METHODS AND APPLICATIONS OF ANALYSIS.
Vol. 9, No. 2, pp. 273–290, June 2002

005

GENERALIZATIONS OF FUETER'S THEOREM *

K.I. KOU[†], T. QIAN[†], AND F. SOMMEN[‡]

Abstract. In relation to the solution of the Vekua system for axial type monogenic functions, generalizations of Fueter's Theorem are discussed. We show that if f is a holomorphic function in one complex variable, then for any underlying space $\mathbf{R}_1^n$ the induced function $\Delta^{k+(n-1)/2} f(x_0+\underline{x})P_k(\underline{x})$, where $P_k(\underline{x})$ is left-monogenic and homogeneous of degree k, is left-monogenic whenever $k+(n-1)/2$ is a non-negative integer. If the space dimension $n+1$ is odd, then the above also holds for k being non-negative integers.

1. Introduction. If $f(z)$ is a holomorphic function in an open set of the upper half complex plane and

$$f(z) = u(s,t) + \mathbf{i}v(s,t), \qquad z = s + \mathbf{i}t,$$

then, Fueter's theorem (see [F]) asserts that in the corresponding region there holds

$$D\Delta\left(u(q_0,|\underline{q}|) + \frac{\underline{q}}{|\underline{q}|}v(q_0,|\underline{q}|)\right) = 0,$$

where $\underline{q} = q_1\mathbf{i}+q_2\mathbf{j}+q_3\mathbf{k}, D = \partial_0+\underline{\partial}, \underline{\partial} = \partial_1\mathbf{i}+\partial_2\mathbf{j}+\partial_3\mathbf{k}, \Delta = \partial_0^2+\partial_1^2+\partial_2^2+\partial_3^2, \partial_i = \frac{\partial}{\partial q_i}, i = 0,1,2,3$. The quaternionic space may be identified with $\mathbf{R}_1^n$ for $n = 3$, where

$$\mathbf{R}_1^n = \{x = x_0 + \underline{x} : x_0 \in \mathbf{R}, \underline{x} \in \mathbf{R}^n\}$$

and

$$\mathbf{R}^n = \{\underline{x} = x_1\mathbf{e}_1 + \cdots x_n\mathbf{e}_n : x_i \in \mathbf{R}, i = 1,\cdots,n\},$$

and $\mathbf{e}_i^2 = -1, \mathbf{e}_i\mathbf{e}_j = -\mathbf{e}_j\mathbf{e}_i, i,j = 1,2,\cdots,n, i<j$ ([BDS]). The quaternionic algebra corresponds to the non-universal Clifford algebra over $\mathbf{R}$ in the above setting with $n = 2$ and $\mathbf{e}_3 = \mathbf{e}_1\mathbf{e}_2$. Fueter's machinery is used to develop singular integral and Fourier multiplier theory on the unit sphere of the quaternionic space in [Q1].

In 1957, Sce extended this result to $\mathbf{R}_1^n$ for n being odd positive integers ([Sc]). He proved that under the same assumptions on f, there holds

$$D\Delta^{(n-1)/2}\left(u(x_0,|\underline{x}|) + \frac{\underline{x}}{|\underline{x}|}v(x_0,|\underline{x}|)\right) = 0,$$

where $D = \partial_0+\underline{\partial}$, $\underline{\partial} = \partial_1\mathbf{e}_1+\cdots+\partial_n\mathbf{e}_n, \Delta = \partial_0^2+\partial_1^2+\cdots+\partial_n^2, \partial_i = \frac{\partial}{\partial x_i}, i = 0,1,\cdots,n$. Using Fourier transformation Qian extended Sce's result to $\mathbf{R}_1^n$ for n being even

*Received May 2, 2002; accepted for publication October 30, 2002.
†Faculty of Science and Technology, University of Macau, P. O. Box 3001 Macau (fsttq@umac.mo).
‡Department of Mathematical Analysis, University of Ghent, B-9000 Gent, Belgium (fs@cage.rug.ac.be).

positive integers ([Q2]). The theme was further developed and found to play a crucial role in the study of Fourier multipliers and singular integrals on the unit sphere of $\mathbf{R}_1^n$ and its Lipschitz perturbations ([Q3]). As example, the study shows that by means of this technique, some topics on the sphere may be reduced to the corresponding ones on the unit circle in the complex plane.

In a recent paper Sommen proved the following result: If $n+1$ is an even positive integer, then

$$D\Delta^{k+(n-1)/2}\left(\left(u(x_0,\underline{x})+\frac{\underline{x}}{|\underline{x}|}v(x_0,|\underline{x}|)\right)P_k(\underline{x})\right)=0,$$

where P_k is any polynomial in $\underline{x}$ of homogeneity k, left-monogenic with respect to the Dirac operator $\underline{\partial}$, viz. $\underline{\partial}P_k(\underline{x})=0$ ([So]). When $k=0$, this reduces to Sce's result.

Below we will briefly write

$$f(x_0+\underline{x})=u(x_0,\underline{x})+\frac{\underline{x}}{|\underline{x}|}v(x_0,|\underline{x}|).$$

The present paper extends Sommen's result, as given in the following

THEOREM 1. *Let f be a holomorphic function in a relatively open set B in the upper half complex plane. Let $P_k(\underline{x})$ be left-monogenic, homogeneous of degree k. Then in the set $\overrightarrow{B}=\{x=x_0+\underline{x}\in\mathbf{R}_1^n\ :\ (x_0,|\underline{x}|)\in B\}$ the function*

$$\Delta^{k+(n-1)/2}[f(x_0+\underline{x})P_k(\underline{x})]$$

is left-monogenic whenever $k+(n-1)/2$ is a non-negative integers. If the space dimension $n+1$ is odd, then the monogenicity of the above expression also holds for k being non-negative integer and $P_k(\underline{x})$ a homogeneous left-monogenic polynomial of degree k.

Note that in Theorem 1 when $n+1$ is odd and $k+(n-1)/2$ is a non-negative integer, then k has to be a "half integer" among $-(n-1)/2,...,-1/2,1/2,3/2,...$ In this case the definition of $P_k(\underline{x})$ is given in §2 below. These cases were studied in detail in the Ph.D. thesis [VL1] and papers [VL2] and [SVL]. Also based on these studies, for k being non-negative integers, $P_k(\underline{x})$ can be non-polynomial. The second conclusion of Theorem 1, however, does require $P_k(\underline{x})$ to be polynomial.

In the text the single letter C will denote constants depending on n and k that may be different from time to time. We will use the notation $\gamma_{j,\alpha},\beta_k$, etc. to denote the constants that are explicitly defined and of the same values throughout the paper.

Acknowledgement: The authors wish to thank Li Banghe who contributed to the authors' understanding to some related knowledge on generalized functions. We also wish to thank Alan McIntosh and Richard Delanghe for their communication on, and encouragement to undertake this work. The second author would like to express his sincere thanks to Richard Delanghe for his support in obtaining a scholarship from Office of Research Policy, Ghent University, Project 01VB6698. The work is partly supported by the 2001 Research Grant No. RG024/00-01S/QT/FST of the University of Macau.

2. The Equations For General Axial Monogenic Functions. The classical Fueter's Theorem and its generalizations obtained in [Sc] and [Q2] provides us with monogenic functions of the axial type

$$A(x_0, r) + \underline{\omega} B(x_0, r) = \Delta^{(n-1)/2} f(x_0 + \underline{x}),$$

whereby $\underline{x} = r\underline{\omega}, r = |\underline{x}|$ and A and B are scalar valued functions. This means that

$$D(A(x_0, r) + \underline{\omega} B(x_0, r)) = 0,$$

where $D = \partial_0 + \underline{\partial}$. In polar coordinates,

$$\underline{\partial} = \underline{\omega}(\partial_r + \frac{1}{r}\Gamma_{\underline{\omega}}),$$

where

$$\Gamma_{\underline{\omega}} = -\underline{x} \wedge \partial_{\underline{x}} = -\sum \mathbf{e}_j \mathbf{e}_k (x_j \partial_k - x_k \partial_j),$$

being the spherical vector derivative (or spin-orbit coupling).

We obtain from $\Gamma_{\underline{\omega}}(\underline{\omega}) = (n-1)\underline{\omega}$ the Vekua-type system

$$\begin{aligned} &\partial_0 A - \partial_r B = ((n-1)/r)B \\ &\partial_0 B + \partial_r A = 0. \end{aligned}$$

More generally, one may consider monogenic functions of the axial type

$$(A(x_0, r) + \underline{\omega} B(x_0, r)) P_k(\underline{\omega}),$$

whereby

$$P_k(r\underline{x}) = r^k P_k(\underline{\omega})$$

is left-monogenic in $\mathbf{R}^n$, i.e. $\underline{\partial} P_k(\underline{x}) = 0$, and homogeneous of degree $k \in \mathbf{Z}^+, \mathbf{Z}^+ = \{0, 1, ...\}$.

It is clear that if P_k is a homogeneous polynomial of degree k then it is spherical harmonic, and, conversely, every spherical harmonic of degree k, S_k, admits a decomposition of the form

$$S_k(\underline{\omega}) = P_k(\underline{\omega}) + \underline{\omega} P_{k-1}(\underline{\omega}).$$

Moreover, the function $\underline{\omega} P_k(\underline{\omega})$, which appears in the above generalized axial system, is the restriction to S^{n-1} of the inverted function $\frac{\underline{x}}{|\underline{x}|^n} P_k(\frac{\underline{x}}{|\underline{x}|^2})$ which is monogenic in $\mathbf{R}^n \setminus \{0\}$ and homogeneous of degree $-(k+n-1)$, i.e. outer spherical monogenic of degree $-(k+n-1)$, and also spherical harmonic of degree $k+1$.

The properties

$$\underline{\partial}(r^k P_k(\underline{\omega})) = 0, \quad \underline{\partial}(r^{-(k+n-1)} \underline{\omega} P_k(\underline{\omega})) = 0$$

imply that

$$\Gamma_{\underline{\omega}} P_k(\underline{\omega}) = -k P_k(\underline{\omega}), \quad \Gamma_{\underline{\omega}}(\underline{\omega} P_k(\underline{\omega})) = (k+n-1)\underline{\omega} P_k(\underline{\omega}).$$

Based on these, the monogenicity of $(A + \underline{\omega} B)P_k(\underline{\omega})$ then requires the functions A and B to satisfy the Vekua system

$$\begin{aligned}\partial_0 A - \partial_r B &= ((k+n-1)/r)B \\ \partial_0 B + \partial_r A &= (k/r)A.\end{aligned}$$

The generalized Fueter's Theorem obtained in [So] provides special solutions to this system of the form

$$\Delta^{k+(n-1)/2}(f(x_0 + \underline{x})P_k(\underline{x})),$$

for the case n being odd. The first question is to generalize this to the case n being even. But there is yet another interesting generalization having to do with the consideration of spherical monogenics P_α of general complex degree of homogeneity $\alpha \in \mathbf{C}$.

Definition. *Let U be an open domain in $\mathbf{R}^n$ of the form $U = \mathbf{R}^+ \times O, O$ being an open subset of S^{n-1}, i.e. $U = \{\underline{x} = r\underline{\omega} \ : \ r > 0, \underline{\omega} \in O\}$. Then a function P_α is called spherical monogenic of degree α in U if $P_\alpha(\underline{x}) = r^\alpha P_k(\underline{\omega})$ and $\underline{\partial} P_\alpha(\underline{x}) = 0$ in U. The restriction $P_\alpha(\underline{\omega})$ of $P_\alpha(\underline{x})$ to S^{n-1} is called spherical monogenic of complex degree α in O.*

The theory of spherical monogenics of complex degree is fully elaborated in the Ph.D. thesis [VL1] and basic information is also contained in [VL2] and [SVL]. In this paper we are using the following basic facts about spherical monogenics of complex degrees.

(i) If a spherical monogenic P_α is a homogeneous polynomial, then certainly $\alpha = k \in \mathbf{Z}^+$. But the converse is NOT true. In fact, there exist spherical monogenics $P_k(\underline{x})$ with $k \in \mathbf{Z}^+$ that are not polynomials. In that case $P_k(\underline{\omega})$ is not globally defined over S^{n-1}, i.e. there are singularities.

(ii) Whenever P_α is spherical monogenic of degree α, $P_\beta(\underline{x}) = \frac{\underline{x}}{|\underline{x}|^n} P_\alpha(\frac{\underline{x}}{|\underline{x}|^2})$ is spherical monogenic of the degree $\beta = -\alpha - n + 1$.

(iii) In particular, spherical monogenics of degree $-(k+n-1)$, $k \in \mathbf{Z}^+$, are the inverse of spherical monogenics of degree $k \in \mathbf{Z}^+$. In case they are the inverse of homogeneous polynomials they are defined everywhere outside the origin, as well as in all points $\underline{\omega} \in S^{n-1}$, i.e. $U = \{\mathbf{R}^+ \times S^{n-1}\}$.

(iv) There are no other spherical monogenics P_α which are defined all over S^{n-1}. Spherical monogenics of degree α not included in the discrete set $\{0, 1, 2, ...\} \cup \{-n + 1, -n, -n-1, ...\}$ can only be defined in proper subsets of S^{n-1}, i.e. they must have singularities. Even for $\alpha = k \in \mathbf{Z}^+$ and for $\alpha = -(k+n-1), k \in \mathbf{Z}^+$ there can be singularities.

EXAMPLE. (The homogeneous Radon kernel)

Let $\underline{t}, \underline{s} \in S^{n-1}$, and $< \underline{t}, \underline{s} >= 0$, i.e. $\underline{T} = \underline{t} + i\underline{s}$ is a nullvector. Let z^α be the generalized power of $z \in \mathbf{C}$ defined for $z \in \mathbf{C} \setminus L$, L being a line issuing from the origin. Then

$$R_\alpha(\underline{x}, \underline{T}) =< \underline{x}, \underline{T} >^\alpha \underline{T},$$

as a function of $\underline{x}$, is spherical monogenic of degree α in a suitable conic region in $\mathbf{R}^n$.

Moreover, in case $\underline{T_1}, \underline{T_2}$ are anti-commuting nullvectors, then the product of the two corresponding Radon kernels

$$< \underline{x}, \underline{T_1} >^\alpha < \underline{x}, \underline{T_2} >^\beta \underline{T_1 T_2}$$

is monogenic whenever it is defined. In the case $\alpha + \beta = 0$ we obtain monogenic functions which are homogeneous of degree 0.

Hence there are plenty of examples and there is no reason why one couldn't consider functions of the more general axial type

$$(A(x_0, r) + \underline{\omega} B(x_0, r)) P_\alpha(\underline{x})$$

defined for $\underline{\omega} \in O$ (open subset of S^{n-1}) and (x_0, r) belongs to an open subset of $\mathbf{R} \times \mathbf{R}^+$. In this case the equations of monogenicity reduce to the Vekua-type system with complex parameter:

$$\begin{aligned}
&\partial_0 A - \partial_r B = ((\alpha + n - 1)/r)B \\
&\partial_0 B + \partial_r A = (\alpha/r)A.
\end{aligned}$$

So here we do not have any restriction to α except for the fact that in case $\beta = -(\alpha + n - 1)$ we can always write $P_\beta(\underline{\omega}) = \underline{\omega} P_\alpha(\underline{\omega})$ and axial functions of the form $(A + \underline{\omega} B) P_\beta(\underline{\omega})$ may be rewritten into the form $(B - \underline{\omega} A) P_\alpha(\underline{\omega})$. It is, therefore, always sufficient to consider only one of the two cases. This allows one to restrict to the values of α for which e.g. $Re(\alpha) \geq -(n-1)/2$ and in particular when we consider restrictions to $\alpha \in \mathbf{Z}$ we only have to consider the case

$$\alpha \in \{-[(n-1)/2], ..., -1, 0, 1, 2, ...\}.$$

Now in this paper we want to produce solutions of the above Vekua system which can be written into the form concerned in the Fueter's Theorem:

$$\Delta^{\alpha + (n-1)/2}(f(x_0 + \underline{x}) P_\alpha(\underline{x})).$$

There are two cases.

Case 1. $\alpha + (n-1)/2$ is a non-negative integer (see §3).

In this case we can generalize the method used by Sce to obtain the Fueter's Theorem showing that the above expression is monogenic.

In case n is odd, we correspondingly consider the values $\alpha = -(n-1)/2, ..., -1, 0, 1, 2, 3, ...$ which are integers including the cases $\alpha = k \in \mathbf{Z}^+$ considered in [Sc] and [So] where $P_k(\underline{\omega})$ are globally defined. This restriction may be released.

In case n is even, we correspondingly deal with $\alpha = -(n-1)/2, ... -1/2, 1/2, 3/2, ...$which are half integers. In this case P_α is never globally defined but for Sce's method that gives no problem.

Case 2. $\alpha + (n-1)/2$ is a non-negative non-integer (see §4).

In this case one has to consider general powers of the Laplace operator defined in terms of integral operators (Fourier multipliers) as in [Q2]. Those operators are non-local pseudodifferential operators and the methods make use the fact that $P_\alpha(\underline{\omega})$ is spherical harmonic. It is natural to restrict ourselves to those cases for which $P_\alpha(\underline{\omega})$ have no singularities on S^{n-1}. Due to the fact that $\Delta^{\alpha+(n-1)/2}$ is non-local, any singularity of $P_\alpha(\underline{\omega})$ would indeed have an effect everywhere. So there are only the cases $\alpha = k \in \mathbf{Z}^+$ or $\alpha = -(k+n-1), k \in \mathbf{Z}^+$. We assume that $P_k(\underline{\omega})$ is globally defined. The space dimension $n+1$ in the case has to be odd as dealt with in §4.

In all other cases the Vekua system with complex powers may still be considered, but the solution of it via a generalized Fueter's Theorem may only work under the above made restrictions.

Note finally that in case $\alpha = -(n-1)/2$, no matter whether n is odd or even, we always assert that functions of the special form $f(x_0 + \underline{x})P_\alpha(\underline{x})$ are monogenic in the appropriate domain in $\mathbf{R}^n_1$.

3. The Case Where $k + (n-1)/2 \in \mathbf{Z}^+$ By Using Sce's Method. In this case we can take over almost literally the proof given in [So] for the restrictive case $k \in \mathbf{Z}^+$ based on Sce's method for $k = 0$ and n odd (For a further development of the method see [QS]).

We just repeat it with the adapted notations. Note that in this case there is no restriction on the dimension $n+1$ but we only assume that $\alpha + (n-1)/2 = s \in \mathbf{Z}^+$.

First note again that locally every holomorphic function $f(x+iy)$ in the real variables x and y may be written as $(\partial_x + i\partial_y)h(x,y)$, where h is harmonic. Then we still have that, where defined,

$$(\partial_0 - \underline{\partial})[h(x_0, r)P_\alpha(\underline{x})] = f(x_0 + \underline{x})P_\alpha(\underline{x}),$$

so that we have to verify the validity of the relation

$$[(\partial_0 + \underline{\partial})(\partial_0 - \underline{\partial})]^{s+1}[h(x_0, r)P_\alpha(\underline{x})] = 0.$$

Let us write

$$h_l(x_0, r)P_\alpha(\underline{x}) = \Delta^l(h(x_0, r)P_\alpha(\underline{x})).$$

That this may be done follows again from the fact that for any scalar function $g(x_0, r)$,

$$\underline{\partial}(g(x_0, r)P_\alpha(\underline{x})) = \underline{\omega}\partial_r(g)P_\alpha(\underline{x})$$

and

$$\begin{aligned}\underline{\partial}(\underline{\omega} g(x_0,r)P_\alpha(\underline{x})) &= \underline{\partial}(g(x_0,r)r^{2\alpha+n-1}[\frac{\underline{x}}{|\underline{x}|^n}P_\alpha(\frac{\underline{x}}{r^2})]) \\ &= -(\partial_r g + ((2\alpha+n-1)/r)g)P_\alpha(\underline{x}).\end{aligned}$$

So, as in [So], with $s = \alpha + (n-1)/2$,

$$\Delta(g(x_0,r)P_\alpha(\underline{x})) = (\partial_0^2 + \partial_r^2 + \frac{2s}{r}\partial_r)[g(x_0,r)]P_\alpha(\underline{x}).$$

From this, one obtains consequently,

$$\begin{aligned}&h_1(x_0,r)P_\alpha(\underline{x}) = \Delta(h(x_0,r)P_\alpha(\underline{x})) = (2s)(1/r\partial_r)(h)P_\alpha(\underline{x}),\\ &h_2(x_0,r)P_\alpha(\underline{x}) = 2s(2s-2)(1/r\partial_r)^2(h)P_\alpha(\underline{x}),\end{aligned}$$

etc. (also see [QS]), and, clearly, after $s+1$ steps we get zero.

Hence the Fueter principle is valid in all cases whereby $s = \alpha + (n-1)/2 \in \mathbf{Z}^+$.

Example. For $s = \alpha + (n-1)/2 = 0$ we obtain monogenic functions of the form $F(x) = f(x_0 + \underline{x})P_\alpha(\underline{x})$, f being holomorphic and $\alpha = -(n-1)/2$. Of special interest for applications are homogeneous monogenic of the form

$$F(x) = (x_0 + \underline{x})^l P_\alpha(\underline{x}), \quad l \text{ being complex,}$$

and in particular the functions

$$F(x) = \frac{(x_0+\underline{x})^l}{<\underline{x}, \underline{T}>^{(n-1)/2}}\underline{T}$$

which for the special values $l = k + (n-1)/2, k \in \mathbf{Z}^+$, provide more examples of spherical monogenics of degree $k \in \mathbf{Z}^+$ which are not polynomial, i.e. which have singularities after being restricted to the unit sphere.

Important Remark. The notation may be somewhat misleading because it refers to replacing the complex variable $x_0 + ir$ by $x_0 + \underline{\omega} r$. For example, if $g(x_0+ir) = if(x_0+ir)$ then one would mistakenly have $g(x_0+\underline{x}) = if(x_0+\underline{x})$ rather than $g(x_0+\underline{x}) = \underline{x}f(x_0+\underline{x})$ as it should be from the definition of $g(x_0+\underline{x})$ in §1, where one should write $g(x_0+ir) = u + iv, u, v$ being functions of x_0 and r and then replace i by $\underline{\omega}$. In this way $\Delta^{\alpha+(n-1)/2}\underline{\omega} f(x_0+\underline{x})P_\alpha(\underline{x})$ is also monogenic whenever f is holomorphic. Another way to look this is that the function $\underline{\omega} f(x_0+\underline{x})$ can also be written in the form $(\partial_0 + \underline{\omega}\partial_r)h(x_0,r)$, where $h(x_0,r)$ being scalar (real or complex) harmonic, and that is what matters in the proof of Fueter's Theorem in the case.

In other words, one can even multiply $f(x_0+\underline{x})$ with both the zero-divisors $(1/2)(1+i\underline{\omega})$ and $(1/2)(1-i\underline{\omega})$ to obtain other suitable input for Fueter's Theorem and the result of these multiplications is given by

$$(1/2)(1+i\underline{\omega})f(x_0-ir), \quad (1/2)(1-i\underline{\omega})f(x_0+ir),$$

$f(z)$ being holomorphic (also see [LMcQ]).

As important examples of this we mention the functions

$$\frac{(1/2)(1 \pm i\underline{\omega})}{(x_0 - a) + i(b \mp r)}$$

leading to the construction of "axial Cauchy kernels" of which the singularities include the codim 2 sphere $x_0 = a, r = b$. They are algebraic functions. Yet, from these functions one may construct

$$\frac{(1/2)(1 - i\underline{\omega})(x_0 - a + ib - ir) + (1/2)(1 + i\underline{\omega})(x_0 - a + ib + ir)}{(x_0 - a + ib)^2 + r^2}$$
$$= \frac{x_0 - a + ib - \underline{x}}{(x_0 - a + ib)^2 + r^2}$$

which is a rational function and becomes singular for $x_0 = a$ and $r = b$. After application of the classical Fueter-Sce Theorem as well as its extended version from [So] it stays rational and the order of the singularities on the codim 2 sphere is given by $2k + n$. Also, in the case where n is odd and $a = -(n-1)/2$, the plane wave function $< \underline{x}, \underline{T} >^{(1-n)/2} \underline{T}$ is rational and the product

$$\frac{(x_0 - a + ib - \underline{x})\underline{T}}{((x_0 - a + ib)^2 + r^2) < \underline{x}, \underline{T} >^{(n-1)/2}}$$

is rational, monogenic and has first order zeros on the sphere $x_0 = a, r = b$ as well as higher order zeros on the plane $< \underline{x}, \underline{T} >= 0$. It is a combination of a Cauchy-kernel and a Radon kernel which plays an important role in the transform analysis of monogenic functionals.

4. The Case $n+1$ Odd By Using Fourier Integral Operator. The method was first used in [Q2] to extend Fueter's and Sce's Theorems to the spaces $\mathbf{R}_1^n$ where $n+1$ is odd. We shall further develop this method in combining with certain intertwining relations between differential operators.

Note that $\Delta^{k+(n-1)/2}$ is defined through a certain Fourier multiplier (see [Q2]). In general, we will adopt the following definition: for $s \geq 0$,

$$\Delta^s g(x) = \mathcal{F}^{-1}((2\pi i|\cdot|)^{2s}\mathcal{F}g(\cdot)),$$

where

$$\mathcal{F}g(\xi) = \int_{\mathbf{R}_1^n} e^{2\pi i<x,\xi>} g(x)dx$$

is the Fourier transform of g, also denoted by $\hat{g}$, and

$$\mathcal{F}^{-1}h(x) = \int_{\mathbf{R}_1^n} e^{-2\pi i<x,\xi>} h(\xi)d\xi$$

is the inverse Fourier transform of h.

If k is a non-negative integer, then the Fourier multiplier definition of $\Delta^{k+(n-1)/2}$ coincides with the pointwise differentiation operator when $n+1$ is even ([Q2]). In the case $n+1$ is odd, our proof shows that the tempered distribution defined through the Fourier multiplier is actually identical to a function.

In this section unless stating exceptionally we always assume that $n+1$ is odd and k is a non-negative integer. We first show that for any $l \in \mathbf{Z}$, where $\mathbf{Z}$ is the set of integers, the function

$$\Delta^{k+(n-1)/2}\left((x_0+\underline{x})^l P_k(\underline{x})\right)$$

is left-monogenic.

We shall first deal with the negative power cases. Owing to the relation ([Q2], [Q3])

$$(x_0+\underline{x})^{-l} = \left(\frac{\overline{x}}{|x|^2}\right)^l = \frac{(-1)^{l-1}}{(l-1)!}\left(\frac{\partial}{\partial x_0}\right)^{l-1}\left(\frac{\overline{x}}{|x|^2}\right), \quad l = 1, 2, \cdots,$$

we are reduced to show that

$$\Delta^{k+(n-1)/2}\left(\frac{\overline{x}}{|x|^2}P_k(\underline{x})\right)$$

is left-monogenic.

Lemma 1. *$Q_{k+1}(x) = \overline{x}P_k(\underline{x})$ is harmonic and homogeneous of degree $k+1$.*

Proof. We observe that

$$\left(\frac{\partial}{\partial x_0}\right)^2 Q_{k+1}(x) = 0.$$

Using Leibniz's formula for second derivative, we obtain

$$\left(\frac{\partial}{\partial x_i}\right)^2 Q_{k+1}(x) = 2\left(\frac{\partial}{\partial x_i}\right)(\overline{x})\left(\frac{\partial}{\partial x_i}\right)P_k(\underline{x}) + \overline{x}\left(\frac{\partial}{\partial x_i}\right)^2 P_k(\underline{x}).$$

Adding together, we arrive

$$\Delta Q_{k+1}(x) = -2\underline{\partial}P_k(\underline{x}) + \overline{x}\underline{\Delta}P_k(\underline{x}) = 0. \qquad \square$$

The related proofs in [Q2] and [Q3] use the following Bochner type relation (for the restricted case $j=\alpha=1$): In the tempered distribution sense (see [St]),

$$\left(\frac{Q_j(\cdot)}{|\cdot|^{j+(n+1)-\alpha}}\right)^{\wedge}(\xi) = \gamma_{j,\alpha}\frac{Q_j(\cdot)(\xi)}{|\xi|^{j+\alpha}}, \qquad j \in \mathbf{Z}^+, 0<\alpha<n+1, \tag{1}$$

where Q_j is a harmonic, homogeneous polynomial of degree j, and

$$\gamma_{j,\alpha} = i^j\pi^{(n+1)/2-\alpha}\frac{\Gamma(j/2+\alpha/2)}{\Gamma(j/2+(n+1)/2-\alpha/2)}.$$

The relation (1) is equivalent to

$$\int_{\mathbf{R}_1^n} \frac{Q_j(x)}{|x|^{j+(n+1)-\alpha}}\hat{\phi}(x)dx = i^j\pi^{(n+1)/2-\alpha}\frac{\Gamma(j/2+\alpha/2)}{\Gamma(j/2+(n+1)/2-\alpha/2)}\int_{\mathbf{R}_1^n}\frac{Q_j(x)}{|x|^{j+\alpha}}\phi(x)dx,$$
$$j\in\mathbf{Z}^+, 0<\alpha<n+1,$$

where ϕ is any function in the Schwartz class in $\mathbf{R}_1^n$.

For the case $\alpha = 0$, the left-hand-side integral of (1) is replaced by the principal value integral (see [St]). For $j = 0$, the relation holds for $\alpha - (n+1) \notin 2\mathbf{Z}^+$ and also $-\alpha \notin 2\mathbf{Z}^+$ (See, for instance, [GS]). Now we need to extend (1) to the cases $\mathrm{Re}(\alpha) > -j,\ j \in \mathbf{Z}^+$.

We write the result in a symmetric way, and we have the extension of the above relation:

Lemma 2. *For $-j < \beta, \alpha < (n+1)+j, \alpha+\beta = n+1, j\in\mathbf{Z}^+$, we have*

$$\pi^{\beta/2}\Gamma(\frac{j+\beta}{2})\int_{\mathbf{R}_1^n}\frac{Q_j(x)}{|x|^{j+\beta}}\hat{\phi}(x)dx = \mathbf{i}^j\pi^{\alpha/2}\Gamma(\frac{j+\alpha}{2})\int_{\mathbf{R}_1^n}\frac{Q_j(x)}{|x|^{j+\alpha}}\phi(x)dx, \qquad (2)$$

where ϕ is any function in the Schwartz class in $\mathbf{R}_1^n$.

Proof. For $0<\alpha<n+1$ the both sides of (2) are holomorphic. For $j\geq 1$ we can show that the relation can be extended to all complex numbers α satisfying $\mathrm{Re}(\alpha) > -j$ through holomorphic continuation. In fact, due to the orthogonality property of spherical harmonics of different degrees, there follows

$$LHS = \lim_{\epsilon\to 0+}\int_{\epsilon<|x|\leq 1}\frac{Q_j(x)}{|x|^{j+(n+1)-\alpha}}\left(\hat{\phi}(x)-\hat{\phi}(0)-\cdots-\frac{(\sum_{i=0}^n x_i\frac{\partial}{\partial x_i})^{j-1}\hat{\phi}(0)}{(j-1)!}\right)dx$$
$$+\int_{|x|>1}\frac{Q_j(x)}{|x|^{j+(n+1)-\alpha}}\hat{\phi}(x)dx$$

that can be holomorphically extended to $\mathrm{Re}(\alpha) > -j$. The right-hand-side can also be holomorphically extend to this region. The proof is complete. $\square$

In Lemma 2, let $\alpha = 2-j$, we have

$$\lim_{\epsilon\to 0+}\int_{|x|>\epsilon}\frac{Q_j(x)}{|x|^{j+(n+1)+j-2}}\hat{\phi}(x)dx = \mathbf{i}^j\pi^{(n+1)/2+(j-2)}\frac{1}{\Gamma((n+1)/2+j-1)}\int_{\mathbf{R}_1^n}\frac{Q_j(x)}{|x|^2}\phi(x)dx.$$

Replacing ϕ by $\Delta^{k+(n-1)/2}\phi$ and j by $k+1$, we obtain

$$\lim_{\epsilon\to 0+}\int_{|x|>\epsilon}\frac{Q_{k+1}(x)}{|x|^{(n+1)+2k}}|x|^{2k+(n-1)}\hat{\phi}(x)dx = \beta_k\int_{\mathbf{R}_1^n}\Delta^{k+(n-1)/2}\left(\frac{Q_{k+1}(x)}{|x|^2}\right)\phi(x)dx,$$

where

$$\beta_k = 2^{1-n-2k}\mathbf{i}^{2-n-k}\pi^{-k-(n-1)/2}\frac{1}{\Gamma((n+1)/2+k)}.$$

That is

$$\int_{\mathbf{R}_1^n} \frac{Q_{k+1}(x)}{|x|^2}\hat{\phi}(x)dx = \beta_k \int_{\mathbf{R}_1^n} \Delta^{k+(n-1)/2}\left(\frac{Q_{k+1}(x)}{|x|^2}\right)\phi(x)dx.$$

Replacing Q_{k+1} by $\overline{x}P_k(\underline{x})$, we have

$$LHS = \int_{\mathbf{R}_1^n} \left(\frac{\bar{\cdot}}{|\cdot|^2}P_k(\underline{\cdot})\right)^{\wedge}(x)\phi(x)dx = \gamma_{1,n}^{-1}\int_{\mathbf{R}_1^n} E*(P_k(\underline{\partial}))\delta)(x)\phi(x)dx,$$

where $E(x) = \frac{\overline{x}}{|x|^{n+1}} = \gamma_{1,n}\left(\frac{\bar{\cdot}}{|\cdot|^2}\right)^{\hat{}}(x)$ is the Cauchy kernel in $\mathbf{R}_1^n$ and δ is the Dirac function. So,

$$\Delta^{k+(n-1)/2}\left(\frac{\overline{x}}{|x|^2}P_k(\underline{x})\right) = \gamma_{1,n}^{-1}\beta_k^{-1}E*(P_k(\underline{\partial})\delta)(x) = \gamma_{1,1}\beta_k^{-1}EP_k(\underline{\partial})(x).$$

This shows that the function

$$\Delta^{k+(n-1)/2}\left(\frac{\overline{x}}{|x|^2}P_k(\underline{x})\right),$$

and, therefore, all the functions

$$\Delta^{k+(n-1)/2}\left(\left(\frac{\overline{x}}{|x|^2}\right)^l P_k(\underline{x})\right), \qquad l\in\mathbf{Z}^+\setminus\{0\},$$

are left-monogenic. We in fact have the identity

$$\begin{aligned}&\Delta^{k+(n-1)/2}\left((x_0+\underline{x})^{-l}P_k(\underline{x})\right) \\ &= \gamma_{1,1}\beta_k^{-1}\frac{(-1)^{l-1}}{(l-1)!}\left(\frac{\partial}{\partial x_0}\right)^{l-1}EP_k(\underline{\partial})(x), \qquad l\in\mathbf{Z}^+\setminus\{0\}.\end{aligned} \tag{3}$$

Now we turn to the non-negative power cases and we are to show that for $l\in\mathbf{Z}^+$,

$$\Delta^{k+(n-1)/2}\left((x_0+\underline{x})^{l}P_k(\underline{x})\right)$$

is left-monogenic. We deal with these cases through an intertwining relation for the operator $D\Delta^{k+(n-1)/2}$.

LEMMA 3. *Let n be an even positive integer. Then, for $s = k+(n-1)/2$, we have*

$$(D\Delta^s)\left(\frac{\overline{x}}{|x|^{(n+1)-2s}}g(x^{-1})\right) = \alpha_{n,s}\frac{x}{|x|^{(n+1)+2s+2}}(D\Delta^s)(g)(x^{-1}), \tag{4}$$

where $\alpha_{n,s}$ is a constant depending on n and s, and g is any infinitely differentiable function in $\mathbf{R}_1^n\setminus\{0\}$.

Temporarily accepting Lemma 3, we proceed with the main proof as follows.

In Lemma 3, set $g(x) = (\frac{\overline{x}}{|x|^2})^l P_k(\underline{x}), \quad l \in \mathbf{Z}^+$. Noticing that $g(x^{-1}) = (-1)^k x^l |x|^{-2k} P_k(\underline{x})$, we obtain

$$(D\Delta^{k+(n-1)/2})((-1)^k x^{l-1} P_k(\underline{x})) = \alpha_{n,s} \frac{x}{|x|^{2n+2k+2}} (D\Delta^{k+(n-1)/2}) \left(\left(\frac{\bar{\cdot}}{|\cdot|^2} \right)^l P_k(\underline{\cdot}) \right) (x^{-1}).$$

In the early part of the proof we have shown that the right-hand-side vanishes, we thus conclude

$$(D\Delta^{k+(n-1)/2}) \left((x_0 + \underline{x})^{l-1} P_k(\underline{x}) \right) = 0, \quad l \in \mathbf{Z}^+.$$

To prove Lemma 3 we need first to study fundamental solutions of the operator $D\Delta^{k+(n-1)/2}$. In [PQ] we provide a list of fundamental solutions of the iterated Dirac operators $\underline{\partial}^l, l \in \mathbf{Z}^+$, in the context $\mathbf{R}^n$. To summarize, a fundamental solution for $\underline{\partial}^l$ is, essentially (i.e. apart from a positive multiple constant depending on n and l),

$$\frac{\underline{x}}{|\underline{x}|^{n-l+1}}, \quad \text{if } l \text{ is odd;} \quad \text{and} \quad \frac{1}{|\underline{x}|^{n-l}}, \quad \text{if } l \text{ is even;}$$

except for the cases where n is even and $l \geq n$ for which a fundamental solution is

$$(c \log |\underline{x}| + d) \frac{\underline{x}}{|\underline{x}|^{n-l+1}}, \text{ if } l \text{ is odd ;}$$

and

$$(c \log |\underline{x}| + d) \frac{1}{|\underline{x}|^{n-l}}, \text{ if } l \text{ is even .}$$

Now we are in the context $\mathbf{R}_1^n$ with $\underline{\partial}$ replaced by D. We are able to prove the following results.

In below we denote $2s = 2k + (n-1)$. So $2s$ may be even or odd. It is even if and only if $n+1$, the dimension of $\mathbf{R}_1^n$, is even.

The following result holds for both the cases n being even and odd.

LEMMA 4. *The operator $D|D|^{2s}$ in $\mathbf{R}_1^n$ has a fundamental solution of the same form as those in the above list for $\underline{\partial}^{2s+1}$ in $\mathbf{R}^{n+1}$, except that the term $\underline{x}$ in the latter is replaced by $\overline{x}$.*

Proof. We discuss two cases.

(i) $2s$ is even (The conclusion to this case will not be used in the following proof).

In that case the Fourier multiplier of a fundamental solution of $D|D|^{2s}$ is, apart from a multiple constant depending only on n and k,

$$\frac{1}{\xi} \frac{1}{|\xi|^{2s}} = \frac{\overline{\xi}}{|\xi|^{2s+2}}.$$

A fundamental solution of the operator $|D|^{2s+2}$ as a radial function is the same as the one in the above list for $\underline{\partial}^{2s+2}$ in $\mathbf{R}^{n+1}$. Denote it by $K(x)$. Note that in the present case the space dimension $n+1$ is even, so the fundamental solution obtained from the list is of the form

$$\frac{1}{|x|^{n-2s-1}},\ 2s+2<n+1; \quad \text{and} \quad (c\log|x|+d)\frac{1}{|x|^{n-2s-1}},\ 2s+2\geq n+1.$$

Then $\overline{D}K$ is a fundamental solution for $D|D|^{2s}$. The function $\overline{D}K$ is seen to be of the desired form

$$\frac{\overline{x}}{|x|^{n-2s+1}},\ 2s+2<n+1; \quad \text{and} \quad (c\log|x|+d)\frac{\overline{x}}{|x|^{n-2s+1}},\ 2s+2\geq n+1.$$

(ii) $2s$ is odd.

In that case

$$\frac{1}{\xi}\frac{1}{|\xi|^{2s}}=\frac{1}{|\xi|}\frac{\overline{\xi}}{|\xi|^{2s+1}}.$$

Now a fundamental solution of the operator $D|D|^{2s-1}$, corresponding to the Fourier multiplier

$$\frac{\overline{\xi}}{|\xi|^{2s+1}},$$

can be deduced through the list (the space dimension $n+1$ now is odd), and is of the form

$$\frac{\overline{x}}{|x|^{n-2s+2}}.$$

Since the Fourier transform of $1/|\xi|$ is Riesz potential $1/|x|^n$, a fundamental solution of $D|D|^{2s}$ may be obtained through the convolution

$$\frac{1}{|\cdot|^n}*\frac{\overline{(\cdot)}}{|\cdot|^{n-2s+2}}$$

in the tempered distribution sense.

We assert that the convolution itself is a locally integrable function away from the origin. In fact, the distribution, after being applied a certain times Laplace operator, becomes a local integrable function away from the origin. The assertion then follows from the corresponding result in generalized function theory (see, e.g., [GS]).

Secondly we note that the convolution as a distribution is of homogeneity of degree $2s-n$. To show this, we have got to show that for any test function ϕ in the Schwartz class, we have

$$(M*N(x),\phi(\frac{x}{\delta}))=\delta^{(n+1)+(2s-n)}(M*N(x),\phi(x)),$$

where M and N represent the generalized functions induced by

$$\frac{1}{|x|^n} \quad \text{and} \quad \frac{\overline{x}}{|x|^{n-2s+2}},$$

respectively.

Denote $\tau_\delta f(x) = f(\delta x)$. Owing to the homogeneous properties of M and N, we have

$$\begin{aligned}(M * N(x), \phi(\frac{x}{\delta})) &= (M * N(x), \tau_{\delta^{-1}}\phi(x)) \\ &= (N(x), M * (\tau_{\delta^{-1}}\phi)(x)) \\ &= \delta(N(x), \tau_{\delta^{-1}}M * \phi(x)) \\ &= \delta^{1+2s}(N, M * \phi) \\ &= \delta^{1+2s}(M * N, \phi).\end{aligned}$$

Next let ρ denotes any rotation about the origin in $\mathbf{R}_1^n$. Let ρ be represented by the matrix (ρ_{ij}) and the role of ρ on x, denoted by $\rho^{-1}x$, be given by the matrix vector multiplication $(\rho_{ij})(x)$, where (x) is understood as a column vector. Denote also by ρ its induced action on functions, $\rho(f)(x) = f(\rho^{-1}x)$. Now, since M is a scalar and N is a vector, the function $M * N$ is vector-valued, homogeneous of degree $2s - n$. Denote the vector-valued function

$$K(x) = M * N(x)$$

that is of homogeneity $n - 2s$. We have, owing to the rotational properties of M and N,

$$\begin{aligned}(\rho\overline{K(x)}, \phi(x) &= (\overline{K(x)}, \rho^{-1}\phi(x)) \\ &= (\overline{N(x)}, M * \rho^{-1}\phi(x)) \\ &= (\overline{N(x)}, \rho^{-1}M * \phi(x)) \\ &= (\rho\overline{N(x)}, M * \phi(x)) \\ &= (\overline{N(\rho^{-1}x)}, M * \phi(x)) \\ &= ((\rho_{ij})(\overline{N(x)}), M * \phi(x)) \\ &= (\rho_{ij})(\overline{N(x)}, M * \phi(x)) \\ &= (\rho_{ij})(\overline{K(x)}, \phi(x)) \\ &= ((\rho_{ij})(\overline{K(x)}), \phi(x)).\end{aligned}$$

We thus have

$$\overline{K}(\rho^{-1}x) = \rho(\overline{K(x)}).$$

Invoking the Lemma of Section 1.2, Chapter 3, [St], to $\overline{K(x)}/|x|^{2s-n}$, we conclude that

$$\overline{K(x)}/|x|^{2s-n} = C\frac{x}{|x|},$$

and thus

$$M * N(x) = C\frac{\overline{x}}{|x|^{n-2s+1}},$$

as desired. $\square$

Proof of Lemma 3. Since now $n+1$ is odd, we only concern the case (ii) in Lemma 4. Set $L = D\Delta^s = D|D|^{2s}$. Its fundamental solution is given by $G(x) = C\frac{\overline{x}}{|x|^{n-2s+1}}$. We have

$$\begin{aligned}
&L^{-1}\left(\frac{(\cdot)}{|(\cdot)|^{(n+1)+2s+2}}(Lg)((\cdot)^{-1})\right)(x^{-1}) \\
&\quad = \int_{\mathbf{R}_1^n} G(x^{-1}-y^{-1})\frac{y^{-1}}{|y^{-1}|^{(n+1)+2s+2}}\frac{1}{|y|^{2n+2}}(Lg)(y)dy \\
&\quad = C\frac{\overline{x^{-1}}}{|x^{-1}|^{n-2s+1}}\int_{\mathbf{R}_1^n}\frac{-\overline{(x-y)}}{|x-y|^{n-2s+1}}\frac{\overline{y^{-1}}}{|y^{-1}|^{n-2s+1}} \\
&\qquad \cdot\frac{y^{-1}}{|y^{-1}|^{(n+1)+2s+2}}\frac{1}{|y|^{2n+2}}(Lg)(y)dy \\
&\quad = C\frac{\overline{x^{-1}}}{|x^{-1}|^{n-2s+1}}\int_{\mathbf{R}_1^n}\frac{\overline{(x-y)}}{|x-y|^{n-2s+1}}(Lg)(y)dy \\
&\quad = C\frac{\overline{x^{-1}}}{|x^{-1}|^{n-2s+1}}g(x).
\end{aligned}$$

This concludes that

$$L\left(\frac{\overline{(\cdot)}}{|\cdot|^{(n+1)-2s}}g((\cdot)^{-1})\right)(x) = C\frac{x}{|x|^{(n+1)+2s+2}}(Lg)(x^{-1}).$$

The proof of Lemma 3 is complete. $\square$

Our next step is to show that the monogenicity of

$$\Delta^{k+(n-1)/2}((x_0+\underline{x})^l P_k(\underline{x})), \quad l \in \mathbf{Z},$$

induces that of

$$\Delta^{k+(n-1)/2}(f(x_0+\underline{x})P_k(\underline{x}))$$

in general, and thus conclude the theorem. Through a translation we may assume that the function f is holomorphic in a disc centered at the origin of the complex plane. We can further assume that the Taylor expansion of f has real coefficients by considering the associated holomorphic functions

$$g(z) = (1/2)(f(z)+\overline{f}(\overline{z})) \quad \text{and} \quad h(z) = (1/2\mathbf{i})(f(z)-\overline{f}(\overline{z})),$$

and the decomposition $f = g + \mathbf{i}h$. We are to show that the series

$$\sum_{l=-\infty}^{-1} c_l z^l \quad \text{and} \quad \sum_{l=-\infty}^{-1} c_l \Delta^{k+(n-1)/2}[(x_0+\underline{x})^l P_k(\underline{x})]$$

have the same convergence radius; and

$$\sum_{l=0}^{\infty} c_l z^l \quad \text{and} \quad \sum_{l=0}^{\infty} c_l \Delta^{k+(n-1)/2}[(x_0+\underline{x})^l P_k(\underline{x})]$$

have the same convergence radius.

For the negative power case, we refer to the quantitative relation (3) and the estimate (3) in Proposition 2, [Q3], and obtain

$$|\Delta^{k+(n-1)/2}[(x_0+\underline{x})^l P_k(\underline{x})]| \leq C(1+|l|)^{n+2k} \frac{1}{|x|^{n+k+|l|-1}}.$$

Based on the estimate, the two series of negative powers have the same convergence radius.

Now we consider the positive power series. Note that now n is even. Observe that $\Delta^s = |D|^{-1}\Delta^{k+n/2}$, and so a fundamental solution of Δ^s is the convolution of the Riesz potential $\frac{1}{|\cdot|^n}$ and a fundamental solution of $\Delta^{k+n/2}$, the latter being of the type (in the odd dimensional space)

$$C\frac{1}{|x|^{(n+1)-2s-1}},$$

where C is a constant depending on only n and k. This enables us to work out, in the spirit of Lemma 4, a fundamental solution of Δ^s of the form

$$C\frac{1}{|x|^{(n+1)-2s}}.$$

Following the proof of Lemma 3, we can deduce the intertwining relation

$$(\Delta^s)\left(\frac{1}{|x|^{(n+1)-2s}} g(x^{-1})\right) = C\frac{1}{|x|^{(n+1)+2s+2}}(\Delta^s)(g)(x^{-1}). \tag{5}$$

Replacing s by $s+1$, for $g(x) = (\frac{\overline{x}}{|x|^2})^l P_k(\underline{x})$, we have $g(x^{-1}) = (-1)^k x^l P_k(\underline{x})$, and

$$\Delta^{(k+1)+(n-1)/2}((-1)^k x^l P_k(\underline{x})) = C\frac{1}{|x|^{2n+2k+2}} \Delta^{(k+1)+(n-1)/2}\left(\left(\frac{\overline{x}}{|x|^2}\right)^l P_k(\underline{x})\right)(x^{-1}).$$

Applying Newton potential and relation (3), we have, in the distribution sense,

$$\Delta^{k+(n-1)/2}(x^l P_k(\underline{x})) = \frac{C}{(l-1)!}\int_{\mathbf{R}_1^n} \frac{1}{|x-y|^{n-1}} \frac{1}{|y|^{2n+2k+2}} \partial_0^{l-1} \Delta E P_k(\underline{x})(y^{-1}) dy.$$

Using the estimate obtained for the negative power terms, we conclude, through an argument concerning homogeneity as in Lemma 4, that

$$|\Delta^{k+(n-1)/2}[(x_0+\underline{x})^l P_k(\underline{x})]| \leq C(1+|l|)^{n+2k}|x|^{l-k-n+1}.$$

The above obtained estimate now guarantees that the two series of positive power entries have the same convergence radius.

REFERENCES

[BDS] F. Brackx, R. Delanghe and F. Sommen, *Clifford Analysis,* Research Notes in Mathematics, Vol. 76, Pitman Advanced Publishing Company, Boston, London, Melbourne (1982).

[F] R. Fueter, *Die Funktionentheorie der Differetialgleichungen* $\Delta u = 0$ *und* $\Delta\Delta u = 0$ *mit vier reellen Variablen,* Comm. Math. Helv., 7 (1935), pp. 307–330.

[GS] I.M. Gel'fand and G.E. Shilov, *Generalized functions,* translated by E. Saletan, Academic Press, New York and London, 1964.

[LMcQ] C. Li, A. McIntosh and T. Qian, *Clifford algebras, Fourier transforms, and singular convolution operators on Lipschitz surfaces,* Revista Matemática Iberoamericana, Vol. 10, No.3, (1994), pp. 665–721.

[PQ] J. Peetre and T. Qian, *Möbius covariance of iterated Dirac operators,* J. Austral. Math. Soc., Series A, 56 (1994), pp. 1–12.

[Q1] T. Qian, *Singular integrals on star-shaped Lipschitz surfaces in the quaternionic space,* Math. Ann., 310 (4) (April, 1998), pp. 601–630.

[Q2] T. Qian, *Generalization of Fueter's result to* $\mathbf{R}^{n+1}$, Rend. Mat. Acc. Lincei, s.9, v.8 (1997), pp. 111–117.

[Q3] T. Qian, *Fourier analysis on star-shaped Lipschitz surfaces,* accepted to appear in J. of Func. Annal.

[QS] T. Qian and F. Sommen, *Deriving harmonic functions in higher dimensional spaces,* preprint.

[Sc] M. Sce, *Osservazioni sulle serie di potenze nei moduli quadratici,* Atti Acc. Lincei Rend. fis., s.8, 23, 1957, pp. 220–225.

[So] F. Sommen, *On a generalization of Fueter's theorem,* Zeitschrift für Analysis und ihre Anwendungen, Journal for Analysis and its Applications, Vol. 19 (2000), N0.4, pp. 899–902.

[St] E.M. Stein, *Singular integrals and differentiability properties of functions,* Princeton University Press, 1970.

[SVL] F. Sommen and P. Van Lancker, *Homogeneous monogenic functions in Euclidean spaces,* Integral Transforms and Special Functions, 1998, Vol. 7, No. 3-4, pp. 285–298.

[VL1] P. Van Lancker, *Clifford Analysis on the Unit Sphere,* Ph.D. thesis, University of Gent, 1996.

[VL2] P. Van Lancker, *Clifford Analysis on the Sphere,* V. Dietrich et al. (eds.), Clifford Algebras and Their Applications in Mathematical Physics, Kluwer Acad. Publ., 1998, pp. 201–215.