The Paley-Wiener theorem in the non-commutative and non-associative octonions

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Abstract The Paley-Wiener theorem in the non-commutative and non-associative octonion analytic function space is proved.

Key Words: octonionic exponential function, octonionic Taylor expansion, Fourier transform, Paley-Weiner theorem

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0 Introduction

A function or signal $f(x) \in L^2(R)$ is said to be frequency band-limited if $\widehat{f}(\omega) \subset [-\Omega, \Omega]$. By Shannon sampling theorem, $f(x)$ is completely determined by its values at the points $t_j = \frac{j\pi}{\Omega}$, $j = 0, \pm1, \pm2, \dots$. Precisely, $f(x)$ has the following series expansion

$$f(x) = \sum_{j=-\infty}^{+\infty} f(j\frac{\pi}{\Omega}) \frac{sin(\Omega x - j\pi)}{\Omega x - j\pi},$$

and the series on the right converges uniformly. The natural frequency $\nu = \frac{\Omega}{2\pi}$ is called the Nyquist frequency.

An entire function defined in the complex plane is said to be of exponential type-σ ($\sigma > 0$) if there is a constant $M > 0$, such that $|F(z)| \leq Me^{\sigma|z|}$, $\forall z \in C$.

By the classical Paley-Wiener theorem, if $F(x) \in L^2(R)$, then $F(x)$ is the restriction of a exponential type-σ function $F(x + iy)$ defined in real line if and only if supp $\widehat{F} \subset [-\sigma, \sigma]$. Moreover, if one of the above conditions holds, then

$$F(z) = \frac{1}{2\pi} \int_{-\sigma}^{\sigma} \widehat{F}(\xi) e^{i(x+iy)} d\xi, \ z = x + iy.$$

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The classical Paley-Wiener theorem deeply reveals the relations between the exponential growth property for holomorphic function $F(x+iy)$ and the size of $\widehat{F}$. It is the theoretical basis of Shannon sampling theorem.

In 1957, by turning high-dimension case into one-dimension, E. M. Stein generalized the classical Paley-Wiener theorem to one for holomorphic functions of several variables[1].

The classical Paley-Wiener theorem was also generalized to Paley-Wiener-Schwartz theorem, which was used to characterize Schwartz test functions and distributions with compact support, and by which Malgrang proved that the partial differential equation $P(D)\delta = u$ with constant coefficients has a fundamental solution.

The proof of the classical Paley-Wiener theorem invokes the Phragmén-Lindelöf theorem, which generalize the Maximum Modulus Theorem for complex analytic function from bounded domain into unbounded domain. But the proof of the latter theorem makes use of the basic fact that the product of two complex analytic functions is still analytic. This fails in quaternionic analysis, octonionic analysis and Clifford analysis.

The study of quaternionic analysis was started from 1930s. With more and more applications, the quaternionic Fourier transform theory now is used in the color image processing.

As high-dimensional generalization of complex analysis and quaternionic analysis, based on the associative Clifford algebras, Clifford analysis have been developed since 1970s. By imbedding R^n into Clifford algebra, R^n then have algebraic and complex structures. Such a Clifford method has been wildly used in mathematics and physics and many other respects.

In 2002, in the Clifford algebra setting, K. I. Kou and T, Qian generalized the classical Paley-Wiener theorem to Clifford monogenic functions and gave some applications[2].

As the largest normed division algebra among the real numbers, complex numbers, quaternions and octonions, octonions are neither commutative nor associative, the analysis theory in octonions then should have a wild-ranging applicable space. But, it is the non-commutativity and the non-associativity that obstruct the progress of the octonion analysis for a long time.

However, some applications of octonions in mathematics and physics were observed fairly earlier. In 1925, Élie Cartan discovered the the symmetry between vectors and spinors in 8-dimensional Euclidean space when describing triality[3]. In 1934, Jordan, Newmann, Wigner noticed its relevance to physics on the foundations of quantum mechanics[4].

Besides their possible role in physics, the octonions are important because they tie together some algebraic structure that otherwise appear as isolated and inexplicable exceptions. Simple Lie algebras are a nice example of this phenomenon. There are 5 exceptional simple Lie algebras, these were discovered by Killing and Cartan in the late 1800s. The 4 of them come from the isometry group of the projective planes over $O, O\otimes C, O\otimes H$ and $O\otimes O$[5]. The remaining one is the automorphism group of the octonions. Another good example is the classification of simple formally real Jordan algebras. The 3×3 hermitian octonionic matrices algebra $\mathcal{H}_3(O)$ consists of the exceptional Jordan algebra. Further more, by using the combination property of octonions, a solution can be given for the ancient sphere packing problem when $n = 8$[6].

The achievements of Clifford analysis make it possible to develop the octonion analysis. In 1995, we began to study octonion analysis systematically, and the basic frame of octonion analysis was formulated[7−15].

Now, people pay more attentions to octonion analysis and its applications. In a 2001 paper[5], J. C. Baez pointed out that, octonions stand at the crossroads of many interesting fields of mathematics. He described their relations to Clifford algebras and spinors, Bott periodicity, projective and Lorentzian geometry, Jordan algebras, and the exceptional Lie groups. he also considered the applications of octonions in quantum logic, special relativity and supersymmetry[6]. At the end of [5], the author posed 14 questions that should be exploited. The first one is to set up an octonionic analogue of the theory of complex analytic functions.

Since Paley-Wiener theorem plays an important role in mathematics and information theory, and it holds for Clifford monogenic functions, it is natural to ask wether it holds for the non-commutative and non-associative octonion analytic functions.

In order to describe the octonionic Paley-Wiener, first of all, we must select a suitable definition of octonionic exponential function: the function itself is left (right) octonion analytic, and also, the Fourier transform defined by this exponential function is also left (right) octonion analytic. The octonion Taylor expansion formula in [11] can not be directly applied to the proof of the theorem, we therefore need to set up another octonion Taylor expansion formula with integral terms. Also, the functions considered in [2] are all defined in R^n with Clifford-values, they depend on the associativity of Clifford algebras in many circumstances, such as $(fg)h = f(gh)$, $((fg)h)k = (fg)(hk) = f((gh)k)$ etc. But these fail to hold for octonion-valued functions. The proof in this paper follows the same line as in [2] with necessary and non-trivial changes overcoming the difficulties arising from the non-associativity. The main effort is devoted to deal with associator by newly developed method in octonion analysis. Such method appears to be particularly for the non-associativity and have never occurred for associative analysis. In the end of the paper, we present a simple application of the octonionic Paley-Wiener theorem.

1 Preliminaries

There are only four normed division algebras: the real numbers R, complex numbers C, quaternions H and octonions O, satisfying the relations $R \subsetneqq C \subsetneqq H \subsetneqq O$. In other words, for any $x, y \in R^n$, if we define a product $x \cdot y$ such that $x \cdot y \in R^n$, and $||x \cdot y|| = ||x||||y||$, and also, for any non-zero vector, its inverse exists, then the only four values of n are $1,\ 2,\ 4,\ 8$, where $||x|| = \sqrt{\sum_1^n x_i^2}$.

Quaternions H are not commutative, while the octonions O are neither commutative nor associative, and, unlike R, C and H, the non-associative octonions can not be embedded into the associative Clifford algebras.

Let $e_0, e_1, \ldots, e_6, e_7$ be the basis elements of octonions O, and

$$W = \{(1,2,3), (1,4,5), (2,4,6), (3,4,7), (2,5,7), (6,1,7,), (5,3,6)\}.$$

Then

$$e_0^2 = e_{0,}\, e_\alpha e_0 = e_0 e_\alpha = e_\alpha,\, e_\alpha^2 = -1, \quad \alpha = 1, 2, \ldots, 7,$$

and for any triple of $(\alpha, \beta, \gamma) \in W$,

$$e_\alpha e_\beta = e_\gamma = -e_\beta e_\alpha,\, e_\beta e_\gamma = e_\alpha = -e_\gamma e_\beta,\, e_\gamma e_\alpha = e_\beta = -e_\alpha e_\gamma.$$

Octonions are also called Cayley numbers. For each $x \in O$, x is of the form $x = \sum_0^7 x_k e_k$, $x_k \in R$. Octonion algebra is an alternative algebra, this means that the subalgebra generated by any two elements is associative.

Call the object $[x,y,z]=(xy)z-x(yz)$ to be a associator of x,y,z, then for any $x,y,z\in O$,

$$\begin{gathered}[x,y,z]=[y,z,x]=[z,x,y],\\ [x,x,y]=0,\\ [x,y,z]=-[y,x,z].\end{gathered}$$

Let $a=a_0e_0+a_1e_1+\ldots+a_7e_7$, $b=b_0e_0+b_1e_1+\ldots+b_7e_7$, where $a_k,b_k\in R$, $k=0,1,\ldots,7$, and let $a=a_0+\vec{A}, b=b_0+\vec{B}$, then $ab=a_0b_0+a_0\vec{B}+b_0\vec{A}+\vec{A}\vec{B}$. Putting for $i,j=1,2,\ldots,7$,

$$A_{ij}=det\begin{pmatrix}a_i & a_j\\ b_i & b_j\end{pmatrix}.$$

Then

$$\begin{aligned}\vec{A}\vec{B} &= -\vec{A}\cdot\vec{B}+e_1(A_{2,3}+A_{4,5}-A_{6,7})+e_2(-A_{1,3}+A_{4,6}+A_{5,7})\\ &+ e_3(A_{1,2}+A_{4,7}-A_{5,6})+e_4(-A_{1,5}-A_{2,6}-A_{3,7})\\ &+ e_5(A_{1,4}-A_{2,7}+A_{3,6})+e_6(A_{1,7}+A_{2,4}-A_{3,5})\\ &+ e_7(-A_{1,6}+A_{2,5}+A_{3,4}).\end{aligned}$$

Using the symbol in [15], we have $\vec{A}\vec{B}=-\vec{A}\cdot\vec{B}+\vec{A}\times\vec{B}$, and

$$(\vec{A}\times\vec{B})\cdot\vec{A}=0,(\vec{A}\times\vec{B})\cdot\vec{B}=0,\ \vec{A}//\vec{B}\Longleftrightarrow\vec{A}\times\vec{B}=0,\ \vec{A}\times\vec{B}=-\vec{B}\times\vec{A}.$$

The conjugate of $x\in O$ is defined by $\overline{x}=\sum_0^7 x_k\overline{e_k}$, where $\overline{e_0}=e_0,\overline{e_j}=-e_j, j=1,2,\ldots,7$. We have $\overline{e_ie_j}=\overline{e_j}\,\overline{e_i},\ i,j=1,2,\ldots,7$, and $\overline{xy}=\overline{y}\,\overline{x}, x\overline{x}=\overline{x}x=\sum_0^7 x_i^2=:|x|^2$. So, if $O\ni x\neq 0$, then $x^{-1}=\frac{\overline{x}}{|x|^2}$.

Although octonions do not obey associative law, they still obey some weakened associative laws, including the so-called R. Moufang identities

$$(uvu)x=u(v(ux)),\ x(uvu)=((xu)v)u,\ u(xy)u=(ux)(yu).$$

Suppose Ω is an open and connected set in $R^8, f:\Omega\longrightarrow O, f(x)=\sum_0^7 e_kf_k(x)$, where $f_k(x)$ are all real-valued functions. The Dirac operator D and its conjugate $\overline{D}$ are the first-order systems of differential operators on $C^\infty(\Omega,O)$, defined, respectively, by

$$D=\sum_0^7 e_k\frac{\partial}{\partial x_k},\quad \overline{D}=\sum_0^7\overline{e_k}\frac{\partial}{\partial x_k}.$$

A function f in $C^\infty(\Omega,O)$ is said to be left (right) octonion analytic on Ω, if

$$Df=\sum_0^7 e_k\frac{\partial f}{\partial x_k}=0\quad (fD=\sum_0^7\frac{\partial f}{\partial x_k}e_k=0).$$

Since $D\overline{D} = \overline{D}D = \triangle_8 = \sum_0^7 \frac{\partial^2}{\partial x_k^2}$, all the components of any left (right) octonion analytic function are harmonic functions.

Let M be an 8-dimensional, compact, oriented C^∞–manifold with boundary ∂M contained in some open connected subset Ω of R^8. For each $x \in \partial M$, let $n(x) = \sum_0^7 n_j e_j$ be the outer unit normal to ∂M at x, $dS(x)$ is the scalar element of surface area on ∂M, and $d\sigma = ndS$, $\omega = n(x)f(x)dS(x)$. Let $\Phi(x-z) = \frac{\overline{x}-\overline{z}}{\omega_7|x-z|^8} =: \sum_0^7 \Phi_s e_s$, with ω_7 the surface area of the unit sphere in R^8.

Theorem A[9,12]. Let M be an 8-dimensional, compact, oriented C^∞–manifold with boundary ∂M contained in some open connected subset Ω of R^8. Then

$$\int_{\partial M} \omega = \int_{\partial M} n(x)f(x)dS(x) = 0$$

for any function f which is left octonion analytic in Ω.

Theorem B[9,12]. M, Ω are as above, $Df = 0, x \in \Omega$. Then for any interior point z of M,

$$\begin{aligned} f(z) &= \int_{\partial M} \Phi(x-z)\,(d\sigma(x)f(x)) \\ &= \int_{\partial M} (\Phi(x-z)d\sigma(x))\,f(x) - \int_M \Sigma_{t=0}^7[\ \Phi(x-z), Df_t(x), e_t\ ]dV, \end{aligned}$$

and for any $z \in \Omega \backslash M$,

$$\int_{\partial M} \Phi(x-z)\,(d\sigma(x)f(x)) = 0,$$

where $dV(x) = dx_0 \wedge \cdots \wedge dx_7$.

Remarks

[**1**] The Clifford algebras A_n are real 2^n dimensional associative algebras, satisfying that $A_0 = R$, $A_1 = C$, $A_2 = H$. But $A_3 \neq O$, and there are a lot of essential differences between Clifford algebras and octonions. Octonions are non-associative and divisible, while Clifford algebras are not divisible but associative; The set of left Clifford monogenic functions becomes a right Clifford module, while the set of left octonion analytic functions can not become a right octonion module. For more references on the Clifford analysis, see [16-18];

[**2**] Denote the complexification of O by O^c, then $x \in O^c \Longleftrightarrow x = \sum_{k=0}^7 x_k e_k$, $x_k \in C$. Note that O^c is no longer a division algebra. We claim that the properties of the associator, the R. Moufang identities, the definition for left (right) octonion analytic functions and the theorems stated above can all be generalized to O^c;

[**3**] Hereafter we make no differences from $x = (x_0, \ldots, x_7) \in R^8$ and $x = \sum_{k=0}^7 x_k e_k \in O$, $x = (x_0, \ldots, x_7) \in C^8$ and $x = \sum_{k=0}^7 x_k e_k \in O^c$.

2 Octonionic exponential function and Taylor expansion formula

Similar to [16], the octonionic exponential function is defined as follows. For $x = x_0 + \underline{x} \in C^8$, $\underline{\xi} = \xi_1 e_1 + \cdots + \xi_7 e_7 \in C^7$, denote

$$e(x, \underline{\xi}) = e^{i<\underline{x},\ \underline{\xi}>} e^{-x_0|\underline{\xi}|} \chi_+(\underline{\xi}) + e^{i<\underline{x},\ \underline{\xi}>} e^{x_0|\underline{\xi}|} \chi_-(\underline{\xi}),$$

where

$$\chi_\pm(\underline{\xi}) = \frac{1}{2}(1 \pm i\frac{\underline{\xi}}{|\underline{\xi}|}).$$

It is easy to very that

$$\chi_- \chi_+ = \chi_+ \chi_- = 0,\ \chi_\pm^2 = \chi_\pm,\ \chi_+ + \chi_- = 1.$$

Remark The definition of octonionic exponential function is a generalization of our usual exponential function. For further generalization of exponential function see [16].

The following assertion may be proved by a direct computation.

Proposition 2.1. Octonionic exponential functions $e(x, \underline{\xi})$ satisfy the following properties

$$e(x, \underline{\xi}) e(y, \underline{\xi}) = e(y, \underline{\xi}) e(x, \underline{\xi}) = e(x + y, \underline{\xi}),$$

$$e(x, -\underline{\xi}) = e(-x, \underline{\xi}),\ e(x, \underline{\xi}) = exp\ i(< \underline{x}, \underline{\xi} > -x_0 \underline{\xi}) = \sum_{k=0} \tfrac{1}{k!}(i(< \underline{x}, \underline{\xi} > -x_0 \underline{\xi}))^k.$$

Proposition 2.2. For any $\underline{\xi} \in C^7, e(x, \underline{\xi})$ is both left and right octonion analytic on $x \in O^c$.

Proof. By direct calculating, we get

$$D_x e^{i<\underline{x},\ \underline{\xi}>} e^{-x_0|\underline{\xi}|} \chi_+(\underline{\xi}) = e^{i<\underline{x},\ \underline{\xi}>} e^{-x_0|\underline{\xi}|} \chi_+(\underline{\xi}) D_x = 0,$$

$$D_x e^{i<\underline{x},\ \underline{\xi}} > e^{x_0|\underline{\xi}|} \chi_-(\underline{\xi}) = e^{i<\underline{x},\ \underline{\xi}} > e^{x_0|\underline{\xi}|} \chi_-(\underline{\xi}) D_x = 0.$$

So $D_x e(x, \underline{\xi}) = e(x, \underline{\xi}) D_x = 0$. Hence $e(x, \underline{\xi})$ is both left and right octonion analytic on $x \in O^c$.

By direct calculating we also get

Proposition 2.3. For any positive integer k, $(i(< \underline{x}, \underline{\xi} > -x_0 \underline{\xi}))^k$ is both left and right octonion analytic on $x \in O^c$.

Note that the functions $x^k, x \in O$ are neither left nor right octonion analytic functions. Therefore, the functions $(i(< \underline{x}, \underline{\xi} > -x_0 \underline{\xi}))^k$ are suitable institutions of z^k for $z \in C$.

There was an example showed that[7]: $f(x)$ is a left octonion analytic function, a is an octonion constant, $f(x)a$ is no longer a left octonion analytic function. But, when $f(x)$ is the exponential function, we have the following result

Proposition 2.4. For any O^c-valued function $g(\xi)$, the function $e(x,\underline{\xi})g(\xi)$ is a left octonion analytic function on $x \in O^c$.

Proof. Since $[\chi_+(\underline{\xi}),\ \chi_-(\underline{\xi}),\ g(\xi)] = 0$,

$$\begin{aligned}
& D_x(e(x,\underline{\xi})g(\xi)) \\
= \; & D_x(e^{i<\underline{x},\ \underline{\xi}>}e^{-x_0|\underline{\xi}|}\chi_+(\underline{\xi})g(\xi)) + D_x(e^{i<\underline{x},\underline{\xi}>}e^{x_0|\underline{\xi}|}\chi_-(\underline{\xi})g(\xi)) \\
= \; & D_x e^{i<\underline{x},\ \underline{\xi}>}e^{-x_0|\underline{\xi}|}(\chi_+(\underline{\xi})g(\xi)) + D_x e^{i<\underline{x},\underline{\xi}>}e^{x_0|\underline{\xi}|}(\chi_-(\underline{\xi})g(\xi)) \\
= \; & \frac{1}{2}(-\mid \underline{\xi} \mid)e^{i<\underline{x},\ \underline{\xi}>}e^{-x_0|\underline{\xi}|}\chi_-(\underline{\xi})(\chi_+(\underline{\xi})g(\xi)) + \frac{1}{2}(\mid \underline{\xi} \mid)e^{i<\underline{x},\ \underline{\xi}>}e^{x_0|\underline{\xi}|}\chi_+(\underline{\xi})(\chi_-(\underline{\xi})g(\xi)) \\
= \; & \frac{1}{2}(-\mid \underline{\xi} \mid)e^{i<\underline{x},\ \underline{\xi}>}e^{-x_0|\underline{\xi}|}(\chi_-(\underline{\xi})\chi_+(\underline{\xi}))g(\xi) + \frac{1}{2}(\mid \underline{\xi} \mid)e^{i<\underline{x},\ \underline{\xi}>}e^{x_0|\underline{\xi}|}(\chi_+(\underline{\xi})\chi_-(\underline{\xi}))g(\xi) = 0,
\end{aligned}$$

the result now follows.

Similarly we can prove that for any O^c-valued function $g(\xi)$, the function $g(\xi)e(x,\underline{\xi})$ is a right octonion analytic function on $x \in O^c$.

The kernel of the Dirac operator $D = \sum_0^7 e_k \frac{\partial}{\partial x_k}$ is given by

$$\Phi(y-x) = \frac{1}{\omega_7}\frac{\overline{y}-\overline{x}}{|y-x|^8} = \frac{1}{6\omega_7}\overline{D_x}\frac{1}{|y-x|^6} = \frac{1}{6\omega_7}\overline{D_y}\frac{1}{|y-x|^6}.$$

For any $\nu \in C, Re\nu > -\frac{1}{2}$, $(1-2tx+x^2)^{-\nu} = \sum_{k=0}^{\infty} C_k^{\nu}(t)x^k$, $\forall\, x,t \in R$. Where C_k^{ν} is the Genenbaurer polynomial of degree k associated with ν[18].

For any $x,y \in R^8$, let $x = |x|\xi, y = |y|\omega, r = \frac{|x|}{|y|}, t =< \xi, \omega >$, then

$$\frac{1}{|x-y|^6} = \frac{1}{|y|^6(1-2tr+r^2)^3} = \sum_{k=0}^{\infty}\frac{|x|^k}{|y|^{6+k}}C_k^3(t).$$

Since the calculation does not involve associativity, similar to [18] we get

$$\frac{\overline{y}-\overline{x}}{|y-x|^8} = \frac{1}{6\omega_7}\sum_{k=0}^{\infty}\frac{1}{|y|^{k+7}}\overline{D_x}(C_{k+1}^3(t)|x|^{k+1}) = \sum_{k=0}^{\infty}\frac{|x|^k}{|y|^{k+7}}C_{8,k}^-(\omega),$$

where $C_{8,k}^-(\omega,\xi) = \frac{1}{6\omega_7}[(k+1)C_{k+1}^3(t) - 6C_k^4(t)(<\omega,\xi> - \overline{\omega}\xi)]$. Similarly, we get

$$\frac{\overline{y}-\overline{x}}{|y-x|^8} = \frac{1}{6\omega_7}\sum_{k=0}^{\infty}|x|^k\overline{D_y}(C_k^3(t)\frac{1}{|y|^{k+6}}) = \sum_{k=0}^{\infty}\frac{|x|^k}{|y|^{k+7}}C_{8,k}^+(\xi,\omega)\overline{\omega},$$

where $C_{8,k}^+(\omega,\xi) = -\frac{1}{6\omega_7}[-(k+6)C_k^3(t) - 6C_{k-1}^4(t)(<\xi,\omega> - \overline{\xi}\omega)]$. Note that, if $|x| < |y|$, then

$$\frac{\overline{y}-\overline{x}}{|y-x|^8} = \Sigma_0^{\infty}\frac{(-1)^k}{k!}(<x,\partial_y>)^k\frac{\overline{y}}{|y|^8},$$

where $< x, \partial_y >= \sum_0^7 x_i \partial_{y_i}$. We thus obtain that for each non-negative integer k and $x \in R^8$, if $|y| > |x|$, then

$$\frac{|x|^k}{|y|^{k+7}} C^-_{8,k}(\omega, \xi)\overline{\xi} = \frac{|x|^k}{|y|^{k+7}} C^+_{8,k}(\xi, \omega)\overline{\omega} = \frac{(-1)^k}{k!}(< x, \partial_y >)^k \frac{\overline{y}}{|y|^8}.$$

Since $\frac{\overline{y}}{|y|^8}(y \neq 0)$ is both left and right octonion analytic, we have

Proposition 2.5. For $x \in R^8$ fixed, $\frac{|x|^k}{|y|^{k+7}} C^-_{8,k}(\omega, \xi)\overline{\xi}$ and $\frac{|x|^k}{|y|^{k+7}} C^+_{8,k}(\xi, \omega)\overline{\omega}$ are all left and right octonion analytic on y when $|y| > |x|$.

Compared with [11], we obtain the octonion Taylor expansion with integral terms

Proposition 2.6. Let $f : O \to O^c$ be a left octonion analytic function in a domain containing $B(0, r) = \{x \in R^8 : |x| < r\}$, then

$$f(x) = \sum_{k=0}^{\infty} \int_{\partial B(0,r)} (P^{(k)}(y^{-1}x)\Phi(y)) \,(n(y)f(y))dS_y, \quad x \in B(0, r)$$

where $P^{(k)}(y^{-1}x) = |y^{-1}x|^k C^+_{8,k}(\xi, \omega)$, dS_y is the area measure on $\partial B(0, r)$.

Proof. By theorem B and the expansion of Cauchy kernel, we get

$$\begin{aligned} f(x) &= \int_{\partial B(0,r)} \Phi(y - x)\,(d\sigma_y f(y)) \\ &= \frac{1}{\omega_7}\int_{\partial B(0,r)} (\sum_{k=0}^{\infty} \frac{|x|^k}{|y|^{k+7}} C^+_{8,k}(\xi, \omega)\overline{\omega})\,(d\sigma_y f(y)) \\ &= \frac{1}{\omega_7}\int_{\partial B(0,r)} (\sum_{k=0}^{\infty} \frac{|x|^k}{|y|^{k}} C^+_{8,k}(\xi, \omega)\frac{\overline{\omega}}{|y|^7})\,(n(y)f(y))dS_y \\ &= \frac{1}{\omega_7}\int_{\partial B(0,r)} (\sum_{k=0}^{\infty} \frac{|x|^k}{|y|^{k}} (C^+_{8,k}(\xi, \omega)\frac{\overline{y}}{|y|^8})\,(n(y)f(y))dS_y \\ &= \int_{\partial B(0,r)} \sum_{k=0}^{\infty} (\frac{|x|^k}{|y|^{k}} C^+_{8,k}(\xi, \omega)\Phi(y))\,(n(y)f(y))dS_y \\ &= \sum_{k=0}^{\infty} \int_{\partial B(0,r)} (P^{(k)}(y^{-1}x)\Phi(y))\,(n(y)f(y))dS_y. \end{aligned}$$

Remark Compared with Clifford analysis in [2], the expression of the integrand is closely connected with associative orders. Similar to [18], we also have the basic fact: $P^{(k)}(y^{-1}x)$ is a polynomial of degree k on x, and $|P^{(k)}(y^{-1}x)| \leq Ck^7(\frac{|x|^k}{|y|^k})$.

3 Paley-Wiener theorem in octonions

The Fourier transform in R^n and the inverse Fourier transform are defined by

$$\mathcal{F}(\underline{\xi}) = \widehat{f}(\underline{\xi}) = \int_{R^n} e^{-i<\underline{x},\underline{\xi}>} f(\underline{x}) d\underline{x},$$

$$\mathcal{F}^{-1}(\underline{x}) = f^{\vee}(\underline{x}) = \frac{1}{(2\pi)^n} \int_{R^n} e^{i<\underline{x},\underline{\xi}>} f(\underline{\xi}) d\underline{\xi}.$$

We then have the following basic facts: $\widehat{\mathcal{T}}(\phi) = \mathcal{T}(\widehat{\phi}), \phi \in \mathcal{S}(R^n)$, where $\mathcal{S}(R^n)$ is the Schwarz class of rapidly decreasing functions, $\mathcal{T}$ is the tempered distribution in $\mathcal{S}(R^n)$, and

$$\widehat{1} = (2\pi)^n \delta, \quad (\xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n})^{\vee} = i^{-|\alpha|} \mathcal{D}^{\alpha} \delta,$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\quad \mathcal{D}^{\alpha} = (\frac{\partial}{\partial x_1})^{\alpha_1} \cdots (\frac{\partial}{\partial x_n})^{\alpha_n}$.

Theorem. Let $f : O \to O^c$ be a left octonion analytic function, $f|_{R^7} \in L^2(R^7)$, and $R > 0$ be a positive number. Then the following two assertions are equivalent:

(**i**) There exists a constant C such that $|f(x)| \leq C e^{R|x|}$, $\forall x \in O$.

(**ii**) supp $\mathcal{F}(f|_{R^7}) \subset B(0, R)$.

Moreover, if one of the above conditions holds, then we have

$$f(x) = \frac{1}{(2\pi)^7} \int_{R^7} e(x, \underline{\xi}) \widehat{(f|_{R^7})}(\underline{\xi}) d\underline{\xi}, \quad x \in O.$$

Lemma (Plemelj formula in octonions)[19]. There exist bounded operators P_+, P_- and C_γ in $L_p(R^7)$ $(1 < p < \infty)$ such that for any $u \in L_p(R^7)$ and almost all $\underline{x} \in R^7$,

$$(P_{\pm} u)(\underline{x}) = \pm \lim_{\delta \to 0^+} \int_{R^7} \Phi(\underline{x} \pm \delta - \underline{y}) \, (n(\underline{y}) u(\underline{y})) dS_{\underline{y}},$$

$$(C_\gamma u)(\underline{x}) = 2 p.v. \int_{R^7} \Phi(\underline{x} - \underline{y}) \, (n(\underline{y}) u(\underline{y})) dS_{\underline{y}},$$

and $P_{\pm} = \frac{1}{2}(\pm C_\gamma + I), \quad I = P_+ + P_-, \quad C_\gamma = P_+ - P_-$.

Remarks

The lemma is the special case of the main result in [19] when the Lipschitz surface is R^n, and the kernel function is Cauchy kernel.

Proof of the theorem. (**ii**) $\Rightarrow$ (**i**) Let $F(x) = \frac{1}{2\pi)^7} \int_{R^7} e(x, \underline{\xi}) \widehat{(f|_{R^7})}(\underline{\xi}) d\underline{\xi}$, then

$$|F(x)| = |\tfrac{1}{(2\pi)^7} \textstyle\int_{R^7} e(x, \underline{\xi}) \chi_{B(0,R)}(\xi) \widehat{(f|_{R^7})}(\underline{\xi}) d\underline{\xi}| \leq C e^{R|x_0|} \|\chi_{B(0,R)}\|_2 \|\widehat{(f|_{R^7})}\|_2 \leq C e^{R|x|},$$

where C denotes the constants, and may be deferent at each appearance.

In what follows, let us prove $f(x) = F(x)$. It is enough to prove that $F(x)$ is a left octonion analytic function. Such a fact is obvious in Clifford analysis, but now, it involves associativity. Denote

$$e(x, \underline{\xi}) = e^{i<\underline{x},\underline{\xi}>} e^{-x_0 |\underline{\xi}|} \chi_+(\underline{\xi}) + e^{i<\underline{x},\underline{\xi}>} e^{x_0 |\underline{\xi}|} \chi_-(\underline{\xi}) =: f_1 + f_2,$$

then

$$F(x) = \frac{1}{(2\pi)^7} \int_{R^7} e(x, \underline{\xi}) \widehat{f|_{R^7}}(\underline{\xi}) d\underline{\xi} =: F_1(x) + F_2(x).$$

$$D_x F_1 = \frac{1}{(2\pi)^7} \int_{R^7} D_x (f_1(x, \underline{\xi}) \widehat{f|_{R^7}}(\underline{\xi})) d\underline{\xi} = \frac{1}{2\pi)^7} \int_{R^7} D_x (e^{i<\underline{x}, \underline{\xi}>} e^{-x_0|\underline{\xi}|} \chi_+(\underline{\xi}) \widehat{f|_{R^7}}(\underline{\xi})) d\underline{\xi}.$$

Note that $e^{i<\underline{x}, \underline{\xi}>} e^{-x_0|\underline{\xi}|}$ are complex numbers, by calculating, we get

$$\begin{aligned} D_x (e^{i<\underline{x}, \underline{\xi}>} e^{-x_0|\underline{\xi}|} \chi_+(\underline{\xi}) \widehat{f|_{R^7}}(\underline{\xi})) &= D_x (e^{i<\underline{x}, \underline{\xi}>} e^{-x_0|\underline{\xi}|} (\chi_+(\underline{\xi}) \widehat{f|_{R^7}}(\underline{\xi}))) \\ &= e^{i<\underline{x}, \underline{\xi}>} e^{-x_0|\underline{\xi}|} (-2|\underline{\xi}|) \chi_-(\underline{\xi}) \, (\chi_+(\underline{\xi}) \widehat{f|_{R^7}}(\underline{\xi})). \end{aligned}$$

Since $\chi_-(\underline{\xi})(\chi_+(\underline{\xi}) \widehat{f|_{R^7}}) = (\chi_-(\underline{\xi}) \chi_+(\underline{\xi})) \widehat{f|_{R^7}} = 0$, we have $D_x F_1 = 0$.

Similarly, we have $D_x F_2 = 0$. We thus show that $F(x)$ is a left octonion analytic function.

(i) $\Rightarrow$ **(ii)** Consider

$$G^+(x) = \frac{1}{(2\pi)^7} \int_{R^7} e^{i<\underline{x}, \underline{\xi}>} e^{-x_0|\underline{\xi}|} \chi_+(\underline{\xi}) f(\underline{\xi}) d\underline{\xi}, \quad x_0 > 0,$$

which is well defined as $f \in L^2(R^7)$. Similarly, we can show that $G^+(x)$ is left octonion analytic for $x_0 > 0$. Substituting $f(\underline{\xi})$ with its Taylor expansion, the identity may be rewritten as

$$\begin{aligned} G^+(x) &= \frac{1}{(2\pi)^7} \int_{R^7} (e^{i<\underline{x}, \underline{\xi}>} e^{-x_0|\underline{\xi}|} \chi_+(\underline{\xi})) \\ &\quad \times (\sum_0^\infty \int_{\partial B(0,r)} (P^{(k)}(y^{-1}\underline{\xi}) \Phi(y)) \, \big(n(y) f(y)\big) dS_y) d\underline{\xi} \\ &= \lim_{N \to \infty} \frac{1}{(2\pi)^7} \int_{R^7} (e^{i<\underline{x}, \underline{\xi}>} e^{-x_0|\underline{\xi}|} \chi_+(\underline{\xi}) \chi_{B(0,N)}) \\ &\quad \times (\sum_0^\infty \int_{\partial B(0,r)} (P^{(k)}(y^{-1}\underline{\xi}) \Phi(y)) \, \big(n(y) f(y)\big) dS_y) d\underline{\xi}, \end{aligned}$$

where r is a positive number. Owing to the uniform convergence property of the series for $|\xi| \leq N$, and dS_y, $d\underline{\xi}$ are all real-valued functions, we have

$$\begin{aligned} G^+(x) &= \lim_{N \to \infty} \frac{1}{(2\pi)^7} \sum_0^\infty \int_{\partial B(0,r)} \int_{R^7} (e^{i<\underline{x}, \underline{\xi}>} e^{-x_0|\underline{\xi}|} \chi_+(\underline{\xi}) \chi_{B(0,N)}) \\ &\quad \times ((P^{(k)}(y^{-1}\underline{\xi}) \Phi(y)) \, (n(y) f(y))) \, dS_y d\underline{\xi}. \end{aligned}$$

Deferent form [2], the integrand here depends heavily on the associative orders. By using the formula $x(yz) = (xy)z - [x, y, z]$ twice, we have

$$\begin{aligned}
& (e^{i<\underline{x},\,\underline{\xi}>}e^{-x_0|\underline{\xi}|}\chi_+(\underline{\xi})\chi_{B(0,N)})\ ((P^{(k)}(y^{-1}\underline{\xi})\Phi(y))(n(y)f(y))) \\
= \ & (e^{i<\underline{x},\,\underline{\xi}>}e^{-x_0|\underline{\xi}|}\chi_+(\underline{\xi})\chi_{B(0,N)}(P^{(k)}(y^{-1}\underline{\xi})\Phi(y)))\ (n(y)f(y)) \\
& -[e^{i<\underline{x},\,\underline{\xi}>}e^{-x_0|\underline{\xi}|}\chi_+(\underline{\xi})\chi_{B(0,N)},\ P^{(k)}(y^{-1}\underline{\xi})\Phi(y),\ n(y)f(y)] \\
= \ & ((e^{i<\underline{x},\,\underline{\xi}>}e^{-x_0|\underline{\xi}|}\chi_+(\underline{\xi})\chi_{B(0,N)}P^{(k)}(y^{-1}\underline{\xi}))\ \Phi(y) \\
& -[e^{i<\underline{x},\,\underline{\xi}>}e^{-x_0|\underline{\xi}|}\chi_+(\underline{\xi})\chi_{B(0,N)},\ P^{(k)}(y^{-1}\underline{\xi}),\ \Phi(y)])\ (n(y)f(y)) \\
& -[e^{i<\underline{x},\,\underline{\xi}>}e^{-x_0|\underline{\xi}|}\chi_+(\underline{\xi})\chi_{B(0,N)},\ P^{(k)}(y^{-1}\underline{\xi})\Phi(y),\ n(y)f(y)].
\end{aligned}$$

Now we must consider these two associators

$$\begin{gathered}
[e^{i<\underline{x},\,\underline{\xi}>}e^{-x_0|\underline{\xi}|}\chi_+(\underline{\xi})\chi_{B(0,N)},\ P^{(k)}(y^{-1}\underline{\xi}),\ \Phi(y)], \\
[e^{i<\underline{x},\,\underline{\xi}>}e^{-x_0|\underline{\xi}|}\chi_+(\underline{\xi})\chi_{B(0,N)},\ P^{(k)}(y^{-1}\underline{\xi})\Phi(y),\ n(y)f(y)].
\end{gathered}$$

We claim that the first associator is equal to zero. Since $e^{i<\underline{x},\,\underline{\xi}>}e^{-x_0|\underline{\xi}|}$ is a complex number, by using the expressions of $\chi_+(\underline{\xi})$ and $P^{(k)}(y^{-1}\underline{\xi})$, it is enough to prove $[\underline{\xi},\ \overline{\underline{\xi}}\eta,\ \Phi(y)] = 0$. Note that $\eta = \frac{y}{|y|}$, $\Phi(y) = \frac{\overline{y}}{\omega_7|y|^8}$, and $y \in \partial B(0,r)$. Using the calculating property of octonions and the R. Moufang identities, it is easy to verify that $[\underline{\xi},\ \overline{\underline{\xi}}y,\ \overline{y}] = [\underline{\xi},\ \underline{\xi}\,\underline{y},\ \underline{y}] = 0$. So we get $[\underline{\xi},\ \overline{\underline{\xi}}\eta,\ \Phi(y)] = 0$.

But, the second associator is not zero. Therefore,

$$\begin{aligned}
& G^+(x) = \lim_{N\to\infty}\frac{1}{(2\pi)^7}\sum_0^\infty\int_{\partial B(0,r)}\int_{R^7} \\
& (((e^{i<\underline{x},\,\underline{\xi}>}e^{-x_0|\underline{\xi}|}\chi_+(\underline{\xi})\chi_{B(0,N)}P^{(k)}(y^{-1}\underline{\xi}))\ \Phi(y))\ (n(y)f(y))(y) \\
& -[e^{i<\underline{x},\,\underline{\xi}>}e^{-x_0|\underline{\xi}|}\chi_+(\underline{\xi})\chi_{B(0,N)},\ P^{(k)}(y^{-1}\underline{\xi})\Phi(y),\ n(y)f(y)])\ dS_y d\underline{\xi}.
\end{aligned}$$

Compared with [2], the associator appears in the integrand here. Let

$$\begin{aligned}
\widetilde{G^+(x)} \quad & = \lim_{N\to\infty}\frac{1}{(2\pi)^7}\sum_0^\infty\int_{\partial B(0,r)}\int_{R^7} \\
& ((e^{i<\underline{x},\,\underline{\xi}>}e^{-x_0|\underline{\xi}|}\chi_+(\underline{\xi})\chi_{B(0,N)}P^{(k)}(y^{-1}\underline{\xi}))\Phi(y))\ (n(y)f(y))(y)\ dS_y d\underline{\xi},
\end{aligned}$$

similar to [2], we can exchange the limit procedure and the summation for $x_0 > R$. Thus for $x_0 > R$, we have

$$\widetilde{G^+(x)} = \sum_0^\infty\frac{1}{(2\pi)^7}\int_{\partial B(0,r)}(\int_{R^7}((e^{i<\underline{x},\,\underline{\xi}>}e^{-x_0|\underline{\xi}|}\chi_+(\underline{\xi})P^{(k)}(y^{-1}\underline{\xi})d\underline{\xi})\ \Phi(y))\ (n(y)f(y)))dS_y.$$

Let $\varphi_m(\underline{\xi})$ be a sequence of functions in $C_0^\infty(R^7)$ such that $\varphi_m(\underline{\xi}) = 0$ if $|\underline{\xi}| \le \frac{1}{m}$; $\varphi_m(\underline{\xi}) = 1$ if $|\underline{\xi}| \ge \frac{2}{m}$, and $0 \le \varphi_m(\underline{\xi}) \le 1$ otherwise.

Obviously, $\varphi_m \to 1$ distributionally . We rewrite $\widetilde{G^+(x)}$ as $x_0 > R$,

$$\begin{aligned}\widetilde{G^+(x)} &= \sum_0^\infty \frac{1}{(2\pi)^7}\int_{\partial B(0,r)} (\lim_{m\to\infty}\int_{R^7} \\ &((e^{i<\underline{x},\underline{\xi}>}e^{-x_0|\underline{\xi}|}\chi_+(\underline{\xi})\varphi_m(\underline{\xi})P^{(k)}(y^{-1}\underline{\xi})d\underline{\xi})\Phi(y)))\,(n(y)f(y))(y)dS_y.\end{aligned}$$

Since $e^{<\underline{x},\cdot>}e^{-x_0|\cdot|}\chi_+(\cdot)\varphi_m(\cdot)\in\mathcal{S}(R^7)$, in the notation of distribution,

$$e^{i<\underline{x},\underline{\xi}>}e^{-x_0|\underline{\xi}|}\chi_+(\underline{\xi})\varphi_m(\underline{\xi})P^{(k)}(y^{-1}\underline{\xi})$$

can be rewritten as

$$\begin{aligned}&P^{(k)}(y^{-1}(\cdot))\,((e^{i<\underline{x},\cdot>}e^{-x_0|\cdot|}\chi_+(\cdot)\varphi_m(\cdot))\\ =\;&\mathcal{F}^{-1}(P^{(k)}(y^{-1}(\cdot))\,(\mathcal{F}(e^{i<\underline{x},\cdot>}e^{-x_0|\cdot|}\chi_+(\cdot)\varphi_m(\cdot))))\\ =\;&i^{-k}(P^{(k)}(y^{-1}\underline{D})\delta)(\mathcal{F}(e^{i<\underline{x},\cdot>}e^{-x_0|\cdot|}\chi_+(\cdot))*\mathcal{F}(\varphi_m)).\end{aligned}$$

Now $\mathcal{F}(e^{i<\underline{x},\cdot>}e^{-x_0|\cdot|}\chi_+(\cdot))=\frac{1}{2}\mathcal{F}(e^{i<\underline{x},\cdot>}e^{-x_0|\cdot|})+\frac{1}{2}\mathcal{F}(e^{i<\underline{x},\cdot>}e^{-x_0|\cdot|}\frac{i(\cdot)}{|\cdot|})$, where

$$\frac{1}{2}\mathcal{F}(e^{i<\underline{x},\cdot>}e^{-x_0|\cdot|})(\underline{\zeta})=\widetilde{c}\frac{x_0}{(x_0^2+|\underline{\zeta}-\underline{x}|^2)^4},$$

with $\widetilde{c}=2^6\pi^3\Gamma(4)$. Thus

$$\begin{aligned}&\frac{1}{2}\mathcal{F}(e^{i<\underline{x},\cdot>}e^{-x_0|\cdot|}\frac{i(\cdot)}{|\cdot|})\\ =\;&\frac{1}{2}\int_{x_0}^\infty \underline{D}_{\underline{x}}\mathcal{F}(e^{i<\underline{x},\cdot>}e^{-t|\cdot|})(\underline{\zeta})dt\\ =\;&\widetilde{c}\int_{x_0}^\infty \underline{D}_{\underline{x}}(\frac{t}{(t^2+|\underline{\zeta}-\underline{x}|^2)^4})dt\\ =\;&\widetilde{c}\frac{\underline{\xi}-\underline{x}}{(x_0^2+|\underline{\zeta}-\underline{x}|^2)^4}.\end{aligned}$$

Hence

$$\mathcal{F}(e^{i<\underline{x},\cdot>}e^{-x_0|\cdot|}\chi_+(\cdot))=\widetilde{c}\frac{\overline{x-\underline{\zeta}}}{|x-\underline{\zeta}|^8}=-\widetilde{c}\Phi(\underline{\zeta}-x).$$

Therefore

$$\begin{aligned}&P^{(k)}(y^{-1}(\cdot))(e^{i<\underline{x},\cdot>}e^{-x_0|\cdot|}\chi_+(\cdot)\varphi_m(\cdot))\\ &=i^{-k}(P^{(k)}(y^{-1}\underline{D})\delta)(\mathcal{F}(e^{i<\underline{x},\cdot>}e^{-x_0|\cdot|}\chi_+(\cdot))*\mathcal{F}(\varphi_m))\\ &=-\widetilde{c}i^{-k}(P^{(k)}(y^{-1}\underline{D})\delta)(\Phi(\cdot-x)*\mathcal{F}(\varphi_m))\\ &=-\widetilde{c}i^{-k}(-1)^k\delta(P^{(k)}(y^{-1}\underline{D})\Phi(\cdot-x)*\mathcal{F}(\varphi_m))\\ &=-\widetilde{c}i^k((P^{(k)}(y^{-1}\underline{D})\Phi(\cdot-x)*\mathcal{F}(\varphi_m))(0).\end{aligned}$$

Since $\mathcal{F}(\varphi_m)) \to (2\pi)^7\delta$, we conclude that

$$\int_{R^7}(e^{i<\underline{x},\underline{\xi}>}e^{-x_0|\underline{\xi}|}\chi_+(\underline{\xi})P^{(k)}(y^{-1}\underline{\xi})d\underline{\xi} = -(2\pi)^7\widetilde{c}i^k(P^{(k)}(y^{-1}\underline{D})\Phi)(-x).$$

Thus for $x_0 > R$, we get

$$\widetilde{G^+(x)} = -\widetilde{c}\sum_0^\infty i^k\int_{\partial B(0,r)}(((P^{(k)}(y^{-1}\underline{D})\Phi)(-x))\Phi(y))\ (n(y)f(y))dS(y).$$

So, for $x_0 > R$, we have

$$\begin{aligned} G^+(x) &= -\widetilde{c}\sum_0^\infty i^k\int_{\partial B(0,r)}(((P^{(k)}(y^{-1}\underline{D})\Phi)(-x))\Phi(y))\,(n(y)f(y))dS_y \\ &- \sum_0^\infty\frac{1}{(2\pi)^7}\int_{\partial B(0,r)}\int_{R^7}[e^{i<\underline{x},\underline{\xi}>}e^{-x_0|\underline{\xi}|}\chi_+(\underline{\xi}),\ P^{(k)}(y^{-1}\underline{\xi})\Phi(y),\ n(y)f(y)]d\underline{\xi}dS_y. \end{aligned}$$

Similar to [2], by showing the series to converge uniformly in any compact set in the region $|x| > R$, $G^+(x)$ can be extended to be a left octonion analytic function in $|x| > R$.

Now we define

$$G^-(x) = \frac{1}{(2\pi)^7}\int_{R^7}e^{i<\underline{x},\underline{\xi}>}e^{x_0|\underline{\xi}|}\chi_- f(\underline{\xi})d\underline{\xi}, \quad x_0 < 0.$$

Similarly, we can prove that $G^-(x)$ is octonion analytic in $x_0 < -R$, and

$$\begin{aligned} G^-(x) &= \widetilde{c}\sum_0^\infty i^k\int_{\partial B(0,r)}((((P^{(k)}(y^{-1}\underline{D})\Phi)(-x))\Phi(y))(n(y)f(y))dS_y \\ &- \sum_0^\infty\frac{1}{(2\pi)^7}\int_{\partial B(0,r)}(\int_{R^7}[e^{i<\underline{x},\underline{\xi}>}e^{x_0|\underline{\xi}|}\chi_-(\underline{\xi}),P^{(k)}(y^{-1}\underline{\xi})\Phi(y),n(y)f(y)])d\underline{\xi}dS_y, \end{aligned}$$

and $G^-(x)$ can be extended to be a left octoinon analytic function in $|x| > R$.

Since $e^{i<\underline{x},\underline{\xi}>}e^{-x_0|\underline{\xi}|}\chi_+(\underline{\xi}) + e^{i<\underline{x},\underline{\xi}>}e^{-x_0|\underline{\xi}|}\chi_-(\underline{\xi}) \in C$, we have

$$\begin{aligned} &[e^{i<\underline{x},\underline{\xi}>}e^{-x_0|\underline{\xi}|}\chi_+(\underline{\xi}),P^{(k)}(y^{-1}\underline{\xi})\Phi(y),n(y)f(y)] \\ + &[e^{i<\underline{x},\underline{\xi}>}e^{-x_0|\underline{\xi}|}\chi_-(\underline{\xi}),P^{(k)}(y^{-1}\underline{\xi})\Phi(y),n(y)f(y)] = 0. \end{aligned}$$

Hence

$$G^+(x_0 + \underline{x}) + G^-(-x_0 + \underline{x}) = 0,\ |x| > R.$$

Now, applying the Parseval's identity to $G^+(x)$ and $G^-(x)$, we obtain that, for $x_0 > R$, G^+ and G^- have the alternative forms

$$\begin{aligned} G^+(x_0 + \underline{x}) &= \int_{R^7}\Phi((x_0 + \underline{x}) - \underline{\xi})\mathcal{F}(f|_{R^7})(-\underline{\xi})d\underline{\xi}, \\ G^-(-x_0 + \underline{x}) &= \int_{R^7}\Phi((-x_0 + \underline{x}) - \underline{\xi})F(f|_{R^7})(-\underline{\xi})d\underline{\zeta}. \end{aligned}$$

From the lemma, if $|x| > R$, then

$$\lim_{x_0 \to 0_+} (G^+(x_0 + \underline{x}) + G^-(-x_0 + \underline{x})) = \mathcal{F}(f|_{R^7})(\underline{\xi}).$$

This gives $\mathcal{F}(f|_{R^7})(\underline{\xi}) = 0$ for $|\underline{x}| > R$. Therefore supp $\mathcal{F}(f|_{R^7})(\underline{\xi}) \subset B(0, R)$.

We notice that $f(x)$ and $\frac{1}{(2\pi)^7} \int_{R^7} e(x, \underline{\xi})\widehat{(f|_{R^7})}(\underline{\xi})d\underline{\xi}$ are all left octonion analytic in R^8 and coincident in R^7, they have to be equal. Thus

$$f(x) = \frac{1}{(2\pi)^7} \int_{R^7} e(x, \underline{\xi})\widehat{(f|_{R^7})}(\underline{\xi})d\underline{\xi}, \quad x \in R^8.$$

This finishes the proof.

In the end of the paper, we present an application of the theorem. We know that the conjugate harmonic systems are the main studying object in the high dimensional Hardy space. Suppose $F = (u_1, \dots, u_n)$ is a vector-valued function defined in a domain of R^n, if F is gradient of some real harmonic function, then F is called the Stein-Weiss conjugate harmonic system. In [10], we showed that: If $(f_0, \dots, f_7)$ is Stein-Weiss conjugate harmonic system in R^8, then $F = -f_0e_0 + f_1e_1 + \ldots + f_7e_7$ is both left and right octonion analytic. We thus have the following result

Corollary. Let $(f_0, \dots, f_7)$ be a Stein-Weiss conjugate harmonic system in R^8, $F|_{R^7} \in L^2(R^7)$. Then $|F(x)| \le Ce^{R|x|}$ if and only if supp $\mathcal{F}(F|_{R^7}) \subset B(0, R)$.

Moreover, if one of the above conditions holds, then

$$F(x) = \frac{1}{(2\pi)^7} \int_{R^7} e(x, \underline{\xi})\widehat{(F|_{R^7})}(\underline{\xi})d\underline{\xi}, \quad x \in O.$$

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