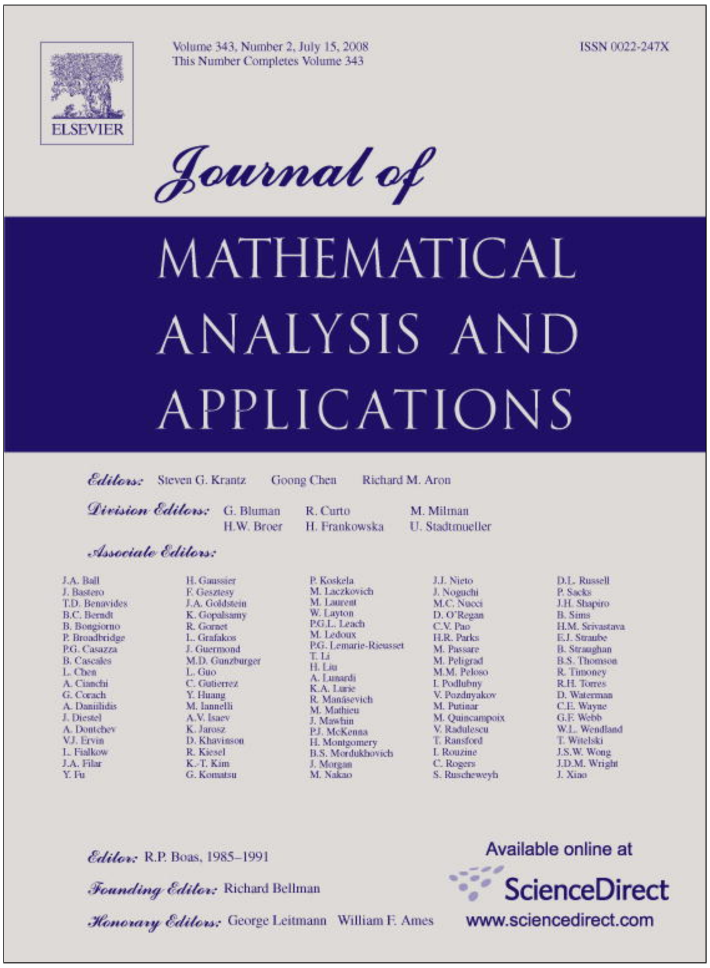
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Cauchy integrals on Lipschitz surfaces in octonionic space ☆

Xing Min Li [a,*], Li Zhong Peng [b], Tao Qian [c]

[a] *Department of Mathematics, School of Mathematical Sciences, South China Normal University, Guangzhou 510631, PR China*
[b] *LMAM, School of Mathematical Sciences, Peking University, Beijing 100871, PR China*
[c] *Faculty of Science and Technology, The University of Macau, 3001 Macau, PR China*

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Abstract

With the newly developed octonion analytic function theory, we confirm the octonionic analogue of the Calderón's conjecture. As application, we obtain the Plemelj formula in octonionic space.

Keywords: Cauchy integrals; Lipschitz surfaces; Octonions; Associator

0. Introduction

Let $z(x) = x + iA(x)$, where $A : R \to R$, $\|A'\|_\infty < \infty$. Then the graph Γ of $z = z(x)$, $x \in R$, is a Lipschitz curve in the complex plane. The classical Cauchy integral of f on Γ is given by

$$(Cf)(x) = p.v. \int_{-\infty}^{+\infty} \frac{f(y)}{z(x) - z(y)}\, dy,$$

and there was a long history to study the Cauchy integral operator on smooth curves, because of the essential technical difficulties, there were no significant progress on Lipschitz curves till 1970s. In 1977, by using complex analysis methods, A.P. Calderón proved the L_2-boundedness of the operator when $\|A'\|_\infty$ is small [5]. He conjectured that, in general, the Cauchy integral operator is bounded in L_2 for all A such that $\|A'\|_\infty < \infty$ [6,7].

The conjecture was first proved by Coifman, McIntosh and Meyer, known as the CMcM theorem [10]. Since then, many alternative proofs, including those for its higher dimensional generalizations, were given [9,11,12,27,28,30].

C. Kenig and Y. Meyer proved that in the one-dimensional case the Calderón's conjecture is equivalent with the square root problem [16].

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* Corresponding author.
E-mail addresses: lxmin57@hotmail.com (X.M. Li), lzpeng@pku.edu.cn (L.Z. Peng), fsttq@umac.mo (T. Qian).

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By embedding R^n into Clifford algebras, using Clifford monogenic function theory, the L_p-boundedness of the Cauchy integral operators on Lipschitz surfaces in Euclidean spaces was proved by M.M. Murray [30] (for surfaces with small Lipschitz constants) and by McIntosh for the general case [27]. The result is also a consequence of the lately proved, and celebrated, $T(b)$ theorem by David, Journé and Semmes [12]. The higher dimensional analogue of the $T(b)$ theorem played an important role in the solution of the Kato square root problem [1].

In [28], McIntosh and Qian studied the bounded holomorphic functional calculus of the Dirac operator on a Lipschitz curve γ, that is an operator algebra equivalent with the convolution operators of certain holomorphic functions ϕ with $u \in L_p(\gamma)$. They obtained the L_p-boundedness of the convolution singular integral operators on γ. An important fact is that when $\varphi = \frac{i}{\pi z}$, the convolution singular integral operator is just the Cauchy integral operator.

In [9], using Clifford monogenic function theory, Chun Li, McIntosh and Semmes generalized the methods and results in [28] to higher dimensional cases. The related Fourier multiplier theory is provided in [8].

The other stream is to study singular integrals on more general curves and surfaces, including "chord-arc" curves and "regular" curves and surfaces, but restricted to kernels of the Cauchy type, or close ones. For this, see, for instance, the related work by D. Jerison, C. Kenig, G. David and S. Semmes.

Let Σ be the Lipschitz surface $\Sigma = \{\mathcal{G}(\mathbf{x})e_0 + \mathbf{x} \in R^{n+1}\colon \mathbf{x} \in R^n\}$, where $\mathcal{G}$ is a real-valued function satisfying $\|\nabla\mathcal{G}\|_\infty \leqslant \tan\omega < \infty$, where $0 \leqslant \omega < \frac{\pi}{2}$. For $0 < \mu < \frac{\pi}{2}$, the open cones $C^0_{\mu^+}$, $C^0_{\mu^-}$ in R^{n+1} are defined by

$$C^0_{\mu^+} = \bigl\{x = x_0 + \mathbf{x} \in R^{n+1}\colon x_0 > -|\mathbf{x}|\tan\mu\bigr\},$$
$$C^0_{\mu^-} = -C^0_{\mu^+}.$$

The Banach spaces $K(C^0_{\mu^\pm})$ consist of the functions ϕ, with values in complex Clifford algebra $C(n)$, and right-monogenic in $C^0_{\mu^\pm}$ for which

$$\|\phi\|_{K(C^0_{\mu^\pm})} = \sigma_n \sup\bigl\{|x|^n\bigl|\phi(x)\bigr|\colon x \in C^0_{\mu^\pm}\bigr\} < \infty,$$

where σ_n is the volume of the unit n-sphere in R^{n+1}. Given $\phi \in K(C^0_{\mu^\pm})$ and $r > 0$, define a function $\underline{\phi}(r)$ by

$$\underline{\phi}_\pm(r) = \int\limits_{|x|=r,\ \pm x_0>0} \phi(x)n(x)\,dS_x,$$

where $n(x)$ is the upward pointing normal to the sphere $\{x \in R^{n+1}\colon |x| = r\}$ and $dS_x = \sqrt{1 + |\nabla\mathcal{G}(\mathbf{x})|^2}\,dx$. In [9], Chun Li, McIntosh and Semmes proved the following result.

Theorem A. *Suppose that $\omega < \mu < \frac{\pi}{2}$ and $1 < p < \infty$. Given $\phi \in K(C^0_{\mu^\pm})$, there is a bounded operator T_ϕ on $L_p(\Sigma, C(n))$ defined for all $u \in L_p(\Sigma, C(n))$ and almost all $x \in \Sigma$, by*

$$\begin{aligned}(T_\phi u)(x) &= \lim_{\delta\to 0^+}\left\{\int\limits_\Sigma \Phi(x \pm \delta - y)n(y)u(y)\,dS_y\right\}\\ &= \lim_{\varepsilon\to 0^+}\left\{\int\limits_{\substack{y\in\Sigma\\ |x-y|\geqslant\varepsilon}} \Phi(x-y)n(y)u(y)\,dS_y + \underline{\phi}_\pm\bigl(\varepsilon n(x)\bigr)u(x)\right\}.\end{aligned}$$

Moreover

$$\bigl\|(T_\phi u)\bigr\|_p \leqslant C_{\omega,\mu,p}\|\phi\|_{K(C^0_{\mu^\pm})}\|u\|_p,$$

where $\underline{\phi}_\pm(\varepsilon n(x))$ is the suitable bounded function extended from $\underline{\phi}_\pm(r)$.

This theorem generalizes the higher dimensional Calderón's conjecture in [27,30].

The purpose of this paper is to present an octonionic analogue of the Calderón's conjecture.

We note that the proofs of Theorem A rely on the following facts:

(1) If $f(x)$ is a right Clifford monogenic function, a is any Clifford constant, then the function $af(x)$ is still a right Clifford monogenic function. That is to say, the set of right Clifford monogenic functions becomes a left Clifford module.
(2) If $g(x)$ is a right Clifford monogenic function, $f(x)$ is any Clifford valued function, then their convolution $f * g$ is also a right Clifford monogenic function.

Since octonions algebra is not associative, these are not true in octonionic analysis [18], however. In this paper, to prove our result, we need an extra condition, along which new methods and some new results are to be explored.

The organization of the paper is as follows. Section 1 is devoted to the necessary preliminaries on octonionic analysis. Section 2 contains the statements of our main results. In Section 3, also as preparation, some technical results in octonions are proved. In Section 4 we prove our main results stated in Section 2.

1. Octonionic analysis

It is well known [2,3,15] that there are only four normed division algebras: the real numbers R, complex numbers C, quaternions H and octonions O, satisfying the relations $R \subsetneqq C \subsetneqq H \subsetneqq O$. In other words, for any $x, y \in R^n$, if we define a product $x \cdot y$ such that $x \cdot y \in R^n$, and $|x \cdot y| = |x||y|$, where $|x| = \sqrt{\sum_1^n x_i^2}$, then the only four values of n are 1, 2, 4, 8. Quaternions algebra H is not commutative, while the octonions algebra O is neither commutative nor associative, and, unlike R, C and H, the non-associative octonions cannot be embedded into the associative Clifford algebras.

Let $e_0, e_1, \ldots, e_6, e_7$ be the basis elements of octonions O, and let

$$W = \{(1,2,3), (1,4,5), (2,4,6), (3,4,7), (2,5,7), (6,1,7,), (5,3,6)\},$$

then the multiplication rules between the basis elements are given as follows [15,23]:

$$e_0^2 = e_0, \quad e_\alpha e_0 = e_0 e_\alpha = e_\alpha, \quad e_\alpha^2 = -1, \quad \alpha = 1, 2, \ldots, 7,$$

and for any triple of $(\alpha, \beta, \gamma) \in W$,

$$e_\alpha e_\beta = e_\gamma = -e_\beta e_\alpha, \quad e_\beta e_\gamma = e_\alpha = -e_\gamma e_\beta, \quad e_\gamma e_\alpha = e_\beta = -e_\alpha e_\gamma.$$

Denote $a = \sum_0^7 a_k e_k$, $b = \sum_0^7 b_k e_k$ ($a_k, b_k \in R$, $k = 0, 1, \ldots, 7$) by $a = a_0 + \vec{A}$, $b = b_0 + \vec{B}$, then $ab = a_0 b_0 + a_0 \vec{B} + b_0 \vec{A} - \vec{A} \cdot \vec{B} + \vec{A} \times \vec{B}$, where

$$\begin{aligned}\vec{A} \times \vec{B} = {} & e_1(A_{2,3} + A_{4,5} - A_{6,7}) + e_2(-A_{1,3} + A_{4,6} + A_{5,7}) \\ & + e_3(A_{1,2} + A_{4,7} - A_{5,6}) + e_4(-A_{1,5} - A_{2,6} - A_{3,7}) + e_5(A_{1,4} - A_{2,7} + A_{3,6}) \\ & + e_6(A_{1,7} + A_{2,4} - A_{3,5}) + e_7(-A_{1,6} + A_{2,5} + A_{3,4}),\end{aligned}$$

and

$$A_{ij} = \det\begin{pmatrix} a_i & a_j \\ b_i & b_j \end{pmatrix}, \quad i, j = 1, 2, \ldots, 7.$$

We have [20,31]

$$(\vec{A} \times \vec{B}) \cdot \vec{A} = 0, \quad (\vec{A} \times \vec{B}) \cdot \vec{B} = 0, \quad \vec{A} // \vec{B} \iff \vec{A} \times \vec{B} = 0, \quad \vec{A} \times \vec{B} = -\vec{B} \times \vec{A}.$$

For each $x \in O$, x is of the form $x = \sum_0^7 x_k e_k$, $x_k \in R$. x_0 is called the scalar part of x, denote it by $Sc\, x$, or $Sc\{x\}$, if ambiguity may arise. Its conjugate is defined by $\overline{x} = \sum_0^7 x_k \overline{e_k}$, where $\overline{e_0} = e_0$, $\overline{e_j} = -e_j$, $j = 1, 2, \ldots, 7$. We have $\overline{e_i e_j} = \overline{e_j}\,\overline{e_i}$, $i, j = 1, 2, \ldots, 7$, and $\overline{xy} = \overline{y}\,\overline{x}$, $x\overline{x} = \overline{x}x = \sum_0^7 x_i^2 =: |x|^2$. So, if $O \ni x \neq 0$, then $x^{-1} = \frac{\overline{x}}{|x|^2}$.

The object $[a, b, c] = (ab)c - a(bc)$ is defined to be the **associator** of a, b, c. Just like the commutator $[a, b] = ab - ba$ to measure the failure of commutativity, the associator is to measure the failure of the associativity.

The octonions obey some weakened associative laws, including the so-called R. Moufang identities [2,15]: for all $x, y, z, u, v \in O$,

$$[\overline{x}, x, y] = 0, \quad [x, x, y] = 0, \quad [x, y, z] = [y, z, x] = [z, x, y],$$
$$[x, z, y] = -[z, x, y], \quad [y, x, z] = -[y, z, x], \quad [y, x, z] = -[z, x, y],$$
$$(uvu)x = u\big(v(ux)\big), \quad x(uvu) = \big((xu)v\big)u, \quad u(xy)u = (ux)(yu).$$

The functions, f, under study will be defined in an open and connected set in R^8, with the form $f(x) = \sum_0^7 e_k f_k(x)$, where $f_k(x)$ $(k = 0, 1, \dots, 7)$, are real-valued functions.

The Dirac operator D and its conjugate $\overline{D}$ are the first-order systems of differential operators on $C^\infty(\Omega, O)$, defined, respectively, by

$$D = \sum_0^7 e_k \frac{\partial}{\partial x_k}, \qquad \overline{D} = \sum_0^7 \overline{e_k} \frac{\partial}{\partial x_k}.$$

A function f in $C^\infty(\Omega, O)$ is said to be left (right) octonion analytic on Ω, if

$$Df = \sum_0^7 e_k \frac{\partial f}{\partial x_k} = 0 \qquad \left(fD = \sum_0^7 \frac{\partial f}{\partial x_k} e_k = 0\right).$$

Let M be an 8-dimensional, compact, oriented C^∞-manifold with boundary ∂M contained in some open connected subset Ω of R^8. Let $d\sigma(x) = \sum_0^7 (-1)^j e_j d\hat{x}_j$, $\nu = f(x)\, d\sigma(x)$ with the exterior derivatives $d\nu = \sum_0^7 (-1)^j \frac{\partial f}{\partial x_j} e_j dx_j \wedge d\hat{x}_j = (f(x)D)\, dV(x)$, where $dV(x) = dx_0 \wedge \cdots \wedge dx_7$ is the volume element on Ω. For each $x \in \partial M$, let $n(x) = \sum_0^7 n_j e_j$ be the outer unit normal to ∂M at x. Then $(-1)^j d\hat{x}_j = n_j(x)\, dS(x)$, where $dS(x)$ is the scalar element of surface area on ∂M, and $d\sigma = n\, dS$, $\nu = f(x)n(x)\, dS(x)$. Let $\Phi(x) = \frac{1}{\omega_7} \frac{\overline{x}}{|x|^8}$ $(x \neq 0)$, where ω_7 is the surface area of the unit sphere in R^8, we have the Cauchy theorem and the Cauchy integral formula for right octonion analytic functions [20].

Theorem B. *Let M be a compact, 8-dimensional, oriented C^∞ manifold in Ω. Then*

$$\int_{\partial M} \nu = \int_{\partial M} f(x)n(x)\, dS(x) = 0$$

for any function f which is right octonion analytic in Ω.

Theorem C. *M, Ω are as above, $fD = 0$, $x \in \Omega$. Then for any interior point z of M,*

$$f(z) = \int_{\partial M} \big(f(x)\, d\sigma(x)\big)\Phi(x - z) = \int_{\partial M} f(x)\big(d\sigma(x)\Phi(x - z)\big) + \int_M \sum_{t=0}^7 \big[e_t, Df_t(x), \Phi(x - z)\big]\, dV,$$

and for any $z \in \Omega \setminus M$, $\int_{\partial M} (f(x)\, d\sigma(x))\Phi(x - z) = 0$.

Remarks. We denote the complexification of O by O^c. Thus, $x \in O^c$ is of the form $x = \sum_{k=0}^7 x_k e_k$, $x_k \in C$. x_0 is still called the scalar part of x and denoted by $Sc\{x\}$. The norm of $x \in O^c$ is defined by $|x| = \sum_0^7 |x_k|^2$, and we can easily show that for any $x, y \in O^c$, $|xy| \leqslant 8^2|x||y|$. Note that O^c is no longer a division algebra, and by checking the proofs, we claim that the theorems stated above are also true in this case.

For other results and more information about Clifford analysis and octonionic analysis, we refer the reader to references [4,13,17–25].

2. Main results

Let Σ denote the Lipschitz surface consisting of points $x = \mathcal{G}(\mathbf{x})e_0 + \mathbf{x} \in O$, where $\mathbf{x} \in R^7$, and $\mathcal{G}$ is a real-valued Lipschitz function which satisfies $\|\nabla\mathcal{G}\|_\infty \leqslant \tan\omega < \infty$, where $0 \leqslant \omega < \pi/2$. If $1 \leqslant p < \infty$, $L_p(\Sigma, O^c)$ is the space of functions $u : \Sigma \to O^c$, which satisfy $\|u\|_p = \{\int_\Sigma |u|^p \, dS_x\}^{1/p} < \infty$, where $dS_x = \sqrt{1 + |\nabla\mathcal{G}(\mathbf{x})|^2}\, dx$. For $0 < \mu < \frac{\pi}{2}$, define the open cones $C^0_{\mu^+}$, $C^0_{\mu^-}$ in R^8 by

$$C^0_{\mu^+} = \{x = x_0 + \mathbf{x} \in R^8\colon\ x_0 > -|\mathbf{x}|\tan\mu\}, \qquad C^0_{\mu^-} = -C^0_{\mu^+}.$$

Denote by $K(C^0_{\mu\pm})$ the spaces of the right octonion analytic functions ϕ in $C^0_{\mu\pm}$, respectively, with

$$\|\phi\|_{K(C^0_{\mu\pm})} = \omega_7 \sup\{|x|^7 |\phi(x)|\colon\ x \in C^0_{\mu\pm}\} < \infty.$$

It may be easily verified that $K(C^0_{\mu\pm})$ are Banach spaces. Given $\phi \in K(C^0_{\mu\pm})$ and $r > 0$, define

$$\underline{\phi}_\pm(r) = \int\limits_{|x|=r,\ \pm x_0>0} \phi(x)n(x)\, dS_x,$$

where $n(x)$ is the upward pointing normal to the sphere $\{x \in R^8\colon\ |x| = r\}$. Note that $\underline{\phi}_\pm(r)$ can be extended to the open cone $T^0_\mu = \{y = y_0 + \mathbf{y} \in R^8\colon\ y_0 > |\mathbf{y}|\cot\mu\}$ by

$$\underline{\phi}_\pm(rt) = \underline{\phi}_\pm(r) - \int\limits_{A(r,t)} \big(\phi(x) + \phi(-x)\big)n(x)\, dS_x, \quad t \in T^0_\mu,\ |t| = 1,\ r > 0,$$

where $A(r,t) = \{x \in R^8\colon\ |x| = r,\ x_0 > 0,\ \langle x, t\rangle < 0\}$, and $n(x)$ is the exterior normal to $A(r,t)$ at the point x, $\langle x, t\rangle$ is the usual inner product in R^8.

Our main results of this paper are as follows.

Theorem 2.1. *Suppose that $0 \leqslant \omega < \mu < \frac{\pi}{2}$ and $1 < p < \infty$, $\phi \in K(C^0_{\mu\pm})$ satisfying that for any constant $a \in O^c$,*

$$\big[a, \phi(x), D\big] = \left[a, \phi(x), \sum_0^7 e_k \frac{\partial}{\partial x_k}\right] = 0, \tag{1}$$

where $[a, \phi(x), D] = (a\phi(x))D - a(\phi(x)D)$. Then there is a bounded operator T_ϕ on $L_p(\Sigma, O^c)$ defined for all $u \in L_p(\Sigma, O^c)$ and almost all $x \in \Sigma$ by

$$\begin{aligned} T_\phi u &= \lim_{\delta\to 0^+} \int\limits_\Sigma \phi(x \pm \delta - y)\big(n(y)u(y)\big)\, dS_y \\ &= \lim_{\varepsilon\to 0^+} \left\{ \int\limits_{y\in\Sigma,\ |x-y|\geqslant\varepsilon} \phi(x-y)\big(n(y)u(y)\big)\, dS_y + \underline{\phi}_\pm\big(\varepsilon n(x)\big)u(x) \right\}, \end{aligned}$$

where $n(x)$ is the upward pointing unit normal vector to Σ which is defined at almost all $x \in \Sigma$.

Moreover

$$\|T_\phi u\|_p \leqslant C_{\omega,\mu,p} \|\phi\|_{K(C^0_{\mu\pm})} \|u\|_p$$

for some constants $C_{\omega,\mu,p}$ which depend only on ω, μ and p.

Theorem 2.2. *Let $\Phi(x) = \frac{1}{\omega_7}\frac{\bar{x}}{|x|^8}$ $(x \neq 0)$, then there are bounded linear Cauchy operators P_+, P_-, and C_Σ on $L_p(\Sigma, O^c)$ $(1 < p < \infty)$, defined for all $u \in L_p(\Sigma, O^c)$ and almost all $x \in \Sigma$ by*

$$(P_\pm u)(x) = \pm \lim_{\delta \to 0^+} \int_\Sigma \Phi(x \pm \delta - y)\big(n(y)u(y)\big)\, dS_y,$$

$$(C_\Sigma u)(x) = 2\, p.v. \int_\Sigma \Phi(x - y)\big(n(y)u(y)\big)\, dS_y.$$

Moreover,

$$P_\pm = \frac{1}{2}(\pm C_\Sigma + I),$$
$$I = P_+ + P_-,$$
$$C_\Sigma = P_+ - P_-.$$

This theorem generalizes the Plemelj formulas in [9,14,32,33], and it played an important role in proving the Paley–Wiener theorem in octonions [26].

3. Technical preparations

Lemma 3.1. *(See [24].) Let e_i, e_j, e_k be any distinct elements of $\{e_1, e_2, \dots, e_7\}$ and $(e_ie_j)e_k \neq \pm 1$. Then $(e_ie_j)e_k = -e_i(e_je_k)$.*

Theorem 3.2. *For any $i, j, k \in \{0, 1, 2, \dots, 7\}$,*

$$[e_i, e_j, e_k] = 0 \quad \Longleftrightarrow \quad ijk = 0,\ \text{or } (i-j)(i-k)(j-k) = 0,\ \text{or } (e_ie_j)e_k = \pm 1.$$

The tedious checking proof of Theorem 3.2 is omitted here.

From Theorem 3.2, we can prove the following theorem, its proof is also omitted.

Theorem 3.3. *For any $i, j, k \in \{0, 1, 2, \dots, 7\}$, $Sc\{[e_i, e_j, e_k]\} = 0$.*

Theorem 3.4. *For any $x, y, z \in O^c$, we have $Sc\{(xy)z\} = Sc\{x(yz)\}$.*

Proof. Let $x = \sum_0^7 x_ie_i$, $y = \sum_0^7 y_je_j$, $z = \sum_0^7 z_ke_k$, where $x_i, y_j, z_k \in C$. By Theorem 3.3,

$$Sc\big\{[x, y, z]\big\} = Sc\Bigg\{\sum_{0 \leqslant i,j,k \leqslant 7} x_iy_jz_k[e_i, e_j, e_k]\Bigg\} = \sum_{0 \leqslant i,j,k \leqslant 7} x_iy_jz_kSc\big\{[e_i, e_j, e_k]\big\} = 0,$$

the result follows. □

For $x = \sum_0^7 x_ke_k$, $y = \sum_0^7 y_ke_k \in O^c$, their inner product is defined by

$$(x, y) = \sum_0^7 x_k\bar{y}_k = (x\bar{y})_0 = (\bar{y}x)_0,$$

where $\bar{y} = \sum_0^7 \overline{y_k}\ \overline{e_k}$, $\bar{y}_k$ is the complex conjugate of y_k. For any $u, v, w \in O^c$, by Theorem 3.4,

$$(uv, w) = \big((uv)\overline{w}\big)_0 = \big(\overline{w}(uv)\big)_0 = \big((\overline{w}u)v\big)_0 = \big((\overline{\bar{u}w})v\big)_0 = (v, \bar{u}w).$$

Thus we obtain the following useful result.

Corollary 3.5. $(uv, w) = (v, \bar{u}w)$ *for any $u, v, w \in O^c$.*

Remark. Corollary 3.5 is critical for the proof of Theorem 3.7 as given below.

From Theorem 3.4, we can also obtain the following interesting result, which generalizes the corresponding result in R^3 to R^7.

Corollary 3.6. *$\vec{A}\cdot(\vec{B}\times\vec{C})=(\vec{A}\times\vec{B})\cdot\vec{C}$ for any $\vec{A},\vec{B},\vec{C}\in R^7$.*

Proof. Let $a=a_0+\vec{A}, b=b_0+\vec{B}, c=c_0+\vec{C}$, where $a_0,b_0,c_0\in R$. Then,

$$\begin{aligned}(ab)c&=(a_0b_0-\vec{A}\cdot\vec{B}+a_0\vec{B}+b_0\vec{A}+\vec{A}\times\vec{B})(c_0+\vec{C})\\&=(a_0b_0-\vec{A}\cdot\vec{B})c_0+(a_0b_0-\vec{A}\cdot\vec{B})\vec{C}+c_0(a_0\vec{B}+b_0\vec{A}+\vec{A}\times\vec{B})\\&\quad-(a_0\vec{B}+b_0\vec{A}+\vec{A}\times\vec{B})\cdot\vec{C}+(a_0\vec{B}+b_0\vec{A}+\vec{A}\times\vec{B})\times\vec{C};\\a(bc)&=(a_0+\vec{A})(b_0c_0-\vec{B}\cdot\vec{C}+b_0\vec{C}+c_0\vec{B}+\vec{B}\times\vec{C})\\&=a_0(b_0c_0-\vec{B}\cdot\vec{C})+a_0(b_0\vec{C}+c_0\vec{B}+\vec{B}\times\vec{C})+(b_0c_0-\vec{B}\cdot\vec{C})\vec{A}\\&\quad-\vec{A}\cdot(b_0\vec{C}+c_0\vec{B}+\vec{B}\times\vec{C})+\vec{A}\times(b_0\vec{C}+c_0\vec{B}+\vec{B}\times\vec{C}).\end{aligned}$$

So, we have

$$\begin{aligned}Sc\big\{(ab)c\big\}&=(a_0b_0-\vec{A}\cdot\vec{B})c_0-(a_0\vec{B}+b_0\vec{A}+\vec{A}\times\vec{B})\cdot\vec{C},\\Sc\big\{a(bc)\big\}&=a_0(b_0c_0-\vec{B}\cdot\vec{C})-\vec{A}\cdot(b_0\vec{C}+c_0\vec{B}+\vec{B}\times\vec{C}).\end{aligned}$$

By invoking Theorem 3.4 we conclude the corollary. □

Let Ω_+ be the domain in R^8 above the surface Σ and Ω_- the one below. Suppose $\tau(x)=\mathrm{dist}(x,\Sigma)$. Then $\mathcal{H}_+=L_2(\Omega_+,O^c,\tau(x)\,dx)$ is a Hilbert space with norm

$$\|f\|_{\mathcal{H}_+}=\left(\iint\limits_{\Omega_+}\big|f(x)\big|^2\tau(x)\,dx\right)^{\frac12}=\sup_{g\in B}\left|\iint\limits_{\Omega_+}\big(f(x),g(x)\big)\tau(x)\,dx\right|,$$

where $B=\{g\in\mathcal{H}_+,$ compactly supported in $\Omega_+\colon\ \|g\|_{\mathcal{H}_+}\leqslant 1\}$. Similarly we define $\mathcal{H}_-$.

Theorem 3.7. *Suppose that $f:\Omega_+\to O^c$ is right octonion analytic and continuous to the boundary Σ. If $|f(x)|\leqslant c/(1+|x|)^7$ and $|\nabla f(x)|\leqslant c/(1+|x|)^8$ for all $x\in\Omega_+\cup\Sigma$, then*

$$\left\{\int\limits_\Sigma\big|f(x)\big|^2\,dS_x\right\}^{\frac12}\leqslant C\big\|\nabla f(x)\big\|_{\mathcal{H}_+},$$

where $\|\nabla f(x)\|^2_{\mathcal{H}_+}=\sum_0^7\|\frac{\partial}{\partial x_j}f\|^2_{\mathcal{H}_+}$.

Proof. By Theorem C and the assumption, for any $t>0$, it is easy to see that $f(x+te_0)$ is the Cauchy integral of its restriction to Σ and $\|f(x+te_0)\|_{L_2(\Sigma,dx)}\to 0$ as $t\to\infty$. So,

$$\begin{aligned}\int\limits_\Sigma\big|f(x)\big|^2\,dS_x&=\int\limits_0^\infty\int\limits_\Sigma\frac{\partial}{\partial t}\big|f(x+te_0)\big|^2\,dS_x\,dt=\int\limits_0^\infty\int\limits_\Sigma\frac{\partial}{\partial t}\big(f(x+te_0),f(x+te_0)\big)\,dS_x\,dt\\&=2Sc\iint\limits_{\Omega_+}\left(\left(\frac{\partial}{\partial x_0}f\right)(x),f(x)\right)dx.\end{aligned}$$

For f and g defined on Ω_+, let

$$B(f,g)=\iint_{\Omega_+}\left(\left(\frac{\partial}{\partial x_0}f\right)(x),g(x)\right)dx,$$

$$N(g)(x)=\sup_{y\in x+\Gamma(\theta)}\left|g(y)\right|,$$

where $\Gamma(\theta)=\{x=x_0+\mathbf{x}\in R^8\colon\ |\mathbf{x}|<x_0\tan\theta\}$ for some $\theta\in(0,\frac{\pi}{2}-\omega)$.

We will first prove that if g is smooth on Ω_+ and satisfies the conditions $|g(x)|\leqslant c/(1+|x|)^7$ and $|\nabla g(x)|\leqslant c/(1+|x|)^8$, then

$$\left|B(f,g)\right|\leqslant C\left(\iint_{\Omega_+}|\nabla f|^2\tau(x)\,dx\right)^{\frac{1}{2}}\left|\|g\|\right|, \tag{2}$$

where

$$\left|\|g\|\right|=\left(\iint_{\Omega_+}|\nabla g|^2\tau(x)\,dx\right)^{\frac{1}{2}}+\left(\int_{\Sigma}N(g)^2\,dS_x\right)^{\frac{1}{2}}.$$

Choose ϕ to be a smooth function on R^7, compactly supported, $\int_{R^7}\phi(\mathbf{x})\,d\mathbf{x}=1$, and define $\phi_\delta(\mathbf{x})=\delta^{-7}\phi(\frac{\mathbf{x}}{\delta})$,

$$\rho(x)=x_0+\mathbf{x}+\phi_{x_0}*\mathcal{G}(\mathbf{x}),\qquad x=x_0+\mathbf{x}\in R^8_+.$$

Following [9], for properly chosen ϕ, ρ defines a bilipschitz map from R^8_+ onto Ω_+ such that $|\nabla\rho|^2x_0\,dx$ is a Carleson measure. Since $fD=0$, $\frac{\partial f}{\partial x_0}=-\sum_1^7\frac{\partial f}{\partial x_k}e_k$, we can rewrite $B(f,g)$ as

$$\iint_{R^8_+}\left(\sum_{i=1}^7 a_j(x)\frac{\partial}{\partial x_j}f\big(\rho(x)\big),g\big(\rho(x)\big)\right)dx,$$

the $a_j(x)\in L_\infty(R^8_+,O^c)$ and $|\nabla a_j(x)|^2x_0\,dx$ are Carleson measures on R^8_+.

By Corollary 3.5, and the methods in [9],

$$\begin{aligned}\left|B(f,g)\right|&=\left|\iint_{R^8_+}\left(\sum_{i=1}^7\Delta^{\frac14}R_j(x)f\big(\rho(x)\big),\Delta^{\frac14}\overline{a_j(x)}g\big(\rho(x)\big)\right)dx\right|\\&\leqslant\left(\iint_{R^8_+}\left|\Delta^{\frac14}f\big(\rho(x)\big)\right|dx\right)^{\frac12}\left(\sum_{i=1}^7\iint_{R^8_+}\left|\Delta^{\frac14}\overline{a_j(x)}g\big(\rho(x)\big)\right|^2\right)^{\frac12},\end{aligned}$$

where R_j denote the Riesz transforms and Δ the negative of the Laplacian on R^7. Similar with [9], we have

$$\left(\iint_{R^8_+}\left|\Delta^{\frac14}f\big(\rho(x)\big)\right|dx\right)^{\frac12}\leqslant C\left(\iint_{\Omega_+}\left|\nabla f(x)\right|^2\tau(x)\,dx\right)^{\frac12},$$

$$\left(\sum_{i=1}^7\iint_{R^8_+}\left|\Delta^{\frac14}\overline{a_j(x)}g\big(\rho(x)\big)\right|^2\right)^{\frac12}\leqslant C\left|\|g\|\right|.$$

Therefore (2) holds. From (2), we have

$$\int_{\Sigma}\left|f(x)\right|^2dS_x=2Sc\,B(f,f)\leqslant C\left(\iint_{\Omega_+}\left|\nabla f(x)\right|^2\tau(x)\,dx\right)^{\frac12}\left|\|f\|\right|.$$

We now claim that

$$\int_{\Sigma} \left|Nf(x)\right|^2 dS_x \leqslant C \int_{\Sigma} \left|f(x)\right|^2 dS_x.$$

Indeed, let

$$C_{\Sigma}(f)(x) = \pm \int_{\Sigma} \big(f(y)\, d\sigma(y)\big)\Phi(x-y) = \pm \int_{\Sigma} \big(f(y)n(y)\big)\Phi(x-y)\, dS_y, \quad x \in \Omega_{\pm},$$

then by Theorem C and the assumption that f decays at ∞ we get $C_{\Sigma}(f) = f$ on Ω_+, $C_{\Sigma}(f) = 0$ on Ω_-. Using the methods in [9], we get

$$\int_{\Sigma} \left|Nf(x)\right|^2 dS_x \leqslant C \int_{\Sigma} \left|Mf(x)\right|^2 dS_x \leqslant C \int_{\Sigma} \left|f(x)\right|^2 dS_x$$

and

$$\left\{ \int_{\Sigma} \left|f(x)\right|^2 dS_x \right\}^{1/2} \leqslant C \left\| \nabla f(x) \right\|_{\mathcal{H}_+},$$

where $Mf(x)$ is the Hardy–Littlewood maximal function of f pointwise on Σ. This finishes the proof of Theorem 3.7. □

Now, assume that f is a function in $\mathcal{H}_+$, with compact support in Ω_+, $\phi \in K(C^0_{\mu^+})$, $j = 0, 1, \ldots, 7$, and $\delta > 0$. Define

$$S_{\phi,\delta,j} f(y) = \iint_{\Omega_+} f(x) \frac{\partial \phi}{\partial x_j}(x - y + \delta)\tau(x)\, dx, \quad y \in \overline{\Omega}_-.$$

Theorem 3.8. *$S_{\phi,\delta,j} f(y)$ is a right octonion analytic function on $\overline{\Omega}_-$.*

Proof. Observe that $\tau(x)\, dx \in R$, denote D by D_y, we have

$$\begin{aligned} \big(S_{\phi,\delta,j} f(y)\big) D_y &= \iint_{\Omega_+} \left(f(x) \frac{\partial \phi}{\partial x_j}(x-y+\delta) \right) D_y \tau(x)\, dx \\ &= \iint_{\Omega_+} \left\{ f(x) \left(\frac{\partial \phi}{\partial x_j}(x-y+\delta) D_y \right) + \left[f(x), \frac{\partial \phi}{\partial x_j}(x-y+\delta), D_y \right] \right\} \tau(x)\, dx. \end{aligned}$$

Since ϕ is a right octonion analytic function, we have

$$\frac{\partial \phi(x-y+\delta)}{\partial x_j} D_y = \frac{\partial}{\partial x_j} \big(\phi(x-y+\delta) D_y\big) = 0.$$

Due to condition (1),

$$\left[f(x), \frac{\partial \phi(x-y+\delta)}{\partial x_j}, D_y \right] = 0, \quad x \in \Omega_+,$$

so we arrive $(S_{\phi,\delta,j} f(y)) D_y = 0$ for all $y \in \overline{\Omega}_-$. This proves the theorem. □

Theorem 3.9. $\|S_{\phi,\delta,j} f(y)\|_{L_2(\Sigma)} \leqslant C \|\phi\|_{K(C^0_{\mu^+})} \|f\|_{\mathcal{H}_+}$.

Proof. Since f is compactly supported we have

$$|S_{\phi,\delta,j} f| \leqslant C/\big(1+|x|\big)^7, \qquad \left|\nabla S_{\phi,\delta,j} f(x)\right| \leqslant C/\big(1+|x|\big)^8,$$

by Theorem 3.8, $S_{\phi,\delta,j}f$ is right octonion analytic on $\overline{\Omega}_-$. From Theorem 3.7, we have

$$\left\| S_{\phi,\delta,j}f(y)\right\|_{L_2(\Sigma)} \leqslant C\left\|\nabla(S_{\phi,\delta,j}f)\right\|_{\mathcal{H}_-}.$$

Since

$$\left|\nabla_y \frac{\partial\phi}{\partial x_j}(x-y)\right| \leqslant \frac{C\|\phi\|_{K(C^0_{\mu+})}}{|x-y|^9} \Big/ \sin^9\left(\frac{\mu-\omega}{2}\right), \quad \forall x-y\in C^0_{\frac{\mu-\omega}{2}},$$

we have

$$\left|\nabla(S_{\phi,\delta,j}f)(y)\right| = \left|\iint_{\Omega_+} f(x)\nabla_y\frac{\partial\phi}{\partial x_j}(x-y+\delta)\tau(x)\,dx\right| \leqslant C\|\phi\|_{K(C^0_{\mu+})}\iint_{\Omega_+}\frac{|f(x)|\tau(x)}{|x-y+\delta|^9}\,dx,$$

where C depends on μ and ω.

Since

$$\sup_{y\in\Omega_-}\iint_{\Omega_+}\frac{\tau(y)^{1/2}\tau(x)^{1/2}}{|x-y+\delta|^9}\,dx \leqslant C,$$

$$\sup_{y\in\Omega_+}\iint_{\Omega_-}\frac{\tau(y)^{1/2}\tau(x)^{1/2}}{|x-y+\delta|^9}\,dy \leqslant C,$$

with C independent of δ (notice $\tau(x)\leqslant|x-y|$, $\tau(y)\leqslant|x-y|$ for $x\in\Omega_+$, $y\in\Omega_-$), by Schur's lemma,

$$\left\|\iint_{\Omega_+}\frac{|f(x)|\tau(x)}{|x-y+\delta|^9}\,dx\right\|_{\mathcal{H}_-} \leqslant C\|f\|_{H_+}.$$

Therefore

$$\left\|S_{\phi,\delta,j}f(y)\right\|_{L_2(\Sigma)} \leqslant C\left\|\nabla(S_{\phi,\delta,j}f)\right\|_{\mathcal{H}_-} \leqslant C\|\phi\|_{K(C^0_{\mu+})}\|f\|_{\mathcal{H}_+}.$$

This proves the theorem. □

The following theorem is a particular case of Lemma 5.4 in [9], whose proof in the octonionic case does not involve associativity, so is omitted here.

Theorem 3.10. *Let*

$$\Phi_t(\mathbf{x}) = \frac{1}{2\sigma_7}\left(\frac{\overline{\mathbf{x}+t}}{|\mathbf{x}+t|^8} - \frac{\overline{\mathbf{x}-t}}{|\mathbf{x}-t|^8}\right) = \frac{1}{\sigma_7}\frac{t}{(|\mathbf{x}|^2+t^2)^4}, \quad t>0,\ \mathbf{x}\in R^7,$$

then for $u\in L_p$, $\Phi_t * u(y) = \int_{R^7}\Phi_t(x-y)u(x)\,dx \to u(y)$ *in* $L_p(\Sigma, O^c)$ $(1<p<\infty)$ *as* $t\to 0$.

4. Proofs of the main results

Proof of Theorem 2.1. We first prove Theorem 2.1 in the case when $\phi\in K(C^0_{\mu+})$. Our proof is an adaptation of the paper [9].

Let us begin with scalar-valued functions $u\in L_p(\Sigma, O^c)$, $1<p<\infty$. Then for $\delta>0$, $W_{\phi_\delta}u(x)$ is right O-analytic on $\overline{\Omega}_+$, where

$$(W_{\phi_\delta}u)(x) = \int_\Sigma \phi(x+\delta-y)u(y)\,dS_y, \quad x\in\overline{\Omega}_+.$$

Suppose first $u\in L_2(\Sigma, O^c)$ with compact support on Σ. It is easy to show that, for a fixed δ, $(W_{\phi_\delta}u)(x)$ is right octonion analytic and continuous to the boundary Σ, satisfying $|(W_{\phi_\delta}u)(x)|\leqslant C/(1+|x|)^7$ and $|\nabla(W_{\phi_\delta}u)(x)|\leqslant C/(1+|x|)^8$ for all $x\in\Omega_+\cup\Sigma$. By Theorem 3.7,

$$\|W_{\phi_\delta}u\|_{L_2(\Sigma)} \leqslant C\big\|\nabla(W_{\phi_\delta}u)\big\|_{H_+} \leqslant C\sum_{j=0}^{7}\sup_{f_j\in B}\left|\iint\limits_{\Omega_+}\left(\frac{\partial}{\partial x_j}(W_{\phi_\delta}u)(x), \overline{f_j}(x)\right)\tau(x)\,dx\right|.$$

From Theorem 3.9,

$$\begin{aligned}\left|\iint\limits_{\Omega_+}\left(\frac{\partial}{\partial x_j}(W_{\phi_\delta}u)(x), \overline{f_j}(x)\right)\tau(x)\,dx\right| &= \left|\iint\limits_{\Omega_+}\int\limits_{\Sigma}\left(\frac{\partial\phi}{\partial x_j}(x-y+\delta)u(y), \overline{f_j}(x)\right)dS_y\,\tau(x)\,dx\right| \\ &= \left|\iint\limits_{\Omega_+}\int\limits_{\Sigma}\left(f_j(x)\frac{\partial\phi}{\partial x_j}(x-y+\delta)\right)_0 u(y)\,dS_y\,\tau(x)\,dx\right| \\ &= \left|\int\limits_{\Sigma}(S_{\phi,\delta,j}f_j)_0(y)u(y)\,dS_y\right| \leqslant \|u\|_{L_2(\Sigma)}\big\|S_{\phi,\delta,j}f_j(y)\big\|_{L_2(\Sigma)} \\ &\leqslant C\|\phi\|_{K(C^0_{\mu^+})}\|f_j\|_{\mathcal{H}_+}\|u\|_{L_2(\Sigma)} \leqslant C\|\phi\|_{K(C^0_{\mu^+})}\|u\|_{L_2(\Sigma)}.\end{aligned}$$

Thus we get $\|W_{\phi_\delta}u\|_{L_2(\Sigma)} \leqslant C\|\phi\|_{K(C^0_{\mu^+})}\|u\|_{L_2(\Sigma)}$ for all $u \in L_2(\Sigma, O^c)$ with compact support. Therefore the operator W_{ϕ_δ} is a Calderón–Zygmund operator, and $\|W_{\phi_\delta}u\|_{L_p(\Sigma)} \leqslant C_p\|\phi\|_{K(C^0_{\mu^+})}\|u\|_{L_p(\Sigma)}$ for all scalar-valued functions $u \in L_p(\Sigma, O^c)$ $(1 < p < \infty)$.

We now prove the same estimate for $T_{\phi_\delta}u$, $u \in L_p(\Sigma, O^c)$, where

$$(T_{\phi_\delta}u)(x) = \int\limits_{\Sigma}\phi(x+\delta-y)\big(n(y)u(y)\big)\,dS_y.$$

Let $n(y)u(y) = \sum_0^7(nu)_k(y)e_k$, where $(nu)_k(y)$ are scalar-valued functions. Then we have

$$\sum_0^7\big\|(nu)_k(y)\big\|_p \leqslant C\big\|n(y)u(y)\big\|_p \leqslant C\big\|\big|n(y)\big|\big|u(y)\big|\big\|_p = C\big\|u(y)\big\|_p,$$

$$\|T_{\phi_\delta}u\|_{L_p(\Sigma)} \leqslant \sum_{k=0}^{7}\big\|W_{\phi_\delta}(nu)_k\big\|_{L_p(\Sigma)} \leqslant C\|\phi\|_{K(C^0_{\mu^+})}\sum_{k=0}^{7}\big\|(nu)_k\big\|_{L_p(\Sigma)} \leqslant C\|\phi\|_{K(C^0_{\mu^+})}\|u\|_{L_p(\Sigma)}.$$

We next prove that $T_{\phi_\delta}u$ converges in the $L_p(\Sigma)$ sense as $\delta \to 0$.

According to Theorem 3.10, we only have to prove $T_{\phi_\delta}(\Phi_t * u)$ converges for fixed $t > 0$ and u compactly supported with Lipschitz continuity.

Note that $dS_y \sim dy$, it is easy to check that for $\mathbf{x} \in R_7$,

$$\big|T_{\phi_\delta}(\Phi_t * u)(\mathbf{x}) - T_{\phi_{\delta'}}(\Phi_t * u)(\mathbf{x})\big| \leqslant C\frac{|\Delta\delta|}{(|\mathbf{x}|^2+t^2)^4},$$

we get $\|T_{\phi_\delta}(\Phi_t * u)(\mathbf{x}) - T_{\phi_{\delta'}}(\Phi_t * u)(\mathbf{x})\|_{L_p(\Sigma)} \to 0$ as $\Delta\delta \to 0$, which means $T_{\phi_\delta}(\Phi_t * u)$ converges in $L_p(\Sigma)$ as $\delta \to 0$. Hence there exists a bounded operator T_ϕ on $L_p(\Sigma)$ given by $T_\phi u = \lim_{\delta\to 0} T_{\phi_\delta}u$, which satisfies the required estimate

$$\|T_\phi u\|_{L_p(\Sigma)} \leqslant C\|\phi\|_{K(C^0_{\mu^+})}\|u\|_{L_p(\Sigma)}.$$

Let $T_\phi u = \sum_0^7(T_{\phi,k}u)e_k$, where

$$(T_{\phi,k}u)(x) = \lim_{\delta\to 0}\int\limits_{\Sigma}\phi(x+\delta-y)(nu)_k(y)\,dS_y$$

are all right octonion analytic functions. Note that $n(x)$ is an O-valued function, for any $x \in O$ and any function $u(x)$, we have

$$\big[n(x)^{-1}, n(x), u(x)\big] = \big[\overline{n(x)}, n(x), u(x)\big] = 0,$$

therefore $(T_{\phi,k}u)(x) = T_\phi(n^{-1}(nu)_k)(x)$.

Thus we get

$$\begin{aligned}\big\|N(T_\phi u)\big\|_{L_p(\Sigma)} &\leqslant C\sum_{k=0}^{7}\big\|N(T_{\phi,k}u)\big\|_{L_p(\Sigma)} \leqslant C\sum_{k=0}^{7}\big\|\big(T_\phi\big(n^{-1}(nu)_k\big)\big)\big\|_{L_p(\Sigma)}\\ &\leqslant C\sum_{k=0}^{7}\big\|\big(n^{-1}(nu)_k\big)\big\|_{L_p(\Sigma)} \leqslant C\|u\|_{L_p(\Sigma)},\end{aligned}$$

where N is the nontangential maximal function.

Then a routine argument shows that the convergence is also almost everywhere. That is

$$T_\phi u(x) = \lim_{\delta\to 0} T_{\phi_\delta} u(x)$$

for almost all $x \in \Sigma$, that is the first equality for T_ϕ in Theorem 2.1.

In order to prove the second equality, we need the following lemma.

Lemma 4.1. *(See [20].) Let M be an 8-dimensional, compact, oriented C^∞-manifold with boundary ∂M contained in some open connected subset Ω of R^8. Then for any O^c-valued $f = \sum_{j=0}^{7} f_j(x)e_j$, $g = \sum_{j=0}^{7} g_j(x)e_j$, we have*

$$\int_M \left\{ f(Dg) + (fD)g - \sum_{j=0}^{7}[e_j, Df_j, g] \right\} dV = \int_{\partial M} f(ng)\, dS$$

where $n = \sum_0^7 n_j e_j$ is the outward unit normal to ∂M at x, and dS is the scalar element of surface area on ∂M.

Suppose that u is a Lip-continuous with a compact support. For any $0 < \varepsilon < 1$,

$$\begin{aligned}(T_{\phi_\delta} u)(x) &= \int_{y\in\Sigma} \phi(x+\delta-y)\big(n(y)u(y)\big)\, dS_y\\ &= \int_{\substack{|x-y|>\varepsilon\\ y\in\Sigma}} \phi(x+\delta-y)\big(n(y)u(y)\big)\, dS_y + \int_{\substack{|x-y|\leqslant\varepsilon\\ y\in\Sigma}} \phi(x+\delta-y)\big(n(y)u(y)\big)\, dS_y,\end{aligned}$$

and

$$\begin{aligned}&\left| \int_{\substack{|x-y|>\epsilon\\ y\in\Sigma}} \phi(x+\delta-y)\big(n(y)u(y)\big)\, dS_y - \int_{\substack{|x-y|>\varepsilon\\ y\in\Sigma}} \big(\phi(x+\delta-y)n(y)\big)\, u(x)\, dS_y \right|\\ &\quad= \left| \int_{\substack{|x-y|>\varepsilon\\ y\in\Sigma}} \phi(x+\delta-y)\big(n(y)u(y)\big)\, dS_y - \int_{\substack{|x-y|>\varepsilon\\ y\in\Sigma}} \phi(x+\delta-y)\big(n(y)u(x)\big)\, dS_y \right.\\ &\qquad \left. - \int_{\substack{|x-y|>\varepsilon\\ y\in\Sigma}} \big[\phi(x+\delta-y), n(y), u(x)\big]\, dS_y \right|,\end{aligned}$$

where the associator is used. It is easy to show that

$$\left| \int_{\substack{|x-y|>\varepsilon\\ y\in\Sigma}} \phi(x+\delta-y)\big(n(y)u(y)\big)\, dS_y - \int_{\substack{|x-y|>\varepsilon\\ y\in\Sigma}} \phi(x+\delta-y)\big(n(y)u(x)\big)\, dS_y \right| \leqslant C\varepsilon\|u'\|_\infty,$$

$$\left| \int_{\substack{|x-y|>\varepsilon\\ y\in\Sigma}} \big[\phi(x+\delta-y), n(y), u(x)\big]\, dS_y \right| \leqslant C\varepsilon.$$

Taking $f=\phi(x+\delta-y)$, $g=1$ in Lemma 4.1, we have

$$\int\limits_{\substack{|x-y|\leqslant\varepsilon\\ y\in\Sigma}}\phi(x+\delta-y)\big(n(y)u(y)\big)\,dS_y=\int\limits_{\substack{|x-y|=\varepsilon\\ y\in\Omega_+}}\phi(x+\delta-y)\big(n(y)u(y)\big)\,dS_y.$$

Owing to Cauchy's theorem and existence of tangential planes of Σ at almost everywhere points on Σ,

$$\lim_{\varepsilon\to 0}\Bigg\{\int\limits_{\substack{|x-y|=\varepsilon\\ y\in\Omega_+}}\phi(x-y)\big(n(y)u(y)\big)\,dS_y-\underline{\phi}_+\big(\varepsilon n(x)\big)\Bigg\}=0\quad\text{a.e.}$$

Hence we get

$$T_\phi u(x)=\lim_{\varepsilon\to 0}\Bigg\{\int\limits_{\substack{|x-y|>\varepsilon\\ y\in\Sigma}}\phi(x-y)\big(n(y)u(y)\big)\,dS_y+\underline{\phi}_+\big(\varepsilon n(x)\big)u(x)\Bigg\}\quad\text{a.e.}$$

This proves that T_ϕ is a Calderón–Zygmund operator, by Theorem 5 in Chapter VII of [29], we obtain that for all $u\in L_p(\Sigma)$,

$$T_\phi u(x)=\lim_{\varepsilon\to 0}\Bigg\{\int\limits_{\substack{|x-y|>\varepsilon\\ y\in\Sigma}}\phi(x-y)\big(n(y)u(y)\big)\,dS_y+\underline{\phi}_+\big(\varepsilon n(x)\big)u(x)\Bigg\}\quad\text{a.e.}$$

A similar proof applies to $\phi\in K(C^0_{\mu^-})$. This completes the proof of Theorem 2.1. □

Remarks. The condition (1) is an extra condition compared with [9]. We can show that (1) holds for $\phi(x)=\Phi(x)=\frac{1}{\omega_7}\frac{\bar{x}}{|x|^8}$ when $x\neq 0$. Therefore, based on our theorem, the octonionic analogue of Calderón's conjecture is true.

We need only to prove that for any $i=1,2,\dots,7$, $[e_i,\Phi,D]=0$. Note that

$$[e_i,\Phi,D]=\sum_{j=0}^{7}\bigg[e_i,\frac{\partial\Phi}{\partial x_j},e_j\bigg].$$

Let $\Phi(x)=\sum_0^7 e_s\Phi_s$, then by using the properties of the associators, we have

$$\sum_{j=0}^{7}\bigg[e_i,\frac{\partial\Phi(x)}{\partial x_j},e_j\bigg]=\sum_{s=0}^{7}\sum_{j=0}^{7}\bigg[e_i,e_s\frac{\partial\Phi_s}{\partial x_j},e_j\bigg]=\sum_{s=1}^{7}\sum_{j=1}^{7}\bigg[e_i,e_s\frac{\partial\Phi_s}{\partial x_j},e_j\bigg],$$

where

$$\Phi_0=\frac{x_0}{\omega_7|x|^8},\qquad \Phi_s=\frac{-x_s}{\omega_7|x|^8},\quad s=1,2,\dots,7,$$
$$\frac{\partial\Phi_s}{\partial x_j}=\frac{8x_sx_j}{\omega_7|x|^{10}},\quad s\neq j,\ s,j=1,2,\dots,7,$$
$$\frac{\partial\Phi_j}{\partial x_j}=\frac{-|x|^2+8(x_j)^2}{\omega_7|x|^{10}},\quad j=1,2,\dots,7.$$

Since

$$\bigg[e_i,\frac{8x_jx_j}{\omega_7|x|^{10}}e_j,e_j\bigg]=0,\qquad \bigg[e_i,\frac{\partial\Phi_j}{\partial x_j}e_j,e_j\bigg]=0,\quad j=1,2,\dots,7,$$

taking $\frac{8x_j x_j}{\omega_7|x|^{10}}$ in place of $\frac{\partial \Phi_j}{\partial x_j}$, we have

$$\left[e_i, \frac{\partial \Phi(x)}{\partial x_j}, e_j\right] = \left[e_i, \sum_{s=0}^{7} \frac{8e_s x_s x_j}{\omega_7|x|^{10}}, e_j\right] = \left[e_i, \sum_{s=1}^{7} \frac{8e_s x_s}{\omega_7|x|^{10}}, x_j e_j\right].$$

Hence $[e_i, \Phi, D] = \sum_{j=0}^{7}[e_i, \frac{\partial \Phi(x)}{\partial x_j}, e_j] = \frac{8}{\omega_7|x|^{10}}[e_i, \sum_1^7 x_s e_s, \sum_1^7 x_j e_j] = 0$.

This shows that (1) holds for $\phi(x) = \Phi(x) = \frac{1}{\omega_7}\frac{\bar{x}}{|x|^8}$ when $x \neq 0$.

Proof of Theorem 2.2. For $\Phi(x) = \frac{1}{\omega_7}\frac{\bar{x}}{|x|^8}$, we have $\underline{\Phi}_{\pm}(\varepsilon n(x)) = \underline{\Phi}_{\pm}(\varepsilon)$, according to [30], $\underline{\Phi}_{\pm}(\varepsilon) = \pm\frac{1}{2}$. The desired results are then the direct corollaries of Theorem 2.1. □

Open problem. Find the necessary and sufficient conditions for an O^c-valued function ϕ, such that for any $a \in O^c$, $[a, \phi(x), D] = 0$.

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