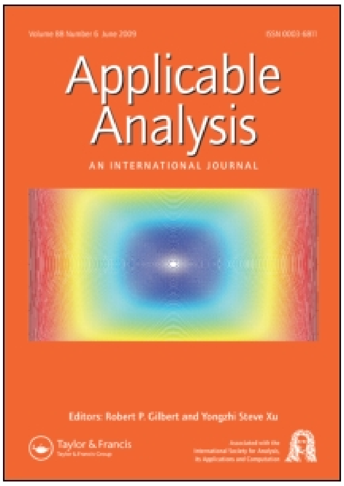

Applicable Analysis
Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t713454076

Sampling theorem and multi-scale spectrum based on non-linear Fourier atoms

Qiuhui Chen [ab]; Tao Qian [c]
[a] School of Informatics, Guangdong University of Foreign Studies, Guangzhou 510420, P. R. China [b] Department of Mathematics, University of Aveiro, Aveiro 3810-193, Portugal [c] Department of Mathematics, University of Macau, Macao via Hong Kong, China

To cite this Article Chen, Qiuhui and Qian, Tao'Sampling theorem and multi-scale spectrum based on non-linear Fourier atoms', Applicable Analysis, 88: 6, 903 — 919
To link to this Article: DOI: 10.1080/00036810903042240
URL: http://dx.doi.org/10.1080/00036810903042240

Applicable Analysis
Vol. 88, No. 6, June 2009, 903–919

Sampling theorem and multi-scale spectrum based on non-linear Fourier atoms

Qiuhui Chen[ab] and Tao Qian[c]*

[a]*School of Informatics, Guangdong University of Foreign Studies, Guangzhou 510420, P. R. China;* [b]*Department of Mathematics, University of Aveiro, Aveiro 3810-193, Portugal;* [c]*Department of Mathematics, University of Macau, Macao via Hong Kong, China*

Communicated by Y. Xu

(*Received 1 April 2009; final version received 7 May 2009*)

This study concerns some new developments of unit analytic signals with non-linear phase. It includes ladder-shaped filter, generalized Sinc function based on non-linear Fourier atoms, generalized sampling theorem for non-bandlimited signals and the notion of multi-scale spectrum for discrete sequences. We first introduce the ladder-shaped filter and show that the impulse response of its corresponding linear time-shift invariant system is the generalized Sinc function as a product of periodic Poisson kernel and Sinc function. Secondly, we establish a Shannon-type sampling theorem based on generalized Sinc function for this type of non-bandlimited signal. We further prove that this type of signal may be holomorphically extended to strips in the complex plane containing the real axis. Finally, we introduce the notion of multi-scale spectrums for discrete sequences and develop the related fast algorithm.

Keywords: Shannon sampling; Fourier atoms; Sinc function; Moebius transform

1. Introduction

The Hilbert transformation is defined by

$$\mathcal{H}f(t) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(x)}{t-x}\,\mathrm{d}x,$$

where the improper integral must converge in a suitable sense. However, the Hilbert transform is well defined for a broad class of functions, namely those in $L^p(\mathbb{R})$ for $1 < p < \infty$ and is a bounded linear operator. The domain of the definition of the Hilbert transform can be extended to the space of bounded functions in $L^\infty(\mathbb{R})$ as well [1]. It is also possible to define the Hilbert transform on the space of tempered distributions as well by an approach due to Gel'fand and Shilov [2], but considerably

*Corresponding author. Email: fsttq@umac.mo

ISSN 0003–6811 print/ISSN 1563–504X online

DOI: 10.1080/00036810903042240
http://www.informaworld.com

more care is needed because of the singularity in the integral. In this article, the improper integral is considered in the principle value meaning.

The Hilbert transform is an important tool in the field of signal processing where it is used to derive the analytic representation of a signal in univariate case [3–6]. Throughout the article, we refer a signal to a real-valued (complex-valued) function but with a real variable. In [7], the author proved that the phase function $\theta_a(t)$ defined by the boundary value

$$e^{i\theta_a(t)} = \frac{e^{it} - a}{1 - \overline{a}e^{it}}$$

of the Möbius transform

$$\tau_a(z) = \frac{z - a}{1 - \overline{a}z}, \quad |a| < 1, \quad |z| < 1, \tag{1.1}$$

satisfies the relations

$$\mathcal{H}(\cos\theta_a(\cdot))(t) = \sin\theta_a(t) \tag{1.2}$$

and

$$\theta_a'(t) = \frac{1 - |a|^2}{1 - 2|a|\cos(t - t_a) + |a|^2} = 2\pi p_a(t), \tag{1.3}$$

where $a = |a|e^{it_a}$ and p_a is the Poisson kernel of the unit circle at a. We further deduced [8,9] that

$$\theta_a(t) := t + 2\arctan\frac{|a|\sin(t - t_a)}{1 - |a|\cos(t - t_a)}, \quad t \in \mathbb{R}, \tag{1.4}$$

which is the sum of a linear part and a 2π-periodic part. Since e^{it} corresponds to the particular case $a = 0$ we have suggested $e^{i\theta_a(t)}$ the name *non-linear Fourier atom associated with a* and devoted to some studies in [7–10]. Some related study of Hardy spaces are developed in [11].

The function $e^{i\theta_a(t)}$ is an analytic signal (see definition, for instance, in [3–6,12,13] with the strictly increasing non-linear instantaneous phase $\theta_a(t)$. As already indicated, the instantaneous frequency of $\cos\theta_a(t)$ or $e^{i\theta_a(t)}$ is 2π-multiple of the fundamental periodic Poisson kernel p_a. The signal $\cos\theta_a(t)$ coincides with the notion of intrinsic mode function in [14,15].

In this article, we study the product of the two well-known functions of which one is the Sinc function defined by

$$\mathrm{Sinc}(t) = \frac{\sin t}{t} = \frac{1}{2}\int_{-\infty}^{\infty} \chi_{[-1,1]}(\xi)e^{it\xi}\mathrm{d}\xi, \tag{1.5}$$

where χ_E stands for the indicator function of the set E in general, and the other is the periodic Poisson kernel. Both of them are theoretically and practically important in signal and harmonic analysis.

We first show that, for real number a with $|a| < 1$, the function (see also (2.5))

$$h_a(t) = \frac{\sqrt{2\pi}}{2} p_a\left(\frac{\pi}{2}t\right)\mathrm{Sinc}\left(\frac{\pi}{2}t\right)$$

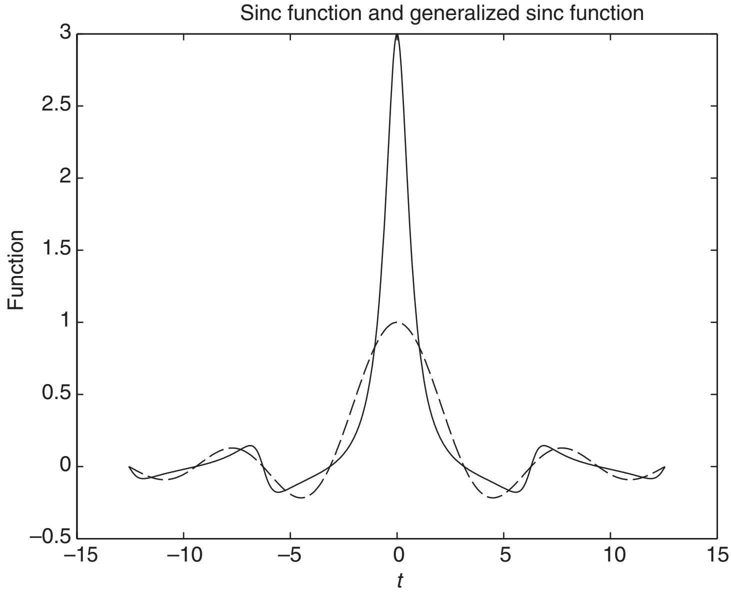

Figure 1. Sinc function (connected by '–') and generalized Sinc function (by real line) with A = 0.5.

is an impulse response of a kind of ladder-shaped filtering process. The latter generalizes the system of ideal low-pass filter. We furthermore prove that the impulse response function h_a is a constant multiple of *generalized Sinc function* (see Figure 1) the latter being defined by

$$\mathrm{Sinc}_a(t) := \frac{\sin\theta_a(t)}{t}, \quad t \in \mathbb{R}. \tag{1.6}$$

We extend the space of bandlimited signals to a certain kind of non-bandlimited signal, and establish the corresponding sampling theorem using the generalized Sinc function. We also prove that the kind of signals can be extended to become analytic functions in strip-shaped domains containing the real axis in the complex plane. Finally, we raise the notion of multi-scale spectrum for data and develop its fast algorithm.

2. Ladder-shaped filter and generalized Sinc functions

We start with a revision for the standard case. We need the definition of the convolution of two functions f and g

$$f * g(t) := \int_{-\infty}^{\infty} f(t-x)\, g(x)\mathrm{d}x, \quad t \in \mathbb{R}.$$

The question of existence of convolution may involve different conditions on f and g. Fortunately, Young's inequality indicates that, for $f \in L^p(\mathbb{R})$ and $g \in L^q(\mathbb{R})$, the convolution determines an $L^r(\mathbb{R})$ function $f*g$ with $1 \le p, q, r \le \infty$ and $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}+1$.

For a continuous linear time-invariant (LTI) system with impulse response $h(t)$, when the input signal is f_{in}, then the output signal f_{out} is the convolution of h and f_{in}, that is

$$f_{out}(t) = \int_{-\infty}^{\infty} f_{in}(t-x)h(x)\mathrm{d}x, \quad t \in \mathbb{R},$$

which yields an equivalent equation in the frequency domain

$$\hat{f}_{out}(\xi) = \sqrt{2\pi}\hat{f}_{in}(\xi)\hat{h}(\xi), \quad \xi \in \mathbb{R},$$

the Fourier transform being defined by

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)e^{-i\xi t}\mathrm{d}t, \quad \xi \in \mathbb{R}.$$

If, in particular, we choose $\hat{h}(\xi)$ to be the indicator function

$$\chi_{[-\frac{\pi}{2},\frac{\pi}{2}]}(\xi) = \begin{cases} 1, & \xi \in [-\frac{\pi}{2}, \frac{\pi}{2}], \\ 0, & \xi \in \mathbb{R} \setminus [-\frac{\pi}{2}, \frac{\pi}{2}], \end{cases}$$

then the output f_{out} is the truncation of f_{in} at the frequency band $[-\frac{\pi}{2}, \frac{\pi}{2}]$, which keeps the lower frequencies of f_{in} and suppresses the higher frequencies. The corresponding impulse response, $h(t)$, is

$$h(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \chi_{[-\frac{\pi}{2},\frac{\pi}{2}]}(\xi)e^{it\xi}\mathrm{d}\xi = \frac{\sqrt{2\pi}}{2}\mathrm{Sinc}\left(\frac{\pi}{2}t\right).$$

This leads to the ideal low-pass filter for the associated discrete LTI system: $h^L_{ideal} = \{h^L_{ideal}(n) : n \in \mathbb{Z}\}$ with

$$h^L_{ideal}(n) = \begin{cases} \frac{1}{2}, & n = 0, \\ 0, & n = 2k, \quad k \in \mathbb{Z} \setminus \{0\}, \\ \frac{(-1)^k}{(2k+1)\pi}, & n = 2k+1, \quad k \in \mathbb{Z}. \end{cases} \tag{2.1}$$

The frequency response of the discrete LTI system is

$$H^L(e^{-i\omega}) = \sum_{n=-\infty}^{\infty} h^L_{ideal}(n)e^{-in\omega}, \quad \omega \in [-\pi, \pi].$$

Through simple calculation, we know that $H^L(e^{-it})$ is the square wave

$$f^{sq}(t) = \begin{cases} 1, & |t| < \frac{\pi}{2}, \\ 0, & \frac{\pi}{2} \le |t| < \pi, \end{cases} \quad \text{and} \quad f^{sq}(t+2\pi) = f^{sq}(t).$$

It is noted that, for the low-pass filter h^L_{ideal} (or H^L), when the input is the unit impulse

$$\delta[n] = \begin{cases} 1, & n = 0, \\ 0, & n \in \mathbb{Z} \end{cases},$$

then the output (discrete impulse response) is the sampling of the continuous impulse response $h(t)=\frac{\sqrt{2\pi}}{2}\mathrm{Sinc}(\frac{\pi}{2}t)$, that is,

$$\sqrt{2\pi}h^{L}_{ideal}(n)=h(n).$$

Now we introduce the non-bandlimited case associated with the generalized Sinc function. To this end, we divide the frequency space $\mathbb{R}$ into parts of *double intervals*

$$I_n:=\Big[-\frac{\pi}{2}(n+1),\ -\frac{\pi}{2}n\Big]\cup\Big[\frac{\pi}{2}n,\ \frac{\pi}{2}(n+1)\Big],\quad n=0,1,\ldots.$$

The Riemann–Legesgue lemma implies that for any integrable signal f the spectrum value $|\hat{f}(\xi)|$ in I_n tends to zero as n tends to ∞. In view of this, we propose the following filtering process: for input signal f_{in}, the output signal f_{out} keeps the frequency information of f_{in} but with different scales in the different frequency bands I_n, namely,

$$\begin{aligned}
&\hat{f}_{out}(\xi)=(1+a)\hat{f}_{in}(\xi),\quad \xi\in I_0,\\
&\hat{f}_{out}(\xi)=a(1+a)\hat{f}_{in}(\xi),\quad \xi\in I_1,\\
&\ldots\\
&\hat{f}_{out}(\xi)=a^n(1+a)\hat{f}_{in}(\xi),\quad \xi\in I_n,\quad n=1,2,\ldots,
\end{aligned}$$

where $a\in(-1,1)$. Correspondingly, the impulse response $h_a(t)$ can be represented in the Fourier domain

$$\hat{h}_a(\xi):=a^n(1+a),\quad \xi\in I_n,\quad n=0,1,2,\ldots, \tag{2.2}$$

where for $a=0$, we temporarily set $0^0=1$. In this notation, the function $\hat{h}_0$ coincides with the indicator function $\chi_{[-\frac{\pi}{2},\frac{\pi}{2}]}$. For general a, the function $\hat{h}_a$ is a step function with ladder shape. Simple calculation provides an alternative form

$$\hat{h}_a(\xi)=(1-a^2)\sum_{n=1}^{\infty}a^{n-1}\chi_{[-\frac{n}{2}\pi,\frac{n}{2}\pi]}(\xi),\quad \xi\in\mathbb{R},$$

from which we get the representation in the time domain

$$h_a(t)=\frac{\sqrt{2\pi}}{2}(1-a^2)\sum_{n=1}^{\infty}a^{n-1}\frac{\sin\frac{n\pi t}{2}}{\frac{\pi t}{2}},\quad t\in\mathbb{R}. \tag{2.3}$$

By recalling the formula

$$\sum_{n=1}^{\infty}r^n\sin nt=\frac{r\sin t}{1-2r\cos t+r^2},\quad |r|<1, \tag{2.4}$$

we have

$$h_a(t)=\frac{\sqrt{2\pi}}{2}\frac{1-a^2}{1-2a\cos\frac{\pi t}{2}+a^2}\frac{\sin\frac{\pi t}{2}}{\frac{\pi t}{2}},\quad t\in\mathbb{R}. \tag{2.5}$$

Note that, apart from a constant multiple $h_a(t)$ is the product of a dilated Sinc function and a periodic Poisson kernel.

On the other hand, the impulse response $h_a(t)$ is a constant multiple of $\frac{\pi}{2}$-dilation of the general Sinc function.

For any complex b in unit disc, straightforward computation gives

$$\frac{z-b}{1-\bar{b}z}=\frac{(z-b)(1-b\bar{z})}{(1-\bar{b}z)(1-b\bar{z})}=\frac{z-b(1+|z|^2)+b^2\bar{z}}{1-2Re(bz)+|b|^2|z|^2}.$$

Letting $b=|b|e^{it_b}$ and $z=e^{it}$,

$$\begin{aligned}\frac{e^{it}-b}{1-\bar{b}e^{it}}&=\frac{e^{it}-2|b|e^{it_b}+|b|^2e^{2it_b}e^{-it}}{1-2Re(|b|e^{i(t-t_b)})+|b|^2}\\&=\frac{\cos t-2|b|\cos t_b+|b|^2\cos(t-2t_b)}{1-2|b|\cos(t-t_b)+|b|^2}+i\frac{\sin t-2|b|\sin t_b-|b|^2\sin(t-2t_b)}{1-2|b|\cos(t-t_b)+|b|^2}.\end{aligned}$$

We obtain that for real numbers a

$$\cos\theta_a(t)=\frac{(1+a^2)\cos t-2a}{1-2a\cos t+a^2}$$
$$\sin\theta_a(t)=\frac{(1-a^2)\sin t}{1-2a\cos t+a^2}$$

Combing the last relation and Equation (2.5) gives

$$h_a(t)=\sqrt{\frac{\pi}{2}}\frac{\sin\theta_a(\frac{\pi t}{2})}{\frac{\pi t}{2}},\quad t\in\mathbb{R},\tag{2.6}$$

that is a constant mutiplication of the $(\pi/2)$-dilated generalized Sinc function defined in (1.6). Noting (2.2) and (2.6), we know that the Fourier transform of $\sqrt{\frac{\pi}{2}}\mathrm{Sinc}_a(\frac{\pi}{2}\cdot)$ is $\hat{h}_a$.

3. Sampling theorems for non-bandlimited signals

We start our discussion from the well-known Whittaker–Kotelnikov–Shannon sampling theorem which states that for any bandlimited function f with $supp\hat{f}\subset[-\Omega,\Omega]$ for an arbitrarily positive number Ω, the signal f can be reconstructed from its sampling sequence $\{f(n\frac{\pi}{\Omega}):n\in\mathbb{Z}\}$ with the Nyquist density $\frac{\Omega}{\pi}$, that is,

$$f(t)=\sum_{n\in\mathbb{Z}}f\left(n\frac{\pi}{\Omega}\right)\frac{\sin(\Omega t-n\pi)}{\Omega t-n\pi},\quad t\in\mathbb{R}.$$

Any bandlimited function f with $supp\hat{f}\subset[-\Omega,\Omega]$ is related to a 2Ω-periodic function

$$M_{f,\Omega}(t):=\frac{\sqrt{2\pi}}{2\Omega}\sum_{n\in\mathbb{Z}}f\left(n\frac{\pi}{\Omega}\right)e^{-i\frac{\pi}{\Omega}nt}.\tag{3.1}$$

The function $M_{f,\Omega}$, however, is well defined not only for Ω-bandlimited functions, but also for any function of sufficient decay rate at ∞, namely the sequence $\{f(n\frac{\pi}{\Omega}):n\in\mathbb{Z}\}$ belongs to l^2. It is easy to see that, for any Ω-bandlimited signal f, the compactly supported function $\hat{f}$ is a pulse in $[-\Omega,\Omega]$ of the 2Ω-periodic function $M_{f,\Omega}$. This suggests us to define the space

$$\mathbb{B}_\Omega := \{ f\in \mathbb{L}^2(\mathbb{R}) : \hat{f}(\xi) = M_{f,\Omega}(\xi)\chi_{[-\Omega,\Omega]}(\xi)\}. \tag{3.2}$$

By the Whittaker–Kotelnikov–Shannon sampling theorem, we know that, f is bandlimited signal with $supp\hat{f}\subset[-\Omega,\Omega]$ if and only if $f\in\mathbb{B}_\Omega$.

Next we will extend the space $\mathbb{B}_\Omega$ of bandlimited signals to certain spaces of non-bandlimited signals. For any function $f\in\mathbb{L}^2(\mathbb{R})$ we know that $M_{f,\Omega}$ is well defined. We define the function $G^a_{f,\Omega}$ by

$$G^a_{f,\Omega}(t) = \begin{cases} (1-a)M_{f,\Omega}(t), & |t|\in[0,\Omega) \\ a(1-a)M_{f,\Omega}(t), & |t|\in[\Omega,2\Omega) \\ \dots, & \dots \\ a^n(1-a)M_{f,\Omega}(t), & |t|\in[n\Omega,(n+1)\Omega), \quad n=0,1,2,\dots. \end{cases} \tag{3.3}$$

The condition $|a|<1$ ensures that

$$\lim_{|t|\to+\infty} G^a_{f,\Omega}(t) = 0.$$

For any $a\in(-1,1)$, denote by $\mathbb{B}^a_\Omega$ the space of signals

$$\mathbb{B}^a_\Omega := \{f\in\mathbb{L}^2(\mathbb{R}) : \hat{f}(\xi) = G^a_{f,\Omega}(\xi),\ \xi\in\mathbb{R}\}. \tag{3.4}$$

For $a=0$,

$$G^0_{f,\Omega}(t) = M_{f,\Omega}(t)\chi_{[-\Omega,\Omega)}(t), \quad t\in\mathbb{R},$$

and $\mathbb{B}^0_\Omega$ is the space of bandlimited signal with $supp\hat{f}\subset[-\Omega,\Omega)$, i.e. $\mathbb{B}^0_\Omega=\mathbb{B}_\Omega$. We establish the following sampling theorem for the space $\mathbb{B}^a_\Omega$ of non-bandlimited signals.

THEOREM 3.1 *A sufficient and necessary condition for a signal f belonging to* $\mathbb{B}^a_\Omega$ *is*

$$f(t) = \frac{1-a}{1+a}\sum_{n\in\mathbb{Z}} f\left(n\frac{\pi}{\Omega}\right)\mathrm{Sinc}_a\left(\Omega\left(t-n\frac{\pi}{\Omega}\right)\right), \quad t\in\mathbb{R} \tag{3.5}$$

where the generalized Sinc *function* $\mathrm{Sinc}_a(t)$, $t\in\mathbb{R}$, *is defined by* (1.6), *and the convergence is in both the* L^2*-norm and the pointwise sense.*

Remark We should point out that the function $\mathrm{Sinc}_a(\pi\cdot)$ is not cardinal. For sampling purpose, we need to normalize it such that $\frac{1-a}{1+a}\mathrm{Sinc}_a(\pi\cdot)$ is a cardinal function.

Proof We first prove the necessity. By the definition of the space $\mathbb{B}^a_\Omega$, we know that any $f\in\mathbb{B}^a_\Omega$ has the following representation in the frequency domain

$$\begin{aligned}\hat{f}(\xi) = G^a_{f,\Omega}(\xi) &= \sum_{l=0}^{\infty} G^a_{f,\Omega}(\xi)(\chi_{[-(l+1)\Omega,-l\Omega]}(\xi) + \chi_{[l\Omega,(l+1)\Omega]})(\xi) \\ &= \frac{1-a}{1+a}\sum_{l=0}^{\infty} a^l\,(1+a)M_{f,\Omega}(\xi)(\chi_{[-(l+1)\Omega,-l\Omega]}(\xi) + \chi_{[l\Omega,(l+1)\Omega]}(\xi)) \\ &= \frac{\sqrt{2\pi}}{2\Omega}\frac{1-a}{1+a}\sum_{n\in\mathbb{Z}} f\left(n\frac{\pi}{\Omega}\right)e^{-i\frac{\pi}{\Omega}n\xi}\sum_{l=0}^{\infty}(1+a)a^l(\chi_{[-(l+1)\Omega,-l\Omega]}(\xi) + \chi_{[l\Omega,(l+1)\Omega]}(\xi)).\end{aligned}$$

Combing this with the equation

$$\sum_{l=0}^{\infty}(1+a)a^l\left(\chi_{[-(l+1)\Omega,-l\Omega]}(\xi)+\chi_{[l\Omega,(l+1)\Omega]}(\xi)\right)=(1-a^2)\sum_{l=1}^{\infty}a^{l-1}\chi_{[-l\Omega,l\Omega]}(\xi)$$

leads to

$$\hat{f}(\xi)=\frac{1-a}{1+a}\sum_{n\in\mathbb{Z}}f\left(n\frac{\pi}{\Omega}\right)e^{-i\frac{\pi}{\Omega}n\xi}\frac{\sqrt{2\pi}}{2\Omega}(1-a^2)\sum_{l=1}^{\infty}a^{l-1}\chi_{[-l\Omega,l\Omega]}(\xi).$$

Applying the inverse Fourier transform to both side of the above equation, noting that the integral

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-i\frac{\pi}{\Omega}n\xi}\chi_{[-l\Omega,l\Omega]}(\xi)e^{i\xi t}\mathrm{d}\xi=\frac{2\Omega}{\sqrt{2\pi}}\frac{\sin(l\Omega(t-\frac{\pi}{\Omega}n))}{\Omega(t-\frac{\pi}{\Omega}n)},$$

we have

$$f(t)=\frac{1-a}{1+a}\sum_{n\in\mathbb{Z}}f\left(n\frac{\pi}{\Omega}\right)(1-a^2)\sum_{l=1}^{\infty}a^{l-1}\frac{\sin(l\Omega(t-\frac{\pi}{\Omega}n))}{\Omega(t-\frac{\pi}{\Omega}n)}.$$

Note that the Equations (2.6) and (2.3) imply that

$$\sin\theta_a(t)=(1-a^2)\sum_{l=1}^{\infty}a^{l-1}\sin lt.$$

We therefore obtain

$$f(t)=\frac{1-a}{1+a}\sum_{n\mathbb{Z}}f\left(n\frac{\pi}{\Omega}\right)\frac{\sin\theta_a(\Omega(t-\frac{\pi}{\Omega}n))}{\Omega(t-\frac{\pi}{\Omega}n)},\quad t\in\mathbb{R},$$

that ends the proof of necessity.

Reversing the above process, we get the proof of the sufficiency part. The L^2-convergence is due to the fact that the system $\{\mathrm{Sinc}_a(\Omega(\cdot-n\frac{\pi}{\Omega})):n\in\mathbb{Z}\}$ is orthogonal in $L^2(\mathbb{R})$. The proof is complete. ■

The following theorem asserts that if $f\in\mathbb{B}_\Omega^a$, $0<|a|<1$, then f may be holomorphically extended to a strip containing the real axis. This is in contrast with bandlimited functions. According to Paley–Wiener Theorem, the latter may be analytically extended to become entire functions in the whole complex plane.

THEOREM 3.2 *If f belongs to $\mathbb{B}_\Omega^a$, then f may be holomorphically extended to the strip*

$$\left\{z=x+iy\;\middle|\;\frac{\log|a|}{\Omega}<y<-\frac{\log|a|}{\Omega},\ -\infty<x<\infty\right\},$$

and, inside the strip, the extended function satisfies the estimate

$$|f(x+iy)|\le\frac{C_{\Omega,a}}{1-e^{2(\log|a|+\Omega|y|)}},$$

where $C_{\Omega,a}$ is a constant depending on Ω and a.

Proof Consider the possible complex numbers z that make the following two integrals $f^+(z)$ and $f^-(z)$ both well defined

$$f^+(z)=\frac{1}{\sqrt{2\pi}}\int_0^\infty e^{iz\xi}G^a_{f,\Omega}(\xi)\mathrm{d}\xi,\quad f^-(z)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^0 e^{iz\xi}G^a_{f,\Omega}(\xi)\mathrm{d}\xi.$$

Since $M_{f,\Omega}$ is 2Ω-periodic, by the definition of $G^a_{f,\Omega}$, $G^a_{f,\Omega}(2n\Omega+\xi)=a^{2n}G^a_{f,\Omega}(\xi)$, $\xi\in[0,2\Omega)$. Writing $\log a=\log|a|+i\arg a$, we have

$$\begin{aligned}f^+(z)&=\frac{1}{\sqrt{2\pi}}\int_0^{2\Omega}\sum_{n=1}^\infty a^{2(n-1)}e^{i[2(n-1)\Omega+\xi]z}G^a_{f,\Omega}(\xi)\mathrm{d}\xi\\&=\frac{1}{\sqrt{2\pi}}\int_0^{2\Omega}e^{i\xi z}\sum_{n=1}^\infty e^{2(n-1)(\log|a|+i\arg a+i\Omega z)}G^a_{f,\Omega}(\xi)\mathrm{d}\xi\\&=\frac{1}{\sqrt{2\pi}}\int_0^{2\Omega}e^{i\xi z}B^a_{z,\Omega}G^a_{f,\Omega}(\xi)\mathrm{d}\xi,\end{aligned}$$

where $B^a_{z,\Omega}$ represents the geometric series in the integral, namely,

$$B^a_{z,\Omega}=\sum_{n=1}^\infty e^{2(n-1)(\log|a|+i\arg a+i\Omega z)}.$$

For $z=x+iy$, due to the relation

$$|e^{2(\log|a|+i\arg a+i\Omega z)}|\le e^{2(\log|a|-\Omega y)},$$

for

$$y>\frac{\log|a|}{\Omega},$$

the geometric series is absolutely convergent to

$$B^a_{z,\Omega}=\frac{1}{1-e^{2(\log|a|+i\arg a+i\Omega z)}}.$$

The function $B^a_{z,\Omega}$ is bounded by

$$\frac{1}{1-e^{2(\log|a|-\Omega y)}}.$$

The factor $B^a_{z,\Omega}$ may be moved out of the integral while $G^a_{f,\Omega}(\xi)$ is integrable and $e^{i\xi z}$ is bounded in the domain of integration. Therefore f^+ is well defined through the integral in the in the half-plane $y>\frac{\log|a|}{\Omega}$ with

$$|f^+(z)|\le\frac{C_{\Omega,a}}{1-e^{2(\log|a|-\Omega y)}}.$$

To show that f^+ is holomorphic in the half-plane, we first write out the difference-quotient

$$\begin{aligned}\frac{f^+(z+\Delta z_n)-f^+(z)}{\Delta z_n}&=\frac{1}{\sqrt{2\pi}}\int_0^{2\Omega}e^{i\xi z}\frac{e^{i\xi\Delta z_n}B^a_{z+\Delta z_n,\Omega}-B^a_{z,\Omega}}{\Delta z_n}G^a_{f,\Omega}(\xi)\mathrm{d}\xi\\&=\frac{1}{\sqrt{2\pi}}\int_0^{2\Omega}e^{i\xi z}\gamma_{\xi,a,\Omega,z}(\Delta z_n)G^a_{f,\Omega}(\xi)\mathrm{d}\xi\end{aligned}$$

with

$$\gamma_{\xi,a,\Omega,z}(\Delta z_n) = \frac{1}{(1-e^{2(\log|a|+i\arg a+i\Omega z)})(1-e^{2i\Omega\Delta z_n}e^{2(\log|a|+i\arg a+i\Omega z)})} \cdot \left(\frac{e^{i\xi\Delta z_n}-1}{\Delta z_n}+\frac{e^{2i\Omega\Delta z_n}-e^{i\xi\Delta z_n}}{\Delta z_n}e^{2(\log|a|+i\arg a+i\Omega z)}\right).$$

For $\xi \in [0, 2\Omega]$ the two functions

$$\frac{e^{i\xi\Delta z_n}-1}{\Delta z_n} \quad \text{and} \quad \frac{e^{2i\Omega\Delta z_n}-e^{i\xi\Delta z_n}}{\Delta z_n}$$

are uniformly bounded as $\Delta z_n \to 0$. Note that

$$\lim_{\Delta z_n\to 0}\gamma_{\xi,a,\Omega,z}(\Delta z_n) = \frac{i\xi + i(2\Omega-\xi)e^{2(\log|a|+i\arg a+i\Omega z)}}{(1-e^{2(\log|a|+i\arg a+i\Omega z)})^2} = i(\xi-2\Omega+2\Omega B^a_{z,\Omega})B^a_{z,\Omega}.$$

The dominated convergence theorem gives

$$(f^+)'(z) = \frac{iB^a_{z,\Omega}}{\sqrt{2\pi}}\int_0^{2\Omega} e^{i\xi z}(\xi-2\Omega+2\Omega B^a_{z,\Omega})G^a_{f,\Omega}(\xi)\mathrm{d}\xi.$$

We therefore conclude that the function f^+ is holomorphic and bounded above any line $\{t+iy \mid -\infty < t < \infty\}$ for $y > \frac{\log|a|}{\Omega}$.

Similarly,

$$|f^-(z)| \le \frac{C_{\Omega,a}}{1-e^{2(\log|a|+\Omega y)}}$$

for

$$y < \frac{-\log|a|}{\Omega},$$

and the function f^- is holomorphic and bounded below any line $\{t+iy \mid -\infty < t < \infty\}$ for $y < \frac{-\log|a|}{\Omega}$.

It follows that restricted in the strip

$$\frac{\log|a|}{\Omega} < y < \frac{-\log|a|}{\Omega},$$

the function $f(z) = f^+(z) + f^-(z)$ is holomorphic and satisfies the estimate

$$|f(z)| \le \frac{C_{\Omega,a}}{1-e^{2(\log|a|+\Omega|y|)}}. \qquad \blacksquare$$

The following theorem extends the sampling formula (3.5) to the strip where $f \in \mathbb{B}^a_\Omega$ is holomorphic.

THEOREM 3.3 *If $f \in \mathbb{B}^a_\Omega$, then for any $z = t+iy$, $\frac{\log|a|}{\Omega} < y < \frac{-\log|a|}{\Omega}$,*

$$f(z) = \frac{1-a}{1+a}\sum_{n\in\mathbb{Z}} f\left(n\frac{\pi}{\Omega}\right)\mathrm{Sinc}_a\left(\Omega\left(z-n\frac{\pi}{\Omega}\right)\right), \tag{3.6}$$

where the convergence is in both the L^2-norm and the pointwise sense.

Proof We show that the generalized Sinc functions in (1.6) may be holomorphically extended to the strip $z = t + iy \mid -\infty < t < \infty, \frac{\log|a|}{\Omega} < y < \frac{-\log|a|}{\Omega}$, and, for n large enough, there holds

$$\left|\mathrm{Sinc}_a\left(\Omega\left(z - n\frac{\pi}{\Omega}\right)\right)\right| \leq \frac{C_{\Omega,a}}{n}. \tag{3.7}$$

We are reduced to proving

$$\left|\sin\theta_a(\Omega z - n\pi)\right| \leq C_{\Omega,a,\delta}, \quad y \in \left(\frac{\log|a|}{\Omega} + \delta, \frac{-\log|a|}{\Omega} - \delta\right), \tag{3.8}$$

where δ is a small positive number satisfying $0 < \delta < \frac{-\log|a|}{\Omega}$. The relation (3.8) is easy to be verified by using the explicit expression of the function $\sin\theta_a$ in terms of Möbius transform. Now as a result of Theorem 3.2 the left-hand side (3.6) may be holomorphically extended to the strip, while each entry on the right-hand side can be holomorphically extended to the same strip, too. Invoking the Cauchy–Schwarz inequality and taking into account the estimate (3.7) and the L^2-convergence of the series $\{f(n\frac{\pi}{\Omega})\}$, the series on the right-hand side uniformly converges in the narrower strips defined through the restricted δ. This shows the pointwise convenience. The L^2-convergence, again, is based on the orthogonality of the entries on the right-hand side. The proof is complete.

If for functions in $\mathbb{B}^a_\Omega, a \neq 0$, one uses the ordinary Shannon sampling, then there is an non-zero error. According to [16] (also see [17]) the L^∞-norm of the error is of the quantity $O(e^{\pi d/h})$, where h is the Nyquist density and d is half of the width of the strip. In our case $h = \pi/\Omega$, $d = -\log|a|/\Omega$, and $O(e^{\pi d/h}) = O(e^{\log|a|})$ which is consistent with Shannon sampling as $a \to 0$ implies $O(e^{\log|a|}) \to 0$.

Finally we consider the corresponding discrete LTI system, for which, we hope that the discrete impulse response is the sampling of the impulse response $h_a(t) = \frac{\sqrt{2\pi}}{2}\frac{\sin\theta_a(\frac{\pi}{2}t)}{\frac{\pi}{2}t}$ for the continuous LTI system. For this discrete LTI system, the filter is $c = \{c_n : n \in \mathbb{Z}\}$ with

$$c_n = \begin{cases} \dfrac{1}{2}\dfrac{1+a}{1-a}, & n = 0, \\ 0, & n = 2k, \quad k \in \mathbb{Z}\setminus\{0\}, \\ \dfrac{1-a^2}{1+a^2}\dfrac{(-1)^k}{(2k+1)\pi}, & n = 2k+1, \quad k \in \mathbb{Z}. \end{cases}$$

The frequency response is

$$H(e^{-i\omega}) = \frac{1}{2}\frac{1+a}{1-a} + \frac{1-a^2}{1+a^2}\sum_{k\in\mathbb{Z}}\frac{(-1)^k}{(2k+1)\pi}e^{-i(2k+1)\omega}, \quad \omega \in [-\pi, \pi].$$

It is easy to establish the relationship between $H(e^{-i\omega})$, $\omega \in [-\pi, \pi]$, and the frequency response $H^L(e^{-i\omega})$, $\omega \in [-\pi, \pi]$ of the LTI system with the ideal low-pass filter, that is,

$$H(e^{-i\omega}) = \frac{a(1+a)}{(1-a)(1+a^2)} + \frac{1-a^2}{1+a^2}H^L(e^{-i\omega}), \quad \omega \in [-\pi, \pi].$$

Simple calculation gives

$$H(e^{-i\omega}) = \begin{cases} \dfrac{(1+a)(1-a+a^2)}{(1-a)(1+a^2)}, & \omega \in (-\frac{\pi}{2}, \frac{\pi}{2}), \\ \dfrac{a(1+a)}{(1-a)(1+a^2)}, & \omega \in [-\pi, -\frac{\pi}{2}] \cup [\frac{\pi}{2}, \pi]. \end{cases}$$

4. Multi-scale spectrum of sequence and implementation

In this section, we introduce a new type of spectrum for sequences, referred to multi-scale spectrum. Different from the traditional Fourier spectrum for sequences, which is essentially suitable for only bandlimited signals, multi-scale spectrum is specifically designed for signals in $\mathbb{B}_{\Omega}^{a}$. Although Fourier spectrum is only suitable for narrow band signal, we apply it to any kind of data. This gives misleading results as we mainly deal with short time signal which is naturally not narrow band signal. We now provide a simple explanation about the difference between Fourier spectrum and multi-scale spectrum. Suppose that we are investigating a continuous signal $f(t)$, $t \in \mathbb{R}$. For computer implementation, we actually deal with adigital signal of length N, namely, $(d_1, d_2, \ldots, d_N)$. By fast Fourier transform (FFT) algorithm, we get the spectrum data $(\hat{d}_1, \hat{d}_2, \ldots, \hat{d}_N)$ of $f(t)$ with the same length as the time domain data. Essentially, in FFT algorithm, we are considering the signal $f(t)$ both time-limited and bandlimited. But the bandwidth of a time-limited signal is infinite. Thus N Fourier spectrum data is not enough to represent the signal $f(t)$ since we cannot get the higher frequency information of $f(t)$ when $f(t)$ has small duration. We can obtain the higher frequency data unless we extend the length of the time domain data of the signal $f(t)$ under study.But by multi-scale spectrum algorithm, we can easily obtain high-frequency information at any band from N data in the time domain.

4.1. *Discrete Fourier transform and FFT revisited*

The definition of multi-scale spectrum is closely connected with sampling theorem of non-bandlimited signals. We need to review some issues related to the definition of discrete Fourier transform (DFT) for sampling data $\{x(n\triangle\}$ of continuous signal $f(t)$ with Nyquist density $\frac{1}{\triangle} = \frac{\Omega}{\pi}$ in classic setting. Sampling theorem suggests engineers to define Fourier spectrum for discrete signal $\{x(n\triangle)\}$ as

$$\hat{x}(\xi) = \sum_n x(n\triangle)e^{-in\triangle\xi}, \quad \xi \in \left[-\frac{\pi}{\triangle}, \frac{\pi}{\triangle}\right]. \tag{4.1.1}$$

The re-constructional formula is given by

$$x(n\triangle) = \frac{\triangle}{2\pi}\int_{-\frac{\pi}{\triangle}}^{\frac{\pi}{\triangle}} \hat{x}(\xi)e^{in\triangle\xi}\mathrm{d}\xi.$$

When the discrete signal $\{x(n\triangle)\}$ is of length N, that is,

$$\{x(n\triangle)\} = \{x(0), x(\triangle), \ldots, x((N-1)\triangle)\},$$

for the purpose of computer implementation, we need to make the spectrum defined in (4.1.1) discrete. This leads to the DFT

$$\hat{x}(\xi_m) = \sum_{n=0}^{N-1} x(n\triangle)e^{-inm\frac{2\pi}{N}}, \quad \text{with} \quad \xi_m = \frac{2\pi}{N\triangle}m, \quad m = 0, 1, \ldots, N-1$$

and the inverse formula

$$x(n\triangle) = \frac{1}{N}\sum_{m=0}^{N-1} \hat{x}(\xi_m)e^{inm\frac{2\pi}{N}}, \quad \text{with} \quad \xi_m = \frac{2\pi}{N\triangle}m, \quad n = 0, 1, \ldots, N-1.$$

It is known that the DFT is an $o(N^2)$ computational procedure. In 1965, Cooley and Tukey [18] raised an algorithm, named the Cooley–Tukey FFT algorithm, which breaks the DFT into smaller DFTs in order to reduce the computational complexity. The Cooley–Tukey FFT algorithm re-expresses the DFT of an arbitrary composite size $N = N_1 N_2$ in terms of smaller DFTs of sizes N_1 and N_2 recursively and thus reduce the computation time to $o(N \log N)$.

4.2. *Multi-scale spectrum*

It is time to define the notion of multi-scale spectrum for discrete sequence. In the last section, we have shown that, for any non-bandlimited signal $f \in \mathbb{B}_\Omega^a$ the generalized sampling theorem holds:

$$f(t) = \frac{1-a}{1+a}\sum_n f(n\triangle)\frac{\sin\theta_a(\frac{\pi}{\triangle}t - n\pi)}{\frac{\pi}{\triangle}t - n\pi}, \quad t \in \mathbb{R}.$$

If we define the ladder-shaped function $g_{a,\Delta}$ (see Figure 2) by

$$g_{a,\triangle}(\xi) = \frac{1-a}{1+a}\begin{cases} 1+a, & \xi \in [-\frac{\pi}{\triangle}, \frac{\pi}{\triangle}) \\ a(1+a), & \xi \in [-\frac{2\pi}{\triangle}, -\frac{\pi}{\triangle}) \cup [\frac{\pi}{\triangle}, \frac{2\pi}{\triangle}) \\ \ldots & \ldots \\ a^n(1+a), & \xi \in [-\frac{(n+1)\pi}{\triangle}, -\frac{n\pi}{\triangle}) \cup [\frac{n\pi}{\triangle}, \frac{(n+1)\pi}{\triangle}) \\ \ldots & \ldots \end{cases} \tag{4.2.1}$$

which has two useful alternative forms

$$g_{a,\triangle}(\xi) = \sum_{n=0}^{\infty} a^n(1-a)\Big(\chi_{[-\frac{(n+1)\pi}{\triangle}, -\frac{n\pi}{\triangle})}(\xi) + \chi_{[\frac{n\pi}{\triangle}, \frac{(n+1)\pi}{\triangle})}(\xi)\Big)$$

and

$$g_{a,\triangle}(\xi) = (1-a)^2\sum_{n=1}^{\infty} a^{n-1}\chi_{[-\frac{n\pi}{\triangle}, \frac{n\pi}{\triangle})}(\xi), \quad \xi \in \mathbb{R},$$

we get that $f \in \mathbb{B}_\Omega^a$ if and only if

$$\hat{f}(\xi) = M_{f,\triangle}(\xi)\, g_a(\xi).$$

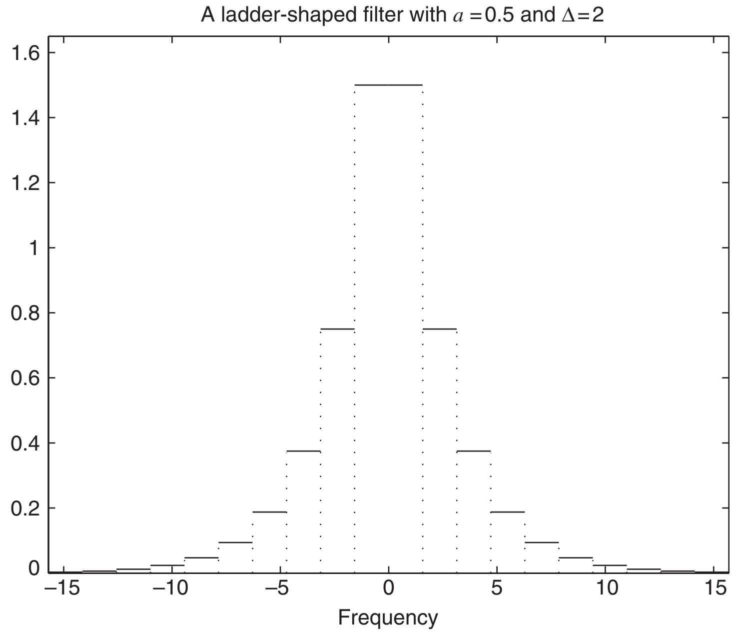

Figure 2. Plot of text color 1l00, 0a00, 0d00 ladder-shaped filter with parameter $a = 0.5$ and $\triangle = 2$.

It is easy to see that the spectrum of the function $f \in \mathbb{B}^a_\Omega$ has the following characteristic property: its high-frequency spectrum is a copy but an exponential-weight compressing of its low-frequency spectrum. We are therefore inspired to define the multi-scale spectrum for the sequence $\{x(n\triangle)\}$

$$\hat{x}^{new}(\xi) = \sum_n x(n\triangle)e^{-in\triangle\xi} g_{a,\triangle}(\xi), \quad \xi \in \mathbb{R}. \tag{4.2.2}$$

THEOREM 4.1 *The function $\hat{x}^{new}$ defined in* (4.2.2) *has the following properties*:

(i) *The function $\hat{x}^{new}$ is neither compactly supported nor periodic*;
(ii) *Riemann–Lebesgue property*:

$$\lim_{|\xi|\to\infty} \hat{x}^{new}(\xi) = 0;$$

(iii) *Inverse formula*:

$$x(n\triangle) = \frac{\triangle}{2\pi(1-a)} \int_{-\frac{\pi}{\triangle}}^{\frac{\pi}{\triangle}} x^{new}(\xi) e^{in\triangle\xi} \mathrm{d}\xi, \quad n \in \mathbb{Z}. \tag{4.2.3}$$

Proof The results (i) and (ii) are a direct conclusion of the definition of the filter $g_{a,\triangle}(\xi)$. To obtain (iii), multiply by $e^{im\triangle}$ on both sides of (4.2.2), integrate over the interval $[-\frac{\pi}{\triangle}, \frac{\pi}{\triangle}]$ and utilize the orthogonality of trigonometric polynomials and the definition of $g_{a,\triangle}(\xi)$ to conclude the formula (4.2.3). ■

The inverse formula states that $\{x(n\triangle\}$ can be reconstructed from its spectrum of low frequencies. Certainly $\{x(n\triangle\}$ can also be obtained by spectrum of high frequencies $\{\xi : |\xi| \in [\frac{n\pi}{\triangle}, \frac{(n+1)\pi}{\triangle})\}$ in different scale n.

4.3. *Fast algorithm for multi-scale spectrum*

It is now right to consider the numerical implementation of multi-scale spectrum for discrete signal $\{x(0),x(\triangle),\dots,x((N-1)\triangle)\}$. We only need to consider the non-negative spectrum since $\hat{x}^{new}(\xi)$ is of the Hermitian property $\overline{\hat{x}^{new}(\xi)} = \hat{x}^{new}(-\xi)$. Set $N_1 = [\frac{N+1}{2}]$ and

$$\xi_m = \frac{2\pi}{N\triangle} m, \quad m = 0, 1, 2, \dots.$$

Note that, for $k = 0, 1, 2, \dots,$

$$\xi_m \in \left[\frac{2k\pi}{\triangle}, \frac{2k\pi+\pi}{\triangle}\right), \quad m = Nk, Nk+1, \dots, Nk+N_1-1$$

and

$$\xi_m \in \left[\frac{2k\pi+\pi}{\triangle}, \frac{2k\pi+2\pi}{\triangle}\right), \quad m = Nk+N_1, Nk+N_1+1, \dots, Nk+N-1.$$

Decomposition algorithm

The spectrum in the low-frequency interval $[0, \frac{2\pi}{\triangle})$ (corresponding to $k=0$) is defined by

$$\begin{cases} \hat{x}^{new}(\xi_m) = (1-a)\sum_{n=0}^{N-1} x(n\triangle)e^{-inm\frac{2\pi}{N}}, & m = 0, 1, \dots, N_1-1 \\ \hat{x}^{new}(\xi_m) = a(1-a)\sum_{n=0}^{N-1} x(n\triangle)e^{-inm\frac{2\pi}{N}}, & m = N_1, N_1+1, \dots, N-1. \end{cases} \tag{4.3.1}$$

The k level detail spectrum ($\xi \in [\frac{2k\pi}{\triangle}, \frac{2k\pi+2\pi}{\triangle})$) is

$$\hat{x}^{new}_k(\xi_m) = a^k \hat{x}^{new}(\xi_{m-Nk}), \quad m = Nk, Nk+1, \dots, Nk+N-1. \tag{4.3.2}$$

Here we denote that $\hat{x}^{new}_0 = \hat{x}^{new}$.

Synthesis algorithm

We now consider the inverse formula. We show that $\{x(n\triangle)\}$ can be reconstructed from low frequency spectrum. In fact, by setting

$$y_m = \frac{1}{1-a}\hat{x}^{new}(\xi_m), \quad m = 0, 1, \dots, N_1-1$$
$$y_m = \frac{1}{a(1-a)}\hat{x}^{new}(\xi_m), \quad m = N_1, N_1+1, \dots, N-1,$$

we obtain

$$x(n\triangle) = \frac{1}{N}\sum_{m=0}^{N-1} y_m e^{inm\frac{2\pi}{N}}. \tag{4.3.3}$$

5. Conclusion

The impulse response of the LTI system corresponding to the ladder-shaped filter is a constant multiplication of the periodic Poisson kernel and Sinc function, from which we define the generalized Sinc function. For any signal whose Fourier transform is the multiplication of an interpolating period function and the ladder-shaped filter, the generalized Shannon-type sampling theorem holds. These kind of signals are non-bandlimited and their high-frequency spectrum are obtained from copying and compressing their low-frequency spectrum. These kind of signals are restrictions on the real line of certain analytic functions in the strip domains parallel to the real axis in the complex plane.

Fourier spectrum for discrete sequences is only suitable for bandlimited signals. For signals of short duration (certainly not bandlimited), FFT algorithm can offer finite frequency information. In practice, however, FFT is mistakenly used to all kinds of signals and thus it may give misleading results [14] . This reason has been stimulating scientists to look for other tools of time-frequency analysis for transient signals such as windowed Fourier transform, Wigner distribution and wavelet, etc. [19]. Multi-scale spectrum is designed for certain non-bandlimited signals characterized by the generalized sampling theorem. Different from Fourier spectrum for discrete sequences, we can obtain frequency information at any band of the original signal. The fast algorithm for multi-scale spectrum is derived based on FFT.

We remark that the associated analytic signal essentially corresponds to the Fourier multiplier $\chi_{\mathbb{R}_+}$ and the associated Hardy function corresponds to the Fourier multiplier $e^{-y|\cdot|}\chi_{\mathbb{R}_+}$. In contrast, the two-sided filter $g_{a,\triangle}$ gives rise to the par-bandlimited signal induced from the original signal that can be holomorphically extended to the strips. We note that $\lim_{\triangle\to\infty}\frac{\sin\theta_a(\triangle t)}{t} = (1+a)\delta(t)$ in the sense of tempered distribution.

Acknowledgement

This work was supported in part by the National Research Foundation for the Doctoral Program of Higher Education of China (SRFDP) under grant 20070512001 and by research grant of the University of Macau No. RG092/04-05S/C118/QT/FST and Macao Sci. and Tech. Develop. Fund 051/2005/A.

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