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Discrete-time analytic signals and Bedrosian product theorems

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ABSTRACT

The aim of this paper is to discuss discrete-time analytic signals and to provide the derivation of Bedrosian product theorem for discrete-time Hilbert transform. With the aid of the continuous-time analytic signals we provide a new approach to produce the discrete-time analytic signals.

1. Introduction

Discrete-time signals are represented mathematically as sequences of numbers. A sequence of numbers x, in which the nth number in the sequence is denoted $x[n]$, is formally written as

$$x = \{x[n]\colon n \in \mathbb{Z}\},$$

where $\mathbb{Z}$ denotes the set of integers.

The instantaneous amplitude and phase are basic concepts in all the questions dealing with modulation of signals appearing especially in communications or information processing [3,4,13]. A purely monochromatic signal such as $a\cos(\omega_0 n + \phi_0)$ cannot transmit any information. For this purpose, a modulation is required. One of the simplest possible is to introduce amplitude modulation [13]. Often, the wave is mathematically modeled as

$$x[n] = a[n]\cos(\phi[n]), \tag{1.1}$$

where $a[n]$ is then said to be the amplitude modulation of the wave, $\phi[n]$ is the phase. However, this representation is clearly not unique, for there are many pairs of sequences that can play the role as amplitude signal. Therefore we need to represent a signal $x[n]$ in a determined way $\{a[n], \phi[n]\}$. For this purpose, we associate with $x[n]$ its analytic signal. We use the discrete Hilbert transform [8,12] to generate the so-called analytic discrete-time signal according to

$$z[n] = x[n] + i\mathcal{H}_d x[n]. \tag{1.2}$$

The $\mathcal{H}_d x[n]$ represents the discrete Hilbert transform of the sequence x defined as:

$$\mathcal{H}_d x[n] = \sum_{m\in\mathbb{Z}} h[n-m]x[m], \tag{1.3}$$

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where

$$h[n] = \begin{cases} \frac{2}{\pi} \frac{\sin^2(\pi \frac{n}{2})}{n}, & n \neq 0, \\ 0, & n = 0. \end{cases} \tag{1.4}$$

In continuous-time signal theory, the associated signal $z(t)$ can be shown to be the boundary value of an analytic function in the upper half plane and thus is called an analytic signal. Although analyticity has no formal meaning for sequence, we will nevertheless apply the same terminology to complex sequence whose imaginary part is the discrete Hilbert transform of its real part [8,12].

Saying that $z[n] = a[n]e^{i\phi[n]}$ is an analytic sequence is equivalent to saying that the discrete Hilbert transform of $x[n] = a[n]\cos(\phi[n])$ is equal to $y[n] = a[n]\sin(\phi[n])$. It is therefore appropriate to make use of the so-called Bedrosian product theorem dealing with the discrete Hilbert transform of a product of two real signals $x_1[n]$ and $x_2[n]$. Under some suitable conditions Bedrosian product theorem says that

$$\mathcal{H}_d(x_1 x_2)[n] = x_1[n]\mathcal{H}_d(x_2)[n]. \tag{1.5}$$

Under suitable conditions for $a[n]$ and $\cos(\phi[n])$ one could have

$$\mathcal{H}_d\big(a[\cdot]\cos\big(\phi[\cdot]\big)\big)[n] = a[n]\mathcal{H}_d\big(\cos\big(\phi[\cdot]\big)\big)[n]. \tag{1.6}$$

If, in addition,

$$\mathcal{H}_d\big(\cos\big(\phi[\cdot]\big)\big)[n] = \sin\big(\phi[\cdot]\big)[n] \tag{1.7}$$

then we have the quadrature signal $a[n]e^{i\phi[n]}$ coincides with its associated analytic signal $x[n] + i\mathcal{H}_d x[n]$.

Discrete Hilbert transforms have played a useful role in signal analysis and have also been of practical importance in various signal processing systems.

The purpose of this paper is twofold. The first is to discuss the notion of an analytic sequence and its use in providing a unified approach to the derivation of Bedrosian product theorems for discrete Hilbert transform. The second is to exhibit a method to deduce analytic sequences through the continuous-time analytic signals. For this purpose we discuss the relations of Hilbert transform and discrete Hilbert transform for band-limited signals.

The writing plan of the paper is as follows. Section 2 is devoted to survey the discrete-time Fourier and discrete Hilbert transform. The analytic sequence is introduced as a complex sequence if its spectrum is zero on the unit circle for $-\pi < \omega < 0$. In Section 3 we establish the Bedrosian product theorems for analytic sequences. A method of producing analytic sequences is presented in Section 4. We conclude that some analytic sequences can be obtained by the aid of the continuous-time analytic signals.

2. Discrete-time Fourier and Hilbert transforms

A discrete-time Fourier transform converts an infinite sequence of data values into a periodic function. Let $z[n]$ be a sequence with n taking all integers. Its discrete-time Fourier transform is the complex-valued periodic function

$$Z\big(e^{i\omega}\big) = \sum_{n=-\infty}^{\infty} z[n]e^{-i\omega n}. \tag{2.1}$$

The sequence can then be represented as (see, e.g. [12])

$$z[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} Z\big(e^{i\omega}\big)e^{i\omega n}\,d\omega, \quad n \in \mathbb{Z}. \tag{2.2}$$

Eqs. (2.1) and (2.2) together form a Fourier representation for the sequence. Eq. (2.2), the inverse Fourier transform, is a synthesis formula. Eq. (2.1), discrete-time Fourier transform, is a series for computing $Z(e^{i\omega})$ from the sequence $z[n]$, i.e., for analyzing the sequence $z[n]$ to determine how much of each frequency component is required to synthesis $z[n]$ using Eq. (2.2).

We consider sequences for which the Fourier transforms are zero on $\omega \in [-\pi, 0)$. Thus, with $z[n]$ denoting the sequence and $Z(e^{i\omega})$ its Fourier transform, we require that

$$Z\big(e^{i\omega}\big) = 0, \quad -\pi \leqslant \omega < 0.$$

The sequence $z[n]$ corresponding to $Z(e^{i\omega})$ must be complex. Therefore, we express $z[n]$ as

$$z[n] = z_R[n] + iz_I[n],$$

where $z_R[n]$ and $z_I[n]$ are real sequences. If $Z_R(e^{i\omega})$ and $Z_I(e^{i\omega})$ denote the Fourier transform of the real sequence $z_R[n]$ and $z_I[n]$, respectively, then

$$Z\big(e^{i\omega}\big) = Z_R\big(e^{i\omega}\big) + iZ_I\big(e^{i\omega}\big).$$

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Alternatively, we can relate $Z_R(e^{i\omega})$ and $Z_I(e^{i\omega})$ directly by

$$Z_I\big(e^{i\omega}\big) = -i\,\mathrm{sgn}(\omega) Z_R\big(e^{i\omega}\big) = H\big(e^{i\omega}\big) Z_R\big(e^{i\omega}\big), \tag{2.3}$$

where

$$H\big(e^{i\omega}\big) = -i\,\mathrm{sgn}(\omega) \tag{2.4}$$

and

$$\mathrm{sgn}(\omega) = \begin{cases} 1, & \text{for } 0 < \omega < \pi, \\ 0, & \text{for } \omega = 0, \\ -1, & \text{for } -\pi < \omega < 0 \end{cases} \tag{2.5}$$

is the signum function.

According to Eq. (2.3), $z_I[n]$ can be obtained by processing $z_R[n]$ with a linear time-invariant discrete-time system with frequency response $H(e^{i\omega})$, as given by Eq. (2.4). This frequency response has unity magnitude, a phase angle of $-\pi/2$ for $0 < \omega < \pi$, and a phase angle of $+\pi/2$ for $-\pi < \omega < 0$. Such a system is called an ideal 90-degree phase shifter. The impulse response $h[n]$ of 90-degree phase shifter, corresponding to the frequency response $H(e^{i\omega})$ given in Eq. (2.4), is

$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{0} ie^{i\omega n}\,d\omega - \frac{1}{2\pi}\int_{0}^{\pi} ie^{i\omega n}\,d\omega = \begin{cases} \frac{2}{\pi}\frac{\sin^2(\pi\frac{n}{2})}{n}, & n \neq 0, \\ 0, & n = 0. \end{cases}$$

Alternatively, when it is clear that we are considering an operation on a sequence, the 90-degree phase shifter is the discrete Hilbert transform.

With $z[n]$ denoting the sequence and $Z(e^{i\omega})$ its discrete-time Fourier transform, we can rewrite the discrete Hilbert transform $\mathcal{H}_d$, see (1.3), as follows

$$\mathcal{H}_d z[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H\big(e^{i\omega}\big) Z\big(e^{i\omega}\big) e^{-i\omega n}\,d\omega, \quad n \in \mathbb{Z}. \tag{2.6}$$

In a style similar to the analog signals, we see that a complex sequence is analytic if its spectrum is zero on the unit circle for $-\pi < \omega < 0$.

Taking into account the fact that $2\sin^2(\pi\frac{n}{2}) = 1 - \cos\pi n$, the discrete Hilbert transform $\mathcal{H}_d z$ of the sequence z can be expressed as

$$\begin{aligned} \mathcal{H}_d z[n] &= \sum_{m\in\mathbb{Z}} h[n-m] z[m] \\ &= \frac{1}{\pi}\sum_{m\in\mathbb{Z}} \left\{\frac{1-\cos\pi(n-m)}{n-m}\right\} z[m] \\ &= \begin{cases} \frac{2}{\pi}\sum_{m \text{ odd}} \frac{z[m]}{n-m}, & \text{for even } n, \\ \frac{2}{\pi}\sum_{m \text{ even}} \frac{z[m]}{n-m}, & \text{for odd } n. \end{cases} \end{aligned}$$

Direct calculations show that the inverse relationship is given by

$$z[m] = \begin{cases} -\frac{2}{\pi}\sum_{n \text{ odd}} \frac{\mathcal{H}_d z[n]}{m-n}, & \text{for even } m, \\ -\frac{2}{\pi}\sum_{n \text{ even}} \frac{\mathcal{H}_d z[n]}{m-n}, & \text{for odd } m. \end{cases}$$

Therefore we deduce that $\mathcal{H}_d^{-1} = -\mathcal{H}_d$, or $\mathcal{H}_d^2 = -\mathcal{I}$, where $\mathcal{I}$ denotes the identity operator.

3. Bedrosian theorems for discrete Hilbert transform

Recall that a sequence of complex samples of the form

$$z[n] = x[n] + iy[n], \quad n \in \mathbb{Z}$$

is called an analytic sequence if $y[n]$ is the discrete Hilbert transform of $x[n]$. This discrete signal does not represent an analytic signal in the sense of the definition of analytic function. The use of this name is justified by the spectral properties.

As trivial example of unit analytic sequences $z[n] = e^{i\omega_0 n}$ we have, for $\omega_0 > 0$,

$$\mathcal{H}_d\big(\cos[\omega_0 k]\big)[n] = \sin[\omega_0 n] \quad \text{and} \quad \mathcal{H}_d\big(\sin[\omega_0 k]\big)[n] = -\cos[\omega_0 n].$$

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An alternative representation of an analytic sequence is in terms of magnitude and phase; i.e., $z[n]$ can be expressed in the specific way in relation to the Hilbert transform as

$$z[n] = a[n]e^{i\phi[n]},$$

where $a[n] = (x^2[n] + y^2[n])^{1/2}$ and

$$\phi[n] = \arctan\left(\frac{y[n]}{x[n]}\right).$$

By the definition of the discrete Hilbert transform, we have $\mathcal{H}_d^2(z)[n] = -z[n]$. On the other hand, for the analytic sequence $z[n]$, we have

$$\mathcal{H}_d z[n] = \mathcal{H}_d x[n] + i\mathcal{H}_d y[n] = y[n] - ix[n] = -iz[n].$$

Furthermore we can easily verify that any complex sequence $z[n]$ satisfying the equation $\mathcal{H}_d z[n] = -iz[n]$ is an analytic sequence. It follows that analytic sequences can be regarded as the eigenfunctions of the discrete Hilbert transform operator corresponding to the eigenvalue $-i$. This observation yields the spectrum characterization of the analytic sequences.

Lemma 3.1. *Let $Z(e^{i\omega})$ be the discrete-time Fourier transform of $z[n]$. Then, $z[n]$ is an analytic sequence if and only if $Z(e^{i\omega}) = 0$ for $-\pi < \omega < 0$.*

Proof. Note that $z[n]$ is an analytic sequence if and only if $\mathcal{H}_d z[n] = -iz[n]$. By taking the discrete-time Fourier transforms on both sides we have from (2.6)

$$H(e^{i\omega})Z(e^{i\omega}) = -iZ(e^{i\omega}).$$

Thus $\mathcal{H}_d z[n] = -iz[n]$ if and only if $Z(e^{i\omega}) = 0$ for $-\pi < \omega < 0$. $\square$

Next, we note that if $z_1[n]$ and $z_2[n]$ are analytic sequences and α, β are complex scalars, then $\alpha z_1[n] + \beta z_2[n]$ is analytic. Moreover, it is clear from Lemma 3.1 that the convolution of $z_1[n]$ and $z_2[n]$:

$$(z_1 * z_2)[n] = \sum_{k\in\mathbb{Z}} z_1[n-k]z_2[k]$$

is also analytic. In particular, the discrete Hilbert transform $\mathcal{H}_d z[n]$ of analytic sequence $z[n]$ is analytic.

We exhibit the Bedrosian product theorem for analytic sequences.

Theorem 3.2. *Suppose that $z_1[n]$ and $z_2[n]$ are complex sequences with discrete-time Fourier transforms $Z_1(e^{i\omega})$ and $Z_2(e^{i\omega})$. Then*

$$\mathcal{H}_d(z_1 z_2)[n] = z_1[n]\mathcal{H}_d(z_2)[n]$$

if there exists a nonnegative number $\sigma < \pi$ such that

$$Z_1(e^{i\omega}) = 0, \quad \text{for } 0 < \sigma < |\omega| < \pi, \quad \text{and} \quad Z_2(e^{i\mu}) = 0, \quad \text{for } 0 < |\mu| \leqslant \sigma < \pi.$$

Proof. In terms of their discrete-time Fourier transform, the product $z_1[n]z_2[n]$ can be rewritten as

$$z_1[n]z_2[n] = \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} Z_1(e^{i\omega})Z_2(e^{i\mu})e^{i(\omega+\mu)n}\,d\omega\,d\mu, \quad n \in \mathbb{Z}. \tag{3.1}$$

For the complex exponential sequence $e_{\omega_0}[n] = \exp[i\omega_0 n]$, $-\pi < \omega_0 \leqslant \pi$, its discrete-time Fourier transform is the periodic impulse train in the distribution sense (see, e.g., [12])

$$E_{\omega_0}(e^{i\omega}) = \sum_{r\in\mathbb{Z}} 2\pi\delta[\omega - \omega_0 + 2\pi r]. \tag{3.2}$$

Here we denote by $\delta[n]$ the discrete-time impulse sequence or Dirac delta impulse:

$$\delta[n] = \begin{cases} 1, & n = 0, \\ 0, & n \neq 0. \end{cases}$$

Consequently we have

$$\begin{aligned}\mathcal{H}_d(e_{\omega_0})[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} H\big(e^{i\omega}\big)E_{\omega_0}\big(e^{i\omega}\big)e^{i\omega n}\,d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\big(-i\,\mathrm{sgn}(\omega)\big)E_{\omega_0}\big(e^{i\omega}\big)e^{i\omega n}\,d\omega.\end{aligned}$$

Because the integration of $E_{\omega_0}(e^{i\omega})$ extends only over one period, from $-\pi<\omega<\pi$, we need include only the $r=0$ term from Eq. (3.2). Thus we compute the discrete-time Hilbert transform of complex exponential sequence as

$$\begin{aligned}\mathcal{H}_d(e_{\omega_0})[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\big(-i\,\mathrm{sgn}(\omega)\big)\,E_{\omega_0}\big(e^{i\omega}\big)e^{i\omega n}\,d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\big(-i\,\mathrm{sgn}(\omega)\big)2\pi\,\delta[\omega-\omega_0]e^{i\omega n}\,d\omega \\ &= -i\,\mathrm{sgn}(\omega_0)\exp(i\omega_0 n). \end{aligned} \tag{3.3}$$

Taking the discrete Hilbert transform on both sides in (3.1) and using relation (3.3) we have, for $n\in\mathbb{Z}$,

$$\mathcal{H}_d(z_1z_2)[n] = \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\big(-i\,\mathrm{sgn}(\omega+\mu)\big)Z_1\big(e^{i\omega}\big)Z_2\big(e^{i\mu}\big)e^{i(\omega+\mu)n}\,d\omega\,d\mu. \tag{3.4}$$

Note that the supports of $Z_1(e^{i\omega})$ and $Z_2(e^{i\mu})$ are $0<|\omega|<\sigma\leqslant\pi$ and $0<\sigma<|\mu|\leqslant\pi$ respectively. It follows that the support of $Z_1(e^{i\omega})Z_2(e^{i\mu})$ is

$$[-\sigma,\sigma]\times[\sigma,\pi]\cup[-\sigma,\sigma]\times[-\pi,-\sigma].$$

Consequently, it is clear that

$$-i\,\mathrm{sgn}(\omega+\mu) = -i\,\mathrm{sgn}(\mu),$$

over the regions of integration in which the integrand in (3.4) is nonvanishing. Thus, we conclude from (3.4) that

$$\begin{aligned}\mathcal{H}_d(z_1z_2)[n] &= \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\big(-i\,\mathrm{sgn}(\mu)\big)Z_1\big(e^{i\omega}\big)Z_2\big(e^{i\mu}\big)e^{i(\omega+\mu)n}\,d\omega\,d\mu \\ &= z_1[n]\frac{1}{2\pi}\int_{-\pi}^{\pi}\big(-i\,\mathrm{sgn}(\mu)\big)Z_2\big(e^{i\mu}\big)e^{i\mu n}\,d\mu \\ &= z_1[n]\mathcal{H}_d z_2[n], \quad n\in\mathbb{Z}.\end{aligned}$$

The proof of the theorem is complete. □

The proof of Theorem 3.2 is in spirit of that of Bedrosian [1]. We can also follow the same line as that of Nuttall and Bedrosian [11] to conclude the following Bedrosian type product theorem:

Theorem 3.3. *Suppose that $z_1[n]$ and $z_2[n]$ are complex sequences with discrete-time Fourier transforms $Z_1(e^{i\omega})$ and $Z_2(e^{i\omega})$. Then*

$$\mathcal{H}_d(z_1z_2)[n] = z_1[n]\mathcal{H}_d(z_2)[n] \tag{3.5}$$

if there exists a nonnegative number $\sigma<\pi$ such that

$$Z_1\big(e^{i\omega}\big)=0, \quad \textit{for } -\pi<\omega<-\sigma, \quad \textit{and} \quad Z_2\big(e^{i\mu}\big)=0, \quad \textit{for } -\pi<\mu<\sigma<\pi.$$

Theorem 3.3 tells us that if $z_1[n]$ and $z_2[n]$ are analytic sequences then the Bedrosian identity (3.5) remains true.

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4. Discrete-time analytic signals

We would like to investigate sequences for which they are analytic. In order to do so we employ the continuous-time analytic signals. A systematic study on analytic signals with nonlinear phase is carried out in Refs. [5,14–16]. We recall that a complex signal $f(t)$ is said to be analytic if its imaginary part is the Hilbert transform of its real part (see, e.g., [6,8,13]), i.e.,

$$f(t) = f_R(t) + i f_I(t),$$

where $f_I(t) = \mathcal{H}(f_R)(t)$ and the Hilbert transform $\mathcal{H}$ is defined as

$$\mathcal{H}(f_R)(t) := \frac{1}{\pi}\,\text{p.v.}\int_{-\infty}^{\infty} \frac{f_R(s)}{t-s}\,ds.$$

Let $\mathbb{R}$ be the field of real numbers and let $\mathbb{C}$ be the field of complex numbers. We denote the upper-half complex plane by $\mathbb{C}^+$. We deal with Hardy spaces H^p of the upper-half complex plane $\mathbb{C}^+$. For $0 < p < \infty$, $f \in H^p$ means that f is analytic in $\mathbb{C}^+$ and that

$$\|f\|_p^p = \sup_{0<y<\infty} \int_{\mathbb{R}} \left|f(t+iy)\right|^p dt < \infty.$$

For $p \geqslant 1$, H^p are Banach spaces. For $p < 1$, H^p are complete metric spaces under the metric $d(f,g) = \|f-g\|_p^p$, (see [7]).

Functions in H^p have nontangential limits almost everywhere on the real line $\mathbb{R}$ and the norms just introduced coincide with the $L^p(\mathbb{R})$ norms of these limits. Thus H^p may be viewed as a closed subspace of the corresponding $L^p(\mathbb{R})$ space. We will not distinguish between the analytic functions in H^p and their boundary limit functions, and we will usually compute norms by resorting to the $L^p(\mathbb{R})$ norms on the boundary.

Qian [14] explored connections between eigenfunctions of Hilbert transformation and functions in Hardy H^p spaces. A complex function $f(t) = \rho(t)(\cos\theta(t) + i\sin\theta(t))$, with $\rho \geqslant 0$ and $\rho \in L^p(\mathbb{R}), 1 \leqslant p \leqslant \infty$, satisfies $\mathcal{H}(\rho(\cdot)\cos\theta(\cdot)) = \rho(t)\sin\theta(t)$ if and only if it is the boundary value of a function in Hardy space H^p of the upper-half complex plane $\mathbb{C}^+$.

To fit to our need, we restate the characterization of analytic signal in terms of boundary values of the functions in the Hardy space H^2 of the upper-half complex plane $\mathbb{C}^+$.

We denote the Fourier transform of a function f by $\widehat{f}$

$$\widehat{f}(\xi) := \int_{-\infty}^{\infty} f(t) e^{-it\xi}\,dt.$$

Therefore we have by the inverse Fourier transform

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}(\xi) e^{it\xi}\,d\xi.$$

We obtain the multiplier form in frequency domain of the Hilbert transform $\mathcal{H}(f)(t)$ of function f

$$\mathcal{H}(f)(t) := \frac{1}{2\pi}\int_{-\infty}^{\infty} (-i\,\mathrm{sgn}\,\xi)\,\widehat{f}(\xi) e^{i\xi t}\,d\xi.$$

If a complex function $f(t) = f_R(t) + i f_I(t)$ is in $L^2(\mathbb{R})$ then $f_R \in L^2(\mathbb{R})$. If $\mathcal{H}(f_R)(t) = f_I(t)$ then we get

$$f(t) = f_R(t) + i f_I(t) = f_R(t) + i\mathcal{H}(f_R)(t) = \frac{1}{\pi}\int_0^{\infty} \widehat{f_R}(\xi) e^{i\xi t}\,d\xi.$$

The function f can be analytically extended to the upper-half complex plane $\mathbb{C}^+$. We denote z the complex variable in the rest of this section. Note that $f_R \in L^2(\mathbb{R})$. If we let $f(z) = \frac{1}{\pi}\int_0^{\infty} \widehat{f_R}(\xi) e^{i\xi z}\,d\xi$, it is well known that $f(z)$ is holomorphic in the upper-half complex plane $\mathbb{C}^+$ and $f(t)$ is the boundary value of $f(z)$ (see, e.g. [10,17]). That means $f \in H^2$ and $f(t)$ is the boundary value of f.

Suppose that $f(t)$ is the boundary value of an H^2 function. Then $\mathrm{supp}(\widehat{f}) \subset (0, +\infty)$ and we have the following representation for f, see, e.g. [10,17],

$$f(t) = \frac{1}{2\pi}\int_0^{\infty} \widehat{f}(\xi) e^{it\xi}\,d\xi.$$

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We may deduce that

$$\begin{aligned} f(t) &= \frac{1}{2\pi}\int_0^\infty \widehat{f}(\xi)e^{it\xi}\,d\xi \\ &= \frac{1}{2}\left\{\frac{1}{2\pi}\int_{-\infty}^\infty (1+\operatorname{sgn}\xi)\widehat{f}(\xi)e^{it\xi}\,d\xi\right\} \\ &= \frac{1}{2}\frac{1}{2\pi}\int_{-\infty}^\infty \widehat{f}(\xi)e^{it\xi}\,d\xi + \frac{1}{2}\frac{1}{2\pi}\int_{-\infty}^\infty \operatorname{sgn}\xi\,\widehat{f}(\xi)e^{it\xi}\,d\xi \\ &= \frac{1}{2}f(t) + i\frac{1}{2}\mathcal{H}(f)(t). \end{aligned} \tag{4.1}$$

Further we write $f(t) = f_R(t) + if_I(t)$ for an H^2 function f. We may deduce from (4.1) that

$$f_R(t) + if_I(t) = f_R(t) + i\mathcal{H}(f_R)(t). \tag{4.2}$$

Comparing the real part with the imaginary in (4.2) yields $\mathcal{H}(f_R)(t) = f_I(t)$.

Summarizing up the discussion above we conclude that

Lemma 4.1. *Let a complex function $f(t) = f_R(t) + if_I(t)$ be in $L^2(\mathbb{R})$. Then $\mathcal{H}(f_R)(t) = f_I(t)$ if and only if f is the boundary value of a function in Hardy space H^2 of the upper-half complex plane $\mathbb{C}^+$.*

In this case we have $\mathcal{H}f(t) = -if(t)$.

We would like to explore connections between the discrete-time Hilbert transform and the continuous-time one. We introduce the space E^p_τ, $p > 0$, of entire functions f of exponential type τ for which

$$\|f\|_p^p = \int_{\mathbb{R}} \left|f(t)\right|^p dt < \infty.$$

E^p_τ, $p > 0$, is clearly a subspace of $L^p(\mathbb{R})$. Recall that an entire function f is of exponential type τ if $f(z) = \mathcal{O}(e^{(\tau+\varepsilon)|z|})$ for all $\varepsilon > 0$. The totality of all entire functions of exponential type at most π that are square integrable on the real axis is know as the Paley–Wiener space and will be designated by E^2.

The celebrated theorem of Paley–Wiener for E^2-functions says: For an entire function f to belong to E^2, it is necessary and sufficient that there exist $\psi \in L^2(-\pi,\pi)$ such that

$$f(z) = \int_{-\pi}^{\pi} \psi(t)e^{izt}\,dt.$$

For $f \in E^p_\tau$ we have Plancherel–Pólya inequality. Let $p, \tau > 0$ and $f \in E^p_\tau$. For $y \in \mathbb{R}$ we have

$$\int_{\mathbb{R}} \left|f(t+iy)\right|^p dt \leqslant e^{p\tau|y|}\int_{\mathbb{R}} \left|f(t)\right|^p dt. \tag{4.3}$$

It follows that the Plancherel–Pólya inequality (4.3) implies that the map $f \to e^{i\pi z}f$ is an isometry from E^p into H^p.

For the sequel we essentially consider E^2 and H^2. The analogous discussion can be considered for the E^p and H^p for $1 < p < \infty$. E^2 is the isometric image of $L^2(-\pi,\pi)$ under the inverse Fourier transform. Central to the E^2 theory is the so-called sinc function

$$\operatorname{sinc}(z) = \frac{\sin \pi z}{\pi z}.$$

We denote by $\chi_{[-\pi,\pi]}(t)$ the characteristic function on $[-\pi,\pi]$. Since $\operatorname{sinc}(z-n)$ is the image of $\chi_{[-\pi,\pi]}(t)e^{-int}/\sqrt{2\pi}$ under inverse Fourier transform, the collection $\{\operatorname{sinc}(z-n)\}_{n\in\mathbb{Z}}$ is an orthonormal basis of E^2. This observation yields the cardinal series representation of a function f in E^2

$$f(t) = \sum_{n\in\mathbb{Z}} f(n)\operatorname{sinc}(t-n). \tag{4.4}$$

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We note that the Hilbert transform of sinc function is

$$\mathcal{H}\,\mathrm{sinc}(t) = \frac{1}{\pi}\,\mathrm{p.v.}\int_{\mathbb{R}} \frac{\mathrm{sinc}(s)}{t-s}\,ds = \frac{1-\cos\pi t}{\pi t}.$$

From the cardinal series representation (4.4) of a function f in E^2 we get

$$\begin{aligned}\mathcal{H}f(t) &= \sum_{n\in\mathbb{Z}} f(n)\frac{1}{\pi}\,\mathrm{p.v.}\int_{\mathbb{R}} \frac{\mathrm{sinc}(s-n)}{t-s}\,ds \\ &= \sum_{n\in\mathbb{Z}} f(n)\frac{1-\cos\pi(t-n)}{\pi(t-n)}.\end{aligned}$$

Finally we arrive at

$$\begin{aligned}\mathcal{H}f(k) &= \sum_{n\in\mathbb{Z}} f(n)\frac{1-\cos\pi(k-n)}{\pi(k-n)} \\ &= \sum_{n\in\mathbb{Z}} h[k-n]f[n] \\ &= \mathcal{H}_d f[k], \quad k\in\mathbb{Z},\end{aligned}$$

where $h[n]$ is defined as in (1.4).

Summarizing up the statements above we form the following lemma (see [9]).

Lemma 4.2. *Let $\mathcal{H}$ and $\mathcal{H}_d$ denote the Hilbert transforms of continuous-time and discrete-time signals respectively. For f in E^2 we have*

$$\mathcal{H}f(k) = \mathcal{H}_d f[k], \quad k\in\mathbb{Z}. \tag{4.5}$$

Relation (4.5) tells us that the Hilbert transform and the discrete Hilbert transform are analogous for E^2 functions. That is to say, for a band-limited signal f whose Fourier transform is supported in the interval $[-\pi,\pi]$, the sampling sequence $\{\mathcal{H}f(n)\}$ at integer points of the Hilbert transform of f is completely determined by the discrete Hilbert transform of the sequence $\{f[n]\}$.

Theorem 4.3. *Let f be an entire function of exponential type π. Then, $\{x[n] = e^{i\pi\frac{n}{2}} f(\frac{n}{2})\}$ is an analytic sequence.*

Proof. It is known that the map $f \to e^{i\pi z} f$ is an isometry from E^2 into H^2, due to the Plancherel–Pólya's inequality [2]. Recall that for functions $f(t)$ and $g(t) = e^{i\pi t} f(t)$ their Fourier transforms $F(\omega)$ and $G(\omega)$ are related with $G(\omega) = F(\omega-\pi)$. If the support of $F(\omega)$ is contained in $[-\pi,\pi]$ then the support of $G(\omega)$ is contained in $[0,2\pi]$. Therefore the support of Fourier transform $2G(2\omega)$ of $g_1(t) = g(t/2)$ is contained in $[0,\pi]$. It follows that $g_1(t) = g(t/2)$ can be expanded to the cardinal series

$$g_1(t) = \sum_{n\in\mathbb{Z}} g_1(n)\,\mathrm{sinc}(t-n) = \sum_{n\in\mathbb{Z}} g_1(n)\frac{\sin\pi(t-n)}{\pi(t-n)}. \tag{4.6}$$

Taking the Hilbert transform on both sides in (4.6), we get

$$\mathcal{H}g_1(t) = \sum_{n\in\mathbb{Z}} g_1(n)\frac{1-\cos\pi(t-n)}{\pi(t-n)}. \tag{4.7}$$

Noticing that $\mathcal{H}(g_1) = -ig_1$ for the analytic signal g_1 we have

$$-ig_1(t) = \sum_{n\in\mathbb{Z}} g_1(n)\frac{1-\cos\pi(t-n)}{\pi(t-n)}. \tag{4.8}$$

Setting $t=k$ in (4.8) we have

$$\begin{aligned}-ig_1(k) &= \sum_{n\in\mathbb{Z}} g_1(n)\frac{1-\cos\pi(k-n)}{\pi(k-n)} \\ &= \sum_{n\in\mathbb{Z}} g_1[n]h[k-n] \\ &= \mathcal{H}_d g_1[k], \quad k\in\mathbb{Z}.\end{aligned}$$

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Owing to $g_1[n] = e^{i\pi \frac{n}{2}} f(\frac{n}{2})$, we have

$$\mathcal{H}_d\left\{e^{i\pi \frac{n}{2}} f\left(\frac{n}{2}\right)\right\}[k] = \mathcal{H}_d g_1[k] = -ig_1(k) = -ie^{i\pi \frac{k}{2}} f\left(\frac{k}{2}\right), \quad k \in \mathbb{Z}.$$

This concludes that $\{x[n] = e^{i\pi \frac{n}{2}} f(\frac{n}{2})\}$ is an analytic sequence. $\square$

Theorem 4.3 exhibits a procedure how to obtain an analytic sequence. We first choose an entire function f of exponential type π and modulate it by $e^{i\pi t}$ and then sample at $n/2$.

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