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A Holomorphic Extension Result*

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In [7] we obtained, as a consequence of the Fourier transform theory developed in the paper, a sufficient and necessary condition on a sequence $(b_n) \in l^\infty$ for the function $\phi(z) = \sum_{n=1}^{\infty} b_n z^n$, $|z| < 1$, to be holomorphically extensible to a heart-shaped region containing the set $\{z \in \mathbf{C} | z \neq 1, |z| = 1\}$, and dominated by $C/|1-z|$ when z is near 1 in the region. This note generalizes this result to the cases when $|b_n| \leq Cn^s$, $-\infty < s < \infty$. It also includes corresponding results for series of negative powers and for Laurent series as well. The theory has applications to singular and fractional integrals on closed Lipschitz curves which are closely related to boundary value problems in Lipschitz domains.

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INTRODUCTION

We will use the following sets in the complex plane $\mathbf{C}$. Set, for $\omega \in (0, \pi/2]$,

$$\mathbf{S}_{\omega,\pm} = \{z \in \mathbf{C} | |\arg(\pm z)| < \omega\},$$

$$\mathbf{W}_{\omega,\pm} = \{z \in \mathbf{C} | |\text{Re}(z)| \leq \pi \text{ and } \text{Im}(\pm z) > 0\} \cup \mathbf{S}_\omega,$$

$$\mathbf{C}_{\omega,\pm} = \{z = \exp(i\eta) \in \mathbf{C} | \eta \in \mathbf{W}_{\omega,\pm}\},$$

respectively, where

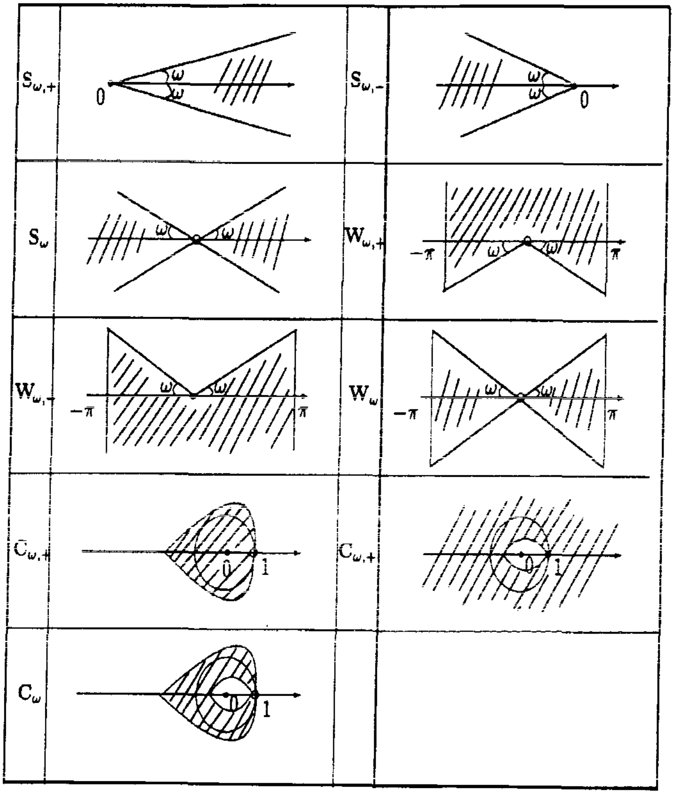
$$\mathbf{S}_\omega = \mathbf{S}_{\omega,+} \cup \mathbf{S}_{\omega,-}.$$

We will also use

$$\mathbf{W}_\omega = \mathbf{W}_{\omega,+} \cap \mathbf{W}_{\omega,-}, \qquad \mathbf{C}_\omega = \mathbf{C}_{\omega,+} \cap \mathbf{C}_{\omega,-}.$$

These sets are illustrated in the diagram on the next page.

*Dedicated to Professor M. T. Cheng.



A set $\mathbf{O}$ in the complex plane is said to be *inner star-shaped with the pole zero*, if $z \in \mathbf{O}$ implies $rz \in \mathbf{O}$ for all $0 < r \leq 1$; and is said to be *outer star-shaped with the pole zero*, if $z \in \mathbf{O}$ implies $rz \in \mathbf{O}$ for all $1 \leq r < \infty$.

For every $\omega \in (0, \pi/2]$, $\mathbf{C}_{\omega,+}$ is a heart-shaped and an inner star-shaped region with the pole zero, while $\mathbf{C}_{\omega,-}$ is the complement set of a heart-shaped region which is an outer star-shaped region with the pole zero.

The following function spaces on the sectors will be used. For $-\infty < s < \infty$,

$$H^s(\mathbf{S}_{\omega,\pm}) = \{b \,:\, \mathbf{S}_{\omega,\pm} \to \mathbf{C} | b \text{ is holomorphic and satisfies}$$
$$|b(z)| \leq C_\mu |z \pm 1|^s \text{ in every } \mathbf{S}_{\mu,\pm},\ 0 < \mu < \omega\},$$

respectively.

For $s = -1, -2, \dots$, we will also use another class of spaces

$$H^s_{\ln}(\mathbf{S}_{\omega,\pm}) = \{b : \mathbf{S}_{\omega,\pm} \to \mathbf{C} | b \text{ is holomorphic and satisfies },$$
$$|b(z)| \le C_\mu |z \pm 2|^s |\ln|z \pm 2|| \text{ in every } \mathbf{S}_{\mu,\pm},\ 0 < \mu < \omega\},$$

respectively.

There are also corresponding function spaces on the double sectors. For $-\infty < s < \infty$,

$$H^s(\mathbf{S}_\omega) = \{b : \mathbf{S}_\omega \to \mathbf{C} | b_\pm \in H^s(\mathbf{S}_{\omega,\pm}),\ \text{ where } b_\pm = b\chi_{\{z \in \mathbf{C} | \pm \mathrm{Re}\, z > 0\}}\},$$

and

$$H^s_{\ln}(\mathbf{S}_\omega) = \{b : \mathbf{S}_\omega \to \mathbf{C} | b_\pm \in H^s_{\ln}(\mathbf{S}_{\omega,\pm}),\ \text{ where } b_\pm = b\chi_{\{z \in \mathbf{C} | \pm \mathrm{Re}\, z > 0\}}\},$$

where χ_E denotes the characteristic function of set E.

Therefore, functions in various H^s and $H^s_{\ln}$ spaces defined above consist of the functions in sectors which are bounded near zero and dominated by $C_\mu |z|^s$ and $C_\mu |z|^s \ln|z|$ at ∞, respectively, in any smaller sectors than those in which the functions are holomorphically defined.

A function given by a Laurent series is said to be *holomorphically defined* in a certain region, if the Laurent series converges to a holomorphic function in the region. In the case, by a theorem of Abel, the power series part then is holomorphically defined in the associated inner star-shaped region with the pole zero, determined by the region in the obvious way, and the negative power series part is holomorphically defined in the associated outer star-shaped region with the pole zero.

Denote, for $s > -1$,

$$K^s(\mathbf{C}_{\omega,\pm}) = \Big\{\phi : \mathbf{C}_{\omega,\pm} \to \mathbf{C} | \phi \text{ is holomorphic and satisfies}$$
$$|\phi(z)| \le C_\mu \frac{1}{|1-z|^{1+s}} \text{ in every } \mathbf{C}_{\mu,\pm},\ 0 < \mu < \omega\Big\},$$

respectively, and

$$K^s(\mathbf{C}_\omega) = \Big\{\phi : \mathbf{C}_\omega \to \mathbf{C} | \phi \text{ is holomorphic and satisfies}$$
$$|\phi(z)| \le C_\mu \frac{1}{|1-z|^{1+s}} \text{ in } \mathbf{C}_\mu,\ \mu \in (0, \omega)\Big\}.$$

We will only give the details of the definitions of $K^s(\mathbf{C}_{\omega,+})$ for $-\infty < s \leq -1$. The definitions of $K^s(\mathbf{C}_{\omega,-})$ and $K^s(\mathbf{S}_\omega)$ for $-\infty < s \leq -1$ can be correspondingly formulated.

We need the following preparation. Assume that

(i) $\underline{b} = (b_n)_{n=0}^{\infty} \in l^{\infty}$;
(ii) $\phi_{\underline{b}}(z) = \sum_{n=0}^{\infty} b_n z^n$ is holomorphically defined in $\mathbf{C}_{\omega,+}$;
(iii) $\phi_{\underline{b}}(1) = \sum_{n=0}^{\infty} b_n$ converges.

Form the difference

$$\begin{aligned}\phi_{\underline{b}}(z) - \phi_{\underline{b}}(1) &= b_1(z-1) + b_2(z^2-1) + \cdots + b_n(z^n-1) + \cdots \\ &= (z-1)\phi_{I(\underline{b})}(z),\end{aligned}$$

where

$$I(\underline{b}) = \left(\sum_{k=n}^{\infty} b_k\right)_{n=1}^{\infty} \in l^{\infty},$$

and

$$\phi_{I(\underline{b})}(z) = \sum_{n=1}^{\infty}\left(\sum_{k=n}^{\infty} b_k\right) z^{n-1}.$$

Then, owing to (ii), $\phi_{I(\underline{b})}$ is holomorphically defined in $\mathbf{C}_{\omega,+}$.

The above formed new sequence $I(b)$ may or may not satisfy the condition (iii). If it satisfies (iii), then it satisfies automatically (i). Therefore the pair $(I(\underline{b}), \phi_{I(\underline{b})})$ satisfies the condition (i), (ii) and (iii). Then one can consider whether the sequence $I(I(\underline{b})) = I^2(\underline{b})$ satisfies (iii) or not, and so on. Denote $I(I^n(\underline{b})) = I^{n+1}(\underline{b})$, and $I^0(\underline{b}) = \underline{b}$. If the above procedure can be applied at most k times, then it happens that pairs $(I^j(\underline{b}), \phi_{I^j(\underline{b})})$, $0 \leq j \leq k$, all satisfy the conditions (i) to (iii), but $I^{k+1}(\underline{b})$ does not satisfy the condition (iii). In this case, we have

$$\phi_{\underline{b}}(z) = \phi_{\underline{b}}(1) + (z-1)\phi_{I(\underline{b})}(1) + \cdots + (z-1)^k \phi_{I^k(\underline{b})}(z). \qquad (1)$$

Now we are ready to introduce the definitions of $K^s(\mathbf{C}_{\omega,+})$, $-\infty < s \leq -1$:

$$K^s(\mathbf{C}_{\omega,+}) = \Big\{\phi_{\underline{b}} : \mathbf{C}_{\omega,+} \to \mathbf{C} | \underline{b} \in l^\infty, \text{ the above procedure can be applied at most } k_s \text{ times, where } k_s = [-s-1] \text{ or } [-s], \text{ depending on whether } s \text{ is an integer or not, respectively, and } |(z-1)^{k_s}\phi_{I^{k_s}(\underline{b})}(z)| \leq \frac{C_\mu}{|z-1|^{1+s}} \text{ in every } \mathbf{C}_{\mu,+},\ 0<\mu<\omega\Big\},$$

where for $\alpha > 0$, $[\alpha] = \max\{n \in Z | n \leq \alpha\}$, the largest integer that does not exceed α.

For $s = -1, -2, \ldots$, we will also consider another class

$$K^s_{\ln}(\mathbf{C}_{\omega,+}) = \Big\{\phi_{\underline{b}} : \mathbf{C}_{\omega,+} \to \mathbf{C} | \underline{b} \in l^\infty, \text{ the above procedure can be applied at most } -s-1 \text{ times, and } |(z-1)^{-s-1}\phi_{I^{-s-1}(\underline{b})}(z)| \leq C\frac{|\ln|z-1||}{|z-1|^{1+s}} \text{ in every } \mathbf{C}_{\mu,+},\ 0<\mu<\omega\Big\}.$$

The above defined various spaces H^s and K^s are all increasing classes along with $s \to \infty$. Now we are ready to state our results. In each of the following statements or argument paragraphs $\pm$ should be understood as either all $+$ or all $-$.

THEOREM 1 *Let* $-\infty < s < \infty$, $s \neq -1; -2, \ldots, b \in H^s(\mathbf{S}_{\omega,\pm})$, *and*

$$\phi(z) = \sum_{n=\pm 1}^{\pm\infty} b(n) z^n.$$

Then $\phi \in K^s(\mathbf{C}_{\omega,\pm})$.

THEOREM 2 *Let* $-\infty < s < \infty$, $\phi \in K^s(\mathbf{C}_{\omega,\pm})$. *Then for every* $\mu \in (0,\omega)$, *there exists a function* $b^\mu \in H^s(\mathbf{S}_{\mu,\pm})$ *such that*

$$\phi(z) = \sum_{n=\pm 1}^{\pm\infty} b^\mu(n) z^n.$$

Moreover, for $s<0$, $z \in \mathbf{S}_{\mu,\pm}$,

$$b^{\mu}(z) = \frac{1}{2\pi}\int_{\lambda_{\pm}(\mu)} \exp(-i\eta z)\phi(\exp(i\eta))\,d\eta, \tag{2}$$

where

$$\begin{aligned}\lambda_{\pm}(\mu) = \{\eta \in \mathbf{W}_{\omega,\pm} | \eta = r\exp(i(\pi \pm \mu)),\\ r \text{ is from } \pi \sec(\mu) \text{ to } 0\text{; and then}\\ \eta = r\exp(\mp i\mu),\ r \text{ is from } 0 \text{ to } \pi \sec(\mu)\},\end{aligned}$$

and, for $s \le 0$, $z \in \mathbf{S}_{\mu,\pm}$,

$$\begin{aligned} b^{\mu}(z) = \frac{1}{2\pi}\lim_{\epsilon\to 0}\Bigg(\int_{l(\epsilon,|z|^{-1})\cup c_{\pm}(|z|^{-1},\mu)\cup\Lambda_{\pm}(|z|^{-1},\mu)}\\ \exp(-i\eta z)\phi(\exp(i\eta))\,d\eta + \phi^{[s]}_{\epsilon,\pm}(z)\Bigg), \end{aligned} \tag{3}$$

where if $r \le \pi$,

$$\begin{aligned} l(\epsilon,r) = \{\eta = x+iy | y=0,\ x \text{ is from } -r \text{ to } -\epsilon,\\ \text{and then from } \epsilon \text{ to } r\},\\ c_{\pm}(r,\mu) = \{\eta = r\exp(i\alpha) | \alpha \text{ is from } \pi \pm \mu \text{ to } \pi, \text{ then from } 0 \text{ to } \mp\mu\}, \end{aligned}$$

and

$$\begin{aligned}\Lambda_{\pm}(r,\mu) = \{\eta \in \mathbf{W}_{\omega,\pm} | \eta = \rho\exp(i(\pi \pm \mu)),\ \rho \text{ is from } \pi \sec(\mu) \text{ to } r;\\ \text{and then } \eta = \rho\exp(\mp i\mu),\ \rho \text{ is from } r \text{ to } \pi \sec(\mu)\},\end{aligned}$$

and if $r > \pi$,

$$\begin{aligned} l(\epsilon,r) = l(\epsilon,\pi), \qquad c_{\pm}(r,\mu) = c_{\pm}(\pi,\mu),\\ \Lambda_{\pm}(r,\mu) = \Lambda_{\pm}(\pi,\mu), \end{aligned}$$

and, in any case,

$$\phi^{[s]}_{\epsilon,\pm}(z) = \int_{L_{\pm}(\epsilon)} \phi(\exp(i\eta))\left(1 + (-i\eta z) + \cdots + \frac{(-i\eta z)^{[s]}}{[s]!}\right)d\eta,$$

where $L_{\pm}(\epsilon)$ *is any contour from* $-\epsilon$ *to* ε *lying in* $\mathbf{C}_{\omega,\pm}$.

THEOREM 3 *Let s be a negative integer.*

1° *If* $b \in H^s(\mathbf{S}_{\omega,\pm})$ *and* $\phi(z) = \sum_{n=\pm 1}^{\pm\infty} b(n)z^n$, *then* $\phi \in K^s_{\ln}(\mathbf{C}_{\omega,\pm})$.

2° *If* $\phi \in K^s_{\ln}(\mathbf{C}_{\omega,\pm})$, *then for every* $\mu \in (0, \omega)$ *there exists a function* b^μ *such that* $b^\mu \in H^s_{\ln}(\mathbf{S}_{\mu,\pm})$, *and*

$$\phi(z) = \sum_{n=\pm 1}^{\pm\infty} b^\mu(n)z^n.$$

Moreover, b^μ *is given by (2).*

The cases "+" and "−" in the above theorems are associated with power series and negative power series, respectively. Combining these results, we obtain the results of the same type on Laurent series (for the case $s = 0$ also see [7]).

THEOREM 4 *Let* $-\infty < s < \infty$, $s \neq -1, -2, \ldots$, $b \in H^s(\mathbf{S}_\omega)$, *and*

$$\phi(z) = \sum_{n=-\infty}^{\infty} b(n)z^n.$$

Then $\phi \in K^s(\mathbf{C}_\omega)$.

There are also the Laurent series counterparts of Theorems 2 and 3 which are left to the interested reader.

Remark 1 For $(b_n)_{n=1}^\infty \in l^\infty$ the series $\phi(z) = \sum_{n=1}^\infty b_n z^n$ is naturally defined and holomorphic in the unit disc. Theorem 1 and **1**° Theorem 3 assert that if $\exists b \in H^s(\mathbf{S}_{\omega,+})$ such that $b_n = b(n)$, then ϕ is holomorphically extensible to $\mathbf{C}_{\omega,+}$, and in any smaller $\mathbf{C}_{\mu,+}$ the function satisfies the estimate given in the definitions of $K^s_{\ln}(\mathbf{S}_{\omega,+})$ or $K^s(\mathbf{S}_{\omega,+})$, depending on whether s is or not a negative integer, respectively. Theorem 2 and **2**° Theorem 3 give the converse results.

Remark 2 Under the assumptions of Theorem 2, the mapping $\phi \to b$ satisfying $\phi(z) = \sum b(n)z^n$ is not a single-valued mapping. In fact, according to Theorem 2, every b^μ, $0 < \mu < \omega$, gives a solution of b, and, if $\mu_1 \neq \mu_2$, then $b^{\mu_1} \neq b^{\mu_2}$ in general (see also Remark 3 below for an example).

Remark 3 In the proof of Theorem 2 we will need the following function space $\tilde{P}^+_\omega$ consisting of all finite linear combinations of holomorphic functions of the form

$$g_n(z) = \begin{cases} 1, & \text{if} \quad z = n; \\ \dfrac{(\exp(i\pi(z-n)) - \exp(-i\pi(z-n)))\exp(-\pi(z-n)\tan\omega)}{2i\pi(z-n)}, & \text{if} \quad z \neq n, \end{cases}$$

where n is a non-negative integer. It is easy to see that

$$|g_n(z)| \leq C_{\mu,n} \frac{\exp(-\pi(\operatorname{Re}(z)\tan\omega - |In(z)|))}{|z+1|}, \quad z \in \mathbf{S}_{\mu,+}, \quad 0 < \mu < \omega.$$

Therefore, $g_n \in \cup_{-\infty<s<\infty} H^s(\mathbf{S}_{\omega,+})$. It is noted that the functions in $\tilde{P}^+$ are just the inverse Fourier transforms of finite polynomials of z given by (2) in Theorem 2. Similarly, we define the space $\tilde{P}^-$ with respect to the negative integers.

Remark 4 The holomorphic extension result Theorem 1 is the best possible in the following sense: if ω is the maximal angle for which $b \in H^s(\mathbf{S}_{\omega,+})$, then ϕ cannot be holomorphically extended to any larger heart-shaped region $\mathbf{C}_{\omega+\delta,+}$, $\delta > 0$ satisfying the corresponding estimates, for, otherwise, it would introduce a contradiction to the maximality of the angle ω, owing to the converse result Theorem 2. Questions along this line can be otherwise formulated. Alan Beardon raised the following questions which still remain open: Whether the heart-shaped region $\mathbf{C}_{\omega,+}$ is the largest region in which the series $\phi(z) = \sum_{n=1}^{\infty} b(n)z^n$ is holomorphically defined with respect to all functions $b \in H^s(\mathbf{S}_{\omega,+})$? And, whether $\partial\mathbf{S}_{\omega,\pm}$ being the natural boundary of a holomorphic function b would imply $\partial\mathbf{C}_{\omega,\pm}$ being the natural boundary of the associated holomorphic function ϕ?

Remark 5 The result **1**° Theorem 3 is consistent with the example $b(z) = z/(1+z^2)$. Albert Baernstein showed me, concerning the case $s = -1$ in **2**° Theorem 3, how to construct a holomorphic function ϕ in the unit disc so that $\phi(z) = O(\ln|z-1|)$ and $\phi'(z) \neq O(1/|z-1|)$, $z \to 1$, using the method of Bloch functions in e.g. [6] and Ahlfors' distortion theorem (see, for example, [1]). He also showed me that it is equivalent to consider the matter in the unit disc instead of in the heart-shaped region, as in the case $s = -1$ the estimates remain unchanged after applying a suitable conformal mapping. It follows, owing to the case $s = 0$ in Theorem 1, that the associated $b(z) \neq O(1/|z|)$ at ∞, which complements **2**° Theorem 3. However, it remains open question the estimates given in **2**° Theorem 3 are the best possible in those cases.

Remark 6 We have restricted ourselves to considering only the first power of the log function in the definitions of $H^s_{\ln}$ and $K^s_{\ln}$ and in Theorem 3. In fact we could generalize the result in 2° of the theorem, with a very same proof, to any kth power of the log function, where k is a positive integer.

Remark 7 A trivial variation of Theorem 1 to Theorem 4 can be obtained in the following way. Denote by $\exp(-i\theta\cdot)$ the function $z \to \exp(i\theta z)$. Define the co-spaces

$$H^{s,\theta}(\mathbf{S}_{\omega,\pm}) = \exp(i\theta\cdot)H^s(\mathbf{S}_{\omega,\pm}), \qquad H^{s,\theta}(\mathbf{S}_\omega) = \exp(i\theta\cdot)H^s(\mathbf{S}_\omega),$$

and

$$K^{s,\theta}(\mathbf{C}_{\omega,\pm}) = \{\phi|\phi \circ \exp(-i\theta\cdot) \in K^s(\mathbf{C}_{\omega,\pm})\}$$

and

$$K^{s,\theta}(\mathbf{S}_\omega) = \{\phi|\phi \circ \exp(-i\theta\cdot) \in K^s(\mathbf{S}_\omega)\}$$

If we change the statements of the theorems by using these spaces with the parameter θ, then the singular point $z = 1$ of the functions ϕ_+, ϕ will now be shifted to the point $z = \exp(i\theta)$ on the unit circle. The proof is an easy exercise, and is left to the reader.

Remark 8 The results corresponding to the case $s = 0$ are obtained as a by-product in [7] in the study of the Fourier transform theory between holomorphic functions defined on sectors. It is proved in [3] that the functions in $K^0(\mathbf{C}_{\omega,\pm})$ and $K^0(\mathbf{S}_\omega)$, acting as kernel functions, all give rise to L^2-bounded operators on those star-shaped Lipschitz curves whose Lipschitz constants are less than $\tan(\omega)$. The class of the singular convolution operators is in fact the H^∞-functional calculus of the Dirac operator $z(d/dz)$ living on the closed curves ([7]). Using conformal mappings, we can deduce a corresponding singular integral theory on arbitrary simply-connected Lipschitz curves. The cases associated with $s \neq 0$ correspond to fractional integrals and differentials on those curves. All those mentioned are closely related to boundary value problems associated with those domains. For related studies, see e.g. [2, 4, 5, 8].

We will only need to prove Theorem 1 to Theorem 3 for the case "+", as the case "−" is similar. §1 will be devoted to proving Theorem 1 and **1**° of Theorem 3. §2 will be devoted to proving Theorem 2 and **2**° of Theorem 3.

The author is grateful to Alan McIntosh for many helpful discussions on this work. Many thanks are due to Joachim Hempel, for his support which has encouraged me to pursue my previous work [7] further and so to obtain the generalizations in this note, and also for the valuable consultations with him throughout the course of the study and the preparation of the paper. Many thanks are due to Albert Baernstein for the time he devoted to answering my questions regarding Remark 5 during his stay in UNE after the annual meeting of the Australian Mathematical Society in July 1994 in Armidale. Especially, his remarks gave impetus to writing up this paper when I was wondering about the result in 2° Theorem 3. I am very grateful to Alan Beardon for his inspirating questions indicated in Remark 4, and, with Joachim Hempel and Imer Bokor together as well, helpful discussions on the iterating process on sequences in formulating the class $C^s(\mathbf{S}_{\omega,+})$, $-\infty < s \leq -1$.

1 PROOFS OF THEOREM 1 AND 1° OF THEOREM 3

In the sequel we will abbreviate $H^s(\mathbf{S}_{\omega,+})$ and $K^s(\mathbf{C}_{\omega,+})$ as H^s_ω and K^s_ω, respectively. We first consider the cases $0 \leq s < \infty$. Define, as in [5],

$$\Psi(z) = \frac{1}{2\pi}\int_{\rho_\theta} \exp(iz\zeta)b(\zeta)\,d\zeta, \qquad z \in \mathbf{V}_{\omega,+},$$

where

$$\mathbf{V}_{\omega,+} = \{z \in \mathbf{C} | \operatorname{Im}(z) > 0\} \cup \mathbf{S}_\omega,$$

and ρ_θ is the ray $r\exp(i\theta)$, $0 < r < \infty$, where θ is chosen so that $\rho_\theta \in \mathbf{S}_{\omega,+}$, and $\exp(iz\zeta)$ is exponentially decaying as $\zeta \to \infty$ along ρ_θ. It is easy to see that Ψ is well defined and holomorphic in $\mathbf{V}_{\omega,+}$. In fact, the definition is independent of specially chosen θ satisfying the required conditions. It is easy to see that for any $\mu \in (0, \omega)$,

$$|\Psi(z)| \leq \frac{C_\mu}{|z|^{1+s}}, \qquad z \in \mathbf{V}_{\mu,+}.$$

We further define the function

$$\Psi^1(z) = \int_{\delta(z)} \Psi(\zeta)\,d\zeta, \qquad z \in \mathbf{S}_{\omega,+},$$

where $\delta(z)$ is any path from $-z$ to z lying in $\mathbf{V}_\omega$. From Cauchy's formula it is easy to deduce, for any $\mu \in (0, \omega)$,

$$|\Psi^1(z)| \leq \frac{C_\mu}{|z|^s}, \qquad z \in \mathbf{S}_{\mu,+}.$$

Define ψ by the Poisson summation formula

$$\Psi(z) = 2\pi \sum_{n=-\infty}^{\infty} \Psi(z + 2n\pi), \qquad z \in \bigcup_{n=-\infty}^{\infty} (2n\pi + \mathbf{W}_{\omega,+}),$$

where the summation takes the following sense: For $s > 0$, the series absolutely and locally uniformly converges to a 2π-periodic holomorphic function, namely ψ, and the function $\phi = \psi \circ \ln / i \in K_{\omega}^{s}$. For $s = 0$ there exists a sequence $(n_k)_1^{\infty}$ such that the partial sum $s_{n_k}(z) = 2\pi \sum_{|n| \leq n_k} \Psi(z + 2n\pi)$ converges locally uniformly to a 2π-periodic function, namely ψ, and the function $\phi = \psi \circ \ln / i \in K_{\omega}^{s}$. It can be shown that different functions Φ defined via different appropriate subsequences (n_k) differ by bounded constants. Using the estimate of Ψ, the proof of the assertion for $s > 0$ is easy. The assertion for $s = 0$ is proved in [7]. To make the paper self-contained, we provide a brief proof of the case and refer the unsatisfied reader to [7]. Consider the decomposition

$$\sum_{k=-n}^{n} \Psi(z + 2k\pi) = \Psi(z) + \sum_{k \neq 0}^{\pm n} (\Psi(z + 2k\pi) - \Psi(2k\pi))$$
$$+ \sum_{k=1}^{n} \Psi^{1'}(2k\pi) = \Psi(z) + \sum_1 + \sum_2, \quad z \in \mathbf{W}_{\mu,+}.$$

We will prove that $\sum_1$ is absolutely convergent and bounded, and $\sum_2$ converges in the sense specified as above, and is also bounded. Therefore, the principle entry of the sum is $\Psi(z)$ which is dominated by $C|z|^{-1}$ as $z \to 0$, and so is the function ψ. Therefore, the function $\phi = \psi \circ \ln / i$ satisfies the desired estimate. To deal with $\sum_1$ we need to use the inequality

$$|\Psi'(z)| \leq \frac{C_\mu}{|z|^{2+s}}, \qquad z \in \mathbf{W}_{\mu,+},$$

deduced by using Cauchy's formula. To deal with $\sum_2$, using the mean value theorem of integration, we have

$$\begin{aligned}\sum_{k=1}^{n} \Psi^{1'}(2k\pi) &= \int_{2\pi}^{2(n+1)\pi} \Psi^{1'}(r)\,dr + \sum_{k=1}^{n} (\Psi^{1'}(2k\pi) \\ &\quad - \text{Re}\,(\Psi^{1'})(\xi_k) - i\ \text{Im}\,(\Psi^{1'})(\eta_k)) \\ &= \Psi^1(2(n+1)\pi) - \Phi^1(2\pi) \\ &\quad + \sum_{k=1} n(\Psi^{1'}(2k\pi) - \text{Re}\,(\Psi^{1'})(\xi_k) - i\ \text{Im}\,(\Psi^{1'})(\eta_k)),\end{aligned}$$

where $\xi_k, \eta_k \in (2k\pi, 2(k+1)\pi)$. Owing to the estimate of Ψ' again, the series part in the above expression converges absolutely. Since the first entry is bounded, by choosing a suitable subsequence (n_k), it tends to a constant with the same bounds. This finishes the proof for the case $s = 0$.

For the cases $s < 0$, we will apply induction to the intervals $-k-1 \leq s < -k$, where $k \geq 0$ is an integer. First consider the case $-1 < s < 0$. Let $b \in H^s_\omega$ and

$$\phi(z) = \sum_{n=1}^{\infty} b(n)z^n, \qquad \phi_0(z) = \sum_{n=1}^{\infty} nb(n)z^n$$

$$z\phi'(z) = \phi_0(z).$$

Since $b \in H^s_\omega$, we have $(\cdot)b(\cdot) \in H^{s+1}_\omega$, where $0 < s+1 < 1$. As proved above, we have $\phi_0 \in K^{s+1}_\mu$, and the series ϕ_0 is locally uniformly convergent. This allows us to integrate the series $\phi_0(z)/z$ term by term. Taking into account that the region $\mathbf{C}_{\omega,+}$ is star-shaped, denoting by $l(0, z)$ the segment from 0 to $1 \approx z = x + iy \in \mathbf{C}_{\mu,+}$, and owing to the estimate on functions in the class K^{s+1}_μ, we have

$$\begin{aligned} |\phi(z)| &\leq \int_{l(0,z)} \left|\frac{\phi_0(\zeta)}{\zeta}\right| |d\zeta| \\ &\leq C_\mu \int_{l(0,z)} \frac{1}{|1-\zeta|^{s+2}} |d\zeta| \\ &\leq C_\mu \int_0^1 \frac{1}{(|1-tx| + t|y|)^{s+2}} dt. \end{aligned}$$

To work out the estimate we consider two cases: $x \leq 1$ and $x > 1$. For $x \leq 1$, the above reduces to

$$\begin{aligned} \left|\int_0^1 \frac{1}{(1 - t(x-|y|))^{s+2}} dt\right| &= \frac{1}{s+1}\frac{1}{x-|y|}\left(\frac{1}{(|1-x|+|y|)^{s+1}} - 1\right) \\ &\leq C_{\mu,s}\frac{1}{|1-z|^{s+1}}, \end{aligned}$$

where we used the condition $z \approx 1$, and so $x \approx 1$, $y \approx 0$.

For the case $x > 1$, since z is in the star-shaped region $\mathbf{C}_{\mu,+}$, there follows $x - 1 = |1-x| \leq (\tan\mu)|y|$, and so $|y| \geq C_\mu(|1-x| + |y|)$. This, together with $x \approx 1$ and $y \approx 0$, gives

$$\begin{aligned}
&\int_0^1 \frac{1}{(|1-tx|+t|y|)^{s+2}}dt \\
&= \int_0^{1/x} \frac{1}{(1-t(x-|y|))^{s+2}}dt + \int_{1/x}^1 \frac{1}{(t(x+|y|)-1)^{s+2}}dt \\
&= \frac{1}{s+1}\left(\frac{2x}{x^2-y^2}\frac{x^{s+1}}{|y|^{s+1}} + \frac{1}{x+|y|}\frac{1}{(|1-x|+|y|)^{s+1}} - \frac{1}{x-|y|}\right) \\
&\le \frac{C_\mu}{|1-z|^{s+1}}.
\end{aligned}$$

For the case $s=-1$, using the previously obtained result for the case $s=0$, a similar argument gives

$$|\phi(z)| \le C_\mu \int_{l(0,z)} \frac{1}{|1-\zeta|}|d\zeta| \le C_\mu |\ln|1-z||,$$

where $z \in \mathbf{C}_{\mu,+}$.

This completes the proof for $-1 \le s < 0$. Our induction hypothesis is:

Let $-k-1 \le s < -k$, where $k \ge 0$ is an integer, and $b \in H_\omega^s$. Defining $\underline{b} = (b(n))_{n=1}^\infty$, we have $\phi_{\underline{b}} \in K_\omega^s$.

Now consider the case $-k-2 \le s < -k-1$, $k \ge 0$ an integer, and $b \in H_\omega^s$. Set

$$\phi(z) = \sum_{n=1}^\infty b(n)z^n, \qquad \phi_0 = \sum_{n=1}^\infty b_0(n)z^n,$$

where $b_0(z) = \sum_{n=0}^\infty b(z+n)$. It is easy to see that $b_0 \in H_\omega^{s+1}$. Since $-k-1 \le s+1 < -k$, it follows, from the induction hypothesis, that $\phi_0 \in K_\omega^{s+1}$. Therefore, $\phi_{I^{[-s-2]}(\underline{b}_0)}$ or $\phi_{I^{[-s-1]}(\underline{b}_0)}$, depending on whether s is or is not an integer, respectively, is holomorphically extensible to $\mathbf{C}_{\omega,+}$, where $\underline{b}_0 = (b_0(n))_{n=1}^\infty$. And, in these cases we have, for $z \in \mathbf{C}_{\mu,+}$,

$$|(z-1)^{[-s-2]}\phi_{I^{[-s-2]}(\underline{b}_0)}(z)| \le C_\mu \frac{|\ln|z-1||}{|z-1|^{s+2}},$$

or

$$|(z-1)^{[-s-1]}\phi_{I^{[-s-1]}(\underline{b}_0)}(z)| \le \frac{C_\mu}{|z-1|^{s+2}},$$

respectively.

Since $I^k \underline{b}_0 = I^{k+1}\underline{b}$ for any integer $k \to 0$, we have that $\phi_{I^k(\underline{b}_0)} = \phi_{I^{k+1}(\underline{b})}$, and so

$$|(z-1)^{[-s-1]}\phi_{I^{[-s-1]}(\underline{b})}(z)| \leq C_\mu \frac{|\ln|z-1||}{|z-1|^{s+1}},$$

or

$$|(z-1)^{[-s]}\phi_{I^{[-s]}(\underline{b})}(z)| \leq \frac{C_\mu}{|z-1|^{s+1}},$$

depending on whether s is or is not an integer, respectively. This proves that $\phi \in K^s_\omega$ when $b \in H^s_\omega$, $-k-2 \leq s < -k-1$. The induction is complete.

2 PROOFS OF THEOREM 2 AND 2° OF THEOREM 3

We will first prove Theorem 2.

Let $\phi \in K^s_\omega$, $-\infty < s < \infty$. The task is to show that the function b^μ defined using (1) or (2) belongs to H^s_μ, and $\phi(z) = \sum_{n=1}^{\infty} b^\mu(n)z^n$.

We first consider the cases $-\infty < s < 0$. It is easy to verify, using the expressions (2), as well as (1) when $s = -1, -2, \ldots$, the corresponding estimates on the given function ϕ, and Cauchy's theorem, that $\lim_{z\to 0} b^\mu(z) = 1/2\pi \int_{-\pi}^{\pi} \phi(\exp(ix))\,dx$, and so b is bounded near zero.

For z near ∞, according to (2), we have

$$b^\mu(z) = \frac{1}{2\pi}\int_{\lambda(\mu)} \exp(i\eta z)\phi(\exp(i\eta))\,d\eta, \qquad z \in \mathbf{S}_{\mu,+}$$

where

$$\lambda(\mu) = \{\eta \in \mathbf{W}_{\omega,+} | \eta = r\exp(i(\pi+\mu)),\ r \text{ is from } \pi \sec(\mu) \text{ to } 0,$$
$$\text{and then } \eta = r\exp(-i\mu),\ r \text{ is from } 0 \text{ to } \pi \sec(\theta)\},$$

where $|\arg(z)| < \mu < \omega$. Let $|\arg(z)| < \theta < \mu$. Owing to the estimate of ϕ and the property of the contour $\lambda(\mu)$, by taking into account the expression (1) and Remark 3 as well for $s = -1, -2, \ldots$, the bounds of the function b^μ are given by

$$|b^\mu(z)| \leq C_\mu \left(|z|^s + \int_0^\infty \exp(-\sin(\mu-\theta)|z|r)\frac{dr}{r^{1+s}}\right) \leq C_{\mu,\theta}|z|^s.$$

Now we consider the cases $0 \le s < \infty$. According to the expression (2), considering first the case $|z|^{-1} \le \pi$ corresponding to $z \approx \infty$, we have,

$$\begin{aligned} b^{\mu}(z) = \frac{1}{2\pi} \lim_{\epsilon \to 0} \Bigg\{ & \left(\int_{\epsilon \le |t| \le |z|^{-1}} \exp(-itz)\phi(\exp(it))\, dt + \phi_{\epsilon}^{[s]}(z) \right) \\ & + \int_{c_+(|z|^{-1},\mu)} \exp(-i\eta z)\phi(\exp(i\eta))\, d\eta \\ & + \int_{\Lambda_+(|z|^{-1},\mu)} \exp(-i\eta z)\phi(\exp(i\eta))\, d\eta \Bigg\} \\ = \frac{1}{2\pi} \lim_{\epsilon \to 0} & \{I_1(\epsilon, z) + I_2(z, \mu) + I_3(z, \mu)\}, \end{aligned}$$

where $|\arg(z)| < \mu < \omega$,

$$c_+(r, \mu) = \{\eta = r\exp(i\alpha) | \alpha \text{ is from } \pi + \mu \text{ to } \pi, \text{ then from } 0 \text{ to } -\mu\},$$

and

$$\begin{aligned} \Lambda_+(r, \mu) = \{\eta \in \mathbf{W}_{\omega,+} | \eta = \rho\exp(i(\pi + \mu)),\ \rho \text{ is from } \pi \sec(\mu) \text{ to } r, \\ \text{and then } \eta = \rho\exp(-i\mu),\ \rho \text{ is from } r \text{ to } \pi \sec(\mu)\}. \end{aligned}$$

We now show that I_1, I_2, I_3 are uniformly dominated by the bounds indicated in the theorem, and $\lim_{\epsilon\to 0} I_1$ exists.

Applying Cauchy's theorem, we have

$$\begin{aligned} I_1(\epsilon, z) = & \int_{\epsilon \le |t| \le |z|^{-1}} \left(\exp(-itz) - 1 - \frac{(-itz)}{1!} - \cdots - \frac{(-itz)^{[s]}}{[s]!} \right) \\ & \times \phi(\exp(it))\, dt + \int_{\epsilon \le |t| \le |z|^{-1}} \left(1 + \frac{(-itz)}{1!} + \cdots + \frac{(-itz)^{[s]}}{[s]!} \right) \\ & \times \phi(\exp(it))\, dt + \phi_{\epsilon,+}^{[s]}(z) \\ = & \int_{\epsilon \le |t| \le |z|^{-1}} \left(\exp(-itz) - 1 - \frac{(-itz)}{1!} - \cdots - \frac{(-itz)^{[s]}}{[s]!} \right) \\ & \phi(\exp(it))\, dt + \phi_{|z|^{-1},+}^{[s]}(z). \end{aligned}$$

Invoking the estimate of ϕ, we have

$$\begin{aligned}
&\left|\int_{\epsilon\le|t|\le|z|^{-1}}\left(\exp(-itz)-1-\frac{(-itz)}{1!}-\cdots-\frac{(-itz)^{[s]}}{[s]!}\right)\phi(\exp(it))\,dt\right|\\
&\quad\le C_\mu\int_{\epsilon\le|t|\le|z|^{-1}}|t|^{[s]+1}|z|^{[s]+1}\frac{1}{|t|^{1+s}}dt\\
&\quad\le C_\mu|z|^{[s]+1}\int_0^{|z|^{-1}}t^{[s]-s}\,dx\\
&\quad= C_\mu|z|^s.
\end{aligned}$$

The argument also shows that $\lim_{\epsilon\to 0} I_1$ exists.

In order to estimate $\phi^{[s]}_{|z|^{-1},+}(z)$, recalling the notation introduced in Theorem 2, we only need to examine a general entry in

$$\int_{L_+(|z|^{-1})}\frac{(-i\eta z)^k}{k!}\phi(\exp(i\eta))\,d\eta,\qquad k=0,1,\ldots,[s],\tag{4}$$

consisting of $\phi^{[s]}_{|z|^{-1},+}(z)$. Choose the contour $L_+(|z|^{-1})$ of the integration to be the upper half circle with radius $|z|^{-1}$ and center 0, we have

$$\begin{aligned}
\left|\int_{L_+(|z|^{-1})}\frac{(-i\eta z)^k}{k!}\phi(\exp(i\eta))\,d\eta\right| &\le C_\mu\int_{L_+(|z|^{-1})}|\eta z|^k|\eta|^{-1-s}|d\eta|\\
&\le C_\mu|z|^s.
\end{aligned}$$

To estimate I_2 we have

$$\begin{aligned}
|I_2(z,\mu)| &\le C_\mu\int_0^\mu \exp(|\eta|\,|z|\sin(\arg(z)+t))|\eta|\frac{dt}{|\eta|^{1+s}}\\
&\le C_\mu|z|^s.
\end{aligned}$$

Now consider I_3. Let $|\arg(z)|<\theta<\mu$, we have

$$\begin{aligned}
|I_3(z,\mu)| &\le C_\mu\int_{\Lambda(|z|^{-1},\mu)}\exp(|\eta|\,|z|\sin(\mu-\theta))\frac{|d\eta|}{|\eta|^{1+s}}\\
&\le C_\mu\int_{|z|^{-1}}^\infty r^{-1-s}\exp(-r|z|\sin(\mu-\theta))\,dr\\
&\le C_{\mu,\theta}|z|^s.
\end{aligned}$$

Now consider the case $|z|^{-1} > \pi$ corresponding to $z \approx 0$. We first show that the integral over the contour $l(\epsilon, \pi)$ is uniformly bounded and has limit as $\varepsilon \to 0$. The argument dealing with $I_1(\epsilon, z)$ for the case $|z|^{-1} \le \pi$ is still valid for the integral over $l(\epsilon, \pi)$, except that the integration contours of the entries in (4) now should be replaced by $L_+(\pi)$. Choose the contour $L_+(\pi)$ of the integration to be the upper half circle with radius π and center 0, we have

$$\left|\int_{L_+(\pi)} \frac{(-i\eta z)^k}{k!}\phi(\exp(i\eta))\,d\eta\right| \le C_\mu \int_{L_+(\pi)} |\eta z|^k |\eta|^{-1-s} |d\eta|$$
$$\le C_\mu |z|^k$$
$$\le C_\mu,$$

where $k = 1, 2, \ldots, [s]$.

To show that the integrals over $c_+(\pi, \mu)$ and $\Lambda_+(\pi, \mu)$ are bounded, we use Cauchy's theorem to change contour of the integration and so to integrate over the set $\{z = x + iy | x = -\pi,\ y$ is from $-\pi\ \tan(\mu)$ to 0, and then $x = \pi,\ y$ from 0 to $-\pi\ \tan(\mu)\}$. It is easy, however, to show that the integral over the last mentioned set is bounded, using only the fact that $\mathrm{Re}\,(z) > 0$.

Now we are left to prove

$$\phi(z) = \sum_{n=1}^{\infty} b^\mu(n) z^n, \qquad -\infty < s < \infty, \quad 0 < \mu < \omega,$$

which is equivalent to prove $b(n) = b^\mu(n)$, $n = 1, 2, \ldots$ in these cases.

Let $r \in (0, 1)$. Using the expression $\phi(rz) = \sum_{n=1}^{\infty} b(n) r^n z^n$ and the absolute convergence of the series in $|z| \le 1$, we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \exp(-itn)\phi(r\exp(it))\,dt = r^n b_n. \tag{5}$$

Now we first deal with the cases $s \ge 0$. Denote $\delta = -\ln(r)$, then $r \to 1-0$ if and only if $\delta \to 0+$. Taking limits $\delta \to 0+$ and $r \to 1-0$ on the left hand side and the right hand side of (5), respectively, the right hand side is equal to b_n, while the left hand side is equal to

$$\lim_{\delta\to 0+} \int_{-\pi}^{\pi} \exp(-itn)\phi(\exp(-\delta + it))\,dt.$$

For every fixed $\epsilon \in (0, \pi)$, the above can be written as

$$\lim_{\delta\to 0+}\left(\int_{0\le|t|\le\epsilon} + \int_{\epsilon\le|t|\le\pi}\right)\exp(-itn)\phi(\exp(-\delta+it))\,dt$$

$$= \lim_{\delta\to 0+}\left(\int_{0\le|t|\le\epsilon}\left(\exp(-itn) - 1 - \frac{-itn}{1!} - \frac{(-itn)^2}{2!} - \cdots - \frac{(-itn)^{[s]}}{[s]!}\right)\right.$$

$$\times\phi(\exp(-\delta+it))\,dt + \int_{L_+(\epsilon)}\left(1 + \frac{-itn}{1!} + \frac{(-itn)^2}{2!} + \cdots + \frac{(-itn)^{[s]}}{[s]!}\right)$$

$$\left.\times\phi(\exp(-\delta+it))\,dt + \int_{\epsilon\le|t|\le\pi}\exp(-itn)\phi(\exp(-\delta+it))\,dt\right)$$

$$= \lim_{\delta\to 0+}\int_{0\le|t|\le\epsilon}\left(\exp(-itn) - 1 - \frac{-itn}{1!} - \frac{(-itn)^2}{2!} - \cdots - \frac{(-itn)^{[s]}}{[s]!}\right)$$

$$\times\phi(\exp(-\delta+it))\,dt + \phi^{[s]}_{\epsilon,+}(n) + \int_{\epsilon\le|t|\le\pi}\exp(-itn)\phi(\exp(it))\,dt, \quad (6)$$

owing to Cauchy's theorem and the fact that the last two integrals are absolutely integrable as $\delta \to 0+$. Invoking the estimate of ϕ, the first integral in the last expression of (6) is dominated, independently of $\delta > 0$, by

$$C_\mu \int_{0\le|t|\le\epsilon} |tn|^{[s]+1}\frac{1}{|t|^{s+1}}\,dt.$$

Taking limit $\epsilon \to 0$ on (6), the integral tends to 0 and (6) reduces to

$$b_n = \lim_{\epsilon\to 0}\left(\int_{\epsilon\le|t|\le\pi}\exp(-itn)\phi(\exp(it))\,dt + \phi^{[s]}_{\epsilon,+}(n)\right),$$

which is equal to (3), therefore to $b^\mu(n)$, by invoking the 2π-periodicity of the integrand function and Cauchy's theorem. The proof for $s \ge 0$ is complete.

For $s < 0$ one can directly take limit $r \to 1-0$ on both sides of (5), owing to the estimate of the function ϕ and the Lebesgue dominated convergence theorem. We therefore have

$$b(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\exp(-itn)\phi(\exp(it))\,dt,$$

which is equal to $b^\mu(n)$, by consequently invoking the 2π-periodicity of the integrand, Cauchy's theorem, and its expression (2).

The proof of Theorem 2 now is complete.

Now we prove **2**° of Theorem 3. Using the expression (2) of the function b^{μ}, it is easy to show that $b(z)$ is bounded near the origin. For large z, invoking the expression (1) and Remark 3, we have, for $|\arg(z)| < \theta < \mu$,

$$|b(z)| \leq C_{\mu}\left(|z|^{s} + \int_{0}^{\infty} \exp(-r|z|\sin(\mu-\theta))|\ln r|r^{-s}\frac{dr}{r}\right)$$
$$\leq C_{\mu}\left(|z|^{s} + |z|^{s}\int_{0}^{\infty} \exp(-r\sin(\mu-\theta))|\ln r - \ln|z||r^{-s}\frac{dr}{r}\right)$$
$$\leq C_{\mu,\theta}|z|^{s}\ln|z|.$$

This proves $b^{\mu} \in H^{s}_{\ln}(\mathbf{S}_{\mu,+})$. The verification of $\phi(z) = \sum_{n=1}^{\infty} b^{\mu}(n)z^{n}$ is similar to the cases $s < 0$ in Theorem 2. The proof is complete.

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