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Two integral operators in Clifford analysis

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ABSTRACT

Segal–Bargmann space $\mathcal{F}_2(\mathbb{C}^n)$ and monogenic Fock space $M_2(\mathbb{R}^{n+1})$ are introduced first. Then, with the help of exponential functions in Clifford analysis, two integral operators are defined to connect $\mathcal{F}_2(\mathbb{C}^n)$ and $M_2(\mathbb{R}^{n+1})$ together. The corresponding integral properties are studied in detail.

1. Introduction

Let $Cl_{0,n}$ be the complex 2^n-dimensional universal Clifford algebra constructed from the basis $\{e_1, e_2, \dots, e_n\}$, under the usual relations

$$e_i e_j + e_j e_i = -2\delta_{i,j}, \quad i, j = 1, \dots, n. \tag{1.1}$$

An element $a \in Cl_{0,n}$ can be represented as $a = \sum_A a_A e_A$, where the coefficients a_A are complex-valued, $e_A = e_{l_1 \dots l_h} = e_{l_1} \dots e_{l_h}$, $A = \{l_1, \dots, l_h\}$ with $1 \leqslant l_1 < \dots < l_h \leqslant n$, $e_0 = e_\emptyset = 1$ is the identity element of $Cl_{0,n}$. The Clifford involution operation $^-$ on $Cl_{0,n}$ is defined, on the basis elements e_A, as

$$\overline{e_j} = -e_j, \quad j = 1, \dots, n; \qquad \overline{e_{l_1 \dots l_k}} = \overline{e_{l_k}} \dots \overline{e_{l_1}}, \tag{1.2}$$

such that $e_j \overline{e_j} = \overline{e_j} e_j = e_0$ and $e_{l_1 \dots l_k} \overline{e_{l_1 \dots l_k}} = \overline{e_{l_1 \dots l_k}} e_{l_1 \dots l_k} = e_0$. In the complex Clifford algebra, the Clifford involution operation $^-$ also acts on the complex coefficients via the usual complex conjugate operation. So if $a = \sum_{A \subset M} a_A e_A$, then $\bar{a} = \sum_{A \subset M} \overline{a_A}\,\overline{e_A}$. The norm of a is $|a| = (a\bar{a})_0^{\frac{1}{2}} = (\sum_A |a_A|^2)^{\frac{1}{2}}$, where for a Clifford number $a = \sum_A a_A e_A$, $(a)_0 = a_0$ (see [1,3]).

Denote $\underline{x} = x_1 e_1 + \dots + x_n e_n \in \mathbb{R}^n$, $x = x_0 + \underline{x} \in \mathbb{R}^{n+1}$, then $|x| = (\sum_{j=0}^n x_j^2)^{\frac{1}{2}}$ and $|\underline{x}| = (\sum_{j=1}^n x_j^2)^{\frac{1}{2}}$. Denote $\underline{\varsigma} \in \mathbb{C}^n$ as $\underline{\varsigma} = \varsigma_1 e_1 + \dots + \varsigma_n e_n$, $\varsigma_j \in \mathbb{C}$, then its Clifford conjugate $\overline{\underline{\varsigma}}$ is given by the Clifford involution operation:

$$\overline{\underline{\varsigma}} = \overline{\varsigma_1}\,\overline{e_1} + \dots + \overline{\varsigma_n}\,\overline{e_n},$$

and $|\underline{\varsigma}| = (\underline{\varsigma}\,\overline{\underline{\varsigma}})^{\frac{1}{2}} = (\overline{\underline{\varsigma}}\,\underline{\varsigma})^{\frac{1}{2}} = (|\varsigma_1|^2 + \dots + |\varsigma_n|^2)^{\frac{1}{2}}$.

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Let $D=\frac{\partial}{\partial x_0}+\sum_{j=1}^n e_j\frac{\partial}{\partial x_j}$ be the Dirac differential operator in $\mathbb{R}^{n+1}$. The $Cl_{0,n}$-valued function $f(x)=\sum_A f_A(x)e_A$ is called *left-monogenic* if $Df(x)=\sum_{j=0}^n\sum_A e_je_A\frac{\partial f_A}{\partial x_j}=0$; and *right-monogenic* if $fD(x)=\sum_{j=0}^n\sum_A e_Ae_j\frac{\partial f_A}{\partial x_j}=0$. If f is both left- and right-monogenic, then we say it is *two-sided monogenic*, or briefly just *monogenic*.

Left- or *right-*, or *monogenic functions* are the generalizations of analytic functions for one complex variable to $\mathbb{R}^{n+1}$. Many basic results of holomorphic functions in one complex variable are extended to these three kinds Clifford monogenic functions [1,3]. In the definition $M_2(\mathbb{R}^{n+1})$ below we concern only two-sided monogenic functions (monogenic functions).

It is clear that the functions, $z_j=x_j-x_0e_j$ $(j=1,\dots,n)$, are the monogenic polynomials of order 1, and other monogenic functions can be constructed by means of symmetric products of z_j, $j=1,\dots,n$. More precisely, denote $\underline{\alpha}=(\alpha_1,\dots,\alpha_n)$, $\alpha_j\geqslant 0$, $|\underline{\alpha}|=\alpha_1+\dots+\alpha_n$, $\underline{\alpha}!=\alpha_1!\dots\alpha_n!$, and $V_{\underline{\alpha}}(x)$ the C-K extension of $\frac{1}{\alpha_1!\dots\alpha_n!}x_1^{\alpha_1}\dots x_n^{\alpha_n}$, then (see [1, pp. 68 and 113])

$$V_{\underline{\alpha}}(x)=\frac{1}{|\underline{\alpha}|!}\sum_{\pi(l_1,\dots,l_{|\underline{\alpha}|})}z_{l_1}\dots z_{l_{|\underline{\alpha}|}},\tag{1.3}$$

is a left- and right-monogenic polynomial of order $|\underline{\alpha}|$, where $\pi(l_1,\dots,l_{|\underline{\alpha}|})$ denote all the distinguished permutations of $(l_1,\dots,l_{|\underline{\alpha}|})$, $l_j\in\{1,\dots,n\}$, with k appearing α_k times in $\{l_1,\dots,l_{|\underline{\alpha}|}\}$, $k=1,\dots,n$. Let $\overline{V_{\underline{\alpha}}(x)}$ be the Clifford conjugate of $V_{\underline{\alpha}}(x)$, it is also given by the symmetric product of $\{\overline{z_j}\colon\ j=1,2,\dots,n\}$.

The polynomials in $\{V_{\underline{\alpha}}(x)\colon\ \alpha_j\geqslant 0;\ \ j=1,\dots,n\}$ are orthogonal to each other with respect to the following inner product (see [1,3])

$$\langle V_{\underline{\alpha}},V_{\underline{\alpha}'}\rangle=\int_{\mathbb{R}^{n+1}}\big(V_{\underline{\alpha}}(x)\overline{V_{\underline{\alpha}'}(x)}\big)_0e^{-n|x|^2}\,dx.\tag{1.4}$$

With the inner product (1.4), let $M_2(\mathbb{R}^{n+1})$ be the separable, infinite-dimensional, Clifford-valued complex Hilbert space generated by the basis $\{V_{\underline{\alpha}}(x)\colon\ k_j=0,1,\dots;\ \ j=1,\dots,n\}$. The norm $\|f\|_M$ induced on $f\in M_2(\mathbb{R}^{n+1})$ is given by

$$\|f\|_M=\bigg(\int_{\mathbb{R}^{n+1}}\big|f(x)\big|^2e^{-n|x|^2}\,dx\bigg)^{\frac12}<\infty.\tag{1.5}$$

$M_2(\mathbb{R}^{n+1})$ is called *monogenic Fock space*. Note that $M_2(\mathbb{R}^{n+1})$ does not consist of monogenic functions, although it is the completion of monogenic polynomials under the Banach norm (1.5). If the weight function $e^{-n|x|^2}$ is replaced by $e^{-|x|^2}$, then the above defined reduces to the monogenic Fock space introduced by J. Cnops and V.V. Kisil in the study of group representations [2].

Segal–Bargmann space $\mathcal{F}_2(\mathbb{C}^n)$ is the Fock space of holomorphic functions in $\mathbb{C}^n$ taking their values in the span of $\{e_0,e_1,\dots,e_n\}$ with complex-valued functions as coefficients (see [4, p. 43]) induced by the inner product based on holomorphic polynomials

$$[F,G]=\int_{\mathbb{C}^n}\big(\overline{G(\underline{\varsigma})}F(\underline{\varsigma})\big)_0e^{-|\underline{\varsigma}|^2}\,d\underline{\varsigma}.\tag{1.6}$$

Thus the norm of $F\in\mathcal{F}_2(\mathbb{C}^n)$ is

$$\|F\|_{\mathcal{F}}=\bigg(\int_{\mathbb{C}^n}\big|F(\underline{\varsigma})\big|^2e^{-|\underline{\varsigma}|^2}\,d\underline{\varsigma}\bigg)^{\frac12}<+\infty,\tag{1.7}$$

where $\underline{\varsigma}=\varsigma_1e_1+\dots+\varsigma_ne_n=\underline{\xi}+i\underline{\eta}\in\mathbb{C}^n$, $|\underline{\varsigma}|=(\sum_{j=1}^n(\underline{\xi}_j^2+\underline{\eta}_j^2))^{\frac12}$. The space $\mathcal{F}_2(\mathbb{C}^n)$ is also a separable, infinite-dimensional, Clifford-valued complex Hilbert space (see Lemma 3.5 below). It is the completion of Clifford-vector-valued holomorphic polynomials under the weighted Banach norm $L_w^2(\mathbb{C}^n,\mathbb{C}^{n+1})$ given in (1.7).

Both $M_2(\mathbb{R}^{n+1})$ and $\mathcal{F}_2(\mathbb{C}^n)$ are complex Hilbert spaces, both being very useful in the group representations. $\mathcal{F}_2(\mathbb{C}^n)$ is the famous Segal–Bargmann space derived from the representation of Heisenberg group [4, p. 43], while $M_2(\mathbb{R}^{n+1})$ is a newly developed one [2,6]. The fact is that monogenic functions are generalizations of holomorphic functions in one complex variable to $\mathbb{R}^{n+1}$ variable, while holomorphic functions are generalizations of holomorphic functions in one variable to $\mathbb{C}^n$ variable. It would be interesting to study the relationships between them, and it is the main goal of this paper to study the connections between $\mathcal{F}_2(\mathbb{C}^n)$ and $M_2(\mathbb{R}^{n+1})$ in terms of the bounded operators between them. In Section 2, the newly developed exponential functions are introduced to connect the monogenic functions and the holomorphic functions together. In Section 3, two integral transforms are defined through the exponential functions. Then the isomorphic properties in terms of the specific bases of $M_2(\mathbb{R}^{n+1})$ and $\mathcal{F}_2(\mathbb{C}^n)$ are established, and the boundedness of the defined isomeric isomorphism operators are discussed. The combinatorial inequality in Lemma 3.3 should have its own interest.

2. Exponential functions in Clifford analysis

Because of the non-commutative properties (1.1) of Clifford algebra $Cl_{0,n}$, there have appeared many generalizations of special functions in Clifford analysis for diverse applications [1,3,7–10]. In [1] (see §15.7.6, p. 131), there is a monogenic exponential function $e(x,\underline{a})$ which is the *C-K extension* of $e^{\langle \underline{x},\underline{a}\rangle}$ for the fixed vector $\underline{a}\in\mathbb{R}^n$. This exponential function is used in the Schrödinger representation of Heisenberg group in the function space $M_2(\mathbb{R}^{n+1})$ (see [2]). In [9,10], F. Sommen considered the extensions of $e^{i\langle \underline{x},\underline{\xi}\rangle}$ to $e(x,\underline{\xi})$ which is monogenic with respect to x while $\underline{\xi}\in\mathbb{R}^n$ is a fixed vector. More recently Alan McIntosh and his collaborators used a similar extension in [7,8] in relation to a study of functional calculus. They extended $e^{i\langle \underline{x},\underline{\xi}\rangle}$ to a complex Clifford-valued exponential functions $e(x,\underline{\varsigma})$, which is monogenic with respect to $x=x_0+\underline{x}\in\mathbb{R}^{n+1}$ and holomorphic with respect to $\underline{\varsigma}=\underline{\xi}+i\underline{\eta}\in\mathbb{C}^n$. By using McIntosh et al.'s idea to Sommen's model, we, in what follows, will use a new exponential function, still denoted by $e(x,\underline{\varsigma})$ (also see [6]), that is monogenic with respect to $x\in\mathbb{R}^{n+1}$ and holomorphic with respect to $\underline{\varsigma}\in\mathbb{C}^n$ respectively.

Let $\underline{\varsigma}=\underline{\xi}+i\underline{\eta}=\sum_{j=1}^n\varsigma_je_j\in\mathbb{C}^n$, and

$$|\underline{\varsigma}|_{\mathbb{C}}^2=\sum_j^n\varsigma_j^2=|\underline{\xi}|^2-|\underline{\eta}|^2+2i\langle\underline{\xi},\underline{\eta}\rangle, \tag{2.1}$$

where $\underline{\xi},\underline{\eta}\in\mathbb{R}^n$, $\varsigma_j=\xi_j+i\eta_j$, and $\langle\underline{\xi},\underline{\eta}\rangle=\sum_j\xi_j\eta_j$, then $|\underline{\varsigma}|_{\mathbb{C}}^2$ is the natural holomorphic extension of $|\underline{\xi}|^2$ from $\mathbb{R}^n$ to $\mathbb{C}^n$ satisfying the relationships $(i\underline{\xi})^2=|\underline{\xi}|^2$ and $(i\underline{\varsigma})^2=|\underline{\varsigma}|_{\mathbb{C}}^2$. If $|\underline{\varsigma}|_{\mathbb{C}}^2\neq 0$, then $|\underline{\varsigma}|_{\mathbb{C}}$ is taken to be any but fixed one of the two square roots of $|\underline{\varsigma}|_{\mathbb{C}}^2$ (see [7,8]). Define

$$e(x_0e_0,\underline{\varsigma})\triangleq\begin{cases}\cos(x_0|\underline{\varsigma}|_{\mathbb{C}})e_0-\sin(x_0|\underline{\varsigma}|_{\mathbb{C}})\frac{\underline{\varsigma}}{|\underline{\varsigma}|_{\mathbb{C}}}, & \text{if } |\underline{\varsigma}|_{\mathbb{C}}^2\neq 0;\\ 1-x_0\underline{\varsigma}, & \text{if } |\underline{\varsigma}|_{\mathbb{C}}^2=0,\end{cases} \tag{2.2}$$

then

$$e(x_0e_0,\underline{\varsigma})=e^{-x_0\underline{\varsigma}}$$

in the Taylor series expansion sense, and

$$e(x,\underline{\varsigma})\triangleq e^{\langle\underline{x},\underline{\varsigma}\rangle}e(x_0e_0,\underline{\varsigma})=e(x_0e_0,\underline{\varsigma})e^{\langle\underline{x},\underline{\varsigma}\rangle} \tag{2.3}$$

where $e^{\langle\underline{x},\underline{\varsigma}\rangle}=e^{\sum_j x_j\varsigma_j}$. Thus $e(x,\underline{\varsigma})$ takes its value in the span of $\{e_0,e_1,\ldots,e_n\}$ with complex-valued functions as its coefficients. It is a natural extension of $e^{\langle\underline{x},\underline{\xi}\rangle}$ which is monogenic with respect to x and holomorphic with respect to $\underline{\varsigma}$ respectively. This can be proved from the power series expansion of the exponential function $e(x,\underline{\varsigma})$ as follows. Define $\underline{\varsigma}^{\underline{\alpha}}=\varsigma_1^{\alpha_1}\ldots\varsigma_n^{\alpha_n}$ and $p(x,\underline{\varsigma})=\sum_{j=1}^n z_j\varsigma_j=\langle\underline{x},\underline{\varsigma}\rangle-x_0\underline{\varsigma}$, where $z_i=x_i-x_0e_i$ $(1\leqslant i\leqslant n)$, then

$$p(x,\underline{\varsigma})^k=\left(\sum_{j=1}^n z_j\varsigma_j\right)^k=k!\sum_{|\underline{\alpha}|=k}V_{\underline{\alpha}}(x)\underline{\varsigma}^{\underline{\alpha}}$$

is a both left- and right-monogenic polynomial of order k (see [1, p. 127]), thus $e(x,\underline{\varsigma})$ in (2.3) can also be written as

$$e(x,\underline{\varsigma})\triangleq\sum_{k=0}^\infty\frac{1}{k!}\left(\sum_{j=1}^n z_j\varsigma_j\right)^k=\sum_{k=0}^\infty\frac{1}{k!}\big(\langle\underline{x},\underline{\varsigma}\rangle-x_0\underline{\varsigma}\big)^k=\sum_{k=0}^\infty\sum_{|\underline{\alpha}|=k}V_{\underline{\alpha}}(x)\underline{\varsigma}^{\underline{\alpha}}, \tag{2.4}$$

the series is point-wise convergent in each variable. Its proof is similar to that of $\exp(x,\mathbf{a})$ in §15.7.6 of [1] (see pp. 131 and 117) and [3] (see p. 175). Similar functions also appeared in [2], but they were not discussed in detail.

By the same procedure as (2.1)–(2.4), other exponential functions are given as follows:

$$e(x,\overline{\underline{\varsigma}})\triangleq\sum_{k=0}^\infty\frac{1}{k!}\left(\sum_{j=1}^n z_j\overline{\varsigma_j}\right)^k=\sum_{k=0}^\infty\sum_{|\underline{\alpha}|=k}V_{\underline{\alpha}}(x)\overline{\underline{\varsigma}}^{\underline{\alpha}}=\begin{cases}e^{\sum_{j=1}^n x_j\overline{\varsigma_j}}(\cos(x_0|\overline{\underline{\varsigma}}|_{\mathbb{C}})e_0+\sin(x_0|\overline{\underline{\varsigma}}|_{\mathbb{C}})\frac{\overline{\underline{\varsigma}}}{|\overline{\underline{\varsigma}}|_{\mathbb{C}}}), & \text{if } |\overline{\underline{\varsigma}}|_{\mathbb{C}}^2\neq 0;\\ e^{\sum_{j=1}^n x_j\overline{\varsigma_j}}(1+x_0\overline{\underline{\varsigma}}), & \text{if } |\overline{\underline{\varsigma}}|_{\mathbb{C}}^2=0,\end{cases} \tag{2.5}$$

where $\overline{\underline{\varsigma}}=\sum_{j=1}^n\overline{\varsigma_j}\overline{e_j}=-\sum_{j=1}^n\overline{\varsigma_j}e_j$, $|\overline{\underline{\varsigma}}|_{\mathbb{C}}^2=(i\overline{\underline{\varsigma}})^2=\sum_{j=1}^n(\overline{\varsigma_j})^2$, $\overline{\underline{\varsigma}}^{\underline{\alpha}}=\overline{\varsigma_1}^{\alpha_1}\ldots\overline{\varsigma_n}^{\alpha_n}$.

$$e(\bar{x},\underline{\varsigma})\triangleq\sum_{k=0}^\infty\frac{1}{k!}\left(\sum_{j=1}^n \overline{z_j}\varsigma_j\right)^k=\sum_{k=0}^\infty\sum_{|\underline{\alpha}|=k}\overline{V_{\underline{\alpha}}(x)}\underline{\varsigma}^{\underline{\alpha}}=\begin{cases}e^{\sum_{j=1}^n x_j\varsigma_j}(\cos(x_0|\underline{\varsigma}|_{\mathbb{C}})e_0+\sin(x_0|\underline{\varsigma}|_{\mathbb{C}})\frac{\underline{\varsigma}}{|\underline{\varsigma}|_{\mathbb{C}}}), & \text{if } |\underline{\varsigma}|_{\mathbb{C}}^2\neq 0;\\ e^{\sum_{j=1}^n x_j\varsigma_j}(1+x_0\underline{\varsigma}), & \text{if } |\underline{\varsigma}|_{\mathbb{C}}^2=0,\end{cases} \tag{2.6}$$

where $\underline{\varsigma}=\sum_{j=1}^n\varsigma_je_j$, $|\underline{\varsigma}|_{\mathbb{C}}^2=(i\underline{\varsigma})^2=\sum_{j=1}^n(\varsigma_j)^2$, $\underline{\varsigma}^{\underline{\alpha}}=\varsigma_1^{\alpha_1}\ldots\varsigma_n^{\alpha_n}$.

Properties of these exponential functions are discussed in [6].

3. Integral operators between $\mathcal{F}_2(\mathbb{C}^n)$ and $M_2(\mathbb{R}^{n+1})$

Assume $f \in M_2(\mathbb{R}^{n+1})$ and $F \in \mathcal{F}_2(\mathbb{C}^n)$. Define

$$(\mathbf{B}f)(\underline{\varsigma}) = \int_{\mathbb{R}^{n+1}} e(\bar{x}, \underline{\varsigma}) f(x) e^{-n|x|^2}\, dx, \tag{3.1}$$

and

$$(\mathbf{A}F)(x) = \int_{\mathbb{C}^n} e(x, \overline{\underline{\varsigma}}) F(\varsigma) e^{-|\underline{\varsigma}|^2}\, d\underline{\varsigma}. \tag{3.2}$$

The main goal of this paper is to study the connections between $\mathcal{F}_2(\mathbb{C}^n)$ and $M_2(\mathbb{R}^{n+1})$ in terms of the operators **A** and **B**.

Lemma 3.1. *Let $V_{\underline{\alpha}}(x)$ be the monogenic polynomial of order $|\underline{\alpha}|$, then*

$$\left|V_{\underline{\alpha}}(x)\right| \leqslant \frac{1}{\alpha_1! \ldots \alpha_n!} \left(x_1^2 + x_0^2\right)^{\alpha_1/2} \cdots \left(x_n^2 + x_0^2\right)^{\alpha_n/2}, \tag{3.3}$$

and

$$\left|e(x, \underline{\zeta})\right| \leqslant e^{\sqrt{x_1^2+x_0^2}|\zeta_1| + \cdots + \sqrt{x_n^2+x_0^2}|\zeta_n|}. \tag{3.4}$$

For the other variations $e(x, \overline{\underline{\zeta}})$ and $e(\bar{x}, \underline{\zeta})$ the same estimate holds.

Proof. By the definition (1.3) of $V_{\underline{\alpha}}(x)$, the number of the distinguished permutations in (1.3) is $\frac{|\underline{\alpha}|!}{\alpha_1! \ldots \alpha_n!}$ (see [1, pp. 68 and 113]). Therefore,

$$\begin{aligned}
\left|V_{\underline{\alpha}}(x)\right| &\leqslant \frac{1}{|\underline{\alpha}|!} \sum_{\pi(l_1,\ldots,l_{|\underline{\alpha}|})} |z_{l_1} \ldots z_{l_{|\underline{\alpha}|}}| \\
&= \frac{1}{|\underline{\alpha}|!} \sum_{\pi(l_1,\ldots,l_{|\underline{\alpha}|})} |z_{l_1} \ldots z_{l_{|\underline{\alpha}|}}| \\
&= \frac{1}{|\underline{\alpha}|!} \frac{|\underline{\alpha}|!}{\alpha_1! \ldots \alpha_n!} |z_1|^{\alpha_1} \ldots |z_n|^{\alpha_n} \\
&= \frac{1}{\alpha_1! \ldots \alpha_n!} \left(x_1^2 + x_0^2\right)^{\alpha_1/2} \cdots \left(x_n^2 + x_0^2\right)^{\alpha_n/2},
\end{aligned}$$

where the property $|z_{l_1} \ldots z_{l_{|\underline{\alpha}|}}| = |z_{l_1}| \ldots |z_{l_{|\underline{\alpha}|}}| = |z_1|^{\alpha_1} \ldots |z_n|^{\alpha_n}$ (see [5, p. 54, Corollary 7.25]) was used. Then

$$\begin{aligned}
\left|e(x, \underline{\zeta})\right| &\leqslant \sum_{k=0}^{\infty} \sum_{|\underline{\alpha}|=k} \left|V_{\underline{\alpha}}(x) \underline{\zeta}^{\underline{\alpha}}\right| \\
&\leqslant \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{|\underline{\alpha}|=k} \frac{|\underline{\alpha}|!}{\alpha_1! \ldots \alpha_n!} \left(\left(x_1^2 + x_0^2\right)|\zeta_1|^2\right)^{\alpha_1/2} \cdots \left(\left(x_n^2 + x_0^2\right)|\zeta_n|^2\right)^{\alpha_n/2} \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} \left(\sqrt{\left(x_1^2 + x_0^2\right)|\zeta_1|^2} + \cdots + \sqrt{\left(x_n^2 + x_0^2\right)|\zeta_n|^2}\right)^k \\
&= e^{\sqrt{x_1^2+x_0^2}|\zeta_1| + \cdots + \sqrt{x_n^2+x_0^2}|\zeta_n|}. \qquad \square
\end{aligned}$$

Remark. As a consequence of Lemma 3.1, the generalized exponential functions $e(\cdot, \underline{\zeta}) \in M_2(\mathbb{R}^{n+1})$ for each fixed $\underline{\zeta} \in \mathbb{C}^n$; and $e(x, \cdot) \in \mathcal{F}_2(\mathbb{C}^n)$ for each fixed $x \in \mathbb{R}^{n+1}$. The estimate (3.4) also allows us to use Fubini's Theorem with the weight functions in the two spaces (see the proofs of Theorems 3.9 and 3.10).

Lemma 3.2. *Let $C_{n,\underline{\alpha}}^2 = \|V_{\underline{\alpha}}\|_M^2 = \int_{\mathbb{R}^{n+1}} \overline{V_{\underline{\alpha}}(x)} V_{\underline{\alpha}}(x) e^{-n|x|^2}\, dx$, $|\underline{\alpha}| > 0$, then*

$$C_{n,\underline{\alpha}}^2 \leqslant \frac{2^{\frac{n+1}{2}} \sqrt{\pi}}{n^{\frac{n}{2}} \underline{\alpha}! |\underline{\alpha}|^{\frac{n-1}{2}}}. \tag{3.5}$$

Proof. In the following, we adopt the multi-index notations as usual. For example, for $\underline{\alpha} = (\alpha_1, \dots, \alpha_n)$, we have $\underline{\alpha}! = \alpha_1! \dots \alpha_n!$, and $\underline{\alpha}!! = (\alpha_1!!) \dots (\alpha_n!!)$. For any $\underline{\alpha}, \underline{\beta} \in \mathbb{N}^n$, where $\mathbb{N}$ stands for the set of non-negative integers, we write $\underline{\beta} \leqslant \underline{\alpha}$ if and only if $\beta_i \leqslant \alpha_i$ for all $1 \leqslant i \leqslant n$. By Lemma 3.1 and simple calculations, we have

$$\begin{aligned} C_{n,\underline{\alpha}}^2 &\leqslant \frac{1}{(\alpha_1! \dots \alpha_n!)^2} \int_{\mathbb{R}^{n+1}} \left(x_1^2 + x_0^2\right)^{\alpha_1} \cdots \left(x_n^2 + x_0^2\right)^{\alpha_n} e^{-n|x|^2}\, dx \\ &= \frac{1}{n^{\frac{n+1}{2}} n^{|\underline{\alpha}|} (\underline{\alpha}!)^2} \int_{\mathbb{R}^{n+1}} \left(x_1^2 + x_0^2\right)^{\alpha_1} \dots \left(x_n^2 + x_0^2\right)^{\alpha_n} e^{-|x|^2}\, dx \\ &= \frac{2^{n+1}}{n^{\frac{n+1}{2}} n^{|\underline{\alpha}|} (\underline{\alpha}!)^2} \int_0^{+\infty} \dots \int_0^{+\infty} \left(x_1^2 + x_0^2\right)^{\alpha_1} \cdots \left(x_n^2 + x_0^2\right)^{\alpha_n} e^{-|x|^2}\, dx_1 \dots dx_n\, dx_0, \end{aligned}$$

and

$$\begin{aligned} \int_0^{+\infty} \left(x_j^2 + x_0^2\right)^{\alpha_j} e^{-x_j^2}\, dx_j &= \sum_{k_j=0}^{\alpha_j} \binom{\alpha_j}{k_j} x_0^{2(\alpha_j - k_j)} \int_0^{+\infty} x_j^{2k_j} e^{-x_j^2}\, dx_j \\ &= \frac{1}{2} \sum_{k_j=0}^{\alpha_j} \binom{\alpha_j}{k_j} \Gamma\left(k + \frac{1}{2}\right) x_0^{2(\alpha_j - k_j)} \\ &= \frac{\sqrt{\pi}}{2} \sum_{k_j=0}^{\alpha_j} \binom{\alpha_j}{k_j} \frac{(2k_j - 1)!!}{2^{k_j}} x_0^{2(\alpha_j - k_j)}. \end{aligned}$$

Then we have

$$\begin{aligned} C_{n,\underline{\alpha}}^2 &\leqslant \frac{2^{n+1}(\sqrt{\pi}/2)^n}{n^{\frac{n+1}{2}} n^{|\underline{\alpha}|} (\underline{\alpha}!)^2} \int_0^{+\infty} \prod_{j=1}^{n} \sum_{k_j=0}^{\alpha_j} \binom{\alpha_j}{k_j} \frac{(2k_j - 1)!!}{2^{k_j}} x_0^{2(\alpha_j - k_j)} e^{-x_0^2}\, dx_0 \\ &= \frac{2(\pi)^{n/2}}{n^{\frac{n+1}{2}} n^{|\underline{\alpha}|} (\underline{\alpha}!)^2} \sum_{k_1=0}^{\alpha_1} \dots \sum_{k_n=0}^{\alpha_n} \frac{\underline{\alpha}!(2\underline{k} - \underline{1})!!}{2^{|\underline{k}|} \underline{k}! (\underline{\alpha} - \underline{k})!} \times \int_0^{\infty} x_0^{2|\underline{\alpha} - \underline{k}|} e^{-x_0^2}\, dx_0 \\ &= \frac{(\pi)^{n/2}}{n^{\frac{n+1}{2}} n^{|\underline{\alpha}|} \underline{\alpha}!} \sum_{\underline{0} \leqslant \underline{k} \leqslant \underline{\alpha}} \frac{(2\underline{k} - \underline{1})!!}{2^{|\underline{k}|} \underline{k}! (\underline{\alpha} - \underline{k})!} \cdot \frac{(2|\underline{\alpha} - \underline{k}| - 1)!!}{2^{|\underline{\alpha} - \underline{k}|}} \\ &\leqslant \frac{(\pi)^{n/2}}{n^{\frac{n+1}{2}} n^{|\underline{\alpha}|} \underline{\alpha}!} \sum_{\underline{0} \leqslant \underline{k} \leqslant \underline{\alpha}} \frac{(2\underline{k} - \underline{1})!!}{(2\underline{k})!!} \cdot \frac{(2|\underline{\alpha} - \underline{k}|)!!}{(\underline{\alpha} - \underline{k})! 2^{|\underline{\alpha} - \underline{k}|}} \\ &\leqslant \frac{(\pi)^{n/2}}{n^{\frac{n+1}{2}} n^{|\underline{\alpha}|} \underline{\alpha}!} \sum_{\underline{0} \leqslant \underline{k} \leqslant \underline{\alpha}} \frac{(|\underline{\alpha} - \underline{k}|)!}{(\underline{\alpha} - \underline{k})!} \\ &\leqslant \frac{(\pi)^{n/2}}{n^{\frac{n+1}{2}} n^{|\underline{\alpha}|} \underline{\alpha}!} \sum_{\underline{0} \leqslant \underline{k} \leqslant \underline{\alpha}} \binom{|\underline{k}|}{\underline{k}}. \end{aligned} \tag{3.6}$$

To obtain an upper bound of $C_{n,\underline{\alpha}}^2$, we need to estimate the sum $T(\underline{\alpha}) = \sum_{\underline{0} \leqslant \underline{k} \leqslant \underline{\alpha}} \binom{|\underline{k}|}{\underline{k}}$ for any multi-index $\underline{\alpha}$. For this, we define $\overline{|\underline{\alpha}|} = \lfloor \frac{|\underline{\alpha}|}{n} \rfloor$, and $r = |\underline{\alpha}| - n\overline{|\underline{\alpha}|}$, where $0 \leqslant r < n$. Define $\widehat{\underline{\alpha}} = (\overline{|\underline{\alpha}|}, \dots, \overline{|\underline{\alpha}|}) + (\underbrace{1, \dots, 1}_{r \text{ times}}, 0 \dots 0)$. Then $|\widehat{\underline{\alpha}}| = n\overline{|\underline{\alpha}|} + r = |\underline{\alpha}|$.

For any multi-index $\underline{\alpha} \in \mathbb{N}^n$, define $d(\underline{\alpha}) = \max\{|\alpha_i - \alpha_j|\colon\ 1 \leqslant i, j \leqslant n\}$. Let $S_\alpha = \{\underline{k} \in \mathbb{N}^n\colon\ \underline{k} \leqslant \alpha\}$. We want to prove the following

Lemma 3.3. *For any multi-index $\underline{\alpha} \in \mathbb{N}^n$, we have*

$$T(\underline{\alpha}) \leqslant T(\widehat{\underline{\alpha}}). \tag{3.7}$$

Moreover, the equality holds if and only if $d(\underline{\alpha}) \leqslant 1$.

Proof. We first prove (3.7) for $n = 2$. For any given $\underline{\alpha} \in \mathbb{N}^2$, we may assume that $\underline{\alpha} = (\alpha_1, \alpha_2)$ such that $\alpha_1 \geqslant \alpha_2$. Observe that if $d(\widehat{\underline{\alpha}}) \leqslant 1$, then the indices $\underline{\alpha}$ and $\widehat{\underline{\alpha}}$ are equal up to a permutation in coordinates, in this case equality of (3.7) holds immediately. In the following, we may assume that $d(\underline{\alpha}) \geqslant 2$, then $\alpha_1 - \alpha_2 \geqslant 2$, and $\alpha_1 > \alpha_1 - 1 \geqslant \overline{|\underline{\alpha}|} \geqslant \alpha_2 + 1 > \alpha_2$. Define affine map $f : \mathbb{R}^2 \to \mathbb{R}^2$, by $f(x, y) = (x-1, y+1)$, hence $d(f(\underline{\alpha})) = |(\alpha_1 - 1) - (\alpha_2 + 1)| = d(\underline{\alpha}) - 2$.

Let $S_0 = S_{\underline{\alpha}} \cap S_{f(\underline{\alpha})}$, $S_1 = S_{\underline{\alpha}} \setminus S_{f(\underline{\alpha})}$ and $S_2 = S_{f(\underline{\alpha})} \setminus S_{\underline{\alpha}}$. Cancelling the common contributions from the multi-nomial coefficients indexed by $\underline{\beta} \in S_0$, we have

$$\begin{aligned} T\big(f(\underline{\alpha})\big) - T(\underline{\alpha}) &= \sum_{\underline{\beta} \in S_{f(\underline{\alpha})}} \binom{|\underline{\beta}|}{\underline{\beta}} - \sum_{\underline{\beta} \in S_{\underline{\alpha}}} \binom{|\underline{\beta}|}{\underline{\beta}} = \sum_{\underline{\beta} \in S_2} \binom{|\underline{\beta}|}{\underline{\beta}} - \sum_{\underline{\beta} \in S_1} \binom{|\underline{\beta}|}{\underline{\beta}} \\ &= \sum_{j=0}^{\alpha_1 - 1} \binom{\alpha_2 + 1 + j}{\alpha_2 + 1} - \sum_{j=0}^{\alpha_2} \binom{\alpha_1 + j}{\alpha_1} \\ &= \left[\sum_{j=\alpha_2+1}^{\alpha_1 - 1} \binom{\alpha_2 + 1 + j}{\alpha_2 + 1} + \sum_{j=d(\underline{\alpha})}^{\alpha_2} \binom{\alpha_2 + 1 + j}{j} - \sum_{j=0}^{\alpha_2} \binom{|\underline{\alpha}| - j}{\alpha_2 - j} \right] + \sum_{j=0}^{d(\underline{\alpha})-1} \binom{\alpha_1 + 1 + j}{j} \\ &= \left[\sum_{j=0}^{d(\underline{\alpha})-2} \binom{|\underline{\alpha}| - j}{\alpha_2 + 1} + \sum_{j=d(\underline{\alpha})-1}^{\alpha_2 - 1} \binom{|\alpha| - j}{\alpha_1 - 1 - j} - \sum_{j=0}^{\alpha_2} \binom{|\underline{\alpha}| - j}{\alpha_2 - j} \right] + \sum_{j=0}^{d(\underline{\alpha})-1} \binom{\alpha_1 + 1 + j}{j} \\ &\geqslant \sum_{j=0}^{d(\underline{\alpha})-1} \binom{\alpha_1 + 1 + j}{j} > 0. \end{aligned} \tag{3.8}$$

It follows from (3.7) and an induction argument that $T(\underline{\alpha}) \leqslant T(\widehat{\underline{\alpha}})$ for any $\underline{\alpha} \in \mathbb{N}^2$.

Back to the proof of the general case of inequality (3.7), where $\alpha \in \mathbb{N}^n$ with $n \geqslant 2$, we can reduce the general case to the case for $n = 2$. In fact, for any given $\underline{\alpha} \in \mathbb{N}^n$, as the values of $T(\underline{\alpha})$ and multi-index $\widehat{\underline{\alpha}}$ do not change if we permutate the components of $\underline{\alpha}$, so with a suitable permutation we may assume that $\alpha_1 \leqslant \cdots \leqslant \alpha_n$. For any given $\underline{\beta} = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n$ we define $\underline{\beta}' = (\beta_2, \ldots, \beta_{n-1}) \in \mathbb{N}^{n-2}$, and $\underline{\beta}'' = (\beta_1, \beta_n) \in \mathbb{N}^2$, hence $\underline{\beta} = (\beta_1, \underline{\beta}', \beta_n)$. In this case $|\underline{\beta}''| = |\underline{\beta}| - |\underline{\beta}'|$, which does not depend on the choices of β_1 and β_n individually.

Then we have

$$\begin{aligned} T(\underline{\alpha}) &= \sum_{\underline{\beta} \leqslant \underline{\alpha}} \binom{|\underline{\beta}|}{\underline{\beta}} = \sum_{\underline{\beta}' \leqslant \underline{\alpha}'} \frac{|\underline{\beta}|!}{\beta_2! \ldots \beta_{n-1}!} \sum_{(\beta_1, \beta_n) \leqslant (\alpha_1, \alpha_n)} \frac{1}{\beta_1! \beta_n!} \\ &= \sum_{\underline{\beta}' \leqslant \underline{\alpha}'} \frac{|\underline{\beta}|!}{\beta_2! \ldots \beta_{n-1}! (|\underline{\beta}| - |\underline{\beta}'|)!} \sum_{\underline{\beta}'' \leqslant \underline{\alpha}''} \frac{(|\underline{\beta}| - |\underline{\beta}'|)!}{\underline{\beta}''!} \\ &= \sum_{\underline{\beta}' \leqslant \underline{\alpha}'} \frac{|\underline{\beta}|!}{\beta_2! \ldots \beta_{n-1}! (|\underline{\beta}| - |\underline{\beta}'|)!} \sum_{\underline{\beta}'' \leqslant \underline{\alpha}''} \frac{|\underline{\beta}''|!}{\underline{\beta}''!} \\ &\leqslant \sum_{\underline{\beta}' \leqslant \underline{\alpha}'} \frac{|\underline{\beta}|!}{\beta_2! \ldots \beta_{n-1}! (|\underline{\beta}| - |\underline{\beta}'|)!} \sum_{\underline{\beta}'' \leqslant \widehat{\underline{\alpha}''}} \frac{|\underline{\beta}''|!}{\underline{\beta}''!} \\ &= T\big((\underline{\alpha}', \widehat{\underline{\alpha}''})\big). \end{aligned}$$

Note that the strict inequality holds if and only if $d(\underline{\alpha}'') > 1$. In this case, $d((\underline{\alpha}', \widehat{\underline{\alpha}''})) < d(\underline{\alpha})$. Inductively, one can show that $T(\underline{\alpha}) < T(\widehat{\underline{\alpha}})$ if $d(\underline{\alpha}) > 1$, which completes the proof of Lemma 3.3. □

It follows, from the definition of the binomial coefficients $\binom{n+1}{r} = \binom{n}{r} + \binom{n}{r-1}$ and an induction argument on n, that we have

$$2\binom{|\widehat{\alpha}|}{\widehat{\alpha}} \leqslant \sum_{\beta \leqslant \widehat{\alpha}} \binom{|\beta|}{\beta} \leqslant 2^n \binom{|\widehat{\alpha}|}{\widehat{\alpha}}. \tag{3.9}$$

In the next lemma we will use Stirling's Inequality: for any natural number n we have

$$n! = \left(\frac{n}{e}\right)^n \sqrt{2\pi n} e^{d_n}, \tag{3.10}$$

where $\frac{1}{12n+1} < d_n < \frac{1}{12n}$.

Lemma 3.4. *For* $|\underline{\alpha}| > 0$, *we have* $\binom{\widehat{\alpha}}{\widehat{\alpha}_1, \ldots, \widehat{\alpha}_n} = \frac{|\alpha|!}{\widehat{\alpha}_1! \ldots \widehat{\alpha}_n!} \leqslant \frac{n^{|\alpha| + \frac{1}{2}}}{(\sqrt{2\pi |\underline{\alpha}|})^{n-1}}$.

Proof. Note that $\widehat{\underline{\alpha}}_i = \overline{|\underline{\alpha}|} + 1$ for $i \leqslant r$, and the remaining $n - r$ ones are given by $\widehat{\underline{\alpha}}_i = \overline{|\underline{\alpha}|}$. Without loss of generality, we assume $r = 0$, and the general case can be argued similarly. It follows from (3.9) that

$$|\underline{\alpha}|! = \left(\frac{|\underline{\alpha}|}{e}\right)^{|\underline{\alpha}|} \sqrt{2\pi |\underline{\alpha}|} e^{d_{|\underline{\alpha}|}} = \left(\frac{n\overline{|\underline{\alpha}|}}{e}\right)^{n\overline{|\underline{\alpha}|}} \sqrt{2\pi n \overline{|\underline{\alpha}|}} e^{d_{|\underline{\alpha}|}},$$

and

$$\widehat{\underline{\alpha}}_i! = \left(\frac{\widehat{\underline{\alpha}}_i}{e}\right)^{\widehat{\underline{\alpha}}_i} \sqrt{2\pi \widehat{\underline{\alpha}}_i} e^{d_{\widehat{\underline{\alpha}}_i}} = \left(\frac{\overline{|\underline{\alpha}|}}{e}\right)^{\overline{|\underline{\alpha}|}} \sqrt{2\pi \overline{|\underline{\alpha}|}} e^{d_{\overline{|\underline{\alpha}|}}}.$$

Then we have

$$\begin{aligned}
\binom{\widehat{\underline{\alpha}}}{\widehat{\underline{\alpha}}_1, \ldots, \widehat{\underline{\alpha}}_n} &= \frac{|\widehat{\underline{\alpha}}|!}{\widehat{\underline{\alpha}}_1! \ldots \widehat{\underline{\alpha}}_n!} = \frac{|\underline{\alpha}|!}{\widehat{\underline{\alpha}}_1! \ldots \widehat{\underline{\alpha}}_n!} \\
&= \left(\frac{n\overline{|\underline{\alpha}|}}{e}\right)^{n\overline{|\underline{\alpha}|}} \sqrt{2\pi n \overline{|\underline{\alpha}|}} e^{d_{|\underline{\alpha}|}} \cdot \left[\left(\frac{e}{\overline{|\underline{\alpha}|}}\right)^{\overline{|\underline{\alpha}|}} \frac{1}{\sqrt{2\pi \overline{|\underline{\alpha}|}}} e^{-d_{\overline{|\underline{\alpha}|}}}\right]^n \\
&= \frac{n^{n\overline{|\underline{\alpha}|}}}{(\sqrt{2\pi}\,)^{n-1}} \frac{\sqrt{n\overline{|\underline{\alpha}|}}}{\sqrt{\overline{|\underline{\alpha}|}}^n} e^{d_{|\underline{\alpha}|} - n d_{\overline{|\underline{\alpha}|}}} \\
&= \frac{n^{|\underline{\alpha}| + \frac{1}{2}}}{(\sqrt{2\pi |\underline{\alpha}|}\,)^{n-1}} e^{\frac{1}{12|\underline{\alpha}|} - \frac{n}{12\overline{|\underline{\alpha}|}+1}} \\
&\leqslant \frac{n^{|\underline{\alpha}| + \frac{1}{2}}}{(\sqrt{2\pi |\underline{\alpha}|}\,)^{n-1}},
\end{aligned}$$

where the last exponential factor is uniformly dominated by 1. □

Now we can complete the proof of Lemma 3.2. In fact, the upper bound of $C_{n,\underline{\alpha}}$ in (3.4) follows from (3.5), (3.8), Lemmas 3.3 and 3.4. □

We will directly use

Lemma 3.5. *(See [4, p. 40].) $\{\underline{\varsigma}^{\underline{\alpha}}\colon\ |\underline{\alpha}| \geqslant 0\}$ is a set of orthogonal basis of $\mathcal{F}_n(\mathbb{C}^n)$ with the coefficient constants*

$$\int_{\mathbb{C}^n} \overline{\underline{\varsigma}}^{\underline{\alpha}} \underline{\varsigma}^{\underline{\alpha}} e^{-|\underline{\varsigma}|^2} \, d\underline{\varsigma} = \pi^n \underline{\alpha}!. \tag{3.11}$$

Next we will show

Lemma 3.6. $\mathbf{B}(V_{\underline{\alpha}})(\underline{\varsigma}) = C_{n,\underline{\alpha}}^2 \underline{\varsigma}^{\underline{\alpha}}$; $\mathbf{A}(\underline{\varsigma}^{\underline{\alpha}})(x) = \pi^n \underline{\alpha}! V_{\underline{\alpha}}(x)$.

Proof. From Remark given immediately after Lemma 3.1, we can exchange the order of integration and summation, and have

$$\begin{aligned}
\mathbf{B}(V_{\underline{\alpha}})(\underline{\varsigma}) &= \int_{\mathbb{R}^{n+1}} \sum_{k=0}^{\infty} \sum_{|\underline{\beta}|=k} \overline{V_{\underline{\beta}}(x)} \underline{\varsigma}^{\underline{\beta}} V_{\underline{\alpha}}(x) e^{-n|x|^2} \, dx \\
&= \int_{\mathbb{R}^{n+1}} \overline{V_{\underline{\alpha}}(x)} V_{\underline{\alpha}}(x) e^{-n|x|^2} \, dx \, \underline{\varsigma}^{\underline{\alpha}} \\
&= C_{n,\underline{\alpha}}^2 \underline{\varsigma}^{\underline{\alpha}}.
\end{aligned}$$

Similarly, $|e(x, \overline{\underline{\varsigma}})|$ has the estimate (3.4) too. The similar reasoning together with Lemma 3.5 gives

$$\begin{aligned}
\mathbf{A}(\underline{\varsigma}^{\underline{\alpha}})(x) &= \int_{\mathbb{C}^n} \sum_{k=0}^{\infty} \sum_{|\underline{\beta}|=k} V_{\underline{\beta}}(x) \overline{\underline{\varsigma}}^{\underline{\beta}} \underline{\varsigma}^{\underline{\alpha}} e^{-|\underline{\varsigma}|^2} \, d\underline{\varsigma} \\
&= \int_{\mathbb{C}^n} V_{\underline{\alpha}}(x) \overline{\underline{\varsigma}}^{\underline{\alpha}} \underline{\varsigma}^{\underline{\alpha}} e^{-|\underline{\varsigma}|^2} \, d\underline{\varsigma} \\
&= \pi^n \underline{\alpha}! V_{\underline{\alpha}}(x). \qquad \square
\end{aligned}$$

Theorem 3.7. *The integral operator* **B** *is a linear bounded operator from $M_2(\mathbb{R}^{n+1})$ to $\mathcal{F}_2(\mathbb{C}^n)$ with a bound to be* $\sqrt{\frac{2^{\frac{n+1}{2}}\pi^{n+\frac{1}{2}}}{n^{\frac{n}{2}}}}$.

Proof. The Hilbert space theory implies that $f \in M_2(\mathbb{R}^{n+1})$ has L^2-convergent Fourier series

$$f(x) = \sum_{k=0}^{+\infty} \sum_{|\underline{\alpha}|=k} f_{\underline{\alpha}} V_{\underline{\alpha}}(x), \tag{3.12}$$

where $f_{\underline{\alpha}}$ are complex constants, and

$$\|f\|_M^2 = \sum_{k=0}^{+\infty} \sum_{|\underline{\alpha}|=k} C_{n,\underline{\alpha}}^2 |f_{\underline{\alpha}}|^2 < \infty. \tag{3.13}$$

The fact $e(\cdot, \underline{\zeta}) \in M_2(\mathbb{R}^{n+1})$ and continuity of the inner product (1.4) imply

$$\mathbf{B}f(\underline{\varsigma}) = \int_{\mathbb{R}^{n+1}} \sum_{k=0}^{+\infty} \sum_{|\underline{\alpha}|=k} f_{\underline{\alpha}} e(\bar{x}, \underline{\varsigma}) V_{\underline{\alpha}}(x) e^{-n|x|^2}\, dx = \sum_{k=0}^{+\infty} \sum_{|\underline{\alpha}|=k} f_{\underline{\alpha}} C_{n,\underline{\alpha}}^2 \underline{\varsigma}^{\underline{\alpha}}.$$

And hence,

$$\|\mathbf{B}f\|_{\mathcal{F}}^2 = \sum_{k=0}^{+\infty} \sum_{|\underline{\alpha}|=k} |f_{\underline{\alpha}}|^2 C_{n,\underline{\alpha}}^4 \underline{\alpha}!\pi^n < \frac{2^{\frac{n+1}{2}}\pi^{n+\frac{1}{2}}}{n^{\frac{n}{2}}} \sum_{k=0}^{+\infty} \sum_{|\underline{\alpha}|=k} |f_{\underline{\alpha}}|^2 C_{n,\underline{\alpha}}^2 = \frac{2^{\frac{n+1}{2}}\pi^{n+\frac{1}{2}}}{n^{\frac{n}{2}}} \|f\|_M^2,$$

where the estimate in Lemma 3.2 was used. □

Theorem 3.8. *The integral operator* **A** *is a linear bounded operator from $\mathcal{F}_2(\mathbb{C}^n)$ to $M_2(\mathbb{R}^{n+1})$ with a bound to be* $\sqrt{\frac{2^{\frac{n+1}{2}}\pi^{n+\frac{1}{2}}}{n^{\frac{n}{2}}}}$.

Proof. $F \in \mathcal{F}_2(\mathbb{C}^n)$ has the Fourier representation

$$F(\underline{\varsigma}) = \sum_{k=0}^{+\infty} \sum_{|\underline{\alpha}|=k} F_{\underline{\alpha}} \underline{\varsigma}^{\underline{\alpha}}, \tag{3.14}$$

where $F_{\underline{\alpha}}$ are complex-valued constants. We have

$$\|F\|_{\mathcal{F}}^2 = \sum_{k=0}^{+\infty} \sum_{|\underline{\alpha}|=k} |F_{\underline{\alpha}}|^2 \pi^n \underline{\alpha}! < \infty. \tag{3.15}$$

The fact that $e(x, \cdot) \in \mathcal{F}(\mathbb{C}^n)$ and continuity of the inner product (1.6) imply

$$(\mathbf{A}F)(x) = \sum_{k=0}^{+\infty} \sum_{|\underline{\alpha}|=k} V_{\underline{\alpha}}(x) F_{\underline{\alpha}} \underline{\alpha}!\pi^n.$$

Then,

$$\|\mathbf{A}F\|_M^2 = \sum_{k=0}^{+\infty} \sum_{|\underline{\alpha}|=k} C_{n,\underline{\alpha}}^2 \left(\pi^n \underline{\alpha}!\right)^2 |F_{\underline{\alpha}}|^2 < \frac{2^{\frac{n+1}{2}}\pi^{n+\frac{1}{2}}}{n^{\frac{n}{2}}} \|F\|_{\mathcal{F}}^2,$$

where the estimate in Lemma 3.2 was used. □

Theorem 3.9. **AB** *is a bounded linear operator from $M_2(\mathbb{R}^{n+1})$ to $M_2(\mathbb{R}^{n+1})$ with the integral kernel*

$$K_M(x, \overline{y}) = \int_{\mathbb{C}^n} e(x, \overline{\underline{\varsigma}}) e(\overline{y}, \underline{\varsigma}) e^{-|\underline{\varsigma}|^2}\, d\underline{\varsigma} = \sum_{k=0}^{+\infty} \sum_{|\underline{\alpha}|=k} \underline{\alpha}!\pi^n V_{\underline{\alpha}}(x) \overline{V_{\underline{\alpha}}(y)}, \tag{3.16}$$

where the series is convergent uniformly in any bounded domain of $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$.

Proof. Using the Fourier expansion of $\mathbf{B}f$ in the proof of Theorem 3.7, and the Fourier expansion of $\mathbf{A}F$ in the proof of Theorem 3.8, we have

$$(\mathbf{AB}f)(x)=\sum_{k=0}^{+\infty}\sum_{|\underline{\alpha}|=k} f_{\underline{\alpha}} C_{n,\underline{\alpha}}^2 \underline{\alpha}!\pi^n V_{\underline{\alpha}}(x),$$

and

$$\|\mathbf{AB}f\|_M^2=\sum_{k=0}^{+\infty}\sum_{|\underline{\alpha}|=k} |f_{\underline{\alpha}}|^2 C_{n,\underline{\alpha}}^6\left(\underline{\alpha}!\pi^n\right)^2<\left(\frac{2^{\frac{n+1}{2}}\pi^{n+\frac{1}{2}}}{n^{\frac{n}{2}}}\right)^2\sum_{k=0}^{+\infty}\sum_{|\underline{\alpha}|=k} |f_{\underline{\alpha}}|^2 C_{n,\underline{\alpha}}^2=\left(\frac{2^{\frac{n+1}{2}}\pi^{n+\frac{1}{2}}}{n^{\frac{n}{2}}}\right)^2\|f\|_M^2,$$

i.e., $(\mathbf{AB}f)(x)\in M_2(\mathbb{R}^{n+1})$. The boundedness of **AB** can, of course, be obtained just by composing the bounded operators **A** and **B**, and the above bound is the product of the bounds of A and B obtained in Theorems 3.7 and 3.8, respectively.

Recalling the estimate for the exponential functions in Lemma 3.1 (and Remark afterwards), Fubini's Theorem allows us to exchange the order of integration, and obtain

$$\begin{aligned}(\mathbf{AB}f)(x)&=\int_{\mathbb{C}^n} e(x,\overline{\underline{\varsigma}})\left(\int_{\mathbb{R}^{n+1}} e(\overline{y},\underline{\varsigma})f(y)e^{-n|y|^2}\,dy\right)e^{-|\underline{\varsigma}|^2}\,d\underline{\varsigma}\\&=\int_{\mathbb{R}^{n+1}}\left(\int_{\mathbb{C}^n} e(x,\overline{\underline{\varsigma}})e(\overline{y},\underline{\varsigma})e^{-|\underline{\varsigma}|^2}\,d\underline{\varsigma}\right)f(y)e^{-n|y|^2}\,dy.\end{aligned}\tag{3.17}$$

So, the kernel function of **AB** is

$$K_M(x,\overline{y})=\int_{\mathbb{C}^n} e(x,\overline{\underline{\varsigma}})e(\overline{y},\underline{\varsigma})e^{-|\underline{\varsigma}|^2}\,d\underline{\varsigma}.\tag{3.18}$$

For fixed x and y, denote the nth partial sums of the series expansions of $F=e(x,\overline{\underline{\varsigma}})$ and $G=e(\overline{y},\underline{\varsigma})$ in $\overline{\underline{\zeta}}$ and $\underline{\zeta}$ by F_n and G_n, respectively. Continuity of the inner product (1.6) implies

$$[F_n,G_n]\to[F,G]$$

that concludes the series expansion for the kernel

$$K_M(x,\overline{y})=\sum_{k=0}^{+\infty}\sum_{|\underline{\alpha}|=k}\underline{\alpha}!\pi^n V_{\underline{\alpha}}(x)\overline{V_{\underline{\alpha}}(y)},\tag{3.19}$$

where the convergence is uniform in any bounded domain of $\mathbb{R}^{n+1}\times\mathbb{R}^{n+1}$.

Because of the estimates of $|V_{\underline{\alpha}}(x)|$ and $\overline{V_{\underline{\alpha}}(y)}$ (see Lemma 3.1), we further obtain a bound of the kernel function by

$$\begin{aligned}\left|K_M(x,\overline{y})\right|&\leqslant\pi^n\sum_{k=0}^{+\infty}\sum_{|\underline{\alpha}|=k}\underline{\alpha}!\frac{1}{\underline{\alpha}!\underline{\alpha}!}\left(x_1^2+x_0^2\right)^{\alpha_1/2}\dots\left(x_n^2+x_0^2\right)^{\alpha_n/2}\left(y_1^2+y_0^2\right)^{\alpha_1/2}\dots\left(y_n^2+y_0^2\right)^{\alpha_n/2}\\&=\pi^n\sum_{k=0}^{+\infty}\frac{1}{k!}\left(\sqrt{\left(x_1^2+x_0^2\right)\left(y_1^2+y_0^2\right)}+\dots+\sqrt{\left(x_n^2+x_0^2\right)\left(y_n^2+y_0^2\right)}\right)^k\\&=\pi^n e^{\sqrt{(x_1^2+x_0^2)(y_1^2+y_0^2)}+\dots+\sqrt{(x_n^2+x_0^2)(y_n^2+y_0^2)}}.\qquad\square\end{aligned}$$

Theorem 3.10. **BA** *is a bounded linear operator from* $\mathcal{F}_2(\mathbb{C}^n)$ *to* $\mathcal{F}_2(\mathbb{C}^n)$ *with the reproducing kernel*

$$K_{\mathcal{F}}(\underline{\varsigma}',\overline{\underline{\varsigma}})=\int_{\mathbb{R}^{n+1}} e(\bar{x},\underline{\varsigma}')e(x,\overline{\underline{\varsigma}})e^{-|x|^2}\,dx=\sum_{k=0}^{+\infty}\sum_{|\underline{\alpha}|=k}C_{n,\underline{\alpha}}^2(\underline{\varsigma}')^{\underline{\alpha}}\overline{\underline{\varsigma}}^{\underline{\alpha}},\tag{3.20}$$

where the series is uniformly convergent in any bounded domain in $\mathbb{C}^n\times\mathbb{C}^n$.

Proof. Assume that $F\in\mathcal{F}_n(\mathbb{C}^n)$ has the representation (3.14) with the norm (3.15), then

$$(\mathbf{BA}F)(\underline{\varsigma})=\sum_{k=0}^{+\infty}\sum_{|\underline{\alpha}|=k}C_{n,\underline{\alpha}}^2\underline{\alpha}!\pi^n F_{\underline{\alpha}}\underline{\varsigma}^{\underline{\alpha}},\tag{3.21}$$

and

$$\|\mathbf{BA}F\|_{\mathcal{F}}^2=\sum_{k=0}^{+\infty}\sum_{|\underline{\alpha}|=k}C_{n,\underline{\alpha}}^4\big(\underline{\alpha}!\pi^n\big)^3|F_{\underline{\alpha}}|^2\leqslant\left(\frac{2^{\frac{n+1}{2}}\pi^{n+\frac{1}{2}}}{n^{\frac{n}{2}}}\right)^2\sum_{k=0}^{+\infty}\sum_{|\underline{\alpha}|=k}\underline{\alpha}!\pi^n|F_{\underline{\alpha}}|^2=\left(\frac{2^{\frac{n+1}{2}}\pi^{n+\frac{1}{2}}}{n^{\frac{n}{2}}}\right)^2\|F\|_{\mathcal{F}}^2,$$

i.e., $(\mathbf{BA}F)(\underline{\varsigma})\in\mathcal{F}_n(\mathbb{C}^n)$.

The same reasoning based on Fubini's Theorem gives

$$(\mathbf{BA}F)(\underline{\varsigma}')=\int_{\mathbb{R}^{n+1}}e(\bar{x},\underline{\varsigma}')\left(\int_{\mathbb{C}^n}e(x,\overline{\underline{\varsigma}})F(\underline{\varsigma})e^{-|\underline{\varsigma}|^2}\,d\underline{\varsigma}\right)e^{-n|x|^2}\,dx=\int_{\mathbb{C}^n}\left(\int_{\mathbb{R}^{n+1}}e(\bar{x},\underline{\varsigma}')e(x,\overline{\underline{\varsigma}})e^{-n|x|^2}\,dx\right)F(\underline{\varsigma})e^{-|\underline{\varsigma}|^2}\,d\underline{\varsigma}.$$

Continuity of the inner product (1.4) implies

$$K_{\mathcal{F}}(\underline{\varsigma}',\overline{\underline{\varsigma}})=\int_{\mathbb{R}^{n+1}}e(\bar{x},\underline{\varsigma}')e(x,\overline{\underline{\varsigma}})e^{-n|x|^2}\,dx=\int_{\mathbb{R}^{n+1}}\sum_{k=0}^{+\infty}\sum_{|\underline{\alpha}|=k}\overline{V_{\underline{\alpha}}(x)}V_{\underline{\alpha}}(x)(\underline{\varsigma}')^{\underline{\alpha}}\overline{\underline{\varsigma}}^{\underline{\alpha}}e^{-n|x|^2}\,dx=\sum_{k=0}^{+\infty}\sum_{|\underline{\alpha}|=k}C_{n,\underline{\alpha}}^2(\underline{\varsigma}')^{\underline{\alpha}}\overline{\underline{\varsigma}}^{\underline{\alpha}}$$

which is uniformly convergent in any bounded domain in $\mathbb{C}^n\times\mathbb{C}^n$.

Since $C_{n,\underline{\alpha}}^2\leqslant\frac{2^{\frac{n+1}{2}}\pi^{n+\frac{1}{2}}}{n^{\frac{n}{2}}}$, we further obtain a bound of the kernel by

$$\begin{aligned}K_{\mathcal{F}}(\underline{\varsigma}',\overline{\underline{\varsigma}})&\leqslant\frac{2^{\frac{n+1}{2}}\pi^{n+\frac{1}{2}}}{n^{\frac{n}{2}}}\sum_{k=0}^{+\infty}\frac{1}{|\underline{\alpha}|!}\sum_{|\underline{\alpha}|=k}\left|\varsigma_1'\varsigma_1\right|^{\alpha_1}\cdots\left|\varsigma_n'\varsigma_n\right|^{\alpha_n}\\&<\frac{2^{\frac{n+1}{2}}\pi^{n+\frac{1}{2}}}{n^{\frac{n}{2}}}\sum_{k=0}^{+\infty}\frac{1}{k!}\big(\left|\varsigma_1'\varsigma_1\right|+\cdots+\left|\varsigma_n'\varsigma_n\right|\big)^k\\&\leqslant\frac{2^{\frac{n+1}{2}}\pi^{n+\frac{1}{2}}}{n^{\frac{n}{2}}}e^{|\varsigma_1'\varsigma_1|+\cdots+|\varsigma_n'\varsigma_n|}.\qquad\Box\end{aligned}$$

Remark 3.11. Since $\frac{2^{\frac{n+1}{2}}\pi^{n+\frac{1}{2}}}{n^{\frac{n}{2}}}$ is less than 1 for n big enough, for such n the integral operators **A**, **B** the composed operators from them are all compress operators.

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