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Co-dimension-p Shannon sampling theorems

YAN YANG* and TAO QIAN

Faculty of Science and Technology, University of Macau, China

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In this article, by defining the generalized co-dimension-p sinc function, the corresponding sinc interpolations (Shannon sampling theorems) are obtained.

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1. Introduction

Sinc function in the real line $\mathbf{R}$ is defined by

$$\operatorname{sinc}(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ixt}\mathrm{dt} = \frac{\sin(\pi x)}{\pi x}.$$

It has a holomorphic extension to the complex plane $\mathbf{C}$, i.e.,:

$$\operatorname{sinc}(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{izt}\mathrm{dt} = \frac{\sin(\pi z)}{\pi z}. \tag{1}$$

Some applications of the sinc function may be found in [1].

For a set A, let χ_A denote the characteristic function of A. Sinc function in $\mathbf{R}^m$ is defined by

$$\begin{aligned}\operatorname{sinc}(\underline{x}) &= (\chi_{[-\pi,\pi]^m})\check{}(\underline{x}) = \frac{1}{(2\pi)^m}\int_{\mathbf{R}^m} e^{i\langle \underline{x},\underline{\xi}\rangle}\chi_{[-\pi,\pi]^m}(\underline{\xi})\mathrm{d}\underline{\xi} \\ &= \prod_{i=1}^{m}\operatorname{sinc}(x_i) = \prod_{i=1}^{m}\frac{\sin(\pi x_i)}{\pi x_i}.\end{aligned}$$

*Corresponding author. Email: ya27406@umac.mo

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As a counterpart generalization of the sinc function (1) to higher-dimensional cases, Kou and Qian extended the sinc function in $\mathbf{R}^m$ to $m+1$-dimensional real variables $\mathbf{R}_1^m$ with the Clifford analysis setting in [2]. The definition of the extended sinc function (we call it *inhomogeneous co-dimension*-1 *sinc function*) is based on the generalized exponential function $e(x,\underline{\xi})$ in $\mathbf{R}_1^m \times \mathbf{R}^m$, extending the classical exponential function $e^{i\langle \underline{x},\underline{\xi}\rangle}$ in $\mathbf{R}^m \times \mathbf{R}^m$. Furthermore, using the inhomogeneous co-dimension-1 sinc function, they obtained the Shannon sampling theorem [2] corresponding to the Paley–Wiener (P–W) theorem in $\mathbf{R}_1^m$ [3]. In this article, we define co-dimension-p sinc functions and prove the corresponding Shannon samplings in relation to the P–W theorems obtained in [4].

2. Preliminaries

For a basic knowledge and notation in relation to the Clifford algebra the readers are referred to [5–7].

Let $\mathbf{e}_1, \ldots, \mathbf{e}_m$ be *basic elements* satisfying $\mathbf{e}_i\mathbf{e}_j + \mathbf{e}_j\mathbf{e}_i = -2\delta_{ij}$, where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise, $i, j = 1, 2, \ldots, m$. Set

$$\mathbf{R}^m = \{\underline{x} = x_1\mathbf{e}_1 + \cdots + x_m\mathbf{e}_m : x_j \in \mathbf{R},\ j = 1, 2, \ldots, m\},$$

and

$$\mathbf{R}_1^m = \{x = x_0 + \underline{x} : x_0 \in \mathbf{R},\ \underline{x} \in \mathbf{R}^m\}.$$

$\mathbf{R}^m$ and $\mathbf{R}_1^m$ are called, respectively, the *homogeneous* and *inhomogeneous* Euclidean spaces.

Elements in $\mathbf{R}^m$ are called *homogeneous vectors* and those of $\mathbf{R}_1^m$ *inhomogeneous vectors* or *vectors*. The real (or complex) Clifford algebra generated by $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m$, denoted by $\mathbf{R}^{(m)}$ (or $\mathbf{C}^{(m)}$), is the universal associative algebra generated by $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m$, over the real (or complex) field $\mathbf{R}$ (or $\mathbf{C}$). A general element in $\mathbf{R}^{(m)}$ (or $\mathbf{C}^{(m)}$), a *Clifford number*, therefore, is of the form $x = \sum_S x_S\mathbf{e}_S$, where for $S \neq \emptyset$, $\mathbf{e}_S$ are *ordered reduced products* of the basis elements and $\mathbf{e}_S = \mathbf{e}_{i_1}\mathbf{e}_{i_2}\cdots\mathbf{e}_{i_l}$, where S runs over all the ordered subsets of $\{1, 2, \ldots, m\}$, namely

$$S = \{1 \leq i_1 < i_2 < \cdots < i_l \leq m\}, \quad 1 \leq l \leq m,$$

and, for $S = \emptyset$, we set $\mathbf{e}_\emptyset = \mathbf{e}_0 = 1$.

The natural inner product between x and y in $\mathbf{C}^{(m)}$, denoted by $\langle x, y\rangle$, is the complex number $\sum_S x_S\overline{y_S}$, where $x = \sum_S x_S\mathbf{e}_S$ and $y = \sum_S y_S\mathbf{e}_S$. The norm associated with this inner product is

$$|x| = \langle x, x\rangle^{1/2} = \left(\sum_S |x_S|^2\right)^{1/2}.$$

The Clifford conjugate of a vector $x = x_0 + \underline{x}$, is defined to be $\overline{x} = x_0 - \underline{x}$. It is easy to verify that if $x \neq 0$ and $x \in \mathbf{R}_1^m$, then x has an inverse, x^{-1}, and

$$x^{-1} = \frac{\overline{x}}{|x|^2}.$$

The *unit sphere* $\{\underline{x} \in \mathbf{R}^m : |\underline{x}| = 1\}$ is denoted by S^{m-1}. We use $B(\underline{x}, r)$ for the open ball in $\mathbf{R}^m$ centered at $\underline{x}$ with radius r, and $\overline{B}(\underline{x}, r)$ for the topological closure of $B(\underline{x}, r)$.

Subsequently we shall study functions defined in the homogeneous space $\mathbf{R}^m$ taking values in $\mathbf{C}^{(m)}$. So, they are of the form $f(\underline{x}) = \sum_S f_S(\underline{x})\mathbf{e}_S$, where f_S are complex-valued functions. We shall use the *Dirac operator*, or the *homogeneous Dirac operator*, $\partial_{\underline{x}}$, where $\partial_{\underline{x}} = (\partial/\partial x_1)\mathbf{e}_1 + \cdots + (\partial/\partial x_m)\mathbf{e}_m$. We define the "left" and "right" roles of the operator $\partial_{\underline{x}}$ by

$$\partial_{\underline{x}} f = \sum_{i=1}^{m} \sum_{S} \frac{\partial f_S}{\partial x_i} \mathbf{e}_i \mathbf{e}_S$$

and

$$f \partial_{\underline{x}} = \sum_{i=1}^{m} \sum_{S} \frac{\partial f_S}{\partial x_i} \mathbf{e}_S \mathbf{e}_i.$$

If $\partial_{\underline{x}} f = 0$ in a domain (open and connected) U, then we say that f is *left-monogenic* in U; and, if $f \partial_{\underline{x}} = 0$ in U, then f is said to be *right-monogenic* in U. Left- or right-monogenic are called *one-sided-monogenic* or simply *monogenic*. The function theories for left- and right-monogenic functions, respectively, are parallel. If f is both left- and right-monogenic, then we say that f is *two-sided-monogenic*.

In $\mathbf{R}_1^m$ we shall use the *inhomogeneous Dirac operator*, or the *generalized Cauchy–Riemann operator*, $\partial_x = \partial_0 + \partial_{\underline{x}}, \partial_0 = (\partial/\partial x_0)$. The concept of monogenic functions in $\mathbf{R}_1^m$ is defined via the inhomogeneous Dirac operator ∂_x in a similar manner. The monogenic function theories for the homogeneous and inhomogeneous spaces, respectively, are analogous.

Let $k \in \mathbf{N}$, where $\mathbf{N}$ denotes the set of non-negative integers. Denote by $M_\ell^+(m, k, \mathbf{C}^{(m)})$ the space of k-homogeneous left-monogenic polynomials in $\mathbf{R}^m$, whose restriction to S^{m-1} is denoted by $\mathcal{M}_\ell^+(m, k, \mathbf{C}^{(m)})$.

The Fourier transform of functions in $\mathbf{R}^m$ is defined by

$$\hat{f}(\underline{\xi}) = \int_{\mathbf{R}^m} e^{-i\langle \underline{x}, \underline{\xi}\rangle} f(\underline{x}) \mathrm{d}\underline{x},$$

and the inverse Fourier transform by

$$\check{g}(\underline{x}) = \frac{1}{(2\pi)^m} \int_{\mathbf{R}^m} e^{i\langle \underline{x}, \underline{\xi}\rangle} g(\underline{\xi}) \mathrm{d}\underline{\xi},$$

where $\underline{\xi} = \xi_1 \mathbf{e}_1 + \cdots + \xi_m \mathbf{e}_m$.

To extend the domain of Fourier transforms to $\mathbf{R}_1^m$, we first need to extend the exponential function $e^{i\langle \underline{x}, \underline{\xi}\rangle}$. Denote, for $x = x_0\mathbf{e}_0 + \underline{x}$,

$$e(x, \underline{\xi}) = e^{i\langle \underline{x}, \underline{\xi}\rangle} e^{-x_0|\underline{\xi}|}\chi_+(\underline{\xi}) + e^{i\langle \underline{x}, \underline{\xi}\rangle} e^{x_0|\underline{\xi}|}\chi_-(\underline{\xi}), \tag{2}$$

where

$$\chi_\pm(\underline{\xi}) = \frac{1}{2}\left(1 \pm i\frac{\underline{\xi}\mathbf{e}_0}{|\underline{\xi}|}\right).$$

It is easy to verify that the functions $\chi_\pm$ satisfy the properties for projections:

$$\chi_-\chi_+ = \chi_+\chi_- = 0, \quad \chi_\pm^2 = \chi_\pm, \quad \chi_+ + \chi_- = 1.$$

As extension of $e(\underline{x}, \underline{\xi}) = e^{i\langle \underline{x}, \underline{\xi}\rangle}$ to $\mathbf{R}_1^m \times \mathbf{R}^m$, it is easy to verify that, for any fixed $\underline{\xi}$, $e(x, \underline{\xi})$ is two-sided-monogenic in $x \in \mathbf{R}_1^m$. The above extension is the inhomogeneous co-dimension-1 CK extension of $e(\underline{x}, \underline{\xi})$ to $\mathbf{R}_1^m$. Replacing $\mathbf{e}_0$ by ϵ_0 in equation (2), where ϵ_0 is a basis element added to the collection $\mathbf{e}_1, \ldots, \mathbf{e}_m$, with $\epsilon_0^2 = -1$ and anti-commutativity with the other $\mathbf{e}_j, j = 1, \ldots, m$, one obtains the homogeneous co-dimension-1 CK extension $e(x_0\epsilon_0, \underline{x}, \underline{\xi})$ of $e^{i\langle \underline{x}, \underline{\xi}\rangle}$ in $\mathbf{R}^{m+1}$. This function $e(x_0\epsilon_0, \underline{x}, \underline{\xi})$ is left-monogenic in $x \in \mathbf{R}^{m+1}$. Generalizations of the exponential function of these types can be first found in the work of Sommen [8], and then in Li *et al.* [7], where $\underline{\xi}$ is further extended to $\underline{\xi} + i\underline{\eta} \in \mathbf{C}^m$.

In [6], the generalized CK extension tells us: If $A_0(\underline{y})$ is an analytic function in $\mathbf{R}^q$, for any k-homogeneous left-monogenic polynomial $P_k(\underline{x})$ in $\mathbf{R}^p$, there exists a unique sequence $(A_l(\underline{y}))_{l>0}$ of analytic functions such that the series

$$\begin{aligned} f_{P_k}(\underline{x}, \underline{y}) &= \sum_{l=0}^{\infty} \underline{x}^l P_k(\underline{x}) A_l(\underline{y}) \\ &= \Gamma\left(k + \frac{p}{2}\right)\left(\frac{r\sqrt{\Delta_{\underline{y}}}}{2}\right)^{-(k+(p/2))}\left[\frac{r\sqrt{\Delta_{\underline{y}}}}{2} J_{k+\frac{p}{2}-1}\left(r\sqrt{\Delta_{\underline{y}}}\right) + \frac{\underline{x}\partial_{\underline{y}}}{2} J_{k+(p/2}\left(r\sqrt{\Delta_{\underline{y}}}\right)\right](P_k(\underline{x})A_0(\underline{y})), \end{aligned} \tag{3}$$

is convergent and its sum f_{P_k} is left-monogenic in any compact set belongs to $\mathbf{R}^p \oplus \mathbf{R}^q$. Where $(\sqrt{\Delta_{\underline{y}}})^2 = \Delta_{\underline{y}}$, the Laplacian in $\underline{y}$, and J_v the Bessel function

$$J_v(u) = \sum_{l=0}^{\infty} \frac{(-1)^l}{2^{2l+v} l! \Gamma(l + v + 1)} u^{2l+v}.$$

We call $f_{P_k}(\underline{x}, \underline{y})$ the *generalized CK extension in relation to* P_k of $A_0(\underline{y})$ and $A_0(\underline{y})$ the *initial value* of $f_{P_k}(\underline{x}, \underline{y})$. In particular, when $k = 0, P_k = 1$, we get $A_0(\underline{y}) = f_{P_k}|_{\mathbf{R}^q}$.

Denote $\mathcal{T}_{P_k}(\mathbf{R}^q)$ the space of all functions of the form (3).

Furthermore, for any left-monogenic function $f(\underline{x}, \underline{y})$ in $\tilde{U}$ belongs to $\mathbf{R}^m = \mathbf{R}^p \oplus \mathbf{R}^q$, where $\underline{x} = r\underline{\omega} \in \mathbf{R}^p$, $\underline{y} \in \mathbf{R}^q$, denote $T_k(f)(\underline{\omega}, \underline{y}) = \lim_{r\to 0} 1/r^k P(k)f(r, \underline{\omega}, \underline{y})$, where $P(k)$ being the projection onto $M_\ell^+(p, k, \mathbf{C}^{(p)})$. It can be decomposed in variable x. By using a basis $P_{k,\alpha}(\underline{\omega})$ for $\mathcal{M}_\ell^+(p, k, \mathbf{C}^{(p)})$,

$$T_k(f)(\underline{\omega}, \underline{y}) = \sum_{\alpha \in A_k} P_{k,\alpha}(\underline{\omega}) T_{k,\alpha}(f)(\underline{y}),$$

where $T_{k,\alpha}(f)(\underline{y})$ are real analytic functions.

Denote the generalized CK extension of $T_{k,\alpha}(f)(\underline{y})$ with $P_{k,\alpha}(\underline{x})$ by $T_{k,\alpha}(\underline{x}, \underline{y})$, then f can be written in a uniquely way as

$$f(\underline{x}, \underline{y}) = \sum_k \sum_{\alpha \in A_k} T_{k,\alpha}(\underline{x}, \underline{y}), \tag{4}$$

where $T_{k,\alpha}(\underline{x}, \underline{y}) = \sum_l \underline{x}^l P_{k,\alpha}(\underline{x}) T_{k,\alpha}^{(l)}(f)(\underline{y})$ and the series (4) converging uniformly on any compact set in $\tilde{U}$. The series (4) is called *the generalized Taylor series* and $T_{k,\alpha}^{(0)}(f)(\underline{y}) = T_{k,\alpha}(f)(\underline{y})$ are called *the initial values* of f.

To extend the domain of Fourier transforms to $\mathbf{R}^p \oplus \mathbf{R}^q$, we also need to extend the exponential function $e^{i\langle \underline{y}, \underline{t}\rangle}$. In [4], for a given k-homogeneous left-monogenic polynomial $P_k(\underline{x})$, we get the generalized CK extension of $e^{i\langle \underline{y}, \underline{t}\rangle}$ in $\mathcal{T}_{P_k}(\mathbf{R}^q)$, $\underline{x} = r\underline{\omega} \in \mathbf{R}^p$, $\underline{y}, \underline{t} \in \mathbf{R}^q$, denoted by

$$\varepsilon_{P_k}^p(\underline{x}, \underline{y}, \underline{t}) = \Gamma\left(k + \frac{p}{2}\right) r^k e^{i\langle \underline{y}, \underline{t}\rangle} \left(\frac{r|\underline{t}|}{2}\right)^{-k-(p/2)+1} \left[I_{k+(p/2)-1}(r|\underline{t}|) + i I_{k+(p/2)}(r|\underline{t}|)\underline{\omega}\frac{\underline{t}}{|\underline{t}|}\right] P_k(\underline{\omega}), \tag{5}$$

where

$$I_v(u) = i^{-v} J_v(iu) = \sum_{k=0}^{\infty} \frac{1}{k!\Gamma(v + k - 1)} \left(\frac{u}{2}\right)^{u+2k},$$

being a kind of Bessel functions. Then $\varepsilon_{P_k}^p(\underline{x}, \underline{y}, \underline{t})$ is left-monogenic in $\mathbf{R}^p \oplus \mathbf{R}^q$.

In particular, when $P_k = 1$, $k = 0$, we have

$$\varepsilon_1^p(\underline{x}, \underline{y}, \underline{t}) = \Gamma\left(\frac{p}{2}\right) e^{i\langle \underline{y}, \underline{t}\rangle} \left(\frac{r|\underline{t}|}{2}\right)^{-(p/2)+1} \left[I_{(p/2)-1}(r|\underline{t}|) + i I_{p/2}(r|\underline{t}|)\underline{\omega}\frac{\underline{t}}{|\underline{t}|}\right].$$

We in particular denote

$$\varepsilon_1^1(x_1\mathbf{e}_1, \underline{y}, \underline{t}) = e(x_1\mathbf{e}_1, \underline{y}, \underline{t}).$$

From [4], we have, for any $u > 0$,

$$\left(\frac{u}{2}\right)^{-v} I_v(u) = \sum_{k=0}^{\infty} \frac{1}{k!\Gamma(v + k + 1)} \left(\frac{u}{2}\right)^{2k} \leq C \sum_{k=0}^{\infty} \frac{u^{2k}}{(2k)!} \leq Ce^u. \tag{6}$$

When $|\underline{t}| \leq \Omega$, we have

$$\begin{aligned}|\varepsilon^{p}_{P_k}(\underline{x}, \underline{y}, \underline{t})| &\leq C\left(\frac{2}{\Omega}\right)^{k}\left(\frac{r\Omega}{2}\right)^{-(p/2)+1}\left[I_{k+\frac{p}{2}-1}(r\Omega) + I_{k+\frac{p}{2}}(r\Omega)\right] \\ &\leq C[I_k(r\Omega) + I_{k+1}(r\Omega)] \\ &\leq Ce^{r\Omega}. \end{aligned} \tag{7}$$

3. Exact interpolation with Shannon sampling in $\mathbf{R}^m = \mathbf{R}^p \oplus \mathbf{R}^q$

Based on the extension of the exponential function given by (5), the *generalized co-dimension-p sinc function* in relation to $P_k(\underline{x}) \in M^{+}_{\ell}(p, k, \mathbf{C}^{(p)})$ is defined by

$$\mathrm{sinc}^{p}_{P_k}(\underline{x}, \underline{y}) = \frac{1}{(2\pi)^q}\int_{\mathbf{R}^q} \varepsilon^{p}_{P_k}(\underline{x}, \underline{y}, \underline{t})\chi_{[-\pi,\pi]^q}(\underline{t})\mathrm{d}\underline{t}. \tag{8}$$

For $h > 0$ fixed, define the *cardinal function* of f to be

$$C(f, h)(\underline{x}, \underline{y}) \equiv \sum_{\underline{k}\in\mathbf{Z}^q} \mathrm{sinc}^{p}_{P_k}\left(\frac{\underline{x}}{h}, \frac{\underline{y} - h\underline{k}}{h}\right) f(h\underline{k}),$$

from equation (8), we have

$$\mathrm{sinc}^{p}_{P_k}\left(\frac{\underline{x}}{h}, \frac{\underline{y} - h\underline{k}}{h}\right) = \frac{h^q}{(2\pi)^q}\int_{[-(\pi/h,(\pi/h]^q} \varepsilon^{p}_{P_k}(\underline{x}, \underline{y} - h\underline{k}, \underline{t})\mathrm{d}\underline{t}. \tag{9}$$

Next, we shall consider the generalized co-dimension-p interpolation via the cardinal function corresponding to the generalized co-dimension-p P–W theorem proved in [4]:

LEMMA 1 [4] (Generalized co-dimension-p P–W theorem) *Let $P_k \in M^{+}_{\ell}(p, k; \mathbf{C}^{(p)})$ be given, F analytic, defined in $\mathbf{R}^q$, taking values in $\mathbf{C}^{(q)}$, which is the complex Clifford algebra generated by $\mathbf{e_{p+1}}, \ldots, \mathbf{e_{p+q}}$, and $F \in L^2(\mathbf{R}^q)$, Ω be a positive real number. Then the following two assertions are equivalent*:

1^0 *F has a homogeneous co-dimensional-p generalized CK extension to $\mathbf{R}^{p+q}$, denoted by f_{P_k}, and there exists a constant C such that*

$$|f_{P_k}(\underline{x}, \underline{y})| \leq Ce^{\Omega|\underline{x}|}, \ \textit{for any}\ \underline{x} \in \mathbf{R}^p,\ \underline{y} \in \mathbf{R}^q.$$

2^0 *supp*$(\hat{F}) \subset \overline{B}(0, \Omega)$.

Moreover, if one of the above conditions holds, then we have

$$f_{P_k}(\underline{x}, \underline{y}) = \frac{1}{(2\pi)^q}\int_{\mathbf{R}^q} \varepsilon^{p}_{P_k}(\underline{x}, \underline{y}, \underline{\xi})\hat{F}(\underline{\xi})\mathrm{d}\underline{\xi},\ \textit{for any}\ \underline{x} \in \mathbf{R}^p, \underline{y} \in \mathbf{R}^q.$$

A function f in $\mathbf{R}^p \oplus \mathbf{R}^q$ is said to be of *exponential type* Ω if

$$|f(\underline{x}, \underline{y})| \leq Ce^{\Omega|\underline{x}|},\ \text{for any}\ \underline{x} \in \mathbf{R}^p, \underline{y} \in \mathbf{R}^q$$

holds.

For any $h > 0$, denote

$$PW_{\mathcal{T}_{P_k}(\mathbf{R}^q)}(\pi/h) = \{f \mid f \in \mathcal{T}_{P_k}(\mathbf{R}^q) \text{ and of exponential type } \pi/h, \text{ the initial value } F \in L^2(\mathbf{R}^q) \text{ and taking values in } \mathbf{C}^{(q)}\}.$$

Particularly, taking $k = 0$, $P_k = 1$ and $p = 1$, we have

$$PW_{\mathbf{R}^{q+1}}(\pi/h) = \{f \mid f \text{ is left-monogenic in } \mathbf{R}^{q+1} \text{and of exponential type } \pi/h,\ f|_{\mathbf{R}^q} \in L^2(\mathbf{R}^q), \text{ and } f|_{\mathbf{R}^q} \text{ taking values in } \mathbf{C}^{(q)}\}.$$

The following theorems characterize the functions in the P–W class $PW_{\mathcal{T}_{P_k}(\mathbf{R}^q)}(\pi/h)$.

Theorem 1 *If $f \in PW_{\mathcal{T}_{P_k}(\mathbf{R}^q)}(\pi/h)$, then for any $\underline{x} \in \mathbf{R}^p, \underline{y} \in \mathbf{R}^q$, we have*

1^0

$$f(\underline{x}, \underline{y}) = \frac{1}{h^q} \int_{\mathbf{R}^q} \operatorname{sinc}^p_{P_k}\left(\frac{\underline{x}}{h}, \frac{\underline{y} - \underline{\xi}}{h}\right) F(\underline{\xi}) \mathrm{d}\underline{\xi}.$$

2^0

$$\frac{1}{(2\pi)^q} \int_{[-\pi/h, \pi/h]^q} |\hat{F}(\underline{y})|^2 \mathrm{d}\underline{y} = \int_{\mathbf{R}^q} |F(\underline{t})|^2 \mathrm{d}\underline{t} = \sum_{\underline{k} \in \mathbf{Z}^q} |F(h\underline{k})|^2, \tag{10}$$

where F is the initial value of f.

Proof 1^0: Since $f \in PW_{\mathcal{T}_{P_k}(\mathbf{R}^q)}(\pi/h)$, according to Lemma 1, we have

$$\begin{aligned} f(\underline{x}, \underline{y}) &= \frac{1}{(2\pi)^q} \int_{\overline{B}(0, \pi/h)} \varepsilon^p_{P_k}(\underline{x}, \underline{y}, \underline{t}) \hat{F}(\underline{t}) \mathrm{d}\underline{t} \\ &= \frac{1}{(2\pi)^q} \int_{[-\pi/h, \pi/h]^q} \varepsilon^p_{P_k}(\underline{x}, \underline{y}, \underline{t}) \hat{F}(\underline{t}) \mathrm{d}\underline{t} \\ &= \frac{1}{(2\pi)^q} \int_{\mathbf{R}^q} \varepsilon^p_{P_k}(\underline{x}, \underline{y}, \underline{t}) \chi_{[-\pi/h, \pi/h]^q}(\underline{t}) \hat{F}(\underline{t}) \mathrm{d}\underline{t}. \end{aligned}$$

By Parseval's theorem and (9), the above is equal to

$$\begin{aligned} &\frac{1}{(2\pi)^q} \int_{\mathbf{R}^q} \widehat{\left[\varepsilon^p_{P_k}(\underline{x}, \underline{y}, \underline{t}) \chi_{[-\pi/h, \pi/h]^q}(\underline{t})\right]}(\underline{\xi}) F(\underline{\xi}) \mathrm{d}\underline{\xi} \\ &= \frac{1}{h^q} \int_{\mathbf{R}^q} \left(\int_{[-\pi/h, \pi/h]^q} \varepsilon^p_{P_k}(\underline{x}, \underline{y} - \underline{\xi}, \underline{t}) \mathrm{d}\underline{t} \right) F(\underline{\xi}) \mathrm{d}\underline{\xi} \\ &= \frac{1}{h^q} \int_{\mathbf{R}^q} \operatorname{sinc}^p_{P_k}\left(\frac{\underline{x}}{h}, \frac{\underline{y} - \underline{\xi}}{h}\right) F(\underline{\xi}) \mathrm{d}\underline{\xi}. \end{aligned}$$

2^0: From Lemma 1, we have

$$F(\underline{t}) = \frac{1}{(2\pi)^q} \int_{B(0,\pi/h)} e^{i\langle \underline{y}, \underline{t}\rangle} \hat{F}(\underline{y}) \mathrm{d}\underline{y} = \frac{1}{(2\pi)^q} \int_{[-\pi/h,\,\pi/h]^q} e^{i\langle \underline{y}, \underline{t}\rangle} \hat{F}(\underline{y}) \mathrm{d}\underline{y}.$$

Considering the Fourier expansion of $\hat{F}$ in the cube $[-\pi/h,\ \pi/h]^q$, we have

$$h^q F(h\underline{k}) = \frac{1}{(2R)^q} \int_{[-R,\,R]^q} e^{i\pi\langle \underline{x}, \underline{k}\rangle/R} \hat{F}(\underline{y}) \mathrm{d}\underline{y} = c_k,$$

where $R = \frac{\pi}{h}$, and c_k are the Fourier coefficients of $\hat{F}$. The Plancherel theorem of Fourier series is

$$\int_{[-R,\,R]^q} |\hat{F}(\underline{y})|^2 \mathrm{d}\underline{y} = (2R)^q \sum_{\underline{k}\in\mathbf{Z}^q} |c_k|^2,$$

and the Plancherel theorem on L^2-functions in $\mathbf{R}^q$ reads

$$\int_{\mathbf{R}^q} |\hat{F}(\underline{y})|^2 \mathrm{d}\underline{y} = \int_{[-R,\,R]^q} |\hat{F}(\underline{y})|^2 \mathrm{d}\underline{y} = (2\pi)^q \int_{\mathbf{R}^q} |F(\underline{t})|^2 \mathrm{d}\underline{t}.$$

So we have

$$\frac{1}{(2\pi)^q} \int_{[-\pi/h,\,\pi/h]^q} |\hat{F}(\underline{y})|^2 \mathrm{d}\underline{y} = \int_{\mathbf{R}^q} |F(\underline{t})|^2 \mathrm{d}\underline{t} = \sum_{\underline{k}\in\mathbf{Z}^q} |F(h\underline{k})|^2. \qquad \blacksquare$$

Corollary 1 *If $f \in PW_{\mathcal{T}_1(\mathbf{R}^q)}(\pi/h)$, then for any $\underline{x} \in \mathbf{R}^p, \underline{y} \in \mathbf{R}^q$, we have*

$$f(\underline{x}, \underline{y}) = \frac{1}{h^q} \int_{\mathbf{R}^q} \operatorname{sinc}_1^p\left(\frac{\underline{x}}{h}, \frac{\underline{y}-\underline{\xi}}{h}\right) f(\underline{\xi}) \mathrm{d}\underline{\xi}.$$

$$\frac{1}{(2\pi)^q} \int_{[-\pi/h,\,\pi/h]^q} |\hat{f}(\underline{y})|^2 \mathrm{d}\underline{y} = \int_{\mathbf{R}^q} |f(\underline{t})|^2 \mathrm{d}\underline{t} = \sum_{\underline{k}\in\mathbf{Z}^q} |f(h\underline{k})|^2.$$

From (8) and the co-dimension-p P–W theorem, we can obtain that $\operatorname{sinc}_{P_k}^p(\underline{x}/h, \underline{y}/h)$ belongs to $PW_{\mathcal{T}_{P_k}(\mathbf{R}^q)}(\sqrt{q}\pi/h)$. Furthermore, we can construct functions in $PW_{\mathcal{T}_{P_k}(\mathbf{R}^q)}(\sqrt{q}\pi/h)$ using the following Theorem.

Theorem 2 *Let $P_k \in M_\ell^+(p,k;\mathbf{C}^{(p)})$ be given, $F \in L^2(\mathbf{R}^q)$ and take values in $\mathbf{C}^{(q)}$. Then $f \in PW_{\mathcal{T}_{P_k}(\mathbf{R}^q)}(\sqrt{q}\pi/h)$, where*

$$f(\underline{x}, \underline{y}) = h^q \int_{\mathbf{R}^q} \operatorname{sinc}_{P_k}^p\left(\frac{\underline{x}}{h}, \frac{\underline{y}-\underline{\xi}}{h}\right) F(\underline{\xi}) \mathrm{d}\underline{\xi}. \tag{11}$$

Proof Applying the Parseval's theorem to the right-hand side of equation (11), owing to equation (9), we have

$$\begin{aligned} f(\underline{x},\underline{y}) &= \frac{h^q}{(2\pi)^q}\int_{\mathbf{R}^q}\widehat{\left[\operatorname{sinc}^p_{P_k}\left(\frac{\underline{x}}{h},\frac{\underline{y}-\underline{\xi}}{h}\right)\right]}(-\underline{t})\hat{F}(\underline{t})\mathrm{d}\underline{t} \\ &= \frac{h^q}{(2\pi)^q}\int_{\mathbf{R}^q} h^{-q}\varepsilon^p_{P_k}(\underline{x},\underline{y},\underline{t})\chi_{[-\pi/h,\pi/h]^q}(\underline{t})\hat{F}(\underline{t})\mathrm{d}\underline{t} \\ &= \frac{1}{(2\pi)^q}\int_{\mathbf{R}^q}\varepsilon^p_{P_k}(\underline{x},\underline{y},\underline{t})\chi_{[-\pi/h,\pi/h]^q}(\underline{t})\hat{F}(\underline{t})\mathrm{d}\underline{t} \\ &= \frac{1}{(2\pi)^q}\int_{[-\pi/h,\pi/h]^q}\varepsilon^p_{P_k}(\underline{x},\underline{y},\underline{t})\hat{F}(\underline{t})\mathrm{d}\underline{t}. \end{aligned}$$

According to the evaluation (7) of $\varepsilon^p_{P_k}$, we have

$$|f(\underline{x},\underline{y})| \le Ce^{\sqrt{q}\pi/h|\underline{x}|},\ \text{ for any } \underline{x}\in\mathbf{R}^p, \underline{y}\in\mathbf{R}^q.$$

By Lemma 1, we get $f(\underline{x},\underline{y}) \in PW_{\mathcal{T}_{P_k}(\mathbf{R}^q)}(\sqrt{q}\pi/h)$. ■

Next, the exact $\operatorname{sinc}_{P_k}$ interpolation of functions in $PW_{\mathcal{T}_{P_k}(\mathbf{R}^q)}(\pi/h)$ is given.

THEOREM 3 *If* $f \in PW_{\mathcal{T}_{P_k}(\mathbf{R}^q)}(\pi/h)$, *then for any* $\underline{x}\in\mathbf{R}^p$, $\underline{y}\in\mathbf{R}^q$,

$$f(\underline{x},\underline{y}) = C(f,h)(\underline{x},\underline{y}) = \sum_{\underline{k}\in\mathbf{Z}^q}\operatorname{sinc}^p_{P_k}\left(\frac{\underline{x}}{h},\frac{\underline{y}-h\underline{k}}{h}\right)F(h\underline{k}), \tag{12}$$

where F is the initial value of f and the series on the right-hand side is absolutely and uniformly convergent for any $\underline{y}\in\mathbf{R}^q$ *and* $\underline{x}$ *belongs to any bounded set in* $\mathbf{R}^p$.

Proof Since $f(\underline{x},\underline{y}) \in PW_{\mathcal{T}_{P_k}(\mathbf{R}^q)}(\pi/h)$, Lemma 1 gives

$$f(\underline{x},\underline{y}) = \frac{1}{(2\pi)^q}\int_{[-\pi/h,\pi/h]^q}\varepsilon^p_{P_k}(\underline{x},\underline{y},\underline{t})\hat{F}(\underline{t})\mathrm{d}\underline{t}. \tag{13}$$

Expanding $\varepsilon^p_{P_k}(\underline{x},\underline{y},\underline{t})$ on the cube $[-\pi/h,\pi/h]^q$ into its multiple Fourier series in q-variables, we have

$$\varepsilon^p_{P_k}(\underline{x},\underline{y},\underline{t}) = \sum_{\underline{k}\in\mathbf{Z}^q} e^{i\langle h\underline{k},\underline{t}\rangle}a_{\underline{k}}(\underline{x},\underline{y}), \tag{14}$$

where

$$\begin{aligned} a_{\underline{k}}(\underline{x},\underline{y}) &= \frac{h^q}{(2\pi)^q}\int_{[-\pi/h,\pi/h]^q}\varepsilon^p_{P_k}(\underline{x},\underline{y},\underline{t})e^{-i\langle h\underline{k},\underline{t}\rangle}\mathrm{d}\underline{t} \\ &= \frac{h^q}{(2\pi)^q}\int_{[-\pi/h,\pi/h]^q}\varepsilon^p_{P_k}(\underline{x},\underline{y}-h\underline{k},\underline{t})\mathrm{d}\underline{t} \\ &= \operatorname{sinc}^p_{P_k}\left(\frac{\underline{x}}{h},\frac{\underline{y}-h\underline{k}}{h}\right) \end{aligned}$$

are the Fourier coefficients of $\varepsilon^p_{P_k}(\underline{x},\underline{y},\underline{t})$.

Substituting the series expansion (14) in the integral (13) and interchanging the order of the summation and the integration due to the L^2-convergence, we have

$$\begin{aligned}
f(\underline{x},\underline{y}) &= \frac{1}{(2\pi)^q}\int_{[-\pi/h,\pi/h]^q}\sum_{\underline{k}\in\mathbf{Z}^q} e^{i\langle h\underline{k},\underline{t}\rangle}\mathrm{sinc}^p_{P_k}\left(\frac{\underline{x}}{h},\frac{\underline{y}-h\underline{k}}{h}\right)\hat{F}(\underline{t})\mathrm{d}\underline{t}\\
&= \sum_{\underline{k}\in\mathbf{Z}^q}\mathrm{sinc}^p_{P_k}\left(\frac{\underline{x}}{h},\frac{\underline{y}-h\underline{k}}{h}\right)\left(\frac{1}{(2\pi)^q}\int_{[-\pi/h,\pi/h]^q} e^{i\langle h\underline{k},\underline{t}\rangle}\hat{F}(\underline{t})\mathrm{d}\underline{t}\right)\\
&= \sum_{\underline{k}\in\mathbf{Z}^q}\mathrm{sinc}^p_{P_k}\left(\frac{\underline{x}}{h},\frac{\underline{y}-h\underline{k}}{h}\right)F(h\underline{k}).
\end{aligned}$$

We next show the uniform convergence of the series on the right side.

In fact, for any positive number M, using the Cauchy–Schwarz inequality, we have

$$\left|\sum_{|\underline{k}|\rangle M}\mathrm{sinc}^p_{P_k}\left(\frac{\underline{x}}{h},\frac{\underline{y}-h\underline{k}}{h}\right)F(h\underline{k})\right| \le \left(\sum_{|\underline{k}|\rangle M}\left|\mathrm{sinc}^p_{P_k}\left(\frac{\underline{x}}{h},\frac{\underline{y}-h\underline{k}}{h}\right)\right|^2\right)^{1/2}\left(\sum_{|\underline{k}|\rangle M}|F(h\underline{k})|^2\right)^{1/2}.$$

Note that the function $\varepsilon^p_{P_k}(\underline{x},\underline{y},\cdot)\in L^2([-\pi/h,\pi/h]^q)$. Using the Bessel inequality and equation (7), for any bounded set $U\in\mathbf{R}^p$, we have

$$\begin{aligned}
\left(\sum_{|\underline{k}|\rangle M}\left|\mathrm{sinc}^p_{P_k}\left(\frac{\underline{x}}{h},\frac{\underline{y}-h\underline{k}}{h}\right)\right|^2\right)^{1/2} &\le \left(\frac{h}{2\pi}\right)^{q/2}\|\varepsilon^p_{P_k}(\underline{x},\underline{y},\cdot)\|_{L^2([-\pi/h,\pi/h]^q)}\\
&\le \left(\frac{h}{2\pi}\right)^{q/2} e^{\sqrt{q}|\underline{x}|\pi/h}\le C<\infty,
\end{aligned}$$

where $\underline{y}\in\mathbf{R}^q$, $\underline{x}\in U$. Owing to the estimate and equation (10), the series in equation (12) is convergent uniformly and absolutely in $U\oplus\mathbf{R}^q$. ∎

The homogeneous co-dimension-p P–W theorem is stated as:

LEMMA 2 [4] (Homogeneous co-dimension-p P–W theorem) *Let F be analytic, defined in $\mathbf{R}^q$, taking values in $\mathbf{C}^{(q)}$, the complex Clifford algebra generated by $\mathbf{e_{p+1}},\ldots,\mathbf{e_{p+q}}$, and $F\in L^2(\mathbf{R}^q)$. Ω is a positive real number. Then the following two assertions are equivalent*:

1^0 *F has a homogeneous co-dimensional-p CK extension to $\mathbf{R}^{p+q}$, denoted by f, and there exists a constant C such that*

$$|f(\underline{x},\underline{y})|\le Ce^{\Omega|\underline{x}|},\ \textit{for any}\ \underline{x}\in\mathbf{R}^p,\ \underline{y}\in\mathbf{R}^q.$$

2^0 *$supp(\hat{F})\subset\overline{B}(0,\Omega)$.*

Moreover, if one of the above conditions holds, we have

$$f(\underline{x},\underline{y})=\frac{1}{(2\pi)^q}\int_{\mathbf{R}^q}\varepsilon^p_1(\underline{x},\underline{y},\underline{\xi})\hat{F}(\underline{\xi})\mathrm{d}\underline{\xi},\ \textit{for any}\ \underline{x}\in\mathbf{R}^p,\underline{y}\in\mathbf{R}^q.$$

Corresponding to the Lemma 2, we have the exact sinc_1^p interpolation of functions in $PW_{\mathcal{T}_1(\mathbf{R}^q)}(\pi/h)$.

Corollary 2 *If* $f \in PW_{\mathcal{T}_1(\mathbf{R}^q)}(\pi/h)$, *then for any* $\underline{x} \in \mathbf{R}^p, \underline{y} \in \mathbf{R}^q$,

$$f(\underline{x}, \underline{y}) = C(f, h)(\underline{x}, \underline{y}) = \sum_{\underline{k} \in \mathbf{Z}^q} \mathrm{sinc}_1^p\left(\frac{\underline{x}}{h}, \frac{\underline{y} - h\underline{k}}{h}\right) f(h\underline{k}),$$

where

$$\mathrm{sinc}_1^p(\underline{x}, \underline{y}) = \frac{1}{(2\pi)^q} \int_{\mathbf{R}^q} \varepsilon_1^p(\underline{x}, \underline{y}, \underline{t}) \chi_{[-\pi, \pi]^q}(\underline{t}) \mathrm{d}\underline{t},$$

and the series on the right-hand side is absolutely and uniformly convergent for any $\underline{y} \in \mathbf{R}^q$ *and* $\underline{x}$ *belongs to any bounded set in* $\mathbf{R}^p$.

Henceforth the article shall deal with the Shannon sampling theorem in relation to the generalized Taylor series.

In [4], the co-dimension-p P–W theorem related to generalized Taylor series reads:

Lemma 3 *Assume that* $f(\underline{x}, \underline{y})$ *is left-monogenic in* $\mathbf{R}^m = \mathbf{R}^p \oplus \mathbf{R}^q$ *with the form* (4). *For any* $k \geq 0$ *and* $\alpha \in A_k$, *let* $T_{k,\alpha}(f)(\underline{y}) = T_{k,\alpha}^{(0)}(f)(\underline{y})$ *be analytic, defined in* $\mathbf{R}^q$, *taking values in* $\mathbf{C}^{(q)}$, *the complex Clifford algebra generated by* $\mathbf{e_{p+1}}, \ldots, \mathbf{e_{p+q}}$, $T_{k,\alpha}(f)(\underline{y}) \in L^2(\mathbf{R}^q)$,

$$\left| \sum_k \sum_\alpha P_{k,\alpha}(\underline{x}) \hat{T}_{k,\alpha}(f)(\underline{\xi}) \right| \leq C e^{\Omega|\underline{x}|}, \text{ for any } \underline{x} \in \mathbf{R}^p, \ \underline{\xi} \in \mathbf{R}^q,$$

where Ω *is a positive real number. Then the following two assertions are equivalent*:

1^0 *There exists a constant* C *such that*

$$|f(\underline{x}, \underline{y})| \leq C e^{\Omega|\underline{x}|}, \text{ for any } \underline{x} \in \mathbf{R}^p, \underline{y} \in \mathbf{R}^q.$$

2^0 $\mathrm{supp}(\hat{T}_{k,\alpha}(f)) \subset \overline{B}(0, \Omega)$, *for any* $k \geq 0$ *and* $\alpha \in A_k$.

Moreover, if one of the above conditions holds, we have

$$f(\underline{x}, \underline{y}) = \sum_k \sum_\alpha T_{k,\alpha}(\underline{x}, \underline{y}) = \frac{1}{(2\pi)^q} \sum_k \sum_\alpha \int_{\mathbf{R}^q} \varepsilon_{P_{k,\alpha}}^p(\underline{x}, \underline{y}, \underline{\xi}) \hat{T}_{k,\alpha}(f)(\underline{\xi}) \mathrm{d}\underline{\xi}, \tag{15}$$

for any $\underline{x} \in \mathbf{R}^p, \underline{y} \in \mathbf{R}^q$ *and the series is converging uniformly on any compact set in* $\mathbf{R}^p \oplus \mathbf{R}^q$.

Next, the Shannon sampling theorem corresponding to the P–W theorem above is obtained. For any $h > 0$, denote

$PW_{\mathbf{R}^p\oplus\mathbf{R}^q}(\pi/h) = \{f|f$ is left-monogenic in $\mathbf{R}^p \oplus \mathbf{R}^q$ with the form (4) and of exponential type π/h, the initial values $T_{k,\alpha}(f)(\underline{y}) \in L^2(\mathbf{R}^q)$ and taking values in $\mathbf{C}^{(q)}$.$\}$

Theorem 4 *If $f \in PW_{\mathbf{R}^p\oplus\mathbf{R}^q}(\pi/h)$, then for any $\underline{x} \in \mathbf{R}^p$, $\underline{y} \in \mathbf{R}^q$,*

$$f(\underline{x}, \underline{y}) = \sum_k \sum_\alpha C\big[T_{k,\alpha}(f), h\big](\underline{x}, \underline{y}), \tag{16}$$

where

$$C\big[T_{k,\alpha}(f), h\big](\underline{x}, \underline{y}) = \sum_{\underline{k}\in\mathbf{Z}^q} \operatorname{sinc}^p_{P_{k,\alpha}}\left(\frac{\underline{x}}{h}, \frac{\underline{y} - h\underline{k}}{h}\right) T_{k,\alpha}(f)(h\underline{k}) \tag{17}$$

and $T_{k,\alpha}(f)(\underline{y})$ are the initial values of f. The series (16) *and* (17) *on the right-hand side are uniformly convergent on any compact set in $\mathbf{R}^p \oplus \mathbf{R}^q$.*

Proof If $f \in PW_{\mathbf{R}^p\oplus\mathbf{R}^q}(\pi/h)$, then f has the form in (15). Using Theorem 3, we obtain

$$T_{k,\alpha}(\underline{x}, \underline{y}) = C\big[T_{k,\alpha}(f), h\big](\underline{x}, \underline{y}) = \sum_{\underline{k}\in\mathbf{Z}^q} \operatorname{sinc}^p_{P_{k,\alpha}}\left(\frac{\underline{x}}{h}, \frac{\underline{y} - h\underline{k}}{h}\right) T_{k,\alpha}(f)(h\underline{k}). \qquad \blacksquare$$

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