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An elementary proof of the Paley–Wiener theorem in $\mathbf{C}^m$†

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We give an elementary proof of the Paley–Wiener theorem in several complex variables using Clifford algebra. Thanks to the extension of the exponential function $e^{i\langle x,\underline{\xi}\rangle}$, where $\underline{x}$ is extended to $\mathbf{R}^{m+1} = \mathbf{R}_1^m$, and $\underline{\xi}$ is extended to $\mathbf{C}^m$, the proof of the theorem in one complex variable based on analytic continuation is closely followed to give a proof for several complex variables. This shows that with the help of Clifford algebras multi-variable cases may be treated similarly as the single variable case.

Keywords: Fourier analysis; Paley–Wiener theorem; Exponential function; Clifford algebra

MSC 2000: 30G35

1. Introduction

The classical Paley–Wiener theorem (see [1]) states that a necessary and sufficient condition for a square–integrable function f to be extendable to an entire function in the complex plane with an exponential type bound

$$|f(z)| \leq Ce^{A|z|}$$

is that $\operatorname{supp}\hat{f} \subset [-A, A]$. Due to the fundamental role of the theorem in harmonic analysis and in complex analysis, people have been seeking for generalizations of the results to higher dimensional cases, including several complex variables [1–3] and several real variables [4,5]. The Clifford algebra setting of $\mathbf{R}^m$, dealt with in [4], could be said to be the precise analogue of the classical case. The latter is further

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†Dedicated to Richard Delanghe on the occasion of his 65th birthday.

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generalized to the case of co-dimension p in [5]. The present work, however, will concentrate on several complex variables.

We introduce some notations. Denote by K any symmetric body in $\mathbf{R}^m$ and let

$$K^* = \{ y \in \mathbf{R}^m \mid \text{ for all } x \in K, x \cdot y \leq 1\}$$

then K^* is called the polar set of K. Further, let

$$\varepsilon(K) = \Big\{ F \mid F \text{ is an entire function in } \mathbf{C}^m, \text{ and for every } \varepsilon > 0,$$
$$\exists \text{ a constant } C_\varepsilon \text{ such that } |F(\underline{\zeta})| \leq C_\varepsilon e^{(1+\varepsilon)\sup_{y\in K^*}|\underline{\zeta}\cdot y|} \Big\}.$$

The Paley–Wiener theorem in $\mathbf{C}^n$ (see [1]) is stated as follows.

THEOREM 1.1 *Suppose that $F \in L^2(\mathbf{R}^m)$. Then F is the Fourier transform of a function vanishing outside a symmetric body K if and only if F is the restriction to $\mathbf{R}^m$ of a function in $\varepsilon(K)$.*

Note that, if we take K to be the ball with radius A centered at the origin, denoted by $B(0, A)$, then $K^* = K$, and the inequality in the definition of $\varepsilon(K)$ reduces to $|F(\underline{\zeta})| \leq C_\varepsilon e^{(1+\varepsilon)|\underline{\zeta}|}$, where $|\underline{\zeta}| = \sqrt{|\underline{\zeta}_1|^2 + \cdots + |\underline{\zeta}_m|^2}$. In that case, we further have

$$F(\underline{\zeta}) = \frac{1}{(2\pi)^m} \int_{B(0,A)} e^{i\langle \underline{x}, \underline{\zeta}\rangle} \hat{F}(\underline{x}) \mathrm{d}\underline{x} \quad \text{for any } \underline{\zeta} \in \mathbf{C}^m.$$

This particular case turns out to be essential in applications.

In [2] a theorem of the same spirit is proved in tensor form. The inequality in the class $\varepsilon(K)$ is replaced by

$$|F(z)| \leq c_\varepsilon \exp[(b_1 + \varepsilon)|z_1| + \cdots + (b_m + \varepsilon)|z_m|]$$

and the ball $B(0, A)$, in which the Fourier transform of F is supported, is replaced by the box $G_b = \{|z_1| \leq b_1, \ldots, |z_m| \leq b_m\}$, where $b_i > 0$, $i = 1, \ldots, m$.

The proofs of the classical Paley–Wiener theorem may be classified into two categories. The first type uses a Phragmén–Lindelöf type result (see [1]), while the other is based on holomorphic continuation of the Laplace transforms of the entire function in the assumption of the theorem (see [6,7]). The Laplace transforms are defined for all directions in the complex plane whose existences are guaranteed by the exponential-type inequality assumed in the theorem. To the author's knowledge none of those proofs have direct generalizations to several complex variables. For several complex variables the existence of any Phragmén–Lindelöf type result is in question. The usual proofs of the Paley–Wiener theorem for several complex variables are based on the case of one complex variable [1,2]. The present study intends to show that the second type of proof [6,7], based on holomorphic continuation, may be adapted to the case of several complex variables, where holomorphic continuation should be replaced by monogenic continuation in the Clifford analysis setting. This new proof, as a direct generalization of the case of one complex variable, strengthens the philosophy that Clifford algebra enables one to treat multi-variables similarly as a single variable.

It is to be mentioned that, recently, Masanori Suwa and Kunio Yoshino proved a new type of Paley–Wiener theorem for several complex variables, involving hyperfunctions supported by convex compact sets in $\mathbf{C}^m$ [8]. The hyperfunctions, as a special type of linear forms on entire functions in $\mathbf{C}^m$, form the replacement of the Fourier transforms with compact support of square-integrable functions in $\mathbf{R}^m$, and the Fourier–Laplace transforms of the representations of the linear forms on dilated heat kernels are the replacement of the entire functions. The proof uses heat kernel estimates (see [3]).

More precisely, let K be a convex and compact set in $\mathbf{C}^m$. Denote by $\mathcal{A}$ the space of entire functions in $\mathbf{C}^m$ and by $\mathcal{A}'(K)$ the space of linear forms μ on $\mathcal{A}$ carried by K, that is, for every neighborhood ω of K,

$$|\mu(\varphi)| \leq C_\omega \sup_\omega |\varphi|, \qquad \varphi \in \mathcal{A}.$$

The elements of $\mathcal{A}'(K)$ are called hyperfunctions supported by K. The Fourier–Laplace transform of $\mu \in \mathcal{A}'(K)$ is denoted by

$$\tilde{\mu}(\underline{\zeta}) = \lim_{t\to 0+} \frac{1}{(2\pi)^{m/2}} \langle \mu_{x,t}, e^{-i\underline{\zeta}x} \rangle,$$

where

$$\mu_{x,t} = \langle \mu_z, E(x-z,t) \rangle, \qquad E(z,t) = (4\pi t)^{-(n/2)} e^{-|z|^2/4t}, \quad z \in \mathbf{C}^m.$$

Then the Paley–Wiener theorem for hyperfunctions reads

THEOREM 1.2 *Let K be a convex compact set in $\mathbf{R}^m$ and let $\mu \in \mathcal{A}'(K)$. Then $\tilde{\mu}(\underline{\zeta})$ is an entire function and for every $\varepsilon > 0$ there exists a constant $C_\varepsilon \geq 0$ such that*

$$|\tilde{\mu}(\underline{\zeta})| \leq C_\varepsilon e^{h_K(\eta)+\varepsilon|\underline{\zeta}|}, \qquad \underline{\zeta} = \xi + i\eta \in \mathbf{C}^m$$

where $h_K(\eta) = \sup_{x\in K}\langle x, \eta \rangle$. Conversely, if $F(\underline{\zeta})$ is an entire function satisfying the above inequality, then there exists a unique $\mu \in \mathcal{A}'(K)$ such that $F(\underline{\zeta}) = \tilde{\mu}(\underline{\zeta})$.

2. Preliminaries

For more details on the basic concepts and notations recalled in this section, we refer to [9,10].

Let $\mathbf{e}_1, \ldots, \mathbf{e}_m$ be basis elements satisfying $\mathbf{e}_i\mathbf{e}_j + \mathbf{e}_j\mathbf{e}_i = -2\delta_{ij}$, where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$, $i,j = 1,2,\ldots,m$. Let

$$\mathbf{R}^m = \{\underline{x} = x_1\mathbf{e}_1 + \cdots + x_m\mathbf{e}_m \mid x_j \in \mathbf{R}, j = 1,2,\ldots,m\}$$

be identified with the usual Euclidean space $\mathbf{R}^m$, and denote

$$\mathbf{R}_1^m = \{x_0 + \underline{x} \mid \ x_0 \in \mathbf{R}, \ \underline{x} \in \mathbf{R}^m\}.$$

Let

$$\mathbf{C}^m = \{\underline{\zeta} = \underline{\xi} + i\underline{\eta} \mid \underline{\xi}, \underline{\eta} \in \mathbf{R}^m\}$$

So, if $\underline{\zeta} \in \mathbf{C}^m$, then $\underline{\zeta} = \sum_{j=1}^m \zeta_j \mathbf{e}_j$, where $\zeta_j = \xi_j + i\eta_j$. The space $\mathbf{C}^m$ is identified with the space of m complex variables in the usual setting. It follows that

$$\underline{\zeta}^2 = -|\underline{\zeta}|_*^2$$

where

$$|\underline{\zeta}|_*^2 = \sum_{j=1}^m \zeta_j^2 = |\underline{\xi}|^2 - |\underline{\eta}|^2 + 2i\langle \underline{\xi}, \underline{\eta}\rangle. \tag{1}$$

Elements in $\mathbf{R}_1^m$ are called vectors, and those in $\mathbf{R}^m$ are called pure vectors. The real (or complex) Clifford algebra generated by $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m$, and denoted by $\mathbf{R}^{(m)}$ (or $\mathbf{C}^{(m)}$), is the associative algebra generated by $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m$ over the real (or complex) field $\mathbf{R}$ (or $\mathbf{C}$). A general element in $\mathbf{R}^{(m)}$, therefore, is of the form $x = \sum_S x_S \mathbf{e}_S$, where $\mathbf{e}_\emptyset = \mathbf{e}_0$, and $\mathbf{e}_S = \mathbf{e}_{i_1}\mathbf{e}_{i_2}, \ldots, \mathbf{e}_{i_l}$, where $S = \emptyset$ or S runs over all the ordered subsets of $\{1, 2, \ldots, m\}$, namely

$$S = \langle j_1, \ldots, j_l\rangle, \quad 1 \le i_1 < i_2 < \cdots < i_l \le m, \quad 1 \le l \le m.$$

The natural inner product between $x = \sum_S x_S \mathbf{e}_S$ and $y = \sum_S y_S \mathbf{e}_S$ in $\mathbf{C}^{(m)}$, denoted by $\langle x, y\rangle$, is the complex number $\sum_S x_S y_S$. The norm of $x \in \mathbf{R}^{(m)}$ associated with this inner product is

$$|x| = \langle x, x\rangle^{1/2} = \left(\sum_S |x_S|^2\right)^{1/2}.$$

The Clifford conjugate of $x_0 + \underline{x}$ is $\overline{x} = x_0 - \underline{x}$. For any non-zero vector x we have

$$x^{-1} = \frac{\overline{x}}{|x|^2}.$$

The unit sphere $\{x \in \mathbf{R}_1^m\colon |x| = 1\}$ is denoted by $\mathbf{S}^m$. We use $B(x, r)$ for the open ball in $\mathbf{R}_1^m$ centered at x with radius r.

In what follows we study functions defined in $\mathbf{R}^m$ or $\mathbf{R}_1^m$ and taking values in $\mathbf{C}^m$. Therefore, they are of the form $f(x) = \sum_S f_S(x)\mathbf{e}_S$, where f_S are complex-valued functions. We involve the Cauchy–Riemann operator

$$D = D_0 + \underline{D}$$

where $D_0 = \partial/\partial x_0$ and $\underline{D} = (\partial/\partial x_1)\mathbf{e}_1 + \cdots + (\partial/\partial x_m)\mathbf{e}_m$ denotes the Dirac operator. For the sake of symmetry, we write $D_0 = (\partial/\partial x_0) = (\partial/\partial x_0)\mathbf{e}_0$. We define the "left" and "right" roles of the operator D by

$$Df = \sum_{i=0}^m \sum_S \frac{\partial f_S}{\partial x_i} \mathbf{e}_i \mathbf{e}_S$$

and

$$fD = \sum_{i=0}^{m} \sum_{S} \frac{\partial f_S}{\partial x_i} \mathbf{e}_S \mathbf{e}_i.$$

If $Df = 0$ or $fD = 0$, then we say that f is left-monogenic or right-monogenic in the corresponding domain (open and connected). If f is both left- and right-monogenic, then we say that f is monogenic. We recall the existence of Cauchy's theorem and Cauchy's formula for left- and right-monogenic functions.

The Fourier transform in $\mathbf{R}^m$ is defined by

$$\hat{f}(\underline{\xi}) = \int_{\mathbf{R}^m} e^{-i\langle \underline{x}, \underline{\xi}\rangle} f(\underline{x}) \mathrm{d}\underline{x}$$

and the inverse Fourier transform is

$$\tilde{g}(\underline{x}) = \frac{1}{(2\pi)^m} \int_{\mathbf{R}^m} e^{i\langle \underline{x}, \underline{\xi}\rangle} g(\underline{\xi}) \mathrm{d}\underline{\xi}. \tag{2}$$

In order to extend $\underline{x}$ in (2) to $x \in \mathbf{R}_1^m$ so that $\tilde{g}(x)$ becomes monogenic of a certain type we first need to extend the exponential function $e^{i\langle \underline{x}, \underline{\xi}\rangle}$. Denote, for $x = x_0 e_0 + \underline{x}$,

$$e(x, \underline{\xi}) = e^{i\langle \underline{x}, \underline{\xi}\rangle} e^{-x_0 |\underline{\xi}|} \chi_+(\underline{\xi}) + e^{i\langle \underline{x}, \underline{\xi}\rangle} e^{x_0 |\underline{\xi}|} \chi_-(\underline{\xi})$$

where

$$\chi_\pm(\underline{\xi}) = \frac{1}{2}\left(1 \pm i \frac{\underline{\xi}}{|\underline{\xi}|}\right).$$

It is easy to verify that $\chi_\pm$ satisfies the properties of a projection operator

$$\chi_- \chi_+ = \chi_+ \chi_- = 0, \quad \chi_\pm^2 = \chi_\pm, \quad \chi_+ + \chi_- = 1.$$

The function $e(x, \underline{\xi})$ is an extension of $e(\underline{x}, \underline{\xi}) = e^{i\langle \underline{x}, \underline{\xi}\rangle}$ onto $\mathbf{R}_1^m \times \mathbf{R}^m$. It is easy to verify that for any fixed $\underline{\xi}$, the function $e(x, \underline{\xi})$ is monogenic in $x \in \mathbf{R}_1^m$. Extensions of this kind can be first found in Sommen's work [3]. McIntosh further extended $\underline{\xi} \in \mathbf{R}^m$ to $\underline{\xi} + i\underline{\eta} \in \mathbf{C}^m$: for $x = x_0 + \underline{x}$, $\underline{\zeta} = \underline{\xi} + i\underline{\eta}$,

$$\begin{cases} e_+(x, \underline{\zeta}) = e^{i\langle \underline{x}, \underline{\zeta}\rangle} e^{-x_0 |\underline{\zeta}|_*} \chi_+(\underline{\zeta}) \\ e_-(x, \underline{\zeta}) = e^{i\langle \underline{x}, \underline{\zeta}\rangle} e^{x_0 |\underline{\zeta}|_*} \chi_-(\underline{\zeta}) \\ e(x, \underline{\zeta}) = e_+(x, \underline{\zeta}) + e_-(x, \underline{\zeta}) = e^{i\langle \underline{x}, \underline{\zeta}\rangle} e^{-x_0 (i\underline{\zeta})} \end{cases}$$

where

$$\chi_\pm(\underline{\zeta}) = \frac{1}{2}\left(1 \pm \frac{i\underline{\zeta}}{|\underline{\zeta}|_*}\right)$$

and $|\underline{\zeta}|_*$ is the square root of the complex number $|\underline{\zeta}|_*^2$ in (1) with $\mathrm{Re}|\underline{\zeta}|_* > 0$ (see [11,12]). Note that $|\underline{\zeta}|_*$ is a complex number.

It is easy to verify that with the extended domain of $\chi_\pm$ there still holds

$$\chi_+\chi_- = \chi_-\chi_+ = 0, \qquad \chi_\pm^2 = \chi_\pm, \qquad \chi_+ + \chi_- = 1.$$

The function $e(x, \underline{\zeta})$ is an extension of $e(\underline{x}, \underline{\xi}) = e^{i\langle \underline{x}, \underline{\xi}\rangle}$ onto $\mathbf{R}_1^m \times \mathbf{C}^m$. It is easy to verify that $e_+(x, \underline{\zeta}), e_-(x, \underline{\zeta})$ and $e(x, \underline{\zeta})$ are monogenic in $x \in \mathbf{R}_1^m$, and holomorphic in $\underline{\zeta} \in \mathbf{C}^m$. As a piece of art they have been proved to be crucial to harmonic analysis in $\mathbf{R}^m$.

It is easy to verify that $e(\underline{x}, \underline{\zeta}) = e^{i\langle \underline{x}, \underline{\zeta}\rangle}$ and

$$|e(\underline{x}, \underline{\zeta})| \leq e^{|\underline{x}||\underline{\eta}|} \tag{3}$$

where $\underline{\zeta} = \underline{\xi} + i\underline{\eta}$.

It is useful to introduce the quantity $|\underline{\zeta}|^* = \sqrt{Re^2|\underline{\zeta}|_* + |\underline{\eta}|^2}$. Then there holds $|\underline{\zeta}|^* = |\underline{\xi}|$ if $\underline{\eta} = 0$ and $|\underline{\zeta}|^* = |\underline{\eta}|$ if $\underline{\xi} = 0$.

LEMMA 2.1 *The non-negative function $|\cdot|^*$: $\mathbf{C}^m \to [0, \infty)$ satisfies the relation $|\underline{\zeta}|^* \leq |\underline{\zeta}|$, where the equality holds if and only if $\underline{\eta} = t\underline{\xi}$, $t \in \mathbf{R}$.*

Proof The inequality is equivalent to

$$\mathrm{Re}^2|\underline{\zeta}|_* \leq |\underline{\xi}|^2.$$

Let $\underline{\xi} = \sum_1^m x_i\mathbf{e}_i$, $\underline{\eta} = \sum_1^m y_i\mathbf{e}_i$. Denote $X = \mathrm{Re}|\underline{\zeta}|_*$, $Y = \mathrm{Im}|\underline{\zeta}|_*$. Through simple computations we are reduced to showing

$$X^2 \leq \sum_1^m x_k^2 \tag{4}$$

under the conditions

$$X^2 - Y^2 = \sum_1^m x_k^2 - \sum_1^m y_k^2 \quad \text{and} \quad XY = \sum_1^m x_k y_k. \tag{5}$$

Using Cauchy–Schwarz's inequality the second condition of (5) implies that

$$X^2Y^2 \leq \sum_1^m x_k^2 \sum_1^m y_k^2. \tag{6}$$

If $X^2 = \sum_1^m x_k^2$, then the first condition in (5) implies $Y^2 = \sum_1^m y_k^2$, and (6) will hold as an equality. This concludes that $\underline{\xi}$ and $\underline{\eta}$ are proportional, and the second equality in (5) also holds. Conversely, if $\underline{\xi}$ and $\underline{\eta}$ are proportional, then the conditions in (5) hold, and $X^2 = \sum_1^m x_k^2$. Now consider a positive increment $X^2 = \sum_1^m x_k^2 + \delta$, $\delta > 0$. In that case the first condition of (5) implies that Y^2 has the same increment and thus the inequality (6) cannot hold. The proof is complete. ■

In the particular case $m=1$ we have $\underline{\zeta}=(\xi+i\eta)\mathbf{e}_1$. Note that $|\underline{\zeta}|_*=\mathrm{sgn}(\xi)(\xi+i\eta)$. By taking $\mathbf{e}_1=-i$, we have

$$\begin{aligned} e(x,\underline{\zeta}) &= e^{-i(x_0+x_1\mathbf{e}_1)((\xi+i\eta)\mathbf{e}_1)} \qquad \text{(the Clifford multiple form)} \\ &= e^{i(x_1+ix_0)(\xi+i\eta)} \qquad \text{(the usual complex variable form)} \\ &= e^{ix_1(\xi+i\eta)}e^{-x_0(\xi+i\eta)} \\ &= e^{ix_1(\xi+i\eta)}e^{-x_0|\underline{\zeta}|_*}\left(\frac{1}{2}+\frac{\xi+i\eta}{2|\underline{\zeta}|_*}\right)+e^{-ix_1(\xi+i\eta)}e^{-x_0|\underline{\zeta}|_*}\left(\frac{1}{2}-\frac{\xi+i\eta}{2|\underline{\zeta}|_*}\right) \\ &= e^{i\langle\underline{x},\eta\rangle}e^{-x_0|\underline{\zeta}|_*}\left(\frac{1}{2}+\frac{i\zeta}{2|\underline{\zeta}|_*}\right)+e^{i\langle\underline{x},\eta\rangle}e^{x_0|\underline{\zeta}|_*}\left(\frac{1}{2}-\frac{i\zeta}{2|\underline{\zeta}|_*}\right) \\ &= e^{i\langle\underline{x},\eta\rangle}e^{-x_0|\underline{\zeta}|_*}\chi_+(\underline{\zeta})+e^{i\langle\underline{x},\eta\rangle}e^{x_0|\underline{\zeta}|_*}\chi_-(\underline{\zeta}) \\ &= e_+(x,\underline{\zeta})+e_-(x,\underline{\zeta}). \end{aligned}$$

For the case $m=1$ we have $|\underline{\zeta}|^*=|\underline{\zeta}|=||\underline{\zeta}|_*|$.

3. The Paley–Wiener theorem in $\mathbf{C}^m$

The following result may be found in [11].

LEMMA 3.1 *Let $\underline{\zeta}=\underline{\xi}+i\underline{\eta}\in\mathbf{C}^m$, and assume that $|\underline{\xi}|\neq 0$. Let*

$$\theta=\tan^{-1}\left(\frac{|\underline{\eta}|}{\mathrm{Re}|\underline{\zeta}|_*}\right)\in\left[0,\frac{\pi}{2}\right).$$

Then

$$|\chi_\pm(\underline{\zeta})|\leq\frac{\sec\theta}{\sqrt{2}}.$$

For the case $m=1$ the norm $|\chi_\pm(\zeta)|\equiv 1/\sqrt{2}$.

Our Paley–Wiener theorem in $\mathbf{C}^m$ is stated as follows.

THEOREM 3.2 *Suppose that $F\in L^2(\mathbf{R}^m)$ and let A be a positive real number. Then the following two conditions are equivalent*:

(i) *$F(\underline{\xi})$ can be extended to a holomorphic function $f(\underline{\zeta})$ and*

$$|f(\underline{\zeta})|\leq ce^{A|\underline{\zeta}|^*} \quad \text{for any } \underline{\zeta}=\underline{\xi}+i\underline{\eta}, \tag{7}$$

(ii) $\mathrm{supp}(\hat{F})\subset B(0,A)$.

Moreover, if one of the above conditions holds, then

$$f(\underline{\zeta})=\int_{B(0,A)}e(\underline{x},\underline{\zeta})\hat{F}(\underline{x})\mathrm{d}\underline{x} \quad \text{for any } \underline{\zeta}\in\mathbf{C}^m. \tag{8}$$

Proof (ii) ⇒ (i): If $\operatorname{supp}(\hat{F}) \subset B(0,A)$, then

$$F(\underline{\xi}) = \frac{1}{(2\pi)^m}\int_{\mathbf{R}^m} e(\underline{x},\underline{\xi})\hat{F}(\underline{x})\mathrm{d}\underline{x} = \frac{1}{(2\pi)^m}\int_{B(0,A)} e(\underline{x},\underline{\xi})\hat{F}(\underline{x})\mathrm{d}\underline{x}.$$

Put

$$f(\underline{\zeta}) = \frac{1}{(2\pi)^m}\int_{B(0,A)} e(\underline{x},\underline{\zeta})\hat{F}(\underline{x})\mathrm{d}\underline{x}, \qquad \underline{\zeta} = \underline{\xi} + i\underline{\eta}.$$

Clearly, f is an entire function in $\mathbf{C}^m$ extending F, and, due to estimate (3),

$$|f(\underline{\zeta})| \le ce^{A|\underline{\eta}|} \le ce^{A|\underline{\zeta}|^*} \quad \text{for any } \underline{\zeta} = \underline{\xi} + i\underline{\eta}.$$

(i) ⇒ (ii): Put $F_\varepsilon(\underline{\xi}) = F(\underline{\xi})e^{-\varepsilon|\underline{\xi}|}$, where $\varepsilon > 0$, $\underline{\xi} \in \mathbf{R}^m$. We will show that

$$\lim_{\varepsilon \to 0^+}\int_{\mathbf{R}^m} F_\varepsilon(\underline{\xi})e^{i\langle\underline{x},\underline{\xi}\rangle}\mathrm{d}\underline{\xi} = 0, \quad |\underline{x}| > A. \tag{9}$$

Now, assume temporarily that (9) holds. Since $\|F_\varepsilon - F\|_2 \to 0$ as $\varepsilon \to 0^+$, the Plancherel theorem implies that $\|\hat{F}_\varepsilon - \hat{F}\|_2 \to 0$ as $\varepsilon \to 0^+$. By taking a suitable subsequence $\epsilon_k \to 0^+$ we have pointwise convergence, and therefore (9) implies $\hat{F}(\underline{x}) = 0$ for $|\underline{x}| > A$, that is, $\hat{F}$ vanishes outside $B(0,A)$.

Hence, we are reduced to proving (9). For each $\mathbf{n} = n_0 + \underline{n} \in \mathbf{S}^m$, let $A_{\mathbf{n}}$ be the half-space in $\mathbf{R}_1^m$ defined by

$$A_{\mathbf{n}} = \{x = x_0 + \underline{x} \in \mathbf{R}_1^m,\ \langle x, \mathbf{n}\rangle > A\}$$

and let $\mathbf{n}(\mathbf{C}^m)$ be the associated surface in $\mathbf{C}^m$, defined by

$$\mathbf{n}(\mathbf{C}^m) = \begin{cases} \{\underline{\zeta} = \underline{\xi} + i\underline{\eta} \in \mathbf{C}^m,\ \underline{\xi} \neq 0,\ |n_0|\underline{\eta} = \operatorname{Re}|\underline{\zeta}|_*\underline{n}\},\ n_0 \neq 0 \\ \{\underline{\zeta} = \underline{\xi} + i\underline{\eta} \in \mathbf{C}^m,\ \operatorname{Re}|\underline{\zeta}|_* = 0,\ \underline{\eta} = |\underline{\eta}|\underline{n}\},\ n_0 = 0. \end{cases}$$

Note that the surfaces stretch to infinity. Since $\underline{\zeta} \in \mathbf{n}(\mathbf{C}^m)$ implies $t\underline{\zeta} \in \mathbf{n}(\mathbf{C}^m)$ for $t > 0$, they are, in fact, cones in $\mathbf{C}^m$. For $m = 1$ the surfaces $\mathbf{n}(\mathbf{C}^m)$ reduce to the boundary of the cone with axis $\mathbf{e}_1$ and opening angle 2θ, where $\theta = \tan^{-1}|n_0|/|\underline{n}|$. The definition of $\mathbf{n}(\mathbf{C}^m)$ shows that opposite values of n_0 correspond to the same surface. On a surface $\mathbf{n}(\mathbf{C}^m)$ the quantities $\operatorname{Re}|\underline{\zeta}|_*$ and $|\underline{\eta}|$ are proportional and the vector $\underline{n}$ indicates the direction (or axis) of the cone. $n_0 = 0$ is a limit case: for $m = 1$ it is a ray.

For $\mathbf{n} \in \mathbf{S}^m$, put

$$\Phi_{\mathbf{n}}(x) = \frac{1}{(2\pi)^m}\int_{\mathbf{n}(\mathbf{C}^m)} f(\underline{\zeta})[e_+(x,\underline{\zeta})\chi_{[n_0]} - e_-(x,\underline{\zeta})\chi_{[-n_0]}]\mathrm{d}\underline{\zeta}$$

where

$$\chi_{[n_0]} = \begin{cases} 1, & n_0 \ge 0 \\ 0, & n_0 < 0. \end{cases}$$

We now show that $\Phi_{\mathbf{n}}(x)$ is right-monogenic in $A_{\mathbf{n}}$.

In fact, when $\mathbf{n} = n_0 + \underline{n} \in \mathbf{S}^m$, $n_0 > 0$, we have

$$\Phi_{\mathbf{n}}(x) = \frac{1}{(2\pi)^m} \int_{\mathbf{n}(\mathbf{C}^m)} f(\underline{\zeta}) e_+(x, \underline{\zeta}) \mathrm{d}\underline{\zeta}. \tag{10}$$

For any $x \in A_{\mathbf{n}}$, $\underline{\zeta} = \underline{\xi} + i\underline{\eta} \in \mathbf{n}(\mathbf{C}^m)$, we have $|\underline{\eta}|/\mathrm{Re}|\underline{\zeta}|_* = |\underline{n}|/n_0$. Since $|\chi_+(\underline{\zeta})| \leq \sec\theta/\sqrt{2} = 1/\sqrt{2}n_0$ (from Lemma 2.1), we have

$$\begin{aligned}
|e_+(x, \underline{\zeta})| &= |e^{i\langle \underline{x}, \underline{\zeta}\rangle} e^{-x_0|\underline{\zeta}|_*} \chi_+(\underline{\zeta})| \\
&\leq c_{n_0} e^{-\langle \underline{x}, \underline{\eta}\rangle - x_0 \mathrm{Re}|\underline{\zeta}|_*} \\
&= c_{n_0} e^{-\{\langle \underline{x}, \underline{n}\rangle + x_0 n_0\}\mathrm{Re}|\underline{\zeta}|_*/n_0} \\
&= c_{n_0} e^{-\langle x, \mathbf{n}\rangle \mathrm{Re}|\underline{\zeta}|_*/n_0} \\
&= c_{n_0} e^{-\langle x, \mathbf{n}\rangle |\underline{\zeta}|^*} \\
&\leq c_{n_0} e^{-(A+\delta))|\underline{\zeta}|^*}
\end{aligned}$$

where $\delta > 0$, $c_{n_0} = 1/\sqrt{2}n_0$. Due to assumption (7) the integrand in (10) decays exponentially, which allows us to differentiate under the integral sign. The monogeneity of e_+ then implies the one of Φ_n. This concludes that $\Phi_{\mathbf{n}}(x)$ is right-monogenic in $A_{\mathbf{n}}$ when $\mathbf{n} \in \mathbf{S}^m$, $n_0 > 0$. The proof of $\Phi_{\mathbf{n}}(x)$ being right-monogenic in $A_{\mathbf{n}}$ for $\mathbf{n} \in \mathbf{S}^m$ and $n_0 < 0$ is proved similarly.

When $\mathbf{n} = \underline{n} \in \mathbf{R}^m$, $|\underline{n}| = 1$,

$$\Phi_{\underline{n}}(x) = \frac{1}{(2\pi)^m} \int_{\underline{n}(\mathbf{C}^m)} f(\underline{\zeta})[e_+(x, \underline{\zeta}) - e_-(x, \underline{\zeta})] \mathrm{d}\underline{\zeta}.$$

For $\underline{\zeta} \in \underline{n}(\mathbf{C}^m)$, $\mathrm{Re}|\underline{\zeta}|_* = 0$, $x \in A_{\mathbf{n}}$, it follows that

$$\begin{aligned}
|e_+(x, \underline{\zeta})| = |e_-(x, \underline{\zeta})| &\leq ce^{-\langle \underline{x}, \underline{\eta}\rangle} = ce^{-\langle \underline{x}, |\underline{\eta}|\underline{n}\rangle} \\
&= ce^{-\{\langle \underline{x}, \underline{n}\rangle\}|\underline{\eta}|} = ce^{-\langle x, \mathbf{n}\rangle |\underline{\eta}|} \\
&= ce^{-\langle x, \mathbf{n}\rangle |\sqrt{\mathrm{Re}^2|\underline{\zeta}|_* + |\underline{\eta}|^2}|} \\
&\leq ce^{-(A+\delta)|\underline{\zeta}|^*}
\end{aligned}$$

for some $\delta > 0$. This implies that $\Phi_{\mathbf{n}}(x)$ is right-monogenic in $A_{\mathbf{n}}$ when $\mathbf{n} = \underline{n} \in \mathbf{R}^m$, $|\underline{n}| = 1$.

For the particular cases $\mathbf{n} = \mathbf{e}_0$ and $\mathbf{n} = -\mathbf{e}_0$, we have

$$\Phi_{\mathbf{e}_0}(x) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^m} f(\underline{\xi}) e_+(x, \underline{\xi}) \mathrm{d}\underline{\xi}, \quad x_0 > 0$$

and

$$\Phi_{-\mathbf{e}_0}(x) = -\frac{1}{(2\pi)^n} \int_{\mathbf{R}^m} f(\underline{\xi}) e_-(x, \underline{\xi}) \mathrm{d}\underline{\xi}, \quad x_0 < 0.$$

$\Phi_{\mathbf{e}_0}(x)$ and $\Phi_{-\mathbf{e}_0}(x)$ are right-monogenic in the extended regions, the respective half planes, because $F \in L^2(\mathbf{R}^m)$.

It is easy to verify that

$$\int_{\mathbf{R}^m} f_\varepsilon(\underline{\xi}) e^{i\langle \underline{x}, \underline{\xi}\rangle} \, d\underline{\xi} = \Phi_{\mathbf{e}_0}(\varepsilon + \underline{x}) - \Phi_{-\mathbf{e}_0}(-\varepsilon + \underline{x}). \tag{11}$$

We thus have to show that the right-hand side of (11) tends to 0 as $\varepsilon \to 0^+$ for $|\underline{x}| > A$.

We show that there is a unique right-monogenic function Φ for $|x| > A$ which coincides with all the functions $\Phi_{\mathbf{n}}$ for $A_{\mathbf{n}}$. Temporarily accepting this, we have, for $|\underline{x}| > A$,

$$\lim_{\varepsilon \to 0^+} \Phi_{\mathbf{e}_0}(\varepsilon + \underline{x}) = \lim_{\varepsilon \to 0^+} \Phi(\varepsilon + \underline{x})$$

$$\lim_{\varepsilon \to 0^+} \Phi_{-\mathbf{e}_0}(\varepsilon + \underline{x}) = \lim_{\varepsilon \to 0^+} \Phi(-\varepsilon + \underline{x})$$

and hence

$$\lim_{\varepsilon \to 0^+} \Phi(\varepsilon + \underline{x}) - \lim_{\varepsilon \to 0^+} \Phi(-\varepsilon + \underline{x}) = \Phi(\underline{x}) - \Phi(\underline{x}) = 0.$$

Thus

$$\lim_{\varepsilon \to 0^+} \int_{\mathbf{R}^m} f_\varepsilon(\underline{\xi}) e^{i\langle \underline{x}, \underline{\xi}\rangle} \mathrm{d}\underline{\xi} = 0, \quad |\underline{x}| > A.$$

We now prove the existence of the unique right-monogenic continuation by showing that any two of the functions $\Phi_{\mathbf{n}}$ agree in an open set of the intersection of their domains. If $\mathbf{n}^1, \mathbf{n}^2 \in \mathbf{S}^m$, and $\angle(\mathbf{n}^1, \mathbf{n}^2) \le \pi/2$, then the set of the inner points of $A_{\mathbf{n}^1} \cap A_{\mathbf{n}^2}$ is non-empty. Let x be such an inner point. Consider

$$\int_{\Gamma_r} f(\underline{\zeta})[e_+(x, \underline{\zeta})\chi_{[n_0]} - e_-(x, \underline{\zeta})\chi_{[-n_0]}]\mathrm{d}\underline{\zeta}$$

where Γ_r is the part of the sphere $\{|\underline{\zeta}| = r | \underline{\zeta} \in \mathbf{C}^m\}$ in $\mathbf{C}^m$ joining the two surfaces $\mathbf{n}^1(\mathbf{C}^m)$ and $\mathbf{n}^2(\mathbf{C}^m)$, and $\mathbf{n}$ is any point on the geodesics of $\mathbf{S}^m$ joining $\mathbf{n}^1$ and $\mathbf{n}^2$. As $r \to \infty$, the integrand decays exponentially, while the surface area of Γ_r grows at most polynomially. We therefore conclude that the integral tends to zero as $r \to \infty$. Using Cauchy's theorem for the surfaces in $\mathbf{C}^m$ we conclude that $\Phi_{\mathbf{n}^1}(x) = \Phi_{\mathbf{n}^2}(x)$. This shows that all $\Phi_{\mathbf{n}}$ extend right-monogenically to each other, to become a unique right-monogenic function in the union of their domains. The union contains the set $|x| > A$. The proof is complete. ∎

An immediate consequence of this theorem is

COROLLARY 3.3. *Assume that $F \in L^2(\mathbf{R}^m)$ and let A be a positive real number. Then the following two conditions are equivalent*:

(i) *$F(\underline{\xi})$ can be extended to an entire function $f(\underline{\zeta})$ and $|f(\underline{\zeta})| \le ce^{A|\underline{\eta}|}$, for any $\underline{\zeta} = \underline{\xi} + i\underline{\eta}$.*

(ii) $\operatorname{supp}(\hat{F}) \subset B(0, A)$.

Moreover, if one of the above conditions holds, we have

$$f(\underline{\zeta}) = \int_{B(0,A)} e(\underline{x}, \underline{\zeta})\hat{F}(\underline{x})\mathrm{d}\underline{x} \quad \text{for any } \underline{\zeta} \in \mathbf{C}^m.$$

Proof In the part (ii) $\Rightarrow$ (i) of the proof of Theorem 1.1, we have obtained that the condition (ii) implies the stronger inequality $|f(\underline{\zeta})| \leq ce^{A|\underline{\eta}|}$. ■

By combining the particular case $K = B(0, A)$ of Theorem 1.1 (see remark after the statement of the theorem), we further have:

COROLLARY 3.4 *Assume that* $F \in L^2(\mathbf{R}^m)$ *and let* A *be a positive real number. The following conditions are equivalent*:

(i) $F(\underline{\xi})$ *can be extended to an entire function* $f(\underline{\zeta})$, *and* $|f(\underline{\zeta})| \leq ce^{A|\underline{\zeta}|}$, *for any* $\underline{\zeta} = \underline{\xi} + i\underline{\eta}$.
(ii) $F(\underline{\xi})$ *can be extended to an entire function* $f(\underline{\zeta})$, *and* $|f(\underline{\zeta})| \leq ce^{A|\underline{\zeta}|^*}$, *for any* $\underline{\zeta} = \underline{\xi} + i\underline{\eta}$.
(iii) $F(\underline{\xi})$ *can be extended to an entire function* $f(\underline{\zeta})$, *and* $|f(\underline{\zeta})| \leq ce^{A|\underline{\eta}|}$, *for any* $\underline{\zeta} = \underline{\xi} + i\underline{\eta}$.
(iv) $\text{supp}(\hat{F}) \subset B(0, A)$.

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