A CLASS OF FOURIER MULTIPLIERS ON STARLIKE LIPSCHITZ SURFACES

PENGTAO LI, IENG TAK LEONG, AND TAO QIAN

Abstract. In this paper, we consider a class of Fourier multipliers whose symbols are controlled by a polynomial on starlike Lipschitz surfaces and get the L^2 boundedness of these operators on Sobolev spaces and their endpoint estimates.

1. Introduction

We first give a brief introduction to the studies of singular integral operators on Lipschitz curves and surfaces. In 1977, in [C], A. P. Calderón proved the L^2 boundedness of the singular Cauchy integral operators on a Lipschitz curve γ, where the Lipschitz constant of γ is small. In 1982, R. R. Coifman, A. McIntosh and Y. Meyer eliminate this restriction in [CMcM]. Ever since then, there have been a number of proofs on the L^2 boundedness of the singular Cauchy integral operator on Lipschitz curves. We refer the reader to the works of R. R. Coifman, P. Jones, S. Semmes, D. Jerison and C. Kenig ([CJS], [CM], [JK]) for further information.

In higher dimensional spaces, the corresponding question is the L^2 boundedness of certain convolution singular integrals on a Lipschitz surface

$$\Sigma = \left\{ g(\underline{x})e_0 + \underline{x} \in \mathbb{R}^{n+1},\ x \in \mathbb{R}^n \right\},$$

where g is a Lipschitz function which satisfies $\|\nabla g\|_\infty \leq \tan\omega < \infty$. In [LMcS], C. Li, A. McIntosh and S. Semmes embed $\mathbb{R}^{n+1}$ in the Clifford algebra $\mathbb{R}_{(n)}$ with identity e_0 and consider convolution singular integrals induced by right monogenic functions ϕ satifying $|\phi(x)| \leq C|x|^{-n}$ on a sector

$$S^0_\mu = \left\{ x = x_0 + \underline{x} \in \mathbb{R}^{n+1} : |x_0| < |\underline{x}| \tan\mu \right\},$$

where $\mu > \omega$. They proved that if there exists a function $\phi_1 \in L^\infty$ such that

$$\phi_1(R) - \phi_1(r) = \int_{r<|x|<R} \phi(x)dx, 0 < r < R < \infty,$$

then the related convolution singular integral operator $T_{(\phi,\ \phi_1)}$ defined by

$$(T_{\phi,\ \phi_1} u)(x) = \lim_{\varepsilon\to 0} \left\{ \int_{y\in\Sigma, |y-x|\geq\varepsilon} \phi(x-y)n(y)u(y)dS_y + \phi_1(\varepsilon n(x))u(x) \right\}$$

is bounded on $L^p(\Sigma)$ for $1 < p < \infty$. Alternative to the method of [LMcS], the work [GLQ] gives a treatment of the same topic by using the martingale method. In [Ta], the corresponding singular integral theory with harmonic kernels is established.

In the classical context, under some conditions, there exists a one to one correspondence between the convolution singular integral operators and their Fourier multiplier forms.

2000 *Mathematics Subject Classification.* Primary 35Q30; 76D03; 42B35; 46E30.

Key words and phrases. Clifford analysis, Fourier multiplier, singular integral, starlike Lipschitz surface, monogenic function.

Such correspondence is generalized to the analytic Cauchy kernels and the related Fourier multipliers ([CM], [LMcQ], [McQ2]). On the Lipschitz curves and surfaces context, by some appropriate definition of Fourier transform on the curves and surfaces, the above convolution operators have alterative representations in terms of Fourier multipliers ([McQ1], [LMcQ]). In [LMcQ], in the spirit of the $H^\infty-$functional calculi of the Dirac operator on the Lipschitz surfaces, C. Li, A. McIntosh and T. Qian extend a function of m real variables monogenically to a function of $m+1$ real variables (with values in complex Clifford algebra) and generalize its Fourier transform holomorphically to a function of m complex variables. In the setting, the Fourier transform b of ϕ can be regarded as, in a generalized sense, the Fourier multiplier representation of $T_{(\phi,\ \phi_1)}$. They proved that the class of bounded linear operators on $L^p(\Sigma)$ form the bounded H^∞ functional calculus of the Dirac differential operator $-iD_\Sigma = \sum_{k=1}^m -ie_k D_{k,\ \Sigma}$ and

$$T_{(\phi,\ \phi_1)} = b(-iD_\Sigma) = b(-iD_{1,\ \Sigma}, -iD_{2,\ \Sigma}, \cdots, -iD_{m,\ \Sigma}),$$

where the operators $D_{k,\Sigma}$ are defined by

$$D_{k,\ \Sigma} = \left(\frac{\partial}{\partial x_k}\right)_\Sigma u,$$

and u is the restriction to Σ of a function U being left monogenic on a neighborhood of Σ. The theory on Lipschitz surfaces is a generalization of the theory on Lipschitz curves developed by A. McIntosh and T. Qian ([McQ1], [McQ2]).

The above studies on singular integrals and Fourier multipliers on infinite Lipschitz curves and surfaces guide us to consider the analogue for the closed curves and surfaces. In a series of works in 1996-2001, the third author studied the Fourier analysis on starlike Lipschitz graphs and established a theory of a class of singular integrals on those surfaces in $\mathbb{R}^n$. For the details, we refer the readers to [Q1]-[Q5].

In the above mentioned works on convolution singular integral operators on either infinite or closed Lipschitz graphs, the multipliers $b(\xi)$ belong to the class $H^\infty(S^c_\mu)$ of bounded and holomorphic functions in sectors, viz.

$$\begin{aligned} H^\infty(S^c_{\mu,\pm}) \quad = \quad & \Big\{b:\ S^c_{\mu,\pm} \to \mathbb{C}:\ b \text{ is holomorphic and satisfies} \\ & |b(z)| \le C_\nu \text{ in every } \ S^c_{\nu,\pm},\ 0<\nu<\mu\Big\}, \end{aligned}$$

and

$$H^\infty(S^c_\mu) = \Big\{b:\ S^c_\mu \to \mathbb{C}:\ b_\pm = b\chi_{\{z\in\mathbb{C}:\ \pm\mathrm{Re}z>0\}} \in H^\infty(S^c_{\mu,\pm})\Big\},$$

where the sectors $S^c_{\mu,\pm}$ and S^c_μ are some cones in the complex plane defined in Section 3. It is natural to ask what happens if $b(\xi)$ is dominated by a polynomial, and if we could establish a similar theory of singular integral operators for such multipliers.

On the other hand, in the recent development of Clifford analysis, there exist some examples that can not be included into the established frame work of singular integrals on the Lipschitz graphs.

Example 1.1. In [E], [ES], D. Eelbode introduces the photogenic Cauchy transform C^α_P on the unit sphere in $\mathbb{R}^m$ when solving the so called photogenic Dirac equation for hyperbolic fundamental solutions having singularities on the nullcone. Before we introduce this transform, we state some background knowledge.

Let $\mathbb{R}^{1,m}$ denote the real orthogonal space with orthogonal basis $B_{1,m}(\varepsilon, e_j) = \{\varepsilon, e_1, \ldots, e_m\}$, endowed with the quadratic form

$$Q_{1,m}(T, \underline{X}) = T^2 - \sum_{j=1}^{m} X_j^2 = T^2 - R^2,$$

where we have put $R = |\underline{X}| = (\sum_{j=1}^{m} X_j^2)^{1/2}$. The orthogonal space $\mathbb{R}^{1,m}$ is called m dimensional space-time, m referring to the number of the spatial dimensions. The space-time Clifford algebra $\mathbb{R}_{1,m}$ is generated by the multiplication rules: $e_i e_j + e_j e_i = -2\delta_{ij}$ for all $1 \le i, j \le m$, $e_i\varepsilon + \varepsilon e_i = 0$ for all i and $\varepsilon^2 = 1$. Vectors in $\mathbb{R}^{1,m}$, i.e. $(m+1)-$tuples $(T, \underline{X})$ or space-time vectors, are identified with $1-$vectors in $\mathbb{R}_{1,m}$ under the canonical map

$$(T, \underline{X}) = (T, X_1, \cdots, X_m) \longmapsto \varepsilon T + \underline{X} \in \mathbb{R}_{1,m}.$$

The Dirac operator on $\mathbb{R}^{1,m}$ is given by the vector derivative $D(T, \underline{X})_{1,m} = \varepsilon\partial_T - \sum_{j=1}^{m} e_j \partial_{X_j}$ which factorizes the wave operator $\square_m = \partial_T^2 - \Delta_m$ on $\mathbb{R}^{1,m}$:

$$\square_m = (\varepsilon\partial_T - \sum_{j=1}^{m} e_j \partial_{X_j})^2.$$

For $\alpha + m \ge 0$ and $\underline{\omega} \in S^{m-1}$, we consider the following photogenic Dirac equation

$$(\varepsilon\partial_T - \partial_{\underline{X}})\mathcal{F}_{\alpha,\underline{\omega}}(T, \underline{X}) = T^{\alpha+m-1}\delta(T\underline{\omega} - \underline{X})$$

and take the transformation:

$$\lambda = T \text{ and } \underline{x} = \frac{\underline{X}}{T} = r\underline{\xi} \in B_m(1), \text{ where } B_m(1) \text{ is the unit ball in } \mathbb{R}^m \text{ and } |\xi| = 1.$$

It was proved by D. Eelbode in [E] that

$$\begin{aligned} \mathcal{F}_\alpha(\underline{x}, \underline{\omega}) &= (2\alpha + m + 1)c(\alpha, m)(\varepsilon + \underline{x})\frac{(1 - r^2)^{\alpha + \frac{m-1}{2}}}{(1 - <\underline{x}, \underline{\omega}>)^{\alpha+m}} \\ &+ (\alpha + m)c(\alpha, m)(\varepsilon + \underline{\omega})\frac{(1 - r^2)^{\alpha + \frac{m+1}{2}}}{(1 - <\underline{x}, \underline{\omega}>)^{\alpha+m+1}}, \end{aligned}$$

where $c(\alpha, m)$ is a constant associated with α and m. Furthermore, let $f(\underline{\omega})$ be arbitrary function defined on the sphere S^{m-1}. For all $\underline{x} \in B_m(1)$, the corresponding photogenic Cauchy transform $C_P^\alpha[f](\underline{x})$ of f is defined by

$$C_P^\alpha[f](\underline{x}) = \frac{1}{\Omega_m}\int_{S^{m-1}} \mathcal{F}_\alpha(\underline{x}, \underline{\omega})\underline{\omega} f(\omega) d\omega,$$

where Ω_m is the surface area of the sphere S^{m-1}.

If we apply this transform C_P^α on the inner and outer spherical monogenic polynomials P_k and Q_k on $\mathbb{R}^n \setminus \{0\}$, respectively, and take their boundary values by letting $r \to 1-$, we can get the boundary values $C_P^\alpha[P_k] \uparrow$ and $C_P^\alpha[Q_k] \uparrow$, as follows

$$\begin{aligned} C_P^\alpha[P_k] \uparrow (\underline{\xi}) &= \frac{\Gamma(\frac{m-1}{2})}{8\pi^{\frac{m-1}{2}}} \frac{(\alpha + m + k)\{(\alpha + m + k - 1) + (k - \alpha)\underline{\xi}\varepsilon\}P_k(\underline{\xi})}{(\alpha + \frac{m+1}{2})(\alpha + \frac{m-1}{2})}, \\ C_P^\alpha[Q_k] \uparrow (\underline{\xi}) &= \frac{\Gamma(\frac{m-1}{2})}{8\pi^{\frac{m-1}{2}}} \frac{(1 + \alpha - k)\{(\alpha - k) + (\alpha + m + k - 1)\underline{\xi}\varepsilon\}Q_k(\underline{\xi})}{(\alpha + \frac{m+1}{2})(\alpha + \frac{m-1}{2})}. \end{aligned}$$

Clearly, the appearance of the terms $k^2P_k(\underline{\xi})$, $kP_k(\underline{\xi})$, $k^2Q_k(\underline{\xi})$, $kQ_k(\underline{\xi})$ implies that for $f \in L^2(S^{m-1})$ the boundary value $C_P^\alpha[f] \uparrow$ does not belong to $L^2(S^{m-1})$ unless we restrict f to be in a smaller space. In [E], the author replaces $L^2(S^{m-1})$ by a certain Sobolev space on the sphere to get the boundedness of $C_P^\alpha[f] \uparrow$. Eelbode's work above inspires us to consider the class of Fourier multipliers $b(\xi)$ satisfying

$$|b(\xi)| \le C|\xi + 1|^s \text{ in some domain for } s > 0,$$

and study the boundedness of the convolution singular integral operators related to these Fourier multipliers.

Remark 1.2. Specially, if we take some special b_k in the definition of the Fourier multipliers (see Definition 4.3 and the remark below), we can see that the multiplier operators become the boundary values of the Cauchy transform on the hyperbolic unit sphere studied in [E] and [ES]. Hence the result on the boundedness of multipliers, obtained in the present paper, generalizes those in [E].

In comparison with the work dealing with photogenic Cauchy transforms C_P^α in Example 1.1, there exist two difficulties in our setting for the Fourier multipliers.

(1) The kernel $\mathcal{F}_\alpha(\underline{x}, \underline{\omega})$ of the Cauchy transforms C_P^α can be deduced from the fundamental solution of the wave operator $\Box_m$. However, for the kernel of our multipliers, such an explicit representation is not available.

(2) On the unit sphere of $\mathbb{R}^n$, the Plancherel theorem holds. After getting the decomposition of $C_P^\alpha(f)$ associated with spherical monogenic, in [E], the author could easily deduce that the function $C_P^\alpha(f)$ belongs to $L^2(S^{m-1})$ for f belongs to some Sobolev space. In new context, i.e. on starlike Lipschitz surfaces, however, there does not exist Plancherel's theorem. The method in [E] is not applicable.

To overcome the mentioned difficulties, we estimate the kernels of multipliers by Fueter's theorem and its generalization (see Section 3 for details). Our estimates show that the kernels of the Fourier multipliers decay as a polynomial with index $-(n + s)$. Our proof is similar to that of Qian in [Q5] with some modifications. In particular, for a negative index s, the term $|x|^s$ is unbounded in the domain $H_{\omega,+}$. Hence after getting the estimate of the kernels on $H_{\omega,-}$, the Kelvin inversion fails to get that on $H_{\omega,+}$ (see Theorem 3.9).

The organization of this paper is as follows. In Section 2, we state some preliminary knowledge, notation and terminology that will be used throughout the paper. In Section 3, using a generalized Fueter's result, we estimate the kernel of the Fourier multiplier operators. In Section 4, by the theory of Hardy spaces on starlike Lipschitz surfaces established in [K], [Mi], we prove the L^2 boundedness and the endpoint estimate of the corresponding singular integral operators.

2. Preliminaries

In this paper we work with the real Clifford algebra $\mathbb{R}^{(n)}$ generated by $e_1, e_2, \cdots e_n$ as its basic vectors, over the real number field $\mathbb{R}$ under the multiplication relations:

$$\begin{aligned} e_0 &= 1; \\ e_i^2 &= -e_0 = -1, \quad 1 \le i \le n; \\ e_ie_j + e_je_i &= 0, \quad i \ne j, 1 \le i, j \le n. \end{aligned}$$

We denote by $\mathbb{R}_1^n$ and $\mathbb{R}^n$ the real linear subspaces of $\mathbb{R}^{(n)}$ generated by $\{e_0, e_1, e_2, \cdots e_n\}$ and by $\{e_1, e_2, \cdots e_n\}$, respectively. A vector in $\mathbb{R}_1^n$ is represented as $x = x_0e_0 + \underline{x}$, where $x_0 \in \mathbb{R}$ and $\underline{x} = x_1e_1 + \cdots + x_ne_n \in \mathbb{R}^n$. Similar to the complex case, we call x_0e_0 and $\underline{x}$ the real and imaginary parts of x. There are two basic operations on these elements:$(e_{i_1}\cdots e_{i_l})^* = e_{i_l}\cdots e_{i_1}$ and $(e_{i_1}\cdots e_{i_l})' = (e_{i_1})'\cdots(e_{i_l})'$, where $(e_0)' = e_0$, $(e_j)' = -e_j, j = 1,\cdots,n$. By linearity, they can be extended to $\mathbb{R}^{(n)}$, $\mathbb{R}^n_1$, $\mathbb{R}^n$ respectively. We define the operation "-" by $\overline{x} = (x^*)'$. If x and y are two elements in $\mathbb{R}^{(n)}$, then we have $\overline{xy} = \overline{y}\,\overline{x}$. If $x = x_0 + \underline{x}$, then $\overline{x} = x_0 - \underline{x}$. If x is a nonzero vector, then its inverse x^{-1} exists, and satisfies: $x^{-1} = \frac{\overline{x}}{|x|^2}$ and $x^{-1}x = xx^{-1} = 1$, where $|x|^2 = x\cdot\overline{x}$.

We also use the complex Clifford algebra $\mathbb{C}^{(n)}$ generated by $e_1,\cdots,e_n$ over the complex number field $\mathbb{C}$, whose elements are also denoted by $x, y, \cdots$. The complex imaginary element i commutes with all the e_j, $j = 0, 1,\cdots,n$ and $i' = -i$. Therefore we can extend the definitions of $*$, $'$ and $-$ from $\mathbb{R}^{(n)}$ to $\mathbb{C}^{(n)}$ respectively.

The natural inner product $\langle x, y\rangle$ between x and y in $\mathbb{C}^{(n)}$ is the complex number $\Sigma_S\, x_S\,\overline{y}_S$, where $x = \Sigma_S\, x_S\, e_S$, $y = \Sigma_S\, y_S\, e_S$ and S runs over all the ordered subsets $(i_1,\cdots,i_l)$ with $i_1 < i_2 < \cdots < i_l$ of the set $\{1, 2,\cdots,n\}$ and $e_S = e_{i_1}\cdots e_{i_l}$. Hence it is natural to define the norm of this inner space as $|x| = \langle x, x\rangle^{1/2} = (\Sigma_S |x_S|^2)^{1/2}$. We can easily prove the inner product and the norm satisfy the parallelogram identity, that is, $\langle x, y\rangle = \frac{1}{4}(|x+y|^2 - |x-y|^2)$. The angle between two vectors x and y, denoted by $\arg(x, y)$, is defined to be $\arccos\langle x, y\rangle/(|x||y|)$, where the function arccos takes values in $[0, \pi)$.

We denote the unit sphere $\{x \in \mathbb{R}^n_1 : |x| = 1\}$ in $\mathbb{R}^n_1$ by $S_{\mathbb{R}^n_1}$ and the unit sphere in $\mathbb{R}^n$ $\{\underline{x} \in \mathbb{R}^n : |\underline{x}| = 1\}$ by $S_{\mathbb{R}^n}$.

We now state some basic analysis in $\mathbb{R}^n_1$. The readers can find the details about the results we list here in [BDS] and [DSS].

The differential operator $D = D_0 + \underline{D}$, where $D_0 = \frac{\partial}{\partial x_0}$, and $\underline{D} = \sum_{k=1}^n \frac{\partial}{\partial x_k} e_k$, applies to C^1-functions f and gives

$$Df = \sum_{k=0}^{n} \frac{\partial f_S}{\partial x_k} e_k e_S$$

and also

$$fD = \sum_{k=0}^{n} \frac{\partial f_S}{\partial x_k} e_S e_k, \text{ where } f = \sum_S f_S\, e_S.$$

In polar coordinate system the Dirac operator D can be decomposed into

$$D = \xi\partial_r - \frac{1}{r}\partial_\xi = \xi\left(\partial_r - \frac{1}{r}\Gamma_\xi\right),$$

where Γ_ξ is a first order differential operator depending only on the angular coordinates known as spherical Dirac operator (see [DSS]).

A C^1-function defined on an open subset of $\mathbb{R}^{n+1}$ with values in $\mathbb{R}_{(n)}$ or $\mathbb{C}_{(n)}$ is called left monogenic if $Df = 0$ and right monogenic if $fD = 0$. All real analytic functions f defined in domains in $\mathbb{R}^n$ have both left- and right-monogenic extensions to domains in $\mathbb{R}^{n+1}$. These two extensions coincide if and only if $Df = fD$, so they coincide if f is scalar valued.

One basic example of both left- and right-monogenic functions is the Cauchy kernel which is defined by $E(x) = \overline{x}/|x|^{n+1}$ on $\mathbb{R}^n_1 \setminus \{0\}$. By this kernel, we can define the Kelvin inversion $I(f)(x) = E(x)f(x^{-1})$. This operation preserves the monogenicity of the functions. Take the unit sphere $S_{\mathbb{R}^n_1}$ for example, for f is a monogenic function defined on the interior of $S_{\mathbb{R}^n_1}$, $I(f)$ is also a monogenic one defined on the exterior of $S_{\mathbb{R}^n_1}$. It is easy to see that $|I(f)(x)| \le |f(x)|/|x|^n$. In Clifford analysis, after getting the estimate of f on the interior of $S_{\mathbb{R}^n_1}$, we often apply the Kelvin inversion to get the related estimate on the exterior part, and vice versa.

3. A class of Fourier multipliers generated by the monomial functions in $\mathbb{R}^n_1$

In this section, we consider a class of Fourier multipliers which are dominated by a polynomial in some domain and then estimate the kernel of the singular integrals associated to the multipliers. We achieve this goal by the generalization of Fueter's theorem which is obtained in [Sc] and [Q4]. The basic idea is construct a relation between a set O in the complex plane $\mathbb{C}$ and the one $\overrightarrow{O}$ in the $n+1$ dimensional space $\mathbb{R}^n_1$, and hence one can study the functions defined on $\overrightarrow{O}$ by the results which have been established on O.

We begin this method with the definition of intrinsic sets.

Definition 3.1. (i) A subset O in the complex plane $\mathbb{C}$ is said to be intrinsic if it is symmetric with respect to the real axis, i.e. O is invariant under the complex conjugate.

(ii) A function f^0 is said to be intrinsic if the domain of f^0 is an intrinsic set in $\mathbb{C}$ and $\overline{f^0(z)} = f^0(\overline{z})$ within its domain.

Functions of the form $\sum c_k(z-a_k)^k,\ k \in \mathbb{Z},\ a_k, c_k \in \mathbb{R}$ are intrinsic functions. If $f = u+iv$, where u and v are real-valued, then f^0 is intrinsic if and only if $u(x,-y) = u(x,y)$ and $v(x,-y) = -v(x,y)$ in their domains.

We consider $\mathbb{R}^n_1$ as $n+1$ dimensional Euclidean space and define the intrinsic sets in $\mathbb{R}^n_1$ as follows.

Definition 3.2. A subset in $\mathbb{R}^n_1$ is said to be intrinsic if it is invariant under all the rotations of $\mathbb{R}^n_1$, considered as $n+1$ dimensional Euclidean space, that keep the e_0-axis fixed. If O is a subset in the complex plane, then in $\mathbb{R}^n_1$, one defines an intrinsic set

$$\overrightarrow{O} = \left\{x \in \mathbb{R}^n_1 : (x_0, |\underline{x}|) \in O\right\},$$

which is called the induced set from O.

Definition 3.3. Let $f^0(z) = u(x,y) + iv(x,y)$ be an intrinsic function defined on an intrinsic set $U \subset \mathbb{C}$. Define a function $\overrightarrow{f^0}$ defined on the induced set $\overrightarrow{U}$ as follows:

$$\overrightarrow{f^0}(x_0 + \underline{x}) = u(x_0, |\underline{x}|) + \frac{\underline{x}}{|\underline{x}|} v(x_0, |\underline{x}|),$$

which is called the function induced from f^0.

We denote by τ the mapping:

$$\tau(f^0) = k_n^{-1} \Delta^{(n-1)/2} \overrightarrow{f^0},$$

where $\Delta = D\overline{D}$ with $\overline{D} = D_0 - \underline{D}$ and $k_n = (2i)^{n-1}\Gamma^2(\frac{n+1}{2})$ is the normalizing constant that makes $\tau((\cdot)^{-1}) = E$. The operator $\Delta^{(n-1)/2}$ is defined via the Fourier multiplier transform on tempered distributions $\mathcal{M} : \mathcal{S}' \to \mathcal{S}'$ induced by the multiplier $m(\xi) = (2\pi i|\xi|)^{n-1}$,

$$\mathcal{M}f = \mathcal{R}(m\mathcal{F}f),$$

where

$$\mathcal{F}f(\xi) = \int_{\mathbb{R}^n_1} e^{2\pi i\langle x,\xi\rangle} f(x)dx$$

and

$$\mathcal{R}h(x) = \int_{\mathbb{R}^n_1} e^{-2\pi i\langle x,\xi\rangle} h(\xi)d\xi.$$

The monomial functions in $\mathbb{R}^n_1$ are defined by

$$P^{(-k)} = \tau((\cdot)^{-k}) \text{ and } P^{(k-1)} = I(P^{(-k)}), k \in \mathbb{Z}^+,$$

where I denotes the Kelvin inversion $I(f)(x) = E(x)f(x^{-1})$.

In [Q5], Qian gets the following proposition, which generalizes the relation between the Cauchy kernel $\frac{1}{z}$ in the complex plane and the Cauchy kernel in $\mathbb{R}^n_1$.

Proposition 3.4. *([Q5]) Let $k \in \mathbb{Z}^+$. Then (1) $P^{(-1)}(x) = E(x)$; (2) $P^{(-k)}(x) = \frac{(-1)^{k-1}}{(k-1)!}(\frac{\partial}{\partial x_0})^{k-1}E(x)$; (3) $P^{(-k)}$ and $P^{(k-1)}$ are monogenic; (4) $P^{(-k)}$ is homogeneous of degree $-n+1-k$ and $P^{(k-1)}$ homogeneous of degree $k-1$; (5) For $c_n = \int_{-\infty}^{\infty}(1+t^2)^{-(n+1)/2}dt$, we have*

$$c_n P^{(-k)}_{n-1}(x_0 + x_1e_1 + \cdots + x_{n-1}e_{n-1}) = \int_{-\infty}^{\infty} P^{(-k)}_n(x)dx_n;$$

(6)$P^{(-k)} = I(P^{(k-1)})$; (7) If n is odd, then $P^{(k-1)} = \tau((\cdot)^{n+k-2})$.

In our analysis, we need the following sets in the complex plane. For $\omega \in (0, \frac{\pi}{2})$, let

$$\begin{aligned}
S^c_{\omega,\pm} &= \left\{z \in \mathbb{C} : |arg(\pm z)| < \omega\right\} \text{ with the angle } \arg(z) \in (-\pi, \pi], \\
S^c_{\omega,\pm}(\pi) &= \left\{z \in \mathbb{C} : |\mathrm{Re}\, z| \le \pi,\ z \in S^c_{\omega,\pm}\right\}, \\
S^c_{\omega} &= S^c_{\omega,+} \cup S^c_{\omega,-} \text{ and } S^c_{\omega}(\pi) = S^c_{\omega,+}(\pi) \cup S^c_{\omega,-}(\pi), \\
W^c_{\omega,\pm}(\pi) &= \left\{z \in \mathbb{C} : |\mathrm{Re}\, z| \le \pi \text{ and } \pm\ \mathrm{Im} z > 0\right\} \cup S^c_{\omega}(\pi), \\
H^c_{\omega,\pm} &= \left\{z = \exp(i\eta) \in \mathbb{C}, \eta \in W^c_{\omega,\pm}(\pi)\right\} \\
H^c_{\omega} &= H^c_{\omega,+} \cap H^c_{\omega,-}.
\end{aligned}$$

We define the Fourier multiplier in the following function spaces

$$\begin{aligned}
K^s(H^c_{\omega,\pm}) \quad = \quad & \Big\{\phi^0 :\ H^c_{\omega,\pm} \to \mathbb{C}, \phi^0 \text{ is holomorphic and} \\
& |\phi^0(z)| \le \frac{C_\mu}{|1-z|^{1+s}} \text{ in every } H^c_{\mu,\pm},\ 0 < \mu < \omega\Big\},
\end{aligned}$$

$$K^s(H^c_{\omega}) = \left\{\phi^0 :\ H^c_{\omega} \to \mathbb{C},\ \phi^0 = \phi^{0,+} + \phi^{0,-}, \phi^{0,\pm} \in K^s(H^c_{\omega,\pm})\right\},$$

$$H^s(S^c_{\omega,\pm}) = \left\{b :\ S^c_{\omega,\pm} \to \mathbb{C},\ b \text{ is holomorphic and } |b(z)| \le C_\mu |z \pm 1|^s \text{ in every } S^c_{\mu,\pm},\ 0 < \mu < \omega\right\}$$

and

$$H^s(S^c_{\omega}) = \left\{b :\ S^c_{\omega} \to \mathbb{C},\ b_\pm = b\chi_{\{z \in \mathbb{C}: \pm Re z > 0\}} \in H^s(S^c_{\omega,\pm})\right\}.$$

By Fueter's method which we state at the beginning of the section, we will work on the "heart-shaped" regions and their complements.

Set

$$\begin{aligned}
H_{\omega,\pm} &= \left\{x \in \mathbb{R}^n_1 : \frac{(\pm \ln |x|)}{\arg(e_0, x)} < \tan\omega\right\} = \overrightarrow{H^c_{\omega,\pm}}, \\
H_{\omega} &= H_{\omega,+} \cap H_{\omega,-} = \left\{x \in \mathbb{R}^n_1 :\ \frac{|\ln |x||}{\arg(e_0, x)} < \tan\omega\right\} = \overrightarrow{H^c_{\omega}}.
\end{aligned}$$

Therefore, the corresponding function spaces in $\mathbb{R}^n_1$ are

$$K^s(H_{\omega,\pm}) = \left\{\phi :\ H_{\omega,\pm} \to \mathbb{C}^{(n)}, \phi \text{ is monogenic and } |\phi(x)| \le \frac{C_\mu}{|1-x|^{n+s}},\ x \in H_{\mu,\pm},\ 0 < \mu < \omega\right\}$$

and

$$K^s(H_{\omega}) = \left\{\phi :\ H_{\omega} \to \mathbb{C}^{(n)},\ \phi = \phi^+ + \phi^-,\ \phi^{\pm} \in K^s(H_{\omega,\pm})\right\}.$$

Now we consider the multiplier $b \in H^s(S^c_{\omega,\pm})$. At first we prove the following lemma, which gives an estimate of the jth derivative of an intrinsic function ϕ^0.

Lemma 3.5. *Suppose $b(z) \in H^s(S^c_{\omega,-})$. For the multiplier defined by $\phi^0(z) = \sum_{k=1}^{\infty} b(-k)z^{-k}$, its jth derivative satisfies*

$$|(\phi^0)^{(j)}(z)| \leq \frac{C}{|1-z|^{s+j+1}},$$

where $z \in H^c_{\mu,-}$, $0 < \mu < \omega$, and j is a positive integer.

Proof. Without loss of generality, we can assume $|b(-k)| \leq |k|^s$ for $b(z) \in H^s(S^c_{\omega,-})$. The case $j = 0$ was treated by T. Qian. In [Q3], the author proves that for $\phi^0(z) = \sum_{k=1}^{\infty} b(-k)z^{-k}$,

$$|\phi^0(z)| \leq \frac{C}{|1-z|^{s+1}}.$$

We will use the same method as in [Q3]. Take the circle $C(z,r)$ with the center z and the radius r. By Cauchy's formula, we can get

$$\left|(\phi^0)^{(j)}(z)\right| \leq \frac{C_j}{2\pi} \int_{C(z,r)} \frac{|\phi^0(\xi)|}{|z-\xi|^{j+1}} |d\xi|.$$

Letting $r = \frac{1}{2}|1-z|$, then $\xi \in C(z,r)$ implies

$$|1-\xi| \geq |1-z| - |z-\xi| = |1-z| - \frac{1}{2}|1-z| = \frac{1}{2}|1-z|.$$

Hence we get

$$\left|(\phi^0)^{(j)}(z)\right| \leq \frac{2j!C_\mu}{\delta^j(\mu)} \frac{1}{|1-z|^{j+s+2}} |1-z| \leq C_{\mu,j} \frac{1}{|1-z|^{j+s+1}}.$$

This completes the proof of Lemma 3.5. □

The next lemma is a useful tool used in the proof of our main result in this section. By this lemma, we can estimate the multipliers in $K^s(H_{\omega,\pm})$ by an inductive argument.

Lemma 3.6. *([Q5], Lemma 1) Let $f^0(z) = u(s,t) + iv(s,t)$ be function holomorphically defined in a relatively open subset U of the upper half complex plane. For $l = 0$, denote $u_0 = u$ and $v_0 = v$. For $l \in \mathbb{Z}^+$, denote*

$$u_l = 2l\frac{1}{t}\frac{\partial u_{l-1}}{\partial t} \text{ and } v_l = 2l\left(\frac{\partial v_{l-1}}{\partial t}\frac{1}{t} - \frac{v_{l-1}}{t^2}\right) = 2l\frac{\partial}{\partial t}(\frac{v_{l-1}}{t}).$$

Then we have

$$\Delta^l \overrightarrow{f^0}(x) = u_l(x_0, |\underline{x}|) + \frac{\underline{x}}{|\underline{x}|} v_l(x_0, |\underline{x}|).$$

Lemma 3.6 enables us to estimate the kernels of the multipliers which are generated by functions in $H^s(S^c_\omega)$ and spherical monogenic functions.

Theorem 3.7. *For $s > 0$, if $b \in H^s(S^c_{\omega,\pm})$ and $\phi(x) = \sum_{k=\pm}^{\pm\infty} b(k)P^{(k)}(x)$, then $\phi \in K^s(H_{\omega,\pm})$.*

Proof. Following Qian's idea in [Q5], we divide the proof into two cases according to the parity of n.

Case 1. n is odd: We assume $n = 2m+1$ and restrict our discussion for $x \approx 1$. By Lemma 3.6, we only need to estimate u_l and v_l, separatively. There exist two subcases to be considered.

Subcase (1.1). $|\underline{x}| > (\delta(\mu)/2^{m+1/2})|1-x|$. In this case, we let $z = x_0 + i|\underline{x}|$. $x \approx 1$ implies $z \approx 1$. We can write $z = s + it$ with $s = x_0$ and $t = |\underline{x}|$ and get $t = |\underline{x}| = |1-z|$.

For $l = 0$, $u_l = u_0 = u$ and $v_l = v_0 = v$. By the estimate of ϕ_0, we have

$$|u_0|,\ |v_0| \leq |\phi_0| \leq \frac{C}{\delta^0(\mu)} \frac{1}{|1-z|^{s+1}}.$$

For $l = 1$ and $t \approx |1 - z|$, we can get

$$|u_1| = \left|2l\frac{1}{t}\frac{\partial u_0}{\partial t}\right| \leq \frac{1}{|1-z|}\frac{1}{|1-z|^{s+2}} = \frac{1}{|1-z|^{s+3}};$$
$$|v_1| = \left|\frac{1}{t}\frac{\partial v_0}{\partial t} - \frac{v_0}{t^2}\right| \leq (\frac{1}{|1-z|}\frac{1}{|1-z|^{s+2}} + \frac{1}{|1-z|^2}\frac{1}{|1-z|^{s+1}}) = \frac{1}{|1-z|^{s+3}}.$$

Because $\Delta^1\phi^0(x) = u_1(x_0, |\underline{x}|) + \frac{\underline{x}}{|\underline{x}|}v_1(x_0, |\underline{x}|)$, we have

$$\left|\Delta^1\phi^0(x)\right| \lesssim \left|u_1(x_0, |\underline{x}|)\right| + \left|\frac{\underline{x}}{|\underline{x}|}v_1(x_0, |\underline{x}|)\right| \lesssim \frac{1}{|1-z|^{s+3}}.$$

Repeating this method m times, we can get, for u_m and v_m,

$$|u_m(x)|,\ |v_m(x)| \lesssim \frac{1}{|1-z|^{s+2m+1}} = \frac{1}{|1-z|^{n+s}}.$$

Subcase (1.2). $|\underline{x}| \leq (\delta(\mu)/2^{m+1/2})|1-x|$. Points x in $H_{\omega,-}$ satisfying $x \approx 1,\ x_0 \leq 1$ belong to $Subcase$ (1.1). Therefore we assume $x_0 > 1$. In the following we will prove the following claim: for $z = s+it \approx 1$, $s > 1$, $z \in H^c_{\mu,-}$ and $|t| \leq (\delta(\mu)/2^{m+1/2}|1-z|)$, then (1) the functions u_l are even functions in the second variable t.

(2) the jth derivations satisfy

$$\left|\frac{\partial^j}{\partial t^j}u_l(s,t)\right| \leq \frac{C_\mu C_l 2^{lj} C_j}{\delta^{2l+j}}\frac{1}{|1-z|^{2l+j+s+1}},$$

where the constant C_j is

$$C_j = \begin{cases} (j+4l)!, & j \text{ even}, \quad (3.1)\\ (j+5l)!, & j \text{ odd}. \quad (3.1') \end{cases}$$

We prove the above two conclusions (1) and (2) by means of induction on l. Clearly for $l = 0$, by Lemma3.5, we have

$$\left|\frac{\partial^j}{\partial t^j}u_0(s,t)\right|,\ \left|\frac{\partial^j}{\partial t^j}v_0(s,t)\right| \leq \left|\frac{\partial^j}{\partial t^j}\phi^0(s,t)\right| \leq \frac{j!}{(\delta(\mu))^j}\frac{1}{|1-z|^{j+s+1}}.$$

Now we assume (1) and (2) hold for $0 \leq l \leq m-1$. Because $u_{l+1} = 2(l+1)(1/t)(\partial u_l/\partial t)(s,t)$ and the assumption that u_l are even, u_{l+1} is an even function. This verifies (1).

For (2), we consider the case j being even first. By the definition and (1), $\partial u_l/\partial t$ is an odd function in the second variable t. We can get

$$\frac{\partial u_l}{\partial t}(s,0) = \frac{\partial^{2k+1}u_l}{\partial^{2k+1}t}(s,0) = 0.$$

By Taylor's expansion, we have

$$\begin{aligned} u_{l+1}(s,t) &= \frac{2(l+1)}{t}\left(\sum_{k=0}^{\infty}\frac{1}{(2k)!}\frac{\partial^{2k+1}u_l}{\partial t^{2k+1}}(s,0)t^{2k} + \sum_{k=0}^{\infty}\frac{1}{(2k+1)!}\frac{\partial^{2k+2}u_l}{\partial t^{2k+2}}(s,0)t^{2k+1}\right)\\ &= \sum_{k=0}^{\infty}\frac{1}{(2k)!}\frac{\partial^{2k+1}u_l}{\partial t^{2k+1}}(s,0)t^{2k}. \end{aligned}$$

Let $k = j/2 + k'$ and notice that $\left(\frac{t}{\delta|1-z|}\right)^{2k'} \le \left(\frac{1}{2^{m+1/2}}\right)^{2k'}$. We get

$$\left|\frac{\partial^j}{\partial t^j}u_{l+1}(s,t)\right| = \left|2(l+1)\sum_{k=j/2}^{\infty}\frac{(2k)(2k-1)\cdots(2k-j+1)}{(2k+1)!}\frac{\partial^{2k+2}u_l}{\partial t^{2k+2}}(s,0)t^{2k-j}\right|$$
$$\le 2(l+1)\sum_{k'=0}^{\infty}\frac{(2k'+j)(2k'+j-1)\cdots(2k'+1)}{(2k'+j+1)!}\frac{C_\mu C_l 2^{l(2k'+j+2)(2k'+j+2+4l)}}{\delta^{2l+2k'+j+2}}\frac{t^{2k'}}{|1-z|^{2l+2k'+j+2+s+1}}$$
$$\le 2(l+1)\frac{C_\mu C_l 2^{l(j+2)}}{\delta^{2(l+1)+j}|1-z|^{2(l+1)+j+1+s}}\sum_{k=0}^{\infty}\frac{(j+2k+2+4l)\cdots(2k+2)}{2^k}.$$

By an estimate of Qian ([Q5]), we get that the series in the last inequality converges and satisfies

$$\sum_{k=0}^{\infty}\frac{(j+2k+2+4l)\cdots(2k+2)}{2^k} \le 2^{j+4l-1}(j+4l+4)!.$$

Finally we have

$$\left|\frac{\partial^j}{\partial t^j}u_{l+1}(s,t)\right| \le 2(l+1)\frac{C_\mu C_l 2^{l(j+2)}}{\delta^{2(l+1)+j}|1-z|^{2(l+1)+j+1+s}}2^{j+4l-1}(j+4l+4)!.$$

Now we estimate $\left|\frac{\partial^j}{\partial t^j}u_{l+1}(s,t)\right|$ for the odd j case. Similar to the proof for the case j being even, by using Taylor's expansion, we have

$$\frac{\partial^j}{\partial t^j}u_{l+1}(s,t) = 2(l+1)t\sum_{k=\frac{j+1}{2}}^{\infty}\frac{2k(2k-1)\cdots(2k+1-j)}{(2k+1)!}\frac{\partial^{2k+2}u_l}{\partial t^{2k+2}}(s,0)t^{2k-1-j}.$$

Let $2k-1-j = 2k'$, we have

$$\left|\frac{\partial^j}{\partial t^j}u_{l+1}(s,t)\right|$$
$$\le 2(l+1)t\sum_{k=0}^{\infty}\frac{(2k+j+1)(2k+j)\cdots(2k+2)}{(2k+j+2)!}\frac{C_\mu C_l 2^{l(2k+3+j)}}{\delta^{2l(2k+3+j)}}\frac{(2k+3+j+5l)!}{|1-z|^{2l+2k+3+j+s+1}}t^{2k}$$
$$\le 2(l+1)\left(\frac{t}{\delta|1-z|}\right)\frac{1}{\delta^{2(l+1)+j}}\frac{C_\mu C_l 2^{l(j+3)}}{|1-z|^{2(l+1)+j+s+1}}$$
$$\sum_{k=0}^{\infty}\frac{(2k+j+1)(2k+j)\cdots(2k+2)}{(2k+j+2)!}2^{kl}(\frac{1}{2^{m+1/2}})^{2k}(2k+3+j+5l)!$$
$$\le 2(l+1)\left(\frac{t}{\delta|1-z|}\right)\frac{1}{\delta^{2(l+1)+j}}\frac{C_\mu C_l 2^{l(j+3)}}{|1-z|^{2(l+1)+j+s+1}}2^{j+5l+4}((j+5l+3)/2)!$$

Let $j=0$ and $l=m$, we have

$$|u_m(s,t)| \le \frac{C_\mu C_0(4m)!}{\delta^{2m}}\frac{1}{|1-z|^{2m+s+1}} \le \frac{C}{|1-z|^{n+s}}.$$

Now we estimate v_m. As before, we divide the discussion into two subcases.

Subcase (1.3). $|\underline{x}| > (\delta(\mu)/2^{m+1/2})$. When $l=0$, by noticing that $|t| \approx |1-z|$, we have $|v_0(s,t)| = |v(s,t)| \lesssim \frac{2C_\mu}{|1-z|^{1+s}}$. For $l=1$, because $|(\phi^0)^j(z)| \le \frac{2j!C_\mu}{\delta^j(\mu)}\frac{1}{|1-z|^{1+j+s}}$, we have

$$|v_1(s,t)| \le \frac{2C_\mu}{\delta(\mu)}\left(\frac{1}{|1-z|^{2+s}}\frac{1}{|1-z|}+\frac{1}{|1-z|^2}\frac{1}{|1-z|^{1+s}}\right) \le C_\mu\frac{1}{|1-z|^{s+3}}.$$

Repeating this m times, we have

$$|v_m(s,t)| \lesssim C_\mu \frac{1}{|1-z|^{2m+1+s}} = \frac{C_\mu}{|1-z|^{n+s}}.$$

Subcase(1.4). $|\underline{x}| \leq (\delta(\mu)/2^{m+1/2})|1-x|$, whence we can assume $x_0 > 1$. We have for $0 \leq l \leq m$,

Claim (1). $v_l(s,t)$ is odd in the second variable t. In fact for $l=0$, $v_0(s,t) = \mathrm{Im}\phi^0(s,t)$.Because $\phi^0(z) = \sum_{k=1}^{\infty} b(-k)z^{-k}$, we have

$$\phi^0(\overline{z}) = \sum_{k=1}^{\infty} b(-k)\overline{z}^{-k} = \overline{\sum_{k=0}^{\infty} b(-k)z^{-k}} = \overline{\phi^0(z)}.$$

Let $\phi^0(z) = u(x,y) + iv(x,y)$ with u and v being real, then

$$u(x,-y) + iv(x,-y) = \overline{u(x,y)} - i\overline{v(x,y)} = u(x,y) - iv(x,y).$$

Hence $v(x,-y) = -v(x,y)$, that is, v_0 is odd.

For $l=1$, $v_1 = 2\frac{\partial}{\partial t}(\frac{v_0}{t})$ is odd because the function (v_0/t) is even. We assume that v_l is odd for $0 \leq l \leq m-1$. Thus

$$v_m = 2m(\frac{1}{t}\frac{\partial v_{m-1}}{\partial t} - \frac{v_{m-1}}{t^2}) \text{ is also odd .}$$

This proves *Claim (1)*.

Claim (2). For $0 \leq l \leq m$,

$$\left|\frac{\partial^j}{\partial t^j} v_l(s,t)\right| \leq \frac{C_\mu C_l C_j j!}{\delta^j} \frac{1}{|1-z|^{2l+j+s+1}},$$

where the constant C_j is defined by

$$C_j = \begin{cases} (j+5l)!, & \text{for } j \text{ even ,} \\ (j+4l)!, & \text{for } j \text{ odd .} \end{cases}$$

For simplicity, we only prove the case for j odd. When $l=0$, the estimate

$$\left|\frac{\partial^j}{t^j} v_0(s,t)\right| \leq \frac{C_\mu C_j j!}{(\delta^j)} \frac{1}{|1-z|^{j+s+1}}$$

is justified by that of $|(\phi^0)^{(j)}|$. Because $v_l(s,t)$ is odd respect to the second variable, $(\partial^{2k} v_l/\partial t^{2k})(s,0) = 0$. By Taylor's expansion, we have

$$v_{l+1}(s,t) \quad = \quad 2(l+1)\frac{1}{t^2}\sum_{k=0}^{\infty}\left(\frac{1}{(2k)!} - \frac{1}{(2k+1)!}\right) t^{2k+1}\frac{\partial^{2k+1} v_l}{\partial t^{2k+1}}(s,0).$$

Let $k = k'+1$ and write $k = k'$, we get

$$\frac{\partial^j v_{l+1}}{\partial t^j}(s,t) = 2(l+1)\sum_{k=0}^{\infty}\frac{2k+2}{(2k+3)!}\frac{\partial^{2k+3} v_l}{\partial t^{2k+3}}(s,0)(2k+1)\cdots(2k+2-j)t^{2k+1-j}.$$

We assume that *Claim (2)* holds for $1 \le l \le m-1$. Let $2k - j = 2k'$, we have, by the fact that $t/(\delta|1-z|) \le 2^{-(m+1/2)}$,

$$\begin{aligned}\left|\frac{\partial^j v_{l+1}}{\partial t^j}(s,t)\right| &\le 2(l+1)\sum_{k=0}^{\infty}\frac{2k+2}{(2k+3)!}(2k+1)\cdots(2k+2-j)\left|\frac{\partial^{2k+3}v_l}{\partial t^{2k+3}}(s,0)\right|t^{2k+1-j}\\ &\le 2(l+1)\frac{1}{2^{m+1/2}}\frac{2^{l(j+3)}}{\delta^{2(l+1)+j}}\frac{1}{|1-z|^{2(l+1)+j+s+1}}\\ &\quad \sum_{k=0}^{\infty}\frac{(2k+j+3+5l)\cdots(2k+j+4)(2k+j+2)\cdots(2k+2)}{2^k}.\end{aligned}$$

This proves Claim (2).

Similarly we can prove the claim for n even and get the desired result. Taking $j=0$ and $l=m$, we get

$$|v_m(s,t)| \le \frac{C_\mu C_m (4m)!}{\delta^{2m}}\frac{1}{|1-z|^{2m+1+s}} \le \frac{C_{\mu,\delta}}{|1-z|^{n+s}}.$$

Now we are ready to study the multipliers defined on the domain $S^c_{\omega,+}$. By the Kelvin inversion, we estimate the function $\phi(x) = \sum_{i=1}^{\infty} b(i)P^{(i)}(x)$ for $b \in H^{s,r}(S^c_{\omega,+})$. We have

$$I(\phi)(x) = \sum_{i=-1}^{-\infty}\tilde{b}(i)P^{(i-1)}(x),$$

where $\tilde{b}(z) = b(-z) \in H^{s,r}(S^c_{\omega,-})$. Since $I(\phi) = \tau(\phi^0)$, where

$$\phi^0(z) = \sum_{i=-1}^{-\infty}\tilde{b}(i)z^{i-1} = \frac{1}{z}\sum_{i=-1}^{-\infty}\tilde{b}(i)z^i \in H^{s,c}_{\omega,-},$$

we have $\phi(x) = I^2(\phi) = E(x)I(\phi)(x^{-1})$ and

$$|\phi(x)| = \left|E(x)I(\phi)(x^{-1})\right| \le \frac{1}{|x|^n}\frac{C_\mu}{|1-x^{-1}|^{n+s}} = \frac{C_\mu|x|^s}{|1-x|^{n+s}}.$$

Because $x \in H_{\nu,+} = \overrightarrow{H^c_{\nu,+}}$, we have $(x_0, |\underline{x}|) \in H^c_{\nu,+}$ and $|x| = (x_0^2 + |\underline{x}|^2)^{1/2} \le 1 + e^{\tan\nu}$. Finally we get $|\phi(x)| \le C_\nu/|1-x|^{n+s}$. This completes the proof of *Case 1.*

Case 2. n is even. As above we only need to estimate the the kernel $\phi(x)$ defined on $H_{\omega,-}$. Let $b \in H^{s,r}(S^c_{\omega,-})$. Consider $\phi(x) = \sum_{k=1}^{\infty} b(-k)P_n^{(-k)}(x)$. Because $n+1$ is odd, we have

$$\begin{aligned}c_{n+1}\phi(x) &= \sum_{k=1}^{\infty} b(-k)\int_{-\infty}^{\infty} P_{n+1}^{(-k)}(x + x_{n+1}e_{n+1})dx_{n+1}\\ &\le c_\mu\int_{-\infty}^{\infty}\frac{1}{|1-(x+x_{n+1}e_{n+1})|^{n+1+s}}dx_{n+1}\\ &= \frac{1}{|1-x|^{n+s}}\int_0^{\infty}\frac{|1-x|d(\frac{x_{n+1}}{|1-x|})}{\left(1+(\frac{x_{n+1}}{|1-x|})^2\right)^{\frac{n+1+s}{2}}}\\ &\le \frac{C}{|1-x|^{n+s}}.\end{aligned}$$

This completes the proof of Theorem 3.7. □

The following corollary can be deduced from Theorem 3.7 immediately.

Corollary 3.8. *Let $s > 0$, $b \in H^s(S^c_\omega)$ and $\phi(x) = (\sum_{i=1}^{\infty} + \sum_{i=-1}^{-\infty})b(i)P^{(i)}(x)$. Then $\phi(x) \in K^s(H_\omega)$.*

For the case $s < 0$, there exists a similar estimate for the function $\phi(x)$ as the one given in the above theorem. In the following theorem, we prove that the conclusion in Theorem 3.7 also holds when the spatial dimension n is odd.

Theorem 3.9. *For $s < 0$, $b \in H^s(S^c_{\omega,\pm})$ and $\phi(x) = \sum_{k=\pm1}^{\pm\infty} b(k)P^{(k)}(x)$, we have $\phi \in K^s(H_{\omega,\pm})$ when the spatial dimension n is odd.*

Proof. Because the index s is negative, we can not directly apply the method in the proof of Theorem 3.7. Precisely, for $s < 0$, the term $|z|^s$ is unbounded when z is near the origin. Therefore after obtaining the estimate of the function $\phi^0(z)$ on the domain $S^c_{\omega,-}$, we find that the Kelvin inversion method is not applicable to get the corresponding estimate on $S^c_{\omega,+}$.

To deal with this case, we estimate the function $\phi(x)$ on $H_{\omega,+}$ and on $H_{\omega,-}$, separately. On the domain $H_{\omega,-}$, the estimate of $\phi(x)$ is the same as that of Theorem3.7. We omit the detail.

For the domain $H_{\omega,+}$, because the Kelvin inversion method is not valid, we need to estimate the intrinsic function $\phi^0(z)$ on $H^c_{\omega,+}$. To achieve this goal, we apply the following Fueter's result (see (7) of Proposition 3.4):

If n is odd, then $P^{(k-1)} = \tau((\cdot)^{n+k-2})$, where the mapping τ denotes the operator $\tau(f^0) = k_n^{-1}\Delta^{(n-1)/2}\overrightarrow{f^0}$ with $\overrightarrow{f^0}(x) = u(x_0, |\underline{x}|) + \frac{\underline{x}}{|\underline{x}|}v(x_0, |\underline{x}|)$.

Now we complete the estimate of the kernel $\phi(x)$. We assume $b \in H^{s,r}(S^c_{\omega,+})$ and consider $\phi(x) = \sum_{k=1}^{\infty} b(k)P^{(k)}(x)$. By Fueter's theorem, we have

$$\phi(x) = \Delta^m\phi^0(x_0, |\underline{x}|), \text{ where } \phi^0(z) = \sum_{k=1}^{\infty} b(k)z^{n+k-1}.$$

For convenience, we write $\phi^0(z) = z^{n-1}\phi_1^0(z)$, where $\phi_1^0(z) = \sum_{k=1}^{\infty} b(k)z^k$. By a result of T. Qian in [Q1], for $b(z) \in H^s(S^c_{\omega,+})$,

$$|\phi_1^0(z)| \lesssim \frac{1}{|1-z|^{1+s}}, \text{ where } z \in H^c_{\omega,+}.$$

Then we have

$$|\phi^0(z)| \leq |z|^{n-1}\frac{1}{|1-z|^{1+s}} \leq \frac{C_\omega}{|1-z|^{1+s}},$$

where in the last inequality, we used the fact that the function $|z|^{n-1}$ is bounded in the domain $H^c_{\omega,+}$. Therefore, repeating the procedure used in Theorem3.7, we can deduce the estimate of the induced function $\phi(x)$ by that of the intrinsic function $\phi^0(z)$ obtained above. This completes the proof. □

As a direct consequence of Theorem 3.9, we have

Corollary 3.10. *For the spatial dimension n being odd, the conclusion of Corollary 3.8 holds for $s < 0$.*

Remark 3.11. In next section, we will see that if $b \in H^s(S^c_\omega)$, $s > 0$, there exists a holomorphic function $b_1(z)$ such that $|b_1(z)| \leq C_\mu$ and

$$\phi(x) = \Gamma_\xi^{s_1}\phi_1(x) \text{ with } s_1 = [s] + 1,$$

where ϕ_1 is the kernel function associated with b_1 in Theorem 3.7. However, by the above method, we can only get the estimate:

$$|\phi(x)| \leq \frac{C}{|1-x|^{n+s_1}},$$

which is not as accurate as that in Theorem 3.7.

4. L^p boundedness of the hyperbolic type Fourier multiplier

In this section, we consider a class of Fourier multipliers on starlike Lipschitz surfaces. A closed surface Σ in $\mathbb{R}^n_1$ is said to be starlike Lipschitz, if it is $n-$dimensional and star sharped about the origin, and there exists a constant $M < \infty$ such that for $x_1, x_2 \in \Sigma$,

$$\frac{\left|\ln |x_1^{-1}x_2|\right|}{\arg(x_1, x_2)} \leq M.$$

We denote by $N = Lip(\Sigma)$ the smallest constant M that makes the above inequality holds.

For $s \in S_{\mathbb{R}^n_1}$, we define the mapping $r_s:\ x \to sxs^{-1}$ for $x \in \mathbb{R}^n_1$. The following lemma will be used when proving Theorem 4.5 below.

Lemma 4.1. *([Q5], Lemma 3) For any $x, y \in \mathbb{R}^n_1$, we have* (1) $|r_s(y^{-1}x)| = |y^{-1}x|$ *and more generally, r_s preserves norms of the elements in $\mathbb{R}^n$ that can be expressed as a product of vectors;* (2) $\langle r_s(x),\ r_s(y)\rangle = \langle x, y\rangle$*;* (3) $\arg(r_s(x), r_s(y)) = \arg(x, y)$*;* (4) $(r_s(y))^{-1}r_s(x) = r_s(y^{-1}x)$*;* (5) *There exists a vector $s \in S_{\mathbb{R}^n_1}$ such that $r_s(y^{-1}x) = |y|^{-1}\tilde{x}$, where $\tilde{x} \in \mathbb{R}^n_1$; Moreover,* $|x-y| = ||y|e_0 - \tilde{x}|$ *and* $\arg(y, x) = \arg(|y|e_0, \tilde{x})$*;* (6) *For the same s as in* (5) *we have $r_s(E(y)) = E(y)$, where $E(y)$ is the Cauchy kernel $\frac{y}{|y|^{n+1}}$.*

By (1) and (5) of Lemma 4.1, we can prove if x' and x belong to a starlike Lipschitz surface with the Lipschitz constant N, then

$$\left(\left|\ln |x^{-1}x'|\right| / \arg(x, x')\right) = \left|\ln ||x|^{-1}\tilde{x}|\right| / \arg(1, |x|^{-1}\tilde{x}) \leq N,$$

that is, $|x|^{-1}\tilde{x} \in H_\omega$. This gives a relation between the sets H_ω and starlike Lipschitz surfaces.

We denote by $\mathcal{M}_k$ the finite dimensional right module of k homogeneous left monogenic functions in $\mathbb{R}^n_1$ and by $\mathcal{M}_{-(k+n)}$ the finite dimensional right module of $-(k+n)$ homogeneous left monogenic functions in $\mathbb{R}^n_1 \setminus \{0\}$. The spaces $\mathcal{M}_k$ and $\mathcal{M}_{-(k+n)}$ are eigenspaces of the left spherical Dirac operator Γ_ξ. We define

$$P_k : f \to P_k(f) \text{ and } Q_k : f \to Q_k(f)$$

the projection operators on $\mathcal{M}_k$ and $\mathcal{M}_{-(k+n)}$, respectively.

The Fourier multipliers are defined on the following test function space:

$$\mathcal{A} = \{f:\ f(x) \text{ is left monogenic in an annual } \rho - s < |x| < l + s \text{ for some } s > 0\}.$$

For $f \in \mathcal{A}$, in the annuals where f is defined, we have the Laurant series expansion

$$f(x) = \sum_{k=0}^{\infty} P_k(f)(x) + \sum_{k=0}^{\infty} Q_k(f)(x).$$

Here we have used the projection operators P_k and Q_k given by

$$\begin{aligned} P_k(f)(x) &= \frac{1}{\Omega_n}\int_\Sigma |y^{-1}x|^k C^+_{n+1,k}(\xi, \eta)E(y)n(y)f(y)d\sigma(y) \\ \text{and } Q_k(f)(x) &= \frac{1}{\Omega_n}\int_\Sigma |y^{-1}x|^{-n-k} C^-_{n+1,k}(\xi, \eta)E(y)n(y)f(y)d\sigma(y), \end{aligned}$$

where $x = |x|\xi$, $y = |y|\eta$ and $n(y)$ is the outward unit normal vector field of Σ at y. Here $C^+_{n+1,k}(\xi,\eta)$ and $C^-_{n+1,k}(\xi,\eta)$ are the functions defined by

$$C^+_{n+1,k}(\xi,\eta)\frac{1}{1-n}\left[-(n+k-1)C_k^{(n-1)/2}(\langle\xi,\eta\rangle)+(1-n)C_{k-1}^{(n+1)/2}(\langle\xi,\eta\rangle)(\langle\xi,\eta\rangle-\overline{\xi}\eta)\right]$$

and

$$C^-_{n+1,k}(\xi,\eta)\frac{1}{n-1}\left[(k+1)C_{k+1}^{(n-1)/2}(\langle\xi,\eta\rangle)+(1-n)C_k^{(n+1)/2}(\langle\eta,\xi\rangle)(\langle\eta,\xi\rangle-\overline{\eta}\xi)\right],$$

where C_k^ν is the Gegenbaur polynomial of degree k associated with ν (see [DSS]).

Now we give the definition of the Fourier multipliers on a starlike Lipschitz surface Σ induced by the sequence $\{b_k\}$, where $b_k = b(k)$ are from a function $b(z)$ belongs to $H^s(S^c_\omega)$. We see that the kernel functions $\phi(x)$ obtained in Theorem 3.7 satisfy $|\phi(x)| \leq C_\mu/|1-x|^{n+s}$ for $s > 0$. The regularity index s indicates that we may not define the multipliers for $f \in L^2(\Sigma)$ as the classical Cauchy integrals. In order to compensate the role of s, we need to restrict our multipliers into some subspaces of $L^2(\Sigma)$. Hence we define the following Sobolev space on the starlike Lipschitz surface Σ.

Definition 4.2. Let $s \in \mathbb{Z}^+ \cup \{0\}$ and Σ be a starlike Lipschitz surface, define the Sobolev norm $\|\cdot\|_{W^{p,s}_{\Gamma_\xi}(\Sigma)}$, $1 \leq p < \infty$, as

$$\|\cdot\|_{W^{p,s}_{\Gamma_\xi}(\Sigma)} = \|f\|_{L^p(\Sigma)} + \sum_{j=0}^{s}\|\Gamma_\xi^j f\|_{L^p(\Sigma)}.$$

The Sobolev space associated with the spherical monogenic operator Γ_ξ is defined as the closure of the class $\mathcal{A}$ under the norm $\|\cdot\|_{W^{p,s}_{\Gamma_\xi}(\Sigma)}$, that is, $\overline{\mathcal{A}}^{\|\cdot\|_{W^{p,s}_{\Gamma_\xi}(\Sigma)}}$.

Now we give the definition of the Fourier multiplier operators. By Definition 4.2, $\mathcal{A}$ is dense in $W^{p,s}_{\Gamma_\xi}$. When defining the Fourier multiplier operators, we assume $f \in \mathcal{A}$ in the next definition.

Definition 4.3. For $\{b_k\}_{k\in\mathbb{Z}}$ is a sequence which satisfies $|b_k| \leq k^s$, we define the hyperbolic Fourier multiplier $M_{(b_k)}$ as follows

$$M_{(b_k)}f(x) = \sum_{k=0}^{\infty} b_k P_k(f)(x) + \sum_{k=0}^{\infty} b_{-k-1}Q_k(f)(x).$$

Remark 4.4. When Σ is the unit sphere, if we take two sequences $\{b_k^{(1)}\}$, $b_k^{(1)} = k^2$ and $\{b_k^{(2)}\}$, $b_k^{(2)} = k$, our Fourier multiplier defined in Definition 4.3 retreats to the boundary values of the photogenic Cauchy integral in the hyperbolic unit sphere which was studied in [E, Chapter 6]. See Example 1.1.

Now for $k \geq 0$, we define

$$\tilde{P}^{(k)}(y^{-1}x) = |y^{-1}x|^k C^+_{n+1,k}(\xi,\eta) \text{ and } \tilde{P}^{(-k-1)}(y^{-1}x) = |y^{-1}x|^{-k-n}C^-_{n+1,k}(\xi,\eta).$$

Then the projection operators P_k and Q_k can be represented as

$$\begin{aligned} P_k(f)(x) &= \frac{1}{\Omega_n}\int_\Sigma \tilde{P}^{(k)}(y^{-1}x)E(y)n(y)f(y)d\sigma(y);\\ Q_k(f)(x) &= \frac{1}{\Omega_n}\int_\Sigma \tilde{P}^{(-k-1)}(y^{-1}x)E(y)n(y)f(y)d\sigma(y). \end{aligned}$$

If we denote by $\tilde{\phi}(y^{-1}x) = \sum_{-\infty}^{\infty} b_k\tilde{P}^{(-k)}(y^{-1}x)$ the kernel functions of the multiplier operators $M_{(b_k)}$ in Definition 4.3, we get the following estimate.

Theorem 4.5. *Let* $\omega \in (\arctan(N), \frac{\pi}{2})$ *and* $b \in H^s(S^c_\omega)$. *Then the kernel function* $\tilde{\phi}(y^{-1}x)E(y)$ *associated with the sequence* $\{b_k\}$ *in the manner given above is monogenically defined in an open neighborhood of* $\Sigma \times \Sigma \setminus \{(x, y) : x = y\}$. *Moreover, in the neighborhood,*

$$|\tilde{\phi}(y^{-1}x)| \leq \frac{C}{|1 - y^{-1}x|^{n+s}}.$$

Proof. The proof of this theorem is similar to that of [Q1, Proposition 7]. We omit the detail. □

For $f \in \mathcal{A}$, the above introduced multiplier $M_{(b_k)}$ is well defined. For $b \in H^s(S^c_\omega)$, we consider the multiplier $M^r_{(b_k)}(f)(x)$ which is defined by

$$M^r_{(b_k)}(f)(x) = \sum_{k=0}^{\infty} b_k P_k(f)(rx) + \sum_{k=0}^{\infty} b_{-k-1} Q_k(f)(r^{-1}x), \ \rho - s < |x| < l + s,$$

where $x \in \Sigma, \ r \approx 1$ and $r < 1$.

We denote by M_1 and M_2 the two sums in above expression of $M^r_{(b_k)}$. Because $b \in H^s(S^c_\omega)$, b is bounded near the origin and $|b(z)| \leq |z|^s$ when $|z| > 1$. We deduce that $|b(z)| \leq |z|^s < |z|^{s_1}$ when $|z| > 1$. Hence $b \in H^{s_1}(S^c_\omega)$ for $s_1 = [s] + 1$. We write $b_1(z) = z^{-s_1} b(z)$ and see that $|b_1(z)| \leq |b(z)/z^{s_1}| \leq C$ implies $b_1(z) \in H^\infty(S^c_\omega)$ which is defined by

$$\begin{aligned} H^\infty(S^c_{\mu,\pm}) \quad = \quad & \{b : \ S^c_{\mu,\pm} \to \mathbb{C} : \ b \text{ is holomorphic and satisfies} \\ & |b(z)| \leq C_\nu \text{ in every } \ S^c_{\nu,\pm}, \ 0 < \nu < \mu\}, \end{aligned}$$

and

$$H^\infty(S^c_\mu) = \left\{ b : \ S^c_\mu \to \mathbb{C} : \ b_\pm = b\chi_{\{z \in \mathbb{C}: \ \pm \mathrm{Re} z > 0\}} \in H^\infty(S^c_{\mu,\pm}) \right\},$$

where the sectors $S^c_{\mu,\pm}$ and S^c_μ.

For M_1, $|b_k| = |b(k)| \leq k^{s_1}$, we take $b_1(z) = z^{-s_1} b(z)$. It is easy to see that $b_1(z)$ is also holomorphic in S^c_ω. Then we have

$$M_1 = \sum_{k=0}^{\infty} b_k P_k(f)(rx) = \sum_{k=0}^{\infty} b_{1,k} k^{s_1} P_k(f)(rx),$$

where $b_{1,k} = b_1(k) = \frac{b_k}{k^{s_1}}$. Because the spaces M_k is the eigenspace of the left spherical Dirac operator Γ_ξ, we have $\Gamma_\xi P_k(f)(rx) = k P_k(f)(rx)$ and

$$M_1 = \sum_{k=0}^{\infty} b_{1,k} \Gamma^{s_1}_\xi P_k(f)(rx) = \Gamma^{s_1}_\xi \left(\sum_{k=0}^{\infty} b_{1,k} P_k(f)(rx) \right).$$

According to a result of [DSS], we give another expression of $P_k(f)$.

$$\begin{aligned} P_k(f)(x) \quad &= \quad \frac{1}{\Omega_n} \int_\Sigma \tilde{P}^k(y^{-1}rx) E(y) n(y) f(y) d\sigma(y) \\ &= \quad \frac{1}{\Omega_n} \int_\Sigma \sum_{|\underline{\alpha}|=k} V_{\underline{\alpha}}(rx) W_{\underline{\alpha}}(y) n(y) f(y) d\sigma(y), \end{aligned}$$

where we used the Cauchy-Kovalevska extension

$$\tilde{P}^{(k)}(y^{-1}x)E(y) = \sum_{|\underline{\alpha}|=k} V_{\underline{\alpha}}(x) W_{\underline{\alpha}}(y),$$

where $V_{\underline{\alpha}}(x) \in \mathcal{M}_k$ and $W_{\underline{\alpha}}(y) \in \mathcal{M}_{-n-k}$ (see [DSS, Chapter 2, (1.15)]). By the above relation, we have

$$\begin{aligned}
\Gamma_\xi P_k(f)(x) &= \frac{1}{\Omega_n}\int_\Sigma \sum_{|\underline{\alpha}|=k} (\Gamma_\xi V_{\underline{\alpha}})(x) W_{\underline{\alpha}}(y) n(y) f(y) d\sigma(y) \\
&= \frac{1}{\Omega_n}\int_\Sigma \sum_{|\underline{\alpha}|=k} k V_{\underline{\alpha}}(x) W_{\underline{\alpha}}(y) n(y) f(y) d\sigma(y) \\
&= \frac{1}{\Omega_n}\int_\Sigma \sum_{|\underline{\alpha}|=k} \frac{k}{n+k-2} V_{\underline{\alpha}}(x)(n+k-2) W_{\underline{\alpha}}(y) n(y) f(y) d\sigma(y) \\
&= \frac{k}{(n+k-2)\Omega_n}\int_\Sigma \sum_{|\underline{\alpha}|=k} V_{\underline{\alpha}}(x)(\Gamma_\eta W_{\underline{\alpha}})(y) n(y) f(y) d\sigma(y).
\end{aligned}$$

Because of the fast decay of the Fourier expansions of functions in $\mathscr{A}$, we have, by integration by parts,

$$\begin{aligned}
M_1 &= \sum_{k=1}^\infty b_{1,k} k^{s_1} P_k(f)(rx) \\
&= \sum_{k=1}^\infty b_{1,k}\left(\frac{k}{n+k-2}\right)^{s_1} \frac{r^k}{\Omega_n}\int_\Sigma \sum_{|\underline{\alpha}|=k} V_{\underline{\alpha}}(x)(\Gamma_\eta^{s_1} W_{\underline{\alpha}})(y) n(y) f(y) d\sigma(y) \\
&= \sum_{k=1}^\infty b_{1,k}\left(\frac{k}{n+k-2}\right)^{s_1} \frac{r^k}{\Omega_n}\int_\Sigma \sum_{|\underline{\alpha}|=k} V_{\underline{\alpha}}(x) W_{\underline{\alpha}}(y) n(y)(\Gamma_\eta^{s_1} f)(y) d\sigma(y).
\end{aligned}$$

Because $|b_{1,k}(\frac{k}{n+k-2})^{s_1}| \le C$, if we denote $b_{1,k}(\frac{k}{n+k-2})^{s_1}$ by $b_{1,k}$ again, we get the singular integral expression of M_1 as follows

$$\begin{aligned}
M_1 &= \sum_{k=1}^\infty b_{1,k} \frac{1}{\Omega_n}\int_\Sigma \tilde{P}^k(y^{-1}rx) E(y) n(y)(\Gamma_\eta^{s_1} f(y)) d\sigma(y) \\
&= \frac{1}{\Omega_n}\int_\Sigma \left(\sum_{k=1}^\infty b_{1,k}\tilde{P}^k(y^{-1}rx)\right) E(y) n(y)(\Gamma_\eta^{s_1} f(y)) d\sigma(y) \\
&= \frac{1}{\Omega_n}\int_\Sigma \tilde{\phi}_1(y^{-1}rx) E(y) n(y)(\Gamma_\eta^{s_1} f(y)) d\sigma(y).
\end{aligned}$$

Similarly, for the term M_2, by the Cauchy-Kovalevska extension again ([DSS, Chapter II, (1.16)]), we have

$$\begin{aligned}
M_2 &= \sum_{k=0}^\infty b_{-k-1} Q_k(f)(r^{-1}x) \\
&= \sum_{k=0}^\infty \frac{b_{-k-1}}{(k+1)^{s_1}}\left(\frac{k+1}{k}\right)^{s_1} \frac{1}{\Omega_n}\int_\Sigma \sum_{|\underline{\alpha}|=k} W_{\underline{\alpha}}(r^{-1}x) k^{s_1} \overline{V}_{\underline{\alpha}}(y) n(y) f(y) d\sigma(y) \\
&= \sum_{k=0}^\infty \frac{b_{-k-1}}{(k+1)^{s_1}}\left(\frac{k+1}{k}\right)^{s_1} \frac{1}{\Omega_n}\int_\Sigma \sum_{|\underline{\alpha}|=k} W_{\underline{\alpha}}(r^{-1}x)(\Gamma_\eta^{s_1}\overline{V}_{\underline{\alpha}})(y) n(y) f(y) d\sigma(y) \\
&= \sum_{k=0}^\infty \frac{b_{-k-1}}{(k+1)^{s_1}}\left(\frac{k+1}{k}\right)^{s_1} \frac{1}{\Omega_n}\int_\Sigma \sum_{|\underline{\alpha}|=k} W_{\underline{\alpha}}(r^{-1}x)\overline{V}_{\underline{\alpha}}(y) n(y)(\Gamma_\eta^{s_1} f)(y) d\sigma(y).
\end{aligned}$$

As before, we still write the term $\frac{b_{-k-1}}{(k+1)^{s_1}}\left(\frac{k+1}{k}\right)^{s_1}$ as b_{-1-k} and get the the singular integral expression of M_2 as follows

$$\begin{aligned} M_2 &= \sum_{k=0}^{\infty} b_{-k-1}\frac{1}{\Omega_n}\int_\Sigma \tilde{P}^{-k-1}(y^{-1}r^{-1}x)E(y)n(y)(\Gamma_\eta^{s_1}f)(y)d\sigma(y)\\ &= \frac{1}{\Omega_n}\int_\Sigma\left(\sum_{k=0}^{\infty} b_{-k-1}\tilde{P}^{-k-1}(y^{-1}r^{-1}x)\right)E(y)n(y)(\Gamma_\eta^{s_1}f)(y)d\sigma(y)\\ &= \frac{1}{\Omega_n}\int_\Sigma \tilde{\phi}_2(y^{-1}r^{-1}x)E(y)n(y)(\Gamma_\eta^{s_1}f(y))d\sigma(y). \end{aligned}$$

Finally we rewrite the multiplier $M^r_{(b_k)}(f)(x)$ as

$$M^r_{(b_k)}(f)(x) = \lim_{r\to 1-}\frac{1}{\Omega_n}\int_\Sigma(\widetilde{\phi_1}(y^{-1}rx)+\widetilde{\phi_2}(y^{-1}r^{-1}x))E(y)n(y)(\Gamma_\xi^{s_1}f)(y)d\sigma(y),$$

where we have used the fact that the series defining $M^r_{b_k}(f)$ uniformly converges as $r\to 1-$ for $f\in\mathcal{A}$.

We get the following boundary value result for $M_{(b_k)}(f)(x)$.

Theorem 4.6. *If $b\in H^s(S^c_\omega)$, then for $f\in\mathcal{A}$ and $x\in\Sigma$, we have*

$$\begin{aligned} M_{(b_k)}(f)(x) &= \lim_{r\to 1-}\frac{1}{\Omega_n}\int_\Sigma(\widetilde{\phi_1}(y^{-1}rx)+\widetilde{\phi_2}(y^{-1}r^{-1}x))E(y)n(y)(\Gamma_\xi^{s_1}f)(y)d\sigma(y)\\ &= \lim_{\varepsilon\to 0}\frac{1}{\Omega_n}\{\int_{|y-x|>\varepsilon, y\in\Sigma}[\widetilde{\phi_1}(y^{-1}x)+\widetilde{\phi_2}(y^{-1}x)]E(y)n(y)(\Gamma_\xi^{s_1}f)(y)d\sigma(y)\\ &+ (\widetilde{\phi_1}(\varepsilon,x)+\widetilde{\phi_2}(\varepsilon,x))f(x)\}. \end{aligned}$$

Here

$$\widetilde{\phi_1}(\varepsilon,x) = \int_{S(\varepsilon,x,+)}\widetilde{\phi_1}(y^{-1}x)E(y)n(y)d\sigma(y)$$

$$\text{and}\quad \widetilde{\phi_2}(\varepsilon,x) = \int_{S(\varepsilon,x,-)}\widetilde{\phi_2}(y^{-1}x)E(y)n(y)d\sigma(y),$$

where $S(\varepsilon,x,\pm)$ is the part of the sphere $|y-x|=\varepsilon$ inside or outside Σ depending on the index of $\widetilde{\phi_i}$ taking $i=1$ or $i=2$.

Proof. The proof of this theorem is similar to that of the classical Plemelj formula for Cauchy integral. For simplicity, we only consider

$$\lim_{r\to 1-} I = \lim_{r\to 1-}\frac{1}{\Omega_n}\int_\Sigma\widetilde{\phi_1}(y^{-1}rx)E(y)n(y)(\Gamma_\xi^{s_1}f)(y)d\sigma(y).$$

Another integral can be dealt with similarly. For a fixed $\varepsilon>0$, the above integral I can be divided into three parts as follows:

$$\begin{aligned} I &= \frac{1}{\Omega_n}\int_\Sigma\widetilde{\phi_1}(y^{-1}rx)E(y)n(y)(\Gamma_\xi^{s_1}f)(y)d\sigma(y)\\ &= \frac{1}{\Omega_n}\int_{y\in\Sigma,|y-x|>\varepsilon}\widetilde{\phi_1}(y^{-1}rx)E(y)n(y)(\Gamma_\xi^{s_1}f)(y)d\sigma(y)\\ &+ \frac{1}{\Omega_n}\int_{y\in\Sigma,|y-x|\le\varepsilon}\widetilde{\phi_1}(y^{-1}rx)E(y)n(y)[(\Gamma_\xi^{s_1}f)(y)-(\Gamma_\xi^{s_1}f)(x)]d\sigma(y)\\ &+ \frac{1}{\Omega_n}\int_{y\in\Sigma,|y-x|\le\varepsilon}\widetilde{\phi_1}(y^{-1}rx)E(y)n(y)d\sigma(y)(\Gamma_\xi^{s_1}f)(x)\\ &=: I_1+I_2+I_3, \end{aligned}$$

where the symbol $\Gamma_\xi f(y)$ denotes the spherical Dirac operator Γ_ξ which operates on the variable η of f with $y = |y|\eta$.

Let $r \to 1-$, the term I_1 tends to

$$\frac{1}{\Omega_n}\int_{y\in\Sigma,|y-x|>\varepsilon} \widetilde{\phi_1}(y^{-1}x)E(y)n(y)(\Gamma_\eta^{s_1} f)(y)d\sigma(y).$$

For the term I_2, because $f \in \mathcal{A}$ implies $\Gamma_\xi^{s_1} f$ is a Lipschitz function, we have

$$\lim_{\varepsilon\to 0}\lim_{r\to 1-} I_2 = \lim_{r\to 1-}\lim_{\varepsilon\to 0}\int_{y\in\Sigma,|y-x|\le\varepsilon} \widetilde{\phi_1}(y^{-1}rx)E(y)n(y)[(\Gamma_\xi^{s_1} f)(y) - (\Gamma_\xi^{s_1} f)(x)]d\sigma(y) = 0.$$

At last we estimate the term I_3. By Cauchy's theorem, for fixed $\varepsilon > 0$, we have

$$\lim_{r\to 1-} I_3 = \lim_{r\to 1-}\int_{y\in\Sigma,|y-x|\le\varepsilon} \widetilde{\phi_1}(y^{-1}rx)E(y)n(y)d\sigma(y)(\Gamma_\xi^{s_1} f)(x) = \widetilde{\phi_1}(\varepsilon, x)(\Gamma_\xi^{s_1} f)(x).$$

This completes the proof.

□

We obtain the L^2 boundedness of the Fourier multipliers by the Hardy spaces of monogenic functions on starlike Lipshcitz surfaces. This idea was pioneered by R. Coifman and Y. Meyer in [CM] and used by T. Qian in [Q2], [Q3], [Q5].

As an useful tool in the research on the boundary value problem on the non-smooth domains, the theories of Hardy spaces on Lipschitz curves and surfaces have attracted the attention of many mathematicians. In the 1980s, D. Jerison and C. Kenig consider the complex variable case in [K], [JK]. In M. Mitrea's book [Mi], the theory of Clifford monogenic Hardy spaces on higher dimensional Lipschitz graphs is introduced.

Let Δ and Δ^c be the bounded and unbounded connected components of $\mathbb{R}_1^n \setminus \Sigma$. For $\alpha > 0$, define the non-tangential approach regions $\Lambda_\alpha(x)$ and $\Lambda_\alpha^c(x)$ to a point $x \in \Sigma$ to be

$$\Lambda_\alpha(x) = \{x \in \Delta,\ |y - x| < (1 + \alpha)dist(y, \Sigma)\}$$

and

$$\Lambda_\alpha^c(x) = \{\, y \in \Delta^c,\ |y - x| < (1 + \alpha)dist(y, \Sigma)\}\,.$$

Let f be defined in Δ (Δ^c). The interior (exterior) non-tangential maximal function $N_\alpha(f)$ is defined by

$$N_\alpha(f)(x) = \sup\{\,|f(y)|\ :\ y \in \Lambda_\alpha(x)(y \in \Lambda_\alpha^c(x))\}\,.$$

For $0 < p < \infty$, the Hardy space $\mathcal{H}^p(\Delta)$ ($\mathcal{H}^p(\Delta^c)$) is defined by

$$\begin{aligned}\mathcal{H}^p(\Delta) &= \{f : f \text{ is left monogenic in } \Delta, \text{ and } N_\alpha(f) \in L^p(\Sigma)\}\,,\\ \mathcal{H}^p(\Delta^c) &= \{f : f \text{ is left monogenic in } \Delta^c, \text{ and } N_\alpha(f) \in L^p(\Sigma)\}\,.\end{aligned}$$

The theory of the monogenic Hardy space in [Mi] implies that for $p > 1$, the $\mathcal{H}^p(\Delta)$ norm of a function is equivalent to the L^p norm of its non-tangential limit on the boundary. A similar conclusion holds for the space $\mathcal{H}^p(\Delta^c)$. Precisely, if $f \in \mathcal{H}^p(\Delta)$ for $p > 1$, we have

$$C_1\|f\|_{\mathcal{H}^p(\Delta)} \le \|f\|_{L^p(\Sigma)} \le C_2\|f\|_{\mathcal{H}^p(\Delta)}.$$

If $f \in \mathcal{M}_k$ with $k \ne -1, -2, \cdots, -n+1$, we have $\Gamma_\xi f(\xi) = kf(\xi)$ because $\mathcal{M}_k$ is the subspace of $k-$homogeneous left monogenic functions. For $f \in \mathcal{A}$, we define $\Gamma(f\,|_\Gamma)$ to be the restriction on Γ of the monogenic extension of $\Gamma_\xi(f\,|_{S_{\mathbb{R}_1^n}})$, then the definition of Γ_ξ can be extended to $\Gamma_\xi : \mathcal{A} \to \mathcal{A}$.

It is well known that, for $p = 2$, the above Hardy spaces $\mathcal{H}^2(\Delta)$ and $\mathcal{H}^2(\Delta^c)$ have an equivalent characterization of the higher order $g-$functions. Taking $\mathcal{H}^2(\Delta)$ for example, we have

Proposition 4.7. *([Mi], [JK]) Suppose that $f \in \mathcal{H}^2(\Delta)$. Then the norm $\|f\|_{\mathcal{H}^2(\Delta)}$ is equivalent to the norm*

$$\left(\int_0^1 \int_\Sigma \left|(\Gamma_\xi^j f)(sx)\right|^2 (1-s)^{2j-1} d\sigma(x) \frac{ds}{s}\right)^{1/2}, \ j = 1, 2, \cdots .$$

As two subspaces of $L^2(\Sigma)$, we can prove Hardy space $\mathcal{H}^2(\Delta)$ and $\mathcal{H}^2(\Delta^c)$ are orthogonal with each other. We state this property in the following proposition.

Proposition 4.8. *([CMcM]) Suppose that $f \in L^2(\Sigma)$. Then there exist $f^+ \in \mathcal{H}^2(\Delta)$ and $f^- \in \mathcal{H}^2(\Delta^c)$ such that their non-tangential boundary limits, still denoted by f^+ and f^-, respectively, lie in $L^2(\Sigma)$, and $f = f^+ + f^-$. The mapping $f \to f^\pm$ are continuous on $L^2(\Sigma)$.*

In [E], D. Eelbode studied the boundary value of the photogenic Cauchy transform C_P^α on the hyperbolic unit sphere. The occurrence of the factors $k^2 P_k(f)$ and $k^2 Q_k(f)$ in Example 1.1 implies that the boundary value $C_P^\alpha[f] \uparrow$ of C_P^α is not a bounded operator from $L^2(S^{m-1})$ to itself. As an alternative, if we restrict the functions into some smaller subspaces of $L^2(S^{m-1})$, we can get the corresponding boundedness.

Now we give the main result of this paper.

Theorem 4.9. *Let $\omega \in (\arctan(N), \frac{\pi}{2})$. If $b \in H^s(S_\omega^c)$, $s > 0$, then with the convention $b(0) = 0$, the multiplier defined in Definition 4.3 can be extended to a bounded operator from $W_{\Gamma_\xi}^{2,s_1}(\Sigma)$ to $L^2(\Sigma)$, where $s_1 = \lceil s \rceil$. Moreover, for the operator norm of the multiplier $\|\cdot\|_{op}$, we have*

$$\|M_{(b(k))}\|_{op} \le C_\nu \left\|\frac{b}{|z+1|^s}\right\|_{L^\infty(S_\nu^c)}, \ \arctan N < \nu < \omega.$$

Proof. Because $f \in W_{\Gamma_\xi}^{2,s_1}(\Sigma) \subset L^2(\Sigma)$, by Proposition 4.8, we have for such a f, $f = f^+ + f^-$, where $f^+ \in \mathcal{H}^2(\Delta)$ and $f^- \in \mathcal{H}^2(\Delta^c)$ such that

$$\|f^\pm\|_{L^2(\Sigma)} \le C_N \|f\|_{W^{2,s_1}(\Sigma)}.$$

By linearity and Theorem 4.6, we can $M_b(f) = M_{b^+} f^+ + M_{b^-} f^-$ where

$$M_{b^\pm} f^\pm(x) = \lim_{r \to -} \int_\Sigma \tilde{\phi}_\pm(r^{\pm 1} y^{-1} x) E(y) n(y) f(y) d\sigma(y), x \in \Sigma.$$

Hence it sufficient to prove

$$\|M_{b^\pm} f^\pm\|_{\mathcal{H}^2} \le C_N \|\Gamma_\xi^{s_1} f^\pm\|_{\mathcal{H}^2}.$$

We prove the above inequality for the part f^+ and omit the symbol " + " in the sequel for simplicity. The treatment of f^- is the same as that of f^+.

By Theorem 4.5, for $b \in H^s(S_\omega^c)$, we have

$$|\tilde{\phi}(y^{-1}x)| \le \frac{C}{|1 - y^{-1}x|^{n+s}}.$$

Therefore we have, by Hölder's inequality,

$$\begin{aligned} |\Gamma_\xi^{1+s_1} M_b f(x)| &\le \left(\int_{\Sigma_{\sqrt{t}}} |\phi(y^{-1}x)| \frac{d\sigma(y)}{|y|^n}\right)^{1/2} \left(\int_{\Sigma_{\sqrt{t}}} |\phi(y^{-1}x)| |\Gamma_\xi^{s_1+1} f(y)|^2 \frac{d\sigma(y)}{|y|^n}\right)^{1/2} \\ &\le C \left(\int_{\Sigma_{\sqrt{t}}} \frac{1}{|1 - y^{-1}x|^{n+s}} \frac{d\sigma(y)}{|y|^n}\right)^{1/2} \left(\int_{\Sigma_{\sqrt{t}}} \frac{1}{|1 - y^{-1}x|^{n+s}} |\Gamma_\xi^{s_1+1} f(y)|^2 \frac{d\sigma(y)}{|y|^n}\right)^{1/2}. \end{aligned}$$

By change of variable, we can get

$$|\Gamma_\xi^{1+s_1} M_b f(x)| \leq C\left(\int_\Sigma \frac{1}{[(1-\sqrt{t})^2+\theta_0^2]^{\frac{n+s}{2}}} d\sigma(y)\right)^{1/2}\left(\int_\Sigma \frac{1}{[(1-\sqrt{t})^2+\theta_0^2]^{\frac{n+s}{2}}} |\Gamma_\xi^{1+s_1} f(y)|^2 d\sigma(y)\right)^{1/2},$$

where the integral in the last inequality satisfies

$$\int_\Sigma \frac{1}{[(1-\sqrt{t})^2+\theta_0^2]^{\frac{n+s}{2}}} d\sigma(y) \leq \int_0^\pi \frac{\sin^{n-1}\theta_0}{[(1-\sqrt{t})^2+\theta_0^2]^{\frac{n+s}{2}}} d\theta_0 \lesssim \frac{1}{(1-\sqrt{t})^s}.$$

Hence by the equivalent characterization given in Proposition 4.7, we have

$$\begin{aligned}
\|M_b f\|^2_{H^2(\Delta)} &\leq \int_0^1 \int_\Sigma |\Gamma_\xi^{1+s_1} M_b f(tx)|^2 (1-t)^{2s_1+1} d\sigma(x) \frac{dt}{t} \\
&\lesssim \int_0^1 \int_\Sigma \frac{1}{(1-\sqrt{t})^s}\left(\int_\Sigma \frac{1}{[(1-\sqrt{t})^2+\theta_0^2]^{\frac{n+s}{2}}} |\Gamma_\xi^{1+s_1} f(\sqrt{t}y)|^2 d\sigma(y)\right)(1-\sqrt{t})^{2s_1+1} d\sigma(x) \frac{dt}{t} \\
&\lesssim \int_0^1 \int_\Sigma |\Gamma_\xi^{1+s_1} f(\sqrt{t}y)|^2 \left(\int_\Sigma \frac{(1-\sqrt{t})^s}{[(1-\sqrt{t})^2+\theta_0^2]^{\frac{n+s}{2}}} d\sigma(x)\right)(1-\sqrt{t}) d\sigma(y) \frac{dt}{t} \\
&\lesssim \int_0^1 \int_\Sigma \left|\Gamma_\xi(\Gamma_\xi^{s_1} f)(\sqrt{t}y)\right|^2 (1-\sqrt{t}) d\sigma(y) \frac{dt}{t} \\
&\lesssim \|\Gamma_\xi^{s_1} f\|_{\mathcal{H}^2(\Delta)},
\end{aligned}$$

where we have used the facts $(1-\sqrt{t})^{2s_1+1-s} = (1-\sqrt{t})^{1+s+2s_1-s} \leq (1-\sqrt{t})^{1+s}$ for $t \in (0,1)$ and

$$\int_\Sigma \frac{(1-\sqrt{t})^s}{[(1-\sqrt{t})^2+\theta_0^2]^{\frac{n+s}{2}}} d\sigma(x) \lesssim (1-\sqrt{t})^s \frac{1}{(1-\sqrt{t})^s} \lesssim C$$

in the forth inequality and Proposition 4.7 in the last one. This completes the proof of Theorem 4.9. □

For the classical convolution singular integral operators T_ϕ on $\mathbb{R}^n$, one of basic facts is the endpoint estimate, that is the weak type $(1,1)$ boundedness. We call an operator T is of weak type $(1,1)$ boundedness on Σ if the following inequality holds for all $\lambda > 0$,

$$|\{x \in \Sigma : \ |T(f)(x)| > \lambda\}| \leq \frac{C}{\lambda}\|f\|_1.$$

In other words, we say the operator is bounded from L^1 to the weak type space WL^1. This is also the case on the finite and infinite Lipschitz graphs. We can refer to [LMcS], [LMcQ], [Q5] and the references therein. By this weak boundedness, we can use the interpolation theory and the duality of the operators to get the L^p boundedness of T_ϕ. In the rest of this section, we study the endpoint estimate of the Fourier multipliers.

Theorem 4.10. *Let $\omega \in (\arg(N), \frac{\pi}{2})$. If $b \in H^s(S^c_\omega)$, $s > 0$ and $b(0) = 0$. Then the multipliers $M_{(b_k)}$ defined by*

$$M_{(b_k)}(f)(x) = \sum_{k=0}^\infty b_k P_k(f)(x) + \sum_{k=0}^\infty b_{-k-1} Q_k(f)(x)$$

are weak type bounded from $W^{1,s_1}_{\Gamma_\xi}(\Sigma)$ to $WL^1(\Sigma)$, where $s_1 = \lceil s \rceil$

Proof. For $b \in H^s(S^c_\omega)$, $|b(z)| \le C|z|^s,\ s > 0$ for $z \in S^c_\omega$. Therefore, it is natural to get

$$\left|\frac{b(z)}{z^s}\right| \le C \text{ with } C \text{ a constant.}$$

On the other hand, $b \in H^s(S^c_\omega)$ implies b is holomorphic in S^c_ω. Then $z^{-s}b(z)$ is also a holomorphic function in S^c_ω. Now for the Fourier multipliers $M_{(b_k)}$, we have

$$\begin{aligned} M_{(b_k)}f(x) &= \sum_{k=0}^{\infty} b_k P_k(f)(x) + \sum_{k=0}^{\infty} b_{-k-1} Q_k(f)(x) \\ &= I + II. \end{aligned}$$

For the sake of convenience, we deal with the term I for example. As before, the term I can be represented as

$$I = \frac{1}{\Omega_n}\int_\Sigma \tilde{\phi}(y^{-1}x)E(y)n(y)f(y)d\sigma(y).$$

If we write $b(z) = z^{s_1}b_1(z)$ with $b_1(z) \in H^\infty(S^c_\omega)$, then the corresponding sequence is $\{b_{1,k}\}$ with the elements $b_k = k^{s_1}b_{1,k}$. Hence we can rewrite the term I as follows

$$I = \sum_{k=0}^{\infty} b_{1,k}k^{s_1}P_k(f)(x).$$

The kernel associated with $M_{b_{1,k}}$ is denoted by $\tilde{\phi}_1(y^{-1}x)E(y)$ and satisfies the estimate

$$\Gamma_\xi(\tilde{\phi}_1(y^{-1}x))E(y) = \sum_{k=1}^{\infty} kb_1(k)\tilde{P}^{(k)}(y^{-1}x)E(y),$$

then we have, by integration by parts,

$$\begin{aligned} I &= \frac{1}{\Omega_n}\int_\Sigma \Gamma^{s_1}_\xi(\tilde{\phi}_1(y^{-1}x))E(y)n(y)f(y)d\sigma(y) \\ &= \frac{1}{\Omega_n}\int_\Sigma \tilde{\phi}_1(y^{-1}x)E(y)n(y)\Gamma^{s_1}_\eta(f)(y)d\sigma(y). \end{aligned}$$

As before, if we take $s = 0$ in Theorem 4.5, $\widetilde{\phi_1}(y^{-1}x)$ satisfies

$$|\widetilde{\phi_1}(y^{-1}x)| \le \frac{C}{|1 - y^{-1}x|^n}.$$

Therefore the multiplier $M_{b_{1,k}}$ comes back to a H^∞−Fourier multiplier on starlike Lipschitz graphs and is of the weak type $(1,1)$ boundedness. Then we have

$$\begin{aligned} \left|\{x \in \Sigma :\ |M_{b_k}f(x)| > \lambda\}\right| &= \left|\left\{x \in \Sigma :\ |M_{b_{1,k}}(\Gamma^{s_1}_\xi f)(x)| > \lambda\right\}\right| \\ &\le \frac{C}{\lambda}\left\|\Gamma^{s_1}_\xi f\right\|_{L^1}. \end{aligned}$$

This completes the proof of this theorem. □

At last, we consider the boundedness of the Fourier multipliers for the case $s < 0$. Let $-n < s < 0$ and $\{b_k\}$ is a sequence which satisfies $|b_k| \le k^s$. We define the Fourier multiplier operator $M_{(b_k)}$ as follows.

$$M_{(b_k)}(f)(x) = \sum_{k=1}^{\infty} b_k P_k(f)(x) + \sum_{k=1}^{\infty} b_{-k-1} Q_k(f)(x).$$

Similar to the case $s > 0$, we can represent this multiplier as

$$M_{(b_k)}(f)(x) = \frac{1}{\Omega_n}\int_\Sigma \widetilde{\phi}(y^{-1}x)E(y)n(y)f(y)d\sigma(y).$$

Here $x \in \Sigma$ and $\widetilde{\phi}(y^{-1}x) = (\sum_{k=1}^{\infty} + \sum_{-\infty}^{-1})b_k\widetilde{P}^{(k)}(y^{-1}x)$, where $\widetilde{P}^{(k)}$ are the polynomials defined by

$$\widetilde{P}^{(k)}(y^{-1}x) = |y^{-1}x|^k C^+_{n+1,k}(\xi,\eta)$$

and

$$\widetilde{P}^{(-k-1)}(y^{-1}x) = |y^{-1}x|^{-k-n} C^-_{n+1,k}(\xi,\eta).$$

To obtain the boundedness of the multipliers, we need estimate the function $\widetilde{\phi}(x)$. By the method of Theorem 3.7, we can prove the kernel $\phi(x) = \sum_{k=-\infty}^{\infty} b_k P^k(x)$ satisfies

$$|\phi(x)| \le \frac{C|x|^s}{|1-x|^{n+s}}, \quad \text{where } x \in H_\omega,$$

then for the kernel $\widetilde{\phi}(y^{-1}x)$ defined above, we can apply the method of [Q1, Proposition 7] to get

$$|\widetilde{\phi}(y^{-1}x)| \le \frac{C|y^{-1}x|^s}{|1-y^{-1}x|^{n+s}}.$$

For every two points x_1, x_2 on the starlike Lipschitz surface, we have $x_2^{-1}x_1 \in H_\omega$, that is, there exist two constants $C_1,\ C_2$ such that $C_1 \le |x_2^{-1}x_1| \le C_2$. Then for every two points $x_1,\ x_2 \in \Sigma$, the equality

$$|x_1| = |x_2x_2^{-1}x_1| = |x_2||x_2^{-1}x_1|$$

implies $C_1|x_1| \le |x_2| \le C_2|x_1|$. In other words, the norms of two points on starlike Lipschitz surface are approximately a constant associated with Σ, which is denoted by C_Σ. Hence we can get the estimate

$$\begin{aligned}|\widetilde{\phi}(y^{-1}x)E(y)n(y)| &\le \frac{C|y^{-1}x|^s}{|1-y^{-1}x|^{n+s}}\frac{1}{|y|^n}\\ &\le \frac{C|x|^s}{|y-x|^{n+s}}\\ &\le \frac{C_\Sigma}{|y-x|^{n+s}}.\end{aligned}$$

Because the Lipschitz surface Σ is a special case of the space of homogeneous type, our Fourier multipliers $M_{(b_k)}f(x)$ can be regarded as the fractional integral operators on the surface Σ. By the classical theory of the fractional integral in the space of homogeneous type, we can get the $L^p - L^q$ boundedness of the Fourier multipliers as follows.

Theorem 4.11. *Let $-n < s < 0$ and $1 \le p < q < \infty$ with $\frac{1}{q} = \frac{1}{p} + \frac{s}{n}$. If $b(z) \in H^s(S^c_\omega)$, the Fourier multiplier on starlike Lipschitz surface defined by*

$$M_{(b_k)}f(x) = \sum_{k=1}^{\infty} b_k P_k(f)(x) + \sum_{k=1}^{\infty} b_{-k-1}Q_k(f)(x),$$

where $b_k = b(k)$ is bounded from $L^p(\Sigma)$ to $L^q(\Sigma)$.

Acknowledgements. The research was supported by the Macao Government Science and Technology Development Fund FDCT/014/2008/A1.

References

[BDS] F. Brackx, R. Delanghe and F. Sommen, *Clifford analysis,* Res. Notes Math., **76** Pitman, London, 1982.

[C] A. P. Caldéron, *Cauchy integral on Lipschitz curves and related operators,* Proc. Nat. Acad. Sc.**74** (1977), 1324-1327.

[CJS] R. Coifman, P. W. Jones and S. Semmes, *Two elementary proofs of the L^2 boundedness of Cauchy integrals on Lipschitz curves*, J. Amer. Math. Soc. **2** (1989), 553-564.

[CM] R. Coifman and Y. Meyer, *"Fourier Analysis of Multilinear Convolution, Calderón's Theorem, and Analysis on Lipschitz Curves,"* Lecture Notes in Mathematics, Vol. **779**, pp. 104-122, Springer-Verlag, Berlin/New York, 1980.

[CMcM] R. Coifman, A. McIntosh and Y. Meyer, *L'intégrale de Cauchy définit un opérateur borné sur L^2 pour les courbes lipschitziennes*, Ann. of Math.**116** (1982), 361-387.

[D] R. Delanghe, *Clifford Analysis: History and Perspective,* Comp. Meth. and Func. Theory **1** (2001), 107-153.

[De] C. A. Deavours, *The quaternion calculus,* Amer. Math. Monthly **80** (1973), 995-1008.

[DSS] R. Delanghe, F. Sommen and V. Souček, *Clifford Alegbra and Spinor Valued Functions*, Kluwer Academic Publishers, Dordrecht, 1992.

[E] D. Eelobde, *Clifford analysis on the hyperbolic unit ball*, PhD-thesis, Ghent, Belgium, 2005.

[ES] D. Eelbode and F. Sommen, *The photogenic Cauchy transform,* Journ. Geom. Phys. **54** (2005), 339-354.

[GLQ] G. Gaudry, R.-L. Long and T. Qian, *A martingale proof of L^2-boundedness of Clifford-valued singular integrals,* Ann. Math. Pura Appl. **165** (1993), 369-394.

[JK] D. Jerison and C. Kenig, *Hardy spaces, A_∞, a singular integrals on chord-arc domains,* Math. Scand. **50** (1982), 221-247.

[K] C. Kenig, *Weighted H^p spaces on Lipschitz domains*, Amer. J. Math. **102** (1980), 129-163.

[LMcQ] C. Li, A. McIntosh and T. Qian, *Clifford alegbras, Fourier transforms, and singular convolution operator s on Lipschitz surfaces,* Rev. Mat. Iberoamericana **10** (1994), 665-721.

[LMcS] C. Li, A. McIntosh and S. Semmens, *Convolution singular integrals on Lipschitz surfaces*, J. Amer. Math. Soc. **5** (1992), 455-481.

[Mc1] A. McIntosh, *Clifford Alegbras and the High Dimensional Cauchy Integral,* Approximation and Function Spaces, Banach Center Publishions Vol. **22**, Pwn-Polish Scientific, Warsaw, 1989.

[Mc2] A. McIntosh, *Operators which have an $H^\infty-$functional calculus*, in "Miniconference on Operator Theory and Partial Differential Equations" Vol.**14**, 210-231, Proc. Centre Math. Analysis, U. N. U., Canberra, 1986.

[Mc3] A. McIntosh, *Clifford alegbra, Fourier theory, singular integrals, and harmonic functions on Lipschitz domains,* in "Clifford Alegbras in Analysis and Related Topics"(J. Ryan, Ed.), Studies in Advanced Mathematics Series, pp. 33-87, CRC Press, Boca Raton, 1996.

[McQ1] A. McIntosh and T. Qian, *Convolution singular integral operators on Lipschitz curves*, in "Proc. of the Special year on harmonic Analysis at Nankai Inst. of Math., Tianjin, China," Lecture Notes in mathematics Vol.**1494**, pp. 142-162, Springer-Verlag, berlin/New York, 1991.

[McQ2] A. McIntosh and T. Qian, *L^p Fourier multipliers on Lipschitz curves,* Trans. Amer. mat. Soc., **333** (1992), 157-176.

[Mi] M. Mitrea, *Clifford Wavelets, Singular Integrals, and Hardy spaces*, Lecture Notes in Mathematics, Vol.**1575**, Springer-Verlag, Berlin/New York, 1994.

[Q1] T. Qian, *A holomorphic extension result,* Complex Variables **32** (1996), 59-77.

[Q2] T. Qian, *Singular integrals with holomorphic kernels and $H^\infty-$Fourier multipliers on star-shaped Lipschitz curves*, Studia Math. **123**, 3 (1997), 195-216.

[Q3] T. Qian, *Singular integrals on star-shaped Lipschitz surfaces in the quaternionic space,* Math. Ann. **310** (1998), 601-630.

[Q4] T. Qian, *Generalization of Fueter's result to $\mathbb{R}^{n+1}$*, Rend. Mat. Acc. Lincei(9) **8** (1997), 111-117.

[Q5] T. Qian, *Fourier analysis on starlike Lipschitz surfaces*, J. Func. Anal. **183** (2001), 370-412.

[Sc] M. Sce, *Osservazioni sulle serie di potenze nei moduli quadratici,* Atti Acc. Lincei Rend. Fis. (8) **23** (1957), 220-225.

[St] E. Stein, *Singular Integrals and Differential Proposities of Functions,* Princeton Univ. Press, Princeton, NJ, 1971.

[Ta] T. Tao, *Convolution Operators on Lipschitz Graphs with Harmonic Kernels,* Adv. Appl. Clifford Algebras, **6** (1996), 207-218.

Department of Mathematics, Faculty of Science and Technology, University of Macau, Block2-B202, Macau, China.
E-mail address: li_ptao@163.com

Department of Mathematics, FST, University of Macau, PO Box 3001, Macau, China
E-mail address: itleong@umac.mo

Department of Mathematics, FST, University of Macau, PO Box 3001, Macau, China
E-mail address: fsttq@umac.mo