CLIFFORD MARTINGALE Φ-EQUIVALENCE BETWEEN $S(f)$ AND f^*

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Abstract. The L^2-norm equivalence between a Clifford martingale f and its square function $S(f)$ plays an important role in the proof of the L^2-boundedness of Cauchy integral operators on Lipschitz graphs and the Clifford $T(b)$ Theorem [2, 4]. This note generalises the result to the Φ-equivalence between the maximal function f^* and $S(f)$, where Φ is a nondecreasing and continuous function from $\mathbb{R}^+$ to $\mathbb{R}^+$, of the moderate growth $\Phi(2u) \leq C_1\Phi(u)$ and satisfies $\Phi(0) = 0$.

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1. Introduction

It is well known that martingale theory plays a remarkable role in analysis, especially in harmonic analysis. Many ideas and methods in harmonic analysis come from, or closely relate to martingale theory. In [2] R. Coifman, P. Jones and S. Semmes gave an elementary proof of the L^2-boundedness of Cauchy integral operators on Lipschitz curves using a martingale approach. However, their proof does not exhaust the effectiveness of using martingale in the problem: it depends on a separate Carleson measure argument. [1] shows that the Carleson measure argument can be replaced by a pure martingale argument.

The idea of [1] then motivated G. Gaudry, R-L. Long and T. Qian to generalise the result of [2] to the higher dimensional cases, and to show that the Clifford $T(b)$ Theorem can be proved in the same spirit [4].

What plays the central role in [4] is the L^2-norm-equivalence between a Clifford martingale and its square function. Since the maximal function f^* is L^2-bounded, this implies the L^2- equivalence between f^* and the square function. This later mentioned result is associated with the function $\Phi(t) = t^2$ (in the sense given in Th.3.3 below). In this note we shall generalise the result to some more general functions Φ.

The remaining part of this section will be devoted to introducing notation and terminology and preliminary knowledge of Clifford algebra. In Section 2 we discuss basic properties of Clifford martingales. In this note our context is a bit more general than that of [4] and our treatment is slightly different. Section 3 proves the main result, viz. the Φ-equivalence.

Let $(\Omega, \mathcal{F}, \nu)$ be a nonnegative σ-finite space, ϕ a bounded Clifford-valued measurable function. Consider the Clifford-valued measure $d\mu = \phi d\nu$. The martingales under study are with respect to $d\mu$ and a family $\{\mathcal{F}_n\}_{-\infty}^{\infty}$ of sub-σ-field satisfying

$$\{\mathcal{F}_n\}_{-\infty}^{\infty} \quad \text{nondecreasing}, \quad \mathcal{F} = \cup\mathcal{F}_n, \quad \cap\mathcal{F}_n = \emptyset, \tag{1.1}$$

$$(\Omega, \mathcal{F}_n, \nu) \quad \text{complete}, \quad \sigma - \text{finite}, \forall n. \tag{1.2}$$

Let $\mathbf{e}_1, ..., \mathbf{e}_n$ be the basic vectors of $\mathbf{R}^d$ satisfying

$$\mathbf{e}^2 = -1, \mathbf{e}_i\mathbf{e}_j = -\mathbf{e}_j\mathbf{e}_i, \quad i \neq j, i, \quad j = 1, 2, ..., d, \tag{1.3}$$

and $\mathbf{R}^{(d)}$ the Clifford algebra over the real number field of dimension 2^d generalized by the increasingly ordered subsets $\mathbf{e}_A$'s of $\{1, \cdots, d\}$ with the identification $\mathbf{e}_A = \mathbf{e}_{j_1} \cdots \mathbf{e}_{j_l}, A = \{j_1, \cdots, j_l\}, 1 \le l \le d, \mathbf{e}_\emptyset = \mathbf{e}_0 = 1$.

We shall use the following norm in $\mathbf{R}^{(d)}$:

$$|\lambda| = (\sum_A \lambda_A^2)^{1/2}, \quad \lambda = \sum_A \lambda_A \mathbf{e}_A. \tag{1.4}$$

For the norm we have the relation

$$|\lambda\mu| \le k|\lambda||\mu|, \quad \forall\lambda, \mu \in \mathbf{R}^{(d)}, \tag{1.5}$$

where k is a constant depending only on the dimension d.

When at least one of λ and μ, say λ, is of the form $\lambda = \sum_{i=0}^d \lambda_i \mathbf{e}_i$, i.e. a vector in $\mathbf{R}^{d+1} \subset \mathbf{R}^{(d)}$ we have

$$k^{-1}|\lambda||\mu| \le |\lambda\mu|. \tag{1.5'}$$

To see this, noticing that if $0 \neq \lambda \in \mathbf{R}^{d+1}$, then the left and right inverse of λ is

$$\lambda^{-1} = \frac{\overline{\lambda}}{|\lambda|^2},$$

we have, for any $\mu \in \mathbf{R}^{(d)}$,

$$|\mu| = |\lambda^{-1}\lambda\mu| \leq k|\lambda^{-1}||\lambda\mu| = k|\lambda|^{-1}|\lambda\mu|$$

which gives (1.5′).

In what follows we often use the fact that for $a = a_1a_2a_3a_4, a_i \in \mathbf{R}^{d+1}$ we have $|a| \approx |a_1||a_2||a_3||a_4|$. Constants with subscripts such as C_0, C_1 will be considered to be the same throughout the paper. Constants C may vary from one line to another, but remain to be the same on the same line.

2. Clifford Conditional Expectation, Clifford Martingale

We begin with the definition of conditional expectation. Let $(\Omega, \mathcal{F}, \nu)$ be a σ-finite measure space, $d\mu = \phi d\nu$ a $\mathbf{R}^{d+1}$-valued measure. If $|\Omega|_\nu = \infty$, we assume that the domain of $d\mu$ is not $\mathcal{F}$ but a subring of $\mathcal{F}$. This does not bring us any trouble when defining conditional expectation. Let $\mathcal{J}$ be a sub-σ-field of $\mathcal{F}$ such that $(\Omega, \mathcal{J}, \nu)$ is σ-finite and complete. Denote the conditional expectations with respect to ν and μ by $\tilde{E}$ and E, respectively. The definition of $\tilde{E}$ is standard:

$$\tilde{E}(\phi|\mathcal{J}) = \sum_{i=0}^{d} \tilde{E}(\phi_i|\mathcal{J})\mathbf{e}_i, \quad \text{with} \quad \phi = \sum_{i=0}^{d} \phi_i \mathbf{e}_i.$$

Thus $\tilde{E}$ enjoys all the good properties of classical conditional expectations.

Assume that ϕ is bounded and $\tilde{E}(\phi|\mathcal{J}) \neq 0, a.e.$ In the sequel, unless otherwise stated, all functions under study will be assumed to be Clifford-valued. We define

$$E^{(l)}(f|\mathcal{J}) = \tilde{E}(\phi|\mathcal{J})^{-1}\tilde{E}(\phi f|\mathcal{J}), \quad f \in L^1_{loc}(\nu), \tag{2.1}$$

$$E^{(r)}(f|\mathcal{J}) = \tilde{E}(f\phi|\mathcal{J})\tilde{E}(\phi|\mathcal{J})^{-1}. \quad f \in L^1_{loc}(\nu), \tag{2.1′}$$

$E^{(l)}$ and $E^{(r)}$ satisfy the following properties.

(a) $E^{(l)}$ is right-Clifford-scalar linear and both left- and right-real-scalar linear, and

$$E^{(l)}(fg|\mathcal{J}) = E^{(l)}(f|\mathcal{J})g, \quad g \text{ is } \mathcal{J} - \text{measurable}.$$

For $E^{(r)}$ similar properties hold.
(b) $E^{(l)}(1|\mathcal{J}) = 1 = E^{(r)}(1|\mathcal{J})$.
(c) Both $E^{(l)}$ and $E^{(r)}$ are $\mathcal{J}$-measurable, and

$$\int_A E^{(l)}(f|\mathcal{J})d_l\mu = \int_A f d_l\mu, \quad \forall A \in \mathcal{J}, \forall f \in L^1(A,\nu), \tag{2.2}$$

$$\int_A E^{(r)}(f|\mathcal{J})d_r\mu = \int_A f d_r\mu, \quad \forall A \in \mathcal{J}, \forall f \in L^1(A,\nu), \tag{2.2$'$}$$

where

$$\int_A f d_l\mu = \int_A \phi f d\nu, \quad \int_A f d_r\mu = \int_A f\phi d\nu. \tag{2.3}$$

To see (2.2), notice that we have

$$d\mu|_{\mathcal{J}} = \tilde{E}(\phi|\mathcal{J})d\nu|_{\mathcal{J}}, \tag{2.4}$$

which follows from

$$\int_A \tilde{E}(\phi|\mathcal{J})d\nu = \int_A d\mu, \quad \forall A \in \mathcal{J}, \nu(A) < \infty.$$

Thus, we have

$$\int_A E^{(l)}(f|\mathcal{J})d_l\mu = \int_A \tilde{E}(\phi|\mathcal{F})\tilde{E}(\phi|\mathcal{F})^{-1}\tilde{E}(\phi f|\mathcal{F})d\nu = \int_A \phi f d\nu = \int_A f d_l\mu.$$

(2.2$'$) can be verified similarly.
(d) When $\mathcal{J}_1 \subset \mathcal{J}_2$, we have, denoting $E^{(l)}$ or $E^{(r)}$ by E,

$$E(E(f|\mathcal{J}_2)|\mathcal{J}_1) = E(f|\mathcal{J}_1). \tag{2.5}$$

For $E = E^{(l)}$, (2.5) is verified as follows.

$$\begin{aligned} E^{(l)}(E^{(l)}(f|\mathcal{J}_2)|\mathcal{J}_1) &= E^{(l)}(\tilde{E}(\phi|\mathcal{J}_2)^{-1}\tilde{E}(\phi f|\mathcal{J}_2)|\mathcal{J}_1) \\ &= \tilde{E}(\phi|\mathcal{J}_1)^{-1}\tilde{E}(\phi\tilde{E}(\phi|\mathcal{J}_2)^{-1}\tilde{E}(\phi f|\mathcal{J}_2)|\mathcal{J}_1) \\ &= \tilde{E}(\phi|\mathcal{J}_1)^{-1}\tilde{E}(\phi f|\mathcal{J}_1) \\ &= E^{(l)}(f|\mathcal{J}_1). \end{aligned}$$

As a consequence of (2.5), we have

$$E(E(f|\mathcal{J}_2) - E(f|\mathcal{J}_1)|\mathcal{J}_1) = 0. \tag{2.6}$$

Now assume that we have a nondecreasing family $\{\mathcal{F}_n\}_{-\infty}^{\infty}$. In the classical case, the martingale differential operators $\tilde{\Delta}_n = \tilde{E}_n - \tilde{E}_{n-1}, \tilde{E}_n = \tilde{E}(\cdot|\mathcal{F}_n)$ are orthogonal:

$$\tilde{E}(\tilde{\Delta}_n f \tilde{\Delta}_m g|\mathcal{F}_k) = 0, n \neq m, n.m \geq k, \forall f, g \in L^2.$$

In the Clifford martingale case, because of the noncommutativity, only the following substitution holds. Let $< \cdot, \cdot >$ denote following pairing:

$$< f, g >= \int_\Omega f\phi g d\nu. \tag{2.7}$$

(e) Let $\{\mathcal{F}_n\}_{-\infty}^{\infty}$ be nondecreasing and $(\Omega, \mathcal{F}_n, \nu)$ be complete and σ-finite, and $\tilde{E}(\phi|\mathcal{F}_n) \neq 0$, a.e. $\forall n$, and $\Delta_n^{(r)}$ and $\Delta_m^{(l)}$ be naturally defined. We have

$$\tilde{E}(\Delta_n^{(r)} f\phi\Delta_m^{(l)} g|\mathcal{F}_k) = 0, \quad n \neq m, n, m \geq k. \tag{2.8}$$

In particular,

$$< \Delta_n^{(r)} f, \Delta_m^{(l)} g >= 0, \quad n \neq m. \tag{2.8$'$}$$

This follows from, if say $n > m \geq k$,

$$\begin{aligned} \tilde{E}(\Delta_n^{(r)} f\phi\Delta_m^{(l)} g|\mathcal{F}_k) &= \tilde{E}(\tilde{E}(\Delta_n^{(r)} f\phi\Delta_m^{(l)} g|\mathcal{F}_{n-1})|\mathcal{F}_k) \\ &= \tilde{E}(\tilde{E}(\Delta_n^{(r)} f\phi|\mathcal{F}_{n-1})\Delta_m^{(l)} g|\mathcal{F}_k) \\ &= 0, \end{aligned}$$

where we used the counterpart of (2.6) for classical conditional expectations. The typical case of ϕ is the case where ϕ is $\mathbf{R}^{d+1}$-valued and $d\mu$ is absolutely continuous with respect to $d\nu$. In this paper we assume the condition $C_0^{-1} \leq |\phi| \leq C_0, a.e.$ Thus we have:

(f) Let $1 \leq p < \infty$ and $\mathcal{J}$ any sub-σ-field under consideration. Then E is L^p-bounded, if and only if

$$C^{-1}C_0^{-1} \leq |\tilde{E}(\phi|\mathcal{J})| \leq CC_0, a.e.$$

The sufficiency of the condition follows from the definition of E, the boundedness of classical martingales and the boundedness of ϕ. The necessity can be proved as in [1]. In fact, if E is bounded in $L^p, 1 \leq p < \infty$, we can let $f = \phi^{-1}g$, where g is any integrable step function. Then the boundedness of E gives

$$\int_\Omega |\tilde{E}(\phi|\mathcal{J})|^{-p}|g|^p d\nu \leq C^p \int_\Omega \frac{|g|^p}{|\phi|^p} d\nu \leq C^p C_0^p \int_\Omega |g|^p d\nu,$$

where, again, we used the boundedness of ϕ. Since g is arbitrary, we conclude the bounds of $\tilde{E}(\phi|\mathcal{J})$.
The case $p = \infty$ is similar.
Now we turn to the investigation of Clifford martingales. Let $(\Omega, \mathcal{F}, \nu)$ be a σ-finite measure space endowed with a nondecreasing family $\{\mathcal{F}_n\}_{-\infty}^{\infty}$ satisfying (1.1) and (1.2). From the property (f), it is natural to assume

$$C_0^{-1} \leq |\tilde{E}(\phi|\mathcal{F}_n)| \leq C_0, a.e., \forall n. \tag{2.9}$$

Let $f = (f_n)_{-\infty}^{\infty}$ be a $\mathbf{R}^{(d)}$-valued process. $(f_n)_{-\infty}^{\infty}$ is said to be a l- or r-martingale, if for $E = E^{(l)}$ or $E = E^{(r)}$, respectively,

$$f_n = E(f_{n+1}|\mathcal{F}_n), a.e. \tag{2.10}$$

For a martingale $f = (f_n)$ (l- or r-), the maximal and the square functions are defined by

$$f_n^* = \sup_{k \leq n} |f_k|, \quad f^* = f_\infty^*, \tag{2.11}$$

$$S_n(f) = (|f_{-\infty}|^2 + \sum_{-\infty}^{n} |\Delta_k f|^2)^{1/2}, \quad S(f) = S_\infty(f), \tag{2.12}$$

where $f_{-\infty} = \lim_{n \to -\infty} f_n$ pointwise.
$f = (f_n)_{-\infty}^{\infty}$is said to be L^p-bounded, $1 \leq p \leq \infty$, if

$$\|f\|_p = \sup_n \|f_n\|_p < \infty. \tag{2.13}$$

All the arguments in the sequel are the same for l- and r-martingales and we use E to represent either $E^{(l)}$ or $E^{(r)}$. We want to show that the maximal operator $*$ is of type p-p for $1 < p \leq \infty$, and weak type 1-1. Moreover, for the case $1 < p \leq \infty$, every L^p-bounded martingale $f = (f_n)_{-\infty}^{\infty}$ is generated by some function $f \in L^p(\nu)$, i.e.

$$f_n = E(f|\mathcal{F}_n), \quad \forall n. \tag{2.14}$$

For $1 \leq p \leq \infty$, all L^p-bounded martingales have pointwise limits $\lim_{n \to \infty} f_n$ and $\lim_{n \to -\infty} f_n$. We state these as propositions.

Proposition 2.1. *Let $1 < p \leq \infty$. Then the maximal operator $*$ is of type p-p and weak type 1-1. For $1 < p \leq \infty$, every L^p-bounded martingale $f = (f_n)_{-\infty}^{\infty}$ is generated by some function $f \in L^p(\nu)$, with $\|f\|_p \approx \sup_n \|f_n\|_p$.*

Proof. Let $f = (f_n)_{-\infty}^{\infty}$ be a martingale, say, for example, a left one. Then

$$f_n = E(f_{n+1}|\mathcal{F}_n) = \tilde{E}(\phi|\mathcal{F}_n)^{-1}\tilde{E}(\phi f_{n+1}|\mathcal{F}_n),$$

$$\begin{aligned} f_n &= E(f_{n+2}|\mathcal{F}_n) = \tilde{E}(\phi|\mathcal{F}_n)^{-1}\tilde{E}(\phi f_{n+2}|\mathcal{F}_n) \\ &= \tilde{E}(\phi|\mathcal{F}_n)^{-1}\tilde{E}(\tilde{E}(\phi f_{n+2}|\mathcal{F}_{n+1})|\mathcal{F}_n), \end{aligned}$$

which means that

$$\tilde{E}(\phi f_{n+1}|\mathcal{F}_n) = \tilde{E}(\tilde{E}(\phi f_{n+2}|\mathcal{F}_{n+1})|\mathcal{F}_n),$$

i.e., $(\tilde{E}(\phi f_{n+1}|\mathcal{F}_n))_{-\infty}^{\infty}$ is a martingale with respect to $(\Omega, \mathcal{F}, \nu, \{\mathcal{F}_n\}_{-\infty}^{\infty})$. It is also L^p-bounded, owing to the relation

$$\tilde{E}(\phi f_{n+1}|\mathcal{F}_n) = \tilde{E}(\phi|\mathcal{F}_n) f_n,$$

which follows from the expression of f_n in the beginning of the proof.
Furthermore, we have

$$\sup_n \|f_n\|_p \approx \sup_n \|\tilde{E}(\phi f_{n+1}|\mathcal{F}_n)\|_p,$$

$$f^* \approx \sup_n |\tilde{E}(\phi f_{n+1}|\mathcal{F}_n)|.$$

So $*$ is of type p-p and weak type 1-1 owing to the corresponding results in the classical case. Now for $1 < p \leq \infty$, for any integer $M > 0$, decomposing $\Omega = \cup \Omega_k, \Omega_k \in \mathcal{F}_{-M}, |\Omega_k| < \infty$. Since for every $k, (\tilde{E}(\phi f_{n+1}|\mathcal{F}_n)\chi_{\Omega_k})_{n \geq -M}$ is a classical martingale, we can obtain some $\phi f \in L^p(\Omega_k, \nu)$ such that on Ω_k

$$\tilde{E}(\phi f_{n+1}|\mathcal{F}_n) = \tilde{E}(\phi f|\mathcal{F}_n), \quad n \geq -M.$$

Thus

$$f_n = \tilde{E}(\phi|\mathcal{F}_n)^{-1}\tilde{E}(\phi f_{n+1}|\mathcal{F}_n) = \tilde{E}(\phi|\mathcal{F}_n)^{-1}\tilde{E}(\phi f|\mathcal{F}_n) = E(f|\mathcal{F}_n), \quad n \geq -M.$$

Letting $M \to \infty$, (2.14) follows. Furthermore, we have

$$\|f\chi_{\Omega_k}\|_p \leq C \sup_n \|f_n \chi_{\Omega_k}\|_p,$$

and

$$\|f\|_p \leq C \sup_n \|f_n\|_p.$$

In addition, $\sup_n \|f_n\|_p \leq C\|f\|_p$ and so $\|f\|_p \approx \sup_n \|f_n\|_p$. The proof of the proposition is complete.
By virtue of the proposition we can identify a L^p-bounded martingale with the function that generalizes the martingale in the sense of (2.14).

Proposition 2.2. *Let $1 \le p \le \infty, f = (f_n)_{-\infty}^{\infty}$ be a L^p-bounded martingale. Then*

$$\lim_{n\to\infty} f_n = f, \text{ for } 1 < p \le \infty, \tag{2.15}$$

where f is the function specified in Prop 2.1 that generalizes $(f_n)_{-\infty}^{\infty}$, and

$$\lim_{n\to\infty} f_n \text{ exists, for } p = 1, \tag{2.15$'$}$$

$$\lim_{n\to-\infty} f_n = 0, \text{ for } 1 \le p < \infty. \tag{2.15$''$}$$

Proof. Let $\Omega = \cup\Omega_k, \Omega_k \in \mathcal{F}_0, |\Omega_k| < \infty, \forall k$. Then both $(\tilde{E}(\phi|\mathcal{F}_n)\chi_{\Omega_k})_{n>0}$ and $(\tilde{E}(\phi f_{n+1}|\mathcal{F}_n)\chi_{\Omega_k})_{n>0}$ are L^p-bounded martingales with respect to $(\Omega_k, \mathcal{F} \cap \Omega_k, \{\mathcal{F}_n \cap \Omega_k\}_{n\ge 0})$, and have their respective limits:

$$\lim_{n\to\infty} \tilde{E}(\phi|\mathcal{F}_n) = \phi, a.e. \text{ on every } \Omega_k,$$

$$\lim_{n\to\infty} \tilde{E}(\phi f_{n+1}|\mathcal{F}_n) = \phi g, a.e. \text{ for some } g \text{ on every } \Omega_k, \text{ and } g = f \text{ if } 1 < p \le \infty.$$

The last two limits conclude (2.15) and (2.15$'$). Now we prove (2.15$''$). Denote $\theta(\omega) = \overline{\lim}_{n\to-\infty}|f_n|$. Then $\theta(\omega) \le f^*(\omega)$, and $\theta(\omega)$ is $\cap\mathcal{F}_n$ measurable. This concludes $\theta(\omega) = a \ge 0$, a.e. By the weak type p-p of $*$, for $1 \le p < \infty$, we have

$$|\{\theta(\omega) > \lambda\}|_\nu \le |\{f^* > \lambda\}|_\nu \le (\frac{C}{\lambda}\|f\|_p)^p, \quad \forall\lambda > 0.$$

So, $a = 0$. This proves the assertion (2.15$''$). The proof of the proposition is complete.

Remark. *In the classical case, for $1 < p < \infty$, the assertion $\lim_{n\to-\infty} f_n = 0$, a.e., was proved in [3].*

3. Φ-Equivalence Between $S(f)$ and f^*

The proof of the Φ-equivalence will refer to the following result.

Theorem 3.1. *There exists a constant C depending only on C_0 in (2.9) such that*

$$C^{-1}\tilde{E}(S(f)^2|\mathcal{F}_0) \le \tilde{E}(|f|^2|\mathcal{F}_0) \le C\tilde{E}(S(f)^2|\mathcal{F}_0). \tag{3.1}$$

For a proof we refer the reader to [4]. It is noted that in the inequalities of the theorem and all the related ones in the sequel the associated constants depend only on C_0 in (2.9), but not on $\{\mathcal{F}_n\}_{-\infty}^{\infty}$, nor on the martingales under consideration. Owing to this, for any integer $M > 0$, the estimates associated with the family $\{\mathcal{F}_n\}_{n\geq -M}$ involve the same constants. Taking limit $M \to \infty$, we conclude the case $\{\mathcal{F}_n\}_{-\infty}^{\infty}$.

Let Φ be a nondecreasing and continuous function from $I\!R^+$ to $I\!R^+$ satisfying $\Phi(0) = 0$ and the moderate growth condition

$$\Phi(2u) \leq C_1\Phi(u), \quad u > 0. \tag{3.2}$$

We shall begin with establishing a Φ-equivalence between $S(f)$ and f^* for those martingales f which are predictably dominated, in the sense

$$|\Delta_n f| \leq D_{n-1}, \quad \forall n, \tag{3.3}$$

where $D = (D_n)$ is a nonnegative nondecreasing and adapted process to $\{\mathcal{F}_n\}$. Still, we need only to consider the case $\{\mathcal{F}_n\}_{n\geq 0}$ (In this case for any process $\lambda = (\lambda_n)_{n\geq 0}$, we add $\lambda_{-1} = 0$, so any f which satisfies (3.3) must satisfy $f_0 = 0$. This is not an essential restriction, of course).

Theorem 3.2. *Let $f = (f_n)_{n\geq 0}$ be a l- or r-martingale satisfying (3.3). Then*

$$\int_\Omega \Phi(S(f))d\nu \leq C\int_\Omega \Phi(f^* + D_\infty)d\nu, \tag{3.4}$$

$$\int_\Omega \Phi(f^*)d\nu \leq C\int_\Omega \Phi(S(f) + D_\infty)d\nu, \tag{3.4$'$}$$

where the involved constants depend only on C_0, C_1.

Proof. We shall use the stopping time argument and the good λ-inequality. Let α be an arbitrary real number that is bigger than 1 and $\beta > 0$ to be determined later and λ be any level. Notice that

$$|f_n| \leq |f_{n-1}| + |\Delta_n f| \leq f_{n-1}^* + D_{n-1} = \rho_{n-1}.$$

Define the stopping time

$$\tau = \inf\{n : \rho_n > \beta\lambda\}$$

and the associated stopping martingale

$$f^{(\tau)} = (f_n^{(\tau)})_{n\geq 0} = (f_{\min(n,\tau)})_{n\geq 0}.$$

Then we have

$$\{\tau < \infty\} = \{\rho_\infty > \beta\lambda\}, \quad f^{(\tau)*} = \sup_n |f_{\min(n,\tau)}| \le f_\tau^* \le \rho_{\tau-1} \le \beta\lambda.$$

Now consider the adapted process $(S_n(f^{(\tau)}))_{n\ge 0}$, and define the stopping time

$$T = \inf\{n : S_n(f^{(\tau)}) > \lambda\}.$$

Then we have

$$\{T < \infty\} = \{S(f^{(\tau)}) > \lambda\}, \quad S_{T-1}(f^{(\tau)}) \le \lambda.$$

Thus, we have

$$\begin{aligned}\{S(f) > \alpha\lambda\} &\subset \{\tau < \infty\} \cup \{\tau = \infty, S_\tau(f)^2 > \alpha^2\lambda^2\} \\ &\subset \{\tau < \infty\} \cup \{S(f^{(\tau)})^2 - S_{T-1}(f^{(\tau)})^2 > (\alpha^2 - 1)\lambda^2\},\end{aligned}$$

and

$$\begin{aligned}&\tilde{E}\ (\chi_{\{S(f^{(\tau)})^2 - S_{T-1}(f^{(\tau)})^2 > (\alpha^2-1)\lambda^2\}}|\mathcal{F}_T) \\ \le\ &\frac{1}{(\alpha^2-1)\lambda^2}\tilde{E}(S(f^{(\tau)})^2 - S_{T-1}(f^{(\tau)})^2|\mathcal{F}_T).\end{aligned}$$

Now consider a new underlying space $(\Omega, \mathcal{F}, \nu, \{\mathcal{J}_n\}_{n\ge 0})$ with $\mathcal{J}_n = \mathcal{F}_{T+n}$, and the martingale

$$g = (g_n)_{n\ge 0} \text{ with } g_n = f_{T+n}^{(\tau)} - f_{T-1}^{(\tau)}.$$

Then we have

$$\Delta_n g = f_{T+n}^{(\tau)} - f_{T-1}^{(\tau)} - (f_{T+n-1}^{(\tau)} - f_{T-1}^{(\tau)}) = \Delta_{T+n} f^{(\tau)}$$

and

$$S(g)^2 = \sum_{n=0}^\infty |\Delta_n g|^2 = \sum_{n=0}^\infty |\Delta_{T+n} f^{(\tau)}|^2 = \sum_{k=T}^\infty |\Delta_k f^{(\tau)}|^2 = S(f^{(\tau)})^2 - S_{T-1}(f^{(\tau)})^2.$$

By invoking Th. 3.1, we obtain

$$\begin{aligned}\tilde{E}(S(f^{(\tau)})^2 - S_{T-1}(f^{(\tau)})^2|\mathcal{F}_T) &= \tilde{E}(S(g)^2|\mathcal{J}_0) \\ &\le C\tilde{E}(|g|^2|\mathcal{J}_0) \\ &= C\tilde{E}(|f^{(\tau)} - f_{T-1}^{(\tau)}|^2|\mathcal{F}_T) \\ &\le C\beta^2\lambda^2.\end{aligned}$$

Now, since $\{S(f^{(\tau)} > \alpha\lambda\} \subset \{T \leq \infty\}$, we have

$$\begin{aligned}
|\{S(f^{(\tau)}) > \alpha\lambda\}|_\nu &\leq \int_{\{T<\infty\}} \chi_{\{S(f^{(\tau)})>\alpha\lambda\}} d\nu \\
&= \int_{\{T<\infty\}} \tilde{E}(\chi_{\{S(f^{(\tau)})>\alpha\lambda\}}|\mathcal{F}_T) d\nu \\
&\leq \int_{\{T<\infty\}} \tilde{E}(\chi_{\{S(f^{(\tau)})^2 - S_{T-1}(f^{(\tau)})^2 > (\alpha^2-1)\lambda^2\}}|\mathcal{F}_T) d\nu \\
&\leq \frac{C\beta^2}{\alpha^2-1} |\{S(f^{(\tau)}) > \lambda\}|_\nu \leq \frac{C\beta^2}{\alpha^2-1} |\{S(f) > \lambda\}|_\nu,
\end{aligned}$$

and hence

$$|\{S(f) > \alpha\lambda\}|_\nu \leq |\{\rho_\infty > \beta\lambda\}|_\nu + \frac{C\beta^2}{\alpha^2-1}|\{S(f) > \lambda\}|_\nu,$$

which is the desired good λ-inequality for the couple $(S(f), f^* + D_\infty)$. The one for the couple $(f^*, S(f) + D_\infty)$ is similar. From them we obtain (3.4) and (3.4$'$).
We can get rid of D_∞ in the following two cases:
(i) Φ is convex;
(ii) $(\Omega, \mathcal{F}, \nu, \{\mathcal{F}_n\}_{-\infty}^{\infty})$ is regular in some sense. For simplicity, we only consider the simplest regularity, i.e. the dyadic type one: each $\mathcal{F}_n$ is atomic whose atom $I^{(n)} = I_1^{(n+1)} + I_2^{(n+1)}$ satisfies $||I_1^{(n+1)}|_\mu| = ||I_2^{(n+1)}|_\mu|$. A little more general regularity as in [5] is applicable to our case. We have

Theorem 3.3. *Under the additional condition (i) on Φ or (ii) on $(\Omega, \mathcal{F}, \nu, \{\mathcal{F}_n\}_{-\infty}^{\infty})$ we have*

$$\int_\Omega \Phi(S(f)) d\nu \approx \int_\Omega \Phi(f^*) d\nu,$$

where all the constants involved in the equivalence depend only on C_0 and C_1.

Proof. First consider $\{\mathcal{F}_n\}_{n\geq 0}$. Davis' decomposition holds in such case: every Clifford martingale $f = (f_n)_{n\geq 0}$ can be decomposed into a sum of two martingales $g = (g_n)_{n\geq 0}$ and $h = (h_n)_{n\geq 0}$ satisfying

$$|\Delta_n g| \leq 4d_{n-1}^*, \quad d_n^* = \sup_{k\leq n} |d_k|, \ d_k = \Delta_k f, \tag{3.5}$$

$$\int_\Omega \Phi(\sum_0^\infty |\Delta_n h|) d\nu \leq C \int_\Omega \Phi(d^*) d\nu, \quad \forall \text{ convex } \Phi. \tag{3.6}$$

(See [6] for the proof of the classical case.) Now for $f = (f_n)_{n\geq 0}$, we have

$$\begin{aligned}\int_\Omega \Phi(S(f))d\nu \;&\leq\; C\int_\Omega \Phi(S(g))d\nu + C\int_\Omega \Phi(S(h))d\nu \\ &\leq\; C\int_\Omega \Phi(g^*) + C\int_\Omega \Phi(d^*) + C\int_\Omega \Phi(\sum_0^\infty |\Delta_n h|)d\nu \\ &\leq\; C\int_\Omega \Phi(f^*)d\nu.\end{aligned}$$

For its reciprocal the proof is similar.

Now consider the dyadic type case. We claim that in the case (3.3) holds for every martingale $f = (f_n)_{-\infty}^{\infty}$ for some suitably defined $D = (D_n)$. In fact,

$$D_{n-1}|_{I^{n-1}} = \sup_{k\leq n}\max(|\Delta_k f||_{I_1^{(k)}}, |\Delta_k f||_{I_2^{(k)}})$$

is a nonnegative, nondecreasing and adapted process such that

$$|\Delta_n f| \leq D_{n-1},$$

and

$$D_\infty \leq C\min(f^*, S(f)).$$

Only the last assertion needs to be verified. In fact,

$$\int_{I^{(k-1)}} \Delta_k f d\mu = 0$$

implies

$$\int_{I_1^{(k)}} \Delta_k f d\mu = -\int_{I_2^{(k)}} \Delta_k f d\mu.$$

This implies

$$\Delta_k f|_{I_1^{(k)}} |I_1^{(k)}|_\mu = -\Delta_k f|_{I_2^{(k)}} |I_1^{(k)}|_\mu,$$

or

$$\frac{|\Delta_k f|_{I_1^{(k)}}|}{|\Delta_k f|_{I_2^{(k)}}|} = \frac{||I_2^{(k)}|_\mu|}{||I_1^{(k)}|_\mu|}.$$

Therefore, on $I^{(k-1)}$

$$\max(|\Delta_k f||_{I_1^{(k)}}, |\Delta_k f||_{I_2^{(k)}}) \leq C|\Delta_k f|,$$

and thus

$$D_\infty \leq C\sup_k |\Delta_k f| \leq C\min(S(f), f^*).$$

The proof is complete.

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