B-Splines of Blaschke Product Type

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Abstract

In this note we construct a class of new splines related to a Blaschke product. They emerge naturally when studying the filter functions of a class of linear time-invariant systems which are related to the boundary values of a Blaschke product for the purpose of sampling non-bandlimited signals using nonlinear Fourier atoms. The new splines generalize the well-known symmetric B-splines. We establish their properties such as integral representation property, a partition of unity property, a recurrence relation and difference property. We also investigate their random behavior. Lastly, our numerical experiments confirm our theories.

Keywords: B-spline, Blaschke product, Fourier transform, sinc function, central limit theorem.

1 Introduction

Recently the problem of defining meaningful *instantaneous frequency* of a real signal, and the interpretation of that quantity have attracted substantial attentions. One common approach, for example see [7], is that the instantaneous frequency is defined to be the derivative of the phase function of a companion *analytic signal* x_a given by the formula

$$x_a := x + \mathrm{i}y,$$

where $y = \mathcal{H}x$ represents the *Hilbert transform* of a signal x which is defined for $t \in \mathbb{R}$, where $\mathbb{R}$ is the set of real numbers, as

$$(\mathcal{H}x)(t) \ := \ \frac{1}{\pi}\,\text{p.v.}\int_{\mathbb{R}} \frac{x(s)}{t-s}ds.$$

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Questions have been raised about the existence of analytic signals with positive instantaneous frequencies. Specifically, for what kind of signals is the above defined instantaneous frequency meaningful in physics? These kind of signals are often called *mono-components* in engineering. Many literatures address this problem, see for example, [1], [2], [7], [12] and [13].

Some examples of mono-components with a strictly increasing nonlinear phase were recently given in [14]. The starting point for this observation is the introduction of an analytic signal, called a *nonlinear Fourier atom*, which is essentially defined by the boundary value of a *Blaschke product* of order one associated with a parameter $a \in \mathbb{R}$ with $|a| < 1$ as

$$\mathrm{e}^{\mathrm{i}\theta_a(t)} := \left. \frac{z-a}{1-az} \right|_{z=\mathrm{e}^{\mathrm{i}t}}.$$

The phase function θ_a has the following explicit decomposition

$$\theta_a(t) = t + 2\arctan \frac{|a|\sin t}{1-|a|\cos t}, \quad t \in \mathbb{R},$$

which indicates that it can be decomposed into a linear part and a periodic part. Moreover, the signal $\cos\theta_a$ is a mono-component, as substantiated by the formula

$$\mathcal{H}(\cos\theta_a) = \sin\theta_a, \quad t \in \mathbb{R}. \tag{1.1}$$

Its instantaneous frequency ω_{ins} is given at $t \in \mathbb{R}$ explicitly in terms of the *Poisson kernel* p_a defined by

$$\omega_{ins}(t) = p_a(t) := \frac{1-a^2}{1-2a\cos t + a^2}, \quad a, t \in \mathbb{R}, |a| < 1,$$

which shows that $\omega_{ins}(t) > 0$ for $t \in \mathbb{R}$.

We point out that this kind of mono-component nonlinear Fourier atoms coincide with the notion of *intrinsic mode functions* used in *empirical mode decompositions* of non-stationary signals [10]. Therefore, there is a trend to use mono-components to represent signals rather than the usual Fourier basis with linear phase. In a recent paper [4], the authors related nonlinear Fourier atoms to a continuous linear time-invariant (LTI) system by allowing the output signal to keep the scaled frequency information of the input signal in different frequency bands. The LTI system has the *impulse response* function $\frac{1}{\sqrt{2\pi}(1+a)}\mathrm{sinc}_a(\cdot)$ with

$$\mathrm{sinc}_a(t) := \frac{\sin\theta_a(t)}{t}, \quad t \in \mathbb{R},$$

the so-called *generalized sinc function* that admits a *Shannon type sampling theorem* for non-bandlimited signals. This LTI system has the piecewise constant function

$$H_a := a^{\lfloor|\cdot|\rfloor}$$

as its *filter function*, where $\lfloor t \rfloor$ is the greatest integer that is no more than the real number t. The function H_a, due to its shape, is called a *ladder-shaped filter*. A LTI system with a filter function associated with a two-parameter Blaschke product was considered in [5], and more generally, a LTI system with a filter function that is an arbitrary spline, in particular, a spline associated with an arbitrary finite order Blaschke product was considered in [3]. Note that H_a and the function $\frac{1}{\sqrt{2\pi}(1+a)}\mathrm{sinc}_a(\cdot)$ form a Fourier transform pair. Here for a function $f \in L^2(\mathbb{R})$, the Fourier transform is defined by

$$\hat{f}(\xi) = \mathcal{F}f(\xi) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x)\mathrm{e}^{-\mathrm{i}x\xi}\,\mathrm{d}x, \quad \xi \in \mathbb{R}.$$

It is not hard to see that the ladder-shaped filter H_a can also be written into a series as

$$H_a(t) = (1-a)\sum_{k\in\mathbb{N}} a^{k-1}\chi_{(-k,k)}(t), \quad t \in \mathbb{R}. \tag{1.2}$$

Here χ_A denotes the characteristic function of a given set A and $\mathbb{N}$ denotes the set of natural numbers.

Noting that $H_a(2\cdot)$ is a generalization of the *ideal low-pass filter*, which is the first order *symmetric B-spline* $\beta_1^0 := \chi_{(-\frac{1}{2},\frac{1}{2})}$, we write

$$\beta_1^a(t) := H_a(2t) = a^{\lfloor |2t| \rfloor}, \quad t \in \mathbb{R}$$

and subsequently the n-th convolution of β_1^a by

$$\beta_n^a(t) := \int_{\mathbb{R}} \beta_{n-1}^a(t-y)\beta_1^a(y)\,\mathrm{d}y, \quad t \in \mathbb{R}.$$

The function β_n^a can be regarded as a class of generalized B-splines related to the parameter a since it is well-known that the n-fold convolution of β_1^0 leads to the n-th symmetric B-splines, which play important roles in the theory of numerical calculation, wavelet and approximation [6, 8, 11, 9].

In this paper, we extend the above idea to a more generalized B-spline that relates to a m-fold Blaschke product. Here, the m-fold Blaschke product for a real vector $\vec{a} := (a_1, \ldots, a_m) \in [0,1)^m$ with pairwise distinct components is defined by

$$B_m(z) := B_{\{a_1,\ldots,a_m\}}(z) = \prod_{j\in\mathbb{N}_m} \frac{z-a_j}{1-a_jz}, \quad z \in \mathbb{C}, \tag{1.3}$$

where, the index set $\mathbb{N}_m := \{1, 2, \ldots, m\}$, and $\mathbb{C}$ denotes the set of complex numbers. For convenience, a normalization constant α_m associated with the m-th order Blaschke product is defined by

$$\alpha_m := B_m(1) - B_m(0) = 1 - (-1)^m \prod_{j\in\mathbb{N}_m} a_j.$$

Definition 1 *The ladder-shaped filter of Blaschke type is the piecewise constant function*

$$\beta_1^{\vec{a}}(t) = \frac{1}{\alpha_m}\sum_{j\in\mathbb{N}_m} \frac{a_j^{\lfloor |2t| \rfloor}}{(1-a_j)B_m'(a_j)}, \quad t \in \mathbb{R}. \tag{1.4}$$

Definition 2 *: The n-th B-spline of Blaschke type (BB-spline) is defined recursively by*

$$\beta_n^{\vec{a}}(t) := \int_{\mathbb{R}} \beta_{n-1}^{\vec{a}}(t-y)\beta_1^{\vec{a}}(y)\,\mathrm{d}y, \quad t \in \mathbb{R}. \tag{1.5}$$

Throughout the paper, we require that $\vec{a} \in [0,1)^m$ in order to ensure that $\beta_1^{\vec{a}}$ can be treated as a density of certain random variable.

The writing plan is as follows. In Section 2, we introduce some basic formulae which are important for the proof of our main results. Section 3 is dedicated to investigating the properties of the B-spline of Blaschke product type. Section 4 focuses on the limiting probability behavior of the BB-spline $\beta_n^{\vec{a}}$ by using the central limit theorem. Section 5 confirms our theories by numerical experiments.

2 Preliminaries

As in [3] by using the boundary value on the unit circle of the Blaschke product B_m, we define the value of the phase function $\theta_{\vec{a}}$ at $t \in \mathbb{R}$ through

$$\mathrm{e}^{\mathrm{i}\theta_{\vec{a}}(t)} := B_m(\mathrm{e}^{\mathrm{i}t}). \tag{2.1}$$

We remark that $\theta_{\vec{a}}(t)$ is related to the phase functions of scalar case by

$$\theta_{\vec{a}}(t) := \sum_{j\in\mathbb{N}_m} \theta_{a_j}(t), \quad t \in \mathbb{R}. \tag{2.2}$$

The function $p_{\vec{a}}$ is defined as the derivative of $\theta_{\vec{a}}$, that is,

$$p_{\vec{a}}(t) := \frac{d}{\mathrm{d}t}\theta_{\vec{a}}(t), \quad t \in \mathbb{R}.$$

The phase function $\theta_{\vec{a}}$ is associated with the generalized sinc function

$$\mathrm{sinc}_{\vec{a}}(t) := \frac{\sin\theta_{\vec{a}}(t)}{t}, \quad t \in \mathbb{R}. \tag{2.3}$$

We will need some properties of Blaschke products. Direct calculation gives us that

$$B_m'(z) = \sum_{l\in\mathbb{N}_m} \frac{1-|a_l|^2}{(1-a_l z)^2}\left(\prod_{k\in\mathbb{N}_m, k\neq l} \frac{z-a_k}{1-a_k z}\right), \tag{2.4}$$

and, consequently,

$$B_m'(a_j) = \frac{1}{1-|a_j|^2} \prod_{k\in\mathbb{N}_m, k\neq j} \frac{a_j-a_k}{1-a_k a_j}.$$

The following two formulae are related to the rational decomposition of a Blaschke product. The first one decomposes B_m into partial fractions as:

$$B_m(z) = \frac{1}{B_m(0)} + \sum_{j\in\mathbb{N}_m} \frac{1}{a_j B_m'(a_j)}\frac{1}{1-a_j z}. \tag{2.5}$$

The second represents B_m as a linear combination of Möbius transforms of various parameters in the form of

$$B_m(z) = \frac{1+(-1)^m}{2} + \sum_{j\in\mathbb{N}_m} \frac{\tau_{a_j}(z)}{(1-a_j^2)B_m'(a_j)}, \tag{2.6}$$

where $\tau_{a_j}(z) = \frac{z-a_j}{1-a_j z}$, $j \in \mathbb{N}_m$. In particular, formulae (2.5) and (2.6) yield

$$\sum_{j\in\mathbb{N}_m} \frac{1}{B_m'(a_j)} \frac{a_j}{(1-a_j)^4} = \frac{1}{2}\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} B_m\right)(1) + \frac{1}{6}\left(\frac{\mathrm{d}^3}{\mathrm{d}z^3} B_m\right)(1) \tag{2.7}$$

and

$$\sin\theta_{\vec{a}}(t) = \sum_{j\in\mathbb{N}_m} \frac{\sin\theta_{a_j}(t)}{(1-a_j^2)B_m'(a_j)}. \tag{2.8}$$

We need the following formulae

$$\sum_{k\in\mathbb{N}} a^{k-1}\sin(kx) = \frac{1}{1-a^2}\sin\theta_a(x), \tag{2.9}$$

$$\sum_{k\in\mathbb{N}} k a^{k-1}\cos(kx) = \frac{1}{1-a^2} p_a(x)\cos\theta_a(x), \tag{2.10}$$

and

$$\lim_{t\to 0}\operatorname{sinc}_{\vec{a}}(t) = \sum_{j\in\mathbb{N}_m} \frac{1}{(1-a_j)^2 B_m'(a_j)} = B_m'(1), \tag{2.11}$$

It can be checked that $\beta_1^{\vec{a}}$ has the decomposition

$$\beta_1^{\vec{a}}(t) = \frac{1}{\alpha_m} \sum_{j\in\mathbb{N}_m}\sum_{k\in\mathbb{N}} \frac{a_j^{k-1}}{B_m'(a_j)} \chi_{(-\frac{k}{2},\frac{k}{2})}(t), \quad t\in\mathbb{R}. \tag{2.12}$$

To end this section, we consider the Fourier transform of $\beta_n^{\vec{a}}$.

Lemma 2.1 *The Fourier transform of $\beta_n^{\vec{a}}$ is*

$$\left(\beta_n^{\vec{a}}\right)^\wedge(\xi) = \frac{1}{\sqrt{2\pi}} \frac{1}{\alpha_m^n} \left(\operatorname{sinc}_{\vec{a}}(\xi/2)\right)^n. \tag{2.13}$$

Proof: It suffices to check that

$$(\beta_1^{\vec{a}})^\wedge(\xi) = \frac{1}{\sqrt{2\pi}} \frac{1}{\alpha_m} \operatorname{sinc}_{\vec{a}}(\xi/2)$$

since $\left(\beta_n^{\vec{a}}\right)^\wedge(\xi) = (2\pi)^{\frac{n-1}{2}} \left((\beta_1^{\vec{a}})^\wedge(\xi)\right)^n$. Applying Fourier transform to both sides of (2.12) gives

$$\begin{aligned}
\left(\beta_1^{\vec{a}}(\cdot)\right)^\wedge(\xi) &= \frac{1}{\alpha_m} \sum_{j\in\mathbb{N}_m}\sum_{k\in\mathbb{N}} \frac{a_j^{k-1}}{B_m'(a_j)} \left(\chi_{(-\frac{k}{2},\frac{k}{2})}(\cdot)\right)^\wedge(\xi) \\
&= \frac{1}{\sqrt{2\pi}\alpha_m} \sum_{j\in\mathbb{N}_m}\sum_{k\in\mathbb{N}} \frac{a_j^{k-1}}{B_m'(a_j)} \left(\frac{\sin(k\xi/2)}{\xi/2}\right).
\end{aligned}$$

Using the formula (2.9), it follows that

$$\left(\beta_1^{\vec{a}}(\cdot)\right)^\wedge(\xi) = \frac{1}{\sqrt{2\pi}\alpha_m}\sum_{j\in\mathbb{N}_m}\frac{1}{(1-a_j^2)B_m'(a_j)}\frac{\sin\theta_{a_j}(\xi/2)}{\xi/2}.$$

Combining this with the formula (2.8), we conclude (2.13) for the case $n=1$ and then (2.13) for any $n\in\mathbb{N}$. $\square$

3 Properties

This section is concerned with the properties of $\beta_n^{\vec{a}}$. Obviously, the spline function $\beta_n^{\vec{a}}$ is supported in $\mathbb{R}$, positive and even.

Next, we will investigate other important properties of $\beta_n^{\vec{a}}$. To this end, we introduce the following notations. For the vector $k\in\mathbb{N}^n$, we denote $k:=(k_1,\ldots,k_n)$ with each $k_r\in\mathbb{N}$ for each $r\in\mathbb{N}_n$, and denote $|k|=\sum_{r\in\mathbb{N}_n}k_r$. Define $\mathcal{A}_n:=\{a_1,\ldots,a_m\}^n$, and the vector $\gamma:=(\gamma_1,\ldots,\gamma_n)\in\mathcal{A}_n$. We also use the multi-index notation $\gamma^k:=\gamma_1^{k_1}\ldots\gamma_n^{k_n}$. We also make the convention that $0^0=1$.

Theorem 3.1 *For any $f\in L^2(\mathbb{R})$ and $n\in\mathbb{N}$, the identity holds*

$$\int_{\mathbb{R}} f(s)\beta_n^{\vec{a}}(s)\,\mathrm{d}s = \frac{1}{\alpha_m^n}\sum_{\gamma\in\mathcal{A}_n}\sum_{k\in\mathbb{N}^n}\frac{k_1\cdots k_n\gamma^{k-1}}{B_m'(\gamma_1)\cdots B_m'(\gamma_n)}\int_{(-\frac{1}{2},\frac{1}{2})^n} f(k\cdot t)\,\mathrm{d}t, \tag{3.1}$$

where the inner product $k\cdot t=k_1t_1+\cdots+k_nt_n$. Specifically, when $m=1$, this formula has a simple form

$$\int_{\mathbb{R}} f(s)\beta_n^{a}(s)\,\mathrm{d}s = \sum_{k\in\mathbb{N}^n}(1-a)^n k_1\cdots k_n a^{|k|-n}\int_{(-\frac{1}{2},\frac{1}{2})^n} f(k\cdot t)\,\mathrm{d}t$$

and certainly

$$\int_{\mathbb{R}} f(s)\beta_n^{0}(s)\,\mathrm{d}s = \int_{(-\frac{1}{2},\frac{1}{2})^n} f(t_1+\cdots+t_n)\,\mathrm{d}t.$$

Proof: We adopt induction. In the case $n=1$, Eq. (3.1) can be concluded from the calculation below by involving the formula (2.12) and the Lebesgue dominated convergence theorem. We have

$$\begin{aligned}\int_{\mathbb{R}} f(s)\beta_1^{\vec{a}}(s)\,\mathrm{d}s &= \int_{\mathbb{R}} f(s)\frac{1}{\alpha_m}\sum_{j\in\mathbb{N}_m}\sum_{k\in\mathbb{N}}\frac{a_j^{k-1}}{B_m'(a_j)}\chi_{[-\frac{k}{2},\frac{k}{2})}(s)\,\mathrm{d}s\\ &= \frac{1}{\alpha_m}\sum_{j\in\mathbb{N}_m}\sum_{k\in\mathbb{N}}\frac{a_j^{k-1}}{B_m'(a_j)}\int_{[-\frac{k}{2},\frac{k}{2})} f(s)\,\mathrm{d}s\\ &= \frac{1}{\alpha_m}\sum_{j\in\mathbb{N}_m}\sum_{k\in\mathbb{N}}\frac{ka_j^{k-1}}{B_m'(a_j)}\int_{(-\frac{1}{2},\frac{1}{2})} f(kt)\,\mathrm{d}t.\end{aligned}$$

Now we show that the induction hypothesis (the case of $n-1$) implies (3.1). By noting the definition of $\beta_n^{\vec{a}}$ and Fubini's theorem, we get

$$\begin{aligned}\int_{\mathbb{R}} f(s)\beta_n^{\vec{a}}(s)\,\mathrm{d}s &= \int_{\mathbb{R}} f(s)\int_{\mathbb{R}} \beta_{n-1}^{\vec{a}}(s-y)\beta_1^{\vec{a}}(y)\,\mathrm{d}y\,\mathrm{d}s \\ &= \int_{\mathbb{R}} \beta_1^{\vec{a}}(y)\int_{\mathbb{R}} f(s)\beta_{n-1}^{\vec{a}}(s-y)\,\mathrm{d}s\,\mathrm{d}y \\ &= \int_{\mathbb{R}} \beta_1^{\vec{a}}(y)\int_{\mathbb{R}} f(s+y)\beta_{n-1}^{\vec{a}}(s)\,\mathrm{d}s\,\mathrm{d}y.\end{aligned}$$

The induction hypothesis leads to

$$\begin{aligned}&\int_{\mathbb{R}} f(s)\beta_n^{\vec{a}}(s)\,\mathrm{d}s \\ = &\int_{\mathbb{R}} \beta_1^{\vec{a}}(y)\frac{1}{\alpha_m^{n-1}}\sum_{\gamma\in\mathcal{A}_{n-1}}\sum_{k\in\mathbb{N}^{n-1}}\frac{k_1\cdots k_{n-1}\gamma^{k-1}}{B_m'(\gamma_1)\cdots B_m'(\gamma_{n-1})}\int_{(-\frac{1}{2},\frac{1}{2})^{n-1}} f(k\cdot t+y)\,\mathrm{d}t\,\mathrm{d}y \\ = &\frac{1}{\alpha_m^{n-1}}\sum_{\gamma\in\mathcal{A}_{n-1}}\sum_{k\in\mathbb{N}^{n-1}}\frac{k_1\cdots k_{n-1}\gamma^{k-1}}{B_m'(\gamma_1)\cdots B_m'(\gamma_{n-1})}\int_{(-\frac{1}{2},\frac{1}{2})^{n-1}}\int_{\mathbb{R}} f(y+k\cdot t)\beta_1^{\vec{a}}(y)\,\mathrm{d}y\,\mathrm{d}t \\ = &\frac{1}{\alpha_m^{n-1}}\sum_{\gamma\in\mathcal{A}_{n-1}}\sum_{k\in\mathbb{N}^{n-1}}\frac{k_1\cdots k_{n-1}\gamma^{k-1}}{B_m'(\gamma_1)\cdots B_m'(\gamma_{n-1})}\int_{(-\frac{1}{2},\frac{1}{2})^{n-1}}\frac{1}{\alpha_m}\sum_{j\in\mathbb{N}_m}\sum_{\ell\in\mathbb{N}}\frac{\ell a_j^{\ell-1}}{B_m'(a_j)} \\ &\int_{(-\frac{1}{2},\frac{1}{2})} f(\ell x+k\cdot t)\,\mathrm{d}x\,\mathrm{d}t \\ = &\frac{1}{\alpha_m^{n}}\sum_{\gamma\in\mathcal{A}_{n}}\sum_{k\in\mathbb{N}^{n}}\frac{k_1\cdots k_{n}\gamma^{k-1}}{B_m'(\gamma_1)\cdots B_m'(\gamma_{n})}\int_{(-\frac{1}{2},\frac{1}{2})^{n}} f(k\cdot t)\,\mathrm{d}t.\end{aligned}$$

□

The next theorem transfers the integral of smooth functions with B_n^a to a n-fold symmetric difference. For $t\in\mathbb{R}$, define the difference operator Δ_t by $\Delta_t f(x) := f(x+\frac{t}{2}) - f(x-\frac{t}{2})$. For $t_j\in\mathbb{R}$, $j\in\mathbb{N}_n$, the operator $\Delta_{t_1\ldots t_n} := \Delta_{t_1}\cdots\Delta_{t_n}$ stands for the composition of Δ_{t_j}, $j\in\mathbb{N}_n$. We point out that $\Delta_{t_1\ldots t_n}$ is independent of the order of the operators Δ_{t_j}, $j\in\mathbb{N}_n$.

Theorem 3.2 *For any $f\in C^n(\mathbb{R})$, the following equation holds*

$$\int_{\mathbb{R}} f^{(n)}(x)\beta_n^{\vec{a}}(x)\,\mathrm{d}x = \frac{1}{\alpha_m^n}\sum_{\gamma\in\mathbb{A}_n}\sum_{k\in\mathbb{N}^n}\frac{\gamma^{k-1}}{B_m'(\gamma_1)\cdots B_m'(\gamma_n)}\Delta_{k_1\ldots k_n} f(0). \tag{3.2}$$

The scalar case corresponds to

$$\int_{\mathbb{R}} f^{(n)}(x)\beta_n^{a}(x)\,\mathrm{d}x = \sum_{k\in\mathbb{N}^n}(1-a)^n a^{|k|-n}\Delta_{k_1\ldots k_n} f(0)$$

and

$$\int_{\mathbb{R}} f^{(n)}(x)\beta_n^{0}(x)\,\mathrm{d}x = \Delta_1^n f(0) = \sum_{j\in\mathbb{N}_n}(-1)^{n-j}\begin{pmatrix} n \\ j\end{pmatrix} f(j-\frac{n}{2}).$$

Proof: We adopt induction again. By using (3.1), we know that

$$\begin{aligned}
\int_{\mathbb{R}} f'(t)\beta_1^{\vec{a}}(t)\,\mathrm{d}t &= \frac{1}{\alpha_m}\sum_{j\in\mathbb{N}_m}\sum_{k\in\mathbb{N}}\frac{ka_j^{k-1}}{B_m'(a_j)}\int_{(-\frac{1}{2},\frac{1}{2})} f'(kt)\,\mathrm{d}t \\
&= \frac{1}{\alpha_m}\sum_{j\in\mathbb{N}_m}\sum_{k\in\mathbb{N}}\frac{a_j^{k-1}}{B_m'(a_j)}\int_{(-\frac{k}{2},\frac{k}{2})} f'(x)\,\mathrm{d}x \\
&= \frac{1}{\alpha_m}\sum_{j\in\mathbb{N}_m}\sum_{k\in\mathbb{N}}\frac{a_j^{k-1}}{B_m'(a_j)}\left(f(\frac{k}{2})-f(-\frac{k}{2})\right) \\
&= \frac{1}{\alpha_m}\sum_{j\in\mathbb{N}_m}\sum_{k\in\mathbb{N}}\frac{a_j^{k-1}}{B_m'(a_j)}\Delta_k f(0),
\end{aligned}$$

which concludes (3.2) in the case $n=1$. Suppose that (3.2) is true for the case $n-1$. We now verify the case n. By the definition of $\beta_n^{\vec{a}}$ and direct calculation, it follows

$$\begin{aligned}
&\int_{\mathbb{R}} f^{(n)}(t)\beta_n^{\vec{a}}(t)\,\mathrm{d}t \\
=\ &\int_{\mathbb{R}} f^{(n)}(t)\int_{\mathbb{R}}\beta_{n-1}^{\vec{a}}(t-y)\beta_1^{\vec{a}}(y)\,\mathrm{d}y\,\mathrm{d}t \\
=\ &\int_{\mathbb{R}}\beta_1^{\vec{a}}(y)\int_{\mathbb{R}}\left(f'\right)^{(n-1)}(t)\beta_{n-1}^{\vec{a}}(t-y)\,\mathrm{d}t\,\mathrm{d}y \\
=\ &\int_{\mathbb{R}}\beta_1^{\vec{a}}(y)\int_{\mathbb{R}}\left(f'\right)^{(n-1)}(t+y)\beta_{n-1}^{\vec{a}}(t)\,\mathrm{d}t\,\mathrm{d}y.
\end{aligned}$$

The induction assumption implies

$$\begin{aligned}
&\int_{\mathbb{R}} f^{(n)}(t)\beta_n^{\vec{a}}(t)\,\mathrm{d}t \\
=\ &\int_{\mathbb{R}}\beta_1^{\vec{a}}(y)\frac{1}{\alpha_m^{n-1}}\sum_{\gamma\in\mathcal{A}_{n-1}}\sum_{k\in\mathbb{N}^{n-1}}\frac{\gamma^{k-1}}{B_m'(\gamma_1)\cdots B_m'(\gamma_{n-1})}\Delta_{k_1\ldots k_{n-1}}f'(y)\,\mathrm{d}y \\
=\ &\frac{1}{\alpha_m^{n-1}}\sum_{\gamma\in\mathcal{A}_{n-1}}\sum_{k\in\mathbb{N}^{n-1}}\frac{\gamma^{k-1}}{B_m'(\gamma_1)\cdots B_m'(\gamma_{n-1})}\int_{\mathbb{R}}(\Delta_{k_1\ldots k_{n-1}}f)'(y)\beta_1^{\vec{a}}(y)\,\mathrm{d}y \\
=\ &\frac{1}{\alpha_m^{n-1}}\sum_{\gamma\in\mathcal{A}_{n-1}}\sum_{k\in\mathbb{N}^{n-1}}\frac{\gamma^{k-1}}{B_m'(\gamma_1)\cdots B_m'(\gamma_{n-1})}\frac{1}{\alpha_m}\sum_{j\in\mathbb{N}_m}\sum_{\ell\in\mathbb{N}}\frac{a_j^{\ell-1}}{B_m'(a_j)}\Delta_\ell\left(\Delta_{k_1\ldots k_{n-1}}f\right)(0) \\
=\ &\frac{1}{\alpha_m^{n}}\sum_{\gamma\in\mathcal{A}_{n}}\sum_{k\in\mathbb{N}^{n}}\frac{\gamma^{k-1}}{B_m'(\gamma_1)\cdots B_m'(\gamma_{n})}\Delta_{k_1\ldots k_n}f(0).
\end{aligned}$$

□

The next theorem indicates that $\beta_m^{\vec{a}}$ satisfies the Strang-Fix condition of order 1.

Theorem 3.3 *The* 1*-periodic function*

$$F_n(t) := \frac{\alpha_m^n}{(B_m'(1))^n} \sum_{k\in\mathbb{Z}} \beta_n^{\vec{a}}(t-k) = 1$$

for any $n \in \mathbb{N}$ *and* $t \in \mathbb{R}$.

Proof: It suffices to show that, for any $n \in \mathbb{N}$, the Fourier coefficients $c_j(F_n) = \delta_{j,0}$. Here δ denotes the Krönecker symbol. We will adopt induction. By using (2.12), we have

$$\begin{aligned}
c_j(F_1) &= \int_{(-\frac{1}{2},\frac{1}{2})} \frac{\alpha_m}{B_m'(1)} \sum_{k\in\mathbb{Z}} \beta_1^{\vec{a}}(t-k)\mathrm{e}^{-\mathrm{i}j2\pi t}\,\mathrm{d}t \\
&= \int_{(-\frac{1}{2},\frac{1}{2})} \frac{\alpha_m}{B_m'(1)} \sum_{k\in\mathbb{Z}} \frac{1}{\alpha_m} \sum_{\lambda\in\mathbb{N}_m} \sum_{l\in\mathbb{N}} \frac{a_\lambda^{l-1}}{B_m'(a_\lambda)} \chi_{(-\frac{l}{2},\frac{l}{2})}(t-k)\mathrm{e}^{-\mathrm{i}j2\pi t}\,\mathrm{d}t \\
&= \frac{1}{B_m'(1)} \sum_{k\in\mathbb{Z}} \sum_{\lambda\in\mathbb{N}_m} \sum_{l\in\mathbb{N}} \frac{a_\lambda^{l-1}}{B_m'(a_\lambda)} \int_{(-\frac{1}{2},\frac{1}{2})} \chi_{(-\frac{l}{2},\frac{l}{2})}(t-k)\mathrm{e}^{-\mathrm{i}j2\pi t}\,\mathrm{d}t \\
&= \frac{1}{B_m'(1)} \sum_{\lambda\in\mathbb{N}_m} \sum_{l\in\mathbb{N}} \sum_{k\in\mathbb{Z}} \frac{a_\lambda^{l-1}}{B_m'(a_\lambda)} \int_{(-\frac{1}{2}-k,\frac{1}{2}-k)} \chi_{(-\frac{l}{2},\frac{l}{2})}(t)\mathrm{e}^{-\mathrm{i}j2\pi t}\,\mathrm{d}t.
\end{aligned}$$

Using the orthonormality of $\{\mathrm{e}^{\mathrm{i}2\pi j\cdot} : j \in \mathbb{Z}\}$ leads to

$$\begin{aligned}
c_j(F_1) &= \frac{1}{B_m'(1)} \sum_{\lambda\in\mathbb{N}_m} \sum_{l\in\mathbb{N}} \sum_{k\in\mathbb{Z}} \frac{a_\lambda^{l-1}}{B_m'(a_\lambda)} \int_{(-\frac{1}{2}-k,\frac{1}{2}-k)\cap(-\frac{l}{2},\frac{l}{2})} \mathrm{e}^{-\mathrm{i}j2\pi t}\,\mathrm{d}t \\
&= \frac{1}{B_m'(1)} \sum_{\lambda\in\mathbb{N}_m} \sum_{l\in\mathbb{N}} \frac{a_\lambda^{l-1}}{B_m'(a_\lambda)} \sum_{-\frac{l-1}{2}\le k\le\frac{l-1}{2}} \delta_{j,0} \\
&= \frac{\delta_{j,0}}{B_m'(1)} \sum_{\lambda\in\mathbb{N}_m} \frac{1}{B_m'(a_\lambda)} \sum_{l\in\mathbb{N}} l a_\lambda^{l-1}.
\end{aligned}$$

Noting that

$$\sum_{k\in\mathbb{N}} k a^{k-1} = \frac{1}{(1-a)^2}$$

and (2.11), we get

$$c_j(F_1) = \frac{\delta_{j,0}}{B_m'(1)} \sum_{\lambda\in\mathbb{N}_m} \frac{1}{(1-a_\lambda)^2 B_m'(a_\lambda)} = \delta_{j,0}.$$

Now, suppose that $c_j(F_{n-1}) = \delta_{j,0}$. We want to verify that $c_j(F_n) = \delta_{j,0}$, which can be done by

$$
\begin{aligned}
c_j(F_n) &= \int_{(-\frac{1}{2},\frac{1}{2})} \frac{\alpha_m^n}{(B'_m(1))^n} \sum_{k\in\mathbb{Z}} \beta_n^{\vec{a}}(t-k)\mathrm{e}^{-\mathrm{i}j2\pi t}\,\mathrm{d}t \\
&= \frac{\alpha_m^n}{(B'_m(1))^n} \int_{(-\frac{1}{2},\frac{1}{2})} \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} \beta_{n-1}^{\vec{a}}(t-k-y)\beta_1^{\vec{a}}(y)\,\mathrm{d}y\ \mathrm{e}^{-\mathrm{i}j2\pi t}\,\mathrm{d}t \\
&= \frac{\alpha_m^n}{(B'_m(1))^n} \int_{\mathbb{R}} \beta_1^{\vec{a}}(y) \int_{(-\frac{1}{2},\frac{1}{2})} \sum_{k\in\mathbb{Z}} \beta_{n-1}^{\vec{a}}(t-k-y)\ \mathrm{e}^{-\mathrm{i}j2\pi t}\,\mathrm{d}t\,\mathrm{d}y \\
&= \frac{\alpha_m}{B'_m(1)} \int_{\mathbb{R}} \beta_1^{\vec{a}}(y) \int_{(-\frac{1}{2},\frac{1}{2})} \frac{\alpha_m^{n-1}}{(B'_m(1))^{n-1}} \sum_{k\in\mathbb{Z}} \beta_{n-1}^{\vec{a}}(t-k-y)\ \mathrm{e}^{-\mathrm{i}j2\pi t}\,\mathrm{d}t\,\mathrm{d}y \\
&= \frac{\alpha_m}{B'_m(1)} \int_{\mathbb{R}} \beta_1^{\vec{a}}(y) \int_{(-\frac{1}{2}-y,\frac{1}{2}-y)} \frac{\alpha_m^{n-1}}{(B'_m(1))^{n-1}} \sum_{k\in\mathbb{Z}} \beta_{n-1}^{\vec{a}}(t-k)\ \mathrm{e}^{-\mathrm{i}j2\pi t}\,\mathrm{d}t\mathrm{e}^{-\mathrm{i}2\pi jy}\,\mathrm{d}y \\
&= \frac{\alpha_m}{B'_m(1)} \int_{\mathbb{R}} \beta_1^{\vec{a}}(y) c_j(F_{n-1})\ \mathrm{e}^{-\mathrm{i}j2\pi y}\,\mathrm{d}y \\
&= \delta_{j,0}\frac{\alpha_m}{B'_m(1)}\sqrt{2\pi}(\beta_1^{\vec{a}})^\wedge(2\pi j) \\
&= \delta_{j,0}\frac{1}{B'_m(1)}\mathrm{sinc}_{\vec{a}}(\pi j).
\end{aligned}
$$

Finally, using (2.11) again, we conclude that $c_j(F_n) = \delta_{j,0}$. □

Next theorem states that the derivative of $\beta_n^{\vec{a}}$ equals to the weighted sum of the symmetric difference of $\beta_{n-1}^{\vec{a}}$.

Theorem 3.4 *For any $t \in \mathbb{R}$, the following identity holds*

$$\frac{d}{\mathrm{d}t}\beta_n^{\vec{a}}(t) = \frac{1}{\alpha_m} \sum_{j\in\mathbb{N}_m} \sum_{k\in\mathbb{N}} \frac{a_j^{k-1}}{B'_m(a_j)} \Delta_k \beta_{n-1}^{\vec{a}}(t). \tag{3.3}$$

The scalar case leads to

$$\frac{d}{\mathrm{d}t}\beta_n^{a}(t) = \sum_{k=1}^{\infty}(1-a)a^{k-1}\Delta_k\beta_{n-1}^{a}(t)$$

and

$$\frac{d}{\mathrm{d}t}\beta_n^{0}(t) = \Delta_1\beta_{n-1}^{0}(t).$$

Proof: Using Eq. (2.12), the dominated convergence theorem, and the direct calculation below

$$
\begin{aligned}
\frac{d}{\mathrm{d}t}\beta_n^{\vec{a}}(t) &= \frac{d}{\mathrm{d}x}\int_{\mathbb{R}} \beta_{n-1}^{\vec{a}}(t-y)\beta_1^{\vec{a}}(y)\,\mathrm{d}y \\
&= \int_{\mathbb{R}} (\beta_{n-1}^{\vec{a}})'(t-y)\frac{1}{\alpha_m}\sum_{j\in\mathbb{N}_m}\sum_{k\in\mathbb{N}}\frac{a_j^{k-1}}{B_m'(a_j)}\chi_{(-\frac{k}{2},\frac{k}{2})}(y)\,\mathrm{d}y \\
&= \frac{1}{\alpha_m}\sum_{j\in\mathbb{N}_m}\sum_{k\in\mathbb{N}}\frac{a_j^{k-1}}{B_m'(a_j)}\int_{(-\frac{k}{2},\frac{k}{2})}(\beta_{n-1}^{\vec{a}})'(t-y)\,\mathrm{d}y \\
&= \frac{1}{\alpha_m}\sum_{j\in\mathbb{N}_m}\sum_{k\in\mathbb{N}}\frac{a_j^{k-1}}{B_m'(a_j)}\int_{(-\frac{k}{2},\frac{k}{2})}(\beta_{n-1}^{\vec{a}})'(t+y)\,\mathrm{d}y \\
&= \frac{1}{\alpha_m}\sum_{j\in\mathbb{N}_m}\sum_{k\in\mathbb{N}}\frac{a_j^{k-1}}{B_m'(a_j)}\int_{(-\frac{k}{2},\frac{k}{2})}\left(\beta_{n-1}^{\vec{a}}(t+y)\right)'\,\mathrm{d}y \\
&= \frac{1}{\alpha_m}\sum_{j\in\mathbb{N}_m}\sum_{k\in\mathbb{N}}\frac{a_j^{k-1}}{B_m'(a_j)}\left(\beta_{n-1}^{\vec{a}}(t+\frac{k}{2})-\beta_{n-1}^{\vec{a}}(t-\frac{k}{2})\right),
\end{aligned}
$$

we conclude (3.3). □

The following theorem states that $\beta_n^{\vec{a}}$ can be alternatively defined by a recursive formula.

Theorem 3.5 *For $n \geq 2$ and $n \in \mathbb{N}$, $\beta_n^{\vec{a}}$ and $\beta_{n-1}^{\vec{a}}$ satisfy the recursive relation for every $t \in \mathbb{R}$,*

$$\beta_n^{\vec{a}}(t) = \frac{1}{\alpha_m}\sum_{j\in\mathbb{N}_m}\sum_{k\in\mathbb{N}}\frac{a_j^{k-1}}{B_m'(a_j)}\left(\frac{t+\frac{nk}{2}}{n-1}\beta_{n-1}^{\vec{a}}(t+\frac{k}{2})+\frac{\frac{nk}{2}-t}{n-1}\beta_{n-1}^{\vec{a}}(t-\frac{k}{2})\right). \tag{3.4}$$

In scalar case, the formula becomes

$$\beta_n^{a}(t) = \sum_{k\in\mathbb{N}}(1-a)a^{k-1}\left(\frac{t+\frac{nk}{2}}{n-1}\beta_{n-1}^{a}(t+\frac{k}{2})+\frac{\frac{nk}{2}-t}{n-1}\beta_{n-1}^{a}(t-\frac{k}{2})\right)$$

and in particular,

$$\beta_n^{0}(t) = \frac{t+\frac{n}{2}}{n-1}\beta_{n-1}^{0}(t+\frac{1}{2})+\frac{\frac{n}{2}-t}{n-1}\beta_{n-1}^{0}(t-\frac{1}{2}).$$

Proof: We note the functions on both sides of Eq. (3.4) are continuous for $n \geq 2$. It suffices to prove this identity by showing the Fourier transform of left-hand side equals to that of the right-hand side. By using (2.13), we obtain

$$
\begin{aligned}
&\sqrt{2\pi}\alpha_m^{n-1}\mathcal{F}\left(\cdot\beta_{n-1}^{\vec{a}}(\cdot+\frac{k}{2})-\cdot\beta_{n-1}^{\vec{a}}(\cdot-\frac{k}{2})\right)(\xi)\\
=\;& \mathrm{i}\frac{d}{d\xi}\left(\mathrm{e}^{\mathrm{i}\frac{k}{2}\xi}\left(\mathrm{sinc}_{\vec{a}}(\frac{\xi}{2})\right)^{n-1}-\mathrm{e}^{-\mathrm{i}\frac{k}{2}\xi}\left(\mathrm{sinc}_{\vec{a}}(\frac{\xi}{2})\right)^{n-1}\right)\\
=\;& -2\frac{d}{d\xi}\left(\sin\frac{k\xi}{2}\left(\mathrm{sinc}_{\vec{a}}(\frac{\xi}{2})\right)^{n-1}\right)\\
=\;& -2\left(\frac{k}{2}\cos\frac{k\xi}{2}\left(\mathrm{sinc}_{\vec{a}}(\frac{\xi}{2})\right)^{n-1}+(n-1)\sin\frac{k\xi}{2}\left(\mathrm{sinc}_{\vec{a}}(\frac{\xi}{2})\right)^{n-2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\mathrm{sinc}_{\vec{a}}(\frac{\xi}{2})\right)\right)\\
=\;& -2\left(\mathrm{sinc}_{\vec{a}}(\frac{\xi}{2})\right)^{n-1}\left(\frac{k}{2}\cos\frac{k\xi}{2}+(n-1)\sin\frac{k\xi}{2}\frac{\frac{\xi}{2}}{\sin\theta_{\vec{a}}(\frac{\xi}{2})}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\mathrm{sinc}_{\vec{a}}(\frac{\xi}{2})\right)\right)\\
=\;& -2\left(\mathrm{sinc}_{\vec{a}}(\frac{\xi}{2})\right)^{n-1}\left(\frac{k}{2}\cos\frac{k\xi}{2}+\frac{n-1}{2}\sin\frac{k\xi}{2}\frac{\frac{\xi}{2}}{\sin\theta_{\vec{a}}(\frac{\xi}{2})}\frac{\cos\theta_{\vec{a}}(\frac{\xi}{2})p_{\vec{a}}(\frac{\xi}{2})\frac{\xi}{2}-\sin\theta_{\vec{a}}(\frac{\xi}{2})}{(\frac{\xi}{2})^2}\right)\\
=\;& -\left(\mathrm{sinc}_{\vec{a}}(\frac{\xi}{2})\right)^{n-1}\left(k\cos\frac{k\xi}{2}+(n-1)\sin\frac{k\xi}{2}\left(\frac{\cos\theta_{\vec{a}}(\frac{\xi}{2})p_{\vec{a}}(\frac{\xi}{2})}{\sin\theta_{\vec{a}}(\frac{\xi}{2})}-\frac{1}{\xi/2}\right)\right).
\end{aligned}
$$

Using the formulae (2.9) and (2.10), it follows

$$
\begin{aligned}
&\sum_{j\in\mathbb{N}_m}\sum_{k\in\mathbb{N}}\frac{a_j^{k-1}}{B_m'(a_j)}\left(k\cos\frac{k\xi}{2}+(n-1)\sin\frac{k\xi}{2}\left(\frac{\cos\theta_{\vec{a}}(\frac{\xi}{2})p_{\vec{a}}(\frac{\xi}{2})}{\sin\theta_{\vec{a}}(\frac{\xi}{2})}-\frac{1}{\xi/2}\right)\right)\\
=\;& \sum_{j\in\mathbb{N}_m}\frac{1}{B_m'(a_j)}\left(\frac{p_{a_j}(\frac{\xi}{2})\cos\theta_{a_j}(\frac{\xi}{2})}{1-a_j^2}+(n-1)\frac{\sin\theta_{a_j}(\frac{\xi}{2})}{1-a_j^2}\left(\frac{\cos\theta_{\vec{a}}(\frac{\xi}{2})p_{\vec{a}}(\frac{\xi}{2})}{\sin\theta_{\vec{a}}(\frac{\xi}{2})}-\frac{1}{\xi/2}\right)\right).
\end{aligned}
$$

Differentiating Eq. (2.8) results in

$$
p_{\vec{a}}(t)\cos\theta_{\vec{a}}(t)=\sum_{j\in\mathbb{N}_m}\frac{p_{a_j}(t)\cos\theta_{a_j}(t)}{(1-a_j^2)B_m'(a_j)}. \tag{3.5}
$$

Consequently, Eqs. (2.8) and (3.5) yield

$$
\begin{aligned}
&\sum_{j\in\mathbb{N}_m}\sum_{k\in\mathbb{N}}\frac{a_j^{k-1}}{B_m'(a_j)}\left(k\cos\frac{k\xi}{2}+(n-1)\sin\frac{k\xi}{2}\left(\frac{\cos\theta_{\vec{a}}(\frac{\xi}{2})p_{\vec{a}}(\frac{\xi}{2})}{\sin\theta_{\vec{a}}(\frac{\xi}{2})}-\frac{1}{\xi/2}\right)\right)\\
=\;& p_{\vec{a}}(\frac{\xi}{2})\cos\theta_{\vec{a}}(\frac{\xi}{2})+(n-1)\sin\theta_{\vec{a}}(\frac{\xi}{2})\left(\frac{\cos\theta_{\vec{a}}(\frac{\xi}{2})p_{\vec{a}}(\frac{\xi}{2})}{\sin\theta_{\vec{a}}(\frac{\xi}{2})}-\frac{1}{\xi/2}\right)\\
=\;& np_{\vec{a}}(\frac{\xi}{2})\cos\theta_{\vec{a}}(\frac{\xi}{2})-(n-1)\mathrm{sinc}_{\vec{a}}(\frac{\xi}{2}).
\end{aligned}
$$

Therefore the Fourier transform of the right-hand side of Eq. (3.4) is

$$
\begin{aligned}
&\mathcal{F}\left(\frac{1}{\alpha_m}\sum_{j\in\mathbb{N}_m}\sum_{k\in\mathbb{N}}\frac{a_j^{k-1}}{B_m'(a_j)}\left(\frac{\cdot}{n-1}\beta_{n-1}^{\vec{a}}(\cdot+\frac{k}{2})-\frac{\cdot}{n-1}\beta_{n-1}^{\vec{a}}(\cdot-\frac{k}{2})\right)\right)(\xi)\\
=\;& -\frac{1}{\sqrt{2\pi}}\frac{1}{\alpha_m^n}\left(\operatorname{sinc}_{\vec{a}}(\frac{\xi}{2})\right)^{n-1}\left(\frac{n}{n-1}p_{\vec{a}}(\frac{\xi}{2})\cos\theta_{\vec{a}}(\frac{\xi}{2})-\operatorname{sinc}_{\vec{a}}(\frac{\xi}{2})\right).
\end{aligned}
$$

On the other hand, by using (2.10) again, the Fourier transform of the right-hand side of Eq. (3.4) is

$$
\begin{aligned}
&\mathcal{F}\left(\frac{1}{\alpha_m}\sum_{j\in\mathbb{N}_m}\sum_{k\in\mathbb{N}}\frac{a_j^{k-1}}{B_m'(a_j)}\frac{nk}{2(n-1)}\left(\beta_{n-1}^{\vec{a}}(\cdot+\frac{k}{2})+\beta_{n-1}^{\vec{a}}(\cdot-\frac{k}{2})\right)\right)(\xi)\\
=\;& \frac{1}{\alpha_m}\sum_{j\in\mathbb{N}_m}\sum_{k\in\mathbb{N}}\frac{a_j^{k-1}}{B_m'(a_j)}\frac{nk}{2(n-1)}\frac{1}{\sqrt{2\pi}}\frac{1}{\alpha_m^{n-1}}\left(\mathrm{e}^{\mathrm{i}\frac{k\xi}{2}}+\mathrm{e}^{-\mathrm{i}\frac{k\xi}{2}}\right)\left(\operatorname{sinc}_{\vec{a}}(\frac{\xi}{2})\right)^{n-1}\\
=\;& \frac{1}{\sqrt{2\pi}\alpha_m^n}\frac{n}{(n-1)}\left(\operatorname{sinc}_{\vec{a}}(\frac{\xi}{2})\right)^{n-1}\sum_{j\in\mathbb{N}_m}\frac{1}{B_m'(a_j)}\sum_{k\in\mathbb{N}}ka_j^{k-1}\cos\frac{k\xi}{2}\\
=\;& \frac{1}{\sqrt{2\pi}\alpha_m^n}\frac{n}{(n-1)}\left(\operatorname{sinc}_{\vec{a}}(\frac{\xi}{2})\right)^{n-1}\sum_{j\in\mathbb{N}_m}\frac{1}{B_m'(a_j)}\frac{1}{1-a_j^2}p_{a_j}(\frac{\xi}{2})\cos\theta_{a_j}(\frac{\xi}{2})\\
=\;& \frac{1}{\sqrt{2\pi}\alpha_m^n}\frac{n}{(n-1)}\left(\operatorname{sinc}_{\vec{a}}(\frac{\xi}{2})\right)^{n-1}p_{\vec{a}}(\frac{\xi}{2})\cos\theta_{\vec{a}}(\frac{\xi}{2}).
\end{aligned}
$$

Finally, we conclude that

$$
\begin{aligned}
&\mathcal{F}\left(\frac{1}{\alpha_m}\sum_{j\in\mathbb{N}_m}\sum_{k\in\mathbb{N}}\frac{a_j^{k-1}}{B_m'(a_j)}\left(\frac{\cdot+\frac{nk}{2}}{n-1}\beta_{n-1}^{\vec{a}}(\cdot+\frac{k}{2})+\frac{\frac{nk}{2}-\cdot}{n-1}\beta_{n-1}^{\vec{a}}(\cdot-\frac{k}{2})\right)\right)(\xi)\\
=\;& \mathcal{F}\left(\frac{1}{\alpha_m}\sum_{j\in\mathbb{N}_m}\sum_{k\in\mathbb{N}}\frac{a_j^{k-1}}{B_m'(a_j)}\left(\frac{\cdot}{n-1}\beta_{n-1}^{\vec{a}}(\cdot+\frac{k}{2})-\frac{\cdot}{n-1}\beta_{n-1}^{\vec{a}}(\cdot-\frac{k}{2})\right)\right)(\xi)\\
&+\mathcal{F}\left(\frac{1}{\alpha_m}\sum_{j\in\mathbb{N}_m}\sum_{k\in\mathbb{N}}\frac{a_j^{k-1}}{B_m'(a_j)}\frac{nk}{2(n-1)}\left(\beta_{n-1}^{\vec{a}}(\cdot+\frac{k}{2})+\beta_{n-1}^{\vec{a}}(\cdot-\frac{k}{2})\right)\right)(\xi)\\
=\;& -\frac{1}{\sqrt{2\pi}}\frac{1}{\alpha_m^n}\left(\operatorname{sinc}_{\vec{a}}(\frac{\xi}{2})\right)^{n-1}\left(\frac{n}{n-1}p_{\vec{a}}(\frac{\xi}{2})\cos\theta_{\vec{a}}(\frac{\xi}{2})-\operatorname{sinc}_{\vec{a}}(\frac{\xi}{2})\right)\\
&+\frac{1}{\sqrt{2\pi}\alpha_m^n}\frac{n}{(n-1)}\left(\operatorname{sinc}_{\vec{a}}(\frac{\xi}{2})\right)^{n-1}p_{\vec{a}}(\frac{\xi}{2})\cos\theta_{\vec{a}}(\frac{\xi}{2})\\
=\;& \frac{1}{\sqrt{2\pi}}\frac{1}{\alpha_m^n}\left(\operatorname{sinc}_{\vec{a}}(\frac{\xi}{2})\right)^{n}=(\beta_n^{\vec{a}})^\wedge(\xi).
\end{aligned}
$$

□

4 Probability Behavior

Define

$$p(t) = \frac{\alpha_m}{B'_m(1)} \beta_1^{\vec{a}}(t), \quad t \in \mathbb{R}.$$

The function p can be considered as a density since $p(t) \geq 0$ for any $t \in \mathbb{R}$ and $\int_{\mathbb{R}} p(t)\,\mathrm{d}t = 1$. In fact, by noting (2.11), we have

$$\begin{aligned}
\int_{\mathbb{R}} p(t)\,\mathrm{d}t &= \frac{\alpha_m}{B'_m(1)} \int_{\mathbb{R}} \frac{1}{\alpha_m} \sum_{j \in \mathbb{N}_m} \sum_{k \in \mathbb{N}} \frac{a_j^{k-1}}{B'_m(a_j)} \chi_{(-\frac{k}{2}, \frac{k}{2})}(t)\,\mathrm{d}t \\
&= \frac{1}{B'_m(1)} \sum_{j \in \mathbb{N}_m} \sum_{k \in \mathbb{N}} \frac{k a_j^{k-1}}{B'_m(a_j)} \\
&= \frac{1}{B'_m(1)} \sum_{j \in \mathbb{N}_m} \frac{1}{(1-a_j)^2 B'_m(a_j)} = 1.
\end{aligned}$$

Let X be a random variable with density p. We first calculate the expected value $E(X)$ and the variance $D(X)$ of X.

Lemma 4.1 *The expectation and variance of X are as follows*

$$E(X) = 0, \qquad D(X) = E(X^2) = \sigma_0^2 := \frac{1}{12} + \frac{B_m^{(2)}(1)}{4B'_m(1)} + \frac{B_m^{(3)}(1)}{12B'_m(1)}.$$

In scalar case, we have $\sigma_0^2 = \frac{1}{12} + \frac{a}{2(1-a)^2}$.

Proof: The vanishingness of the first moment of X is due to the evenness of the density p. Thus $D(X) = E(X^2)$. Using the definition of the density p, we obtain that

$$\begin{aligned}
\sigma_0^2 &= \int_{\mathbb{R}} t^2 p(t)\,\mathrm{d}t \\
&= \frac{\alpha_m}{B'_m(1)} \int_{\mathbb{R}} t^2 \frac{1}{\alpha_m} \sum_{j \in \mathbb{N}_m} \sum_{k \in \mathbb{N}} \frac{a_j^{k-1}}{B'_m(a_j)} \chi_{(-\frac{k}{2}, \frac{k}{2})}(t)\,\mathrm{d}t \\
&= \frac{1}{B'_m(1)} \sum_{j \in \mathbb{N}_m} \sum_{k \in \mathbb{N}} \frac{a_j^{k-1}}{B'_m(a_j)} \int_{(-\frac{k}{2}, \frac{k}{2})} t^2\,\mathrm{d}t \\
&= \frac{1}{12B'_m(1)} \sum_{j \in \mathbb{N}_m} \frac{1}{B'_m(a_j)} \sum_{k \in \mathbb{N}} k^3 a_j^{k-1}.
\end{aligned}$$

Using the equation

$$\sum_{k \in \mathbb{N}} k^3 a^{k-1} = \frac{a^2 + 4a + 1}{(1-a)^4}$$

and the two identities (2.11) and (2.7), we have

$$\begin{aligned}\sigma_0^2 &= \frac{1}{12B'_m(1)}\sum_{j\in\mathbb{N}_m}\frac{1}{B'_m(a_j)}\frac{a_j^2+4a_j+1}{(1-a_j)^4}\\ &= \frac{1}{12B'_m(1)}\sum_{j\in\mathbb{N}_m}\frac{1}{B'_m(a_j)}\left(\frac{1}{(1-a_j)^2}+\frac{6a_j}{(1-a_j)^4}\right)\\ &= \frac{1}{12}+\frac{1}{2B'_m(1)}\sum_{j\in\mathbb{N}_m}\frac{1}{B'_m(a_j)}\frac{a_j}{(1-a_j)^4}\\ &= \frac{1}{12}+\frac{B_m^{(2)}(1)}{4B'_m(1)}+\frac{B_m^{(3)}(1)}{12B'_m(1)},\end{aligned}$$

which completes the proof of this lemma. □

Let $X_n, n\in\mathbb{N}$ be a sequence of independent and identically distributed (iid) random variables with density p. Denote by $Y_n=\sum_{j=1}^n X_j$. The density p_{Y_n} of Y_n is as follows

$$p_{Y_n}(t)=\underbrace{p*\cdots*p}_{n}(t)=\left(\frac{\alpha_m}{B'_m(1)}\right)^n\beta_n^{\vec{a}}(t),\quad t\in\mathbb{R}.$$

Normalize the random variable Y_n by

$$Z_n=\frac{Y_n-E(Y_n)}{\sqrt{D(Y_n)}}=\frac{Y_n}{\sqrt{n}\sigma_0}.$$

The density of Z_n is

$$p_{Z_n}(x)=\sqrt{n}\sigma_0 p_{Y_n}(\sqrt{n}\sigma_0 x)=\sqrt{n}\sigma_0\left(\frac{\alpha_m}{B'_m(1)}\right)^n\beta_n^{\vec{a}}(\sqrt{n}\sigma_0 x),\quad x\in\mathbb{R}.$$

The central limit theorem tells us that the limiting distribution of Z_n is the normalized Gaussian distribution. We therefore obtain the asymptotic formula for the spline $\beta_n^{\vec{a}}$.

Theorem 4.2 *There hold the following two asymptotic formulae for spline $\beta_n^{\vec{a}}$ in time domain and frequency domain, respectively,*

$$\lim_{n\to\infty}\sqrt{n}\sigma_0\left(\frac{\alpha_m}{B'_m(1)}\right)^n\beta_n^{\vec{a}}(\sqrt{n}\sigma_0 t)=\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{t^2}{2}},\quad t\in\mathbb{R}\tag{4.1}$$

and

$$\lim_{n\to\infty}\frac{1}{\sqrt{2\pi}}\left(\frac{1}{B'_m(1)}\right)^n\left(\mathrm{sinc}_{\vec{a}}\left(\frac{\xi}{2\sqrt{n}\sigma_0}\right)\right)^n=\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{\xi^2}{2}},\quad \xi\in\mathbb{R}.\tag{4.2}$$

Moreover, the limit in (4.1) is uniformly convergent in $\mathbb{R}$ in the pointwise sense and convergent in the $L^q(\mathbb{R})$ sense for any $q\in[2,\infty]$, while the convergence in (4.2) is both in the pointwise sense and in the $L^p(\mathbb{R})$ sense for any $p\in[1,\infty)$.

Proof: Noting that $p_{\vec{a}}(0) = B'_m(1)$, $p'_{\vec{a}}(0) = 0$ and the Taylor formula of $\sin\theta_{\vec{a}}$

$$\sin\theta_{\vec{a}}(t) = p_{\vec{a}}(0)t + \frac{p_{\vec{a}}^{(2)}(0) - p_{\vec{a}}^3(0)}{3!}t^3 + O(t^4)$$

for any t in certain neighborhood of the origin, we get

$$\frac{1}{B'_m(1)}\sin\theta_{\vec{a}}(t) = t + \frac{p_{\vec{a}}^{(2)}(0) - p_{\vec{a}}^3(0)}{3!B'_m(1)}t^3 + O(t^4)$$

and

$$\frac{1}{B'_m(1)}\mathrm{sinc}_{\vec{a}}(t) = 1 - \frac{p_{\vec{a}}^3(0) - p_{\vec{a}}^{(2)}(0)}{3!B'_m(1)}t^2 + O(t^3).$$

Noting that

$$p_{\vec{a}}^{(2)}(0) = p_{\vec{a}}^3(0) - \left(B_m^{(3)}(1) + 3B_m^{(2)}(1) + B'_m(1)\right),$$

it follows

$$\begin{aligned}\frac{1}{B'_m(1)}\mathrm{sinc}_{\vec{a}}(t) &= 1 - \frac{B_m^{(3)}(1) + 3B_m^{(2)}(1) + B'_m(1)}{3!B'_m(1)}t^2 + O(t^3)\\ &= 1 - 2\sigma_0^2 t^2 + O(t^3),\end{aligned}$$

which yields

$$\frac{1}{B'_m(1)}\mathrm{sinc}_{\vec{a}}\left(\frac{t}{2\sigma_0\sqrt{n}}\right) = 1 - \frac{1}{2n}t^2 + O\left(\frac{t^3}{\sqrt{n^3}}\right).$$

The above equation holds for any $t \in \mathbb{R}$ since $\mathrm{sinc}_{\vec{a}}$ is real analytic (The convergence radius of its Taylor series is infinite). Therefore

$$\lim_{n\to\infty}\left[\frac{1}{B'_m(1)}\mathrm{sinc}_{\vec{a}}\left(\frac{t}{2\sigma_0\sqrt{n}}\right)\right]^n = \mathrm{e}^{-\frac{t^2}{2}},$$

which implies the pointwise convergence in (4.2).

Lemma 2 in [15] shows that, for any $t \in \mathbb{R}$ and for any $n \in \mathbb{N}\setminus\{1\}$,

$$\mathrm{sinc}\left(\frac{t}{\sqrt{n}}\right) \le \left(1 - \mathrm{rect}\left[\frac{t}{2}\right]\right)\frac{2}{(\pi t)^2} + \mathrm{e}^{-t^2},$$

where $\mathrm{sinc}t := \sin t/t$. Noticing that

$$\frac{1}{B'_m(1)}\mathrm{sinc}_{\vec{a}}(t) = \frac{1}{B'_m(1)}p_{\vec{a}}(t)\mathrm{sinc}(t),$$

we find that $\left(\frac{1}{B'_m(1)}\mathrm{sinc}_{\vec{a}}\left(\frac{\cdot}{2\sigma_0\sqrt{n}}\right)\right)^n$ is dominated by a $L^p(\mathbb{R})$ function for any $n \in \mathbb{N}\setminus\{1\}$ and for any given $p \in [1,\infty)$. This proves the $L^p(\mathbb{R})$ convergence of (4.2) for $p \in [1,\infty)$ by the Lebesgue dominated convergence theorem.

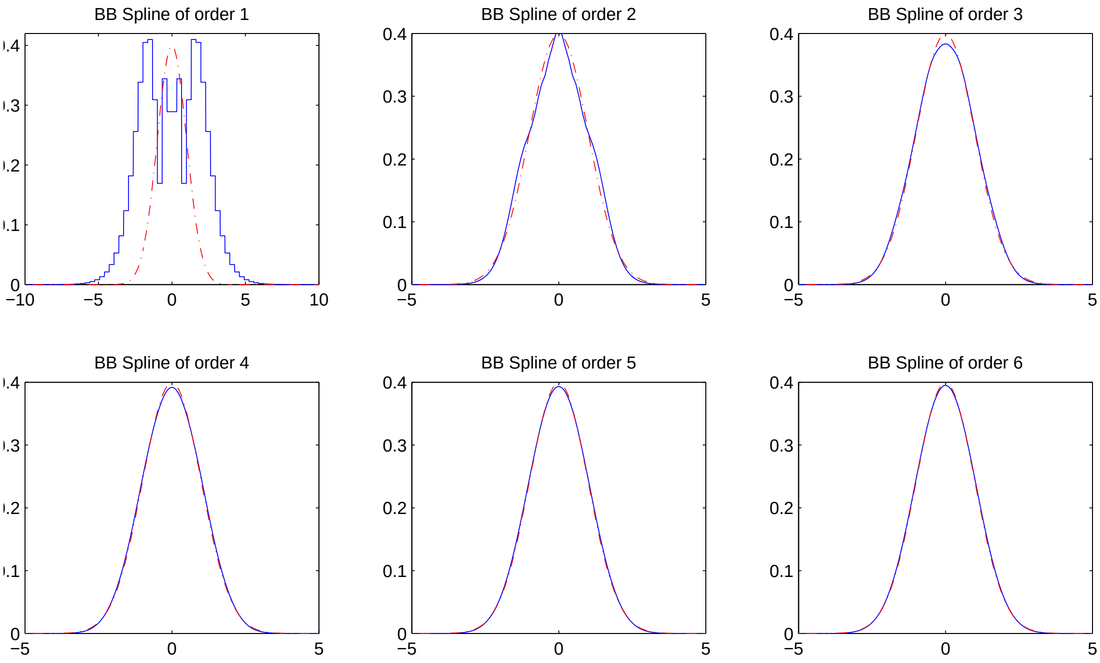

Figure 5.1: Normalized BB splines of various orders with $\vec{a} = [.2, .3, .5, .7]$ converging to the standard Gaussian. Solid (blue) line: BB-splines; Dash-dotted (red) line: standard Gaussian.

Since the Fourier transformation is a bounded linear operator from $L^p(\mathbb{R})$ to $L^q(\mathbb{R})$ for any $p \in [1, 2]$ and $p^{-1} + q^{-1} = 1$, the $L^q(\mathbb{R})$ convergence with $q \in [2, \infty]$ in (4.1) follows.

The point-wise convergence in (4.1) is a direct consequence of the following inequality

$$|\mathcal{F}h_n(x) - \mathcal{F}h(x)| \leq ||\mathcal{F}h_n - \mathcal{F}h||_{L^\infty(\mathbb{R})} \leq ||h_n - h||_{L^1(\mathbb{R})} \to 0$$

for any convergent sequence $\{h_n\}$ in $L^1(\mathbb{R})$. $\square$

5 Numerical Experiments

In this section, we conduct numerical experiments to confirm our previous theories. In Figs. 5.1 and 5.2 we plot the normalized *BB-splines* of various orders and various powers of the *general sinc* functions according to the functions on the left-hand sides of Eqs. (4.1) and (4.2), respectively. From the two figures we clearly see that both the BB-splines and the powers of the general sinc converge to the standard Gaussian quickly.

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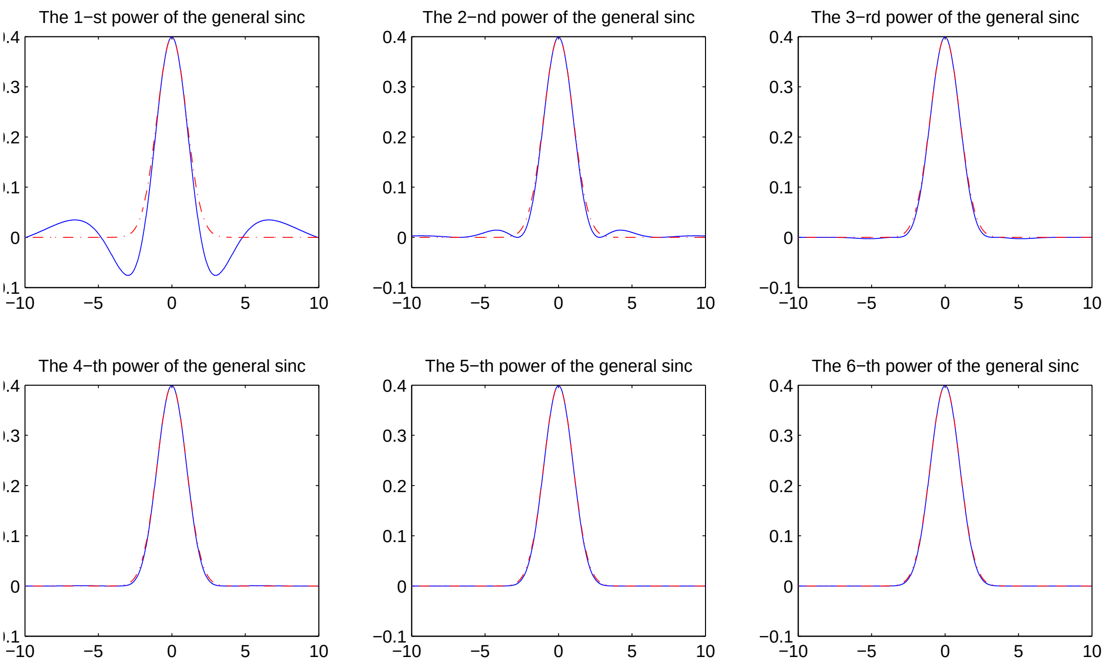

Figure 5.2: Normalized various powers of the general sinc functions with $\vec{a} = [.2, .3, .5, .7]$ converging to standard Gaussian. Solid (blue) line: Powers of the general sinc function; Dash-dotted (red) line: standard Gaussian.

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