathbb{R}_0^{n+1}\right)$*, then it has the unique solution (6.22) if and only if (6.23) holds.*

(3) *Let* $-n < m < -1$ *and* $r = -(m+1)$*. If* $g \in \widehat{H}_{r,0}\left(\mathbb{R}_0^{n+1}\right)$*, then it has the unique solution (6.32).*

(4) *Let* $m \leq -n$ *and* $r = -(m+1)$*. If* $g \in \widehat{H}_{r,0}\left(\mathbb{R}_0^{n+1}\right)$*, then it has the unique solution (6.32) when the* C_{-m-1}^{n} *conditions (6.94) are fulfilled.*

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