

**Funzioni di variabile complessa.** — *Generalization of Fueter's result to  $\mathbf{R}^{n+1}$ .* Nota di TAO QIAN, presentata (\*) dal Socio E. Vesentini.

ABSTRACT. — Fueter's result (see [6, 8]) on inducing quaternionic regular functions from holomorphic functions of a complex variable is extended to Euclidean spaces  $\mathbf{R}^{n+1}$ . It is then proved to be consistent with M. Sce's generalization for  $n$  being odd integers [6].

KEY WORDS: Clifford analysis; Harmonic analysis; Complex analysis; Singular integrals; Fourier multiplier.

RIASSUNTO. — *Generalizzazione del risultato di Fueter allo spazio  $\mathbf{R}^{n+1}$ .* Il risultato di Fueter [6, 8] sulle funzioni regolari determinate in funzioni olomorfe di una variabile complessa viene esteso allo spazio euclideo  $\mathbf{R}^{n+1}$ . Viene poi dimostrata, per  $n$  intero dispari, la compatibilità con la generalizzazione di M. Sce [6].

We will be working in  $\mathbf{R}^{n+1}$ , the real-linear span of  $e_0, e_1, \dots, e_n$ , where  $e_0$  is identical with 1 and  $e_i e_j + e_j e_i = -2\delta_{ij}$ .  $\mathbf{R}^{n+1}$  is embedded into the Clifford algebra  $\mathbf{R}^{(n)}$  generated by  $e_1, \dots, e_n$ . A typical element in  $\mathbf{R}^{n+1}$  is denoted  $x = x_0 + \underline{x}$ , where  $x_0 \in \mathbf{R}$  and  $\underline{x} = x_1 e_1 + \dots + x_n e_n$ ,  $x_j \in \mathbf{R}$ . If  $x \neq 0$ , then its inverse  $x^{-1}$  exists:  $x^{-1} = \bar{x} |x|^{-2}$ , where  $\bar{x} = x_0 - \underline{x}$ . We will study  $\mathbf{R}^{n+1}$ -variable and Clifford-valued functions and the concepts of left- and right-monogeneity are introduced via the Dirac operator  $D = \frac{\partial}{\partial_0} + e_1 \frac{\partial}{\partial x_1} + \dots + e_n \frac{\partial}{\partial x_n}$  in the usual way. In this *Note*, a function is said to be monogenic if it is both left- and right-monogenic. The Cauchy kernel stands for  $E(x) = \bar{x} / |x|^{n+1}$  and the Kelvin inversion of a function  $f$  is  $I(f)(x) = E(x)f(x^{-1})$ . The symbol  $\mathbf{Z}$  and  $\mathbf{Z}^+$  denote the sets of all integers and positive integers, respectively.

We will use Fourier transform of functions  $f$  on  $\mathbf{R}^{n+1}$  defined by

$$\mathcal{F}(f)(\xi) = \int_{\mathbf{R}^{n+1}} e^{2\pi i \langle x, \xi \rangle} f(x) dx,$$

and the result (see [7])

$$(1) \quad \mathcal{F}\left(\frac{P_k(\cdot)}{|\cdot|^{k+n+1-\alpha}}\right)(\xi) = \gamma_{k,\alpha} \frac{P_k(\xi)}{|\xi|^{k+\alpha}},$$

(\*) Nella seduta del 7 marzo 1997.

where  $0 < \alpha < n + 1$ ,  $k \in \mathbf{Z}^+$ ,  $P_k$  is a homogeneous harmonic polynomial of degree  $k$ , and

$$\gamma_{k,\alpha} = i^k \pi^{(n+1)/2 - \alpha} \Gamma(k/2 + \alpha/2) / \Gamma(k/2 + (n+1)/2 - \alpha/2),$$

( $\Gamma$  denotes the ordinary gamma function).

The inverse Fourier transform of a function  $g$ , denoted by  $\mathcal{R}(g)$ , is defined by

$$\mathcal{R}(g) = \int_{\mathbf{R}^{n+1}} e^{-2\pi i(x, \xi)} g(\xi) d\xi.$$

Functions in the Schwarz class have Fourier transforms which are still in the class, and in this case the Fourier inversion formula holds:  $\mathcal{R}\mathcal{F}(f) = f$ . In the sequel, Fourier and inverse Fourier transforms will be used in the distribution sense.

For a function  $g$  defined in  $\mathbf{R}^{n+1}$  one can introduce a Fourier multiplier transform  $M_g$  by  $M_g f = \mathcal{R}(g\mathcal{F}f)$ . It is easy to verify that the Laplace differential operator is identical with the Fourier multiplier transform induced by  $-4\pi^2 |\xi|^2$ .

Let  $f^0$  be a complex-valued function defined in an open set  $O$  in the upper half complex plane. Write  $f^0 = u + iv$ , where  $u$  and  $v$  are real-valued. Denote, for  $x \in \vec{O}$ ,

$$\vec{f}^0(x) = u(x_0, |x|) + \frac{x}{|x|} v(x_0, |x|),$$

where

$$\vec{O} = \{x \in \mathbf{R}^{n+1} : (x_0, |x|) \in O\} \subseteq \mathbb{C}^+ \text{ upper complex plane}$$

$\vec{f}^0$  is said to be the *induced function* from  $f^0$ , and  $\vec{O}$  the *induced set* from  $O$ .

We will be working with functions of the form

$$g(x) = p(x_0, |x|) + i \frac{x}{|x|} q(x_0, |x|),$$

where  $p$  and  $q$  are real-valued. We will call  $p$  and  $q$  the real and the imaginary parts of  $g$ , respectively.

The concepts of intrinsic sets and functions naturally fit into our theory. In the complex plane  $\mathbb{C}$  a set is said to be *intrinsic*, if it is open and symmetric with respect to the real axis; and a function  $f^0$  an *intrinsic function* if it is defined in an intrinsic set and satisfies  $f^0(z) = f^0(\bar{z})$  within its domain (see [5]). In the notation  $f^0 = u + iv$ , the above condition is equivalent to requiring that  $u$  is even and  $v$  is odd in their second argument. In particular,  $v(x_0, 0) = 0$ , i.e.  $f^0$  is real-valued if restricted to the real line in its domain.

Denote by  $\tau$  the mapping

$$\tau(f^0) = \Delta^{(n-1)/2} \vec{f}^0,$$

where  $f^0$  is any holomorphic intrinsic function and the differential operation in the distribution sense. Here we adopt the convention that  $\vec{f}^0 = 0$  outside the induced set  $\vec{O}$ .

Note that for  $n \in \mathbf{Z}^+$  being odd, the operator  $\Delta^{(n-1)/2}$  is a pointwise differential operator, while for  $n \in \mathbf{Z}^+$  being even, it is the Fourier multiplier operator induced by  $(2\pi i |\xi|)^{n-1}$  mapping some functions to merely distributions.

OBSERVATION. If  $b$  is a complex function defined in an intrinsic set, then  $g^0(z) = (b(z) + \bar{b}(\bar{z}))/2$ , and  $h^0(z) = (b(z) - \bar{b}(\bar{z}))/2i$  both are intrinsic, defined in the same set, and  $b = g^0 + ih^0$ .

The observation enables us to extend the domain of  $\tau$  to complex functions  $b$  defined in intrinsic sets but not necessarily intrinsic themselves. We define, for such a function  $b$ :  $\tau(b) = \tau(g^0) + i\tau(h^0)$ . The mapping  $\tau$  extended in such a way is linear under addition and real-scalar multiplication. By virtue of the observation and the extended definition of  $\tau$  we may concentrate only on holomorphic intrinsic functions. A characterization of such functions is that coefficients of their Laurent series expansions in annuli centered at real points in their domains are all real. Our study (see [2, 4]) shows that it is essential to concentrate in the functions  $\tau((\cdot)^{-k})$ ,  $k \in \mathbb{Z}$ .

Define, for  $k \in \mathbb{Z}^+$ ,

$$P^{(-k)} = \tau((\cdot)^{-k}), \quad P^{(k-1)} = I(P^{(-k)}).$$

We have

THEOREM 1. Let  $k \in \mathbb{Z}^+$ . Then (i)  $P^{(-k)}$  and  $P^{(k-1)}$  both are monogenic; (ii)  $P^{(-k)}$  is homogeneous of degree  $-n + 1 - k$  and  $P^{(k-1)}$  is homogeneous of degree  $(k-1)$ ; (iii) if  $n$  is odd, then  $P^{(k-1)} = \tau((\cdot)^{n+k-2})$ .

The underlying idea of Theorem 1 is to explore the similarity between Clifford analysis and complex analysis of one variable. Some close similarity between the quaternionic and the complex analysis of one variable has been established in [4] via the corresponding relation  $z^k \rightarrow P^{(k)}$ . An analogous theory for general Clifford algebras is studied in [2].

In the quaternionic space, which is not quite identical with our case  $n = 3$  for the former is a full algebra but the latter is not, Fueter's result states that  $\tau$  maps holomorphic functions of a complex variable to quaternionic regular functions (see e.g. [8]). M. Sce proved that, if  $n$  is odd, then  $\tau$  maps holomorphic functions defined in open sets of the upper half complex plane to Clifford monogenic functions [6] which generalizes Fueter's result. The assertion (iii) shows that, for  $n$  being odd, our generalization through the Kelvin inversion is consistent with Sce's on the functions  $f^0(z) = z^k$ ,  $k \in \mathbb{Z}$ . For  $n$  being even, however, Fueter or Sce's device in terms of the differential operator  $\Delta^{(n-1)/2}$  can only be generalized, using Fourier multiplier transform, to the power functions of negative powers, i.e. to  $f^0(z) = z^k$ ,  $-k \in \mathbb{Z}^+$ ; while the other half corresponding to nonnegative powers cannot be generalized using the differential operator. This can be seen, e.g. from the following example: for  $f^0(z) = z$ ,  $\tau(z)$  is a distribution but not a function. The setting of using the Kelvin inversion to define  $P^{(k)}$ ,  $k \geq 0$ , is suggested by the author's earlier work [4] where the setting is used for the convenience of proving estimates. Having proved the assertions (i) and (ii) the author got to know through J. Ryan about Sce's result and the reference [6]. The proof of (iii) is similar to the proof of the analogous relation in the quaternions case (see [4]), but uses, in a slightly developed form, some techniques of [6]. It would be interesting to see alterna-

tive proofs of (iii) simpler than what we give in below, e.g. using conformal covariance of intertwining differential operators like those used in [3].

PROOF. (i) Using the Fourier transform result cited above and the relation

$$(\cdot)^{\rightarrow-k}(x) = \left( \frac{\bar{x}}{|x|^2} \right)^k = \frac{(-1)^{k-1}}{(k-1)!} \left( \frac{\partial}{\partial x_0} \right)^{k-1} \left( \frac{\bar{x}}{|x|^2} \right),$$

we have

$$\begin{aligned} P^{(-k)}(x) &= \tau((\cdot)^{\rightarrow-k})(x) = \frac{(-1)^{k-1}}{(k-1)!} \left( \frac{\partial}{\partial x_0} \right)^{k-1} \mathcal{R}\mathcal{F} \left( \Delta^{(n-1)/2} \frac{\bar{x}}{|x|^2} \right) = \\ &= \frac{(-1)^{k-1}}{(k-1)!} \left( \frac{\partial}{\partial x_0} \right)^{k-1} \mathcal{R} \left( \gamma_{1,n} (2\pi i |\xi|)^{n-1} \frac{\bar{\xi}}{|\xi|^{1+n}} \right) = \\ &= \frac{(-1)^{k-1}}{(k-1)!} \left( \frac{\partial}{\partial x_0} \right)^{k-1} \gamma_{1,n}^2 (2\pi i)^{n-1} \frac{\bar{x}}{|x|^{1+n}} = \frac{(-1)^{k-1}}{(k-1)!} \kappa_n \left( \frac{\partial}{\partial x_0} \right)^{k-1} E(x), \end{aligned}$$

where we let  $\kappa_n = (2\pi i)^{n-1} \gamma_{1,n}^2 = (2i)^{n-1} \Gamma^2((n+1)/2)$ . This implies that  $P^{(-k)}$  are monogenic for  $k \in \mathbf{Z}^+$ . The monogeneity of  $P^{(k)}$  is from the property of the Kelvin inversion (see [1]) or a special case of Bojarski's result on intertwining Dirac operators, for which we refer the reader to [3].

(ii) It is a consequence of the expression of  $P^{(-k)}$  obtained above and the property of the Kelvin inversion.

(iii) Let  $n = 2m + 1$ . We have  $\kappa_n = (-1)^m 2^{2m} (m!)^2 = (-1)^m ((2m)!!)^2$ . We will use the mathematical induction. The case  $k = 1$  reduces to verifying  $\Delta^m(x^{2m}) = (-1)^m (2m)!!$ . The following lemma will be used.

LEMMA 1. Let  $f^0(z) = u(x_0, y) + iv(x_0, y)$  be a function holomorphically defined in an open set  $U$  of the upper half complex plane. Denote  $u_0 = u$ ,  $v_0 = v$ , and, for  $s \in \mathbf{Z}^+$ ,

$$u_s = 2s \frac{\partial u_{s-1}}{\partial y} \frac{1}{y}, \quad v_s = 2s \left( \frac{\partial v_{s-1}}{\partial y} \frac{1}{y} - \frac{v_{s-1}}{y^2} \right) = 2s \frac{\partial}{\partial y} \frac{v_{s-1}}{y}.$$

Then

$$\Delta^s \vec{f}^0(x) = u_s(x_0, |\underline{x}|) + \frac{\underline{x}}{|\underline{x}|} v_s(x_0, |\underline{x}|), \quad x_0 + i|\underline{x}| \in U.$$

The lemma may be proved using the mathematical induction via a computation of  $\Delta(u_{s-1} + iv_{s-1})$  invoking the following relation proved in [6]:

$$\frac{\partial u_{s-1}}{\partial x_0} = \frac{\partial v_{s-1}}{\partial y} + 2(s-1) \frac{v_{s-1}}{y}, \quad \frac{\partial u_{s-1}}{\partial y} = -\frac{\partial v_{s-1}}{\partial x_0}.$$

We will frequently use the following formula given in [6]: For any function

$f^0 = u + iv$  and  $r \in \mathbf{Z}^+$ ,

$$(2) \quad (\vec{f}^0)^r(x) = \sum_{l=0}^{[r/2]} (-1)^l \binom{r}{2l} u^{r-2l} v^{2l} + \frac{x}{|x|} \sum_{l=0}^{[r/2]} (-1)^l \binom{r}{2l+1} u^{r-2l-1} v^{2l+1},$$

where  $\binom{r}{l}$  are binomial coefficients with the convention that  $\binom{r}{l} = 0$  for  $l > r$ , and  $[s]$  the largest integer that does not exceed  $s$ .

Using formula (2) for  $f^0(z) = z$ ,  $r = 2m$  and Lemma 1, one immediately deduces  $\Delta^m(x^{2m}) = (-1)^m ((2m)!!)^2$  which proves the case  $k = 1$ . Now assuming  $P^{(k)} = \tau((\cdot)^{n+k-1})$ , we need to show  $P^{(k+1)} = \tau((\cdot)^{n+k})$ . The verification is a bit cumbersome. First, this reduces to proving

$$(3) \quad \frac{-1}{k+1} \frac{\partial}{\partial x_0} (I(\Delta^m((\cdot)^{2m+k}))) = I(\Delta^m((\cdot)^{2m+k+1})),$$

where  $k \in \mathbf{Z}^+$  or  $k = 0$ .

Using formula (2) and Lemma 1, we have

$$\begin{aligned} \Delta((\cdot)^{2m+k})(x) &= \\ &= (2m)!! \left[ \sum_{l=0}^{m+[k/2]} (-1)^l \binom{2m+k}{2l} (2l)(2l-2)\dots(2l-2m+2) x_0^{2m+k-2l} y^{2l-2m} + \right. \\ &\quad \left. + \frac{x}{y} \sum_{l=0}^{m+[k/2]} (-1)^l \binom{2m+k}{2l+1} (2l)(2l-2)\dots(2l-2m+2) x_0^{2m+k-2l-1} y^{2l+1-2m} \right], \end{aligned}$$

where we have put  $y = |x|$ .

By applying the Kelvin inversion, *i.e.* replacing  $x_0, y$  and  $x/y$  by  $x_0|x|^{-2}, y|x|^{-2}$  and  $-x/y$ , respectively, the above becomes

$$(4) \quad (2m)!! \frac{\bar{x}}{|x|^{n+2k+1}} \cdot \left[ \sum_{l=0}^{m+[k/2]} (-1)^l \binom{2m+k}{2l} (2l)(2l-2)\dots(2l-2m+2) x_0^{2m+k-2l} y^{2l-2m} + \right. \\ \left. + \frac{x}{y} \sum_{l=0}^{m+[k/2]} (-1)^{l+1} \binom{2m+k}{2l+1} (2l)(2l-2)\dots(2l-2m+2) x_0^{2m+k-2l-1} y^{2l+1-2m} \right].$$

Applying the differential operator  $[-1/(k+1)] \partial / \partial x_0$  to the expression (4), we obtain

$$(5) \quad \frac{-(2m)!!}{k+1} E(x) \frac{1}{|x|^{2k+2}} \left\{ \left( -(n+2k)x_0 + \frac{x}{y} y \right) [\dots] + (x_0^2 + y^2) \frac{\partial}{\partial x_0} [\dots] \right\},$$

where  $[\dots]$  is as  $[\dots]$  in (4).

Now

$$\begin{aligned} & \left( -(n+2k)x_0 + \frac{x}{y}y \right) [\dots] = \\ & = \left\{ \sum_{l=0}^{m+[k/2]} (-1)^{l+1} \binom{2m+k}{2l} (n+2k)(2l)(2l-2) \dots (2l-2m+2) x_0^{2m+k-2l+1} y^{2l-2m} + \right. \\ & + \left. \sum_{l=0}^{m+[k/2]} (-1)^l \binom{2m+k}{2l+1} (2l)(2l-2) \dots (2l-2m+2) x_0^{2m+k-2l-1} y^{2l+1-2m+1} \right\} + \\ & + \frac{x}{y} \left\{ \sum_{l=0}^{m+[k/2]} (-1)^{l+1} \binom{2m+k}{2l+1} (n+2k)(2l)(2l-2) \dots (2l-2m+2) x_0^{2m+k-2l} y^{2l+1-2m} + \right. \\ & + \left. \sum_{l=0}^{m+[k/2]} (-1)^l \binom{2m+k}{2l} (2l)(2l-2) \dots (2l-2m+2) x_0^{2m+k-2l} y^{2l+1-2m} \right\}, \end{aligned}$$

and,

$$\begin{aligned} (x_0^2 + y^2) \frac{\partial}{\partial x_0} [\dots] & = \\ & = \left\{ \sum_{l=0}^{m+[k/2]} (-1)^l \binom{2m+k}{2l} (2l)(2l-2) \dots (2l-2m+2)(2m+k-2l) \cdot \right. \\ & \cdot (x_0^{2m+k-2l+1} y^{2l-2m} + x_0^{2m+k-2l-1} y^{2l-2m+2}) \left. \right\} + \\ & + \frac{x}{y} \left\{ \sum_{l=0}^{m+[k/2]} (-1)^{l+1} \binom{2m+k}{2l+1} (2l)(2l-2) \dots (2l-2m+2)(2m+k-2l-1) \cdot \right. \\ & \cdot (x_0^{2m+k-2l} y^{2l+1-2m} + x_0^{2m+k-2l-2} y^{2l+1-2m+2}) \left. \right\}. \end{aligned}$$

By comparing the coefficients of a general nomial  $x_0^{2m+k+1-2l} y^{2l-2m}$  in the real part of (5) with those in the real part of  $I(\Delta^m((\cdot)^{2m+k+1}))(x) = E(x)(\Delta^m((\cdot)^{2m+k+1}))(x^{-1})$ , the later being of the expression (4) but with  $k+1$  in place of  $k$ , we are reduced to verifying

$$(6) \quad -2l(n+2k) \binom{2m+k}{2l} + (2m-2l) \binom{2m+k}{2l-1} + 2l(2m+k-2l) \binom{2m+k}{2l} + \\ + (2m-2l)(2m+k-2l+2) \binom{2m+k}{2l-2} = -(k+1)2l \binom{2m+k+1}{2l}.$$

Using the relation

$$(s-l) \binom{s}{l} = (l+1) \binom{s}{l+1},$$

the second and fourth entries on the left hand side of (6) add up to

$$(7) \quad 2l(2m - 2l) \binom{2m+k}{2l-1},$$

while the first and third to

$$(8) \quad -2l(2l+k+1) \binom{2m+k}{2l} = [-4l^2 - 2l(k+1)] \binom{2m+k}{2l} = \\ = -2l(2m+k-2l+1) \binom{2m+k}{2l-1} - 2l(k+1) \binom{2m+k}{2l}.$$

Combining (7) with the right hand side of (8) and using the relation

$$\binom{s}{l} + \binom{s}{l-1} = \binom{s+1}{l},$$

we obtain (6). The verification of the equality between the imaginary part of (5) and that of  $I(\Delta^m((\cdot)^{2m+k+1}))$  is similar, and the proof of (iii) is complete.

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