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Two families of unit analytic signals with nonlinear phase

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Abstract

This paper focuses on constructing two families of unit analytic signals with nonlinear phase. The first is the 2π-periodic extension of the nonlinear Fourier atoms, viz. $\{e^{i\theta_a(t)} : |a| < 1, t \in \mathbb{R}\}$, where $\theta_a'(t)$ is the Poisson kernel of the unit circle associated with a in the unit disc in the complex plane and satisfies $\theta_a(t + 2\pi) = \theta_a(t) + 2\pi$; and the second consists of $\{e^{i\phi_a(t)} : |a| < 1, t \in \mathbb{R}\}$, that are the images of the nonlinear Fourier atoms under Cayley transform. These unit analytic signals are mono-components based on which one can define meaningful instantaneous frequency. The pairs $(1, \theta_a(t))$ and $(1, \phi_a(t))$ form canonical pairs. The real signals $\cos\theta_a(t)$ corresponding to the first family coincide with the notion of normalized intrinsic mode functions. We finally point out that, starting from nonlinear Fourier atoms, the Gram–Schmidt procedure leads to Laguerre bases.

Keywords: Hilbert transform; Nonlinear and non-stationary signal; Möbius transform; Cayley transform; Instantaneous frequency

1. Introduction

A cosine modulation $f(t) = \rho\cos(\omega_0 t + \phi_0) = a\cos\phi(t)$ has the frequency ω_0 that is the derivative of the linear phase $\phi(t) = \omega_0 t + \phi_0$. To generalize this notion, any real signal $f(t)$ may be written into an amplitude–frequency modulation with an amplitude $\rho(t)$ and a time varying phase $\phi(t)$, where $\phi(t)$ is, in general, nonlinear:

$$f(t) = \rho(t)\cos\phi(t) \quad \text{with } \rho(t) \geq 0. \tag{1.1}$$

By analytic signal approach one can determine a unique such decomposition (1.1) by the following procedure. Let $A(f)$ be the *analytic signal associated with* f with the characteristic property

$$\widehat{A(f)}(\omega) = \begin{cases} 2\hat{f}(\omega) & \text{if } \omega \geq 0 \\ 0 & \text{if } \omega < 0. \end{cases}$$

Equivalently,

$$A(f)(t) = f(t) + i\mathcal{H}f(t),$$

where $\mathcal{H}$ stands for Hilbert transform defined through the principal value integral

$$\mathcal{H}f(t) = \frac{1}{\pi}\,\text{p.v.}\int_{-\infty}^{\infty}\frac{f(x)}{t-x}dx = \lim_{\epsilon\to 0}\frac{1}{\pi}\int_{|x-t|>\epsilon}\frac{f(x)}{t-x}dx.$$

The analytic signal $A(f)(t)$ is complex-valued which can be represented in the quadrature form

$$A(f)(t) = \rho(t)e^{i\phi(t)}.$$

Under the conditions $\phi'(t)$ is non-negative or non-positive for all t, then the quantities $\rho(t)$ and $\phi'(t)$ are called the *instantaneous amplitude* and *instantaneous frequency*, of the real signal $f(t)$, respectively. From the above procedure we see that if these concepts can be defined in some cases, then they are uniquely defined. Especially, if the amplitude of the analytic signal $A(f)$ is of constant 1, we call it *unit analytic signal*. The corresponding modulation

$$f(t) = \rho(t)\cos\phi(t)$$

is called the *canonical amplitude–phase modulation,* or *canonical modulation* for short.

The notion of instantaneous amplitude and frequency, however, is not valid for *multi-components* (see, e.g., [3,9,

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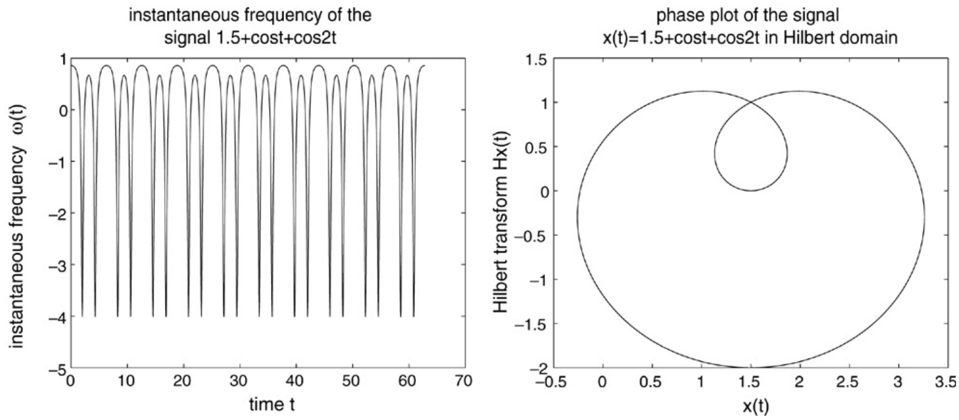

Fig. 1. Left: the instantaneous frequency of the signal $f(t) = 1.5 + \cos t + \cos 2t$. Right: the phase plot of the signal $f(t) = 1.5 + \cos t + \cos 2t$ in Hilbert domain.

10,27,7,8,16,13,14,5,15]). The latter corresponds to the cases where either the condition $\theta'(t) \geq 0$ or the condition $\theta'(t) \leq 0$ hold for all t. For instance, the corresponding $\theta'(t)$ of the signal $f(t) = 1.5 + \cos t + \cos 2t$ obtained through its analytic signal sometimes positive and sometimes negative. In such cases we call the quantities $\theta'(t)$ the *pseudo-instantaneous frequency*, and the corresponding amplitude $\rho(t)$ the *pseudo-instantaneous amplitude*, respectively, of the signal. The left plot of Fig. 1 illustrates the pseudo-instantaneous frequency in time–frequency domain of this signal $f(t)$. It tells us that the pseudo-instantaneous frequency of the signal $f(t) = 1.5 + \cos t + \cos 2t$ does not have physical meaning. The right plot of Fig. 1 presents the phase plot in Hilbert domain (the horizontal axis is $x(t)$ and the vertical axis is $Hx(t)$) of the signal $f(t) = 1.5 + \cos t + \cos 2t$ which is not a closed Jordan curve. From these we know that the signal is not a mono-component but a multi-component. This suggests to decompose multi-components into a sum of mono-components to which meaningful instantaneous frequency may be defined.

The *empirical mode decomposition* (see [17]) is an algorithm designed for decomposing a complicated transient signal into a sum of *intrinsic mode functions*. The authors in [17] expect that for every intrinsic mode function a meaningful instantaneous frequency can be defined. They describe the intrinsic mode functions by the characterizing zero-crossing, extremals and local symmetry of the signal under study, that is, the number of the extremal and the number of the zero crossings must either equal or differ at most by one, and, at any point, the mean value of the envelope defined by the local maxima and the envelope defined by the local minima is zero. A large amount of literature discusses the intrinsic mode functions, see, for instance, [1,3,5,9,20,21]. The key issue is: under what conditions the quadrature signal $A(t) := \rho(t)e^{i\phi(t)}$ coincides with an analytic signal $A(f)(t)$? The relation $A(t) = A(f)(t)$ is equivalent to

$$\mathcal{H}(\rho(t)\cos\theta(t)) = \rho(t)\sin\theta(t). \tag{1.2}$$

The appearance of Eq. (1.2) seems to suggest connections with the Bedrosian and Nuttall theorems (see [1,20]). In [21], a pair $(\rho(t), \theta(t))$ of amplitudes and phases is called a canonical pair if (1.2) holds with monotonous phase $\theta(t)$. Our interest here is only limited to a study of canonical pairs $(1, \theta(t))$.

In this paper, we will construct two families of unit analytic signals and study their time–frequency properties. We will be based on nonlinear Fourier atoms (see [25,22,24]) and Cayley transform. The nonlinear Fourier atoms $e^{in\theta_a(t)}$, $a \in D$, where D is the open unit disc in the complex plane, are time-limited functions defined in $[-\pi, \pi]$. They are first noted in [25,22]. It was proved that the nonlinear Fourier atoms behave nicely with *circular Hilbert transform*. We will show that the periodic nonlinear Fourier atoms $e^{i\theta_a(t)}$, $t \in \mathbb{R}$, where $\theta_a(t)$ satisfy $\theta_a(t + 2k\pi) = \theta_a(t) + 2k\pi$, $k = 0, \pm 1, \pm 2, \ldots$ behave nicely with Hilbert transform, which implies that $e^{i\theta_a(t)}$, $t \in \mathbb{R}$ are mono-components and meaningful instantaneous frequencies may be defined through them. The spectrum of each $e^{i\theta_a(t)}$ is impulses at all non-negative integers with exponentially decreasing components. The phase of each $e^{i\theta_a(t)}$ can be decomposed into a sum of a linear part and a periodic part. The second family of unit signals are images, $e^{\phi_a(t)}$, of the time-limited nonlinear Fourier atoms mapped by Cayley transform. Through calculation, we find that its corresponding real signal has the form $\frac{p_1(t)}{p_2(t)}$ for some polynomials p_1 and p_2, and its phase is of the form $\arctan \frac{q_1(t)}{q_2(t)}$ for some polynomials q_1 and q_2. It is proved that the spectrum is a sum of a negative impulse at the zero point and an exponentially decaying multiple of the Heaviside function. We also relate the nonlinear Fourier atoms to Laguerre filters and Laguerre bases.

The writing plan is as follows. Section 2 is devoted to reviewing circular Hilbert transform and its properties, dealing with Hilbert transform of time-limited signals. In Section 3, we construct two kinds of unit analytic signals. Section 4 is contributing to a study of the relationship of nonlinear Fourier atoms and Laguerre bases. Section 5 deals with the spectrum of the two families of unit analytic signals. Some conclusions are drawn in Section 6.

2. Hilbert transform of time-limited signals

Let $f(t)$ be a time-limited signal with finite energy in the time interval $[-\pi, \pi]$, viz., a square-integrable function. It can be expended into an $L^2([-\pi, \pi])$-convergent Fourier series

$$f(t) = \sum_{n\in\mathbb{Z}} c_n(f)e^{int}$$

where the Fourier coefficients $c_n(f)$ are defined by

$$c_n(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\mathrm{e}^{-\mathrm{i}nt}\mathrm{d}t.$$

Such a signal may be 2π-periodically extended to the whole real line, still denoted by $f(t)$. We formally define its Hilbert transform by

$$\begin{aligned}\mathcal{H}f(t) &= \frac{1}{\pi}\lim_{\epsilon\to 0}\lim_{N\to\infty}\int_{(-(2N+1)\pi,(2N+1)\pi)\cap\{|x-t|>\epsilon\}}\frac{f(x)}{t-x}\mathrm{d}x\\ &= \frac{1}{\pi}\lim_{\epsilon\to 0}\lim_{N\to\infty}\\ &\quad\times\int_{(-\pi,\pi)\cap\{|x-t|>\epsilon\}}\left(\sum_{k=-N}^{N}\frac{1}{t-x-2k\pi}\right)f(x)\mathrm{d}x\\ &= \lim_{\epsilon\to 0}\int_{(-\pi,\pi)\cap\{|x-t|>\epsilon\}}\cot\left(\frac{t-x}{2}\right)f(x)\mathrm{d}x.\end{aligned}$$

Note that the transform is defined through principal integrals at both infinity and locally at the point t. The principal integral at infinity introduces the symmetric summation in integers k and the corresponding infinite series uniformly converges to the continuous function $\cot\left(\frac{t-x}{2}\right)$ for any fixed $\epsilon > 0$.

We show that the Hilbert transform $\mathcal{H}(f)$ is well defined, and

$$\begin{aligned}\mathcal{H}f(t) &= \lim_{\epsilon\to 0}\frac{1}{2\pi}\int_{(-\pi,\pi)\cap\{|x-t|>\epsilon\}}\cot\left(\frac{t-x}{2}\right)f(x)\mathrm{d}x\\ &= \sum_{n\in\mathbb{Z}} -\mathrm{i}\,\mathrm{sgn}(n)\,c_n(f)\mathrm{e}^{\mathrm{i}nt}, \quad \text{a.e.}\end{aligned} \tag{2.1}$$

This is, in fact, an example of the theory established in [23].

To prove these, we investigate the function

$$f(t,y) = \frac{1}{\pi}\lim_{N\to\infty}\int_{-(2N+1)\pi}^{(2N+1)\pi}\frac{f(x)}{x-(t+\mathrm{i}y)}\mathrm{d}x, \quad y>0.$$

It can be shown (see Example (i), [23]) that

$$\begin{aligned}f(t,y) &= \frac{1}{\pi}\lim_{N\to\infty}\int_{-\pi}^{\pi}\sum_{k=-N}^{N}\frac{1}{x-(t+\mathrm{i}y+2k\pi)}f(x)\mathrm{d}x\\ &= \frac{-\mathrm{i}}{2\pi}\int_{-\pi}^{\pi}\frac{1+\mathrm{e}^{-y}\mathrm{e}^{\mathrm{i}(t-x)}}{1-\mathrm{e}^{-y}\mathrm{e}^{\mathrm{i}(t-x)}}f(x)\mathrm{d}x.\end{aligned}$$

From [23], we have, in the L^2 sense,

$$\lim_{y\to 0} f(t,y) = \frac{1}{2}f(t) + \frac{1}{2}\mathcal{H}f(t). \tag{2.2}$$

The convergence is also in the point-wise sense for a.e. t.

On the other hand,

$$\frac{1+\mathrm{e}^{-y}\mathrm{e}^{\mathrm{i}(t-x)}}{1-\mathrm{e}^{-y}\mathrm{e}^{\mathrm{i}(t-x)}} = \frac{1}{2}P_{\mathrm{e}^{-y}}(t-x) + \mathrm{i}\frac{1}{2}Q_{\mathrm{e}^{-y}}(t-x),$$

where $P_r(t)$ and $Q_r(t)$ are, respectively, the Poisson kernel and conjugate Poisson kernel at the point $r\mathrm{e}^{\mathrm{i}t}$, with the Fourier series expansions

$$P_r(t) = \sum_{n=-\infty}^{\infty} r^{|n|}\mathrm{e}^{\mathrm{i}nt}$$

$$Q_r(t) = \sum_{n=-\infty}^{\infty} -\mathrm{i}\,\mathrm{sgn}(n)\,r^{|n|}\mathrm{e}^{\mathrm{i}nt},$$

where $\mathrm{sgn}(n)$ is the signum function

$$\mathrm{sgn}\,(n) = \begin{cases} 1, & n = 1, 2, \ldots\\ -1, & n = -1, -2, \ldots\\ 0, & n = 0.\end{cases}$$

From Parseval's identity, the Fourier series representation of $f(t,y)$, defined through a convolution integral, is

$$\begin{aligned}f(t,y) &= \frac{1}{2}\sum_{n=-\infty}^{\infty} r^{|n|}c_n(f)\mathrm{e}^{\mathrm{i}nt}\\ &\quad + \mathrm{i}\frac{1}{2}\sum_{n=-\infty}^{\infty} -\mathrm{i}\,\mathrm{sgn}(n)\,r^{|n|}c_n(f)\mathrm{e}^{\mathrm{i}nt}, \quad r = \mathrm{e}^{-y}.\end{aligned}$$

Letting $y\to 0$, in the L^2 convergence sense, we have

$$\lim_{y\to 0} f(t,y) = \frac{1}{2}f(t) + \mathrm{i}\frac{1}{2}\sum_{n=-\infty}^{\infty} -\mathrm{i}\,\mathrm{sgn}(n)c_n(f)\mathrm{e}^{\mathrm{i}nt}. \tag{2.3}$$

Comparing (2.2) and (2.3), we conclude (2.1).

For functions in $L^2([-\pi,\pi])$ the 2π-periodic functions $\mathcal{H}(f)$ defined through (2.1) may be regarded as functions defined in $[-\pi,\pi]$, and the corresponding singular integral operator is called *circular Hilbert transform,* denoted by

$$\mathcal{H}_c f(t) = \frac{1}{\pi}\,\text{p.v.}\int_{-\pi}^{\pi}\cot\left(\frac{t-x}{2}\right)f(x)\mathrm{d}x.$$

The Fourier multiplier associated with $\mathcal{H}_c$ is $-\mathrm{i}\,\mathrm{sgn}(n)$. That is

$$\mathcal{H}_c f(t) = \sum_{n\in\mathbb{Z}} -\mathrm{i}\,\mathrm{sgn}(n)\,c_n(f)\mathrm{e}^{\mathrm{i}nt}.$$

The circular Hilbert transform $\mathcal{H}_c$ is bounded from $L^p([-\pi,\pi])$ to $L^p([-\pi,\pi])$, for $1 < p < \infty$, weak-type bounded from $L^1([-\pi,\pi])$ to $L^1([-\pi,\pi])$, and bounded from $L^\infty([-\pi,\pi])$ to $BMO([-\pi,\pi])$ (see, e.g., [11,18]).

The circular Hilbert transform is closely related to $\mathcal{Z}$ transform of digital signals. We can give a simple explanation for this. For a digital signal $c = \{c_n : n\in\mathbb{Z}\}$, its $\mathcal{Z}$ transform is defined by

$$\mathcal{Z}c = \sum_{n\in\mathbb{Z}} c_n z^n$$

with $z = \mathrm{e}^{\mathrm{i}t}$ on a unit circle. If we define a transform h_c for digital signal c by

$$h_c\{c_n\} = \{-\mathrm{i}\,\mathrm{sgn}(n)\,c_n\},$$

then

$$\mathcal{H}_c f(t) = \mathcal{Z}h_c\{c_n(f)\}.$$

It is easy to see that $\mathcal{H}_c\,\mathrm{e}^{\mathrm{i}nt} = -\mathrm{i}\,\mathrm{sgn}(n)\,\mathrm{e}^{\mathrm{i}nt}$, which means that the transform $\mathcal{H}_c$ does not change the shape of a harmonic wave and corresponds to the property of $\frac{\pi}{2}$-phase-shift of the Hilbert operator $\mathcal{H}$. Consequently, we have that

$$\mathcal{H}_c\,(\cos nt) = \sin nt, \quad \mathcal{H}_c\,(\sin nt) = -\cos nt.$$

The counterparts of Bedrosian's theorems in the present setting are discussed in [25,22]. Suppose that $f_1, f_2 \in L^2([-\pi,\pi])$. Then

$$\mathcal{H}_c[f_1 f_2] = f_1 \mathcal{H}_c[f_2]$$

if there exists a non-negative integer N such that

$$c_n(f_1) = 0, \quad |n| > N \quad \text{and}$$
$$c_n(f_2) = 0, \quad |n| \le N.$$

There is a formulation of a corresponding circular analytic signal theory for time-limited signal. Consider

$$f(t) + \mathrm{i}\mathcal{H}_c f(t) = c_0 + 2\sum_{k=1}^{\infty} c_k \mathrm{e}^{\mathrm{i}kt}.$$

It is easy to show that c_k are bounded as $f \in L^2([-\pi,\pi])$. As a consequence, the series $c_0 + 2\sum_{k=1}^{\infty} c_k z^k$ converges to an analytic function for $|z| < 1$. By writing $z = r\mathrm{e}^{\mathrm{i}t}, 0 \le r < 1$, it can be shown that $f(t) + \mathrm{i}\mathcal{H}_c f(t)$ is the boundary value of the above defined analytic function in the unit disc. We call $A(f)(t) := f(t) + \mathrm{i}\mathcal{H}_c f(t)$ the *circular analytic signal* of the time-limited signal $f(t)$.

We write

$$f(t) + \mathrm{i}\mathcal{H}_c f(t) = \rho(t)\,\mathrm{e}^{\mathrm{i}\theta(t)},$$

where

$$\rho(t) = \sqrt{f^2(t) + (\mathcal{H}_c f(t))^2}, \quad \theta(t) = \arctan\frac{\mathcal{H}_c f(t)}{f(t)}.$$

We call $\rho(t)$ and $\theta(t)$ *circular amplitude* and *circular phase* of the time-limited signal $f(t)$. In particular, when the circular amplitude is 1, we call $\mathrm{e}^{\mathrm{i}\theta(t)}$ *circular unit analytic signal.*

3. Two families of unit analytic signals

In this section, we will introduce two families of unit analytic signals on the real line of which the first is a periodic extension of the nonlinear Fourier atoms and the second is the images of the nonlinear Fourier atoms under Cayley transformation. We first discuss the relationship between circular analytic signals and analytic signals. The basic idea is borrowed from [25], where Cayley transformation plays a crucial role.

The Cayley transform

$$\omega = \frac{\mathrm{i} - z}{\mathrm{i} + z}$$

is a conformed mapping from the unit disc $\{z : |z| < 1\}$ to the upper-half-complex plane $\{z : \operatorname{Im} z > 0\}$, which also extends to their boundaries to become a bijective, bi-continuous, and strictly increasing under their canonical parameterizations $z = \mathrm{e}^{\mathrm{i}t}, 0 \le t \le 2\pi$, and $w = s, -\infty < s < \infty$. Consequently, if $f(z)$ is an analytic function in the unit disc, then $f(\frac{\mathrm{i}-z}{\mathrm{i}+z})$ is an analytic function in the upper-half-complex plane. By recalling the facts that circular analytic signals and analytic signals are the boundary values of analytic functions in the unit disk and in the upper-half-complex plane, respectively, we know that $\lim_{r\to 1-} f(z), z = r\mathrm{e}^{\mathrm{i}s}$, and $\lim_{y\to 0+} f(\frac{\mathrm{i}-z}{\mathrm{i}+z}), z = t + \mathrm{i}y$, are circular analytic signal and analytic signal, respectively, if $f(z)$ is a certain analytic function on the unit disc. The boundary correspondence is based on the mapping

$$\cos s + \mathrm{i}\sin s = \frac{\mathrm{i} - t}{\mathrm{i} + t} = \frac{1 - t^2}{1 + t^2} + \mathrm{i}\frac{2t}{1 + t^2},$$

which maps the real line to the unit circle. By rewriting

$$\cos s = \frac{1 - t^2}{1 + t^2} = \frac{1 - \tan^2\frac{s}{2}}{1 + \tan^2\frac{s}{2}} \tag{3.1}$$

and

$$\sin s = \frac{2t}{t^2 + 1} = \frac{2\tan\frac{s}{2}}{1 + \tan^2\frac{s}{2}}, \tag{3.2}$$

we have

$$s = 2\arctan t.$$

Therefore, if $F(s)$ is a certain circular analytic signal, that is

$$F(s) := \cos\theta(s) + \mathrm{i}\sin\theta(s), \quad s \in [-\pi,\pi]$$

with

$$\mathcal{H}_c(\cos\theta(\cdot))(s) = \sin\theta(s).$$

Then

$$f(t) := \cos\theta(2\arctan t) + \mathrm{i}\sin\theta(2\arctan t), \quad t \in (-\infty,\infty)$$

is a unit analytic signal with

$$\mathcal{H}\cos\theta(2\arctan(\cdot))(t) = \sin\theta(2\arctan t). \tag{3.3}$$

We need to study nonlinear Fourier atoms as boundary values of Möbius transforms in the unit disc. Let $a = |a|\mathrm{e}^{\mathrm{i}t_a}$ with $|a| < 1$. The nonlinear Fourier atoms $\{\mathrm{e}^{\mathrm{i}\theta_a(s)} : |a| < 1, s \in [-\pi,\pi]\}$ with

$$\theta_a(s) = s + 2\arctan\frac{|a|\sin(s - t_a)}{1 - |a|\cos(s - t_a)}, \quad s \in [-\pi,\pi], \tag{3.4}$$

were first noted in [25,22]. The decomposition (3.4) of $\theta_a(s)$ into a linear part and a periodic part appeared in [6]. From [22], we know that the real part and imaginary part of $\mathrm{e}^{\mathrm{i}n\theta_a(s)}$ form a pair of Hilbert transforms, that is,

$$\mathcal{H}_c\cos\theta_a(s) = \sin\theta_a(s).$$

Consequently, the signal $\mathrm{e}^{\mathrm{i}\theta_a(s)}, s \in [-\pi,\pi]$, is a circular unit analytic signal. Fig. 2 illustrates the real part $\cos\theta_{1/2}(s), s \in [-\pi,\pi]$, corresponding to $a = \frac{1}{2}$ in the general setting, and its phase plot in the circular Hilbert domain.

The relation (3.4) is obtained from the boundary value $\mathrm{e}^{\mathrm{i}\theta_a(s)}$ of the Möbius transform

$$\tau_a(z) = \frac{z - a}{1 - \bar{a}z}. \tag{3.5}$$

To see this, we note that

$$\theta_a(s) - s = 2\arctan\frac{|a|\sin(s - t_a)}{1 - |a|\cos(s - t_a)}$$

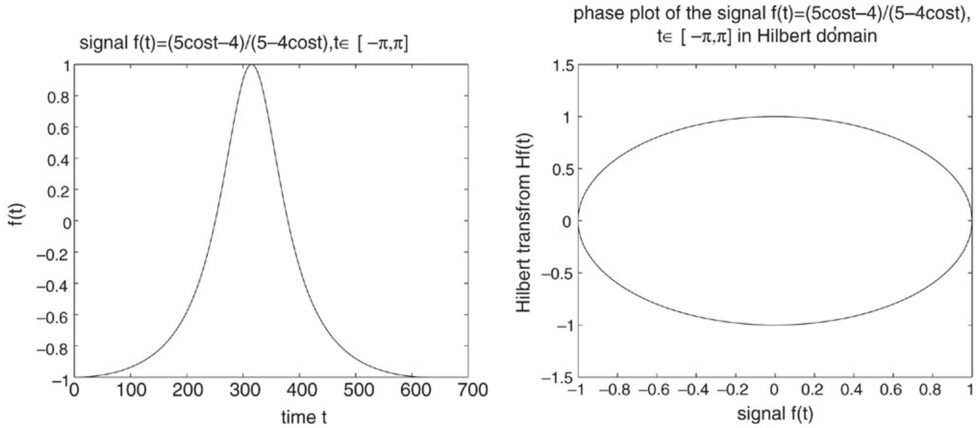

Fig. 2. Left: the plot of the time-limited signal $\cos\theta_{1/2}(s) = (5\cos s - 4)/(5 - 4\cos s)$, $s \in [-\pi, \pi]$. Right: the phase plot of the signal $\cos\theta_{1/2}(s)$ in circular Hilbert domain.

$$= 2\mathrm{Arg}\, A(s) = \mathrm{Arg}\, \frac{A(s)}{\bar{A}(s)}$$

with $A(s) = 1 - |a|\mathrm{e}^{\mathrm{i}(t_a - s)}$. Noting that $|\frac{A(s)}{\bar{A}(s)}| = 1$, we obtain that

$$\mathrm{e}^{\mathrm{i}\theta_a(s)} = \mathrm{e}^{\mathrm{i}s}\frac{A(s)}{\bar{A}(s)} = \mathrm{e}^{\mathrm{i}s}\frac{1 - |a|\mathrm{e}^{\mathrm{i}(t_a - s)}}{1 - |a|\mathrm{e}^{\mathrm{i}(s - t_a)}} = \frac{\mathrm{e}^{\mathrm{i}s} - |a|\mathrm{e}^{\mathrm{i}t_a}}{1 - |a|\mathrm{e}^{\mathrm{i}(s - t_a)}} = \frac{\mathrm{e}^{\mathrm{i}s} - |a|\mathrm{e}^{\mathrm{i}t_a}}{1 - |a|\mathrm{e}^{-\mathrm{i}t_a}\mathrm{e}^{\mathrm{i}s}} = \frac{\mathrm{e}^{\mathrm{i}s} - a}{1 - \bar{a}\mathrm{e}^{\mathrm{i}s}}.$$

We thus arrive at

$$\mathrm{e}^{\mathrm{i}\theta_a(s)} = \frac{\mathrm{e}^{\mathrm{i}s} - a}{1 - \bar{a}\mathrm{e}^{\mathrm{i}s}}.$$

From this equation, the circular instantaneous frequency $\omega_{\mathrm{ins}}(s)$ of $\mathrm{e}^{\mathrm{i}\theta_a(s)}$ can be easily obtained

$$\omega_{\mathrm{ins}}(s) = \frac{1 - |a|^2}{1 - 2|a|\cos(s - t_a) + |a|^2}$$

which is just the Poisson kernel (see [11] or [26]) in the disc. We can further obtain an explicit representation for $\cos\theta_a(s)$:

$$\cos\theta_a(s) = \frac{\cos s - 2|a|\cos t_a + |a|^2\cos(s - 2t_a)}{1 + |a|^2 - 2|a|\cos(s - t_a)}. \tag{3.6}$$

If, in particular, a is a real number whose absolute value is less than 1, then $\cos\theta_a(s)$ may be simplified into

$$\cos\theta_a(s) = \frac{\left(1 + |a|^2\right)\cos s - 2|a|}{1 + |a|^2 - 2|a|\cos s}.$$

We now study the first family of unit analytic signals. From the above discussion, we know that $\mathrm{e}^{\mathrm{i}\theta_a(s)}, s \in [-\pi, \pi]$, are circular analytic signals. This includes that $\cos\theta_a(s)$ and $\sin\theta_a(s)$ satisfy the circular Hilbert relation, viz.

$$\mathcal{H}_c\cos\theta_a(s) = \sin\theta_a(s), \quad s \in [-\pi, \pi].$$

Setting $\theta_a(s + 2k\pi) = 2k\pi + \theta_a(s)$, the time-limited function $\mathrm{e}^{\mathrm{i}\theta_a(s)}$ is 2π-periodically extended to a function in the whole real line. We denote it by $\mathrm{e}^{\mathrm{i}\theta_a(t)}, t \in \mathbb{R}$. It is interesting that the signals $\mathrm{e}^{\mathrm{i}\theta_a(t)}, t \in \mathbb{R}$, are unit analytic signals with periodic and positive instantaneous frequency. Consequently, the signals $\cos\theta_a(t)$ are mono-component with frequency modulation. To see this, we need to show that the Hilbert transform of $\cos\theta_a(t)$ is the corresponding sine function $\sin\theta_a(t)$, i.e.,

$$\mathcal{H}\cos\theta_a(t) = \sin\theta_a(t), \quad t \in \mathbb{R}. \tag{3.7}$$

But according to Section 2, the local and global Riemann improper integral defining Hilbert transform of $\cos\theta_a(t)$ is well defined, and we have (3.7).

Fig. 3 illustrates the signal $f(t) = \cos\theta_{1/2}(t), t \in \mathbb{R}$, and its phase plot in Hilbert domain. The difference between Figs. 2 and 3 is that the phase plot wends infinitely many times in the latter, but once in the former. The phase and the instantaneous frequency of $\cos\theta_a(t)$ are

$$\theta_a(t) = t + 2\arctan\frac{|a|\sin(t - t_a)}{1 - |a|\cos(t - t_a)}, \quad t \in \mathbb{R},$$

and

$$\omega_{\mathrm{ins}}(t) = \frac{1 - |a|^2}{1 - 2|a|\cos(t - t_a) + |a|^2}, \quad t \in \mathbb{R},$$

respectively. Fig. 4 illustrates the plots of the phase and the instantaneous frequency of the signal $\cos\theta_{1/2}(t), t \in \mathbb{R}$. Finally, we point out that the signal $\cos\theta_a(t), t \in \mathbb{R}$, is a normalized intrinsic mode function described in [17] since the envelopes of the signal are the lines $(t, 1)$ and $(t, -1)$ with $t \in \mathbb{R}$ which ensures the local symmetry and there is only one oscillatory between any two neighboring zeros. Noting that the phase $\theta_a(t)$ is a strictly increasing function, the zero-crossings of $\cos\theta_a(t)$ are

$$\left\{t : t = \theta_a^{-1}\left(2k\pi \pm \frac{\pi}{2}\right)\right\}.$$

When a is a real number with $|a| < 1$, we know that

$$\theta_a^{-1}\left(2k\pi \pm \frac{\pi}{2}\right) = 2k\pi \pm \arccos\frac{2|a|}{1 + |a|^2}.$$

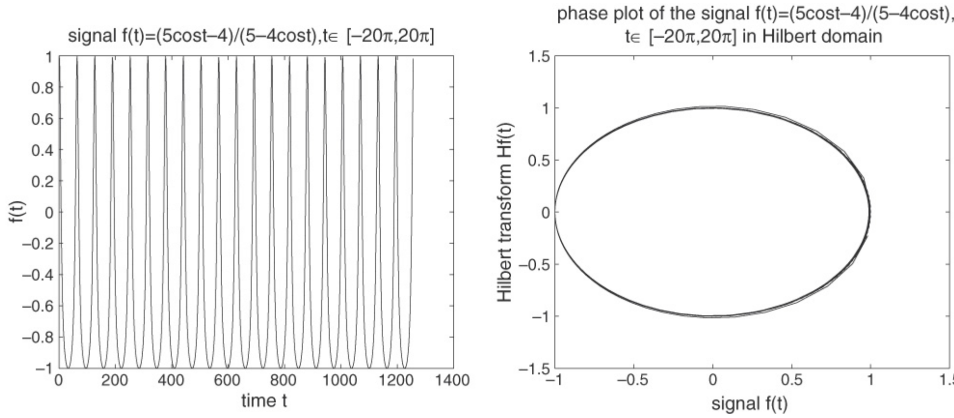

Fig. 3. Left: the plot of the signal $\cos\theta_{1/2}(t) = (5\cos t - 4)/(5 - 4\cos t)$, $t \in \mathbb{R}$. Right: the phase plot of the signal $\cos\theta_{1/2}(t)$ in Hilbert domain.

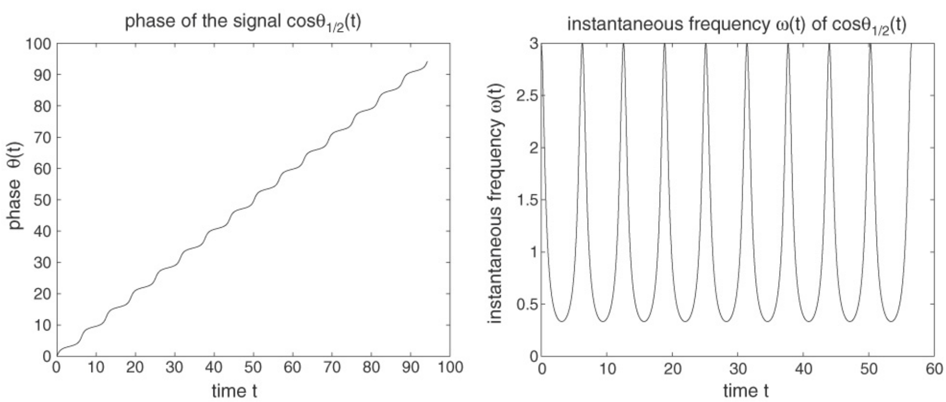

Fig. 4. Left: the plot of the phase of $\cos\theta_{1/2}(t)$. Right: the plot of the instantaneous frequency of $\cos\theta_{1/2}(t)$.

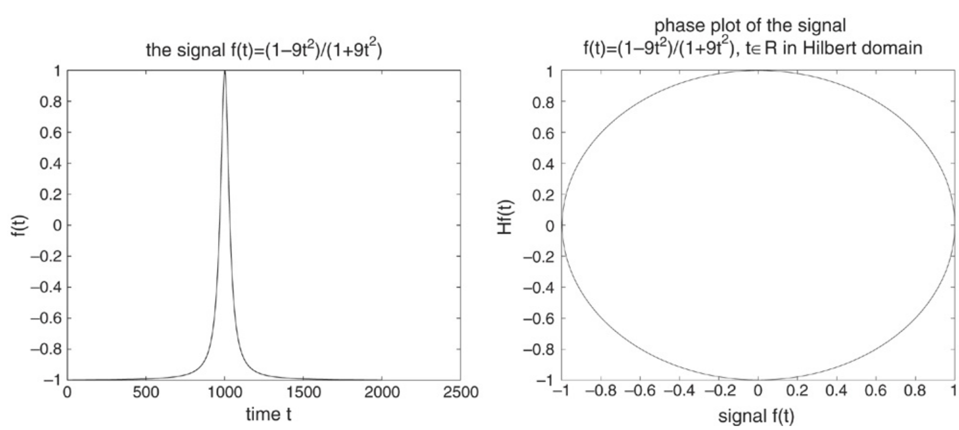

Fig. 5. The signal $\cos\phi_{1/2}(t) = (1 - 9t^2)/(1 + 9t^2)$, $t \in \mathbb{R}$ and its phase plot in Hilbert domain.

Now, we turn to discuss the second family and explain how to generate unit analytic signals on the line from the circular analytic signals $e^{i\theta(s)}$, $s \in [-\pi, \pi]$, based on Cayley transformation.

From (3.6), we know that

$$\cos\theta_a(t) = \frac{-2|a|\cos t_a + (1 + |a|^2\cos 2t_a)\cos t + |a|^2 \sin 2t_a \sin t}{1 + |a|^2 - 2|a|\cos t_a \cos t - 2|a|\sin t_a \sin t},$$
$$t \in [-\pi, \pi].$$

Combining this with (3.1) and (3.2), we obtain a family of new functions

$$\cos\phi_a(t) := \cos\theta_a(2\arctan t)$$

defined on the whole real line. Eq. (3.3) implies that

$$\mathcal{H}\cos\phi_a(t) = \sin\phi_a(t), \quad t \in \mathbb{R},$$

from which we know that $e^{i\phi_a(t)}$ is also a unit analytic signal.

Fig. 5 illustrates the plot of the signal $\cos\phi_{1/2}(t)$, $t \in \mathbb{R}$, and its phase plot in Hilbert domain.

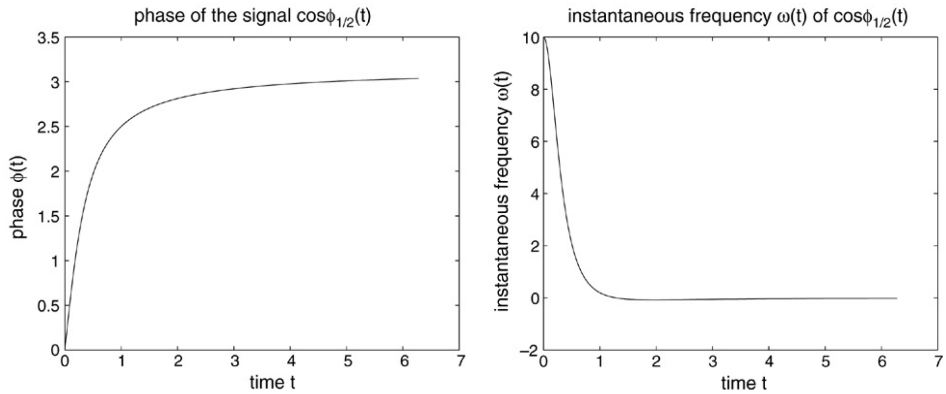

Fig. 6. Left: the plot of the phase of $\cos\phi_{1/2}(t)$. Right: the plot of the instantaneous frequency of $\cos\phi_{1/2}(t)$.

Write $\Psi(t_a) := 1+|a|^2+2|a|\cos t_a$. The signal $\cos\phi_a(t)$ has the explicit form

$$\begin{aligned}\cos\phi_a(t) &= \frac{-2|a|\cos t_a + (1+|a|^2\cos 2t_a)\frac{1-t^2}{t^2+1} + |a|^2(\sin 2t_a)\frac{2t}{t^2+1}}{1+|a|^2-2|a|(\cos t_a)\frac{1-t^2}{t^2+1} - 2|a|(\sin t_a)\frac{2t}{t^2+1}} \\ &= -\frac{1+|a|^2\cos 2t_a + 2|a|\cos t_a}{\Psi(t_a)t^2 - 4|a|(\sin t_a)t + (\Psi(t_a)-4|a|\cos t_a)}\, t^2 \\ &\quad + \frac{2|a|^2(\sin 2t_a)}{\Psi(t_a)t^2 - 4|a|(\sin t_a)t + (\Psi(t_a)-4|a|\cos t_a)}\, t \\ &\quad + \frac{1+|a|^2\cos 2t_a - 2|a|\cos t_a}{\Psi(t_a)t^2 - 4|a|(\sin t_a)t + (\Psi(t_a)-4|a|\cos t_a)}.\end{aligned}$$

In particular, when a is a positive number less than 1, we get the representation

$$\cos\phi_a(t) = \frac{(1-|a|)^2 - (1+|a|)^2t^2}{(1-|a|)^2+(1+|a|)^2t^2}.$$

The phase $\phi_a(t)$ of the signal $\cos\phi_a(t)$ is

$$\phi_a(t) = \theta_a(2\arctan t), \qquad t\in\mathbb{R}.$$

Noting the decomposition (3.4) of $\theta_a(s)$, we arrive at

$$\phi_a(t) = 2\arctan t + 2\arctan\frac{|a|\sin(2\arctan t - t_a)}{1-|a|\cos(2\arctan t - t_a)}. \tag{3.8}$$

Next, we will offer a simpler representation of the phase $\phi_a(t)$ for the convenience of our further discussion. In fact, we have

$$\phi_a(t) = 2\arctan t + 2\arctan\frac{P(f)(t)}{P(g)(t)} \tag{3.9}$$

with two polynomials $P(f)(t)$ and $P(g)(t)$ of degree 2, namely,

$$P(f)(t) = -|a|\sin t_a + (2|a|\cos t_a)t + (|a|\sin t_a)t^2$$

and

$$g_a(t) = 1-|a|\cos t_a + (2|a|\sin t_a)t + (1+|a|\cos t_a)t^2.$$

In particular, when a is a positive real number less than 1, the formula (3.9) becomes

$$\phi_a(t) = 2\arctan t + 2\arctan\frac{2at}{1-a+(1+a)t^2}.$$

The formula (3.9) can be obtained by direct calculation based on (3.8) and by using the facts

$$\cos(2\arctan t - t_a) = \frac{1-t^2}{1+t^2}\cos t_a + \frac{2t}{1+t^2}\sin t_a$$

and

$$\sin(2\arctan t - t_a) = \frac{2t}{1+t^2}\cos t_a - \frac{1-t^2}{1+t^2}\sin t_a.$$

The instantaneous frequency of the signal $\cos\phi_a(t)$, $t\in\mathbb{R}$ is equal to the Poisson kernel on the line:

$$\omega_{\text{ins}}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\theta_a(2\arctan t) = \frac{1}{\pi}\frac{h_a}{(t-t_a)^2+h_a^2} = P_{h_a}(t-t_a),$$

where $h_a = \frac{1-|a|^2}{1+2|a|\cos t_0+|a|^2}$, $t_a = \frac{2|a|\sin t_0}{1+2|a|\cos t_0+|a|^2}$ and $a=|a|\mathrm{e}^{\mathrm{i}t_0}$. Fig. 6 illustrates the plots of the phase and the instantaneous frequency of the signal $\cos\phi_{1/2}(t)$, $t\in\mathbb{R}$.

4. Gram–Schmidt procedure: From nonlinear Fourier atoms to Laguerre bases

The time–frequency properties of the nonlinear Fourier atoms $\{\mathrm{e}^{\mathrm{i}\theta_a(s)} : |a|<1\}$ were studied in [6]. It was shown that for any fixed a, $|a|<1$, the family of functions $\{\mathrm{e}^{\mathrm{i}n\theta_a(s)} : n\in\mathbb{Z}\}$ forms a Riesz basis for the space $L^2(-\pi,\pi)$ with the upper bound $2\pi\frac{1+|a|}{1-|a|}$ and the lower bound $2\pi\frac{1-|a|}{1+|a|}$, respectively, and $\{g(t-n)\mathrm{e}^{\mathrm{i}m\theta_a(t)} : n\in\mathbb{Z}\}$ forms a frame of $L^2(R)$ for some suitable windowed functions g. A general sampling theorem is also obtained for the nonlinear Fourier atoms on the line $\{\mathrm{e}^{\mathrm{i}n\theta_a(t)} : n\in\mathbb{Z}\}$, where $\mathrm{e}^{\mathrm{i}\theta_a(t)}$, $t\in\mathbb{R}$, is considered as periodic extension of $\mathrm{e}^{\mathrm{i}\theta_a(s)}$, $s\in[-\pi,\pi]$. In this section, we will point out that the Gram–Schmidt procedure of this kind of Riesz basis will lead to the orthonormal system with amplitude and frequency modulation in the space of $L^2([-\pi,\pi])$

$$\{1\}\cup\{\rho(t)\cos\psi_n(t) : n\geq 1\}\cup\{\rho(t)\sin\psi_n(t) : n\geq 1\}, \tag{4.1}$$

where

$$\rho(t)=\left(\frac{1-|a|^2}{1-2|a|\cos(t-t_a)+|a|^2}\right)^{\frac{1}{2}}, \tag{4.2}$$

the positive square root of Poisson kernel and

$$\psi_n(t)=nt+(2n-1)\arctan\frac{|a|\sin(t-t_a)}{1-|a|\cos(t-t_a)} \tag{4.3}$$

for any complex number $a=|a|\mathrm{e}^{\mathrm{i}t_a}$ with $|a|<1$. It is easy to see that the case $a=0$ corresponds to the classic Fourier bases $\{1,\cos nt,\sin nt,n=1,2,\ldots\}$. We should point out not only in the sense of mathematics is this generalization, but also emphasizes much more on physical aspects. We will offer a brief explanation for it. We will see that this system is orthonormal in $L^2([0,2\pi])$. Thus any complicated signal can be expanded into different components with time-varying frequency, that is,

$$f(t)=\sum_{n=-\infty}^{\infty}c_n(f)\rho(t)\mathrm{e}^{\mathrm{i}\psi_n(t)}$$

with $c_n(f)=\langle f,\rho(\cdot)\mathrm{e}^{\mathrm{i}\psi_n(\cdot)}\rangle$. Compared with the classic case, the generalized Fourier bases have advantages for the time–frequency analysis of nonlinear and non-stationary signals.

A simple calculation offers us that

$$\frac{\mathrm{e}^{\mathrm{i}t}}{\mathrm{e}^{\mathrm{i}t}-a}=\left.\frac{|z|^2-\bar{a}z}{|z|^2-2\mathrm{Re}(a\bar{z})+|a|^2}\right|_{z=\mathrm{e}^{\mathrm{i}t}} =\frac{1}{\sqrt{1-2|a|\cos(t-t_a)+|a|^2}}\mathrm{e}^{-\mathrm{i}\arctan\frac{|a|\sin(t-t_a)}{1-|a|\cos(t-t_a)}}$$

and

$$\left(\frac{\mathrm{e}^{\mathrm{i}t}-a}{1-\bar{a}\mathrm{e}^{\mathrm{i}t}}\right)^n=\left(\mathrm{e}^{\mathrm{i}t}\frac{1-|a|\mathrm{e}^{\mathrm{i}(t_a-a)}}{1-|a|\mathrm{e}^{\mathrm{i}(t-t_a)}}\right)^n =\mathrm{e}^{\mathrm{i}n(t+2\arctan\frac{|a|\sin(t-t_a)}{1-|a|\cos(t-t_a)})}.$$

Thus, this system (4.1) has the complex form $\{e_n(t):t\in[-\pi,\pi]:n\in\mathbb{Z}\}$ of $L^2([-\pi,\pi])$ with $e_0(t)=1$ and

$$e_n(t)=\sqrt{1-|a|^2}\frac{\mathrm{e}^{\mathrm{i}t}}{\mathrm{e}^{\mathrm{i}t}-a}\left(\frac{\mathrm{e}^{\mathrm{i}t}-a}{1-\bar{a}\mathrm{e}^{\mathrm{i}t}}\right)^n,\quad t\in[-\pi,\pi] \tag{4.4}$$

for $n\neq 0$ and for any fixed complex number $a=|a|\mathrm{e}^{\mathrm{i}t_a}$ with $|a|<1$. Obviously, the real variable function $e_n(t)$ is the boundary value on the unit circle $\{z:|z|=1\}$ of the complex variable function

$$E_n(z)=\sqrt{1-|a|^2}\frac{z}{z-a}\left(\frac{z-a}{1-\bar{a}z}\right)^n. \tag{4.5}$$

To accommodate convention in signal processing and control theory literature, the substitution $z\to z^{-1}$ gives us that

$$E_n(z^{-1})=\sqrt{1-|a|^2}\frac{1}{1-az}\left(\frac{1-az}{z-\bar{a}}\right)^n.$$

Note that $E_n(z^{-1})$ is just the z-transforms of Laguerre sequences discussed in [4], which is called a Laguerre filter, an IIR filter with rational coefficients for $a\neq 0$. This function $E_n(z^{-1})$ is also used in system identification [19,28,12,2], which leads to so-called Laguerre Models.

The following theorem shows that the system (4.1) can be obtained from a Gram–Schmidt procedure of the Riesz bases $\left\{(\frac{\mathrm{e}^{\mathrm{i}t}-a}{1-\bar{a}\mathrm{e}^{\mathrm{i}t}})^n:n\in\mathbb{Z}\right\}$.

Theorem 4.1. *Suppose that a is a complex number with $|a|<1$ and $\tau(z)$ is defined in (3.5). Then the orthogonal system*

$$\left\{e_n(t):e_0(t)=1,e_n(t)=\sqrt{1-|a|^2}\frac{\mathrm{e}^{\mathrm{i}t}}{\mathrm{e}^{\mathrm{i}t}-a}\left(\frac{\mathrm{e}^{\mathrm{i}t}-a}{1-\bar{a}\mathrm{e}^{\mathrm{i}t}}\right)^n,\ n=1,2,\ldots,\right\}$$

can be derived by a Gram–Schmidt procedure from the stable system $\left\{\tau_a^n(\mathrm{e}^{\mathrm{i}t})\right\}_{n=0}^{\infty}$.

Proof. Denote by $v_n=\tau_a^n(\mathrm{e}^{\mathrm{i}t})$. Starting from $\{v_0,v_1,\ldots\}$, the Gram–Schmidt procedure can generate an orthogonal system $\{e_n\}$ by the recursion

$$G_n:=v_n-\sum_{k=0}^{n-1}\langle v_n,e_k\rangle e_k$$

and the normalization

$$e_n:=\frac{G_n}{\|G_n\|}.$$

Here the inner product and norm are in the meaning of the space $L^2([-\pi,\pi])$. We will show that e_n is just $e_n(t)$. We adopt induction. Obviously $e_0=g_0=1$ since $v_0=1$. Thus, $G_1=v_1-\langle v_1,e_0\rangle e_0$ which is equivalent to

$$G_1=\tau_a(\mathrm{e}^{\mathrm{i}t})-\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\mathrm{e}^{\mathrm{i}t}-a}{1-\bar{a}\mathrm{e}^{\mathrm{i}t}}\mathrm{d}t=\tau_a(\mathrm{e}^{\mathrm{i}t})+a =(1-|a|^2)\frac{\mathrm{e}^{\mathrm{i}t}}{1-\bar{a}\mathrm{e}^{\mathrm{i}t}}.$$

Note that the norm of G_1 can be obtained by the following calculation

$$\|G_1\|^2=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{(1-|a|^2)\mathrm{e}^{\mathrm{i}t}}{1-\bar{a}\mathrm{e}^{\mathrm{i}t}}\frac{(1-|a|^2)\mathrm{e}^{-\mathrm{i}t}}{1-a\mathrm{e}^{-\mathrm{i}t}}\mathrm{d}t =\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{(1-|a|^2)^2\mathrm{e}^{\mathrm{i}t}}{(1-\bar{a}\mathrm{e}^{\mathrm{i}t})(\mathrm{e}^{\mathrm{i}t}-a)} =\frac{1}{2\pi\mathrm{i}}\oint_{|z|=1}\frac{(1-|a|^2)^2}{(1-\bar{a}z)(z-a)}\mathrm{d}z=1-|a|^2.$$

Therefore, $e_1=e_1(t)$. Now we consider the case with $n=2$. In this case, we have

$$G_2=v_2-\langle v_2,e_0\rangle e_0-\langle v_2,e_1\rangle e_1.$$

We need to calculate the inner product $\langle v_2,e_j\rangle$ with $j=0,1$. For the convenience of readers, we list the calculation as follows

$$\langle v_2,e_0\rangle=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{\mathrm{e}^{\mathrm{i}t}-a}{1-\bar{a}\mathrm{e}^{\mathrm{i}t}}\right)^2\mathrm{d}t$$

$$= \frac{1}{2\pi \mathrm{i}} \oint_{|z|=1} \frac{(z-a)^2}{(1-\bar{a}z)^2 z} \mathrm{d}z = a^2$$

and

$$\begin{aligned}\langle v_2, e_1\rangle &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\frac{\mathrm{e}^{\mathrm{i}t}-a}{1-\bar{a}\mathrm{e}^{\mathrm{i}t}}\right)^2 \sqrt{1-|a|^2} \frac{\mathrm{e}^{-\mathrm{i}t}}{1-a\mathrm{e}^{-\mathrm{i}t}} \mathrm{d}t \\ &= \frac{\sqrt{1-|a|^2}}{2\pi} \int_{-\pi}^{\pi} \frac{\mathrm{e}^{\mathrm{i}t}-a}{(1-\bar{a}\mathrm{e}^{\mathrm{i}t})^2} \mathrm{d}t \\ &= \frac{\sqrt{1-|a|^2}}{2\pi \mathrm{i}} \oint_{|z|=1} \frac{z-a}{(1-\bar{a}z)^2 z} \mathrm{d}z = -a\sqrt{1-|a|^2}.\end{aligned}$$

Thus we get the explicit representation of G_2

$$\begin{aligned}G_2 &= \left(\frac{\mathrm{e}^{\mathrm{i}t}-a}{1-\bar{a}\mathrm{e}^{\mathrm{i}t}}\right)^2 - a^2 + a(1-|a|^2)\frac{\mathrm{e}^{\mathrm{i}t}}{1-\bar{a}\mathrm{e}^{\mathrm{i}t}} \\ &= \frac{1}{(1-\bar{a}\mathrm{e}^{\mathrm{i}t})^2}\Big((\mathrm{e}^{\mathrm{i}t}-a)^2 - a^2(1-\bar{a}\mathrm{e}^{\mathrm{i}t})^2 \\ &\quad + a(1-|a|^2)\mathrm{e}^{\mathrm{i}t}(1-\bar{a}\mathrm{e}^{\mathrm{i}t})\Big) \\ &= (1-|a|^2)\mathrm{e}^{\mathrm{i}t}(\mathrm{e}^{\mathrm{i}t}-a)\frac{1}{(1-\bar{a}\mathrm{e}^{\mathrm{i}t})^2}.\end{aligned}$$

The norm of G_2 can be derived from the following calculation

$$\begin{aligned}&\|G_2\|^2 \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{(1-|a|^2)\mathrm{e}^{\mathrm{i}t}(\mathrm{e}^{\mathrm{i}t}-a)}{(1-\bar{a}\mathrm{e}^{\mathrm{i}t})^2} \\ &\quad \times \frac{(1-|a|^2)\mathrm{e}^{-\mathrm{i}t}(\mathrm{e}^{-\mathrm{i}t}-\bar{a})}{(1-a\mathrm{e}^{-\mathrm{i}t})^2}\mathrm{d}t \\ &= \frac{(1-|a|^2)^2}{2\pi}\int_{-\pi}^{\pi} \frac{\mathrm{e}^{\mathrm{i}t}}{(1-\bar{a}\mathrm{e}^{\mathrm{i}t})(\mathrm{e}^{\mathrm{i}t}-a)}\mathrm{d}t \\ &= \frac{(1-|a|^2)^2}{2\pi \mathrm{i}}\oint_{|z|=1} \frac{1}{(1-\bar{a}z)(z-a)}\mathrm{d}z = 1-|a|^2.\end{aligned}$$

This implies that $e_2 = \frac{G_2}{\|G_2\|} = e_2(t)$. Now suppose that $e_n = e_n(t)$ for $n \geq 1$. We shall show that $e_{n+1} = e_{n+1}(t)$. Note that

$$G_{n+1} = v_{n+1} - \langle v_{n+1}, e_0\rangle e_0 - \sum_{j=1}^{n} \langle v_{n+1}, e_j\rangle e_j$$

with

$$\langle v_{n+1}, e_0\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{\mathrm{e}^{\mathrm{i}t}-a}{1-\bar{a}\mathrm{e}^{\mathrm{i}t}}\right)^{n+1}\mathrm{d}t = (-a)^{n+1}$$

and

$$\begin{aligned}\langle v_{n+1}, e_j\rangle &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{\mathrm{e}^{\mathrm{i}t}-a}{1-\bar{a}\mathrm{e}^{\mathrm{i}t}}\right)^{n+1} \\ &\quad \times \sqrt{1-|a|^2}\frac{\mathrm{e}^{-\mathrm{i}t}}{\mathrm{e}^{-\mathrm{i}t}-\bar{a}}\left(\frac{\mathrm{e}^{-\mathrm{i}t}-\bar{a}}{1-a\mathrm{e}^{-\mathrm{i}t}}\right)^j \mathrm{d}t \\ &= \frac{\sqrt{1-|a|^2}}{2\pi}\int_{-\pi}^{\pi}\left(\frac{\mathrm{e}^{\mathrm{i}t}-a}{1-\bar{a}\mathrm{e}^{\mathrm{i}t}}\right)^{n+1-j}\frac{1}{1-\bar{a}\mathrm{e}^{\mathrm{i}t}}\mathrm{d}t \\ &= \frac{\sqrt{1-|a|^2}}{2\pi \mathrm{i}}\oint_{|z|=1}\left(\frac{z-a}{1-\bar{a}z}\right)^{n+1-j}\frac{\mathrm{d}z}{(1-\bar{a}z)z} \\ &= (-a)^{n+1-j}\sqrt{1-|a|^2}\end{aligned}$$

for $j = 1, 2, \ldots, n$. Then G_{n+1} equals to

$$\begin{aligned}G_{n+1} &= \tau_a^{n+1}(\mathrm{e}^{\mathrm{i}t}) - (-a)^{n+1} - \sum_{j=1}^{n}\sqrt{1-|a|^2}(-a)^{n+1-j} \\ &\quad \times \sqrt{1-|a|^2}\frac{\mathrm{e}^{\mathrm{i}t}}{\mathrm{e}^{\mathrm{i}t}-a}\tau_a^j(\mathrm{e}^{\mathrm{i}t}) \\ &= \tau_a^{n+1}(\mathrm{e}^{\mathrm{i}t}) - (-a)^{n+1} - (1-|a|^2)(-a)^{n+1}\frac{\mathrm{e}^{\mathrm{i}t}}{\mathrm{e}^{\mathrm{i}t}-a} \\ &\quad \times \sum_{j=1}^{n}\left(-\frac{1}{a}\right)^j \tau_a^j(\mathrm{e}^{\mathrm{i}t}).\end{aligned}$$

We need to calculate the summation of the right side of the above equation

$$\begin{aligned}&(1-|a|^2)(-a)^{n+1}\frac{\mathrm{e}^{\mathrm{i}t}}{\mathrm{e}^{\mathrm{i}t}-a}\sum_{j=1}^{n}\left(-\frac{1}{a}\right)^j\tau_a^j(\mathrm{e}^{\mathrm{i}t}) \\ &= (1-|a|^2)(-a)^{n+1}\frac{\mathrm{e}^{\mathrm{i}t}}{\mathrm{e}^{\mathrm{i}t}-a}\frac{-\frac{1}{a}\tau_a(\mathrm{e}^{\mathrm{i}t}) - (-\frac{1}{a}\tau_a(\mathrm{e}^{\mathrm{i}t}))^{n+1}}{1+\frac{1}{a}\tau_a(\mathrm{e}^{\mathrm{i}t})} \\ &= (1-|a|^2)(-a)^{n+1}\frac{\mathrm{e}^{\mathrm{i}t}}{\mathrm{e}^{\mathrm{i}t}-a}\frac{-\tau_a(\mathrm{e}^{\mathrm{i}t}) - a(-\frac{1}{a}\tau_a(\mathrm{e}^{\mathrm{i}t}))^{n+1}}{a+\tau_a(\mathrm{e}^{\mathrm{i}t})} \\ &= (1-|a|^2)(-a)^{n+1}\frac{\mathrm{e}^{\mathrm{i}t}}{\mathrm{e}^{\mathrm{i}t}-a}\frac{-\tau_a(\mathrm{e}^{\mathrm{i}t}) - a(-\frac{1}{a}\tau_a(\mathrm{e}^{\mathrm{i}t}))^{n+1}}{\frac{(1-|a|^2)\mathrm{e}^{\mathrm{i}t}}{1-\bar{a}\mathrm{e}^{\mathrm{i}t}}} \\ &= (-a)^{n+1}\frac{1}{\tau_a(\mathrm{e}^{\mathrm{i}t})}\left(-\tau_a(\mathrm{e}^{\mathrm{i}t}) - a\left(-\frac{1}{a}\right)^{n+1}\tau_a^{n+1}(\mathrm{e}^{\mathrm{i}t})\right) \\ &= -(-a)^{n+1} - a\tau_a^n(\mathrm{e}^{\mathrm{i}t}).\end{aligned}$$

Therefore, we get that

$$\begin{aligned}G_{n+1} &= \tau_a^{n+1}(\mathrm{e}^{\mathrm{i}t}) + a\tau_a^n(\mathrm{e}^{\mathrm{i}t}) = \tau_a^{n+1}(\mathrm{e}^{\mathrm{i}t})\left(1 + a\frac{1}{\tau_a(\mathrm{e}^{\mathrm{i}t})}\right) \\ &= \frac{(1-|a|^2)\mathrm{e}^{\mathrm{i}t}}{\mathrm{e}^{\mathrm{i}t}-a}\left(\frac{\mathrm{e}^{\mathrm{i}t}-a}{1-\bar{a}\mathrm{e}^{\mathrm{i}t}}\right)^{n+1}.\end{aligned}$$

The norm of G_{n+1} is easy to get

$$\begin{aligned}\|G_{n+1}\|^2 &= \frac{(1-|a|^2)^2}{2\pi}\int_{-\pi}^{\pi}\frac{\mathrm{e}^{\mathrm{i}t}}{(\mathrm{e}^{\mathrm{i}t}-a)(1-\bar{a}\mathrm{e}^{\mathrm{i}t})}\mathrm{d}t \\ &= \frac{(1-|a|^2)^2}{2\pi \mathrm{i}}\oint_{|z|=1}\frac{\mathrm{d}z}{(1-\bar{a}z)(z-a)} = 1-|a|^2.\end{aligned}$$

This implies that $e_{n+1} = e_{n+1}(t)$. The proof of this theorem is completed. □

5. Spectra of unit analytic signals

In this section, we will study the Fourier transform of two families of unit analytic signals and discuss the nonnecessity of Bedrosian conditions. We first need to calculate the Fourier transforms of the real signals $\cos\theta_a(t),\ t \in \mathbb{R}$, and $\cos\phi_a(t),\ t \in \mathbb{R}$. This leads us to evaluate the two integrals

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\cos\theta_a(t)\mathrm{e}^{\mathrm{i}\xi t}\mathrm{d}t \quad \text{and} \quad \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\cos\phi_a(t)\mathrm{e}^{\mathrm{i}\xi t}\mathrm{d}t.$$

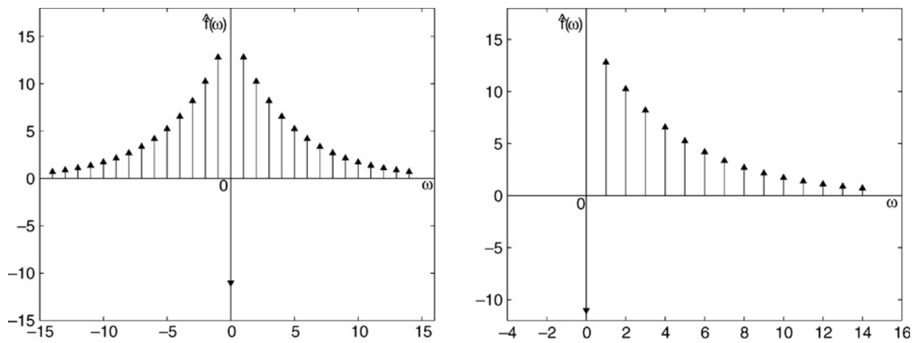

Fig. 7. The Fourier transforms of the real signal $\cos\theta_{1/2}(t)$ and the unit analytic signal $e^{i\theta_{1/2}(t)}$.

To avoid the difficulty of direct calculation of the two integrals, we recall some properties of $\cos\theta_a(t)$ and $\cos\phi_a(t)$. We now consider the first integral. Note that the analytic signal $e^{i\theta_a(t)}$ is the boundary value of a Möbius transform which has the following series expansion

$$\begin{aligned}\frac{z-a}{1-\bar a z} &= \frac{1}{\bar a}\frac{(\bar a z-1)+(1-\bar a a)}{1-\bar a z}\\ &= -\frac{1}{\bar a}+\frac{1-\bar a a}{\bar a}\frac{1}{1-\bar a z}\\ &= -a+\frac{1-|a|^2}{\bar a}\sum_{k=1}^{\infty}\bar a^k z^k,\end{aligned}$$

and that the convergence domain of the above series is $\{z : |z| < |a|^{-1}\}$. We get

$$e^{i\theta_a(t)} = \frac{e^{it}-a}{1-\bar a e^{it}} = -a+\frac{1-|a|^2}{\bar a}\sum_{k=1}^{\infty}\bar a^k e^{ikt}.$$

Hence the Fourier transform of the unit analytic signal $e^{i\theta_a(t)}$ is

$$\begin{aligned}&\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{i\theta_a(t)}e^{-i\xi t}\,dt\\ &\quad = -\frac{a}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-i\xi t}\,dt+\frac{1-|a|^2}{\sqrt{2\pi}\bar a}\sum_{k=1}^{\infty}\bar a^k\int_{-\infty}^{\infty}e^{-it(\xi-k)}\,dt\\ &\quad = -\sqrt{2\pi}\,a\delta(\xi)+\frac{\sqrt{2\pi}(1-|a|^2)}{\bar a}\sum_{k=1}^{\infty}\bar a^k\delta(\xi-k).\end{aligned}$$

Fig. 7 illustrates the plots of the Fourier transform of the real signal $\cos\theta_{1/2}(t)$, $t\in\mathbb{R}$, and the unit analytic signal $e^{i\theta_{1/2}(t)}$, $t\in\mathbb{R}$.

Now it is easy to see that the Fourier transform of the unit analytic signal $e^{i\theta_a(t)}$ is a one-sided 1-periodic impulse signal with exponentially decreasing impulse components. As a consequence, the Fourier transform of $\cos\theta_a(t)$ is

$$\begin{aligned}&\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\cos\theta_a(t)e^{-i\xi t}\,dt\\ &\quad = -\sqrt{2\pi}\,(\mathrm{Re}\,a)\,\delta(\xi)+\frac{\sqrt{2\pi}(1-|a|^2)}{2a}\sum_{k=1}^{\infty}a^k\delta(\xi+k)\\ &\qquad +\frac{\sqrt{2\pi}(1-|a|^2)}{2\bar a}\sum_{k=1}^{\infty}\bar a^k\delta(\xi-k).\end{aligned}$$

Now we turn to calculate the Fourier transform of the unit analytic signal $e^{i\phi_a(t)}$ with real number $|a|<1$. This is reduced to the calculation of the integral

$$\int_{-\infty}^{\infty}\frac{(1-|a|)^2-(1+|a|)^2t^2}{(1-|a|)^2+(1+|a|)^2t^2}e^{-i\xi t}\,dt.$$

To this end, we first calculate the integral $\int_{-\infty}^{\infty}\frac{e^{-i\xi t}}{t^2+B^2}\,dt$ for any positive number B. This is related to the Fourier transform of the Poisson kernel, or the Laplace integrals, and we have

$$\int_{-\infty}^{\infty}\frac{e^{-i\xi t}}{t^2+B^2}\,dt=\frac{\pi}{B}e^{-B|\xi|}.$$

This, combined with the identity

$$\begin{aligned}&\int_{-\infty}^{\infty}\frac{(1-|a|)^2-(1+|a|)^2t^2}{(1-|a|)^2+(1+|a|)^2t^2}e^{-i\xi t}\,dt\\ &\quad = -\int_{-\infty}^{\infty}e^{-i\xi t}\,dt+\frac{2(1-|a|)^2}{(1+|a|)^2}\int_{-\infty}^{\infty}\frac{e^{-i\xi t}}{t^2+\left(\frac{1-|a|}{1+|a|}\right)^2}\,dt,\end{aligned}$$

implies that

$$\begin{aligned}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\cos\phi_a(t)e^{-i\xi t}\,dt &= -\sqrt{2\pi}\,\delta(\xi)\\ &\quad +\frac{\sqrt{2\pi}(1-|a|)}{(1+|a|)}e^{-\frac{1-|a|}{1+|a|}|\xi|}.\end{aligned}$$

Recalling that $\mathcal{H}\cos\phi_a(t)=\sin\phi_a(t)$ and that the analytic signal $e^{i\phi_a(t)}$ has positive spectrum, we get that the Fourier transform of the unit analytic signal $e^{i\phi_a(t)}$ is

$$\begin{aligned}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{i\phi_a(t)}e^{-i\xi t}\,dt &= -\sqrt{2\pi}\delta(\xi)\\ &\quad +\frac{2\sqrt{2\pi}(1-|a|)}{(1+|a|)}e^{-\frac{1-|a|}{1+|a|}\xi}H(\xi),\end{aligned}$$

where $H(\xi)$ is the Heaviside function.

Fig. 8 illustrates the plots of the Fourier transform of the real signal $\cos\phi_{1/2}(t)$, $t\in\mathbb{R}$, and the unit analytic signal $e^{i\phi_{1/2}(t)}$, $t\in\mathbb{R}$.

From discussions above we know that the spectra of $e^{i\theta_a(t)}$ and $e^{i\phi_a(t)}$ contain nontrivial impulses at the origin. The following theorem shows that $\rho(t)\cos\psi_n(t)$ and $\rho(t)\sin\psi_n(t)$ form a Hilbert transform pair. This shows that the spectrum condition in Bedorisian theorem is not necessary.

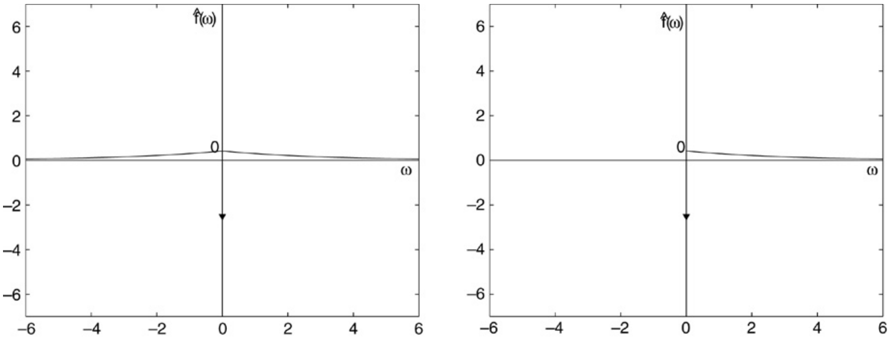

Fig. 8. The Fourier transforms of the real signal $\cos\phi_{1/2}(t)$ and the unit analytic signal $\mathrm{e}^{\mathrm{i}\phi_{1/2}(t)}$.

Theorem 5.1. *Suppose that ρ and ψ are defined in (4.2) and (4.3). For any positive integer n, the following equation holds*

$$\mathcal{H}\left(\rho(\cdot)\cos\psi_n(\cdot)\right)(t) = \rho(t)\sin\psi_n(t), \quad t \in \mathbb{R}. \tag{5.1}$$

Proof. We have known that $\rho(t)\cos\psi_n(t)$ is the real part of the boundary value on the unit circle of the complex variable function $E_n(z)$, which is an analytic function in the domain $\{z : |z| \le |a|^{-1}\}$. Note that

$$\frac{z}{z-a}\left(\frac{z-a}{1-\bar{a}z}\right)^n = \sum_{l=0}^{n-1}\sum_{k=0}^{\infty}\binom{n-1}{l}\binom{n-1+k}{k} \times (-1)^{n-1-l}\bar{a}^{n-1-l+k}z^{k+l+1}.$$

We can expand $E_n(\mathrm{e}^{\mathrm{i}t})$ in series form

$$E_n(\mathrm{e}^{\mathrm{i}t}) = \sum_{m=1}^{\infty} b_k \mathrm{e}^{\mathrm{i}mt}$$

with

$$b_m = \sum_{\triangle}\binom{n-1}{l}\binom{n-1+k}{k}(-1)^{n-1-l}\bar{a}^{n-1-l+k}.$$

Here $\triangle = \{(l,k) \in \mathbb{Z}^2 : l+k = m-1, l = 0,\ldots,n-1, k = 0,1,\ldots\}$. Combining this with the property of $\frac{\pi}{2}$ phase-shift of Hilbert transform, that is, $\mathcal{H}(\mathrm{e}^{\mathrm{i}\cdot})(t) = -\mathrm{i}\mathrm{e}^{\mathrm{i}t}$, leads to

$$\mathcal{H}E_n(\mathrm{e}^{\mathrm{i}\cdot})(t) = -\mathrm{i}\sum_{k=1}^{\infty} b_m \mathrm{e}^{\mathrm{i}mt} = -\mathrm{i}E_n(\mathrm{e}^{\mathrm{i}t}).$$

Comparing the real part with imaginary part of the above equation gives that $\mathcal{H}\rho(\cdot)\cos\psi_n(\cdot)(t) = \rho(t)\sin\psi_n(t)$. The proof of the theorem is finished. □

6. Conclusions

Analytic signals are the boundary values of certain analytic functions in the upper-half-complex plane. The real part and the imaginary part of an analytic signal form a pair of Hilbert transform. Hilbert transform of a time-limited signal is a circular Hilbert transform. Any circular analytic signal is the boundary value of an analytic function on the unit disc. Möbius transforms are analytic functions on the unit disc whose boundary values on the unit circle lead to a family of circular unit analytic signals $\mathrm{e}^{\mathrm{i}\theta_a(s)}$, $s \in [-\pi,\pi]$. The real part and the imaginary part of a signal $\mathrm{e}^{\mathrm{i}\theta_a(s)}$, $s \in [-\pi,\pi]$, form a circular Hilbert transform pair. Moreover, the real part and the imaginary part of the signal $\mathrm{e}^{\mathrm{i}\theta_a(t)}$, $t \in \mathbb{R}$, also form a Hilbert transform pair. The signal $\cos\theta_a(t)$ is mono-component in [3] and normalized intrinsic mode function in [17]. A Cayley transform transfers circular unit analytic signals to unit analytic signals on the line. Consequently, the output signal $\mathrm{e}^{\mathrm{i}\phi_a(t)}$, $t \in \mathbb{R}$, of the circular unit analytic signal $\mathrm{e}^{\mathrm{i}\theta_a(s)}$, $s \in [-\pi,\pi]$, acting by the Cayley transform is a family of unit analytic signals. The signal $\mathrm{e}^{\mathrm{i}\phi_a(t)}$, $t \in \mathbb{R}$, has strictly increasing time-varying phase and non-negative spectrum. The Gram–Schmidt procedure of $\{\mathrm{e}^{\mathrm{i}n\theta_a(t)}\}_{n\in\mathbb{Z}}$ leads to Laguerre bases.

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