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Analytic unit quadrature signals with nonlinear phase

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Abstract

The notion of intrinsic mode functions (IMFs) in the algorithm of Hilbert-Huang transform (HHT) [N.E. Huang, Z. Shen, S.R. Long, M.C. Wu, H.H. Shih, Q. Zheng, N.-C. Yen, C.C. Tung, H.H. Liu, The empirical mode decomposition and the Hilbert spectrum for nonlinear and non-stationary time series analysis, Proc. R. Soc. London A 454 (1998) 903–995] is essentially an engineering description in relation to mono-components of nonlinear and non-stationary signals. In this note we prove a version of Bedrosian's theorem on the unit circle. We give a sufficient condition together with an example for nonlinear phases $\theta(t)$ that make the unit quadrature signals $e^{i\theta(t)}$ to be analytic. We also establish a corresponding relationship between the periodic and non-periodic signals on the whole time range.

Keywords: Hilbert-Huang transform; Nonlinear and non-stationary signal; Möbius transform; Bedrosian's theorem

1. Introduction

The frequency of non-stationary signals varies with time. The traditional Fourier analysis, however, can not expose the time-varying property of frequency of non-stationary signals. This is due to the basic fact that in Fourier analysis a general signal is superposition of harmonic waves of which each has a constant frequency. In mathematics Fourier transform is a kind of univariate representation of signals in the time domain or the frequency domain separately, and thus does not enjoy the time-frequency localization. The latter leads to the study of windowed Fourier transform and wavelet transform, which are bivariate representations of signals in time and frequency domains simultaneously, and offer finite time-frequency localization [4]. They, on the other hand, both have the shortcoming that they use fixed time-frequency atoms to match a large variety of signals. It would be often the case that these fixed time-frequency atoms are not the *intrinsic components* of the signal under study. The ideal method of time-frequency analysis would be decomposition adapting a signal into certain basic intrinsic components which are mono-components [2,3]. For those components,

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one can define meaningful *instantaneous frequency* and furthermore construct time-frequency distribution.

Recently, Norden E. Huang presented a new time-frequency algorithm for nonlinear and non-stationary signal analysis: Hilbert-Huang transform (HHT) (see [8,6,7]). By using the algorithm of empirical mode decomposition (EMD), any multi-component can be decomposed into a finite sum of intrinsic mode functions (IMFs), which are essentially mono-components. The notion of IMF defined by Huang plays a crucial role in the HHT algorithm. The original concept of IMFs is an engineering description: The local maximums and minimums take turn to occur, and between a pair of adjacent local extremes, the signal is monotone and passes through the zero only once, and is of the local symmetry, i.e., the mean of any adjacent pair of upper and lower envelopes is of the zero value. Experiments show that IMFs behave nicely with Hilbert transform in the following sense [6,8]: Each term of the IMFs in the EMD, regarded as mono-component of the signal, is the real part of a complex-valued signal $f(t) = a(t)\,\mathrm{e}^{\mathrm{i}\theta(t)}$ satisfying the equation $\mathcal{H}(f)(t) = -\mathrm{i}\,f(t)$, where $\mathcal{H}(f)$ is the Hilbert transform of $f(t)$ on the real line, defined by

$$\mathcal{H}f(t) = \frac{1}{\pi}\,\text{v.p.}\int_{-\infty}^{\infty}\frac{f(x)}{t-x}\mathrm{d}x. \qquad (1.1)$$

Functions satisfying the equation $\mathcal{H}(f)(t) = -\mathrm{i}\,f(t)$ are called *analytic signals*. Through this representation *instantaneous frequencies* may be defined as the derivative of phase $\theta(t)$. This inspires us to ask: How to characterize functions a and θ such that the quadrature signal $a(t)\,\mathrm{e}^{\mathrm{i}\theta(t)}$ is analytic? In this note we restrict ourselves mainly to the unit circle (corresponding to the periodic case) and the unit quadrature case (corresponding to $a \equiv 1$). We provide a sufficient condition on non-linear functions $\theta(t)$ giving rise to analytic signals, i.e., satisfying $\mathcal{H}(\mathrm{e}^{\mathrm{i}\theta(t)}) = -\mathrm{i}\,\mathrm{e}^{\mathrm{i}\theta(t)}$, or, equivalently, $\mathcal{H}(\cos\theta(t)) = \sin\theta(t)$. An example is presented for such signals. There is a close relation between analytic signals and Bedrosian's theorem [1,10]. The unit circle (periodic) version of a Bedrosian's theorem is proved.

2. Hilbert transform on the circle

We study periodic signals on the whole time range, or, equivalently, study signals defined on compact intervals. In the case here is a corresponding analytic signal theory in the same pattern as in HHT. Without loss of generality, we assume that a signal $f(t)$ is defined in $[-\pi,\pi]$. We further assume that $f \in L^2([-\pi,\pi])$. Between f and its associated Fourier series there holds

$$f(t) = \sum_{k=-\infty}^{\infty} c_k\,\mathrm{e}^{\mathrm{i}kt},$$

where $c_n = c_n(f)$ is the nth Fourier coefficient of f, and the convergence is in the L^2-norm sense. Based on Carleson's Theorem the equivalence holds also in the almost everywhere point-wise convergence sense.

The Hilbert transform of $f(t), t \in [-\pi,\pi]$, or on the unit circle, is defined through Fourier multiplier, by

$$\tilde{\mathcal{H}}(f)(t) = -\mathrm{i}\sum_{k=-\infty}^{\infty}\mathrm{sgn}(k)c_k\,\mathrm{e}^{\mathrm{i}kt}, \qquad (2.1)$$

where $\mathrm{sgn}(k)$ is the signum function

$$\mathrm{sgn}(k) = \begin{cases} 1, & k = 1, 2, \ldots \\ -1, & k = -1, -2, \ldots \\ 0, & k = 0. \end{cases} \qquad (2.2)$$

Accordingly, we have

$$f(t) + \mathrm{i}\tilde{\mathcal{H}}(f)(t) = c_0 + 2\sum_{k=1}^{\infty} c_k\,\mathrm{e}^{\mathrm{i}kt}. \qquad (2.3)$$

It is easy to show that c_k are bounded as $f \in L^2([-\pi,\pi])$. As consequence, the series $c_0 + 2\sum_{k=1}^{\infty} c_k z^k$ converges to an analytic function for $|z| < 1$. By writing $z = r\,\mathrm{e}^{\mathrm{i}t}$, $0 \le r < 1$, this further implies that $f(t) + \mathrm{i}\tilde{\mathcal{H}}(f)(t)$ is the boundary value of the above defined analytic function in the unit disc.

We note that the circular Hilbert transform $\tilde{\mathcal{H}}$ has the property

$$\tilde{\mathcal{H}}^2(f)(t) = -f(t) + a, \qquad (2.4)$$

where a is a complex number.

It is known (see, for instance, [12] or [9]) that, based on the Fourier series version of Parseval's formula, the above defined circular Hilbert transform has an alternative form as a singular integral:

$$\tilde{\mathcal{H}}f(t) = \frac{1}{2\pi}\lim_{\epsilon\to 0^+}\int_{\epsilon<|s|\le\pi}\cot\left(\frac{s}{2}\right)f(t-s)\,\mathrm{d}s, \quad \text{a.e.} \qquad (2.5)$$

Using its Fourier multiplier definition one easily gets, for instance,

$$\tilde{\mathcal{H}}(\cos nt) = \sin nt.$$

Indeed, $\cos nt = 1/2(\mathrm{e}^{-\mathrm{i}nt} + \mathrm{e}^{-\mathrm{i}nt})$. Hence,

$$\begin{aligned}\tilde{\mathcal{H}}(\cos nt) &= \frac{-\mathrm{i}}{2}(\mathrm{sgn}(-n)\mathrm{e}^{-\mathrm{i}nt} + \mathrm{sgn}(n)\mathrm{e}^{\mathrm{i}nt})\\ &= \frac{-\mathrm{i}}{2}2\mathrm{i}\sin nt = \sin nt.\end{aligned}$$

Similarly, $\tilde{\mathcal{H}}(\sin nt) = -\cos nt$.

3. Analytic signals on circle

Suppose that with $f \in L^2([-\pi,\pi])$ we obtained $\tilde{\mathcal{H}}f$, and, as deduced in the previous section, $f(t) + \mathrm{i}\tilde{\mathcal{H}}f(t)$ is the boundary value of an analytic function in the unit disc. We write

$$f(t) + \mathrm{i}\tilde{\mathcal{H}}f(t) = \rho(t)\,\mathrm{e}^{\mathrm{i}\theta(t)}, \tag{3.1}$$

where

$$\rho(t) = \sqrt{f^2(t) + (\tilde{\mathcal{H}}f(t))^2}, \quad \cos\theta(t) = \frac{f(t)}{\rho(t)},$$

$$\text{and} \quad \sin\theta(t) = \frac{\tilde{\mathcal{H}}f(t)}{\rho(t)}. \tag{3.2}$$

For any real-valued function $f \in L^2([-\pi,\pi])$, we may associate it with a complex-valued function, $\mathcal{A}[f](t)$, defined, in the above notation, by

$$\mathcal{A}[f](t) = \rho(t)\,\mathrm{e}^{\mathrm{i}\theta(t)}. \tag{3.3}$$

We call $\mathcal{A}[f](t)$ the *analytic signal associated with f.* From (2.4) one easily deduces that

$$\tilde{\mathcal{H}}\mathcal{A}[f](t) = -\mathrm{i}(\mathcal{A}[f](t) - a),$$

where a is a complex number. On the other hand, if $F = f + \mathrm{i}g$ satisfies

$$\tilde{\mathcal{H}}F = -\mathrm{i}(F - a),$$

then modular constants $\tilde{\mathcal{H}}f = g$.

The following Bedrosian's theorem in the circular case is expected.

Theorem 3.1. *Suppose that* $f_1, f_2 \in L^2([-\pi,\pi])$. *Then*

$$\mathcal{A}[f_1 f_2] = f_1\mathcal{A}[f_2] \tag{3.4}$$

if there exists $K \in \mathbb{Z}^+ \cup \{0\}$ *such that*

$$c_n(f_1) = 0, \quad |n| > K \quad \text{and} \quad c_n(f_2) = 0, \quad |n| \le K.$$

Proof. Let

$$f_1(t) = \sum_{n=-K}^{K} c_n(f_1)\,\mathrm{e}^{\mathrm{i}nt}, \quad \text{and}$$

$$f_2(t) = \left(\sum_{m=-\infty}^{-K-1} + \sum_{m=K+1}^{\infty}\right) c_m(f_2)\,\mathrm{e}^{\mathrm{i}mt}.$$

Since $\mathcal{A}[f_1 f_2]$ keeps all $c_n(f_1 f_2)$, $n \ge 0$ and kills all $c_n(f_1 f_2)$, $n < 0$ we have

$$\begin{aligned}\mathcal{A}[f_1 f_2] &= \sum_{n+m=0} c_n(f_1)c_m(f_2)\\ &\quad + 2\sum_{k=1}^{\infty}\left(\sum_{n+m=k} c_n(f_1)c_m(f_2)\right)\mathrm{e}^{\mathrm{i}kt}\\ &= 2\sum_{k=1}^{\infty}\sum_{n=-K}^{K} c_n(f_1)c_{k-n}(f_2)\,\mathrm{e}^{\mathrm{i}kt}.\end{aligned}$$

Note that the Fourier coefficients $c_m(f_2)$ for $m < -K$ do not play any role in the last expression. On the other hand,

$$\begin{aligned}&f_1\mathcal{A}[f_2]\\ &\quad = \left(\sum_{n=-K}^{K} c_n(f_1)\,\mathrm{e}^{\mathrm{i}nt}\right) 2\left(\sum_{m=K+1}^{\infty} c_m(f_2)\,\mathrm{e}^{\mathrm{i}mt}\right)\\ &\quad = 2\sum_{k=1}^{\infty}\sum_{n=-K}^{K} c_n(f_1)c_{k-n}(f_2)\,\mathrm{e}^{\mathrm{i}kt}.\end{aligned}$$

Therefore, we have $\mathcal{A}[f_1 f_2] = f_1\mathcal{A}[f_2]$, and then we complete the proof. □

Corollary 3.2. *Let* f_1 *and* f_2 *be real-valued. If*

$$c_n(f_1) = 0 \quad \textit{for} \quad n < -K$$

and

$$c_m(\mathcal{A}[f_2]) = 0, \quad \textit{for} \quad m \le K.$$

Then $\mathcal{A}[f_1 f_2] = f_1 \mathcal{A}[f_2]$.

Proof. Since f_1, f_2 are real-valued, their Fourier coefficients satisfy the Hermitian relation $\overline{c_n(f_i)} = c_{-n}(f_i)$. The assertion then follows from Theorem 3.1. □

The last corollary shows that if $f(t) = \rho(t)\cos\theta(t)$ and the amplitude ρ has low frequencies and $\cos\theta(t)$ has high frequencies such that the two frequencies ranges are disjoint, then

$$\tilde{\mathcal{H}}(\rho(t)\cos\theta(t)) = \rho(t)\tilde{\mathcal{H}}(\cos\theta(t)).$$

We therefore are interested in finding those phases $\theta(t)$ for which

$$\theta'(t) \ge 0, \quad \text{and} \quad \tilde{\mathcal{H}}(\cos\theta(t)) = \sin\theta(t),$$

as in such cases the signal

$$\rho(t)\,\mathrm{e}^{\mathrm{i}\theta(t)}$$

is an analytic signal.

In the sequel by *unit analytic signals on the circle* we mean those

$$f(t) = \mathrm{e}^{\mathrm{i}\theta(t)}, \quad \text{with} \quad \theta'(t) \ge 0,$$
$$\text{and} \quad \tilde{\mathcal{H}}(\cos\theta(t)) = \sin\theta(t). \tag{3.5}$$

The concept *unit analytic signals on the line* is defined similarly where the circular Hilbert transform $\tilde{\mathcal{H}}$ is replaced by the Hilbert transform $\mathcal{H}$ (see (1.1)) on the real line.

4. Existence of non-trivial unit analytic signals

As trivial example of unit analytic signals we have, for any integer n,

$$\begin{aligned}\tilde{\mathcal{H}}(\cos(nt+b)) &= \tilde{\mathcal{H}}(\cos nt\cos b - \sin nt \sin b)\\ &= \cos b\tilde{\mathcal{H}}(\cos nt) - \sin b\tilde{\mathcal{H}}(\sin nt)\\ &= \cos b \sin nt + \sin b\cos nt\\ &= \sin(nt+b).\end{aligned}$$

They are the cases with linear phases. For non-linear unit analytic signals we have the following result.

Theorem 4.1. *Let a be a complex number such that $|a| < 1$, and $\tau_a(z) = (z-a)/(1-\bar{a}z)$ be the corresponding Möbius transform. Then the unimodular function $\tau_a(\mathrm{e}^{\mathrm{i}t})$, $t \in [-\pi, \pi]$, is a unit analytic signal.*

Note that τ_a conformally maps the unit disc to the unit disc, a to 0, and the unit circle to the unit circle. Thus the parametric function $\tau_a(\mathrm{e}^{\mathrm{i}t})$ is of modular one.

Proof. Write $\tau_a(\mathrm{e}^{\mathrm{i}t}) = \mathrm{e}^{\mathrm{i}\theta_a(t)}$. From the knowledge of Möbius transform we know that the parametric function $\theta_a(t)$ is strictly increasing and $\theta(\pi) - \theta(-\pi) = 2\pi$. The function is in fact absolutely continuous and the derivative satisfies

$$\frac{1}{2\pi}\frac{\mathrm{d}\theta_a(t)}{\mathrm{d}t} = \frac{1}{2\pi}\frac{1-|a|^2}{1-2|a|\cos(t-t_a)+|a|^2} =: p_a(t) > 0,$$

where $a = |a|\,\mathrm{e}^{\mathrm{i}t_0}$, and p_a is the Poisson kernel for the point a (see [5]).

The function $\tau_a(z)$ is analytic in an open neighborhood of the closed unit disc, thus it can be expressed by the Cauchy integral over the unit circle:

$$\tau_a(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\mathrm{e}^{\mathrm{i}\theta_a(s)}}{\mathrm{e}^{\mathrm{i}s}-z}\mathrm{d}s, \quad z = r\,\mathrm{e}^{\mathrm{i}t}.$$

Letting $r \to 1^-$, by the Plemelj's formula, we have

$$\mathrm{e}^{\mathrm{i}\theta(t)} = \tfrac{1}{2}\,\mathrm{e}^{\mathrm{i}\theta(t)} + \tfrac{1}{2}\mathrm{i}\tilde{\mathcal{H}}\,\mathrm{e}^{\mathrm{i}\theta(t)}.$$

The last equality is further reduced to

$$\mathrm{e}^{\mathrm{i}\theta(t)} = \mathrm{i}\tilde{\mathcal{H}}\,\mathrm{e}^{\mathrm{i}\theta(t)}.$$

Substituting $\mathrm{e}^{\mathrm{i}\theta(t)}$ with $\cos\theta_a(t) + \mathrm{i}\sin\theta_a(t)$ in the above equality, and comparing the real- and the imaginary-parts of the obtained equality, we arrive

$$\tilde{\mathcal{H}}\cos\theta_a(t) = \sin\theta_a(t),$$

as desired, and, even further,

$$\tilde{\mathcal{H}}\sin\theta_a(t) = -\cos\theta_a(t).$$

The proof is complete. □

For a further generalization of the theory please see [11].

We present an example here. Taking $a = 1/2$ in Theorem 4.1, we have

$$e^{i\theta_a(t)} = c(t) + is(t),$$

where

$$c(t) = \frac{5\cos t - 4}{5 - 4\cos t}, \quad s(t) = \frac{3\sin t}{5 - 4\cos t}.$$

The theorem asserts that the function $c(t) + is(t)$ is a unit analytic signal. As verification of the theorem, now we prove this fact using elementary computation. We shall show

(i) $c^2(t) + s^2(t) = 1$;
(ii) $s'(t) = h(t)c(t), \quad h(t) \geq 0$;
(iii) $\tilde{\mathcal{H}}(c(t)) = s(t)$.

Under (i)–(iii), we may write

$$c(t) = \cos\theta(t), \quad s(t) = \sin\theta(t),$$

$$\theta(t) = \int_0^t h(u)\,du, \quad \tilde{\mathcal{H}}(\cos\theta(t)) = \sin\theta(t).$$

Now,

$$c^2(t) + s^2(t) = \frac{(5\cos t - 4)^2 + (3\sin t)^2}{(5 - 4\cos t)^2} = \frac{25\cos^2 t + 16 - 40\cos t + 9\sin^2 t}{25 + 16\cos^2 t - 40\cos t}.$$

On replacing 16 by $16\sin^2 t + 16\cos^2 t$ in the numerator, we have $c^2(t) + s^2(t) = 1$. This proves (i).

Next, we have

$$s'(t) = \frac{d}{dt}\left(\frac{3\sin t}{5 - 4\cos t}\right) = \frac{15\cos t - 12}{(5 - 4\cos t)^2}.$$

Therefore

$$\frac{s'(t)}{c(t)} = \frac{3}{5 - 4\cos t} > 0.$$

To show (iii), we point out that

$$c(t) = -\tfrac{1}{2} + \tfrac{3}{4}\cos t + \cdots + \tfrac{3}{2^{n+1}}\cos nt + \cdots,$$
$$s(t) = \tfrac{3}{4}\sin t + \cdots + \tfrac{3}{2^{n+1}}\sin nt + \cdots.$$

The circular Hilbert transform may be applied term by term to the series $c(t)$, and we then obtain (iii). The term by term operation is justified by the L^2-boundedness of the circular Hilbert transform (Riesz Theorem) and by the convergence in the L^2-sense of series representing the function $c(t)$. Figs. 1 and 2 illustrate the plots of the signal $f(t) = (5\cos t - 4)/(5 - 4\cos t)$ and its phase in the Hilbert domain, in one period and in several periods, respectively.

From (i) to (iii), we know that, the phase

$$\theta(t) = \arctan\frac{\tilde{\mathcal{H}}c(t)}{c(t)} = \arctan\frac{3\sin t}{5\cos t - 4},$$

satisfies that

$$\theta'(t) = h(t), \quad \cos\theta(t) = c(t), \quad \text{and}$$

$$\tilde{\mathcal{H}}\cos\theta(t) = \tilde{\mathcal{H}}c(t) = s(t) = \sin\theta(t).$$

That is to say that $e^{i\theta(t)}$ is an analytic signal.

5. Signals on the whole time range

This section will be based on the Cayley transform that conformally maps the upper-half-complex plane to the unit disc

$$\omega = \frac{z - i}{z + i}.$$

It maps the real line to the unit circle via

$$\omega = \frac{t - i}{t + i} = \frac{t^2 - 1}{t^2 + 1} + i\frac{2t}{t^2 + 1}.$$

On letting

$$\cos s = \frac{t^2 - 1}{t^2 + 1} = -\frac{1 - \tan^2(s/2)}{1 + \tan^2(s/2)},$$

$$\text{and} \quad \sin s = \frac{2t}{t^2 + 1} = \frac{2\tan(s/2)}{1 + \tan^2(s/2)},$$

we have

$$s = 2\tan^{-1} t.$$

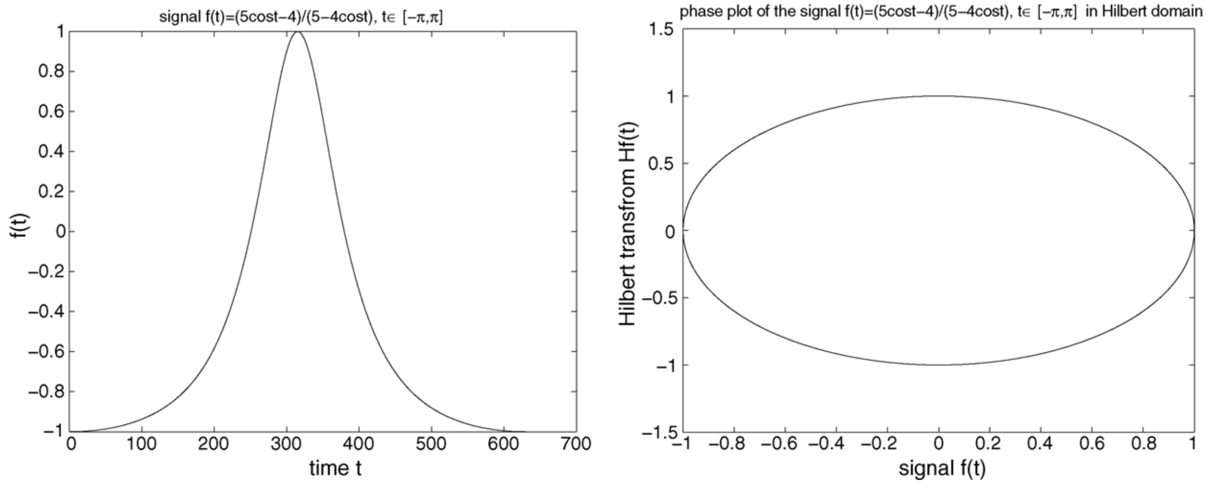

Fig. 1. The signal $f(t) = (5\cos t - 4)/(5 - 4\cos t)$ in one period and its phase plot in Hilbert domain.

Owing to the conformal property of the mapping analytic functions on the upper-half-plane are mapping to analytic functions on the unit disc, and, vice versa. So, if $F(s) = \cos\theta(s) + \mathrm{i}\sin\theta(s)$ is a unit analytic function defined on the unit circle, then

$$f(t) = \cos\theta(2\tan^{-1} t) + \mathrm{i}\sin\theta(2\tan^{-1} t)$$

is a unit analytic function on the upper-half-plane, and we have

$$\mathcal{H}(\cos\theta(2\tan^{-1} t)) = \sin\theta(2\tan^{-1} t).$$

Now we use the example in the previous section to obtain a unit analytic signal on the real line. Note that from $s = 2\tan^{-1} t$ we have

$$f(t) = c(2\tan^{-1} t) = \frac{1 - 9t^2}{1 + 9t^2}, \quad t \in \mathbb{R}.$$

The Hilbert transform of $f(t)$ is

$$\mathcal{H}f(t) = s(2\tan^{-1} t) = \frac{6t}{1 + 9t^2}, \quad t \in \mathbb{R}.$$

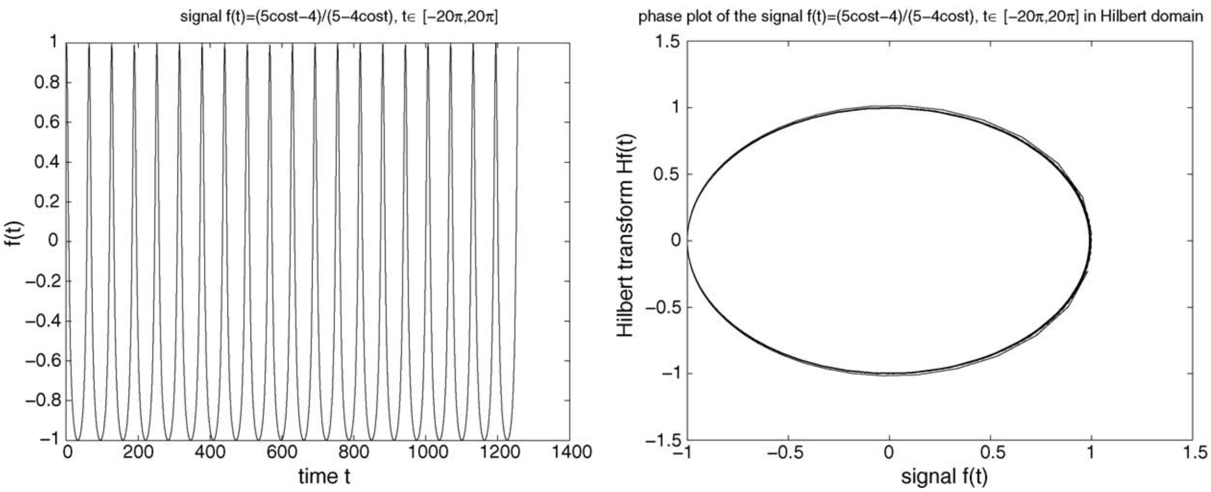

Fig. 2. The signal $f(t) = (5\cos t - 4)/(5 - 4\cos t)$ in several periods and its phase plot in Hilbert domain.

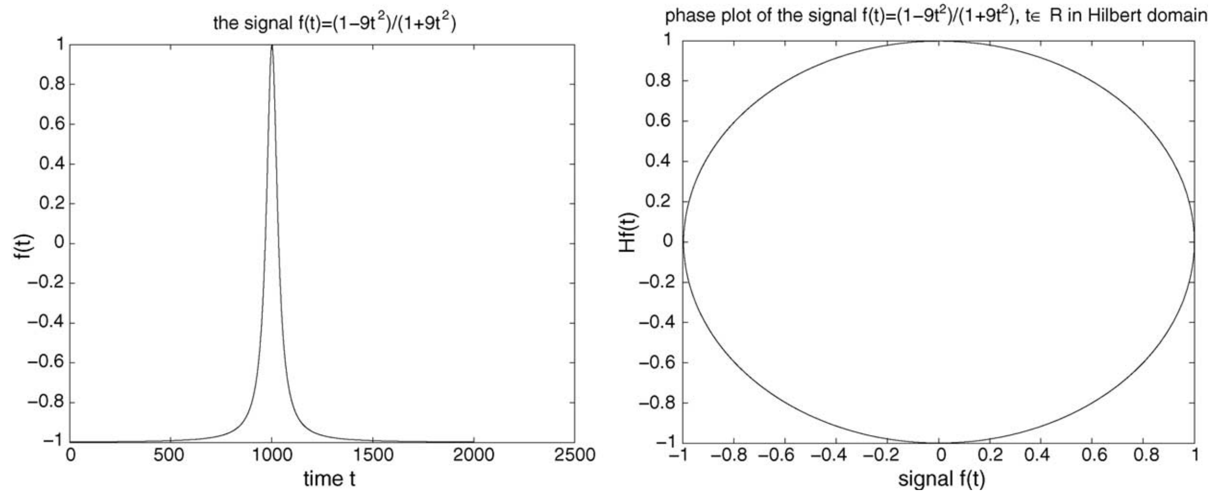

Fig. 3. The signal $f(t) = (1-9t^2)/(1+9t^2)$, $t \in \mathbb{R}$ and its phase plot in Hilbert domain.

As verification the last relation can be obtained from direct calculation:

$$\frac{1}{\pi}\,\text{v.p.}\int_{-\infty}^{\infty}\frac{1-9x^2}{1+9x^2}\frac{1}{t-x}\mathrm{d}x$$

$$=\frac{2}{\pi}\,\text{v.p.}\int_{-\infty}^{\infty}\frac{1}{(9x^2+1)(t-x)}\mathrm{d}x=\frac{18}{\pi(9t^2+1)}$$

$$\times\,\text{v.p.}\int_{-\infty}^{\infty}\left(\frac{x}{9x^2+1}+\frac{t}{9x^2+1}+\frac{1}{t-x}\right)\mathrm{d}x$$

$$=\frac{18t}{\pi(9t^2+1)}\,\text{v.p.}\int_{-\infty}^{\infty}\frac{1}{9x^2+1}\mathrm{d}x=\frac{6t}{1+9t^2}.$$

The phase is

$$\phi(t)=\arctan\frac{6t}{1-9t^2},\quad t\in\mathbb{R}.$$

Its derivative is the Poisson kernel of the real line and thus is always positive. Fig. 3 illustrates the plots of the signal $f(t) = (1-9t^2)/(1+9t^2)$ and its phase in the Hilbert domain.

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