Journal of Functional Analysis **189**, 227–241 (2002)
doi:10.1006/jfan.2001.3848, available online at http://www.idealibrary.com on IDEAL®

The Paley–Wiener Theorem in $\mathbf{R}^n$ with the Clifford Analysis Setting[1]

Kit-Ian Kou and Tao Qian

Faculty of Science and Technology, University of Macau, P.O. Box 3001, Macau, Macau
E-mail: fstkik@umac.mo, fsttq@umac.mo

Communicated by M. Vergne

Received February 27, 2001; revised July 24, 2001; accepted July 26, 2001

We prove the Paley–Wiener Theorem in the Clifford algebra setting. As an application we derive the corresponding result for conjugate harmonic functions.

Key Words: Fourier analysis; Paley–Wiener theorem; conjugate harmonic system; Clifford analysis.

INTRODUCTION

Higher dimensional extensions of the Paley–Wiener Theorem have been studied, for instance, in [1, 6, 11, 14, 16]. In [16] a corresponding extension is obtained by imbedding $\mathbf{R}^n$ into $\mathbf{C}^n$ and by reducing it to the one complex variable case. The present work uses the imbedding of $\mathbf{R}^n$ into the real-Clifford algebra $\mathbf{R}^{(n)}$ (see the notation in Section 1). The latter imbedding provides $\mathbf{R}^n$ with a global complex structure in analogy with the imbedding of $\mathbf{R}$ into the complex plane. Under this frame we present in this note the precise analogue of the classical Paley–Wiener Theorem which has been targeted by others. In [1] results of the same kind are obtained of which either stronger conditions are imposed (see [1, 30.10]) or weaker conclusions, namely, in the distribution sense, are obtained (see [1, 30.19]). In [11] a set of results is obtained in which the pointwise estimate in the usual Paley–Wiener Theorem is replaced by an integral inequality.

It is well known that the classical Paley–Wiener Theorem has important applications to a wide range of topics in function theory of one complex variable and approximation of one real variable, etc. As an example, in the Shannon sampling and interpolation using the sinc functions, the sampling

[1] The study was supported by a Research Grant of the University of Macau, RG002/00-01W/QT/FST.

0022-1236/02 $35.00

scale is determined by the constant R (see Section 2, Theorem 2.1) appearing in the exponential part of the estimate for the holomorphic function under study [17]. Owing to the analogous complex structure in $\mathbf{R}^n$ induced by the Dirac operator (see Section 1), the Paley–Wiener Theorem proved in this note offers the same applications to topics in several real variables.

In Section 1 we provide the basic knowledge of Clifford analysis used in the paper. In Section 2 we formulate and prove the Paley–Wiener Theorem. Our proof is guided by the one for the classical Paley–Wiener Theorem cited in [19]. In Section 3 we show that the concept of monogenic functions is a natural way to represent conjugate harmonic systems. As an application, we present a new result on conjugate harmonic systems.

Some alternative proofs of the classical Paley–Wiener Theorem invoke the Phragmén–Lindelöf Theorem in one complex variable (see, for instance, [3, 16]). The proof of the latter theorem involves products of complex analytic functions and makes use of the fact that the product of two analytic functions is still analytic. This fails in the Clifford setting. In general, products of monogenic functions are no longer monogenic. It would be interesting, however, to see the generalization of the Phragmén–Liwdelöf Theorem in the Clifford analysis setting, and accordingly, a proof of the Paley–Wiener Theorem using the generalized Phragmén–Liwdelöf Theorem.

1. PRELIMINARIES

Most of the basic knowledge and notation recalled in this section are referred to [1, 2, 4].

Let $\mathbf{e}_1, ..., \mathbf{e}_n$ be *basic elements* satisfying $\mathbf{e}_i\mathbf{e}_j + \mathbf{e}_j\mathbf{e}_i = -2\delta_{ij}$, where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise, $i, j = 1, 2, ..., n$. Let

$$\mathbf{R}^n = \{\underline{x} = x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n : x_j \in \mathbf{R}, j = 1, 2, ..., n\}$$

be identical with the usual Euclidean space $\mathbf{R}^n$, and

$$\mathbf{R}_1^n = \{x_0 + \underline{x} : x_0 \in \mathbf{R}, \underline{x} \in \mathbf{R}^n\}.$$

An element in $\mathbf{R}_1^n$ is called a *vector*. The real (complex) Clifford algebra generated by $\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_n$, denoted by $\mathbf{R}^{(n)}$ ($\mathbf{C}^{(n)}$), is the associative algebra generated by $\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_n$, over the real (complex) field $\mathbf{R}$ ($\mathbf{C}$). A general element in $\mathbf{R}^{(n)}$, therefore, is of the form $x = \sum_S x_S \mathbf{e}_S$, where $\mathbf{e}_S = \mathbf{e}_{i_1}\mathbf{e}_{i_2}\cdots\mathbf{e}_{i_l}$, and S runs over all the ordered subsets of $\{1, 2, ..., n\}$, namely

$$S = \{1 \leqslant i_1 < i_2 < \cdots < i_l \leqslant n\}, \qquad 1 \leqslant l \leqslant n.$$

The natural inner product between x and y in $\mathbf{C}^{(n)}$, denoted by $\langle x, y\rangle$, is the complex number $\sum_S x_S \overline{y_S}$, where $x = \sum_S x_S \mathbf{e}_S$ and $y = \sum_S y_S \mathbf{e}_S$. The norm associated with this inner product is

$$|x| = \langle x, x\rangle^{\frac{1}{2}} = \left(\sum_S |x_S|^2\right)^{\frac{1}{2}}.$$

If $x, y, ..., u$ are vectors, then

$$|xy \cdots u| = |x|\, |y| \cdots |u|.$$

The conjugate of a vector $x = x_0 + \underline{x}$ is defined as $\bar{x} = x_0 - \underline{x}$. It is easy to verify that $0 \neq x \in \mathbf{R}_1^n$ implies

$$x^{-1} = \frac{\bar{x}}{|x|^2}.$$

The *unit sphere* $\{x \in \mathbf{R}_1^n : |x| = 1\}$ is denoted by S^n. We use $B(x, r)$ for the open ball in $\mathbf{R}_1^n$ centered at x with radius r.

In below we will study functions defined in $\mathbf{R}^n$ or $\mathbf{R}_1^n$ taking values in $\mathbf{C}^{(n)}$. So, they are of the form $f(x) = \sum_S f_S(x)\, \mathbf{e}_S$, where f_S are complex-valued functions. We will be using the *Dirac operator*

$$D = D_0 + \underline{D},$$

where $D_0 = \partial/\partial x_0$ and $\underline{D} = (\partial/\partial x_1) = \mathbf{e}_1 + \cdots + (\partial/\partial x_n)\, \mathbf{e}_n$. To be symmetric, we also write $D_0 = \partial/\partial x_0 = (\partial/\partial x_0)\, \mathbf{e}_0$, with $\mathbf{e}_0 = 1$. We define the "left" and "right" roles of the operators D by

$$Df = \sum_{i=0}^{n} \sum_S \frac{\partial f_S}{\partial x_i}\, \mathbf{e}_i \mathbf{e}_S$$

and

$$fD = \sum_{i=0}^{n} \sum_S \frac{\partial f_S}{\partial x_i}\, \mathbf{e}_S \mathbf{e}_i.$$

If $Df = 0$ in a domain (open and connected) Ω, then we say that f is *left-monogenic* in Ω; and, if $fD = 0$ in Ω, we say that f in *right-monogenic* in Ω. If f is both left- and right-monogenic, then we say that f is *monogenic*.

The Cauchy Theorem holds in the present case: Let Ω be a domain of Lipschitz boundary $\partial\Omega$ and g be right- and f be left-monogenic in a neighborhood of $\Omega \cup \partial\Omega$. Then

$$\int_{\partial\Omega} g(y)\, n(y)\, f(y)\, d\sigma(y) = 0,$$

where $n(y)$ is the outward unit normal to the surface $\partial\Omega$ at y and $d\sigma(y)$ is the area measure. We also have the Cauchy Formulas. Under the above assumptions,

$$g(x)=\frac{1}{\omega_n}\int_{\partial\Omega} g(y)\,n(y)\,E(y-x)\,d\sigma(y), \qquad x\in\Omega$$

and

$$f(x)=\frac{1}{\omega_n}\int_{\partial\Omega} E(y-x)\,n(y)\,f(y)\,d\sigma(y), \qquad x\in\Omega,$$

where

$$E(x)=\frac{\bar{x}}{|x|^{n+1}}$$

is the *Cauchy kernel*, and $\omega_n=2\pi^{(n+1)/2}/\Gamma(\frac{n+1}{2})$ is the area of the n-dimensional unit sphere S^n in $\mathbf{R}_1^n$.

We will use the Taylor expansion of left-monogenic functions: If f is left-monogenic in a domain containing $B(0,r)\cup\partial B(0,r)$, then

$$f(x)=\sum_{k=0}^{\infty}\frac{1}{\omega_n}\int_{\partial B(0,r)} P^{(k)}(y^{-1}x)\,E(y)\,n(y)\,f(y)\,d\sigma(y), \qquad x\in B(0,r), \quad (1)$$

where

$$P^{(k)}(y^{-1}x)=|y^{-1}x|^k\,C^+_{n+1,k}(\xi,\eta), \quad (2)$$

and

$$C^+_{n+1,k}(\xi,\eta)=\frac{1}{1-n}\Big[-(n+k-1)\,G_k^{\frac{n-1}{2}}(\langle\xi,\eta\rangle) + (1-n)\,G_{k-1}^{\frac{n+1}{2}}(\langle\xi,\eta\rangle)(\langle\xi,\eta\rangle-\bar{\xi}\eta)\Big], \quad (3)$$

where $x=|x|\,\xi$, $y=|y|\,\eta$, and G_k^ν is the Gegenbauer polynomial of degree k associated with ν (see [2]).

The function in (2) being a function of $y^{-1}x$ can be seen from (3) and the relations

$$\langle\xi,\eta\rangle=\frac{\langle y^{-1}x,1\rangle}{|y^{-1}x|} \qquad \text{and} \qquad \bar{\xi}\eta=\left(\frac{y^{-1}x}{|y^{-1}x|}\right)^{-1}. \quad (4)$$

We note that in (1) the integral region $\partial B(0, r)$ can be changed to any $\partial B(0, \rho)$ with $0<\rho<r$ (see [18, 2]).

The Taylor expansion (1) is originated by [10] and, independently by [9], and was followed by various versions later on (see [1, 2] for instance). The form (1) is taken from [2] combined with a recent study on the form in [7, 8].

We correspondingly have Taylor expansions at points different from the origin, and those for right-monogenic functions. We also have Laurent expansions of one-sided or two-sided monogenic functions on annulus. In the present paper, we only use Taylor expansions at the origin, and we will be based on the following facts:

- $|P^{(k)}(y^{-1}x)| \leqslant C_n k^n(|x|^k/|y|^k)$ (established by combining estimates (8) and (9) of [10, p. 431]), where C_n stands for a constant depending on the dimension n but not k.
- $P^{(k)}(y^{-1}x)$ is a polynomial in x of degree k (see [2, 18]).

The Fourier transform in $\mathbf{R}^n$ is defined by

$$\mathscr{F}(f)(\underline{\xi}) = \int_{\mathbf{R}^n} e^{-i\langle \underline{x}, \underline{\xi}\rangle} f(\underline{x})\, d\underline{x}$$

and the inverse Fourier transform is defined by

$$\mathscr{F}^{-1}(g)(\underline{x}) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} e^{i\langle \underline{x}, \underline{\xi}\rangle} g(\underline{\xi})\, d\underline{\xi}.$$

Here $\underline{\xi} = \xi_1 \mathbf{e}_1 + \cdots + \xi_n \mathbf{e}_n$. To extend the Fourier transform to $\mathbf{R}_1^n$, we need first to extend the exponential function $e^{i\langle \underline{x}, \underline{\xi}\rangle}$. Denote, for $x = x_0 + \underline{x}$,

$$e(x, \underline{\xi}) = e^{i\langle \underline{x}, \underline{\xi}\rangle} e^{-x_0 |\underline{\xi}|} \chi_+(\underline{\xi}) + e^{i\langle \underline{x}, \underline{\xi}\rangle} e^{x_0 |\underline{\xi}|} \chi_-(\underline{\xi}),$$

where

$$\chi_\pm(\underline{\xi}) = \frac{1}{2}\left(1 \pm i\frac{\underline{\xi}}{|\underline{\xi}|}\right).$$

It is easy to verify that

$$\chi_-\chi_+ = \chi_+\chi_- = 0, \qquad \chi_\pm^2 = \chi_\pm, \qquad \chi_+ + \chi_- = 1.$$

The function $e(x, \underline{\xi})$ is obviously an extension of $e(\underline{x}, \underline{\xi}) = e^{i\langle \underline{x}, \underline{\xi}\rangle}$ onto $\mathbf{R}_1^n \times \mathbf{R}^n$. It is easy to verify that $e(x, \underline{\xi})$ is monogenic in $x \in \mathbf{R}_1^n$ for any fixed $\underline{\xi}$. Generalizations of the exponential function of this kind can be first found in Sommen's work [12, 13], and then in [4], where $\underline{\xi}$ is further extended to $\underline{\zeta} = \underline{\xi} + i\underline{\eta} \in \mathbf{C}^n$.

It is well known that if $f \in L^2(\mathbf{R}^n)$, then $f = f^+ + f^-$, where f^+ is the boundary value of a function in the Hardy space H^2 in the upper-half-space, and f^- is the boundary value of a function in the Hardy space H^2 in the lower-half-space (see [4, 5]). The monogenic Hardy functions, still denoted by f^+ and f^-, in the upper and lower half spaces are, in fact, given by

$$f^{\pm}(x) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} e^{i\langle \underline{x}, \underline{\xi} \rangle} e^{\mp x_0 |\underline{\xi}|} \chi_{\pm}(\underline{\xi})\, \mathscr{F}(f|_{\mathbf{R}^n})(\underline{\xi})\, d\underline{\xi}, \qquad \pm x_0 > 0,$$

respectively.

In [1] a Clifford valued generalized function theory is developed. Below we will adopt the definition that T is called a *tempered distribution*, if T is a continuous linear functional from $\mathscr{S}(\mathbf{R}^n)$ to $\mathbf{C}^{(n)}$, where $\mathscr{S}(\mathbf{R}^n)$ is the Schwarz class of rapidly decreasing functions. This is equivalent with the one defined in [1] using modules but it also enables us to quickly define Fourier transforms of tempered distributions, by

$$\mathscr{F}(T)(\varphi) = T(\mathscr{F}(\varphi)), \qquad \forall \varphi \in \mathscr{S}(\mathbf{R}^n),$$

which is just to perform Fourier transform on each of the components of the distribution. We will use the results

$$\mathscr{F}(1) = (2\pi)^n\, \delta, \qquad \mathscr{F}^{-1}(\xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}) = i^{-|\alpha|} D^\alpha \delta,$$

where $\alpha = (\alpha_1, ..., \alpha_n)$, $D^\alpha = (\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_n)^{\alpha_n}$ and δ is the usual Dirac δ function.

2. THE PALEY–WIENER THEOREM

The theorem is stated as follows

THEOREM 2.1. *Let $f\colon \mathbf{R}_1^n \to \mathbf{C}^{(n)}$ be left-monogenic in $\mathbf{R}_1^n$, $f|_{\mathbf{R}^n} \in L^2(\mathbf{R}^n)$, and $R > 0$ be a positive number. Then the following two assertions are equivalent:*

(i) *There exists a constant C such that*

$$|f(x)| \leqslant Ce^{R|x|}, \qquad \forall x \in \mathbf{R}_1^n.$$

(ii) $\operatorname{supp} \mathscr{F}(f|_{\mathbf{R}^n}) \subset B(0, R)$.

Moreover, if one of the above conditions holds, then we have

$$f(x) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} e(x, \underline{\xi})\, \mathscr{F}(f|_{\mathbf{R}^n})(\underline{\xi})\, d\underline{\xi}, \qquad x \in \mathbf{R}^n_1.$$

Proof. (ii) ⇒ (i). Assume that (ii) holds. Let

$$F(x) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} e(x, \underline{\xi})\, \mathscr{F}(f|_{\mathbf{R}^n})(\underline{\xi})\, d\underline{\xi}.$$

Denote by $\chi_{B(0,R)}$ the characteristic function of $B(0, R)$. Since supp $\mathscr{F}(f|_{\mathbf{R}^n}) \subset B(0, R)$, we have

$$F(x) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} e(x, \underline{\xi})\, \chi_{B(0,R)}(\underline{\xi})\, \mathscr{F}(f|_{\mathbf{R}^n})(\underline{\xi})\, d\underline{\xi}.$$

The Hölder inequality then implies

$$|F(x)| \leqslant Ce^{R|x_0|}\, \|\chi_{B(0,R)}\|_2\, \|\mathscr{F}(f|_{\mathbf{R}^n})\|_2 \leqslant Ce^{R|x|}.$$

Since $f(\underline{x}) = F(\underline{x})$ in $\mathbf{R}^n$ and both are left-monogenic in $\mathbf{R}^n_1$, we conclude that $f(x) = F(x)$. Thus $f(x)$ is of the desired estimate.

(i) ⇒ (ii). Assume that (i) holds. Consider

$$G^+(x) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} e^{i\langle \underline{x}, \underline{\xi}\rangle} e^{-x_0|\underline{\xi}|} \chi_+(\underline{\xi})\, f(\underline{\xi})\, d\underline{\xi}, \qquad x_0 > 0, \tag{5}$$

which is well defined as $f \in L^2(\mathbf{R}^n)$. It is easy to show that $G^+(x)$ is left-monogenic for $x_0 > 0$. Substituting $f(\underline{\xi})$ by its Taylor series (1), the identity (5) may be rewritten as

$$\begin{aligned} G^+(x) &= \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} e^{i\langle \underline{x}, \underline{\xi}\rangle} e^{-x_0|\underline{\xi}|} \chi_+(\underline{\xi}) \\ &\quad \times \left(\sum_{k=0}^{\infty} \frac{1}{\omega_n} \int_{\partial B(0,r)} P^{(k)}(y^{-1}\underline{\xi})\, E(y)\, n(y)\, f(y)\, d\sigma(y) \right) d\underline{\xi} \\ &= \lim_{N \to \infty} \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} e^{i\langle \underline{x}, \underline{\xi}\rangle} e^{-x_0|\underline{\xi}|} \chi_+(\underline{\xi})\, \chi_{B(0,N)}(\underline{\xi}) \\ &\quad \times \left(\sum_{k=0}^{\infty} \frac{1}{\omega_n} \int_{\partial B(0,r)} P^{(k)}(y^{-1}\underline{\xi})\, E(y)\, n(y)\, f(y)\, d\sigma(y) \right) d\underline{\xi}, \end{aligned}$$

where r is any positive number. Owing to the uniform convergence property of the series for $|\underline{\xi}| \leqslant N$, we have

$$G^+(x) = \lim_{N\to\infty} \frac{1}{(2\pi)^n} \sum_{k=0}^{\infty} \frac{1}{\omega_n} \\ \times \int_{\partial B(0,r)} \left(\int_{\mathbf{R}^n} e^{i\langle \underline{x}, \underline{\xi}\rangle} e^{-x_0 |\underline{\xi}|} \chi_+(\underline{\xi})\, \chi_{B(0,N)}(\underline{\xi})\, P^{(k)}(y^{-1}\underline{\xi})\, d\underline{\xi} \right) \\ \times E(y)\, n(y)\, f(y)\, d\sigma(y). \tag{6}$$

We now prove that for $x_0 > 0$, we can exchange the order of taking the limit $N \to \infty$ and taking the summation $\sum_{k=0}^{\infty}$, by showing that the series is dominated by an absolutely convergent one independent of N for $x_0 > R$. Accepting that, we will consequently have

$$G^+(x) = \sum_{k=0}^{\infty} \frac{1}{(2\pi)^n \omega_n} \int_{\partial B(0,r)} \\ \times \left(\int_{\mathbf{R}^n} e^{i\langle \underline{x}, \underline{\xi}\rangle} e^{-x_0 |\underline{\xi}|} \chi_+(\underline{\xi})\, P^{(k)}(y^{-1}\underline{\xi})\, d\underline{\xi} \right) \\ \times E(y)\, n(y)\, f(y)\, d\sigma(y), \qquad x_0 > R. \tag{7}$$

In fact, using the bounds of $P^{(k)}(y^{-1}\underline{\xi})$, and that of $f(y)$, and the spherical coordinates, we have

$$\frac{1}{(2\pi)^n \omega_n} \left| \int_{\partial B(0,r)} \left(\int_{\mathbf{R}^n} e^{i\langle \underline{x}, \underline{\xi}\rangle} e^{-x_0 |\underline{\xi}|} \chi_+(\underline{\xi})\, P^{(k)}(y^{-1}\underline{\xi})\, d\underline{\xi} \right) \right. \\ \left. \times E(y)\, n(y)\, f(y)\, d\sigma(y) \right| \\ \leqslant C_n k^n \int_{\mathbf{R}^n} e^{-x_0 |\underline{\xi}|}\, |\underline{\xi}|^k\, r^{-k} r^{-n} r^n e^{Rr}\, d\underline{\xi} \\ = C_n k^n \frac{e^{Rr}}{r^k} \int_0^{\infty} e^{-x_0 s} s^{k+n-1}\, ds \\ = C_n k^n \frac{e^{Rr}}{r^k} \frac{(k+n-1)!}{x_0^{k+n}}.$$

The last inequality holds for any $r > 0$. Taking the minimum value of the last expression with respect to r, we have that the series in $G^+(x)$ is dominated by

$$C_n \sum_{k=0}^{\infty} k^n (k+n-1)! \left(\frac{e}{k}\right)^k R^k \frac{1}{x_0^{n+k}} = \frac{C_n}{x_0^n} \sum_{k=0}^{\infty} d_k \frac{1}{x_0^k}, \tag{8}$$

where

$$d_k = k^n(k+n-1)!\left(\frac{e}{k}\right)^k R^k.$$

Using Stirling's formula, we conclude that

$$\overline{\lim_{k\to\infty}}\,(d_k)^{\frac{1}{k}} = R.$$

Using Hadamard's criterion, the convergence radius of the associated power series is R^{-1}. Correspondingly, the series (8) converges for $x_0 > R$. Now we have justified that we can exchange the limit procedure $N \to \infty$ and the summation $\sum_{k=0}^{\infty}$ in (6) if $x_0 > R$, and thus (7) holds for $x_0 > R$.

Let $\varphi_m(\underline{\xi})$ be a sequence of functions in $C_0^\infty(\mathbf{R}^n)$ such that $\varphi_m(\underline{\xi}) = 0$ if $|\underline{\xi}| \leqslant \frac{1}{m}$ and $\varphi_m(\underline{\xi}) = 1$ if $|\underline{\xi}| \geqslant \frac{2}{m}$ and $0 \leqslant \varphi_m(\underline{\xi}) \leqslant 1$ otherwise. Obviously, $\varphi_m \to 1$ distributionally. We rewrite $G^+(x)$ as

$$\begin{aligned} G^+(x) = {} & \frac{1}{(2\pi)^n}\sum_{k=0}^{\infty}\frac{1}{\omega_n} \\ & \times \int_{\partial B(0,r)}\left(\lim_{m\to\infty}\int_{\mathbf{R}^n} e^{i\langle \underline{x},\underline{\xi}\rangle} e^{-x_0|\underline{\xi}|}\chi_+(\underline{\xi})\,\varphi_m(\underline{\xi})\,P^{(k)}(y^{-1}\underline{\xi})\,d\underline{\xi}\right) \\ & \times E(y)\,n(y)\,f(y)\,d\sigma(y), \qquad x_0 > R. \end{aligned}$$

Since $e^{i\langle \underline{x},\cdot\rangle}e^{-x_0|\cdot|}\chi_+(\cdot)\,\varphi_m(\cdot) \in \mathscr{S}(\mathbf{R}^n)$, the inside integral can be rewritten in the notation of distribution:

$$\begin{aligned} & P^{(k)}(y^{-1}(\cdot))(e^{i\langle \underline{x},\cdot\rangle}e^{-x_0|\cdot|}\chi_+(\cdot)\,\varphi_m(\cdot)) \\ & = \mathscr{F}^{-1}(P^{(k)}(y^{-1}(\cdot))(\mathscr{F}(e^{i\langle \underline{x},\cdot\rangle}e^{-x_0|\cdot|}\chi_+(\cdot)\,\varphi_m(\cdot)))) \\ & = i^{-k}(P^{(k)}(y^{-1}\underline{D})\,\delta)(\mathscr{F}(e^{i\langle \underline{x},\cdot\rangle}e^{-x_0|\cdot|}\chi_+(\cdot)) * \mathscr{F}(\varphi_m)). \end{aligned} \tag{9}$$

Now

$$\mathscr{F}(e^{i\langle \underline{x},\cdot\rangle}e^{-x_0|\cdot|}\chi_+(\cdot)) = \frac{1}{2}\mathscr{F}(e^{i\langle \underline{x},\cdot\rangle}e^{-x_0|\cdot|}) + \frac{1}{2}\mathscr{F}\left(e^{i\langle \underline{x},\cdot\rangle}e^{-x_0|\cdot|}\frac{i(\cdot)}{|\cdot|}\right),$$

where

$$\begin{aligned} \frac{1}{2}\mathscr{F}(e^{i\langle \underline{x},\cdot\rangle}e^{-x_0|\cdot|})(\underline{\zeta}) & = \frac{1}{2}\int_{\mathbf{R}^n} e^{-i\langle \underline{\zeta},\underline{\xi}\rangle}e^{i\langle \underline{x},\underline{\xi}\rangle}e^{-x_0|\underline{\xi}|}\,d\underline{\xi} \\ & = \frac{1}{2}\int_{\mathbf{R}^n} e^{-i\langle \underline{\zeta}-\underline{x},\underline{\xi}\rangle}e^{-x_0|\underline{\xi}|}\,d\underline{\xi} \\ & = \tilde{c}\,\frac{x_0}{(x_0^2+|\underline{\zeta}-\underline{x}|^2)^{\frac{n+1}{2}}}, \end{aligned}$$

where $\tilde{c} = 2^{n-1}\pi^{(n-1)/2}\Gamma(\frac{n+1}{2})$.

We subsequently have

$$\frac{1}{2}\mathscr{F}\left(e^{i\langle \underline{x},\cdot\rangle}e^{-x_0|\cdot|}\frac{i(\cdot)}{|\cdot|}\right)(\underline{\zeta})=\frac{1}{2}\int_{x_0}^{\infty}\underline{D}_{\underline{x}}\mathscr{F}(e^{i\langle \underline{x},\cdot\rangle}e^{-t|\cdot|})(\underline{\zeta})\,dt$$

$$=\tilde{c}\int_{x_0}^{\infty}\underline{D}_{\underline{x}}\left(\frac{t}{(t^2+|\underline{\zeta}-\underline{x}|^2)^{\frac{n+1}{2}}}\right)dt$$

$$=\tilde{c}\,\frac{\underline{\zeta}-\underline{x}}{(x_0^2+|\underline{\zeta}-\underline{x}|^2)^{\frac{n+1}{2}}}.$$

Hence

$$\mathscr{F}(e^{i\langle \underline{x},\cdot\rangle}e^{-x_0|\cdot|}\chi_+(\cdot))(\underline{\zeta})=\tilde{c}\,\frac{\overline{x-\underline{\zeta}}}{|x-\underline{\zeta}|^{n+1}}=-\tilde{c}E(\underline{\zeta}-x). \tag{10}$$

(Note that this computation may be omitted if one directly uses the corresponding result in [4].) Therefore, (9) becomes

$$\begin{aligned}&-\tilde{c}i^{-k}(P^{(k)}(y^{-1}\underline{D})\,\delta)(E(\cdot-x)*\mathscr{F}(\varphi_m))\\&\quad=-\tilde{c}i^{-k}(-1)^k\,\delta((P^{(k)}(y^{-1}\underline{D})\,E)(\cdot-x)*\mathscr{F}(\varphi_m))\\&\quad=-\tilde{c}i^{k}((P^{(k)}(y^{-1}\underline{D})\,E)(\cdot-x)*\mathscr{F}(\varphi_m))(0).\end{aligned}$$

Since $\mathscr{F}(\varphi_m)\to(2\pi)^n\,\delta$, we conclude that

$$\int_{\mathbf{R}^n}e^{i\langle \underline{x},\underline{\xi}\rangle}e^{-x_0|\underline{\xi}|}\chi_+(\underline{\xi})\,P^{(k)}(y^{-1}\underline{\xi})\,d\underline{\xi}=-(2\pi)^n\,\tilde{c}i^k(P^{(k)}(y^{-1}\underline{D})\,E)(-x).$$

Thus for $x_0>R$, we have

$$G^+(x)=-\tilde{c}\sum_{k=0}^{\infty}\frac{i^k}{\omega_n}\int_{\partial B(0,r)}(P^{(k)}(y^{-1}\underline{D})\,E)(-x)\,E(y)\,n(y)\,f(y)\,d\sigma(y). \tag{11}$$

We next point out that the series expression of $G^+(x)$ in (11) for $x_0>R$ can be monogenically extended to all $x\in\mathbf{R}_1^n$ with $|x|>R$.

In fact, invoking the estimate

$$|(P^k(y^{-1}\underline{D})\,E)(-x)|\leqslant C_nk^n\frac{1}{|x|^{n+k}}\frac{1}{|y|^k},$$

we can proceed as before with a general entry of the series (11), and we obtain that the series (11) is dominated by

$$\tilde{c} \sum_{k=0}^{\infty} k^n (k+n-1)! \left(\frac{e}{k}\right)^k R^k \frac{1}{|x|^{n+k}}.$$

The same argument then implies that the series (11) converges uniformly in any compact set in the region $|x| > R$ and thus the sum function is left-monogenic for $|x| > R$.

Now we define

$$G^-(x) = \frac{1}{(2\pi)^n} \int_{-\infty}^{\infty} e^{i\langle \underline{x}, \underline{\xi} \rangle} e^{x_0 |\underline{\xi}|} \chi_-(\underline{\xi})\, f(\underline{\xi})\, d\underline{\xi}, \qquad x_0 < 0, \tag{12}$$

that is left-monogenic for $x_0 < 0$. Using the same procedure we can first show that for $-x_0 > R$,

$$G^-(x) = \tilde{c} \sum_{k=0}^{\infty} \frac{i^k}{\omega_n} \int_{\partial B(0,r)} (P^{(k)}(y^{-1}\underline{D})\, E)(-x)\, E(y)\, n(y)\, f(y)\, d\sigma(y), \tag{13}$$

and then $G^-(x)$ can be monogenically extended to $|x| > R$ using the series expansion (13). We will be content with only pointing out how the negative sign in the beginning of formula (11) drops off in the case of (13).

When we compute $\mathscr{F}(e^{i\langle \underline{x}, \cdot \rangle} e^{x_0 |\cdot|} \chi_-(\cdot))$, with $x_0' = -x_0 > 0$, we first write it as $\mathscr{F}(e^{i\langle \underline{x}, \cdot \rangle} e^{-x_0' |\cdot|} \chi_-(\cdot))$. Then, as before, we have

$$\frac{1}{2} \mathscr{F}(e^{i\langle \underline{x}, \cdot \rangle} e^{-x_0' |\cdot|})(\underline{\zeta}) = \tilde{c} \frac{x_0'}{(x_0'^2 + |\underline{\zeta} - \underline{x}|^2)^{\frac{n+1}{2}}}.$$

We accordingly have

$$\frac{1}{2} \mathscr{F}\left(e^{i\langle \underline{x}, \cdot \rangle} e^{-x_0' |\cdot|} \left(-i \frac{(\cdot)}{|\cdot|} \right) \right)(\underline{\zeta}) = -\tilde{c} \frac{\underline{\zeta} - \underline{x}}{(x_0'^2 + |\underline{\zeta} - \underline{x}|^2)^{\frac{n+1}{2}}}.$$

Putting together, we have

$$\mathscr{F}(e^{i\langle \underline{x}, \cdot \rangle} e^{x_0 |\cdot|} \chi_-(\cdot))(\underline{\zeta}) = -\tilde{c} \frac{\overline{x - \underline{\zeta}}}{|x - \underline{\zeta}|^{n+1}} = \tilde{c} E(\underline{\zeta} - x). \tag{14}$$

Now we show that G^+ and G^- have the alternative forms

$$G^+(x_0+\underline{x})=\frac{1}{\omega_n}\int_{\mathbf{R}^n} E((x_0+\underline{x})-\underline{\zeta})\,\mathscr{F}(f|_{\mathbf{R}^n})(-\underline{\zeta})\,d\underline{\zeta}$$

and

$$G^-(-x_0+\underline{x})=-\frac{1}{\omega_n}\int_{\mathbf{R}^n} E((-x_0+\underline{x})-\underline{\zeta})\,\mathscr{F}(f|_{\mathbf{R}^n})(-\underline{\zeta})\,d\underline{\zeta},$$

respectively. In fact, owing to Parseval's identity

$$\int_{\mathbf{R}^n} h(\underline{\xi})\,g(\underline{\xi})\,d\underline{\xi}=\int_{\mathbf{R}^n} \mathscr{F}(h)(\underline{\zeta})\,\mathscr{F}(g)(-\underline{\zeta})\,d\underline{\zeta},$$

and the identity (10), we have

$$\begin{aligned}G^+(x_0+\underline{x})&=\frac{1}{(2\pi)^n}\int_{\mathbf{R}^n} e^{i\langle \underline{x},\,\underline{\xi}\rangle}e^{-x_0\,|\underline{\xi}|}\chi_+(\underline{\xi})\,f(\underline{\xi})\,d\underline{\xi}\\&=\frac{1}{(2\pi)^n}\int_{\mathbf{R}^n} \mathscr{F}(e^{i\langle \underline{x},\,\cdot\rangle}e^{-x_0\,|\cdot|}\chi_+(\,\cdot\,))(\underline{\zeta})\,\mathscr{F}(f|_{\mathbf{R}^n})(-\underline{\zeta})\,d\underline{\zeta}\\&=\frac{1}{\omega_n}\int_{\mathbf{R}^n} E((x_0+\underline{x})-\underline{\zeta})\,\mathscr{F}(f|_{\mathbf{R}^n})(-\underline{\zeta})\,d\underline{\zeta}.\end{aligned}$$

The last step uses the relation $1/\omega_n=\tilde{c}/(2\pi)^n$. The expression for G^- can be proved similarly by using (14). The Plemelj formula (see [4]) then gives

$$\lim_{x_0\to 0+}(G^+(x_0+\underline{x})+G^-(-x_0+\underline{x}))=\mathscr{F}(f|_{\mathbf{R}^n})(-\underline{x}).$$

This, together with the series expressions (11) and (13) for $|x|>R$, gives $\mathscr{F}(f|_{\mathbf{R}^n})(\underline{x})=0$ for $|\underline{x}|>R$. Therefore supp $\mathscr{F}(f|_{\mathbf{R}^n})\subset B(0,R)$.

To show

$$f(x)=\frac{1}{(2\pi)^n}\int_{\mathbf{R}^n} e(x,\underline{\xi})\,\mathscr{F}(f|_{\mathbf{R}^n})(\underline{\xi})\,d\underline{\xi},\qquad x\in\mathbf{R}^n_1,$$

we notice that the left-hand side is equal to the right-hand side if $x_0=0$. Since both sides are left-monogenic in $\mathbf{R}^n_1$ and coincident in $\mathbf{R}^n$, they have to be equal. ∎

3. AN APPLICATION TO CONJUGATE HARMONIC SYSTEM IN $\mathbf{R}_1^n$

If an ordered set of $n+1$ functions $u_0(x_0, x_1, ..., x_n)$, $u_1(x_0, x_1, ..., x_n)$, ..., $u_n(x_0, x_1, ..., x_n)$ satisfies the relations

$$\begin{cases} \sum_{j=0}^{n} \frac{\partial u_j}{\partial x_j} = 0 \\ \frac{\partial u_k}{\partial x_j} = \frac{\partial u_j}{\partial x_k}, \qquad 0 \leqslant k < j \leqslant n, \end{cases}$$

then it is called a *conjugate harmonic system* (see [15, 16]). Below we denote

$$U = -u_0 + u_1\mathbf{e}_1 + \cdots + u_n\mathbf{e}_n.$$

PROPOSITION 3.1. *An ordered set of functions $u_0, u_1, ..., u_n$ is a conjugate harmonic system if and only if the corresponding vector-valued function U is monogenic.*

Proof. Denote $\underline{u} = u_1\mathbf{e}_1 + \cdots + u_n\mathbf{e}_n$. Then

$$\begin{aligned} DU &= (D_0 + \underline{D})(-u_0 + \underline{u}) \\ &= \left(-D_0 u_0 - \sum_{j=1}^{n} \frac{\partial u_j}{\partial x_j}\right) + (D_0\underline{u} - \underline{D}u_0) + \sum_{1 \leqslant k < j \leqslant n} \left(\frac{\partial u_k}{\partial x_j} - \frac{\partial u_j}{\partial x_k}\right)\mathbf{e}_j\mathbf{e}_k. \end{aligned}$$

So, $DU = 0$ if and only if

$$\begin{cases} \sum_{j=0}^{n} \frac{\partial u_j}{\partial x_j} = 0 \\ \frac{\partial u_k}{\partial x_j} = \frac{\partial u_j}{\partial x_k}, \qquad 0 \leqslant k < j \leqslant n. \end{cases}$$

The right-monogenity is proved similarly. ▮

The proposition indicates that the Clifford algebra frame of $\mathbf{R}^n$ is a natural one to study Hardy spaces in relation to the space (see [5]). The following is an immediate consequence of Theorem 2.1.

THEOREM 3.1. *Let $u_0, u_1, ..., u_n$ be a conjugate harmonic system in $\mathbf{R}_1^n$. Let $U|_{\mathbf{R}^n} \in L^2(\mathbf{R}^n)$. Then*

$$|U(x)| \leqslant Ce^{R|x|}$$

if and only if

$$\operatorname{supp} \mathscr{F}(U)(0, \cdot) \subset B(0, R),$$

where $\mathscr{F}(U)(0, \underline{\xi}) = \mathscr{F}(U|_{\mathbf{R}^n})(\underline{\xi})$. *Moreover, if one of the above conditions holds, then*

$$U(x) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} e(x, \underline{\xi})\, \mathscr{F}(U)(0, \underline{\xi})\, d\underline{\xi}.$$

ACKNOWLEDGMENTS

Special thanks are due to F. Sommen who read the first draft of the note and gave valuable comments. The authors also thank A. McIntosh, J. Ryan, and K. S. Lau for helpful discussions. The second author thanks K. S. Lau for his kind invitation for the author to visit the Chinese University of Hong Kong. During the visit some progress was made on this study.

REFERENCES

1. F. Brackx, R. Delanghe, and F. Sommen, "Clifford analysis," Research Notes in Mathematics, Vol. 76, Pitman, Boston/London/Melbourne, 1982.
2. R. Delanghe, F. Sommen, and V. Soucek, "Clifford Algebras and Spinor Valued Functions: A Function Theory for Dirac Operator," Kluwer Academic, Dordrecht, 1992.
3. Y. Katznelson, "An Introduction to Harmonic Analysis," 2nd ed., Dover, New York, 1976.
4. C. Li, A. McIntosh, and T. Qian, Clifford algebras, Fourier transforms, and singular convolution operators on Lipschitz surfaces, *Rev. Mat. Iberoamericana* **10** (1994), 665–721.
5. M. Mitrea, "Clifford Wavelets, Singular Integrals, and Hardy Spaces," Lecture Notes in Mathematics, Vol. 1575, Springer-Verlag, New York/Berlin, 1994.
6. M. Plancherel and G. Polya, Fonctions entières et intégrales de Fourier multiplier, *Math. Helv.* **9** (1937), 224–248.
7. T. Qian, Generalization of Fueter's result to $\mathbf{R}^{n+1}$, *Rend. Mat. Acc. Lincei* **8** (1997), 111–117.
8. T. Qian, Singular integrals on star-shaped Lipschitz surfaces in the quaternionic space, *Math. Ann.* **310** (1998), 601–630.
9. J. Ryan, Clifford analysis with elliptic and quasi-elliptic functions, *Appl. Anal.* **13** (1982), 151–171.
10. F. Sommen, Spherical monogenic functions and analytic functionals on the unit sphere, *Tokyo J. Math.* **4** (1981).
11. F. Sommen, Hypercomplex Fourier and Laplace transforms, II, *Complex Variables* **1** (1983), 209–238.
12. F. Sommen, Plane waves, biregular functions and hypercomplex Fourier analysis, SRNI, 5–12 January, 1985, *Rend. Circ. Mat. Palermo (2) Suppl.* **9** (1985), 205–219.

13. F. Sommen, Microfunctions with values in a Clifford algebra, II, *Sci. Papers College Arts Sci. Univ. Tokyo* **36** (1986), 15–37.
14. E. Stein, Functions of exponential type, *Ann. of Math.* **65** (1957), 582–592.
15. E. Stein, "Singular Integrals and Differentiability Properties of Functions," Princeton Univ. Press, Princeton, NJ, 1970.
16. E. Stein and G. Weiss, "Introduction to Fourier Analysis on Euclidean Spaces," Princeton Univ. Press, Princeton, NJ, 1987.
17. F. Stenger, "Numerical Methods Based on Sinc and Analytic Functions," Springer-Verlag, New York, 1993.
18. A. Sudbery, Quaternionic analysis, *Math. Proc. Cambridge Philos. Soc.* **85** (1979), 199–225.
19. A. Timan, "Theory of Approximation of Functions of a Real Variable," Fizmatgiz, Moscow, 1960; English translation, Internat. Ser. Monogr. Pure Appl. Math., Vol. 34, pp. 582–592, Macmillan, New York, 1963.