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CHARACTERIZATION OF BOUNDARY VALUES OF FUNCTIONS IN HARDY SPACES WITH APPLICATIONS IN SIGNAL ANALYSIS

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ABSTRACT. In time-frequency analysis Hilbert transformation is used to define analytic signals based on which meaningful instantaneous amplitude and instantaneous frequency are defined. In relation to this background we study characteristic properties of the real-valued measurable functions $\rho(t)$ and $\theta(t)$, $t \in \mathbf{R}$, such that

$$H(\rho(\cdot)\cos\theta(\cdot))(t) = \rho(t)\sin\theta(t), \quad \rho(t) \geq 0,$$

where H is the Hilbert transformation on the line. A weaker form of this equation is

$$H(\rho(\cdot)c(\cdot))(t) = \rho(t)s(t), c^2 + s^2 = 1, \quad \rho(t) \geq 0.$$

We prove that a characterization of a triple (ρ, c, s) satisfying the equation with $\rho \in L^p(\mathbf{R})$, $1 \leq p \leq \infty$, is that $\rho(c + is)$ is the boundary value of an analytic function in the Hardy space $H^p(\mathbf{C}^+)$ in the upper-half complex plane $\mathbf{C}^+$. We will be dealing with parameterized and non-parameterized solutions combined with the cases $\rho \equiv 1$ and $\rho \not\equiv 1$. The counterpart theory in the unit disc is formulated first. The upper-half complex plane case is solved by converting it to the unit disc through Cayley transform. Examples in relation to signal analysis are constructed.

1. Introduction. Hilbert transform of a nice function f on the real line is defined by the principal value singular integral

$$Hf(t) = \text{p.v.}\, \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{t-x}\, f(x)\, dx. \tag{1}$$

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If $f \in L^p(\mathbf{R})$, $1 \le p < \infty$, then Hf is well defined [**13**]. The Fourier multiplier form of the Hilbert transformation, when appropriate, is

$$Hf(t) = \frac{1}{2\pi}\int_{\mathbf{R}} e^{i\xi t}(-i\,\mathrm{sgn}\,\xi)\hat{f}(\xi)\,d\xi, \tag{2}$$

where $-i\,\mathrm{sgn}$ is the corresponding Fourier multiplier, and sgn is the signum function taking values $1, -1$ or 0 for $\xi > 0$, $\xi < 0$ or $\xi = 0$, respectively. In this paper, the Fourier transform of a function f in $L^1(\mathbf{R})$ is defined by

$$\hat{f}^(\xi) = \int_{-\infty}^{\infty} e^{-it\xi} f(t)\,dt.$$

It can be easily observed from the Fourier multiplier form of the Hilbert transformation that $H^2 = -I$, where I denotes the *identity operator*. In Section 4, to define the Hilbert transform of essentially bounded functions, we will adopt the definition of Hilbert transformation for distributions. In below all functions discussed are assume to be measurable in the Lebesgue sense.

In time-frequency analysis to define instantaneous amplitude and frequency of a given signal, that is a function, f, one first formulates the associated *analytic signal*, $f+iHf$, denoted by Af and then further writes

$$Af(t) = f(t) + iHf(t) = \rho(t)e^{i\theta(t)},$$

where

$$\rho(t) = \sqrt{f^2(t) + (Hf(t))^2}, \quad \cos\theta(t) = \frac{f(t)}{\sqrt{|f(t)|^2 + |Hf(t)|^2}}.$$

As a consequence,

$$f(t) = \rho(t)\cos\theta(t), \tag{3}$$

is called the *canonical modulation* of f, and the related pair (ρ, θ) is called the *canonical pair*. Note that, depending on what type of function θ one wants, the angular parametrization θ may not always be possible. If not possible, the above can be written as

$$f(t) + iHf(t) = \rho(t)(c(t) + is(t)), \quad \rho(t) \ge 0, \quad c^2(t) + s^2(t) = 1,$$

and

$$f(t) = \rho(t)c(t).$$

In case an angular parametrization

$$c(t) = \cos\theta(t), \quad \theta'(t) \geq 0, \quad \text{a.e.} \tag{4}$$

exists, then the corresponding $\rho(t)$ and $\theta(t)$ are defined to be the *instantaneous amplitude* and the *instantaneous phase* of f, respectively, and $\theta'(t)$ the *instantaneous frequency*, at time t. Since cosine is periodic, such a function θ is unique under the *minimum increase* property. That is, let $\Theta_i(t)$, $i \in I$, be the collection of all the non-decreasing functions satisfying (4). Then obviously

$$\theta(t) = \min\{\Theta_i(t) : i \in I\}$$

is also one of them. Such defined θ is called the *minimum angular parametrization*, or the *angular parametrization* of f.

The construction of analytic signals gives rise to the singular integral equation

$$H(\rho(\cdot)\cos\theta(\cdot)) = \rho(t)\sin\theta(t), \quad \rho(t) \geq 0, \quad \theta'(t) \geq 0, \quad \text{a.e.} \tag{5}$$

When the existence of angular parametrization θ is unknown, in equation (5) the functions $\cos\theta(t)$ and $\sin\theta(t)$ are replaced by two real-valued functions $c(t)$ and $s(t)$:

$$H(\rho(\cdot)c(\cdot))(t) = \rho(t)s(t), \quad \rho(t) \geq 0, \quad c^2(t) + s^2(t) = 1, \quad \text{a.e.} \tag{6}$$

Due to the relation $H^2 = -I$, if a pair (ρ, θ) is a solution of (5), then the pair is also a solution of

$$H(\rho(\cdot)\sin\theta(\cdot)) = -\rho(t)\cos\theta(t). \tag{7}$$

The latter is called the *conjugate equation* of (5). Similarly the conjugate equation of (6) is

$$H(\rho(\cdot)s(\cdot))(t) = -\rho(t)c(t), \quad \rho(t) \geq 0, \quad c^2(t) + s^2(t) = 1, \quad \text{a.e.} \tag{8}$$

In the HHT algorithm [**4**] a signal is expanded into a series of terms, each is of the form $\rho(t)\cos\theta(t)$, called *intrinsic mode functions* (IMFs). The IMFs are expected to satisfy the relation (5).

In what follows we call the equations (6) and (10) the *non-parameterized equations*, and their solutions *non-parameterized solutions*, and for the equations (5) and (9) below, the *parameterized equations* and *parameterized solutions*, respectively.

The cases $\rho \equiv 1$ is intrinsically related to Bedrosian's theorem [**1**]. The theorem asserts that if $\mathrm{supp}\hat{f} \subset [-\alpha,\alpha], \mathrm{supp}\,\hat{g} \subset \mathbf{R}\setminus[-\alpha,\alpha]$, then

$$H(fg) = fHg.$$

Now assume that ρ and $\cos\theta(t)$, in the positions of f and g above, satisfy the spectrum requirements of Bedrosian's theorem. Then we have

$$H(\rho(\cdot)\cos\theta(\cdot))(t) = \rho(t)H(\cos\theta(\cdot))(t).$$

So, in this case, the Hilbert transform relation in (5) holds if and only if

$$H(\cos\theta(\cdot))(t) = \sin\theta(t). \tag{9}$$

If ρ and c satisfy the theorem in the same pattern, then (6) holds if and only if

$$Hc(t) = s(t), \quad c^2 + s^2 = 1. \tag{10}$$

In such a way we are reduced to solve (9) and (10) in the respective cases.

Note that the conjugate equations of (9) and (10) are

$$H(\sin\theta(\cdot))(t) = -\cos\theta(t) \tag{11}$$

and

$$Hs(t) = -c(t), \quad c^2 + s^2 = 1. \tag{12}$$

Our task with this study is to characterize the solutions of (5), (6), (9) and (10). We will be interested in the characterization in terms of boundary values of the functions in the Hardy spaces. For completeness we will also state some related results in the literature.

The definition of analytic signals is closely related to the Plemelj theorem of boundary values of Cauchy integrals: If $F(z)$ is the Cauchy integral of a function $f \in L^p(\mathbf{R})$, $1 < p < \infty$, or a function f with Hölder continuity and compact support, then F is analytic in the upper-half complex plane, $\mathbf{C}^+$ (in fact, F is in the Hardy H^p space) and

$$\lim_{y \to 0+} F(x+iy) = (1/2)f(x) + i(1/2)Hf(x), \quad \text{a.e.}$$

In the literature the equation (10) has been studied in [**8**] in which solutions of the Blaschke product type, in the upper-half complex plane, together with a factor of linear phase are obtained.

In Section 2 we recall preliminary knowledge of analytic functions in the Nevanlina classes in the unit disc and in the upper-half complex plane. In Sections 3 and 4 we prove, in the unit disc and the upper-half complex plane, respectively, that the non-parameterized solutions coincide with the boundary values of functions in the H^p spaces. The results for $1 \le p < \infty$, and $p = \infty$ in the unit disc are essentially known. We, however, provide new formulation and systematic treatment. The result for $p = \infty$ in the upper-half complex plane is new, where we adopt the distributional Hilbert transformation. In Section 4 we also show that any distribution represented by a function in the Hardy H^p space, $1 \le p \le \infty$, has nonnegative spectrum. In Section 5 we characterize a class of parameterized solutions of (9) and (5). Those are basically finite Blaschke products. Some aspects of the "atomic" cases arising from Möbius transforms are discussed. In Section 6 we construct, from certain basic holomorphic mappings, and the composition and multiplication laws, a class of parameterized solutions that contains the class studied in [**8**] as a proper subclass.

2. Preliminary knowledge on holomorphic function spaces. We will recall basic definitions and results of the Nevanlina classes N and the Hardy spaces H^p, $0 < p \le \infty$. The main references of this part are [**2, 3, 14**].

We will deal with two contexts the unit disc $\mathbf{D}$ and the upper-half complex plane $\mathbf{C}^+$ together, and will use the notation $\mathbf{S}$ for either $\mathbf{D}$

or $\mathbf{C}^+$. Under this convention, the notation $N(\mathbf{S})$, or N in brief, will stand for the Nevanlina class in either $\mathbf{D}$ or $\mathbf{C}^+$. If a paragraph or sentence only refers to one of these two contexts, then we will specify $\mathbf{S}$ to be $\mathbf{D}$ or $\mathbf{C}^+$. For instance, by $N(\mathbf{D})$ we mean the Nevanlina class in $\mathbf{D}$.

We will use the parametrization $e^{it} : A \le t \le 2\pi + A$, $A \in \mathbf{R}$, for the circle $\partial\mathbf{D}$ and $t : -\infty < t < \infty$, for $\partial\mathbf{C}^+ = \mathbf{R}$. We call them the *natural* or *canonical parametrizations* of $\mathbf{D}$ and $\mathbf{C}^+$. Below, unless otherwise stated, we always assume that we are using the canonical parametrization.

Denote the class of analytic (holomorphic) functions defined in $\mathbf{S}$ by $H(\mathbf{S})$. It is easy to prove that if $f \in H(\mathbf{S})$, then $\log|f|$ is subharmonic in $\mathbf{S}$ [**3**]. Jensen's inequality in relation to convex functions further implies that, if $f \in H(\mathbf{S})$, then $|f|^p$, $0 < p < \infty$ and $\log^+|f| = \max(\log|f|, 0)$ are subharmonic.

Among analytic functions with non-tangential boundary values, those in the Nevanlina class are of particular interest.

Definition 2.1. A function f in $H(\mathbf{S})$ is said to be in the Nevanlina class $N(\mathbf{S})$ if the subharmonic function $\log^+|f|$ has a harmonic majorant.

It is proved that functions in $N(\mathbf{S})$ at almost all points of $\partial\mathbf{S}$ have non-tangential boundary limits. Below we use the notation $f_{\partial\mathbf{S}}$ for the boundary value function of $f \in N(\mathbf{S})$.

Note that if $f \in N$, then

$$\log|f_{\partial\mathbf{S}}| \in L^1(\partial\mathbf{S}),$$

and

$$\log|f(z)| = \int_{\partial S} P_z(t)\, d\mu(t), \tag{13}$$

where $P_z(t)$ is the Poisson kernel in $\mathbf{S}$, and, if $\mathbf{S} = \mathbf{C}^+$, then $P_z(t) = P_z^{\mathbf{C}^+}(t)$, where

$$P_z^{\mathbf{C}^+}(t) = P_y(x-t) = \frac{1}{\pi}\frac{y}{(x-t)^2 + y^2} = \frac{1}{\pi}\,\mathrm{Im}\,\frac{1}{t-z}, \quad z = x+iy; \tag{14}$$

and, if $\mathbf{S} = \mathbf{D}$, then $P_z(t) = P_z^{\mathbf{D}}(t)$, where

$$
P_z^{\mathbf{D}}(t) = p_r(\theta - t) = \frac{1}{2\pi} \frac{1 - r^2}{1 - 2r\cos(\theta - t) + r^2} \tag{15}
$$
$$
= \frac{1}{2\pi} \operatorname{Re} \frac{e^{it} + z}{e^{it} - z}, \quad z = re^{i\theta},
$$

and

$$
d\mu(t) = \log |f_{\partial \mathbf{S}}|\, dt + d\mu_s(t), \tag{16}
$$

where $d\mu_s(t)$ is a finite Borel signal measure singular to dt, the latter being the Lebesgue measure on $\partial\mathbf{S}$. We recall that two signal measures ν_1 and ν_2 on the same σ-algebra are said to be singular to each other if there is a set E in the σ-algebra such that $|\nu_1|(E) = 0$ and $|\nu_2|(E^c) = 0$, where E^c denotes the complement set of E. Based on this definition, the support of $d\mu_s$ is contained in a Lebesgue null set of $\partial\mathbf{S}$.

Definition 2.2. Define

$$
N^+(\mathbf{S}) = \left\{ f \in N(\mathbf{S}) : \log|f(z)| \le \int_{\partial\mathbf{S}} \log|f_{\partial\mathbf{S}}(t)| P_z(t)\, dt \right\}.
$$

The relations (13) and (16) imply that $f \in N^+$ if and only if $f \in N$ and $d\mu_s \le 0$. An example of $f \in N \setminus N^+$ is given in Section 3 in which f^{-1} is a singular function.

For $0 < p < \infty$, define the Hardy spaces

$$
H^p(\mathbf{D}) = \left\{ f : f \in H(\mathbf{D}),\ \|f\|_p = \sup_{0<r<1} \left\{ \frac{1}{2\pi} \int_{\partial\mathbf{D}} |f(re^{it})|^p\, dt \right\}^{1/p} < \infty \right\},
$$

and

$$
H^p(\mathbf{C}^+) = \left\{ f : f \in H(\mathbf{C}^+),\ \|f\|_p = \sup_{0<y<\infty} \left\{ \int_R |f(t+iy)|^p\, dt \right\}^{1/p} < \infty \right\}.
$$

For $p = \infty$, define

$$H^\infty(\mathbf{D}) = \Big\{f : f \in H(\mathbf{D}),\ \|f\|_\infty = \sup_{z\in\mathbf{D}} |f(z)| < \infty\Big\},$$

and

$$H^\infty(\mathbf{C}^+) = \Big\{f : f \in H(\mathbf{C}^+), \|f\|_\infty = \sup_{w\in\mathbf{C}^+} |f(w)| < \infty\Big\}.$$

For $p \geq 1$, H^p are Banach spaces. For $p < 1$, H^p are complete metric spaces under the metric

$$d(f, g) = \|f - g\|_p^p.$$

Functions in the Hardy spaces H^p, $0 < p \leq \infty$, satisfy the inequality in Definition 2.2 and therefore belong to N^+. They are, in fact, proper subclasses of N^+. The classes $N(\mathbf{D})$ and $N(\mathbf{C}^+)$, their subclasses $N^+(\mathbf{D})$ and $N^+(\mathbf{C}^+)$ and the Hardy H^∞-spaces are all invariant under conformal mappings. The Hardy spaces H^p, $0 < p < \infty$, however, do not enjoy this property.

Since $H^p \subset N^+$, $0 < p \leq \infty$, there exist non-tangential boundary limits for functions $f \in H^p$. Denote the boundary value function by $f_{\partial\mathbf{S}}$ or simply f if no confusion arises. There exists a one-to-one correspondence between the functions in H^p and their boundary values. The H^p-norm of f is identical to the L^p-norm of the boundary function, f. Based on this we still denote the boundary value of $f \in H^p$ by f and the L^p-norm of the boundary value by $\|f\|_p$. If $f \in L^p(\partial\mathbf{S})$, $1 \leq p \leq \infty$, then f is the boundary value of a function in H^p if and only if the Poisson integral of f is in H^p, [**3**, Corollary 3.2, Chapter II].

We have the relation

$$H^p(\mathbf{S}) \subset N^+(\mathbf{S}) \subset N(\mathbf{S}), \quad 0 < p \leq \infty.$$

The monotones of the spaces $H^p(\mathbf{D})$ implies the finer inclusion relation

$$H^\infty(\mathbf{D}) \subset H^{p+s}(\mathbf{D}) \subset H^p(\mathbf{D}) \subset N^+(\mathbf{D}) \subset N(\mathbf{D}), \quad 0 < p, s < \infty. \tag{17}$$

We have

$$N^+(\mathbf{S}) \cap L^p(\mathbf{S}) = H^p(\mathbf{S}), \quad 0 < p \le \infty, \tag{18}$$

where $L^p(\mathbf{S})$ denotes the class of the functions in $N(\mathbf{S})$ with boundary values in $L^p(\partial\mathbf{S})$.

For Hardy spaces, we have

$$H^r(\mathbf{S}) \cap L^p(\mathbf{S}) \subset H^p(\mathbf{S}), \quad 0 < p \le \infty; \tag{19}$$

and, in $\mathbf{D}$ and for $r < p$, we have precisely

$$H^r(\mathbf{D}) \cap L^p(\mathbf{D}) = H^p(\mathbf{D}), \quad 0 < p \le \infty. \tag{20}$$

The assertion (19) for $\mathbf{S}$ is proved through the set inclusion relation:

$$H^r(\mathbf{S}) \cap L^p(\mathbf{S}) \subset N^+(\mathbf{S}) \cap L^p(\mathbf{S}) = H^p(\mathbf{S}),$$

while the assertion (20) for $\mathbf{D}$ is proved by the monotones of $H^p(\mathbf{D})$.

The Cayley transform is

$$z = \kappa(w) = \frac{i-w}{i+w}.$$

It maps conformally $\mathbf{C}^+$ to $\mathbf{D}$. The mapping is one to one that extends continuously as a one-to-one mapping from $\mathbf{R} \cup \{\infty\}$ to $\partial\mathbf{D}$ (also see Section 5). The inverse mapping is

$$w = i\,\frac{1-z}{1+z}.$$

The correspondence between the boundaries is

$$e^{it} = \frac{i-s}{i+s}, \quad -\infty < s < \infty,\ -\pi < t < \pi,$$

that implies

$$s = \tan(t/2), \quad \text{or} \quad t = 2\arctan s.$$

The H^∞ spaces are conformally invariant under the mapping κ and its inverse. For $0 < p < \infty$ the mapping κ maps $H^p(\mathbf{C}^+)$ to $H^p(\mathbf{D})$, but not vice versa. The conformal invariant relation, as a matter of fact, remains true in the cases $0 < p < \infty$ up to a conformal weight. In fact, we have $F(w) = (\pi^{-1/p}/(z+i)^{2/p})f(\kappa(w)) \in H^p(\mathbf{C}^+)$ if and only if $f(z) \in H^p(\mathbf{D})$, and $\|F\|_p = \|f\|_p$, $0 < p < \infty$. The conformal weight is from the Jacobian relation $|dz| = |dz/dw||dw|$.

The inequality in Definition 2.2 implies that if $f \in N^+$ and $f(e^{it}) = 0$ in some interval of positive Lebesgue measure on $\partial\mathbf{S}$, then f is identical with 0.

We now recall the counterpart Hilbert transformation theory on the unit circle $\partial\mathbf{D}$.

Let $f \in L^2(\partial\mathbf{D})$ and $f(t) = \sum_{k=-\infty}^{\infty} c_k e^{ikt}$ be its Fourier series. The convergence of the series is in the L^2-sense, and, due to Carleson's theorem, as well as in the point-wise convergence sense. Then the *circular Hilbert transform* of f is defined by, see [**3**],

$$\widetilde{H}f(t) = -i \sum_{k=-\infty}^{\infty} \operatorname{sgn}(k) c_k e^{ikt}. \tag{21}$$

Denote $f^+(t) = \sum_{k=0}^{\infty} c_k e^{ikt}$. The function f^+ is the boundary value of the analytic function $\sum_{k=0}^{\infty} c_k z^k$ in $\mathbf{D}$. The latter is in the Hardy space $H^2(\mathbf{D})$ that can be directly verified using the definition of the Hardy space and the Plancherel identity. We have

$$f + i\widetilde{H}f = c_0 + 2\sum_{k=1}^{\infty} c_k e^{ikt} = 2f^+ - c_0, \tag{22}$$

being the boundary value of the function $F(z) = 2\sum_{k=0}^{\infty} c_k z^k - c_0$, $F(0) = c_0$, in the Hardy H^2-space of the unit disc. We also have

$$\widetilde{H}^2 f = -f + c_0. \tag{23}$$

Therefore, $\widetilde{H}^2 = -I + c$, $c \in \mathbf{C}$.

By taking $f = f^+$ in (22), we have

$$\widetilde{H}f^+ = -if^+ + ic_0. \tag{24}$$

This relation holds for functions in all the Hardy $H^p(\mathbf{D})$, $1 \le p \le \infty$, spaces.

The circular Hilbert transform has a singular integral representation (see, for instance [**7**]):

$$\widetilde{H}f(t) = \frac{1}{2\pi} \text{ p.v.} \int_{-\pi}^{\pi} \cot\left(\frac{t-s}{2}\right) f(s)\, ds, \quad \text{a.e.} \tag{25}$$

Using the Fourier multiplier definition of $\widetilde{H}$ in (21) and the relation $\cos kt = 1/2(e^{-ikt} + e^{ikt})$ we have $\widetilde{H}\cos kt = \sin kt$, and similarly $\widetilde{H}\sin kt = -\cos kt$. Treating $\cos kt$ as a periodic function of t on the real line we can prove $H\cos kt = \sin kt$ using either the distributional definition, see Section 4, or the relation (2) via the Fourier transform of e^{ikt}, the Dirac-δ function. The latter, in particular, is widely used by engineers. In Remark 1 of Section 5 we offer a third and non-distributional proof of the relation. We show that if $\theta(t)$ is 2π-periodic and $\widetilde{H}\cos\theta(t) = \sin\theta(t)$, then $H\cos\theta(t) = \sin\theta(t)$.

In the circular case the parametric equation (5) and the non-parametric (6) are replaced by

$$\widetilde{H}(\rho(\cdot)\cos\theta(\cdot)) = \rho(t)\sin\theta(t), \quad \rho(t) \geq 0, \quad \theta'(t) \geq 0, \quad \text{a.e.} \tag{26}$$

and

$$\widetilde{H}(\rho c)(t) = \rho(t)s(t), \quad \rho(t) \geq 0, \quad c^2(t) + s^2(t) = 1, \quad \text{a.e.} \tag{27}$$

When $\rho \equiv 1$, they are reduced to

$$\widetilde{H}(\cos\theta(\cdot))(t) = \sin\theta(t), \quad \theta'(t) \geq 0, \tag{28}$$

and

$$\widetilde{H}c(t) = s(t), \quad c^2(t) + s^2(t) = 1, \quad \text{a.e.} \tag{29}$$

Due to the relation $\widetilde{H}^2 = -I + c$, the *conjugate equations* of (26), (27), (29) and (28) are, respectively,

$$\widetilde{H}(\rho(\cdot)\sin\theta(\cdot)) = -\rho(t)\cos\theta(t), \quad \rho(t) \geq 0, \quad \theta'(t) \geq 0, \quad \text{a.e.,} \tag{30}$$

$$\widetilde{H}(\rho s)(t) = -\rho(t)c(t), \quad \rho(t) \geq 0, \quad c^2(t) + s^2(t) = 1, \quad \text{a.e.,} \tag{31}$$

$$\widetilde{H}(\sin\theta(\cdot))(t) = -\cos\theta(t), \tag{32}$$

and

$$\widetilde{H}s(t) = -c(t), \quad c^2(t) + s^2(t) = 1, \quad \text{a.e.} \tag{33}$$

It is a consequence of Section 3 that in the last two conjugate equations $c \in \mathbf{D}$.

In the following when we say that mod (modulo) an additive constant the functions c, s satisfy (29) and (33), we mean that there exists a complex number $a \in \mathbf{D}$ such that c, s satisfy the equations

$$\widetilde{H}c = s - \mathrm{Im}\,(a), \quad \widetilde{H}s = -c + \mathrm{Re}\,(a), \quad c^2 + s^2 = 1, \quad a \in \mathbf{D}. \tag{34}$$

The same terminology is valid to the equation pairs (28) and (32), (10) and (12), and (9) and (11).

In [**8**] the following concept is introduced.

Definition 2.3. If $f(e^{it}) = c(t) + is(t)$, $c^2 + s^2 = 1$ and c, s satisfy, mod an additive constant, the equations (29) and (33), then $f(e^{it})$ is said to be *phase function* or *phase signal*.

3. Non-parameterized solutions on the unit circle. A complex-valued function is said to be *unimodular* if its value has the modulus 1 almost everywhere in its domain. A unimodular function f may be written $f = c + is$, $c^2 + s^2 = 1$, almost everywhere. According to the context there will be no ambiguity, and c and s in that case represent functions. In some other places the notation c stands for constants and s denotes an independent variable in $\mathbf{R}$ and sometimes represents singular functions in $\mathbf{D}$.

The class of inner functions in $\mathbf{S}$ is denoted by IN $(\mathbf{S})$ and defined by

$$\mathrm{IN}(\mathbf{S}) = \{f \in H^\infty(\mathbf{S}) : f_{\partial\mathbf{S}} \text{ is unimodular on } \partial\mathbf{S}\}.$$

The class of outer functions in $\mathbf{S}$ is denoted by OU $(\mathbf{S})$ and defined by

$$\mathrm{OU}\,(\mathbf{S}) = \left\{ f \in N^+(\mathbf{S}) : \log|f(z)| = \int_{\partial\mathbf{S}} \log|f_{\partial\mathbf{S}}(t)| P_z(t)\,dt \right\}.$$

Alternatively, an outer function is a function in $H(\mathbf{S})$ that does not have zeros in $\mathbf{S}$, and the harmonic function $\log|f(z)|$ is the Poisson integral of some function $k(t)$ satisfying

$$\int_{\partial\mathbf{S}} |k(t)|W(t)\,dt < \infty,$$

where in the notation of (14) and (15), if $\mathbf{S} = \mathbf{C}^+$, then $W(t) = P_1^{\mathbf{C}^+}(t) = (1/\pi)(1/1+t^2)$; and, if $\mathbf{S} = \mathbf{D}$, then $W(t) = P_0^{\mathbf{D}}(t) = p_0(t) = (1/2\pi)$.

Jensen's inequality implies that $f \in H^p(\mathbf{S})$ if and only if $f \in \mathrm{OU}\,(\mathbf{S})$ and $\exp k(t) \in L^p(\mathbf{R})$, $0 < p \leq \infty$.

A Blaschke product in $\mathbf{D}$ is a function in $H(\mathbf{D})$ of the form

$$b(z) = cz^m \prod_{|z_n|\neq 0} \frac{-\overline{z}_n}{|z_n|}\,\frac{z - z_n}{z - \overline{z}_n z},$$

where c is a unimodular constant, and the infinite product is convergent if and only if

$$\sum(1 - |z_n|) < \infty. \tag{35}$$

If b is a Blaschke product, then $b \in H^\infty(\mathbf{D})$ with $|b(z)| \leq 1$ and unimodular on $\partial\mathbf{D}$. If $E \in \partial\mathbf{D}$ is the set of accumulation points of the zeros z_n, then $b(z)$ is extendable to become an analytic function on the complement of the set $E \cup \{1/\overline{z}_n : n = 1, 2, \dots\}$ in the complex plane. In particular, $b(z)$ is analytic across each arc on $\partial\mathbf{D} \setminus E$. On the other hand, the function $|b(z)|$ does not extend continuously from $\mathbf{D}$ to any point of E [**3**, Theorem 6.1, Chapter 2].

Denote by $S(\mathbf{S})$ the class of *singular functions*. Singular functions have a constructive definition: $s \in S(\mathbf{D})$ if

$$s(z) = \exp\left\{-\int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z}\,d\mu(t)\right\}, \tag{36}$$

where $d\mu(t)$ is a finite positive Borel measure singular to the Lebesgue measure dt. Since any Borel measure finite on all compact sets is automatically a regular Borel measure, by invoking the decomposition theorem of regular Borel measures we have the unique decomposition $d\mu = d\mu_c + d\mu_d$, where $d\mu_d$ is a discrete measure having positive mass only on at most countably many points t_n such that $\sum_n d\mu_d(t_n) < \infty$, and $d\mu_c$ is a singular, finite and continuous measure, that is, $d\mu_c(t) = 0$ for all points t. Some related knowledge can be found in [**14**, Chapters IV and VII].

Properties of non-constant singular functions include

(i) $s(z)$ has no zeros in $\mathbf{D}$;

(ii) $|s(z)| < 1$ in $\mathbf{D}$;

(iii) $|s(z)| = 1$, almost everywhere on $\partial\mathbf{D}$; and

(iv) $s(0) > 0$.

Note that (ii) holds due to the relation

$$\log|s(z)| = -\int P_z(\theta)\, d\mu(\theta) < 0.$$

We will use factorization of functions in $N(\mathbf{D})$.

(i) If $f \in N(\mathbf{D})$, then apart from unimodular multiple constants, f can be uniquely factorized into the form

$$f(z) = b(z)g(z)s_1(z)s_2^{-1}(z),$$

where $b(z)$ is a Blaschke product, $g(z)$ is an outer function, s_1, s_2 are singular functions;

(ii) $f \in N^+(\mathbf{D})$ if and only if s_2 is a unimodular constant;

(iii) $f \in H^p(\mathbf{D})$, $0 < p \leq \infty$, if and only if s_2 is a unimodular constant and $g \in H^p(\mathbf{D})$; and

(iv) $f \in \mathrm{IN}\,(\mathbf{D})$ if and only if s_2, g both are unimodular constants, i.e., $f(z) = b(z)s_1(z)$.

Theorem 3.1. *Let $f(e^{it}) = c(t) + is(t)$, $0 \leq t < 2\pi$, be unimodular where functions c and s are real-valued. Then mod an additive constant in $\mathbf{D}$ the functions c and s satisfy the equations* (29) *and* (33) *if and only if f is the boundary value of an inner function in $\mathbf{D}$.*

Proof. Let f be a non-constant inner function in $\mathbf{D}$. Then f has a unimodular non-tangential boundary value $f(e^{it}) = c(t) + is(t)$. Let

$c_0 = f(0)$. Since

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} P_z(\theta) f(e^{i\theta})\, d\theta,$$

where

$$P_z(\theta) = \mathrm{Re}\, \frac{e^{i\theta} + z}{e^{i\theta} - z},$$

[**3**, Corollary 3.2, Chapter 2], we have $|f(0)| \leq \|f|_{\partial \mathbf{D}}\|_\infty = 1$. The maximal modulus principle then implies that $|f(0)| < 1$, and hence $|c_0| < 1$. Since f restricted to $\partial \mathbf{D}$ is the boundary value of a function in $H^\infty(\mathbf{D})$, the Fourier coefficients c_n for negative n are all zero. Thus, according to (22), $\widetilde{H} f(e^{i(\cdot)}) = -if(e^{i(\cdot)}) + ic_0$. In the last equality substituting $f(e^{it}) = c(t) + is(t)$ and comparing the real and imaginary parts, we obtain

$$\widetilde{H} c = s - \mathrm{Im}\,(c_0), \quad \text{and} \quad \widetilde{H} s = -c + \mathrm{Re}\,(c_0).$$

Next we assume that $f(e^{it})$ is unimodular, $f(e^{i(\cdot)}) = c + is$, where the real-valued functions c and s satisfy the relation (34). Expanding the bounded function $c(t)$ into its Fourier series, we have

$$c(t) = \sum_{k=-\infty}^{\infty} c_k e^{ikt},$$

where the convergence is in the L^2-sense on $\partial \mathbf{D}$, and we have, in particular, $\{c_k\} \in l^2$. Since $s(t) = \widetilde{H} c(t) + \mathrm{Im}\,(a)$, the Fourier expansion of $s(t)$ is

$$s(t) = -i \sum_{k=-\infty}^{\infty} \mathrm{sgn}\,(k) c_k e^{ikt} + \mathrm{Im}\,(a).$$

Therefore, the Fourier series of $f(e^{it})$ is

$$f(e^{it}) = [i\, \mathrm{Im}\,(a) + c_0] + 2 \sum_{k=1}^{\infty} c_k e^{ikt}.$$

Define

$$f(z) = [i\, \mathrm{Im}\,(a) + c_0] + 2 \sum_{k=1}^{\infty} c_k z^k.$$

Since $\sum_{k=1}^{\infty}|c_k|^2<\infty$, we have $f\in H^2(\mathbf{D})$ and $f\in H^2(\mathbf{D})\cap L^\infty(\mathbf{D})=H^\infty(\mathbf{D})$. Therefore, $f\in \mathrm{IN}\,(\mathbf{D})$ with the unimodular boundary value $f(e^{it})$. The second equation in (34) implies $c_0=\mathrm{Re}(a)$, therefore $f(0)=a$. The proof is complete. □

Functions of the type s^{-1}, $s\in S(\mathbf{D})$ satisfy the following properties

(i) $s^{-1}(z)$ is a well defined analytic function in D;

(ii) $|s^{-1}(z)|\geq 1$;

(iii) $|s^{-1}(z)|=1$ almost everywhere on ∂D;

(iv) $s^{-1}(0)>0$; and

(v) $s^{-1}(z)$ has non-tangential unimodular boundary limits on ∂D.

Functions of the type s^{-1} are in the Nevanlina class $N(\mathbf{D})$, unimodular on $\partial\mathbf{D}$, but are not analytic phase functions. This may be easily observed. In fact, since $s(e^{it})=e^{i\theta(t)}$ is a phase signal, then $e^{-i\theta(t)}$ must not be a phase signal for, otherwise, $\sin\theta(t)=\widetilde{H}\cos(\theta(t))=\widetilde{H}\cos(-\theta(t))=\sin(-\theta(t))$, which is a contradiction.

As a consequence of the above observation, functions of the type s^{-1}, although holomorphic with non-tangential boundary limits a.e., are not Cauchy integrals of their boundary values.

The simplest example of singular functions is

$$s_0=\exp\left(\frac{z+1}{z-1}\right)$$

generated by the point mass at 1 in (36). Simple computation shows that

$$|s_0^{-1}(re^{it})|=\frac{1+r^2}{1-r^2},\quad \text{for}\quad t=\arccos\frac{2r}{1+r^2}.$$

This proves $s_0^{-1}\notin H^\infty(\mathbf{D})$. Detailed computation shows that for every ζ, $0<|\zeta|<1$, $s_0(z)=\zeta$ infinitely often in every neighborhood of $z=1$ [**3**].

Next we discuss the cases $f\in N(\mathbf{D})$, $f(e^{it})=c(t)+is(t)$, $c^2(t)+s^2(t)\not\equiv 1$. We first note that the proof of Theorem 3.1 does not rely on the unimodular property of the boundary function $f(e^{it})$ and it is valid if the unimodular condition is replaced by the condition

$f(e^{it}) = \rho(t)(c(t) + is(t)) \in L^\infty(\partial\mathbf{D})$, where $c^2 + s^2 = 1$. We, in fact, have the following result for $1 \leq p \leq \infty$.

Theorem 3.2. *Let $1 \leq p \leq \infty$ and function $f(e^{it}) = \rho(t)(c(t) + is(t))$, $0 \leq t < 2\pi$, where $\rho \geq 0$, c and s are real-valued and $c^2 + s^2 = 1$. Then $\rho \in L^p([0, 2\pi])$ and ρc and ρs satisfy,* mod *an additive constant, the equations* (27) *and* (31) *if and only if f is the boundary value of some function in $H^p(\mathbf{D})$.*

Proof. The following proof is also an alternative proof of Theorem 3. Assume that f is a non-constant function in H^p, $1 \leq p \leq \infty$. Then it has a non-tangential boundary value $f(e^{it}) = \rho(t)(c(t) + is(t)) \in L^p([0, 2\pi])$. The function f has a representation, see [**3**],

$$f(z) = \int_{\partial\mathbf{D}} \mathrm{Re}\,(f(e^{it}))S(e^{it}, z)d(e^{it}) + i\,\mathrm{Im}\,(f(0)), \tag{37}$$

where $S(\zeta, z) = [1/(2\pi i)]\,[(\zeta + z)/(\zeta - z)](1/\zeta)$ is the Schwarz kernel.

Writing $z = re^{i\alpha}$, we have

$$S(e^{it}, z)d(e^{it}) = [p_r(t - \alpha) + iq_r(t - \alpha)]dt,$$

where $p_r(t - \alpha)$ is the Poisson kernel defined in (15), and $q_r(t - \alpha)$ is its conjugate defined by

$$q_r(t - \alpha) = \frac{1}{2\pi}\,\mathrm{Im}\,\frac{e^{i\alpha} + z}{e^{i\alpha} - z} = \frac{1}{2\pi}\,\frac{2r\sin(\alpha - t)}{1 - 2r\cos(\alpha - t)}. \tag{38}$$

Taking limit $r \to 1 - 0$ in (37), from the properties of the Poisson and the conjugate Poisson kernels, we have, see [**3**, Chapter III],

$$f(e^{i\alpha}) = \rho(\alpha)c(\alpha) + i\widetilde{H}(\rho(\cdot)c(\cdot))(\alpha) + i\,\mathrm{Im}\,(f(0)).$$

The last equality is simplified to

$$\rho s = \widetilde{H}(\rho c) + \mathrm{Im}\,(f(0)).$$

By applying $\widetilde{H}$ to both sides of the above relation, taking into account $\widetilde{H}(\mathrm{Im}\{f(0)\}) = 0$ and $\widetilde{H}^2 c = -c + \mathrm{Re}(f(0))$, we obtain

$$\widetilde{H}(\rho s) = -\rho c + \mathrm{Re}(f(0))$$

as desired.

Next we assume that $\rho \in L^p([0, 2\pi])$, $f(e^{it}) = \rho(t)(c(t) + is(t))$, $c^2 + s^2 = 1$ and, mod an additive constant, $\widetilde{H}(\rho c) = \rho s$, $\widetilde{H}(\rho s) = -\rho c$. We are to show that f is the boundary value of a function in $H^p(\mathbf{D})$, $1 \le p \le \infty$.

We first note, in the notation of (15) and (38) that there hold

$$\widetilde{H}((\rho c) * p_r) = (\widetilde{H}(\rho c)) * p_r = (\rho c) * q_r. \tag{39}$$

The relations may be proved by using Fourier multiples. They imply

$$[(\rho c) * p_r] + i\widetilde{H}[(\rho c) * p_r] = (\rho c) * (p_r + iq_r). \tag{40}$$

Alternatively,

$$[(\rho c) * p_r] + i\widetilde{H}[(\rho c) * p_r] = [(\rho c) + i(\rho s)] * p_r = f(e^{i(\cdot)}) * p_r. \tag{41}$$

The function in (40) is the Cauchy integral of a function in L^p, and thus one in $H^p(\mathbf{D})$. On the other hand, (41) shows that the H^p-function has the boundary value $f(e^{i(\cdot)})$. The proof is complete. $\square$

In view of Theorems 3.1 and 3.2, to find solutions of (29) and (27) is to find inner functions and H^p-functions. For completeness, we cite, without proof, some sufficient conditions for functions to be in $\mathrm{IN}\,(\mathbf{D})$ and $H^p(\mathbf{D})$-spaces [**3**, Chapter II].

Theorem 3.3. *Let f be in $L^p(\partial\mathbf{D})$, $1 \le p \le \infty$. Then f is in H^p thus satisfies* (29) *if one of the following conditions holds.*

(i) *The Poisson integral of f is analytic in $\mathbf{D}$.*

(ii)
$$\int_0^{2\pi} e^{int} f(e^{it})\, dt = 0, \quad n = 1, 2, \ldots$$

(iii) *For all functions* $g \in H_0^q(\mathbf{D})$,

$$\int_0^{2\pi} fg\,dt = 0,$$

where

$$H_0^q = \left\{ g \in H^q(\mathbf{D}) : g(0) = \frac{1}{2\pi} \int g(e^{it})\,dt = 0 \right\},$$

where $q = p/(p-1)$.

(iv) *On* $|z| > 1$,

$$\frac{1}{2\pi i} \int \frac{f(\zeta)\,d\zeta}{\zeta - z} = 0.$$

(v) *For* $p = \infty$, *there exists a uniformly bounded sequence of analytic polynomials* $p_n(z)$ *such that* $p(e^{i\theta}) \to f(e^{i\theta})$ *almost everywhere.*

4. Theory in $\mathbf{C}^+$. Blaschke products in $\mathbf{C}^+$ are obtained from Blaschke products in $\mathbf{D}$ via the Cayley transform $z = (i-w)/(i+w)$. They are of the form

$$B(w) = c\left(\frac{w-i}{w+i}\right) \prod_{w_n \neq i} \frac{|w_n^2+1|}{w_n^2+1}\,\frac{w-w_n}{w-\overline{w}_n},$$

where c is a constant, $|c| = 1$, and the condition (35) of the zeros becomes

$$\sum \frac{y_n}{1+|w_n|^2} < \infty, \quad w_n = x_n + iy_n.$$

Through the Cayley transform the classes of inner functions, outer functions and singular functions in $\mathbf{D}$ are all changed to the corresponding function classes in $\mathbf{C}^+$.

Before performing change of variables under the Cayley transform we first write a singular function in $\mathbf{D}$ in the form

$$s(z) = \exp\left\{ \frac{z-1}{z+1}\,\mu(\pi) - \int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\,d\mu_1(\theta) \right\},$$

where $d\mu_1$ is $d\mu$ in (36) diminished by the mass $\mu(\pi)$ at $\theta = \pi$.

Changing variables under the inverse Cayley transform $w = i(1-z/1+z)$, followed by the change of variable $\lambda = \tan(\theta/2)$, we have

$$S(w) = s(z^{-1}(w)) = \exp i\left\{w\mu(\pi) + \int_R \frac{w\lambda+1}{\lambda-w}\,\frac{2}{1+\lambda^2}\,d\nu(\lambda)\right\},$$

where $2/(1+\lambda^2)\,d\nu(\lambda)$ is a finite positive Borel measure on $\mathbf{R}$. It is usually further written as

$$S(w) = \exp i\left\{w\mu(\pi) + \int_R \left(\frac{1}{\lambda-w} - \frac{\lambda}{\lambda^2+1}\right)d\nu(\lambda)\right\}.$$

We note that the quantity $\mu(\pi)$ is, in fact, the mass at the infinity after the change of variables under the Cayley transform [**2**].

There are corresponding factorization results for functions in the Nevanlina class of $\mathbf{C}^+$. In particular, for any inner function $F \in IN(\mathbf{C}^+)$, we have $F = BS$, where B is a Blaschke product and S a singular function, with the representations given above.

We need to define Hilbert transformation for bounded functions. We adopt the distributional definition which is naturally related to boundary values of functions in $H^\infty(\mathbf{R})$.

Denote by $\mathcal{D}$ the space of infinitely differentiable functions with compact support on the line, and $\mathcal{D}'$ the space of continuous linear functionals on $\mathcal{D}$, viz. the space of distributions.

Definition 4.1. Let T be a distribution and u a harmonic function in the upper-half complex plane. If

$$\langle T, \phi\rangle = \lim_{y\to 0+} \int_{-\infty}^{\infty} u(x,y)\phi(x)\,dx, \quad \phi \in \mathcal{D},$$

then u is said to be a *harmonic representation of* T.

Obviously, a distribution may have more than one harmonic representation. The following result is known (see, for instance [**5**] or [**6**]).

Theorem 4.1. *Let T be a distribution and U one of its harmonic representations. Let V be any harmonic conjugate of U. Then V is a harmonic representation of some distribution, S.*

Definition 4.2. Any distribution S in Theorem 4.1 is called a *Hilbert transform* of T.

For a chosen harmonic representation of T its harmonic conjugates are not unique. As a consequence the above defined Hilbert transform is unique only up to an additive constant. The relation $H^2 = -I$ now is changed to $H^2 = -I + [c]$, where $[c]$ denotes the class of constants.

On $\mathbf{R}$ the analogous result to Theorem 3.1 reads

Theorem 4.2. *Let F be a unimodular function, $F(s) = C(s)+iS(s)$, $-\infty < s < \infty$, where functions C and S are real-valued. Then C and S, mod additive constants, satisfy the equations* (10) *and* (12) *if and only if F is the boundary value of an inner function in $\mathbf{C}^+$.*

Proof. Let F be an inner function in $\mathbf{C}^+$. Write $F = U + iV$, where U and V are real-valued, and thus are bounded and harmonic. In the pointwise sense,

$$\lim_{y\to 0+} U(s,y) = C(s), \quad \lim_{y\to 0+} V(s,y) = S(s), \quad \text{a.e.}$$

Lebesgue's dominated convergence theorem implies that, for any $\phi \in \mathcal{D}$,

$$\lim_{y\to 0+} \int_{-\infty}^{\infty} U(s,y)\phi(s)\,ds = \int_{-\infty}^{\infty} C(s)\phi(s)\,ds$$

and

$$\lim_{y\to 0+} \int_{-\infty}^{\infty} V(s,y)\phi(s)\,ds = \int_{-\infty}^{\infty} S(s)\phi(s)\,ds.$$

Since V is a harmonic conjugate of U, we obtain

$$HC = S,$$

as desired.

Next we assume $F = C + iS$, $C^2 + S^2 = 1$ and, in the distribution sense, $HC = S$. We will show that F is the boundary value of an inner function in $\mathbf{C}^+$. The idea is to transfer the problem to the unit disc via Cayley transform. We divide the proof into a few steps.

Step 1. Let $U(s,h) = P_h * C(s)$, the Poisson integral of C. It is bounded and harmonic in $\mathbf{C}^+$. Let $V(s,h)$ be any, but fixed, harmonic conjugate of U. Temporarily we take for granted that V is bounded. Then the function $F(s,h) = U(s,h) + iV(s,h)$ is a bounded holomorphic function in $\mathbf{C}^+$, and thus it has a non-tangential boundary value, denoted by $U(s,0) + iV(s,0)$, where due to the property of the Poisson integral, $U(s,0) = C(s)$. Since V is bounded and harmonic, distributionally $V(s,h) \to V(s,0)$. Since $HC = S$, we have, for a constant c_0, $V(s,0) = S(s) + c_0$. Hence $F(s) = C(s) + iS(s)$ is the boundary value of the bounded analytic function $U + i(V - c_0)$ in $\mathbf{C}^+$. Since $F(s)$ is unimodular, $U + i(V - c_0)$ is an inner function, as desired. In below we devote ourselves to showing that $V(s,h)$ is bounded.

Step 2. We will use Cayley transform $z : \mathbf{C}^+ \to \mathbf{D}$, $z = (i - w)/(w + i)$, where on their boundaries we have the relation $e^{it} = (i - s)/(s + i)$ that gives rise to the parametrizations $s = \tan t/2$, $-\pi < t < \pi$, and $t = 2 \arctan s$, $-\infty < s < \infty$. Under the transform we have $u(z) + iv(z) = U(w) + iV(w)$, $f(e^{it}) = c(t) + is(t) = C(s) + iS(s) = C(\tan(t/2)) + iS(\tan(t/2))$. The functions $U + iV$ and $u + iv$ have the same range. Since both Cayley transform and its inverse are conformal, and thus preserve the harmonicity of the real and the imaginary parts of analytic functions, we are reduced to showing that v is bounded in $\mathbf{D}$.

Step 3. Since $c(t)$, the bounded boundary value of u, is in $L^2(\partial\mathbf{D})$, the integral

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z}\, c(\theta)\, d\theta = u(re^{it}) + v_1(re^{it}), \quad z = re^{it},$$

is a function in $H^2(\mathbf{D})$ with L^2-non-tangential boundary value $u(e^{it}) + v_1(e^{it})$. As consequence, there exists a constant c_1 such that $v_1 = v + c_1$. We are therefore reduced to show that v_1 is bounded in $\mathbf{D}$.

Step 4. From the Cayley transform in Step 2 and that $S(s)$ is the distributional Hilbert transform of $C(s)$ in $\mathbf{C}^+$, we have for all C^∞ functions ϕ on $\partial\mathbf{D}$ with compact support away from the point -1,

$$\lim_{r \to 1-} \int v(re^{i\theta})\phi(\theta)\, d\theta = \int S(\tan\theta/2)\phi(\theta)\, d\theta. \tag{42}$$

Since in the $L^2(\partial\mathbf{D})$-norm sense we have $v_1(re^{i\cdot}) \to v_1(e^{i\cdot})$ as $r \to 1$, for such functions ϕ we also have

$$\lim_{r\to 1-} \int v_1(re^{i\theta})\phi(\theta)\,d\theta = \int v_1(e^{i\theta})\phi(\theta)\,d\theta. \tag{43}$$

Recalling the relation $v_1 = v + c_1$ and comparing (42) and (43), we have

$$v_1(e^{it}) = S(\tan t/2) + c_1, \quad \text{a.e.}$$

Step 5. The above shows that the analytic function $u + i(v_1 - c_1)$, as a function in $H^2(\mathbf{D})$, has unimodular boundary value $C(\tan t/2) + iS(\tan t/2)$. The relation $H^2 \cap L^\infty = H^\infty$ in $\mathbf{D}$ implies that $u+i(v_1-c_1)$ is an inner function, and hence v_1 is bounded in $\mathbf{D}$. The proof is complete. $\square$

The inconvenience with $\mathbf{C}^+$ is that the spaces $L^p(\mathbf{R})$, as well as the spaces $H^p(\mathbf{C}^+)$, do not enjoy the set inclusion relation (17), and, especially, functions in $L^\infty(\mathbf{R})$ are not integrable. Cayley transform is used in the proof of the above theorem to facilitate convenience. With essentially the same proof as Theorem 4.2 may be extended to $F = \rho C + i\rho S \in L^\infty(\mathbf{R})$. For the cases $1 \le p < \infty$, one uses a similar proof as in Theorem 3.2, where the Schwarz kernel is replaced by the Cauchy kernel. We have

Theorem 4.3. *Let $1 \le p \le \infty$ and $F(s) = \rho(s)(C(s) + iS(s))$, $-\infty < s < \infty$, where $\rho \ge 0$, C and S are real-valued with $C^2 + S^2 = 1$. Then $\rho \in L^p(\mathbf{R})$ and, ρC and ρS, mod additive constants, satisfy the equation* (6) *and* (8) *if and only if F is the boundary value of a function in $H^p(\mathbf{C}^+)$.*

As in the circle case, we cite, without proof, a number of sufficient conditions for $L^p(\mathbf{R})$ functions to be in $H^p(\mathbf{C}^+)$, see [**3**, Chapter II].

Theorem 4.4. *Let $F = \rho C + i\rho S \in L^p(\mathbf{R})$, $1 \le p \le \infty$. Then $F \in H^p(\mathbf{C}^+)$ if one of the following conditions holds.*

(i) *The Poisson integral of F is analytic in $\mathbf{C}^+$.*

(ii) *When* $p < \infty$,

$$\int \frac{F(s)}{s-z}\, ds = 0, \quad \operatorname{Im} z < 0,$$

and when $p = 0$,

$$\int_{-\infty}^{\infty} F(s)\left(\frac{1}{s-z} - \frac{1}{s-z_0}\right) ds = 0, \quad \operatorname{Im} z < 0,$$

where z_0 *is any fixed point in the lower half plane.*

(iii) *For all functions* G *in*$H^q(\mathbf{C}^+), q = p/(p-1)$,

$$\int_{-\infty}^{\infty} FG\, ds = 0.$$

(iv) *For* $1 \le p \le 2$, (*so that the Fourier transform is defined on* L^p *by Plancherel's theorem*),

$$\hat{F}(\xi) = \lim_{N\to\infty} \int_{-N}^{N} F(s)e^{-2\pi i\xi s}\, ds = 0$$

almost everywhere on $\xi < 0$.

Definition 4.3. Let T be a distribution and $f(x+iy)$ an analytic function in $\mathbf{C}^+$ such that, for any $\phi \in \mathcal{D}$,

$$\langle T, \phi\rangle = \lim_{y\to 0+} \int_{\mathbf{R}} f(x+iy)\phi(x)\, dx,$$

then we say that T is an *upper-Hardy distribution* and $f(x+iy)$ is is an analytic representation of T. In such a case we may write $T = T^+$.

Let T be the tempered distribution represented by the boundary value of a function in $H^p(\mathbf{C}^+)$. From Definition 4.3, $T = T^+$ is an upper-Hardy distribution. The following theorem asserts that $\hat{T^+}$, the Fourier transform of T^+, has positive spectrum in the sense specified in the following theorem.

Theorem 4.5. *If T^+ is the tempered upper-Hardy distribution represented by the boundary value of a function in $H^p(\mathbf{C}^+)$, $1 \le p \le \infty$, then* $\operatorname{supp}\{\hat{T^+}\} \subset [0, \infty)$, *that is,*

$$\langle \hat{T^+}, \phi \rangle = 0, \quad \text{for all } \phi \in \mathcal{D} \text{ such that } \operatorname{supp} \phi \subset (-\infty, 0].$$

We need two technical lemmas.

Lemma 4.1. *For any $f \in L^p(\mathbf{R})$, $1 \le p \le 2$ and $\phi \in \mathcal{S}(\mathbf{R})$, where $\mathcal{S}(\mathbf{R})$ stands for the Schwartz space of rapidly decreasing functions, we have*

$$\int_{\mathbf{R}} f \hat{\phi}\, dx = \int_{\mathbf{R}} \hat{f} \phi\, dx.$$

Proof. For $p = 1, 2$ the results are standard. For $1 < p < 2$, let $f_n \in L^1 \cap L^p(\mathbf{R})$ such that $f_n \to f$ in L^p. For every n we have

$$\int_{\mathbf{R}} f_n \hat{\phi}\, dx = \int_{\mathbf{R}} \hat{f}_n \phi\, dx,$$

For $1/p + 1/p' = 1$, on one hand,

$$\Big| \int_{\mathbf{R}} (f_n - f) \hat{\phi}\, dx \Big| \le \|f_n - f\|_p \|\hat{\phi}\|_{p'} \to 0;$$

and, on the other hand, from the Hausdorff-Young inequality,

$$\Big| \int_{\mathbf{R}} (\hat{f}_n - \hat{f}) \phi\, dx \Big| \le \|\hat{f}_n - \hat{f}\|_{p'} \|\phi\|_p \le \|f_n - f\|_p \|\phi\|_p \to 0.$$

Taking limit $n \to \infty$, we conclude the desired identity. The proof is complete. $\square$

Lemma 4.2. *Let $f \in H^p(\mathbf{C}^+)$, $1 \le p \le 2$. Then* $\operatorname{supp} \hat{f} \subset [0, \infty)$.

Proof. The assertion for $p = 1$ is proved in [**3**, Lemma 3.7, Chapter II]. For a proof of the case $p = 2$ see, for instance, [**12**, Theorem 19.2, Chapter 19]. We provide a uniform treatment for the cases $1 < p \leq 2$, using the classes $\mathcal{U}_N$ defined in [**3**, Corollary 3.3, Chapter II]. Now let $f \in H^p$, $1 < p \leq 2$. Then the above cited Corollary 3.3 implies that there exists a sequence $\{f_n\}_{n=1}^{\infty}$ in $H^1(\mathbf{C}^+) \cap H^p(\mathbf{C}^+)$ such that $f_n \to f \in H^p(\mathbf{C}^+)$. Using the same notation for the boundary values of f and f_n, the result for $p = 1$ and the Hausdorff-Young inequality imply

$$\left(\int_{-\infty}^{0} |\hat{f}(\xi)|^{p'} \, d\xi \right)^{1/p'} \leq \|\hat{f} - \hat{f}_n\|_{p'} \leq \|f - f_n\|_p \longrightarrow 0.$$

It follows that $\hat{f}(\xi) = 0$, almost everywhere $\xi \in (-\infty, 0]$. The proof is complete. □

Proof of Theorem 4.5. First consider the cases $1 \leq p \leq 2$. Let T^+ be a tempered distribution with a holomorphic representation $f \in H^p(\mathbf{C}^+)$. Let ϕ be any function in $\mathcal{D}$ with supp $\phi \subset (-\infty, 0]$. From Lemma 4.2, for any fixed $y > 0$, supp$\hat{f}(\cdot + y) \subset [0, \infty)$. By using Lemma 4.1 we have

$$\begin{aligned} \langle \hat{T^+}, \phi \rangle &= \lim_{y \to 0+} \int_{-\infty}^{\infty} f(t + iy) \hat{\phi}(t) \, dt \\ &= \lim_{y \to 0+} \int_{-\infty}^{\infty} \hat{f}(t + iy) \phi(t) \, dt \\ &= 0. \end{aligned}$$

Below we assume $2 < p \leq \infty$ is fixed and f is the H^p- function representing T^+. Let $g_\delta(t)$ be a function in $\mathcal{D}$ that is even, taking the value 1 for $|t| \leq 1/\delta$, zero for $|t| \geq 2/\delta$ and between 1 and zero for $1/\delta < |t| < 2/\delta$. Denote by $g_\delta(z)$ its Cauchy integral that is in $H^r(\mathbf{C}^+)$, $1 < r < \infty$, whose boundary value, by the Plemelj theorem, is $(1/2)g_\delta(t) + i(1/2)Hg_\delta(t)$. We claim that there exists $q \in (1, 2]$, such that the product function $g_\delta(z)f(z) \in H^q(\mathbf{C}^+)$. For $p < \infty$, taking $1 < q < 2 < p$ and $s = p/q > 1$, Hölder's inequality implies

$$\begin{aligned} \int_{\mathbf{R}} |g_\delta(x + iy) f(x + iy)|^q \, dx \leq &\left(\int_{\mathbf{R}} |g_\delta(x + iy)|^{qs'} \, dx \right)^{1/s'} \\ &\times \left(\int_{\mathbf{R}} |f(x + iy)|^{qs} \, dx \right)^{1/s} < \infty. \end{aligned}$$

If $p=\infty$ we can take $q=2$ as $g_\delta(z)\in H^2(\mathbf{C}^+)$.

Let $\phi\in\mathcal{D}$ be of compact support in $(-\infty,0]$. Temporarily accepting that for every fixed $y>0$,

$$\lim_{\delta\to 0+}\int_{-\infty}^{\infty} g_\delta(t+iy)f(t+iy)\hat\phi(t)\,dt=\frac{1}{2}\int_{-\infty}^{\infty} f(t+iy)\hat\phi(t)\,dt. \tag{44}$$

The Parseval's relation proved in Lemma 4.1 implies

$$\begin{aligned}\langle \hat{T^+},\phi\rangle &= \langle T^+,\hat\phi\rangle\\ &=\lim_{y\to 0+}\int_{-\infty}^{\infty} f(t+iy)\hat\phi(t)\,dt\\ &=2\lim_{y\to 0+}\lim_{\delta\to 0+}\int_{-\infty}^{\infty} g_\delta(t+iy)f(t+iy)\hat\phi(t)\,dt\\ &=2\lim_{y\to 0+}\lim_{\delta\to 0+}\int_{-\infty}^{\infty} (g_\delta(\cdot+iy)f(\cdot+iy))\hat{}\,(t)\phi(t)\,dt.\end{aligned}$$

Now as a function in $H^q(\mathbf{C}^+)$, $1<q\le 2$, the support of $(g_\delta(\cdot+iy)\times f(\cdot+iy))\hat{}$ is contained in $[0,\infty)$, as proved in Lemma 4.2, and hence $\langle \hat{T^+},\phi\rangle=0$ as desired.

Now we are left to show (44). Let y be fixed. Set

$$g_\delta(z)=\frac{1}{2}\,P_y * g_\delta(t)+i\,\frac{1}{2}\,Q_y * g_\delta(t),\quad z=t+iy,$$

$$P_y(t)=\frac{1}{\pi}\,\frac{y}{t^2+y^2},\quad Q_y(t)=\frac{1}{\pi}\,\frac{t}{t^2+y^2}.$$

The properties of the Poisson kernel imply that $P_y * g_\delta(t)$, as $\delta\to 0$, converges to the identity function 1 point-wisely, and is bounded uniformly in δ and t. The Lebesgue dominated convergence theorem implies

$$\lim_{\delta\to 0+}\int_{-\infty}^{\infty} P_y * g_\delta(t)f(t+iy)\hat\phi(t)\,dt=\int_{-\infty}^{\infty} f(t+iy)\hat\phi(t)\,dt.$$

For the fixed y the functions $Q_y * g_\delta(t)$ are not uniformly bounded as a function of t as $\delta\to 0$. We, however, through explicit computation, can show

$$\lim_{\delta\to 0+}\int_{-\infty}^{\infty} Q_y * g_\delta(t)f(t+iy)\hat\phi(t)\,dt=0. \tag{45}$$

Write

$$Q_y * g_\delta(t) = \frac{1}{\pi}\int_{|x|\le 1/\delta} \frac{t-x}{(t-x)^2+y^2}\,dx + \frac{1}{\pi}\int_{1/\delta<|x|\le 2/\delta} \frac{(t-x)g_\delta(x)}{(t-x)^2+y^2}\,dx = I_1^{(\delta)}(t,y) + I_2^{(\delta)}(t,y).$$

We first estimate I_1. Without loss of generality, we may assume $t > 0$. Simple computation gives

$$I_1^{(\delta)} = \frac{1}{\pi}\int_{t-1/\delta}^{t+1/\delta} \frac{x}{x^2+y^2}\,dx = \frac{1}{2\pi}\log\left(\frac{(t+1/\delta)^2+y^2}{(t-1/\delta)^2+y^2}\right).$$

First note that, for a fixed y as $\delta \to 0$ in the point-wise manner,

$$\lim_{\delta\to 0} I_1^{(\delta)}(t,y) = 0.$$

Now we show that it is also uniformly bounded as $\delta \to 0$. Set

$$\alpha_1^{(\delta)}(t,y) = \frac{(t+1/\delta)^2+y^2}{(t-1/\delta)^2+y^2}.$$

For each $\delta > 0$ the function takes the value 1 at $t = 0$, and 1 again as $t \to \infty$. As t leaves zero and goes the infinity, the function first increases to reach its maximum, and then decreases. The maximum value happens at $t_\delta = \sqrt{\delta^2+y^2}/\delta$ with value

$$\frac{1}{\pi}\frac{(\delta t_\delta+1)^2+\delta^2 y^2}{(\delta t_\delta-1)^2+\delta^2 y^2} \longrightarrow \frac{1}{\pi}\frac{(y+1)^2}{(y-1)^2}, \quad \text{as} \quad \delta \to 0.$$

This shows that, in restricting $y \in (0,1/2)$ as $\delta \to 0$, the functions $\alpha_1^{(\delta)}(t,y)$, and therefore $I_1^{(\delta)}(t,y)$ as well, are uniformly bounded in δ and t. Then the Lebesgue dominated convergence theorem may be used to conclude

$$\lim_{\delta\to 0+}\int_{-\infty}^{\infty} I_1^{(\delta)}(t,y) f(t+iy)\hat{\phi}(t)\,dt = 0.$$

Next we study $I_2^{(\delta)}$. Since g_δ is even, $I_2^{(\delta)}$ is reduced to

$$\frac{1}{\pi}\int_{1/\delta}^{2/\delta}\left(\frac{t-x}{(t-x)^2+y^2}+\frac{t+x}{(t+x)^2+y^2}\right)g_\delta(x)\,dx$$
$$=\frac{2t}{\pi}\int_{1/\delta}^{2/\delta}\frac{y^2+t^2-x^2}{(y^2+t^2+x^2-2xt)(y^2+t^2+x^2+2xt)}\,dx.$$

With the assumption $t>0$ the above is dominated by

$$\frac{2t}{\pi}\int_{1/\delta}^{2/\delta}\frac{1}{(x-t)^2+y^2}\,dx=\frac{2t}{\pi}\,\alpha_2^{(\delta)}(t,y).$$

It is easy to verify that, for the fixed $y\in(0,1/2)$, the function $\alpha_2^{(\delta)}(t,y)$ is uniformly bounded and tends to zero as $\delta\to 0$. Since $(2t/\pi)\hat{\phi}(t)$ is a function in the Schwartz class, we finally obtain, by the Lebesgue dominated convergence theorem,

$$\lim_{\delta\to 0+}\int_{-\infty}^{\infty} I_2^{(\delta)}(t,y)f(t+iy)\hat{\phi}(t)\,dt=0.$$

The proof is complete. $\square$

5. Parameterized solutions. The parameterized cases are essentially finite Blaschke products. The results of this section are generalizations of [**10**].

We have the following theorem.

Theorem 5.1. *Assume that θ is a continuous function on* $\mathbf{R}$ *strictly increasing with $m(\theta([0,2\pi]))=2\pi n$, where m stands for the Lebesgue measure. Then the following two conditions are equivalent.*

(i) *$d\theta(t)$ is a sum of a number of n harmonic measures on the unit circle.*

(ii)

$$\widetilde{H}\cos\theta(t)=\sin\theta(t)-(-1)^n\mathrm{Im}\left(\prod_{k=1}^{n}a_k\right) \tag{46}$$

and

$$\widetilde{H}\sin\theta(t) = -\cos\theta(t) + (-1)^n \mathrm{Re}\left(\prod_{k=1}^{n} a_k\right)$$

for some $a_k \in D$, $k = 1, \ldots, n$.

Proof. Let $\iota_a(z) = e^{i\theta_0}(z-a)/(1-\overline{a}z)$, $a \in \mathbf{D}$, be a Möbius transform of the disc $\mathbf{D}$ transforming a to 0 whose the boundary value is defined by

$$e^{i\theta_a(t)} = \iota_a(e^{it}) = \frac{e^{it}-a}{1-\overline{a}e^{it}}, \quad 0 \le t \le 2\pi.$$

Then, see [**10**] or [**3**],

$$\frac{1}{2\pi}\frac{d\theta_a(t)}{dt} = \frac{1}{2\pi}\frac{1-|a|^2}{1-2|a|\cos(t-\theta_0)+|a|^2} = p_{|a|}(t-\theta_0) \tag{47}$$

is the Poisson kernel of the disc at $a = |a|e^{i\theta_0}$. Therefore, $d\theta_a$ is a harmonic measure.

Note that τ_a is an inner function in $\mathbf{D}$. Thus by Theorem 3.1 its boundary value satisfies the equation

$$\widetilde{H}\cos\theta_a(t) = \sin\theta_a(t) + \mathrm{Im}\,(a) \quad \text{and} \quad \widetilde{H}\sin\theta_a(t) = -\cos\theta_a(t) - \mathrm{Re}\,(a).$$

This corresponds to the case $n = 1$ (the *atomic case*).

Now let $d\theta(t)$ be a sum of a number of n harmonic measures on the unit circle with respect to $a_k \in \mathbf{D}$, $k = 1, \ldots, n$. The function $e^{i\sum_{k=1}^{n}\theta_{a_k}(t)}$ is the boundary value of the Blaschke product $b(z) = \prod_{k=1}^{n}\tau_{a_k}(z)$, $a_k \in \mathbf{D}$, the latter being an inner function in $\mathbf{D}$. Since $b(0) = (-1)^n\prod_{k=1}^{n} a_k$, by invoking the relation (24), we have

$$\widetilde{H}e^{i\theta(t)} = -ie^{i\theta(t)} + i(-1)^n\prod_{k=1}^{n} a_k.$$

The desired relations in (ii) then follow.

(ii) $\to$ (i). The assumptions on $\theta(t)$ imply that $e^{i\theta(t)} \in L^2(\partial\mathbf{D})$ and

$$i\widetilde{H}e^{i\theta(t)} = e^{i\theta(t)} + a.$$

Therefore,

$$e^{i\theta(t)} + i\widetilde{H}e^{i\theta(t)} = 2e^{i\theta(t)} + a. \tag{48}$$

The lefthand side of (48), due to (22), is equal to

$$2(e^{i\theta(t)})^{+} - c_0, \tag{49}$$

while the righthand side of (48) is equal to

$$2(e^{i\theta(t)})^{+} + 2(e^{i\theta(t)})^{-} + a. \tag{50}$$

Comparing (49) and (50), we have

$$-c_0 = (2e^{i\theta(t)})^{-} + a,$$

and hence

$$c_0 = -a, \quad \text{and} \quad (2e^{i\theta(t)})^{-} = 0.$$

The last two relations show that $e^{i\theta(t)}$ itself is the boundary value of an analytic function, f, in $\mathbf{D}$, with $f(0) = -a$.

Next we show $f(\mathbf{D}) \subset \mathbf{D}$. First, $f \in H^2$ and $f|_{\partial\mathbf{D}}(t) = e^{i\theta(t)} \in L^\infty$ imply, due to (20), $f \in H^\infty$. Since $f|_{\partial D}$ is unimodular, we obtain that f is an inner function. From the factorization theorem of inner functions, we have $f = cbs$, where c is a constant with $|c| = 1$, b is a Blaschke product and s is a singular function. The fact that any Möbius transform maps $\mathbf{D}$ into $\mathbf{D}$ implies $b(\mathbf{D}) \subset \mathbf{D}$. As for any non-constant singular function s, we have

$$\log|s(z)| = -\int P_z(\theta)\,d\mu(\theta) < 0,$$

where $d\mu$ is a nonnegative Borel measure, we have $s(\mathbf{D}) \subset \mathbf{D}$, and thus also $f(\mathbf{D}) \subset \mathbf{D}$.

Since f is an inner function, it is the Poisson integral of its boundary value $e^{i\theta(t)}$ [**3**, Corollary 3.2, Chapter II]. This, together with the fact that the boundary value is continuous, implies that f is continuously extended $\partial\mathbf{D}$. This further implies that f only has at most a finite number of zeros in $\mathbf{D}$. As a matter of fact, if there were infinitely many zeros, then there would exist a cluster point, z_0, of the zeros. If $z_0 \in \mathbf{D}$, then f is identical to *zero*; and, if $z_0 \in \partial\mathbf{D}$, then f cannot be everywhere unimodular on $\partial\mathbf{D}$. In this case the argument principle can be used for f on the boundary of $\mathbf{D}$ to conclude that f has exactly n zeros in $\mathbf{D}$, together with multiplicities. Let b be the finite Blaschke product formed from the zeros of f; then $s = f/b$ is the singular function in the corresponding factorization of f. Note that f/b is continuously extended to all points of $\partial\mathbf{D}$. This will imply that the singular function s must be trivial and thus equal to a unimodular constant. In fact, a non-trivial singular functions cannot be continuously extended to the closure of the support of the singular measure on $\partial\mathbf{D}$ defined through $d\mu$ in (36) [**3**, Theorem 6.2, Chapter II]. The proof is complete. □

We present below an example of a nonlinear phase as a parameterized solution of (29). Taking $n = 1$ and $a = 1/2$ in Theorem 5.1, we have

$$e^{i\theta_a(t)} = c(t) + is(t),$$

where

$$c(t) = \frac{5\cos t - 4}{5 - 4\cos t}, \quad s(t) = \frac{3\sin t}{5 - 4\cos t}.$$

The theorem asserts that $\widetilde{H}c = s$, $c^2 + s^2 = 1$ and $c(t)$ is well parameterized by $c(t) = \cos\theta_a(t)$, where

$$e^{i\theta_a(t)} = \tau_a(e^{it}) = \frac{e^{it} - a}{1 - \overline{a}e^{it}},$$

with $\theta_a'(t) = p_{|a|}(t) > 0$, where $p_{|a|}$ is the Poisson kernel of $\mathbf{D}$ at $z = |a|$ (for more details see [**11**]).

Note that $c(t) = \cos\theta_a(t)$ from a Möbius transform may be viewed as a periodic function on $\mathbf{R}$. In Section 6, Remark 1, we provide a non-distributional proof of periodization of solutions of (29) on $\mathbf{D}$ including the one in the following theorem. All these, however, are particular cases of Theorem 4.2.

Theorem 5.2 *For $a \in \mathbf{D}$, treating $\cos\theta_a(t)$ as 2π-periodic on the line, we have*

(51)

$$H\cos\theta_a(t) = \sin\theta_a(t) + \mathrm{Im}\,(a) \quad \text{and} \quad H\sin\theta_a(t) = -\cos\theta_a(t) - \mathrm{Re}\,(a).$$

For $a \in D$, writing $a = |a|e^{it_a}$, it is easy to verify

$$\tau_a(z) = e^{it_a}\tau_{|a|}(ze^{-it_a}), \quad \theta_a(t) = t_a + \theta_{|a|}(t - t_a) \tag{52}$$

and

$$P_a^{\mathbf{D}}(t) = P_{|a|}^{\mathbf{D}}(t - t_a) = p_{|a|}(t - t_a).$$

We define

$$L_a^2(\partial D) = \left\{ f : \partial D \to \mathbf{C} \mid \left(\int_0^{2\pi} |f(t)|^2 P_a(t)\,dt \right)^{1/2} < \infty \right\}, \tag{53}$$

and denote by $\|f\|_a = \left(\int_0^{2\pi} |f(t)|^2 P_a(t)\,dt \right)^{1/2}$ the norm of $f \in L_a^2(\partial D)$. We call $L_a^2(\partial D)$ the weighted L^2-space associated with a. The space $L_a^2(\partial D)$ is a Hilbert space equipped with the inner product

$$\langle f, g \rangle_a = \int_0^{2\pi} f(t)\overline{g(t)} P_a(t)\,dt.$$

Note that, if $a = 0$, then all these reduce to the standard case on ∂D.

We have the following

Theorem 5.3. *Let $a \in D$ and $\mathcal{F}_a = \{1/\sqrt{2\pi}e^{in\theta_a(t)}\}_{n=-\infty}^{\infty}$. Then*

(i) *$\mathcal{F}_a$ is a weighted orthonormal system with weight P_a in $[0, 2\pi]$.*

(ii) *The Plancherel theorem holds for the system. Especially, the system is complete in L_a^2.*

(iii) *For the system $\mathcal{F}_a$ the Carleson theorem holds.*

(iv) *The mapping $\theta_a(t)$ preserves the Hardy spaces inside and outside the unit circle.*

The proof is by change of variable, also see [**10**].

In view of Theorem 5.3 every $a \in \mathbf{D}$ corresponds to a trigonometric system. For different numbers a in $\mathbf{D}$, the shapes of the graphs of $\cos\theta_a(t)$ and those of $\sin\theta_a(t)$ are different (see [**11**] for graphical examples). It is observed that the closer the complex number $|a|$ gets to $1-0$, the sharper the graph of $\cos\theta_a(t)$ is. The generalized trigonometric systems are expected to be better suited and adaptable, along with different choices of a, to nonlinear and non-stationary time-frequency analysis.

To study the counterpart theory on the real line we again use the Cayley transform $w(z) = (i-z)/(i+z)$ that maps the real line to the unit circle through

$$w(s) = \frac{1-s^2}{1+s^2} + i\,\frac{2s}{1+s^2}.$$

Setting $t = 2\tan^{-1} s$, the above reads $w(s) = \cos t + i\sin t$, where $t \in (-\pi,\pi)$, $s \in (-\infty,\infty)$. Now, if $f(t) = \cos\theta(t) + i\sin\theta(t)$ is the boundary value of an inner function inside D, then

$$F(s) = \cos\theta(2\tan^{-1} s) + i\sin\theta(2\tan^{-1} s)$$

is the boundary value of the corresponding inner function in the upper-half complex plane. Invoking Theorem 4.2, we have

$$H[\cos\theta(2\tan^{-1} s)] = \sin\theta(2\tan^{-1} s).$$

Based on the same principle, we have

Theorem 5.4. *Assume that Θ is a function on $\mathbf{R}$ that is continuous, strictly increasing with $m(\Theta(\mathbf{R})) = 2\pi n$. Then, the following two conditions are equivalent.*

(i) *$d\Theta(s)$ is a sum of a number of n harmonic measures on the line.*

(ii)

$$H\cos\Theta(s) = \sin\Theta(s).$$

Moreover, if (i) *or* (ii) *holds, then $\Theta(s) = \theta(2\tan^{-1} s)$, where θ is a function satisfying the conditions in Theorem* 5.1.

Proof. In view of the Cayley transform, the proof is converted to the unit disc case. We only note that with the single factor case $\Theta(s) = \theta_a(2\tan^{-1} s)$, the derivative $(d/ds)\Theta(s)$ is a Poisson kernel on the line. In fact,

$$\frac{d}{ds}\,\Theta(s) = \frac{1}{\pi}\,\frac{h_0}{(s-s_0)^2 + h_0^2} = P_{h_0}(s-s_0),$$

where

$$h_0 = \frac{1-|a|^2}{1+2|a|\cos t_0 + |a|^2}, \quad s_0 = \frac{2|a|\sin t_0}{1+2|a|\cos t_0 + |a|^2}$$

and $a = |a|e^{it_0}$.

Analogous to the weighted trigonometric series on the circle we can formulate a weighted Fourier transform theory on the line.

Define

$$L_a^2(\mathbf{R}) = \left\{ f : \mathbf{R} \to \mathbf{C} \mid \left(\int_{-\infty}^{\infty} |f(t)|^2 \tilde{P}_a(t)\,dt \right)^{1/2} < \infty \right\}, \tag{55}$$

where $\widetilde{P}_a$ is the 2π-periodic extension of the circular Poisson kernel P_a associated with $a \in \mathbf{D}$.

Denote

$$\|f\|_a = \left(\int_{-\infty}^{\infty} |f(t)|^2 \tilde{P}_a(t)\,dt \right)^{1/2}$$

the norm of $f \in L_a^2(\mathbf{R})$.

The space $L_a^2(\mathbf{R})$ forms a Hilbert space under the inner product

$$\langle f, g\rangle_a = \int_{-\infty}^{\infty} f(t)\overline{g(t)}\tilde{P}_a(t)\,dt.$$

Note that, if $a = 0$, then all those just defined reduce to the standard case on $\mathbf{R}$.

Define the new Fourier transformation by

$$F_a(f)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\xi\theta_a(t)} f(t)\widetilde{P}_a(t)\,dt.$$

For the generalized case we have the corresponding Plancherel theorem and the Fourier inversion theorem that are all deducible from the classical case through change of variable. There is also an associated Poisson summation formula that reads, [**10**],

$$\sum f(t_0 + 2k\pi) = \sum F_a(f)(k).$$

6. Constructive examples of inner functions. Based on the following holomorphic mappings

(i) The Cayley transform $\mathbf{C}^+ \to \mathbf{D}$ defined by

$$z = \kappa(w) = \frac{i - w}{i + w}.$$

The mapping $\kappa : \mathbf{C}^+ \to \mathbf{D}$ is univalent and onto.

(ii) The mappings $\mathbf{C}^+ \to \mathbf{D}$

$$\varepsilon_L(z + 2L) = \varepsilon_L(z), \quad L > 0.$$

They are onto but not univalent. They are periodic, satisfying $\varepsilon_L \times (z + 2L) = \varepsilon_L(z)$. Denote by $[\varepsilon]$ the class of such mappings.

(iii) Möbius transforms $\mathbf{D} \to \mathbf{D}$

$$\tau_a(z) = \frac{-\overline{a}}{|a|} \frac{z - a}{1 - \overline{a}z}, \quad a \in \mathbf{D}.$$

The mappings are univalent and onto. We denote by $[\tau]$ the class of Möbius transforms.

(iv) The mappings $\mathbf{C}^+ \to \mathbf{C}^+$

$$\mu_{a,b,c,d}(z) = \frac{az + b}{cz + d}, \quad a, b, c, d \text{ real numbers, and } ad - bc > 0.$$

The conformal mappings are univalent and onto. We denote the class of such mappings by $[\mu]$.

(v) The mappings $\mathbf{C}^+ \to \mathbf{D}$

$$\nu_a(z) = \frac{|a^2 + 1|}{a^2 + 1} \frac{w - a}{w - \overline{a}}, \quad a \in \mathbf{C}^+ \setminus \{i\}.$$

The mappings are univalent and onto. We denote the class of the mappings by $[\nu]$. Clearly, $\kappa \in [\nu]$.

(vi) Denote by $[b]$ and $[B]$ the classes of Blaschke products in $\mathbf{D}$ and $\mathbf{C}^+$, respectively.

(vii) Denote by $[f]$ and $[F]$ the classes of inner functions in $\mathbf{D}$ and $\mathbf{C}^+$, respectively.

We can construct functions in $[F]$ from the above listed elementary ones. Some examples are given in the following theorem.

Theorem 6.1. *We have*

(i) $[\tau] \circ [\varepsilon] \subset [b] \circ [\varepsilon] \subset [f] \circ [\varepsilon] \subset [F]$.

(ii) $[\nu] = \kappa \circ [\mu] = [\tau] \circ \kappa \subset [b] \circ \kappa = [B] \subset [f] \circ \kappa = [F]$.

(iii) $[\tau] \circ [\varepsilon] \circ [\mu] \subset [b] \circ [\varepsilon] \circ [\mu] \subset [f] \circ [\varepsilon] \circ [\mu] \subset [F]$.

(iv) *Products of functions in the classes in* (i)–(iii) *are functions in* $[F]$.

Proof. The constrictions are based on the fact that compositions and multiplications of inner functions are still inner functions.

We conclude the paper by drawing a number of remarks to the theorem.

Remark 1. The class $[b] \circ [\varepsilon]$ in assertion (i) consists of the phase signals on $\mathbf{R}$ of the form

$$F_1(t) = e^{i \sum \theta_{a_k}(\pi t/L_k)}, \quad a_k \in \mathbf{D}, \quad L_k > 0,$$

where we take the convention that $\theta_{a_k}(t + 2\pi) = \theta_{a_k}(t) + 2\pi$, and thus each factor in the above product is periodic. The smallest class in (i) is $[\tau] \circ \varepsilon_\pi$ that is studied in Theorem 5.2.

We now cite a non-distributional proof for the boundary values of the functions in $[f] \circ [\varepsilon]$ being phase signals. Take $L = \pi$ and f an inner function in $\mathbf{D}$. Then we can write $f \circ e^{it} = c(t) + is(t)$, where $c(t)$ and $s(t)$ are 2π-periodic. Owing to the periodic property of $c(t)$ the principal value integral in the definition of Hilbert transform is well defined, and we have

$$\text{p.v. } \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{x-t}\, c(t)\, dt = \text{p.v. } \frac{1}{\pi}\int_{0}^{2\pi} \sum_{k=-\infty}^{\infty} \frac{1}{x-t+2k\pi}\, c(t)\, dt$$
$$= \text{p.v. } \frac{1}{2\pi}\int_{0}^{2\pi} \cot\left(\frac{x-t}{2}\right) c(t)\, dt,$$

where the identity

$$\lim_{N\to\infty} \sum_{k=-N}^{N} \frac{1}{x-t+2k\pi} = \frac{1}{2}\cot\left(\frac{x-t}{2}\right)$$

may be found, for instance, in [**9**]. Due to $f \in \mathbf{N}(\mathbf{D})$, we have

$$Hc(t) = \widetilde{H}c(t).$$

Similarly,

$$Hs(t) = \widetilde{H}s(t).$$

Remark 2. The functions in the class $[\nu]$ of the assertion (ii) is discussed in Theorem 5.4. They are the atomic cases of the class $[B]$. The class $[B]$ together with a factor of linear phase is studied in Picinbono [**8**]. As analyzed in Section 4, any linear-phase signal $e^{i\omega_0 t}$ comes from the mass at the infinity of the Borel measure in the integral representation of the singular function. In fact, the whole class of functions $[b] \circ [\varepsilon]$, in assertions (i) and (iii), are from singular functions. To see this, we note that functions $[b] \circ [\varepsilon]$ are periodic but functions in $[B]$ are not. The role of singular functions therefore is by no means "singular."

Remark 3. It is easy to see that if $f_1, f_2, \ldots, f_n$ are inner functions, then the composition $f_1 \circ \cdots \circ f_n$ is also an inner function. The question is: by doing composition consecutively, do we always get new types of inner functions? The answer is negative in view of Frostman's theorem. Indeed, Frostman's theorem asserts that if f is an inner function in $\mathbf{D}$, then the composition of f with a Möbius transform

$$f_\zeta = \frac{f(z)-\zeta}{1-\overline{\zeta}\, f(z)}$$

is a Blaschke product, except possibly some ζ in a set of capacity zero. Frostman's theorem in our notation is $[\tau] \circ [f] \subset [b]$. On the real line Frostman's theorem reads $[\tau] \circ [F] \subset [B]$.

Remark 4. In spite of the argument made in Remark 3, assertion (iii), however, does contain new types of phase signals. The simplest example is of the form $e^{i\mu(s)} = e^{i(as+b)/(cs+d)}$ that are not contained in any subclasses in (i) and (ii). These functions are not periodic but with infinitely many oscillations. They are not Blaschke products as the associated analytic phase signal do not have zeros.

Remark 5. We may construct complicated phase signals based on the product rule stated in assertion (iv). For instance, by multiplying the basic phase signals in (i), (ii) and (iii), we obtain, as long as convergent, the phase signal

$$e^{i\theta_0} \prod_{k=1}^{\infty} \frac{z_k - z}{z - \overline{z}_k} \exp\left(i \sum_{k=1}^{\infty} \theta_{w_k} \left(\frac{a_k z + b_k}{c_k z + d_k} \right) \right),$$

where θ_0 is a real constant, z_k, $k = 1, 2, \dots$, are complex numbers in the upper-half complex plane, w_k, $k = 1, 2, \dots$ are complex numbers in the unit disc, and for each k, the real numbers a_k, b_k, c_k, d_k satisfy $a_k d_k - b_k c_k > 0$.

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