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Fourier Analysis on Starlike Lipschitz Surfaces

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DEDICATED TO ALAN MCINTOSH ON THE OCCASION OF HIS 60TH BIRTHDAY

A theory of a class of singular integrals on starlike Lipschitz surfaces in $\mathbf{R}^n$ is established. The class of singular integrals forms an operator algebra identical to the class of bounded holomorphic Fourier multipliers, as well as to the Cauchy–Dunford bounded holomorphic functional calculus of the spherical Dirac operator. The study proposes a new method inducing Clifford holomorphic functions from holomorphic functions of one complex variable, by means of which problems on the sphere are reduced to those on the unit circle. © 2001 Academic Press

Key Words: functional calculus; Dirac operator; the unit sphere in $\mathbf{R}^n$; Fourier multiplier; singular integral.

0. INTRODUCTION

The aim of this work is to establish the bounded holomorphic functional calculus of the spherical Dirac operator on starlike Lipschitz surfaces with the equivalent forms as Fourier multipliers and singular integral operators in the context. The study provides explicit formulas expressing the integral kernels in terms of the associated Fourier multipliers, and vice versa. In the operator algebra, the existence of an inverse operator may be easily determined, and, if exists, it can be explicitly computed. This implies applications of the theory to certain boundary value problems and singular integral equations related to non-smooth domains ([V], [Ke2], [LMcQ], [Mc3]).

It is a continuation of the study on closed curves and surfaces being proceeded in [Q1–6] and [GQW]. That is a further development of the study of Cauchy's integrals (See, for instance, [C], [CM], [Ke1], [CMcM], [DJS], [Mc1], [CJS]) and of operator algebras of singular integral operators (See [McQ1–2], [LMcS], [GLQ], [LMcQ], [Ta], [Mc3]) on one- and multi-dimensional Lipschitz graphs.

The operator boundedness results of this paper is of the same kind as in [LMcS] on Lipschitz graphs. The method of [LMcS] follows one of the

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two methods of [CJS]. The basic method we use to deal with starlike **curves** and **n-torus** is Poisson summation ([Q1], [GQW], [Q3-4]). By using that method we are reduced to the previously established theory on one- and multi-dimensional Lipschitz graphs ([McQ1–2], [LMcQ], [LMcS], [GLQ], [Ta]). On the higher dimensional sphere and its Lipschitz perturbations, as studied in this paper, however, no Poisson summation method are available. New methods has to be explored (Also see [Q5]). Restricted on the sphere what is proved is the bounded holomorphic Fourier multiplier theorem. Although there have been stronger results on the sphere ([Str]), on its Lipschitz perturbations, however, our result is the strongest as we know.

As the first step to approach $\mathbf{R}^n$ is the Hamilton quaternionic space studied in [Q5]. In that case there is Fueter's result ([Su]) that can be directly adopted and the estimates at the singularity point are not too difficult to handle (see [Q5]).

In order to develop the theory in $\mathbf{R}^n$ we need to extend the machinery used in [Q5] based on Fueter's result. Besides the generalization of Fueter's and Sce's results, and subsequently the estimate of the induced singular integral kernels (Theorem 1), a fair amount of details are involved due to the fact that, not like the quaternionic space, now $\mathbf{R}^n$ is not an algebra. There are also details involved in separately dealing with the odd and even dimensions, as well as in getting rid of the difference between $\mathbf{R}^n$ and $\mathbf{R}^n_1$.

It would be appropriate to mention an alternative direction generalizing the initial study of Cauchy's operators on Lipschitz graphs in comparison with the present study. That is to study more general curves and surfaces rather than the Lipschitz ones, but restricted to consider only Cauchy's kernel or closely related ones. For this, see, for instance, [JK], [Da1–2], [Ke2], [Se1–4]. The fact is that if we study more general surfaces, then we need to place more restrictions on the convolution kernels, in order to make the induced convolution operators well behaved. In some extreme cases the Cauchy kernel is the only well behaved nontrivial kernel ([Q1], [GQ]). The nature of the present study, however, is to the opposite: we consider only Lipschitz curves and surfaces that allow a larger variety of bounded singular integral operators to exist. The operators under our study form an operator algebra on the curves and surfaces.

The writing plan is as follows. Section 1 is devoted to preliminary knowledge, notation and terminology that will be used throughout the paper. Section 2 generalizes Fueter's result and introduces our monomial functions in $\mathbf{R}^n_1$. The functions play the same role in $\mathbf{R}^n_1$ as the functions $f^0(z)=z^k$'s do in the complex plane. We then prove the main technical result Theorem 1. In Section 3 we develop the operator theory on starlike Lipschitz surfaces in $\mathbf{R}^n_1$. We define the Fourier multipliers and the corresponding singular integral operators and prove the L^2-boundedness. Section 4 devotes to the third version of the operators, viz. the bounded holomorphic Cauchy–Dunford functional calculus of the spherical Dirac operator. In Section 5 we indicate how the theory in $\mathbf{R}_1^n$ can be adapted in order to get an analogous theory in the symmetric space $\mathbf{R}^n$.

The author expresses his sincere thankfulness to Alan McIntosh for his constant support in various forms to this study. This direction of study has been under his strong influence. Sincere thanks are due to John Ryan who introduced the author to Clifford analysis including Sce's result and the reference [Sc] which have been crucial to this project. John Ryan contributed valuable discussions regarding the subject during my visit to the University of Arkansas in 1996. Both McIntosh and Ryan read a manuscript of this paper and made valuable comments.

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1. PRELIMINARIES

We briefly recall what will be used in the paper. We shall be working in the real Clifford algebra $\mathbf{R}^{(n)}$ generated by $\mathbf{e}_1, ..., \mathbf{e}_n$, called *basic vectors*, over the real number field. Denote by $\mathbf{R}_1^n$ and $\mathbf{R}^n$ the linear subspaces of $\mathbf{R}^{(n)}$ spanned by $\mathbf{e}_0, \mathbf{e}_1, ..., \mathbf{e}_n$ and by $\mathbf{e}_1, ..., \mathbf{e}_n$, respectively, where $\mathbf{e}_0$ is the algebraic unit element, i.e. $\mathbf{e}_0 = 1$, and $\mathbf{e}_i\mathbf{e}_j + \mathbf{e}_j\mathbf{e}_i = -2\delta_{ij}$, $1 \leqslant i,\ j \leqslant n$. Elements of $\mathbf{R}^{(n)}$ are denoted by $x, y, ...$ and called *Clifford numbers*. An element in $\mathbf{R}_1^n$ is called a *vector* and of the form $x = x_0\mathbf{e}_0 + \underline{x}$, where $x_0 \in \mathbf{R}$, $\underline{x} = x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n \in \mathbf{R}^n$. $x_0\mathbf{e}_0$ and $\underline{x}$ are called the real and the imaginary parts of x, respectively. In the notation of [DSS], $\mathbf{R}_1^n = \mathbf{R}^{0,n} \oplus \mathbf{R}$, $\mathbf{R}^n = \mathbf{R}^{0,n}$. Define two operations on the basic elements: $(\mathbf{e}_{i_1} \cdots \mathbf{e}_{i_l})^* = \mathbf{e}_{i_l} \cdots \mathbf{e}_{i_1}$ and $(\mathbf{e}_{i_1} \cdots \mathbf{e}_{i_l})' = (\mathbf{e}_{i_1})' \cdots (\mathbf{e}_{i_l})'$, where $(\mathbf{e}_0)' = \mathbf{e}_0$, $(\mathbf{e}_j)' = -\mathbf{e}_j$, $j = 1, ..., n$, and extend them by linearity to $\mathbf{R}^{(n)}$, and hence to $\mathbf{R}_1^n$ and $\mathbf{R}^n$. By combining them we define a third operation $^-$ by $\bar{x} = (x^*)'$. If x and y are two Clifford numbers in $\mathbf{R}^{(n)}$, then we have $\overline{xy} = \bar{y}\bar{x}$. If $x = x_0 + \underline{x}$, then $\bar{x} = x_0 - \underline{x}$. If x is a vector and $x \neq 0$, then its inverse x^{-1} exists: $x^{-1} = \bar{x}/|x|^2$ and $x^{-1}x = xx^{-1} = 1$. We also use the complex Clifford algebra $\mathbf{C}^{(n)}$ generated by $\mathbf{e}_1, \ldots, \mathbf{e}_n$ over the complex number field, whose elements are also denoted by $x, y, \ldots$. The complex imaginary element $\mathbf{i}$ commutes with all the $\mathbf{e}_j$, $j=0, 1, \ldots, n$ and $\mathbf{i}' = -\mathbf{i}$. So we extend the definitions of $*$ and $'$ and therefore $^-$ to $\mathbf{C}^{(n)}$. The natural inner product between x and y in $\mathbf{C}^{(n)}$, denoted by $\langle x, y\rangle$, is the complex number $\sum_S x_S \overline{y_S}$, where $x=\sum_S x_S \mathbf{e}_S$, $y=\sum_S y_S \mathbf{e}_S$, S runs over all the ordered subsets $(i_1, i_2, \ldots, i_l)$, $i_1 < i_2 < \cdots < i_l$, of the set $\{1, 2, \ldots, n\}$ and $\mathbf{e}_S = \mathbf{e}_{i_1}\mathbf{e}_{i_2}\cdots\mathbf{e}_{i_l}$. The norm associated with this inner product is $|x| = \langle x, x\rangle^{1/2} = (\sum_S |x_S|^2)^{1/2}$. The norm and inner product satisfy the relation $\langle x, y\rangle = \frac{1}{4}(|x+y|^2 - |x-y|^2)$. So, if a transform in $\mathbf{C}^{(n)}$ preserves the norm, then it also preserves the inner product. If $x, y, \ldots, u$ are vectors, then $|xy\cdots y| = |x|\,|y|\cdots|u|$. The angle between two vectors x and y, denoted by $\arg(x, y)$, is defined to be $\arccos \langle x, y\rangle/(|x|\,|y|)$, where the inverse function arccos takes values in $[0, \pi)$. The concept of angle can be extended to any two elements in $\mathbf{R}^{(n)}$ with the same definition, as both the inner product and the norm are applicable to elements in $\mathbf{R}^{(n)}$. By *the unit sphere of* $\mathbf{R}_1^n$ we mean the set $\{x \in \mathbf{R}_1^n : |x| = 1\}$, denoted by $S_{\mathbf{R}_1^n}$. The unit sphere $\{\underline{x} \in \mathbf{R}^n : |\underline{x}| = 1\}$ in $\mathbf{R}^n$ is denoted by $S_{\mathbf{R}^n}$. We use $B_{\mathbf{X}}(x, \delta)$ for the ball in the metric space X centered at x with radius δ. The substitutions of X are $\mathbf{R}_1^n$, $\mathbf{R}^n$ and $\mathbf{C}$ in the sequel. If $X = \mathbf{C}$, then x is replaced by z and balls are called discs.

From Sections 2 to 4 we shall be working with $\mathbf{R}_1^n$-variable and $\mathbf{C}^{(n)}$-valued functions. The concepts of left- and right- monogenity are introduced in the usual way via the Dirac operator $D = D_0 + \underline{D}$, where $D_0 = (\partial/\partial x_0)$, $\underline{D} = (\partial/\partial x_1)\,\mathbf{e}_1 + \cdots + (\partial/\partial x_n)\,\mathbf{e}_n$. $\underline{D}$ is called the homogeneous Dirac operator in $\mathbf{R}^n$. In this paper a function is said to be *monogenic*, if it is both left- and right-monogenic. The Cauchy kernel stands for $E(x) = \bar{x}/|x|^{n+1}$ and the Kelvin inversion $I(f)(x) = E(x)\, f(x^{-1})$. We assume the reader to be familiar with Cauchy's Theorem and Cauchy's Formula in the form as exhibited in, e.g. [BDS], or [LMcQ], or [DSS].

The integers and positive integers are denoted by $\mathbf{Z}$ and $\mathbf{Z}^+$, respectively. Notations C, C_v, will be used for constants which may vary from one occurrence to the next. Subscripts, such as v in C_v and Σ in C_Σ, are used to stress the dependence of the constants. If in a definition or statement the notation $\pm$ is used more than once, then the definition or statement is meant to be valid for two symmetric cases: one is for all the $\pm$ being replaced by $+$; and the other for all being replaced by $-$. According to this convention, we shall sometimes need to write $\mp$ as $-(\pm)$.

2. MONOMIAL FUNCTIONS IN $\mathbf{R}_1^n$

The concept of intrinsic function naturally fits into our theory. A set in the complex plane $\mathbf{C}$ is said to be *intrinsic* if it is symmetric with respect to the real-axis; and a function f^0 is said to be *intrinsic* if the domain of f^0 is an intrinsic set and $\overline{f^0(z)} = f^0(\bar{z})$ in its domain. A set in $\mathbf{R}_1^n$ is said to be *intrinsic* if it does not change under the rotations of $\mathbf{R}_1^n$, considered as $n+1$ dimensional Euclidean space, that keep the $\mathbf{e}_0$-axis unchanged. If O is a set in the complex plane, then $\vec{O} = \{x \in \mathbf{R}_1^n : (x_0, |\underline{x}|) \in O\}$ is called the *induced set from O*. It is clear that an induced set is always an intrinsic set in $\mathbf{R}_1^n$. Functions of the form $\sum c_k(z-a_k)^k$, $k \in \mathbf{Z}$, a_k, $c_k \in \mathbf{R}$ are intrinsic functions. If $f^0 = u + \mathbf{i}v$, where u and v are real-valued, then f^0 is intrinsic if and only if $u(x, -y) = u(x, y)$, $v(x, -y) = -v(x, y)$ in its domain. In particular, $v(x, 0) = 0$, i.e., f^0 is real-valued if it is restricted to the real line in its domain. For more information on intrinsic functions in the complex plane and in the quaternionic space, we refer the reader to [Ri] and [Tu].

Let $f^0(z) = u(x, y) + \mathbf{i}v(x, y)$ be an intrinsic function defined on an intrinsic set $U \subset \mathbf{C}$. We may induce a function $\overrightarrow{f^0}$ from f^0, defined on the induced set $\vec{U}$, as follows:

$$\overrightarrow{f^0}(x) = u(x_0, |\underline{x}|) + \frac{\underline{x}}{|\underline{x}|} v(x_0, |\underline{x}|), \qquad x \in \vec{U}. \tag{1}$$

The function $\overrightarrow{f^0}$ will be called the *induced function from* f^0.

Let us first assume f^0 to be of the form z^k, $k \in \mathbf{Z}$, and denote by τ the mapping

$$\tau(f^0) = \kappa_n^{-1} \Delta^{(n-1)/2} \overrightarrow{f^0}, \tag{2}$$

where $\Delta = D\bar{D}$, $\bar{D} = D_0 - \underline{D}$ and $\kappa_n = (2\mathbf{i})^{n-1} \Gamma^2(\frac{n+1}{2})$ the normalizing constant that makes $\tau((\cdot)^{-1}) = E$ (see the proof of Proposition 1).

The operator $\Delta^{(n-1)/2}$ is defined via the Fourier multiplier transform on tempered distributions M: $\mathscr{S}' \to \mathscr{S}'$ induced by the multiplier $m(\xi) = (2\pi\mathbf{i}\,|\xi|)^{n-1}$:

$$Mf = \mathscr{R}(m\mathscr{F}f),$$

where

$$\mathscr{F}f(\xi) = \int_{\mathbf{R}_1^n} e^{2\pi\mathbf{i}\langle x, \xi\rangle} f(x)\, dx$$

and

$$\mathscr{R}h(x) = \int_{\mathbf{R}_1^n} e^{-2\pi\mathbf{i}\langle \xi, x\rangle} h(\xi)\, d\xi.$$

It is noted that both the Fourier transform $\mathscr{F}$ and its inverse $\mathscr{R}$ are defined on tempered distributions via pairing with rapidly decreasing functions ([Yo]).

If n is an odd integer, then $\Delta^{(n-1)/2}$ reduces to an ordinary differential operator that was first studied by M. Sce ([Sc], [Q5]).

The *monomial functions in* $\mathbf{R}_1^n$ are defined to be

$$P^{(-k)} = \tau((\cdot)^{-k}), \quad P^{(k-1)} = I(P^{(-k)}), \qquad k \in \mathbf{Z}^+.$$

We shall write $P_n^{(k)}$ for the sequence $P^{(k)}$ defined in $\mathbf{R}_1^n$ if it is necessary to emphasize the dimension n.

We have

PROPOSITION 1. *Let* $k \in \mathbf{Z}^+$. *Then* (i) $P^{-1} = E$; (ii) $P^{(-k)}(x) = (-1)^{k-1}/(k-1)!\,(\partial/\partial x_0)^{k-1} E(x)$; (iii) $P^{(-k)}$ *and* $P^{(k-1)}$ *both are monogenic*; (iv) $P^{(-k)}$ *is homogeneous of degree* $-n+1-k$ *and* $P^{(k-1)}$ *homogeneous of degree* $k-1$; (v) $c_n P_{n-1}^{(-k)}(x_0 + x_1\mathbf{e}_1 + \cdots + x_{n-1}\mathbf{e}_{n-1}) = \int_{-\infty}^{\infty} P_n^{(-k)}(x)\,dx_n$, *where* $c_n = \int_{-\infty}^{\infty} (1+t^2)^{-((n+1)/2)}\,dt$; (vi) $P^{(-k)} = I(P^{(k-1)})$; (vii) *if* n *is odd, then* $P^{(k-1)} = \tau((\cdot)^{n+k+2})$.

Remark 1. The definition of the monomial functions together with the properties proved in Proposition 1 provides a generalization of Fueter's result for quaternions. The latter asserts that if $f^0(z) = u(x, y) + \mathbf{i}v(x, y)$ is holomorphically defined in a relatively open set O of the upper half complex plane, then the function $\Delta(\overrightarrow{f^0}(q))$ is regular (i.e., quaternionic monogenic) for $q \in \vec{O}$, where Δ is the Laplacian in four variables q_0, q_1, q_2, q_3. Sce generalized the result in 1957 to $\mathbf{R}_1^n$ for n being odd integers. The assertions (iii) and (vii) amount to re-producing Sce's result for z^k, $k \in \mathbf{Z}$. The assertion (vii), in particular, shows that, if n is odd, then $P^{(k-1)}$ may be alternatively and consistently defined by using the operator τ, instead of using the Kelvin inversion.

A first development of this theme and a proof of (vii) is contained in [Q6]. The assertions (i)–(vi) will be concerned in the present paper, of which brief proofs will now be outlined.

Proof. Using the Fourier transform result on rational homogeneous functions with harmonic numerators (see [St]) and the relation

$$\overrightarrow{(\cdot)^{-k}}(x) = \left(\frac{\bar{x}}{|x|^2}\right)^k = \frac{(-1)^{k-1}}{(k-1)!}\left(\frac{\partial}{\partial x_0}\right)^{k-1}\left(\frac{\bar{x}}{|x|^2}\right),$$

we have

$$\begin{aligned} P^{(-k)}(x) = \tau((\cdot)^{-k})(x) &= \kappa_n^{-1} \frac{(-1)^{k-1}}{(k-1)!} \left(\frac{\partial}{\partial x_0}\right)^{k-1} M\left(\frac{\overline{(\cdot)}}{|\cdot|^2}\right) \\ &= \kappa_n^{-1} \frac{(-1)^{k-1}}{(k-1)!} \left(\frac{\partial}{\partial x_0}\right)^{k-1} \mathscr{R}\left(\gamma_{1,n}(2\pi \mathbf{i}\,|\xi|)^{n-1} \frac{\bar{\xi}}{|\xi|^{1+n}}\right) \\ &= \kappa_n^{-1} \frac{(-1)^{k-1}}{(k-1)!} \left(\frac{\partial}{\partial x_0}\right)^{k-1} \gamma_{1,n}^2 (2\pi \mathbf{i})^{n-1} \frac{\bar{x}}{|x|^{1+n}} \\ &= \kappa_n^{-1} \frac{(-1)^{k-1}}{(k-1)!} \kappa_n \left(\frac{\partial}{\partial x_0}\right)^{k-1} E(x), \end{aligned}$$

where we have let $\kappa_n = (2\pi\mathbf{i})^{n-1}\gamma_{1,n}^2 = (2\mathbf{i})^{n-1}\Gamma^2(\frac{n+1}{2})$. This implies that $P^{(-k)}$ is monogenic for all $k \in \mathbf{Z}^+$. The monogeneity of $P^{(k-1)}$ and the homogeneity of $P^{(-k)}$ and $P^{(k-1)}$ are easy consequences of the established expression and the properties of the Kelvin inversion. This completes the proofs of (i) to (iv). The assertion (v) follows from (i) and (ii) and the identity

$$c_n P_{n-1}^{(-1)}(x_0 + x_1\mathbf{e}_1 + \cdots + x_{n-1}\mathbf{e}_{n-1}) = \int_{-\infty}^{\infty} P_n^{(-1)}(x)\, dx_n,$$

proved through a direct computation. (vi) follows from the relation $I^2 =$ identity.

The assertion (ii) of Proposition 1 implies

Proposition 2. *The monomials satisfy the following estimates: for* $k \in \mathbf{Z}^+$,

$$|P^{(-k)}(x)| \leqslant C_n k^n |x|^{-(n+k-1)}, \qquad |x| > 1, \tag{3}$$

and

$$|P^{(k)}(x)| \leqslant C_n k^n |x|^k, \qquad |x| < 1. \tag{4}$$

It has the following

Corollary 1.

$$E(x-1) = P^{(-1)}(x) + P^{(-2)}(x) + \cdots + P^{(-k)}(x) + \cdots, \qquad |x| > 1, \tag{5}$$

and

$$E(1-x) = P^{(0)}(x) + P^{(1)}(x) + \cdots + P^{(k)}(x) + \cdots, \qquad |x| < 1. \tag{6}$$

Proof. (5) follows from the Taylor expansion of $E(x-1)$ and the estimate (3). Owing to the relation $I(E(\cdot - 1))(x) = E(x)\,E(x^{-1}-1) = E(1-x)$ and the estimate (4), applying the Kelvin inversion to both sides of (5), we obtain (6).

Noticing that $\tau(\frac{1}{z-1}) = E(x-1)$, the relation (5) is obtained, at least formally, by applying the mapping τ term by term to the series

$$\frac{1}{z-1} = \frac{1}{z} + \frac{1}{z^2} + \cdots + \frac{1}{z^k} + \cdots, \qquad |z| > 1;$$

and (6) is obtained similarly from

$$\frac{1}{1-z} = 1 + z + z^2 + \cdots + z^k + \cdots, \qquad |z| < 1.$$

To develop this aspect we recall some standard terminology and convention. A series of the form $\sum_{k=-\infty}^{\infty} c_k(z-a)^k$, c_k, $a \in \mathbf{C}$, is called a *Laurent series* at a. If $c_k = 0$ for all $k < 0$, it is also called a *power or Taylor series*, and if $c_k = 0$ for all $k \geqslant 0$, a *principal series*. For a, $c_k \in \mathbf{R}$ the series $\phi(x) = \sum c_k P^{(k)}(x - a\mathbf{e}_0)$ and $f^0(z) = \sum c_k(z-a)^k$ will be said to be *associated* to each other and the relation is denoted by $\phi = \Upsilon f^0$. The notation is also valid for a pair of functions defined through associated series. We define the *function* $f^0 = \sum c_k(z-a)^k$ to be the holomorphic extension with the largest open connected domain, called *holomorphic domain*, from the function originally defined through the power series in its convergence disk. The same convention applies to principal series. A function defined through a Laurent series is the one holomorphically defined in the intersection set of the holomorphic domains of its Taylor and principal series parts. Adopting this convention, the series $\sum_{k=1}^{\infty} z^k + \sum_{-\infty}^{-1} - z^k = -1 + \frac{2}{1-z}$ defines a function holomorphic in $\mathbf{C} \setminus \{1\}$. The convention also applies to functions defined through $\sum c_k P^{(k)}(x - a\mathbf{e}_0)$, but using "monogenic" in place of "holomorphic". An example is that

$$\sum_{k=1}^{\infty} P^{(k)}(x) + \sum_{-\infty}^{-1} - P^{(k)}(x)$$

defines a function monogenic everywhere except $x = 1$. It follows, from (5) and (6), that it is the function $2E(1-x)$, and, thus, $\Upsilon(-1 + \frac{2}{1-z}) = 2E(1-x)$.

For non-intrinsic series the following proposition is observed.

PROPOSITION 3. *If the function f^0 is defined in an intrinsic set, then both the functions $g^0(z)=\frac{1}{2}(f^0(z)+\overline{f^0(\bar{z})})$ and $h^0(z)=\frac{1}{2\mathbf{i}}(f^0(z)-\overline{f^0(\bar{z})})$ are intrinsic, defined in the same intrinsic set, and $f^0=g^0+\mathbf{i}h^0$.*

The proposition suggests extending Υ by

$$\Upsilon(f^0)=\Upsilon(g^0)+\mathbf{i}\Upsilon(h^0).$$

The functions f^0 and $\Upsilon(f^0)$ are said to be associated to each other. In this manner it is easy to see that, for $a\in\mathbf{R}$, $c_k\in\mathbf{C}$, we have $f^0(z)=\sum_{-\infty}^{\infty} c_k(z-a)^k=g^0+\mathbf{i}h^0$, where $g^0(z)=\sum_{-\infty}^{\infty} Re(c_k)(z-a)^k$ and $h^0(z)=\sum_{-\infty}^{\infty} Im(c_k)(z-a)^k$, and $\sum_{-\infty}^{\infty} c_kP^{(k)}(x-a\mathbf{e}_0)$ is associated with f^0.

The following is a consequence of Proposition 2.

PROPOSITION 4. *If $a\in\mathbf{R}$, $c_k\in\mathbf{C}$ and $\sum_{k=\pm1}^{\pm\infty} c_k(z-a)^k$ is absolutely convergent in $|(z-a)^{\pm1}|<r$, then $\sum_{k=\pm1}^{\pm\infty} c_kP^{(k)}(x-a\mathbf{e}_0)$ is absolutely convergent in $|(x-a\mathbf{e}_0)^{\pm1}|<r$.*

Owing to Proposition 4 the mapping τ may be extended to Laurent series. Note that, if f^0 represents a principal series, then $\tau(f^0)=\Upsilon(f^0)$; and, if $f^0=\sum_{k=0}^{\infty} c_k(z-a)^k$ representing a power series and the dimension n is odd, then $\tau(\sum_{k=0}^{\infty} c_k(z-a)^k)=(\sum_{k=n-1}^{\infty} c_kP^{(k-n+1)}(x-a\mathbf{e}_0)$, exhibiting a shift of coefficients. In the sequel we shall use the correspondence Υ rather than τ. This is especially convenient as we always use the Kelvin inversion to reduce power series to principal series.

In the following we shall call the series of the form $\sum c_k(z-a)^k$, a, $c_k\in\mathbf{R}$ *intrinsic series*.

For the case of n being an odd integer there is a direct relation between the holomorphic domain of an intrinsic series in the complex plane and the monogenic domain of its associated series in $\mathbf{R}_1^n$.

PROPOSITION 5. *Let $\sum c_k(z-a)^k$ be an intrinsic series whose holomorphic domain is an open intrinsic set, O, then the associated series $\sum c_kP^{(k)}(x-a\mathbf{e}_0)$ in $\mathbf{R}_1^n$ for n odd can be monogenically extended to the intrinsic set $\vec{O}$.*

Proof. Write $n=2m+1$. First we consider the principal series case. Let $f^0=\sum_{k=-\infty}^{-1} c_k(z-a)^k$ be an intrinsic principal series with the convergence disc $B_{\mathbf{C}}(a,\delta)$. For $x\in B_{\mathbf{R}_1^n}(a\mathbf{e}_0,\delta)$, we have

$$
\begin{aligned}
\Upsilon(f^0)(x) &= \sum_{-\infty}^{-1} c_k P^{(k)}(x - a\mathbf{e}_0) \\
&= \kappa_n \sum_{k=-\infty}^{-1} c_k \overrightarrow{\Delta^m(\cdot - a)^k}\,(x) \\
&= \kappa_n \Delta^m \left(\overrightarrow{\sum_{k=-\infty}^{-1} c_k(\cdot - a)^k\,(x)} \right) \\
&= \kappa_n \Delta^m(\overrightarrow{f^0}),
\end{aligned}
$$

where change of order of differentiation and summation is justified by Proposition 4. Now since f^0 can be holomorphically extended to O, by invoking Sce's result on the pointwise monogeneity for n odd ([Sce]), the function $\Upsilon(f^0)(x)$ can be monogenically extended to at least $\vec{O}$.

Let now f^0 be an intrinsic power series holomorphically defined in an intrinsic open set O. Denoting by $I^{\mathbf{c}}$ the Kelvin inversion in the complex plane, we have that $I^{\mathbf{c}} f^0$ is an intrinsic principal series holomorphically defined in the intrinsic set $O^{-1} = \{z \in \mathbf{C} : z^{-1} \in O\}$. Then the assertion for power series follows from what is proved for principal series together with the relations $I^{\mathbf{c}2} = \text{identity}$, $I^2 = \text{identity}$ and $\vec{O}^{-1} = \overrightarrow{O^{-1}}$.

The assertion for Laurent series follows from what have been proved for principal and power series. The proof is complete.

We shall use the following sets in the complex plane. Set, for $\omega \in (0, \frac{\pi}{2})$,

$$\mathbf{S}^{\mathbf{c}}_{\omega,\pm} = \{z \in \mathbf{C} : |\arg(\pm z)| < \omega\},$$

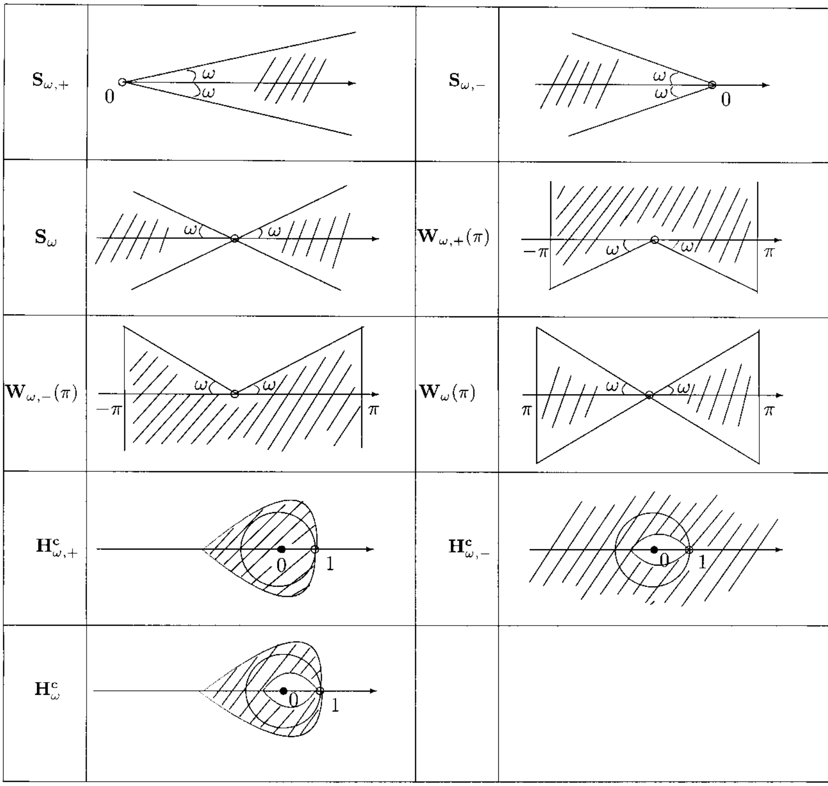
where the angle $\arg(z)$ of the complex number z takes values in $(-\pi, \pi]$,

$$
\begin{aligned}
\mathbf{S}^{\mathbf{c}}_{\omega,\pm}(\pi) &= \{z \in \mathbf{C} : |\mathrm{Re}(z)| \leqslant \pi,\ z \in \mathbf{S}^{\mathbf{c}}_{\omega,\pm}\}, \\
\mathbf{S}^{\mathbf{c}}_{\omega} &= \mathbf{S}^{\mathbf{c}}_{\omega,+} \cup \mathbf{S}^{\mathbf{c}}_{\omega,-}, \\
\mathbf{S}^{\mathbf{c}}_{\omega}(\pi) &= \mathbf{S}^{\mathbf{c}}_{\omega,+}(\pi) \cup \mathbf{S}^{\mathbf{c}}_{\omega,-}(\pi), \\
\mathbf{W}^{\mathbf{c}}_{\omega,\pm}(\pi) &= \{z \in \mathbf{C} : |\mathrm{Re}(z)| \leqslant \pi \text{ and } \pm\mathrm{Im}(z) > 0\} \cup \mathbf{S}^{\mathbf{c}}_{\omega}(\pi), \\
\mathbf{H}^{\mathbf{c}}_{\omega,\pm} &= \{z = \exp(i\eta) \in \mathbf{C} : \eta \in \mathbf{W}^{\mathbf{c}}_{\omega,\pm}(\pi)\},
\end{aligned}
$$

and

$$\mathbf{H}^{\mathbf{c}}_{\omega} = \mathbf{H}^{\mathbf{c}}_{\omega,+} \cap \mathbf{H}^{\mathbf{c}}_{\omega,-}.$$

These sets are illustrated in the diagram. $\mathbf{W}^{\mathbf{c}}_{\omega,+}(\pi)$ and $\mathbf{W}^{\mathbf{c}}_{\omega,-}(\pi)$ are "W" and "M" shaped regions, respectively. $\mathbf{H}^{\mathbf{c}}_{\omega,+}$ is a heart-shaped region, and the complement of $\mathbf{H}^{\mathbf{c}}_{\omega,-}$ is a heart-shaped region. With the obvious meaning we shall sometimes write $\mathbf{H}^{\mathbf{c}}_{\omega,\pm} = e^{i\mathbf{W}^{\mathbf{c}}_{\omega,\pm}(\pi)}$.



The following function spaces will be used:

$$K(\mathbf{H}^{\mathbf{c}}_{\omega,\pm})=\Big\{\phi^0\colon \mathbf{H}^{\mathbf{c}}_{\omega,\pm}\to\mathbf{C}\colon \phi^0 \text{ is holomorphic and satisfies}$$

$$|\phi^0(z)|\leqslant\frac{C_\mu}{|1-z|} \text{ in every } \mathbf{H}^{\mathbf{c}}_{\mu,\pm},\ 0<\mu<\omega\Big\},$$

$$K(\mathbf{H}^{\mathbf{c}}_{\omega})=\{\phi^0\colon \mathbf{H}^{\mathbf{c}}_{\omega}\to\mathbf{C}\colon \phi^0=\phi^{0,+}+\phi^{0,-},\ \phi^{0,\pm}\in K(\mathbf{H}^{\mathbf{c}}_{\omega,\pm})\},$$

$$H^\infty(\mathbf{S}^{\mathbf{c}}_{\omega,\pm})=\{b\colon \mathbf{S}^{\mathbf{c}}_{\omega,\pm}\to\mathbf{C}\colon b \text{ is holomorphic and satisfies}$$

$$|b(z)|\leqslant C_\mu \text{ in every } \mathbf{S}^{\mathbf{c}}_{\mu,\pm},\ 0<\mu<\omega\},$$

and

$$H^\infty(\mathbf{S}^{\mathbf{c}}_{\omega})=\{b\colon \mathbf{S}^{\mathbf{c}}_{\omega}\to\mathbf{C}\colon b_\pm=b\chi_{\{z\in\mathbf{C}\,:\,\pm\mathrm{Re}\,z>0\}}\in H^\infty(\mathbf{S}^{\mathbf{c}}_{\omega,\pm})\}.$$

Remark 2. The sets and function spaces introduced above naturally fit into our theory for closed curves and surfaces. Using those and related ones, we have built up our integral theories on infinite Lipschitz graphs in a series of earlier work [McQ1–2], [LMcQ], [LMcS], [GLQ] and [Ta], on starlike Lipschitz curves in the complex plane in [Q1] and [GQW], on the n-torus and its Lipschitz perturbations in [Q3] and [Q4] and on starlike Lipschitz surfaces in the quaternionic space in [Q5]. Informally, we may just recall that $H^\infty(\mathbf{S}^{\mathbf{c}}_{\omega,\pm})$ and $H^\infty(\mathbf{S}^{\mathbf{c}}_{\omega})$ are spaces of Fourier multipliers and $K(\mathbf{H}^{\mathbf{c}}_{\omega,\pm})$ and $K(\mathbf{H}^{\mathbf{c}}_{\omega})$ are spaces of kernels of singular integrals. On the Fourier multiplier side, this is consistent with the fact that the closure of $\mathbf{S}^{\mathbf{c}}_{\omega}$ contains the spectrum of the spherical Dirac operator on a Lipschitz curves or surfaces whose Lipschitz constants are less than $\tan(\omega)$. On the singular integral side, in the complex plane for instance, we consider convolution integrals of the form $\int_\gamma \phi(z\eta^{-1})\, f(\eta)\, \frac{d\eta}{\eta}$, $z\in\gamma$, on starlike Lipschitz curves γ whose Lipschitz constants are less than $\tan(\omega)$. It is easy to verify that the condition $z, \eta\in\gamma$ implies $z\eta^{-1}\in\mathbf{H}^{\mathbf{c}}_{\omega}$ for $\omega>\arctan(N)$ (see [Q1], [GQW]). This requires that our kernel functions ought to be defined in $\mathbf{H}^{\mathbf{c}}_{\omega}$.

In $\mathbf{R}^n_1$ we shall be working on heart-shaped regions or their complements

$$\mathbf{H}_{\omega,\pm}=\left\{x\in\mathbf{R}^n_1\colon \frac{(\pm\ln|x|)}{\arg(\mathbf{e}_0, x)}<\tan\omega\right\}=\overrightarrow{\mathbf{H}^{\mathbf{c}}_{\omega,\pm}},$$

and

$$\mathbf{H}_\omega=\mathbf{H}_{\omega,+}\cap\mathbf{H}_{\omega,-}=\overrightarrow{\mathbf{H}^{\mathbf{c}}_{\omega}}.$$

That is

$$\mathbf{H}_\omega=\left\{x\in\mathbf{R}^n_1: \frac{|\ln|x||}{\arg(\mathbf{e}_0, x)}<\tan\omega\right\}.$$

Remark 3. The reason for using these sets on surfaces is the same as that described in Remark 2 for star-shaped Lipschitz curves. Precisely, we shall be working on convolution singular integral on star-shaped Lipschitz surfaces and the kernel functions ought to be defined in $\mathbf{H}_\omega$. The following observation for the complex plane case motivated the definition of $\mathbf{H}_\omega$: It is easy to show that a star-shaped Lipschitz curve has the parameterisation $\gamma=\gamma(x)=e^{i(x+iA(x))}$, where $A=A(x)$ is a 2π-periodic Lipschitz function. Assume that the Lipschitz constant of γ is less than $\tan\omega$. Then for

$z = \exp i(x + \mathbf{i}A(x))$, $\eta = \exp \mathbf{i}(y + \mathbf{i}A(y))$, we have $z\eta^{-1} = \exp \mathbf{i}((x-y) + \mathbf{i}(A(x) - A(y)))$. This implies that

$$\frac{|\ln |z\eta^{-1}||}{\arg (z\eta^{-1}, 1)} = \frac{|A(x) - A(y)|}{|x-y|} < \tan \omega.$$

In $\mathbf{R}_1^n$ we shall be using the function spaces

$$K(\mathbf{H}_{\omega,\pm}) = \{\phi\colon \mathbf{H}_{\omega,\pm} \to \mathbf{C}^{(n)} : \phi \text{ is monogenic and satisfies } |\phi(x)| \leqslant C_\mu/|1-x|^n,\ x \in \mathbf{H}_{\mu,\pm},\ 0 < \mu < \omega\},$$

and

$$K(\mathbf{H}_\omega) = \{\phi\colon \mathbf{H}_\omega \to \mathbf{C}^{(n)} : \phi = \phi^+ + \phi^-,\ \phi^\pm \in K(\mathbf{H}_{\omega,\pm})\}.$$

Now we are ready to state our main technical result.

THEOREM 1. *If* $b \in H^\infty(\mathbf{S}^{\mathbf{c}}_{\omega,\pm})$ *and* $\phi(x) = \sum_{k=\pm 1}^{\pm\infty} b(k)\, P^{(k)}(x)$, *then* $\phi \in K(\mathbf{H}_{\omega,\pm})$.

Remark 4. In [Q1] we prove the following interesting holomorphic extension result: Let $b \in H^\infty(\mathbf{S}^{\mathbf{c}}_{\omega,\pm})$ and $f^0(z) = \sum_{k=\pm 1}^{\pm\infty} b(k)\, z^k$. Then f^0, originally defined inside or outside the unit circle corresponding to the case "+" or "−", respectively, can be holomorphically extended to $\mathbf{H}^{\mathbf{c}}_{\omega,\pm}$ satisfying the estimate

$$|f^0(z)| \leqslant \frac{C_\mu}{|1-z|}, \qquad z \in \mathbf{H}^{\mathbf{c}}_{\mu,\pm}, \qquad 0 < \mu < \omega.$$

We also prove that the converse result holds. The holomorphic extension result has significant applications to the singular integral theory on starlike Lipschitz curves ([Q1], [GQW]). Theorem 1 is the counterpart result in the $\mathbf{R}_1^n$ context.

Proof for n odd. Let $n = 2m+1$. Owing to Proposition 3 we are reduced to prove the theorem for b in $H^{\infty,\mathbf{r}}(\mathbf{S}^{\mathbf{c}}_{\omega,\pm})$, where

$$H^{\infty,\mathbf{r}}(\mathbf{S}^{\mathbf{c}}_{\omega,\pm}) = \{b \in H^\infty(\mathbf{S}^{\mathbf{c}}_{\omega,\pm}) : b|_{\mathbf{R}\cap \mathbf{S}^{\mathbf{c}}_{\omega,\pm}} \text{ is real-valued})\}.$$

In fact, in the decomposition $b = g^0 + \mathbf{i}h^0$ both g^0 and h^0 belong to $H^{\infty,\mathbf{r}}(\mathbf{S}^{\mathbf{c}}_{\omega,\pm})$ and are bounded by the bounds of b. We shall first consider the case "−", and then use the Kelvin inversion to conclude the case "+".

Now assume $b \in H^{\infty, \mathbf{r}}(\mathbf{S}^{\mathbf{c}}_{\omega, -})$ and consider $\phi(x) = \sum_{k=1}^{\infty} b(-k)\, P^{(-k)}(x) = \Delta^m \phi^0(x_0, |\underline{x}|)$, where $\phi^0(z) = \sum_{k=1}^{\infty} b(-k)\, z^{-k}$. It is a basic result of [Q1] that $\phi^0 \in K(\mathbf{H}^{\mathbf{c}}_{\omega, -})$. We can further derive, by using Cauchy's formula (see Lemma 6 of [Q5]), that

$$|(\phi^0)^{(j)}(z)| \leqslant \frac{2j!\, C_\mu}{\delta^j(\mu)} \frac{1}{|1-z|^{1+j}}, \qquad z \in \mathbf{H}^{\mathbf{c}}_{\mu, -}, \quad 0 < \mu < \omega, \quad j \in \mathbf{Z}^+ \cup \{0\}, \tag{7}$$

where C_μ is the constant in the definition of $K(\mathbf{H}^{\mathbf{c}}_{\omega, -})$, $\delta(\mu) = \min \{\frac{1}{2}, \tan(\omega - \mu)\}$.

Proposition 5 then asserts that ϕ is monogenically defined in $\mathbf{H}_{\omega, -}$. We are left to show that

$$|\phi(x)| \leqslant \frac{C_\mu}{|1-x|^n}, \qquad x \in \mathbf{H}_{\mu, -} = \overrightarrow{\mathbf{H}^{\mathbf{c}}_{\mu, -}}, \qquad 0 < \mu < \omega.$$

The following Lemma summaries the techniques used by Sce ([Sc]).

Lemma 1. *Let $f^0(z) = u(s, t) + \mathbf{i} v(s, t)$ be a function holomorphically defined in a relatively open set U of the upper half complex plane. Denote $u_0 = u$, $v_0 = v$, and, for $l \in \mathbf{Z}^+$,*

$$u_l = 2l \frac{1}{t} \frac{\partial u_{l-1}}{\partial t}, \qquad v_l = 2l \left(\frac{\partial v_{l-1}}{\partial t} \frac{1}{t} - \frac{v_{l-1}}{t^2} \right) = 2l \frac{\partial}{\partial t} \left(\frac{v_{l-1}}{t} \right).$$

Then

$$\Delta^l \overrightarrow{f^0}(x) = u_l(x_0, |\underline{x}|) + \frac{\underline{x}}{|\underline{x}|} v_l(x_0, |\underline{x}|), \qquad x_0 + \mathbf{i}\, |\underline{x}| \in U.$$

In order to prove the estimate we only need to consider the points $x \approx 1$ in the region $\mathbf{H}_{\omega, -}$. We shall deal with two cases.

Case 1. $|\underline{x}| > (\delta(\mu)/2^{m+1/2})\, |1-x|$.

Owing to Lemma 1, this reduces to study u_l and v_l in the region $\mathbf{H}^{\mathbf{c}}_{\omega, -}$ with the conditions that $z \approx 1$ and $|t| \approx |1-z|$. We shall later substitute $z = s + it$, $s = x_0$, $t = |\underline{x}|$. We observe that $u = u_0$, $v = v_0$ and $\frac{1}{t}$ all are of the magnitude $1/|1-z|$, and taking derivative with respect to t to each of them will reduce the power by one in the magnitude and so get the magnitude $1/|1-z|^2$. To obtain u_1, for instance, starting from u_0, we first take the derivative and then divide the result by t, leading to the magnitude $1/|1-z|^3$. Repeating this procedure up to m times to get u_m, we obtain the magnitude $1/|1-z|^{2m+1} = 1/|1-z|^n$. The estimate for v_m is proved similarly.

Case 2. $|\underline{x}| \leqslant (\delta(\mu)/2^{m+1/2})\ |1-x|$.

Points in $\mathbf{H}_{\omega,\,-}$ satisfying $x \approx 1$, $x_0 \leqslant 1$, belong to Case 1 and have been considered. Now we assume $x_0 > 1$.

Owing to Lemma 1, we need to show, for any $0 < \mu < \omega$,

$$|u_m(s,t)| + |v_m(s,t)| \leqslant \frac{C_{\mu,\,m}}{|1-z|^n}, \qquad z = s + \mathbf{i}t \in \mathbf{H}^{\mathbf{c}}_{\mu,\,-}.$$

We shall first study u_l, $0 \leqslant l \leqslant m$. The proof will involve partial derivatives of u_l with respect to its second argument. We claim that for $z = s + \mathbf{i}t \approx 1$, $s > 1$, $z \in \mathbf{H}^{\mathbf{c}}_{\mu,\,-}$, $\delta = \delta(\mu)$ and $|t| \leqslant (\delta/2^{m+1/2})\ |1-z|$, there hold

(i) u_l is even with respect to its second argument; and

(ii) for any integer $0 \leqslant j < \infty$,

$$\left|\frac{\partial^j}{\partial t^j} u_l(s,t)\right| \leqslant \frac{C_\mu C_l 2^{lj} (j+4l)!}{\delta^{2l+j}} \frac{1}{|1-z|^{2l+j+1}}, \qquad j \text{ even},$$

and

$$\left|\frac{\partial^j}{\partial t^j} u_l(s,t)\right| \leqslant \frac{C_\mu C_l 2^{lj} (j+5l)!}{\delta^{2l+j}} \frac{1}{|1-z|^{2l+j+1}}, \qquad j \text{ odd}.$$

We shall use mathematical induction on l. The assertions for $l = 0$ are from the corresponding properties of ϕ^0.

Now assume (i) and (ii) hold for the indices l: $0 \leqslant l \leqslant m-1$. We shall verify that they remain to hold for the next index $l+1$.

The assertion (i) for the index $l+1$ follows from the definition of u_{l+1} and the assertion (i) for the index l.

Now we prove (ii) for $l+1$. Since $u_l(s,t)$ is an even function with respect to t, $\partial u_l/\partial t$ is odd with respect to t. This implies $(\partial u_l/\partial t)(s,0) = 0$, and the same reasoning gives $((\partial^{2k+1} u_l)/(\partial t^{2k+1}))(s,0) = 0$ for $k \in \mathbf{Z}^+ \cup \{0\}$. For small t the Taylor expansion of $(\partial u_l/\partial t)(s,t)$ as $t = 0$ is

$$u_{l+1}(s,t) = \frac{2(l+1)}{t} \frac{\partial u_l}{\partial t}(s,t) = 2(l+1) \sum_{k=0}^{\infty} \frac{\dfrac{\partial^{2k+2} u_l}{\partial t^{2k+2}}(s,0)}{(2k+1)!} t^{2k}.$$

Taking derivatives with respect to t up to j times, for j even we obtain

$$\frac{\partial^j}{\partial t^j} u_{l+1}(s, t) = 2(l+1) \sum_{k=j/2}^{\infty} \frac{\partial^{2k+2} u_l}{\partial t^{2k+2}}(s, 0) \frac{(2k)(2k-1)\cdots(2k-j+1)}{(2k+1)!} t^{2k-j}.$$

Using the induction hypothesis (ii) for the index l and changing the index k to $j/2+k$, we have

$$\begin{aligned}\left|\frac{\partial^j}{\partial t^j} u_{l+1}(s, t)\right| &\leqslant 2(l+1) \frac{C_\mu C_l 2^{l(j+2)}}{\delta^{2(l+1)+j} |1-z|^{2(l+1)+j+1}} \\ &\quad \times \sum_{k=0}^{\infty} \frac{(j+4l+2k+2)!\, 2^{2kl}}{(j+2k+1)!} \\ &\quad \times (j+2k)\cdots(2k+1) \left(\frac{t}{\delta\,|1-z|}\right)^{2k} \\ &\leqslant 2(l+1) \frac{C_\mu C_l 2^{l(j+2)}}{\delta^{2(l+1)+j} |1-z|^{2(l+1)+j+1}} \\ &\quad \times \sum_{k=0}^{\infty} \frac{(j+4l+2k+2)\cdots(2k+2)}{2^k},\end{aligned}$$

where we have used the condition $\frac{t}{\delta\,|1-z|} \leqslant 1/2^{m+1/2}$.

The last series is evaluated in the following

LEMMA 2.

$$\sum_{k=0}^{\infty} \frac{(j+4l+2k+2)\cdots(2k+2)}{2^k} = 2^{j+4l+3} \left(\frac{j+4l+2}{2}\right)!. \tag{8}$$

Proof. Denote by s the sum of the series. Then $\frac{1}{2}s$ is the sum of the series obtained by multiplying the original series, term by term, by $\frac{1}{2}$. The common trick of evaluating the sum of the difference series then gives

$$s = 2(j+4l+2) \sum_{k=0}^{\infty} \frac{(j+4l+2k)\cdots(2k+2)}{2^k}.$$

Repeating the procedure up to $\frac{j+4l+2}{2}$ times, we obtain

$$s = 2^{(j+4l+2)/2}(j+4l+2)!!\, 2 = 2^{j+4l+3} \left(\frac{j+4l+2}{2}\right)!.$$

The proof is complete.

In order to simplify the expression of the constant C_l, we rather use the following weaker estimate derived from Lemma 2:

$$\sum_{k=0}^{\infty} \frac{(j+4l+2k+2)\cdots(2k+2)}{2^k} \leqslant 2^{j+4l-1}(j+4(j+1))!.$$

This last estimate gives the desired estimate for $|(\partial^j/\partial t^j)\, u_{l+1}(s,t)|$ with $C_l = l!\, 2^{3l(l-1)}$.

For the case j being odd a similar estimate gives

$$\begin{aligned}\left|\frac{\partial^j}{\partial t^j} u_{l+1}(s,t)\right| &\leqslant 2(l+1)\, \frac{C_\mu C_l 2^{l(j+2)}}{\delta^{2(l+1)+j}\, |1-z|^{2(l+1)+j+1}}\, \frac{t}{\delta\, |1-z|} \\ &\quad \times \sum_{k=0}^{\infty} \frac{(j+5l+2k+3)!\, 2^{2kl}}{(j+2k+2)!} \\ &\quad \times (j+2k+1)\cdots(2k+3)\left(\frac{t}{\delta\, |1-z|}\right)^{2k} \\ &\leqslant 2(l+1)\, \frac{C_\mu C_l 2^{l(j+2)}}{\delta^{2(l+1)+j}\, |1-z|^{2(l+1)+j+1}}\, \frac{1}{2^{m+1/2}} \\ &\quad \times \sum_{k=0}^{\infty} \frac{(j+5l+2k+3)\cdots(2k+3)}{2^k} \\ &\leqslant \frac{C_\mu C_{l+1} 2^{(l+1)\,j}(j+5(l+1))!}{\delta^{2(l+1)+j}}\, \frac{1}{|1-z|^{2(l+1)+j+1}}\end{aligned}$$

with an appropriate constant C_l.

Letting $l=m$, $j=0$, we obtain the desired estimate for u_m.

Now we study v_m and still consider the two cases $|\underline{x}| > (\delta(\mu)/2^{m+1/2})\, |1-x|$ and $|\underline{x}| \leqslant (\delta(\mu)/2^{m+1/2})\, |1-x|$. The first case is the easy part and may be dealt with by using a similar argument as used for u_m. For the second case we shall prove: For $0 \leqslant l \leqslant m$, $z = s + \mathbf{i}t \approx 1$, $s > 1$, $z \in \mathbf{H}^{\mathbf{c}}_{\mu,-}$, $0 < \mu < \omega$, and $|t| \leqslant (\delta/2^{m+1/2})\, |1-z|$,

(i) v_l is odd with respect to its second argument;

(ii) for any integer $0 \leqslant j < \infty$

$$\left|\frac{\partial^j}{\partial t^j} v_l(s,t)\right| \leqslant \frac{C_\mu C_l 2^{lj}(j+5l)!}{\delta^{2l+j}}\, \frac{1}{|1-z|^{2l+j+1}}, \qquad j \text{ even},$$

and

$$\left|\frac{\partial^j}{\partial t^j} v_l(s, t)\right| \leqslant \frac{C_\mu C_l 2^{lj}(j+4l)!}{\delta^{2l+j}} \frac{1}{|1-z|^{2l+j+1}}, \qquad j \text{ odd.}$$

We shall use mathematical induction and the proof is similar to that for μ_l.

For $l=0$ (i) and (ii) are consequences of the corresponding properties of ϕ^0.

Now assume (i) and (ii) to hold for the index l: $0 \leqslant l \leqslant m-1$. We are to verify that they hold for $l+1$.

The assertion (i) for $l+1$ follows from the definition of v_{l+1} and the assertion (i) for the index l.

Now we prove (ii) for $l+1$. Since $v_l(s, t)$ is an odd function with respect to t, we have $(\partial^{2k} v_l(s, 0))/\partial t^{2k} = 0$ for $k \in \mathbf{Z}^+ \cup \{0\}$ and so its Taylor expansion in t at $t=0$ reads

$$v_l(s, t) = \sum_{k=0}^{\infty} \frac{\frac{\partial^{2k+1} v_l(s, 0)}{\partial t^{2k+1}}}{(2k+1)!} t^{2k+1}.$$

Hence,

$$t \frac{\partial v_l(s, t)}{\partial t} = \sum_{k=0}^{\infty} \frac{\frac{\partial^{2k+1} v_l(s, 0)}{\partial t^{2k+1}}}{(2k)!} t^{2k+1}$$

and

$$\begin{aligned} v_{l+1}(s, t) &= 2(l+1) \frac{t \frac{\partial v_l}{\partial t} - v}{t^2} \\ &= 2(l+1) \sum_{k=0}^{\infty} \frac{2k+2}{(2k+3)!} \frac{\partial^{2k+3} v_l(s, 0)}{\partial t^{2k+3}} t^{2k+1}. \end{aligned}$$

Taking derivative with respect to t up to j times, discussing the two cases that j is even and odd, similar methods as used above give the desired estimates for $l+1$.

Taking $l=m$ and $j=0$ in the estimate for $|\frac{\partial^j}{\partial t^j} v_l(s, t)|$, we obtain the desired estimate for v_m.

Now consider the case "+". Assume $b \in H^{\infty, \mathbf{r}}(\mathbf{S}^{\mathbf{c}}_{\omega, +})$ and $\psi(x) = \sum_{i=1}^{\infty} b(i) P^{(i)}(x)$. The Kelvin inversion implies that $I(\psi)(x) = \sum_{i=-1}^{-\infty} b'(i) P^{(i-1)}(x)$, where $b'(z) = b(-z) \in H^{\infty, \mathbf{r}}(\mathbf{S}^{\mathbf{c}}_{\omega, -})$. Since $I(\psi) = \tau(\psi^0)$, where

$\psi^0(z)=\sum_{i=-1}^{-\infty} b'(i)\, z^{i-1}=\frac{1}{z}\sum_{i=-1}^{-\infty} b'(i)\, z^i \in \mathbf{H}^{\mathbf{c}}_{\omega,\,-}$, the conclusions for the above considered case "$-$" all apply to $I(\psi)$. Using the relation $\psi=I^2(\psi)=E(q)\, I(\psi)(x^{-1})$ and the fact that $x\in \mathbf{H}_{\nu,\,+}$ if and only if $x^{-1}\in \mathbf{H}_{\nu,\,-}$, we have

$$|\psi(x)|=|E(x)\, I(\psi)(x^{-1})|\leqslant \frac{1}{|x|^n}\, C_\nu \frac{1}{|1-x^{-1}|^n}$$

$$=C_\nu \frac{1}{|1-x|^n}, \qquad x\in \mathbf{H}_{\nu,\,+}.$$

This concludes the case $b\in H^{\infty,\,\mathbf{r}}(\mathbf{S}^{\mathbf{c}}_{\omega,\,+})$ and the proof for n being odd is complete.

Proof for n even. The same argument reduces the case "$+$" to the case "$-$". Let $b\in H^{\infty,\,\mathbf{r}}(\mathbf{S}^{\mathbf{c}}_{\omega,\,-})$ and consider $\phi(x)=\sum_{k=1}^{\infty} b(-k)\, P_n^{(-k)}(x)$. Now $n+1$ is odd and so the conclusions obtained in the first part applies to $n+1$. From (v) of Proposition 1 we obtain

$$c_{n+1}\phi(x)=\int_{-\infty}^{\infty} \sum_{k=1}^{\infty} b(-k)\, P_{n+1}^{(-k)}(x+x_{n+1}\mathbf{e}_{n+1})\, dx_{n+1},$$

where the function ϕ is monogenically defined in $\mathbf{H}_{\omega,\,-}$, which is the intersection of $\mathbf{R}^n_1$ and the corresponding $\mathbf{H}_{\omega,\,-}$ set in $\mathbf{R}^{n+1}_1$, and changing of order of integration and summation is justified by the right order of decaying of $P^{(k)}_{n+1}$ at the infinity. We further have

$$|c_{n+1}\phi(x)|\leqslant C_\nu \int_{-\infty}^{\infty} \frac{1}{|1-(x+x_{n+1}\mathbf{e}_{n+1})|^{n+1}}\, dx_{n+1}$$

$$\leqslant C_\nu \frac{1}{|1-x|^n}, \qquad x\in \mathbf{H}_{\nu,\,-}.$$

Corollary 2. *Let* $b\in H^\infty(\mathbf{S}^{\mathbf{c}}_\omega)$ *and* $\phi(x)=\sum_{i=-\infty}^{\infty}{}' \, b(i)\, P^{(i)}(x)$. *Then* $\phi\in K(\mathbf{H}_\omega)$.

Remark 5. The holomorphic extension result noted in Remark 4 can be extended to the cases where b is holomorphic, bounded near the origin, satisfying $|b(z)|\leqslant C_\nu\, |z|^s$, $|z|>1$, for $-\infty<s<\infty$ in smaller sectors $\mathbf{S}^{\mathbf{c}}_{\nu,\,\pm}$ ([Q2])). Theorem 1 can also be extended to those cases. Details will not be included here, but the following result, with a proof similar to that of Theorem 1 (see [Q2]), will be used in the proof of Proposition 8 (also see Lemma 5).

THEOREM 2. *Let* $-\infty < s < \infty$, $s \neq -1, -2, ...,$ *and* b *a holomorphic function in* $\mathbf{S}^{\mathbf{c}}_{\omega,\pm}$ *satisfying the estimates*

$$|b(z)| \leqslant C_\mu |z \pm 1|^s, \qquad \text{in every } \mathbf{S}^{\mathbf{c}}_{\mu,\pm}, \quad 0 < \mu < \omega.$$

Then $\phi(x) = \sum_{i=\pm 1}^{\pm\infty} b(i) P^{(i)}(x)$ *can be monogenically extended to* $\mathbf{H}_{\omega,\pm}$ *satisfying*

$$|\phi^{\pm}(x)| \leqslant C_\mu \left\| \frac{b(\cdot)}{|(\cdot) \pm 1|^s} \right\|_{L^\infty(\mathbf{S}^{\mathbf{c}}_{\mu'})} \frac{1}{|1-x|^{s+n}},$$

$$x \in \mathbf{H}_{\mu,\pm}, \qquad 0 < \mu < \mu' < \omega.$$

2. SINGULAR INTEGRALS AND FOURIER MULTIPLIERS

A surface Σ is said to be a starlike Lipschitz surface, if it is n-dimensional and star-shaped about the origin, and there exists a constant $M < \infty$ such that $x, x' \in \Sigma$ implies that

$$\frac{|\ln |x^{-1}x'||}{\arg(x, x')} \leqslant M.$$

The minimum value of M is called the Lipschitz constant of Σ, denoted by $N = \mathrm{Lip}(\Sigma)$.

Since locally $\ln |x^{-1}x'| = \ln(1 + (|x^{-1}x'| - 1))| \approx (|x^{-1}x'| - 1) \approx |x^{-1}| (|x'| - |x|) \approx (|x'| - |x|)$, the above defined sense of Lipschitz is consistent with the ordinary sense.

Let $s \in \mathbf{S}_{\mathbf{R}^n_1}$ and we consider the mapping $r_s \colon x \to sxs^{-1}$, $x \in \mathbf{R}^n_1$. Although r_s does not preserve $\mathbf{R}^n_1$, it enjoys the following properties.

LEMMA 3. *For any* $x, y \in \mathbf{R}^n_1$, *we have* (i) $|r_s(y^{-1}x)| = |y^{-1}x|$ *and more generally* r_s *preserves norms of the elements in* $\mathbf{R}^{(n)}$ *that can be expressed as a product of vectors*; (ii) $\langle r_s(x), r_s(y)\rangle = \langle x, y\rangle$; (iii) $\arg(r_s(x), r_s(y)) = \arg(x, y)$; (iv) $(r_s(y))^{-1} r_s(x) = r_s(y^{-1}x)$; (v) *There exists a vector* $s \in \mathbf{S}_{\mathbf{R}^n_1}$ *such that* $r_s(y^{-1}x) = |y|^{-1} \tilde{x}$, *where* $\tilde{x} \in \mathbf{R}^n_1$. *Moreover*, $|x - y| = ||y|\, \mathbf{e}_0 - \tilde{x}|$ *and* $\arg(y, x) = \arg(|y|\, \mathbf{e}_0, \tilde{x})$; *and* (vi) *For the same* s *as in* (v) *we have* $r_s(E(y)) = E(y)$.

Proof. (i) is an immediate consequence of the property of the norm defined in Section 1. (ii) is a consequence of (i) owing to the relation between the inner product and the norm in $\mathbf{C}^{(n)}$. (iii) is a consequence of (i) and (ii). (iv) is trivial. To prove (v), we introduce a new basic vector $\mathbf{e}'$ such that $\mathbf{e}'^2 = 1$ and $\mathbf{e}'\mathbf{e}_i = -\mathbf{e}_i\mathbf{e}'$, $i = 1, ..., n$. Let $\mathbf{f}_0 = \mathbf{e}'$, $\mathbf{f}_i = \mathbf{e}_i\mathbf{f}_0$, $i = 1, ..., n$.

We have $\mathbf{f}_i^2=1$, $\mathbf{f}_i\mathbf{f}_j=-\mathbf{f}_j\mathbf{f}_i$, $0\leqslant i, j\leqslant n$, $i\neq j$. So $\{\mathbf{f}_j\}_{j=0}^n$ forms a basis of type $(n+1, 0)$. It is a basis of $\mathbf{R}^{n+1}=\mathbf{R}^{n+1,0}=\{x_0\mathbf{f}_0+\cdots+x_n\mathbf{f}_n\colon x_j\in\mathbf{R}, j=0, 1, ..., n\}$. Owing to the property of the Clifford group in $\mathbf{R}^{n+1}$, we can choose $s\in\mathbf{R}_1^n$ such that the mapping: $(\cdot)\to(s\mathbf{f}_0)(\cdot)(s\mathbf{f}_0)^{-1}$ on $\mathbf{R}^{n+1}$ maps $y\mathbf{f}_0$ to $\mathbf{f}_0\,|y|$ (see [DSS]). The same mapping maps $x\mathbf{f}_0$ to, namely, $\mathbf{f}_0\tilde{x}$, where $\tilde{x}\in\mathbf{R}_1^n$. Therefore, we have $r_s(y^{-1}x)=[(s\mathbf{f}_0)(y\mathbf{f}_0)(s\mathbf{f}_0)^{-1}]^{-1}[(s\mathbf{f}_0)(x\mathbf{f}_0)(s\mathbf{f}_0)^{-1}]=(\mathbf{f}_0\,|y|)^{-1}(\mathbf{f}_0\tilde{x})=|y|^{-1}\,\tilde{x}$. Since the mappings induced by elements in the Clifford group preserve distance between vectors, we have $|x-y|=|y\mathbf{f}_0-x\mathbf{f}_0|=|\mathbf{f}_0\,|y|-\mathbf{f}_0\tilde{x}|=|\,|y|\,\mathbf{e}_0-\tilde{x}|$. By virtue of (iii), $\arg(y, x)=\arg(r_s(y), r_s(x))=\arg(\mathbf{f}_0\,|y|, \mathbf{f}_0\tilde{x})=\arg(|y|\,\mathbf{e}_0, \tilde{x})$. The proof of (vi) is proceeded as follows. $r_s(E(y))=\frac{1}{|y|^{n-1}}s(y^{-1}\mathbf{e}_0)\,s^{-1}=\frac{1}{|y|^{n-1}}(|y|^{-1}\,\mathbf{f}_0)(\mathbf{f}_0\mathbf{e}_0\widetilde{\mathbf{e}_0})$, where $\widetilde{\mathbf{e}_0}=(s\mathbf{f}_0)(\mathbf{f}_0)(s\mathbf{f}_0)^{-1}=s\mathbf{f}_0s^{-1}=\mathbf{f}_0\,\frac{\bar{y}}{|y|}$, where the last identity is deduced from $(s\mathbf{f}_0)(y\mathbf{f}_0)(s\mathbf{f}_0)^{-1}=\mathbf{f}_0\,|y|$. Substituting the expression of $\widetilde{\mathbf{e}_0}$, we obtain $r_s(E(y))=E(y)$.

Remark 6. We explain how the sets $\mathbf{H}_\omega$ are related to starlike Lipschitz surfaces. Lemma 3 implies that, by choosing an appropriate $s\in S_{\mathbf{R}_1^n}$, $\ln|x^{-1}x'|=\ln|r_s(x^{-1}x')|=\ln|\,|x|^{-1}\,\tilde{x}|$. On the other hand, $\arg(x, x')=\arg(|x|\,\mathbf{e}_0, \tilde{x})=\arg(\mathbf{e}_0, |x|^{-1}\,\tilde{x})$. So, if x and x' belong to a starlike Lipschitz surface with Lipschitz constant N, then $(|\ln|x^{-1}x'||/\arg(x, x'))=(|\ln|\,|x|^{-1}\,\tilde{x}|/\arg(1, |x|^{-1}\,\tilde{x}))\leqslant N$. This implies that $|x|^{-1}\,\tilde{x}\in\mathbf{H}_\omega$ for any $\omega\in(\arctan(N), \frac{\pi}{2})$ (also see the proof of Proposition 7).

We shall be working on a fixed star-shaped Lipschitz surface Σ with Lipschitz constant N and we assume that $\omega\in(\arctan(N), \frac{\pi}{2})$.

Denote

$$\rho=\min\{|x|\colon x\in\Sigma\}\qquad\text{and}\qquad \imath=\max\{|x|\colon x\in\Sigma\}.$$

Without loss of generality we can assume $\rho<1<\imath$.

We shall be working on $L^2(\Sigma)=L^2(\Sigma, d\sigma)$, where $d\sigma$ is the surface area measure. The norm of $f\in L^2(\Sigma)$ is denoted by $\|f\|$.

Coifman–McIntosh–Meyer's Theorem (CMcM' Theorem) asserts that on any Lipschitz surface Σ the Cauchy integral operator

$$C_\Sigma f(x)=p.v.\,\frac{1}{\Omega_n}\int_\Sigma E(x-y)\,n(y)\,f(y)\,d\sigma(y),$$

where $n(y)$ is the outward normal of Σ at $y\in\Sigma$, Ω_n the surface area of the n-dimensional unit sphere $\mathbf{S}_{\mathbf{R}_1^n}$, can be extended to a bounded operator in $L^2(\Sigma)$ ([CMcM], [Mc1]).

We shall be using

$$\mathscr{A}=\{f\colon f(x)\text{ is left-monogenic in }\rho-s<|x|<\imath+s\text{ for some }s>0\}$$

as the class of our test functions. It is a consequence of CMcM's Theorem that $\mathscr{A}$ is dense in $L^2(\Sigma)$ (see [CM], also [GQW] and [Q5]).

Now assume that $f \in \mathscr{A}$. In the annulus where f is defined we have the Laurent series expansion

$$f(x) = \sum_{k=0}^{\infty} P_k(f)(x) + \sum_{k=0}^{\infty} Q_k(f)(x),$$

where for $k \in \mathbf{Z}^+ \cup \{0\}$, $P_k(f)$ belongs to the finite dimensional right module M_k of k-homogeneous left-monogenic functions in $\mathbf{R}_1^n$, and $Q_k(f)$ belongs to the finite dimensional right module $M_{-(k+n)}$ of $-(k+n)$-homogeneous left-monogenic functions in $\mathbf{R}_1^n \setminus \{0\}$. The spaces M_k and M_{-k} are eigenspaces of the left-spherical Dirac operator and the mappings $P_k\colon f \to P_k(f)$ and $Q_k\colon f \to Q_k(f)$ are the projection operators on M_k and $M_{-(k+n)}$, respectively. If f is k-homogeneous spherical harmonic, $k \geqslant 1$, then $f = f^+ + f^-$, where $f^+ \in M_k$ and $f^- \in M_{-k+1-n}$. It is noted that the spaces M_k, $k = -1, -2, \ldots, -n+1$, do not exist ([DSS]). We shall postpone introducing more details of the Dirac operator to the later part of this section.

Formally one may consider the Fourier multiplier operator induced by a bounded sequence (b_k) defined by

$$M_{(b_k)} f(x) = \sum_{k=0}^{\infty} b_k P_k(f)(x) + \sum_{k=0}^{\infty} b_{-k-1} Q_k(f)(x).$$

It is easy to see that $M_{(b_k)}\colon \mathscr{A} \to \mathscr{A}$ is a linear operator and one may ask whether $M_{(b_k)}$ extends to a bounded operator in $L^2(\Sigma)$. If Σ is a sphere, then the boundedness is an immediate consequence of the Plancherel theorem under merely the condition that (b_k) is a bounded sequence (see, e.g. [DSS]). If Σ is a starlike Lipschitz surface, then the condition is not sufficient. We constructed an example in [Q1] showing that in $\mathbf{R}_1^1 = \mathbf{C}$ there is an unbounded operator of this sort.

As the main result of the paper we shall prove

THEOREM 3. *Let* $\omega \in (\arctan(N), \frac{\pi}{2})$. *If* $b \in H^\infty(\mathbf{S}_\omega^{\mathbf{c}})$, *then with the convention* $b(0) = 0$, *the above defined* $M_{(b(k))}$ *can be extended to a bounded operator from* $L^2(\Sigma)$ *to* $L^2(\Sigma)$. *Moreover,*

$$\|M_{(b(k))}\|_{L^2(\Sigma) \to L^2(\Sigma)} \leqslant C_\nu \, \|b\|_{L^\infty(\mathbf{S}_\nu^{\mathbf{c}})}, \qquad \arctan(N) < \nu < \omega.$$

Remark 7. The boundedness is of the same sort as CMcM's Theorem. Since the surfaces are of the homogeneous type in the sense of the doubling measure condition, the L^2-boundedness implies the L^p-boundedness for $1<p<\infty$, as well as the weak-type (1, 1) boundedness, by virtue of the standard Calderón–Zygmund techniques (see [St], [Da2]).

In order to prove the theorem we need to employ the singular integral convolution expression of the operator. We shall start with the same type expressions of the projection operators P_k and Q_k.

According to Section 1.6.4 of Chapter 2 of [DSS], we have, in the annulus where f is defined,

$$P_k(f)(x)=\frac{1}{\Omega_n}\int_\Sigma |y^{-1}x|^k\, C^+_{n+1,k}(\xi,\eta)\, E(y)\, n(y)\, f(y)\, d\sigma(y)$$

and

$$Q_k(f)(x)=\frac{1}{\Omega_n}\int_\Sigma |y^{-1}x|^{-k-n}\, C^-_{n+1,k}(\xi,\eta)\, E(y)\, n(y)\, f(y)\, d\sigma(y),$$

where $x=|x|\,\xi$, $y=|y|\,\eta$,

$$C^+_{n+1,k}(\xi,\eta)=\frac{1}{1-n}\big[-(n+k-1)\, C_k^{(n-1)/2}(\langle\xi,\eta\rangle) + (1-n)\, C_{k-1}^{(n+1)/2}(\langle\xi,\eta\rangle)(\langle\xi,\eta\rangle-\bar{\xi}\eta)\big],$$

and

$$C^-_{n+1,k}(\xi,\eta)=\frac{1}{n-1}\big[(k+1)\, C_{k+1}^{(n-1)/2}(\langle\xi,\eta\rangle) + (1-n)\, C_k^{(n+1)/2}(\langle\eta,\xi\rangle)(\langle\eta,\xi\rangle-\bar{\eta}\xi)\big],$$

where C_k^ν is the Gegenbauer polynomial of degree k associated with ν.

It is noted that $C^\pm_{n+1,k}$ are functions of $y^{-1}x$. This may be seen from the relations

$$\langle\xi,\eta\rangle=\frac{\langle y^{-1}x, 1\rangle}{|y^{-1}x|},\qquad \bar{\eta}\xi=\frac{y^{-1}x}{|y^{-1}x|}\qquad\text{and}\qquad \bar{\xi}\eta=\left(\frac{y^{-1}x}{|y^{-1}x|}\right)^{-1}.$$

Now for $k\in\mathbf{Z}^+\cup\{0\}$, define

$$\tilde{P}^{(k)}(y^{-1}x)=|y^{-1}x|^k\, C^+_{n+1,k}(\xi,\eta)$$

and

$$\tilde{P}^{(-k-1)}(y^{-1}x)=|y^{-1}x|^{-k-n}\, C^-_{n+1,k}(\xi,\eta).$$

It is noted that both $\widetilde{P}^{(k)}$ and $\widetilde{P}^{(-k-1)}$ are defined in the two-forms $\mathbf{R}_1^n \times \mathbf{R}_1^n$, and $\widetilde{P}^{(k)}(y^{-1}x)\,E(y)$ and $\widetilde{P}^{(-k-1)}(y^{-1}x)\,E(y)$ are monogenic functions in both variables x and y (see the sections on Laurent series in [DSS]). If, in particular, $y=1$, then the above functions in the two forms reduce to $P^{(k)}(x)$ and $P^{(-k-1)}(x)$, respectively. This is seen by comparing the Taylor and Laurent expansions of $E(x-1)$ and $E(1-x)$ and (5) and (6)). The domains of $\widetilde{P}^{(k)}$ and $\widetilde{P}^{(-k-1)}$ may be extended to $\mathbf{R}^{(n)} \times \mathbf{R}^{(n)}$, as inner product and vector product both can be extended to the latter product space.

Using these notations, we have, for $k \in \mathbf{Z}^+ \cup \{0\}$, $f \in \mathscr{A}$,

$$P_k(f)(x) = \frac{1}{\Omega_n} \int_\Sigma \widetilde{P}^{(k)}(y^{-1}x)\,E(y)\,n(y)\,f(y)\,d\sigma(y)$$

and

$$Q_k(f)(x) = \frac{1}{\Omega_n} \int_\Sigma \widetilde{P}^{(-k-1)}(y^{-1}x)\,E(y)\,n(y)\,f(y)\,d\sigma(y).$$

Accordingly, we have

$$f(x) = \sum_{k=-\infty}^{\infty} \frac{1}{\Omega_n} \int_\Sigma \widetilde{P}^{(k)}(y^{-1}x)\,E(y)\,n(y)\,f(y)\,d\sigma(y).$$

Remark 8. This is consistent with the convolution integral expressions of the projection operators in the complex and quaternionic contexts. Indeed, if f^0 is a holomorphic function in the annulus $\rho - s < |z| < \iota + s$ in $\mathbf{C}$, σ a star-shaped Lipschitz curve in the annulus, then the Laurent series of f^0 is given by

$$f^0(z) = \sum_{k=-\infty}^{\infty} \frac{1}{2\pi} \int_\sigma (\eta^{-1}z)^k\, f^0(\eta)\, \frac{d\eta}{\eta}.$$

See [Q5] for the analogue in the quaternionic space. In each of these contexts we write the projection operators as convolution integral operators using the natural multiplicative structure of the underlying space. The difference in the $\mathbf{R}_1^n$ context with the previous ones is that now the kernel functions are defined in the two-forms in $\mathbf{R}_1^n \times \mathbf{R}_1^n$.

The above defined functions $\widetilde{P}^{(k)}$ enjoy the following property.

PROPOSITION 6. *For any $s \in \mathbf{S}_{\mathbf{R}_1^n}$ we have $\widetilde{P}^{(k)}(r_s(y^{-1}x)) = r_s(\widetilde{P}^{(k)}(y^{-1}x))$.*

Proof. This may be deduced from (i), (ii), (iv) of Lemma 3 and the facts that r_s is the identity on scalars.

We shall call

$$\tilde{\phi}(y^{-1}x) = \sum_{-\infty}^{\infty} b_k \tilde{P}^{(k)}(y^{-1}x)$$

the kernel function associated with the multiplier operator $M_{(b_k)}$.

PROPOSITION 7. *Let $\omega \in (\arctan(N), \frac{\pi}{2})$ and $b \in H^\infty(\mathbf{S}^{\mathbf{c}}_\omega)$. Then the kernel function $\tilde{\phi}(y^{-1}x)\, E(y)$ associated with the sequence $(b(k))$ in the manner given above is monogenically defined in an open neighborhood of $\Sigma \times \Sigma \backslash \{(x, y): x = y\}$. Moreover, in the neighborhood,*

$$|\tilde{\phi}(y^{-1}x)| \leqslant \frac{C}{|1 - y^{-1}x|^n}.$$

Proof. Let us first consider the left-monogeneity with respect to x. Choosing $s \in S_{\mathbf{R}_1^n}$ as in Lemma 3, applying the mapping r_s term by term to the entries of the series $\tilde{\phi}(y^{-1}x)\, E(y)$ and using the relation $I = r_{s^{-1}} r_s$ and Lemma 3, we have $\tilde{\phi}(y^{-1}x)\, E(y) = r_{s^{-1}}(\tilde{\phi}(|y|^{-1}\, \tilde{x})\, E(y))$. Denote $D_{\tilde{x}} = (\partial/\partial \tilde{x}_0)\, \mathbf{e}_0 + (\partial/\partial \tilde{x}_1)\, \mathbf{e}_1 + \cdots + (\partial/\partial \tilde{x}_n)\, \mathbf{e}_n$, where every $\tilde{x}_k$ is a linear combination of x_i, the components of x, whose coefficients are determined by the chosen $s \in S_{\mathbf{R}_1^n}$ based on the relation $(s\mathbf{f}_0)(x\mathbf{f}_0)(s\mathbf{f}_0)^{-1} = \mathbf{f}_0 \tilde{x}$. Since $\tilde{x} = s^{-1}xs^{-1}$, we have $Ds^{-1}E(\tilde{x}) = Ds^{-1}(s\bar{x}s/|x|^{n+1}) = 0$. Therefore, $Ds^{-1} = p(s)\, D_{\tilde{x}}$, where $p(s)$ is a rational function in $\mathbf{S}$. Since now $D(\tilde{\phi}(y^{-1}x)\, E(y)) = (Ds^{-1})(\phi(|y|^{-1}\, \tilde{x})\, E(y)\, s = (p(s)\, D_{\tilde{x}})(\phi(|y|^{-1}\, \tilde{x})\, E(y)\, s$, by invoking Theorem 1 and Remark 6, we conclude that for any fixed $y', x' \in \Sigma$ and $x' \neq y'$, $\tilde{\phi}(y^{-1}x)\, E(y)$ is left-monogenically defined in a neighborhood U of x' where $y' \in U$, and $\tilde{\phi}(y^{-1}x)$ satisfies the desired estimate with the constant C depending on the size of the neighborhood.

Now consider the right-monogeneity of $\tilde{\phi}(y^{-1}x)\, E(y)$ with respect to x. It follows from the relation $E(y)\, E(1 - xy^{-1}) = E(x - y) = E(1 - y^{-1}x)\, E(y)$ that

$$E(y)\, \tilde{P}^{(k)}(xy^{-1}) = \tilde{P}^{(k)}(y^{-1}x)\, E(y).$$

We then conclude

$$E(y)\, \tilde{\phi}(xy^{-1}) = \tilde{\phi}(y^{-1}x)\, E(y)$$

(also see e.g. [BDS] or [DSS]). This enables us to consider $E(y)\,\tilde{\phi}(xy^{-1})$ instead of $\tilde{\phi}(y^{-1}x)\,E(y)$. A similar argument as above establishes the right-monogeneity in x of the function.

Now we consider the monogeneity in y of the function $\tilde{\phi}(y^{-1}x)\,E(y)$. We claim that the function is also of the form $\psi(x^{-1}y)\,E(x)$, where $\tilde{\psi}$ is a function like $\tilde{\phi}$ associated with a certain bounded holomorphic function. To show this we recall the relation

$$C^{-}_{n+1,k}(\xi,\eta)\,\bar{\eta}=C^{+}_{n+1,k}(\eta,\xi)\,\bar{\xi}$$

(see page 183 formula (1.12) of [DSS]) that implies

$$\tilde{P}^{(k)}(y^{-1}x)\,E(y)=\tilde{P}^{(-k-1)}(x^{-1}y)\,E(x).$$

So, if $\tilde{\phi}$ is defined through $b\in H^{\infty}(\mathbf{S}^{\mathbf{c}}_{\omega})$ by $\tilde{\phi}(x)=\sum'\,b(k)\,P^{(k)}(x)$, then $\tilde{\psi}(y)=\sum_{k\neq-1}b'(k)\,P^{(k)}(y)$, where $b'(z)=b(-z-1)$. The function b' is similar to the function b and the proof of Theorem 1 can be modified to show that the function $\tilde{\psi}$ enjoys the same properties as $\tilde{\phi}$ does. Having proved this, the monogeneity in y follows from the conclusions established in the early part of the proof. The proof is complete.

Using Theorem 2 for $s=1$ in stead of Theorem 1 in the above proof, we obtain

PROPOSITION 8. *Let $\omega\in(\arctan(N),\frac{\pi}{2})$ and b holomorphic in $\mathbf{S}^{\mathbf{c}}_{\omega}$, bounded near the origin and satisfying $|b(z)|\leqslant C_{\mu}\,|z|$ at ∞ in $\mathbf{S}^{\mathbf{c}}_{\mu}$, $0<\mu<\omega$. Then the kernel function $\tilde{\phi}(y^{-1}x)\,E(y)$ associated with the sequence $(b(k))$ is monogenically defined in both x and y in a neighborhood of $\Sigma\times\Sigma\backslash\{x=y\}$. Moreover,*

$$|\tilde{\phi}(y^{-1}x)|\leqslant\frac{C}{|1-y^{-1}x|^{n+1}},$$

For $b\in H^{\infty}(\mathbf{S}^{\mathbf{c}}_{\omega})$ we write briefly $M_{(b(k))}=M_b$, i.e.

$$M_b f(x)=\sum_{k=1}^{\infty} b(k)\,P_k(f)(x)+\sum_{k=1}^{\infty} b(-k)\,Q_{k-1}(f)(x).$$

Now for $x\in\Sigma$, $r\approx 1$ and $r<1$, consider the function

$$\begin{aligned}M^r_b f(x)&=\sum_{k=1}^{\infty} b(k)\,P_k(f)(rx)+\sum_{k=1}^{\infty} b(-k)\,Q_{k-1}(f)(r^{-1}x)\\&=P^r(x)+Q^r(x),\qquad \rho-x<|x|<\imath+s.\end{aligned}$$

Using the convolution expressions of the projections, we have

$$\begin{aligned} P^r(x) &= \sum_{k=1}^{\infty} b(k) \frac{1}{\Omega_n} \int_{\Sigma} \tilde{P}^{(k)}(y^{-1}rx)\, E(y)\, n(y)\, f(y)\, d\sigma(y) \\ &= \frac{1}{\Omega_n} \int_{\Sigma} \left(\sum_{k=1}^{\infty} b(k)\, \tilde{P}^{(k)}(y^{-1}rx) \right) E(y)\, n(y)\, f(y)\, d\sigma(y) \\ &= \frac{1}{\Omega_n} \int_{\Sigma} \tilde{\phi}^{+}(y^{-1}rx)\, E(y)\, n(y)\, f(y)\, d\sigma(y), \end{aligned}$$

where $\tilde{\phi}^{+} = \sum_{k=1}^{\infty} b(k)\, \tilde{P}^{(k)}$. Similarly, we have

$$Q^r(q) = \frac{1}{\Omega_n} \int_{\Sigma} \tilde{\phi}^{-}(y^{-1}r^{-1}x)\, E(y)\, n(y)\, f(y)\, d\sigma(y),$$

where $\tilde{\phi}^{-} = \sum_{k=-\infty}^{-1} b(k)\, \tilde{P}^{(k)}$.

Since the series defining $M_b^r f$ uniformly converges as $r \to 1-$, we can exchange the order of taking limit and summation, and obtain

$$\begin{aligned} M_b f(x) = \lim_{r \to 1-} \frac{1}{\Omega_n} \int_{\Sigma} & (\tilde{\phi}^{+}(y^{-1}rx) + \tilde{\phi}^{-}(y^{-1}r^{-1}x)) \\ & \times E(y)\, n(y)\, f(y)\, d\sigma(y). \end{aligned}$$

THEOREM 4. *If $b \in H^{\infty}(\mathbf{S}^{\mathbf{c}}_{\omega})$, then for any $f \in \mathscr{A}$ and $x \in \Sigma$, we have*

$$\begin{aligned} M_b f(x) &= \lim_{r \to 1-} \frac{1}{\Omega_n} \int_{\Sigma} (\tilde{\phi}^{+}(y^{-1}rx) + \tilde{\phi}^{-}(y^{-1}r^{-1}x)) \\ &\quad \times E(y)\, n(y)\, f(y)\, d\sigma(y) \\ &= \lim_{\varepsilon \to 0} \frac{1}{\Omega_n} \left\{ \int_{|y-x| > \varepsilon,\, y \in \Sigma} \tilde{\phi}(y^{-1}x)\, E(y)\, n(y)\, f(y)\, d\sigma(y) \right. \\ &\quad \left. + \tilde{\phi}^{1}(\varepsilon, x)\, f(x) \right\}, \end{aligned}$$

where $\tilde{\phi} = \tilde{\phi}^{+} + \tilde{\phi}^{-}$ is the function associated with b as specified in Corollary 2 and $\tilde{\phi}^{1}$ the bounded continuous function: $\tilde{\phi}^{1} = \tilde{\phi}^{+,1} + \tilde{\phi}^{-,1}$, where

$$\tilde{\phi}^{\pm,1}(\varepsilon, x) = \int_{S(\varepsilon, x, \pm)} \tilde{\phi}^{\pm}(y^{-1}x)\, E(y)\, n(y)\, d\sigma(y),$$

where $S(\varepsilon, x, \pm)$ is the part of the sphere $|y-x| = \varepsilon$ inside or outside Σ, depending on $\pm$ taking $+$ or $-$.

Proof. We shall only consider the "+" half of the equality corresponding to the decompositions $\tilde{\phi}=\tilde{\phi}^{+}+\tilde{\phi}^{-}$ and $\tilde{\phi}^{1}=\tilde{\phi}^{+,1}+\tilde{\phi}^{-,1}$, as the "−" half can be dealt with similarly. For a fixed $\varepsilon>0$, the integral can be decomposed into

$$\lim_{r\to 1-}\left\{\int_{|y-x|>\varepsilon,\, y\in\Sigma}\tilde{\phi}^{+}(y^{-1}rx)\,E(y)\,n(y)\,f(y)\,d\sigma(y)\right.$$
$$\left.+\int_{|y-x|\leqslant\varepsilon,\, y\in\Sigma}\tilde{\phi}^{+}(y^{-1}rx)\,E(y)\,n(y)\,f(y)\,d\sigma(y)\right\}.$$

As $r\to 1-$, the first part tends to

$$\int_{|y-x|>\varepsilon,\, y\in\Sigma}\tilde{\phi}^{+}(y^{-1}x)\,E(y)\,n(y)\,f(y)\,d\sigma(y).$$

The second part can be further decomposed into

$$\int_{|y-x|\leqslant\varepsilon,\, y\in\Sigma}\tilde{\phi}^{+}(y^{-1}rx)\,E(y)\,n(y)(f(y)-f(x))\,d\sigma(y)$$
$$+\int_{|y-x|\leqslant\varepsilon,\, y\in\Sigma}\tilde{\phi}^{+}(y^{-1}rx)\,E(y)\,n(y)\,d\sigma(y)\,f(x).$$

As $\varepsilon\to 0$, the first integral tends to zero uniformly with respect to $r\to 1-$; invoking Cauchy's theorem, for a fixed ε, the second integral tends to $\tilde{\phi}^{+,1}(\varepsilon, x)\,f(x)$ as $r\to 1-$. The proof is complete.

The proof of Theorem 3 adopts the idea of [CM]. To proceed, some preparations on Hardy spaces of monogenic functions and geometry related to the surface Σ will be needed (see [Mi] for the theory of Clifford monogenic Hardy spaces on higher-dimensional Lipschitz graphs).

Let Δ and Δ^{c} be the bounded and unbounded connected components of $\mathbf{R}_1^n\backslash\Sigma$. For $\alpha>0$, define the *non-tangential approach regions* $\Lambda_\alpha(x)$ and $\Lambda_\alpha^c(x)$ to a point $x\in\Sigma$ to be

$$\Lambda_\alpha(x)=\Lambda_\alpha(x,\Delta)=\{x\in\Delta : |y-x|<(1+\alpha)\operatorname{dist}(y,\Sigma)\},$$

and

$$\Lambda_\alpha^c(x)=\Lambda_\alpha(x,\Delta^c)=\{x\in\Delta^c : |y-x|<(1+\alpha)\operatorname{dist}(y,\Sigma)\}.$$

It is easy to show, similarly to the complex variable case considered in [K1] and [JK], that there exists a positive constant α_0, depending on the Lipschitz constant of Σ only, such that $\Lambda_\alpha(x)\subset\Delta$ and $\Lambda_\alpha^c(x)\subset\Delta^c$ for $0<\alpha<\alpha_0$ and all $x\in\Sigma$. The following argument is independent of specially chosen $\alpha\in(0,\alpha_0)$. We choose and fix α from now on.

Let f be defined in Δ. The *interior non-tangential maximal function* $N_\alpha(f)$ is defined by

$$N_\alpha(f)(x) = \sup\{|f(x)|: y \in \Lambda_\alpha(x)\}, \qquad x \in \Sigma.$$

The *exterior non-tangential maximal function* $N^c_\alpha(f)$ is defined similarly.

For $0 < p_0 < \infty$, the (left-) Hardy space $H^{p_0}(\Delta)$ is defined by

$$H^{p_0}(\Delta) = \{f\colon f \text{ is left-monogenic in } \Delta, \text{ and } N_\alpha(f) \in L^{p_0}(\Sigma)\}.$$

If $f \in H^{p_0}(\Delta)$, then $\|f\|_{H^{p_0}(\Delta)}$ is defined as the L^{p_0} norm of $N_\alpha(f)$ on Σ.

The space $H^{p_0}(\Delta^c)$ is defined similarly, except that the functions in $H^{p_0}(\Delta^c)$ are assumed to vanish at the infinity. Similarly to the monogenic Hardy space case studied in [Mi], one can prove

PROPOSITION 9. *If $f \in H^{p_0}(\Delta)$, $p_0 > 1$, then the non-tangential limit of f,*

$$\lim_{y \to x,\ y \in \Lambda_\alpha(x)} f(y)$$

exists almost everywhere with respect to the surface measure on Σ. Still using f to denote the limit function, we have

$$C_{N, p_0} \|f\|_{H^{p_0}(\Delta)} \leqslant \|f\|_{L^{p_0}(\Sigma)} \leqslant C'_{N, p_0} \|f\|_{H^{p_0}(\Delta)},$$

where C_{N, p_0}, C'_{N, p_0} depend on the Lipschitz constant N and p_0.

In other words, for $p_0 > 1$, the $H^{p_0}(\Delta)$ norm of a function is equivalent to the L^{p_0} norm of its non-tangential limit on the boundary. A similar result holds for functions in the Hardy space associated with Δ^c.

In polar coordinate system the Dirac operator D can be decomposed into

$$D = \zeta \partial_r - \frac{1}{r} \partial_\zeta = \zeta \left(\partial_r - \frac{1}{r} \Gamma_\zeta \right),$$

where Γ_ζ is a first order differential operator depending only on the angular coordinates known as *spherical Dirac operator* (see [DSS] or [L]). It is known that

$$\Gamma_\zeta f(\zeta) = k f(\zeta), \qquad f \in M_k, \tag{9}$$

where M_k, $k \neq -1, -2, ..., -n+1$, is the subspace of k-homogeneous left-monogenic functions. For $f \in \mathscr{A}$ we define $\Gamma_\zeta(f|_\Sigma)$ to be the restriction on Σ of the monogenic extension of $\Gamma_\zeta(f|_{\mathbf{S}_{\mathbf{R}^n_1}})$. The definition of Γ_ζ can be extended to $\Gamma_\zeta\colon \mathscr{A} \to \mathscr{A}$.

The following result on norm equivalence of higher order g-functions of $f \in H^2(\Delta)$ and the proof of it both are similar to those for Lipschitz graphs studied in [Mi] (also see [JK]). The counterpart result holds for $f \in H^2(\Delta^c)$.

PROPOSITION 10. *Suppose that $f \in H^2(\Delta)$. Then the norm $\|f\|_{H^2(\Delta)}$ is equivalent to the norm*

$$\int_0^1 \int_\Sigma |(\Gamma_\zeta^j f)(sx)|^2 (1-s)^{2j-1} \, d\sigma(x) \frac{ds}{s}, \qquad j=1, 2, ...$$

The following is equivalent to CMcM's Theorem on Σ ([CMcM]).

PROPOSITION 11. *Suppose that $f \in L^2(\Sigma)$. Then there exist $f^+ \in H^2(\Delta)$ and $f^- \in H^2(\Delta^c)$ such that their non-tangential boundary limits, still denoted by f^+, f^-, respectively, lie in $L^2(\Sigma)$, and $f = f^+ + f^-$. The mappings $f \to f^\pm$ are continuous on $L^2(\Sigma)$.*

It is easy to see that if $f \in \mathscr{A}$, then the natural decomposition of f into its power series part and principal series part is identical to the decomposition given in Proposition 11.

Denote by Σ_s, $0 < s < 1$, the surface $\{sx \colon x \in \Sigma\}$.

LEMMA 4. *Let $x_0 \in \Sigma$, $0 < s < 1$, and $x = sx_0$. Then there exists a constant C_Σ such that*

$$|1 - y^{-1}x| \geqslant C_\Sigma \{(1-\sqrt{s})^2 + \theta^2\}^{1/2}, \qquad y \in \Sigma_{\sqrt{s}},$$

where $\theta = \arg(x, y)$.

Proof. It is equivalent to prove

$$|y - x| \geqslant C_\Sigma \sqrt{s} \, \{(1-\sqrt{s})^2 + \theta^2\}^{1/2}, \qquad y \in \Sigma_{\sqrt{s}}.$$

Let $x_0 = r_0 \xi$, $y = r\eta$, $x_1 = \sqrt{s}\, x_0 \in \Sigma_{\sqrt{s}}$, where $\xi, \eta \in \mathbf{S}_{\mathbf{R}_1^n}$. A direct computations shows

$$\begin{aligned} |y - x|^2 &= r^2 \left[(1-\beta)^2 + 4\beta \sin^2 \frac{\theta}{2} \right] \\ &\geqslant C_\Sigma s[(1-\beta)^2 + \beta\theta^2], \end{aligned} \tag{10}$$

where $\beta = \frac{sr_0}{r}$.

If s is small, then β is small and $1 - \beta$ has a positive lower bound. Since the right hand side of the desired inequality is bounded from above, it is dominated by a constant multiple of $1-\beta$. We thus obtain the desired estimate.

Now assume that s is close to, but less than 1. In this case β has a positive lower bound. We have two subcases to consider. Denote $r_1 = |x_1| = \sqrt{s}\, r_0$.

(i) $\frac{r_1}{r} \leqslant s^{-1/4}$. In this case $\beta \leqslant s^{1/4}$ and so $1-\beta \geqslant 1 - s^{1/4} > C(1-\sqrt{s})$. The desired estimate then follows.

(ii) $\frac{r_1}{r} > s^{-1/4}$. In this case

$$\ln(s^{-1/4}) < \ln\left(\frac{r_1}{r}\right) \leqslant N\theta,$$

where we have used the fact that $\Sigma_{\sqrt{s}}$ is Lipschitz with the Lipschitz constant N, and so

$$\theta > \frac{1}{4N} \ln(s) \geqslant \frac{1}{4N} (1-\sqrt{s}).$$

Therefore,

$$\theta > \frac{1}{2}\theta + \frac{1}{8N}(1-\sqrt{s}).$$

Substituting into (10) and ignoring the entry related to $1-\beta$, we obtain the desired estimate.

Proof of Theorem 3. Let $f \in \mathscr{A}$. Using the decomposition of f defined in Proposition 11, we have $f = f^+ + f^-$, where $f^+ \in H^2(\Delta)$, $f^- \in H^2(\Delta^c)$, $\|f^\pm\|_{L^2(\Sigma)} \leqslant C_N \|f\|_{L^2(\Sigma)}$. We also have $M_b f = M_{b^+} f^+ + M_{b^-} f^-$, where

$$M_{b^\pm} f^\pm(x) = \lim_{r \to 1-} \int_\Sigma \tilde{\phi}^\pm (r^{\pm 1} y^{-1} x)\, E(y)\, n(y)\, f(y)\, d\sigma(y), \qquad x \in \Sigma.$$

$M_{b^\pm} f^\pm$ can be left-monogenically extended to Δ and Δ^c by using

$$M_{b^\pm} f^\pm(x) = \int_\Sigma \tilde{\phi}^\pm (y^{-1} x)\, E(y)\, n(y)\, f(y)\, d\sigma(y),$$

for $x \in \Delta$ and $x \in \Delta^c$, respectively.

Owing to Proposition 9, it suffices to show

$$\|M_{b^\pm} f^\pm\|_{H^2} \leqslant C_N \|f^\pm\|_{H^2}.$$

We will now prove the inequality for the case "+", as the case "−" may be dealt with similarly. We will suppress the superscript "+" in below for simplicity. Using the Taylor series expansion of f, and that of $M_b f$, we can prove that Γ_ζ commutes with M_b. To prove this we exchange the order of taking differentiation Γ_ζ and infinite summation which is justified by the fast decay of Fourier expansions of functions in $\mathscr{A}$. As consequence of the commutativity we have, for $x \in \Delta$,

$$\Gamma_\zeta M_b f(x) = \frac{1}{\Omega_n} \int_\Sigma \tilde{\phi}(y^{-1}x)\, E(y)\, n(y)\, \Gamma_\zeta f(y)\, d\sigma(y).$$

We also have

$$\Gamma_\zeta^2 M_b f(x) = \frac{1}{\Omega_n} \int_\Sigma \Gamma_\zeta(\tilde{\phi}(y^{-1}x))\, E(y)\, n(y)\, \Gamma_\zeta f(y)\, d\sigma(y)$$

that is justified by the following

LEMMA 5. *If* $\nu \in (\arctan(N), \omega)$, *then*

$$|\Gamma_\zeta(\tilde{\phi}(y^{-1}x))| \leqslant C_\nu \frac{1}{|1 - y^{-1}x|^{n+1}}, \qquad y \in \Sigma, \qquad x \in \Delta.$$

Proof. In the expansion

$$\tilde{\phi}(y^{-1}x)\, E(y) = \sum_{k=1}^{\infty} b(k)\, \tilde{P}^{(k)}(y^{-1}x)\, E(y),$$

substituting

$$\tilde{P}^{(k)}(y^{-1}x)\, E(y) = \sum_{|\underline{\alpha}| = k} V_{\underline{\alpha}}(x)\, W_{\underline{\alpha}}(y),$$

where $V_{\underline{\alpha}} \in M_k$, $Q_{\underline{\alpha}} \in M_{-n-k}$ (see formula (1.15) of p. 184, Chapter 2, [DSS]), and applying Γ_ζ with respect to x to the series, we obtain

$$\Gamma_\zeta(\tilde{\phi}(y^{-1}x))\, E(y) = \sum_{k=1}^{\infty} kb(k)\, \tilde{P}^{(k)}(y^{-1}x)\, E(y).$$

The right hand side series is associated with the multiplier $b'(z) = zb(z)$. Applying Proposition 8, we conclude the lemma.

Now we continue the proof of Theorem 3. For $x \in \Sigma_s$ changing the contour in the integral expressing $\Gamma_\zeta^2 M_b ff(x)$ and using Lemma 5 and 4, we have,

$$
\begin{aligned}
|\Gamma_\zeta^2 M_b f(x)| &\leqslant C\left(\int_{\Sigma_{\sqrt{s}}} |\Gamma_\zeta(\phi(y^{-1}x))| \frac{d\sigma(y)}{|y|^n}\right)^{1/2} \\
&\quad \times \left(\int_{\Sigma_{\sqrt{s}}} |\Gamma_\zeta(\phi(y^{-1}x))|\, |\Gamma_\zeta f(y)|^2 \frac{d\sigma(y)}{|y|^n}\right)^{1/2} \\
&\leqslant C\left(\int_{\Sigma_{\sqrt{s}}} \frac{1}{|1-y^{-1}x|^{n+1}} \frac{d\sigma(y)}{|y|^n}\right)^{1/2} \\
&\quad \times \left(\int_{\Sigma_{\sqrt{s}}} \frac{1}{|1-y^{-1}x|^{n+1}} |\Gamma_\zeta f(y)|^2 \frac{d\sigma(y)}{|y|^n}\right)^{1/2} \\
&\leqslant C\left(\int_{\Sigma} \frac{1}{((1-\sqrt{s})^2+\theta_0^2)^{(n+1)/2}}\, d\sigma(y)\right)^{1/2} \\
&\quad \times \left(\int_{\Sigma} \frac{1}{((1-\sqrt{s})^2+\theta_0^2)^{(n+1)/2}} |\Gamma_\zeta f(\sqrt{s}\, y)|^2\, d\sigma(y)\right)^{1/2},
\end{aligned}
$$

where θ_0 is the angle between $x \in \Sigma_s$ and $y \in \Sigma$.

Since

$$
\begin{aligned}
\int_{\Sigma} \frac{1}{((1-\sqrt{s})^2+\theta_0^2)^{(n+1)/2}}\, d\sigma(y) &\leqslant C\int_0^{\pi} \frac{\sin^{n-1}\theta_0}{((1-\sqrt{s})^2+\theta_0^2)^{(n+1)/2}}\, d\theta_0 \\
&\leqslant C\int_0^{\pi} \frac{\theta_0^{n-1}}{((1-\sqrt{s})^2+\theta_0^2)^{(n+1)/2}}\, d\theta_0 \\
&= C\frac{1}{1-\sqrt{s}},
\end{aligned}
$$

using Proposition 10 for $j = 1, 2$, we have

$$
\begin{aligned}
\|M_b f\|_{H^2(\Delta)}^2 &\approx \int_0^1 \int_{\Sigma} |\Gamma_\zeta^2(M_b f)(sx)|^2 (1-s)^3\, d\sigma(x)\, \frac{ds}{s} \\
&\leqslant C\int_0^1 \int_{\Sigma} \frac{1}{1-\sqrt{s}}\left(\int_{\Sigma} \frac{1}{((1-\sqrt{s})^2+\theta_0^2)^{(n+1)/2}}\right. \\
&\quad \left. \times |\Gamma_\zeta f(\sqrt{s}\, y)|^2\, d\sigma(y)\right)(1-\sqrt{s})^3\, d\sigma(x)\, \frac{ds}{s}
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant C \int_0^1 \int_\Sigma |\Gamma_\zeta f(\sqrt{s}\, y)|^2 \\
&\quad \times \left(\int_\Sigma \frac{1-\sqrt{s}}{((1-\sqrt{s})^2+\theta_0^2)^{(n+1)/2}}\, d\sigma(x) \right) (1-\sqrt{s})\, d\sigma(y)\, \frac{ds}{s} \\
&\leqslant C \int_0^1 \int_\Sigma |\Gamma_\zeta f(\sqrt{s}\, y)|^2 \,(1-\sqrt{s})\, d\sigma(y)\, \frac{ds}{s} \\
&\leqslant C \int_0^1 \int_\Sigma |\Gamma_\zeta f(sy)|^2 \,(1-s)\, d\sigma(y)\, \frac{ds}{s} \\
&\approx C\, \|f\|^2_{H^2(\Delta)}.
\end{aligned}
$$

The bounds of the operator norm $\|M_b\|$ can be derived from the proof of Lemma 5 and the estimates obtained in Theorem 2. The proof is complete.

Remark 9. As in the standard cases the Hilbert transform on the unit sphere and on star-shaped Lipschitz surfaces is defined via the Fourier multiplier $b(z) = -i\,\mathrm{sgn}(z)$, where $\mathrm{sgn}(z)$ is the signum function that takes the value $+1$ for $\mathrm{Re}(z) > 0$ and the value -1 for $\mathrm{Re}(z) < 0$. The associated singular integral is given by the kernel

$$
\begin{aligned}
\frac{1}{\Omega_n}\, \tilde{\phi}(y^{-1}x)\, E(y) &= \frac{1}{\Omega_n} \sum_{k=-\infty}^{\infty}{}' -\mathbf{i}\,\mathrm{sgn}(k)\, \tilde{P}^{(k)}(y^{-1}x)\, E(y) \\
&= -\frac{2\mathbf{i}}{\Omega_n}\, E(1-y^{-1}x)\, E(y) = -\frac{2\mathbf{i}}{\Omega_n}\, E(y-x).
\end{aligned}
$$

When $y = 1$, the above reduces to $-\frac{2\mathbf{i}}{\Omega_n} E(1-x) = \frac{1}{\Omega_n} \Upsilon(\sum_{k=-\infty}^{\infty}{}' -\mathbf{i}\,\mathrm{sgn}(k)\, z^k)$, as deduced in Section 2.

4. HOLOMORPHIC FUNCTIONAL CALCULUS OF THE SPHERICAL DIRAC OPERATOR

We wish to point out that the class of the bounded operator M_b studied in Section 2 constitutes a functional calculus of Γ_ζ, and is, in fact, identical to the Cauchy–Dunford bounded holomorphic functional calculus of Γ_ζ. For a discussion in relation to domains of Dirac operators on Lipschitz graphs we refer the reader to [LMcQ] and [Mc3].

The operators M_b enjoy the following properties, and thus the class M_b, $b \in H^\infty(\mathbf{S}^{\mathbf{c}}_\omega)$, is called a bounded holomorphic functional calculus.

Let $N=\text{Lip}(\Sigma)$, $\arctan(N)<\omega<\frac{\pi}{2}$, $1<p_0<\infty$, $b, b_1, b_2\in H^\infty(\mathbf{S}^{\mathbf{c}}_\omega)$, and $\alpha_1, \alpha_2\in\mathbf{C}$. Then

$$\|M_b\|_{L^{p_0}(\Sigma)\to L^{p_0}(\Sigma)}\leqslant C_{p_0,\,\nu}\,\|b\|_{L^\infty(\mathbf{S}^{\mathbf{c}}_\nu)},\qquad \arctan(N)<\nu<\omega;$$

$$M_{b_1b_2}=M_{b_1}\circ M_{b_2};$$

$$M_{\alpha_1b_1+\alpha_2b_2}=\alpha_1M_{b_1}+\alpha_2M_{b_2}.$$

The first assertion is obtained in Remark 7. The second and the third are derived by using Laurent series expansions of the test functions.

Denote by

$$R(\lambda, \Gamma_\zeta)=(\lambda I-\Gamma_\zeta)^{-1}$$

the resolvent operator of Γ_ζ at $\lambda\in\mathbf{C}$. We show that for non-real λ, $R(\lambda, \Gamma_\zeta)=M_{1/(\lambda-(\cdot))}$. In fact, owing to the relation (9), the Fourier multiplier $\lambda-k$ is associated with the operator $\lambda I-\Gamma_\zeta$, and therefore the Fourier multiplier $(\lambda-k)^{-1}$ is associated with $R(\lambda, \Gamma_\zeta)$. The property of the functional calculus in relation to the boundedness then asserts that, for $1<p_0<\infty$,

$$\|R(\lambda, \Gamma_\zeta)\|_{L^{p_0}(\Sigma)\to L^{p_0}(\Sigma)}\leqslant\frac{C_\nu}{|\lambda|},\qquad \lambda\notin\mathbf{S}^{\mathbf{c}}_\nu.$$

Owing to this estimate, for $b\in\mathbf{S}^{\mathbf{c}}_\omega$ with good decays at both zero and the infinity, the Cauchy–Dunford integral

$$b(\Gamma_\zeta)\,f=\frac{1}{2\pi i}\int_\Pi b(\lambda)\,R(\lambda, \Gamma_\zeta)\,d\lambda\,f$$

defines a bounded operator, where Π is a path consisting of four rays: $\{s\exp(i\theta)$: s is from ∞ to $0\}\cup\{s\exp(-i\theta)$: s is from 0 to $\infty\}\cup\{s\exp(i(\pi+\theta))$: s is from ∞ to $0\}\cup\{s\exp(i(\pi-\theta))$: s is from 0 to $\}$, and $\arctan(N)<\theta<\omega$. The functions b of this sort form a dense subclass of $H^\infty(\mathbf{S}^{\mathbf{c}}_\omega)$ in the sense specified in the Convergence Lemma of McIntosh in [Mc2]. Using the lemma, we can extend the definition given by the Cauchy–Dunford integral and define a functional calculus $b(\Gamma_\zeta)$ on general functions $b\in H^\infty(\mathbf{S}^{\mathbf{c}}_\omega)$.

Now we show $b(\Gamma_\zeta)=M_b$. Assume again that b has good decays at both zero and the infinity, and $f\in\mathscr{A}$. Then change of order of integration and summation in the following chain of equalities may be justified, and we have

$$
\begin{aligned}
b(\Gamma_\zeta)\, f(x) &= \frac{1}{2\pi i} \int_\Pi b(\lambda)\, R(\lambda, \Gamma_\zeta)\, d\lambda\, f(x) \\
&= \frac{1}{2\pi i} \int_\Pi b(\lambda) \sum_{k=-\infty}^{\infty}{}' (\lambda - k)^{-1} \\
&\quad \times \frac{1}{\Omega_n} \int_\Sigma \tilde{P}^{(k)}(y^{-1}x)\, E(y)\, n(y)\, f(y)\, d\sigma(y)\, d\lambda \\
&= \sum_k{}' \left(\frac{1}{2\pi i} \int_\Pi b(\lambda)(\lambda - k)^{-1}\, d\lambda \right) \\
&\quad \times \frac{1}{\Omega_n} \int_\Sigma \tilde{P}^{(k)}(y^{-1}x)\, E(y)\, n(y)\, f(y)\, d\sigma(y) \\
&= \sum_k{}' b(k) \frac{1}{\Omega_n} \int_\Sigma \tilde{P}^{(k)}(y^{-1}x)\, E(y)\, n(y)\, f(y)\, d\sigma(y) \\
&= M_b f(y).
\end{aligned}
$$

Denote by $P^\pm$ the projection operators $P^\pm f = f^\pm$ as defined in Proposition 11. It follows from the norm estimate of the resolvent $R(\lambda, \Gamma_\zeta)$ that $\Gamma_\zeta P^\pm$ are *type*-ω operators (see [Mc2]).

The operators $\Gamma_\zeta P^\pm$ and Γ_ζ are identical to their *dual operators* on $L^2(\Sigma)$ in the *dual pair* $(L^2(\Sigma), L^2(\Sigma))$ under the bilinear pairing

$$\langle\!\langle f, g \rangle\!\rangle = \frac{1}{\Omega_n} \int_\Sigma f(x)\, n(x)\, g(x)\, d\sigma(x).$$

That is

$$\langle\!\langle \Gamma_\zeta P^\pm f, g \rangle\!\rangle = \langle\!\langle f, \Gamma_\zeta P^\pm g \rangle\!\rangle$$

and

$$\langle\!\langle \Gamma_\zeta f, g \rangle\!\rangle = \langle\!\langle f, \Gamma_\zeta g \rangle\!\rangle.$$

These can be easily derived from Parseval's identity

$$\sum_{k=0}^{\infty} \sum_{|\underline{\beta}|=k} \lambda_{\underline{\beta}} \lambda'_{\underline{\beta}} + \mu_{\underline{\beta}} \mu'_{\underline{\beta}} = \frac{1}{\Omega_n} \int_{\mathbf{S}^n_{\mathbf{R}_1}} f(x)\, n(x)\, g(x)\, d\sigma(x),$$

in the notation of p. 193 of [DSS], and the relation (9).

Analogous conclusions hold for the Banach space dual pairs $(L^{p_0}(\Sigma), L^{p_0'}(\Sigma))$, $1 < p_0 < \infty$, $\frac{1}{p_0} + \frac{1}{p_0'} = 1$, under the same form of bilinear pairings.

For Hilbert and Banach space properties of the operators $\Gamma_\zeta P^{\pm}$ and Γ_ζ we refer the reader to the general study on type-ω operators in [Mc2] and [CDMcY].

5. THE ANALOGOUS THEORY IN $\mathbf{R}^n$

We outline how to establish an analogous theory in the symmetric Euclidean spaces $\mathbf{R}^n = \{\underline{x} = x_1 e_1 + \cdots + x_n e_n : x_i \in \mathbf{R}\}$.

In $\mathbf{R}^n$ the Cauchy kernel is $\underline{E}(\underline{x}) = \bar{x}/|\bar{x}|^n$ and the Dirac operator is $\underline{D} = (\partial/\partial x_1)\, \mathbf{e}_1 + \cdots + (\partial/\partial x_n)\, \mathbf{e}_n$. We also have Cauchy's Theorem and Cauchy's Formula ([DSS]). Corresponding to the formula (5), we have

$$\underline{E}(\underline{x} - \mathbf{e}_1) = \underline{P}^{(-1)}(\underline{x}) + \underline{P}^{(-2)}(\underline{x}) + \cdots + \underline{P}^{(-k)}(\underline{x}) + \cdots, \qquad |\underline{x}| > 1. \tag{11}$$

In the relation

$$\underline{E}(\underline{x} - \underline{y}15) = \frac{\bar{x} - \bar{y}}{|\bar{x} - \bar{y}|^n} = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(k-1)!} \langle \underline{y}, \nabla_{\underline{x}} \rangle^{k-1} \frac{\bar{x}}{|\bar{x}|^n}, \tag{12}$$

where $\nabla_{\underline{x}} = (\partial/\partial x_1, ..., \partial/\partial x_n)$, letting $\underline{y} = \mathbf{e}_1$, we obtain

$$\underline{P}^{(-k)}(\underline{x}) = \frac{(-1)^{k-1}}{(k-1)!} \left(\frac{\partial}{\partial x_1} \right)^{k-1} \underline{E}(\underline{x}).$$

From the Taylor series theory we know that the general entries of the infinite series (12) is monogenic in both $\underline{x}$ and $\underline{y}$ with respect to $\underline{D}$. So $\underline{P}^{(-k)}(\underline{x})$'s are monogenic. Define

$$\underline{P}^{(k-1)} = I(\underline{P}^{(-k)}), \qquad k \in \mathbf{Z}^+,$$

where $\underline{I}$ is the Kelvin inversion: $\underline{I}(f)(\underline{x}) = \underline{E} f(\underline{x}^{-1})$. The property of the Kelvin inversion asserts that $\underline{P}^{(k-1)}$'s are monogenic. It can be easily verified that Proposition 1 holds when we replace $P^{(k)}$ by $\underline{P}^{(k)}$, x by $\underline{x}$ and n by $n-1$.

There are corresponding objects like the heart shaped regions $\mathbf{H}_{\omega, \pm}$ in the context, namely

$$\underline{\mathbf{H}}_{\omega, \pm} = \left\{ \underline{x} \in \mathbf{R}^n : \frac{(\pm \ln |\mathbf{e}_1 \underline{x}|)}{\arg(\mathbf{e}_1, \underline{x})} < \tan \omega \right\},$$

and

$$\underline{\mathbf{H}}_{\omega}=\underline{\mathbf{H}}_{\omega,+}\cap\underline{\mathbf{H}}_{\omega,-}.$$

That is

$$\underline{\mathbf{H}}_{\omega}=\left\{x\in\mathbf{R}^{n}:\frac{|\ln|\mathbf{e}_1\underline{x}||}{\arg(\mathbf{e}_1,\underline{x})}<\tan\omega\right\}.$$

We use the functions spaces

$$K(\underline{\mathbf{H}}_{\omega,\pm})=\Big\{\underline{\phi}\colon \underline{\mathbf{H}}_{\omega,\pm}\to\mathbf{C}^{(n)}:\underline{\phi}\text{ is monogenic and satisfies}$$

$$|\underline{\phi}(\underline{x})|\leqslant\frac{C_\mu}{|\mathbf{e}_1-\underline{x}|^{n-1}},\ 0<\mu<\omega\Big\},$$

and

$$K(\underline{\mathbf{H}}_{\omega})=\{\underline{\phi}\colon\underline{\mathbf{H}}^{\mathbf{c}}_{\omega}\to\mathbf{C}^{(n)}:\underline{\phi}=\underline{\phi}^{+}+\underline{\phi}^{-},\ \underline{\phi}^{\pm}\in K(\underline{\mathbf{H}}_{\omega,\pm})\}.$$

Like Theorem 1, the following is the main technical result.

THEOREM 1′. *If* $b\in H^{\infty}(\mathbf{S}^{\mathbf{c}}_{\omega,\pm})$ *and* $\underline{\phi}(\underline{x})=\sum_{k=\pm1}^{\pm\infty}b(k)\,\underline{P}^{(k)}(\underline{x})$, *then* $\underline{\phi}\in K(\underline{\mathbf{H}}_{\omega,\pm})$.

We shall postpone its proof to the end of the section. The analogous result to Theorem 2 also holds.

THEOREM 2′. *Let* $-\infty<s<\infty$, $s\neq-1,-2,...,$ *and* b *a holomorphic function in* $\mathbf{S}^{\mathbf{c}}_{\omega,\pm}$ *satisfying the estimates*

$$|b(z)|\leqslant C_\mu\,|z\pm1|^{s},\qquad \textit{in every }\mathbf{S}^{\mathbf{c}}_{\mu,\pm},\quad 0<\mu<\omega.$$

Then $\underline{\phi}(\underline{x})=\sum_{i=\pm1}^{\pm\infty}b(i)\,\underline{P}^{(i)}(\underline{x})$ *can be monogenically extended to* $\underline{\mathbf{H}}_{\omega,\pm}$ *satisfying*

$$|\underline{\phi}^{\pm}(\underline{x})|\leqslant C_\mu\left\|\frac{b(\cdot)}{|(\cdot)\pm1|^{s}}\right\|_{L^{\infty}(\mathbf{S}^{\mathbf{c}}_{\mu'})}\frac{1}{|\mathbf{e}_1-\underline{x}|^{s+n-1}},$$

$$\underline{x}\in\underline{\mathbf{H}}_{\mu,\pm},\qquad 0<\mu<\mu'<\omega.$$

A surface $\underline{\Sigma}$ in $\mathbf{R}^n$ is aid to be a *starlike Lipschitz surface*, if it is $(n-1)$-dimensional and star-shaped about the origin, and there exists a constant $M<\infty$ such that $\underline{x}, \underline{x}' \in \underline{\Sigma}$ implies that

$$\frac{|\ln |\underline{x}^{-1}\underline{x}'||}{\arg(\underline{x}, \underline{x}')} \leqslant M.$$

The minimum value of M is called the Lipschitz constant of $\underline{\Sigma}$, denoted by $\underline{N}=\mathrm{Lip}(\underline{\Sigma})$.

We shall use the class

$$\mathscr{A}=\{f\colon f(\underline{x}) \text{ is left-monogenic in } \underline{\rho}-s<|\underline{x}|<\underline{\imath}+s \text{ for some } s>0\},$$

where $\underline{\rho}=\inf\{|\underline{x}|\colon \underline{x}\in\underline{\Sigma}\}$ and $\underline{\imath}=\sup\{|\underline{x}|\colon \underline{x}\in\underline{\Sigma}\}$. It is a consequence of CMcM's Theorem that $\mathscr{A}$ is dense in $L^2(\underline{\Sigma})$.

For $f\in\mathscr{A}$ we have the expansion

$$f(\underline{x})=\sum_{k=0}^{\infty} \underline{P}_k(f)(\underline{x})+\sum_{k=0}^{\infty} \underline{Q}_k(f)(\underline{x}),$$

where for $k\in\mathbf{Z}^+\cup\{0\}$, $\underline{P}_k(f)$ belongs to the finite dimensional right module $\underline{M}_k$ of k-homogeneous left-monogenic functions defined in $\mathbf{R}^n$, and $\underline{Q}_k(f)$ belongs to the finite dimensional right module $\underline{M}_{-k-n+1}$ of $-(k+n-1)$-homogeneous left-monogenic functions defined in $\mathbf{R}^n\backslash\{0\}$. The spaces $\underline{M}_k$ and $\underline{M}_{-k-n+1}$ are eigenspaces of the associated spherical left-Dirac operator $\underline{\Gamma}_{\underline{\zeta}}$, defined by

$$\underline{D}=\underline{\zeta}\partial_r-\frac{1}{r}\partial_{\underline{\zeta}}=\underline{\zeta}\left(\partial_r-\frac{1}{r}\underline{\Gamma}_{\underline{\zeta}}\right).$$

It is known that

$$\underline{\Gamma}_{\underline{\zeta}} f(\underline{\zeta})=kf(\underline{\zeta}), \qquad f\in\underline{M}_k.$$

We consider the Fourier multiplier operator induced by a bounded sequence (b_k), defined by

$$\underline{M}_{(b_k)}f(\underline{x})=\sum_{k=0}^{\infty} b_k\underline{P}_k(f)(\underline{x})+\sum_{k=0}^{\infty} b_{-k-1}\underline{Q}_k(f)(\underline{x}).$$

There are analogous singular integral expressions in the present case as in Theorem 4. There is also an analogous Hardy H^2 space theory in the case. Based on these, together with Theorem 1′ and Theorem 2′, we can adapt the method for proving Theorem 3 in the $\mathbf{R}_1^n$ case, step by step, to prove the following

THEOREM 3. *Let* $\omega \in (\arctan(\underline{N}), \frac{\pi}{2})$. *If* $b \in H^\infty(\mathbf{S}^{\mathbf{c}}_\omega)$, *then with the convention* $b(0)=0$, *the above defined* $\underline{M}_b = \underline{M}_{(b(k))}$ *extends to a bounded operator from* $L^2(\underline{\Sigma})$ *to* $L^2(\underline{\Sigma})$. *Moreover,*

$$\|\underline{M}_{(b(k))}\|_{L^2(\underline{\Sigma}) \to L^2(\underline{\Sigma})} \leqslant C_\nu \|b\|_{L^\infty(\mathbf{S}^{\mathbf{c}}_\nu)}, \qquad \arctan(\underline{N}) < \nu < \omega.$$

Through the proof of Theorem 3′ we can show that the class of Fourier multiplier operators $\underline{M}_b$ is identical to a certain class of singular integral operators (see Theorem 4).We can also show, using the indications given in Section 4, that the class is also identical to the Cauchy–Dunford bounded holomorphic functional calculus of the spherical Dirac operator $\underline{\Gamma}_\zeta$.

The reader may have noticed that almost all objects in $\mathbf{R}^n$ are obtained by simply putting underlines to the corresponding objects in $\mathbf{R}^n_1$, except those in which $\mathbf{e}_0 = 1$ is replaced by $\mathbf{e}_1$. In accordance with this, only Theorem 1′ and Theorem 2′ need to be proved. Thanks to the fact that $\mathbf{R}^n$ can be reduced to $\mathbf{R}^{n-1}_1$ we are able to easily produce the theory in $\mathbf{R}^n$. As in the $\mathbf{R}^n_1$ case we only give a proof of Theorem 1′.

Proof of Theorem 1′. As in the proof of Theorem 1, the case $b \in H^\infty(\mathbf{S}^{\mathbf{c}}_{\omega,\pm})$ is reduced to the case $b \in H^{\infty,\mathbf{r}}(\mathbf{S}^{\mathbf{c}}_{\omega,\pm})$, and the case $b \in H^{\infty,\mathbf{r}}(\mathbf{S}^{\mathbf{c}}_{\omega,+})$ is reduced to the case $b \in H^{\infty,\mathbf{r}}(\mathbf{S}^{\mathbf{c}}_{\omega,-})$.

Let $b \in H^{\infty,\mathbf{r}}(\mathbf{S}^{\mathbf{c}}_{\omega,-})$.We have

$$\begin{aligned}
\underline{\phi}(\underline{x}) &= \sum_{k=1}^{\infty} b(-k)\, \underline{P}^{(-k)}(\underline{x}) = \sum_{k=1}^{\infty} b(-k) \frac{(-1)^{k-1}}{(k-1)!} \left(\frac{\partial}{\partial x_1}\right)^{k-1} \underline{E}(\underline{x}) \\
&= -\mathbf{e}_1 \sum_{k=1}^{\infty} b(-k) \frac{(-1)^{k-1}}{(k-1)!} \left(\frac{\partial}{\partial x_1}\right)^{k-1} \left(\frac{x_1 - x_2\mathbf{g}_1 - \cdots - x_n\mathbf{g}_{n-1}}{|x_1 + x_2\mathbf{g}_1 + \cdots + x_n\mathbf{g}_{n-1}|^n}\right) \\
&= -\mathbf{e}_1 \sum_{k=1}^{\infty} b(-k) \frac{(-1)^{k-1}}{(k-1)!} \left(\frac{\partial}{\partial x_1}\right)^{k-1} E(\tilde{x}) \\
&= -\mathbf{e}_1 \tilde{\phi}(\tilde{x}),
\end{aligned}$$

where $\mathbf{g}_i = \mathbf{e}_{i+1}\mathbf{e}_1^{-1}$, $i = 1, ..., n-1$ are basic vectors like $\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_{n-1}$ and $\tilde{x} = x_1 + x_2\mathbf{g}_1 + \cdots + x_n\mathbf{g}_{n-1}$, a vector in $\mathbf{R}^{n-1}_1$. We also have

$$\underline{D} = \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\mathbf{g}_1 + \cdots + \frac{\partial}{\partial x_n}\mathbf{g}_{n-1}\right)\mathbf{e}_1 = \tilde{D}\mathbf{e}_1,$$

where $\tilde{D}$ is the Dirac operator in $\mathbf{R}^{n-1}_1$. We hence conclude that $\underline{\phi}$ is left-monogenic with respect to $\underline{D}$ in $\mathbf{R}^n$ if and only if $\tilde{\phi}$ is left-monogenic with respect to $\tilde{D}$ in $\mathbf{R}^{n-1}_1$. The heart-shaped regions $\underline{\mathbf{H}}_{\omega,\pm}$ are identical to those in $\mathbf{R}^{n-1}_1$ with $\mathbf{e}_1$ being replaced by 1. The desired left-monogeneity and the estimate then follow from those of Theorem 1. The right-monogeneity can be proved similarly, with the only difference that $\mathbf{e}_1$ is factorized out of $\underline{E}(\underline{x})$ from the right hand side, and of $\underline{D}$ from the left hand side, and define $\mathbf{g}_i = \mathbf{e}_1^{-1}\mathbf{e}_{i+1}$. The proof is complete.

The idea of the work is summarized as follows. The key and the innovative method is to establish a corresponding relationship, through generalizing Fueter's and Sce's results ([Sc], [Q6]), between the basic functions of one complex variable: ..., z^{-2}, z^{-1}, 1, z, z^2, ... and the functions $P^{(k)}$ of one Clifford variable, called monomials. We then show that the monomials give rise to, through convolution integrals, the projections of functions on annuli onto the spaces of k-homogeneous monogenics. This is shown to be of exactly the same as in the one complex variable case: The projections there are given by convolution integrals using ..., z^{-2}, z^{-1}, 1, z, z^2, ... as kernels. The significant is that we are now able to estimate kernels of the form of infinite series of the monomials (Theorem 1). This turns to be the usual case on the sphere. It is remarkable that we achieve the estimates without using special functions such as the Gegenbauer polynomials as usually involved in the eigenspace decompositions (also see [BDS], [DSS] and [L]). The desired properties of the kernels are reduced, through the corresponding relationship, to a holomorphic extension result of Laurent series of one complex variable obtained in [Q1] (also see [Q2], [Kh]). The theory established in this paper thus supports the philosophy that results that hold on the unit circle would also hold on the unit sphere via the corresponding relationship between the basic function sequences in the two spaces.

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