On Sets of Zeroes of Clifford Algebra-Valued Polynomials

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Abstract : In this note we study zeroes of Clifford algebra-valued polynomials. We prove that if such a polynomial has only real coefficients, then it has two types of zeroes: the real isolated zeroes and the spherical conjugate ones. The total number of zeroes does not exceed the degree of the polynomial. We also present a technique for computing the zeroes.

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1. Introduction

There has been an ample amount of literature discussing zeros of polynomials. This note concerns about polynomials of one hyper-complex-variable (hyper-complex valued polynomials), while the coefficients may be real, or complex, or hyper-complex constants. By hyper-complex numbers we mean quaternions, octonions, several complex variables or Clifford numbers. Niven in [3, 4] first studied zeroes of hyper-complex polynomials which further led to the article by Eilenberg and Niven [5] where a fundamental theorem for quaternionic polynomials was established. In [6], they proved that any quaternionic polynomial of degree $n \geq 1$ has at least one zero and there should be two types of zeroes: They are either isolated or spherical ones. In [7] zeroes of quaternionic and octonionic polynomials with real coefficients are also studied. In that paper, the authors found that the root-set of such a polynomial is a union of a finite number of codimension 2 Euclidean spheres together with a finite number of real points. In [8], roots of polynomials with bicomplex coefficients are studied. To the authors knowledge, in the higher dimensional cases under the structure of Clifford algebra, there has been no such results.

In this article, we first study zeroes of Clifford algebra-valued polynomials. For the special cases, we consider paravector-valued polynomials with real coefficients. Using a technical method, we introduce a one-to-one correspondence between such a polynomial and a complex polynomial and then extend the results in [6].

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We first give some basic knowledge in relation to Clifford algebra ([1,2]). Let $\mathbf{e}_1, ..., \mathbf{e}_m$ be *basic elements* satisfying $\mathbf{e}_i\mathbf{e}_j + \mathbf{e}_j\mathbf{e}_i = -2\delta_{ij}$, where $\delta_{ij} = 1$ if $i = j$; and $\delta_{ij} = 0$ otherwise, $i, j = 1, 2, \cdots, m$. Let

$$\mathbf{R}^m = \{\underline{x} = x_1\mathbf{e}_1 + \cdots + x_m\mathbf{e}_m : x_j \in \mathbf{R}, j = 1, 2, \cdots, m\}$$

be identical with the usual Euclidean space $\mathbf{R}^m$, and

$$\mathbf{R}_1^m = \{x = x_0\mathbf{e}_0 + \underline{x} : x_0 \in \mathbf{R}, \underline{x} \in \mathbf{R}^m\}, \text{ where } \mathbf{e}_0 = 1.$$

An element in $\mathbf{R}_1^m$ is called a *paravector*. For $x \in \mathbf{R}_1^m$, it consists of a scalar part and a vector part. We use the dotations

$$x_0 = \mathrm{Sc}(x), \ \underline{x} = \mathrm{Vec}(x).$$

The real (or complex) Clifford algebra generated by $\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_m$, denoted by $\mathbf{R}^{(m)}$ (or $\mathbf{C}^{(m)}$), is the associative algebra generated by $\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_m$ over the real (or complex) field $\mathbf{R}$ (or $\mathbf{C}$). A general element in $\mathbf{R}^{(m)}$ (or $\mathbf{C}^{(m)}$), therefore, is of the form $x = \sum_S x_S\mathbf{e}_S$, where $\mathbf{e}_S = \mathbf{e}_{i_1}\mathbf{e}_{i_2}\cdots\mathbf{e}_{i_l}, x_S \in \mathbf{R}$ (or $\mathbf{C}$), and S runs over all the ordered subsets of $\{1, 2, \cdots, m\}$, namely

$$S = \{1 \le i_1 < i_2 < \cdots < i_l \le m\}, \quad 1 \le l \le m.$$

We define the conjugation of $\mathbf{e}_S$ to be $\overline{\mathbf{e}}_S = \overline{\mathbf{e}}_{i_l}\cdots\overline{\mathbf{e}}_{i_1}, \overline{\mathbf{e}}_j = -\mathbf{e}_j$. This induces the Clifford conjugate $\overline{x} = x_0 - \underline{x}$ of a paravector $x = x_0 + \underline{x}$.
The product between x and y in $\mathbf{R}_1^m$, denoted by xy is split into three parts: a scalar part, a vector part and a bivector part, that is

$$xy = (x_0y_0 + \underline{x}\cdot\underline{y}) + (x_0\underline{y} + y_o\underline{x}) + \underline{x}\wedge\underline{y},$$

where

$$\underline{x}\cdot\underline{y} = -\sum_{i=1}^m x_iy_i,$$

$$\underline{x}\wedge\underline{y} = \sum_{i=1}^m\sum_{j=i+1}^m (x_iy_j - x_jy_i)\mathbf{e}_i\mathbf{e}_j.$$

In particular,

$$xx = {x_0}^2 - \sum_{i=1}^m {x_i}^2 + 2x_0\underline{x} = 2x_0x - |x|^2,$$

where

$$|x|^2 = x\overline{x} = \sum_{i=0}^m {x_i}^2.$$

In the following, the so-called Clifford-Heaviside functions

$$P^{\pm}(\underline{x}) = \frac{1}{2}(1 \pm \mathbf{i}\frac{\underline{x}}{|\underline{x}|}), \mathbf{i} \text{ is the imaginary unit in } \mathbf{C}$$

will play an important role, which were first introduced by Sommen in [9] and McIntosh in [10]. Introducing spherical coordinates in $\mathbf{R}^m$, we have $\underline{x} = r\underline{\omega}, r = |\underline{x}| \in [0, \infty),\ \underline{\omega} \in S^{m-1}$, where S^{m-1} is the unit sphere in $\mathbf{R}^m$. Thus,

$$P^{\pm}(\underline{\omega}) = \frac{1}{2}(1 \pm \mathbf{i}\underline{\omega}).$$

They are self adjoint mutually orthogonal primitive idempotents:

$$P^{+}(\underline{\omega}) + P^{-}(\underline{\omega}) = 1,\ P^{+}(\underline{\omega})P^{-}(\underline{\omega}) = P^{-}(\underline{\omega})P^{+}(\underline{\omega}) = 0,\ (P^{\pm}(\underline{\omega}))^2 = P^{\pm}(\underline{\omega}).$$

Furthermore, we have

$$P^{\pm}(\underline{\omega})\underline{\omega} = \underline{\omega}P^{\pm}(\underline{\omega}) = \mp\mathbf{i}P^{\pm}(\underline{\omega})$$

and thus

$$P^{\pm}(\underline{\omega})x = xP^{\pm}(\underline{\omega}) = P^{\pm}(x_0 \mp \mathbf{i}|\underline{x}|). \tag{1}$$

2. The set of zeroes of a Clifford algbra-valued polynomial

In this section, we will consider the following polynomials of degree $n \geq 1$ with paravector variable $x \in \mathbf{R}_1^m$ and paravector coefficients,

$$R_n(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, \tag{2}$$

and

$$L_n(x) = x^n a_n + x^{n-1}a_{n-1} + \cdots + x a_1 + a_0, \tag{3}$$

where $\{a_0, a_1, \cdots, a_n\} \subseteq \mathbf{R}_1^m$ and $a_n \neq 0$.

Since

$$\overline{R_n(\overline{x})} = x^n\overline{a_n} + x^{n-1}\overline{a_{n-1}} + \cdots + x\overline{a_1} + \overline{a_0},$$

it is enough to consider only one of the two polynomials.

First, we show that every polynomial R_n has a special representation based on which our study is carried out.

For $x = x_0 + \underline{x} \in \mathbf{R}_1^m$, we have

$$\begin{aligned} x^2 &= 2x_0 x - |x|^2 \\ &= 2\mathrm{Sc}(x)x - |x|^2. \end{aligned}$$

Then

$$\begin{aligned} x^3 = x^2 x &= \left(2\mathrm{Sc}(x)x - |x|^2\right)x \\ &= 2\mathrm{Sc}(x)x^2 - |x|^2 x \\ &= \left[(2\mathrm{Sc}(x))^2 - |x|^2\right]x - 2\mathrm{Sc}(x)|x|^2 \\ &\ \ \vdots \end{aligned}$$

Going like this, we obtain the bi-axial form formula:

$$x^n = A_n(x)x + B_n(x),$$

where A_n and B_n are real-valued functions of x defined by the recurrent formulas:

$$\begin{aligned} A_{n+1}(x) &= 2\mathrm{Sc}(x)A_n(x) - |x|^2 A_{n-1}(x) \\ B_{n+1}(x) &= -|x|^2 A_n(x), \end{aligned}$$

where

$$\begin{aligned} A_1(x) &= 1 \\ A_2(x) &= 2\mathrm{Sc}(x) \\ B_1(x) &= 0 \\ B_2(x) &= -|x|^2. \end{aligned}$$

Therefore,

$$\begin{aligned} R_n(x) &= a_n[A_n(x)x + B_n(x)] + a_{n-1}[A_{n-1}(x)x + B_{n-1}(x)] + \cdots + a_1 x + a_0 \\ &= A(x)x + B(x), \end{aligned}$$

where

$$A(x) = \sum a_i A_i(x), \ B(x) = \sum a_i B_i(x).$$

Note that, like the coefficients, $A(x)$ and $B(x)$ are paravector-valued.

Remark 2.1 As a matter of fact, given any $x \in \mathbf{R}_1^m$, $A_i(x)$ and $B_i(x)$ depend not on x but on its scalar part x_0 and the modulus of its vector part $|\underline{x}|$.
Thus, we have

Lemma 2.1 *If two paravectors $x = x_0 + \underline{x}$, $y = y_0 + \underline{y}$ with $x_0 = y_0, |\underline{x}| = |\underline{y}|$, then $A_i(x) = A_i(y)$, $B_i(x) = B_i(y)$ and hence $A(x) = A(y)$, $B(x) = B(y)$. In particular, $A(x) = A(\overline{x})$, $B(x) = B(\overline{x})$.*

We are interested in the properties of the zeroes of both polynomials (2). Let us see the following examples:

Example 1 For $x^2 + 1 = 0$, the zeroes in $\mathbf{R}_1^m$ are all the vectors of modulus 1 in $\mathbf{R}^m$. Thus its solutions are all the points on the unit sphere.

Example 2 Equation $x^2 + \mathbf{e}_2 = 0$ has two isolated zeroes in $\mathbf{R}_1^m$,

$$x_{1,2} = \pm\frac{\sqrt{2}}{2}(1 - \mathbf{e}_2).$$

Example 3 Equation $x^2 + \mathbf{e}_3 x + \mathbf{e}_2 = 0$ has no zeroes in $\mathbf{R}_1^m$.

Remark 2.2 In [6], we know that any quaternionic polynomials of degree $n \geq 1$ has at least one zero. While, in higher dimensional cases under the structure of Clifford algebra, when $n \geq 1$, none of the polynomials (2) and (3) have zeroes.

Next, we will consider structure of the set of zeroes of a polynomial of the form (2) when it has zeroes.

Definition 2.1 *If $w_0 = \alpha + \mathrm{Vec}(w_0)$ and $w_1 = \alpha + \mathrm{Vec}(w_1)$ are two different paravectors with $|\mathrm{Vec}(w_0)| = |\mathrm{Vec}(w_1)|$, then they are said to be spherical conjugate to each other.*

Proposition 2.1 *Assume that $w_0 = \alpha + \mathrm{Vec}(w_0)$ and $w_1 = \alpha + \mathrm{Vec}(w_1)$ are spherical conjugate to each other, and they both are zeroes of the polynomial* (2), *then any paravector that is spherical conjugate to w_0 is also a zero of* (2).

Proof For w_0 and w_1, we have

$$R_n(w_0) = 0 = A(w_0)w_0 + B(w_0)$$

$$R_n(w_1) = 0 = A(w_1)w_1 + B(w_1)$$

and

$$A(w_0) = A(w_1),\ B(w_0) = B(w_1).$$

Hence $A(w_0)w_0 + B(w_0) = A(w_0)w_1 + B(w_0)$, which implies that

$$A(w_0)\left[\mathrm{Vec}(w_0) - \mathrm{Vec}(w_1)\right] = 0.$$

Since w_0 and w_1 are two different zeroes, the above means that $A(w_0) = 0$ and then, from the recurrent formula, $B(w_0) = 0$.
For any $w = \alpha + \mathrm{Vec}(w)$ with $|\mathrm{Vec}(w)| = |\mathrm{Vec}(w_0)|$, using Lemma 2.1 we obtain

$$A(w) = A(w_0),\ B(w) = B(w_0).$$

Therefore, $R_n(w) = A(w)w + B(w) = A(w_0)w + B(w_0) = 0$. This completes the proof.

Corollary 2.1 *Let w_0 be a zero of the polynomial* (2) *such that $\mathrm{Vec}(w_0) \neq 0$ and that $\overline{w_0}$ is also a zero of it, then any spherical conjugate paravector to w_0 is also a zero of it.*

Definition 2.2 *Given a polynomial $R_n(x)$, then any of its zeroes generating a family of zeroes that are spherical conjugate to each other is called a spherical zero. A zero that is not spherical is called an isolated zero.*

Proposition 2.2 *The number of the isolated non-scalar zeroes of the polynomial (2) is less or equal to n.*

Proof Let $w_1, w_2, \cdots, w_{n+1}$ be different non-scalar isolated zeroes of R_n. Considering the polynomial

$$L_n(x) = \overline{R_n(\overline{x})} = x^n\overline{a_n} + x^{n-1}\overline{a_{n-1}} + \cdots + x\overline{a_1} + \overline{a_0} = \sum_{k=0}^{n} x^k\overline{a_k},$$

using the properties (1) of $P^+(\underline{\omega})$, we have

$$\begin{aligned} P^+(\underline{\omega})L_n(x) &= \sum_{k=0}^{n} P^+(\underline{\omega})x^k\overline{a_k} \\ &= \sum_{k=0}^{n} P^+(\underline{\omega})(x_0 - \mathbf{i}|\underline{x}|)^k\overline{a_k} \\ &= P^+(\underline{\omega})L_n(x_0 - \mathbf{i}|\underline{x}|). \end{aligned}$$

Similarly,

$$R_n(x)P^+(\underline{\omega}) = R_n(x_0 - \mathbf{i}|\underline{x}|)P^+(\underline{\omega}).$$

Hence

$$\begin{aligned} P^+(\underline{\omega})L_n(x)R_n(x)P^+(\underline{\omega}) &= P^+(\underline{\omega})L_n(x_0 - \mathbf{i}|\underline{x}|)R_n(x_0 - \mathbf{i}|\underline{x}|)P^+(\underline{\omega}) \\ &= P^+(\underline{\omega})F_{2n}(x_0 - \mathbf{i}|\underline{x}|)P^+(\underline{\omega}), \end{aligned}$$

where

$$\begin{aligned} F_{2n}(x_0 - \mathbf{i}|\underline{x}|) &= L_n(x_0 - \mathbf{i}|\underline{x}|)R_n(x_0 - \mathbf{i}|\underline{x}|) \\ &= \left[\sum_{k=0}^{n}(x_0 - \mathbf{i}|\underline{x}|)^k\overline{a_k}\right]\left[\sum_{k=0}^{n} a_k(x_0 - \mathbf{i}|\underline{x}|)^k\right] \\ &= \overline{a_0}a_0 + (\overline{a_0}a_1 + \overline{a_1}a_0)(x_0 - \mathbf{i}|\underline{x}|) + \cdots + \overline{a_n}a_n(x_0 - \mathbf{i}|\underline{x}|)^{2n} \\ &= \sum_{k=0}^{2n}(\sum_{i=0}^{k}\overline{a_i}a_{k-i})(x_0 - \mathbf{i}|\underline{x}|)^{2k} \end{aligned}$$

is a polynomial of $x_0 - \mathbf{i}|\underline{x}|$ of order $2n$ with real coefficients.

Similar computation gives

$$P^-(\underline{\omega})L_n(x)R_n(x)P^-(\underline{\omega}) = P^-(\underline{\omega})F_{2n}(x_0 + \mathbf{i}|\underline{x}|)P^-(\underline{\omega}).$$

If $R_n(w_i) = 0$ $(i = 1, 2, \cdots, n+1)$, then

$$\begin{aligned} 0 &= P^+(\underline{w_i})L_n(w_i)R_n(w_i)P^+(\underline{w_i}) \\ &= P^+(\underline{w_i})F_{2n}\left[\mathrm{Sc}(w_i) - \mathbf{i}|\mathrm{Vec}(w_i)|\right]P^+(\underline{w_i}) \\ &= F_{2n}\left[\mathrm{Sc}(w_i) - \mathbf{i}|\mathrm{Vec}(w_i)|\right](P^+(\underline{w_i}))^2 \\ &= F_{2n}\left[\mathrm{Sc}(w_i) - \mathbf{i}|\mathrm{Vec}(w_i)|\right]P^+(\underline{w_i}). \end{aligned}$$

Since F_{2n} is a complex-valued polynomial and $P^+(\underline{w_i}) \neq 0$, we have

$$F_{2n}\left[\mathrm{Sc}(w_i) - \mathbf{i}|\mathrm{Vec}(w_i)|\right] = 0.$$

Likewise, $F_{2n}\left[\mathrm{Sc}(w_i) + \mathbf{i}|\mathrm{Vec}(w_i)|\right] = 0$. Therefore, $F_{2n}(z)$ has $2n+2$ zeroes, which is a contradiction. This completes the proof.

We further have

Proposition 2.3 *The number of the isolated zeroes of the polynomial (2) is less or equal to n.*

Proof Let $w_1, w_2, \cdots, w_p (1 \leq p \leq n+1)$ be different scalar zeroes and $w_{p+1}, \cdots, w_{n+1}$ be different non-scalar isolated zeroes of R_n. Then the polynomial can be written as

$$R_n(x) = R_{n-p}(x)(x - w_1) \cdots (x - w_p).$$

If $R_n(w_i) = 0$, $p+1 \leq i \leq n+1$, note that $w_i - w_j$ have inverse paravectors, where $p+1 \leq i \leq n+1, 1 \leq j \leq p$, then $R_{n-p}(w_i) = 0$. That means equation $R_{n-p}(x) = 0$ has $n-p+1$ non-scalar isolated zeroes, which is a contradiction to Proposition 2.2. This completes the proof.

Theorem 2.1 *The number of the isolated zeroes together with the double number of the spherical zeroes of polynomial* (2) *does not exceed the degree of the polynomial.*

We will prove it in next section.

Next section, as a special cases, we will study sets of zeroes of polynomials with real coefficients.

3. Sets of zeroes of polynomials with real coefficients

Consider the polynomial

$$Q_n(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, \tag{4}$$

with $a_i \in \mathbf{R}$.

Using Corollary 2.1, we can obtain the following conclusion:

Proposition 3.1 *Any zero of* (4) *with the form* $w_0 = \alpha + \mathrm{Vec}(w_0),\ \mathrm{Vec}(w_0) \neq 0$, *is a spherical zero.*

Proof According to Corollary 2.1, it will suffice to show that $\overline{w_0}$ is also a zero of (4).

In fact, based on the previous discussion, $Q_n(x)$ has the representation

$$Q_n(x) = A(x)x + B(x),$$

where, based on the real-coefficients assumption, $A(x)$ and $B(x)$ are real numbers.

If $Q_n(w_0) = 0$, then

$$A(w_0)\alpha + B(w_0) = 0,\ A(w_0)\mathrm{Vec}(w_0) = 0,$$

which implies $A(w_0) = B(w_0) = 0$.

From Lemma 2.1, we have

$$A(w_0) = A(\overline{w_0}),\ B(w_0) = B(\overline{w_0}).$$

Therefore, $Q_n(\overline{w_0}) = A(\overline{w_0})\overline{w_0} + B(\overline{w_0}) = 0$.

From Proposition 3.1, we obtain

Corollary 3.1 *$Q_n(x)$ has no isolated non-scalar zeroes.*

Next, we will introduce a method to solve the equation $Q_n(x) = 0$.

For $Q_n(x) = \sum_{k=0}^{n} a_k(x_0 + \underline{x})^k$, $a_k \in \mathbf{R}$, we have

$$P^+(\underline{\omega})Q_n(x) = Q_n(x)P^+(\underline{\omega}) = Q_n(x_0 - \mathbf{i}|\underline{x}|)P^+(\underline{\omega})$$

$$P^-(\underline{\omega})Q_n(x) = Q_n(x)P^-(\underline{\omega}) = Q_n(x_0 + \mathbf{i}|\underline{x}|)P^-(\underline{\omega}).$$

Using the properties of $P^{\pm}(\underline{\omega})$, we have

$$Q_n(x) = Q_n(x)[P^+(\underline{\omega}) + P^-(\underline{\omega})] = Q_n(x_0 - \mathbf{i}|\underline{x}|)P^+(\underline{\omega}) + Q_n(x_0 + \mathbf{i}|\underline{x}|)P^-(\underline{\omega}).$$

Thus

$$\begin{aligned} Q_n(x) = 0 &\iff Q_n(x_0 - \mathbf{i}|\underline{x}|)P^+(\underline{\omega}) + Q_n(x_0 + \mathbf{i}|\underline{x}|)P^-(\underline{\omega}) = 0 \\ &\iff Q_n(x_0 - \mathbf{i}|\underline{x}|)P^+(\underline{\omega}) = 0 \text{ and } Q_n(x_0 + \mathbf{i}|\underline{x}|)P^-(\underline{\omega}) = 0 \\ &\iff Q_n(x_0 - \mathbf{i}|\underline{x}|) = 0 \text{ and } Q_n(x_0 + \mathbf{i}|\underline{x}|) = 0 \\ &\iff Q_n(z) = 0. \end{aligned}$$

Note Note that $Q_n(z) = 0$ is an equation of real coefficients. It, therefore, has complex conjugate roots.

From the above discussion, we can obtain the conclusion as follows:

Corollary 3.2 *If $\alpha \pm \mathbf{i}\beta, \beta > 0$, is a solution of $Q_n(z) = 0$, then $\alpha + \beta\underline{\omega}$ is a spherical zero of $Q_n(x)$.*

Proposition 3.2 *Given a polynomial $Q_n(x)$ of real coefficients. Then there exists a one-to-one correspondence between its real isolated zeroes and the real roots of $Q_n(z)$, as well as a one-to-one correspondence between the spherical zeroes of $Q_n(x)$ and the pairs of complex conjugate zeroes of $Q_n(z)$.*

Using Proposition 3.2, we have

Theorem 3.1 *Given a polynomial $Q_n(x)$ with real coefficients, it has at least one zero. The zeroes are either isolated real ones or spherical zeroes. A pair of complex conjugate roots of $Q_n(x)$ with multiplicity $k, 2k \leq n$, corresponds to a single spherical zero. The* *number of the isolated real zeroes together with the double number does not exceed the degree of the polynomial.*

Examples 1 For $x^2 - px + q = 0$, where $p, q \in \mathbf{R}$, considering $z^2 - pz + q = 0$, we have

$$z_{1,2} = \frac{p \pm \sqrt{p^2 - 4q}}{2}.$$

If $p^2 - 4q \geq 0$, then $x_{1,2} = \frac{p \pm \sqrt{p^2-4q}}{2}$ are zeroes of $x^2 - px + q = 0$. If $p^2 - 4q < 0$, then $x = \frac{p}{2} + \frac{\sqrt{4q-p^2}}{2}\underline{\omega}$ is a spherical zero of $x^2 - px + q = 0$.

From Example 1, we obtain that

Corollary 3.3 *There is a one-to-one correspondence between the spherical zero $x = \alpha + r\underline{\omega}$ and the equations of real coefficients $[x^2 - 2\alpha x + (r^2 + \alpha^2)]^k = 0, k = 1, 2, \cdots$.*

Examples 2 For $x^3 + 2x^2 + x + 2 = 0$, considering $z^3 + 2z^2 + z + 2 = 0$, we have

$$z_{1,2} = \pm \mathbf{i}, z_3 = -2.$$

Then zeroes of $x^3 + 2x^2 + x + 2 = 0$ are $x_1 = \underline{\omega}$ and $x_2 = -2$.

Example 3 For $x^4 + x^2 + 1 = 0$, considering $z^4 + z^2 + 1 = 0$, we have

$$z_{1,2} = \frac{1}{2}(1 \pm \sqrt{3}\mathbf{i})$$

$$z_{3,4} = -\frac{1}{2}(1 \pm \sqrt{3}\mathbf{i}).$$

The zeroes of $x^4 + x^2 + 1 = 0$ are $x_{1,2} = \pm\frac{1}{2}(1 + \sqrt{3}\underline{\omega})$.

Before proving Theorem 2.1, we need a Lemma.

Lemma 3.1 *If $\alpha_1 + r_1\underline{\omega}, \alpha_2 + r_2\underline{\omega}, \cdots, \alpha_p + r_p\underline{\omega}, (1 \leq p \leq [\frac{n}{2}])$ are different spherical zeroes of R_n with multiplicity $k_1, \cdots, k_p$, then the polynomial can be written as*

$$R_n(x) = R_{n-2k_p}(x)[x^2 - 2\alpha_1 x + (r_1^2 + \alpha_1^2)]^{k_1} \cdots [x^2 - 2\alpha_p x + (r_p^2 + \alpha_p^2)]^{k_p}.$$

Proof For $R_n(x) = A(x)x + B(x)$. If $w_1 = \alpha_1 + r_1\underline{\omega}$ is a spherical zero of $R_n(x)$, then $\overline{w_1}$ is also a zero of it. As the proof of Proposition 2.1, we have $A(w_1) = A(\overline{w_1}) = 0$ and $B(w_1) = B(\overline{w_1}) = 0$. Using the properties of $A(x)$ and $B(x)$, we obtain that $A(\alpha_1 \pm \mathbf{i}r_1) = 0$ and $B(\alpha_1 \pm \mathbf{i}r_1) = 0$. Therefore, $R_n(\alpha_1 \pm \mathbf{i}r_1) = 0$. We have

$$R_n(z) = R_{n-2}(z)[z - (\alpha_1 + \mathbf{i}r_1)][z - (\alpha_1 - \mathbf{i}r_1)] = R_{n-2}(z)[z^2 - 2\alpha_1 z + (r_1^2 + \alpha_1^2)].$$

Hence $R_n(x) = R_{n-2}(x)[x^2 - 2\alpha_1 x + (r_1^2 + \alpha_1^2)]$. Using a finite number of iterations of this procedure leads to the conclusion. This completes the proof.

Proof of Theorem 2.1 Let $\alpha_1 + r_1\underline{\omega}, \alpha_2 + r_2\underline{\omega}, \cdots, \alpha_p + r_p\underline{\omega}, (1 \leq p \leq [\frac{n}{2}])$ be different spherical zeroes and $w_{2p+1}, \cdots, w_{n+1}$ be different isolated zeroes of R_n. From Lemma 3.1, we know that the polynomial can be written as

$$R_n(x) = R_{n-2k_p}(x)[x^2 - 2\alpha_1 x + (r_1^2 + \alpha_1^2)]^{k_1} \cdots [x^2 - 2\alpha_p x + (r_p^2 + \alpha_p^2)]^{k_p}$$

for some integers $k_1, k_2, ..., k_p$, and thus the degree of $R_{n-2k_p}(x)$ is less than $n - 2p + 1$. If $R_n(w_i) = 0$, $2p+1 \leq i \leq n+1$, note that $w_i^2 - 2\alpha_j w_i + (r_j^2 + \alpha_j^2)$, $2p+1 \leq i \leq n+1, 1 \leq j \leq p$ have inverse vectors, then $R_{n-2k_p}(w_i) = 0$. That means equation $R_{n-2k_p}(x) = 0$ has $n - 2p + 1$ isolated zeroes, which is a contradiction to Proposition 2.3. This completes the proof.

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