Analytic Phase Derivatives, All-Pass Filters and Signals of Minimum Phase

Pei Dang*, Tao Qian

***Abstract*—It is accepted knowledge that inner functions and outer functions in complex analysis correspond, respectively, to all-pass filters and signals of minimum phase. The knowledge, however, has not been justified for general inner and outer functions. In digital signal processing the correspondence and related results are based on studies of rational functions. In this paper, based on the recent result on positivity of phase derivatives of inner functions, we establish the theoretical foundation for all-pass filters and signals of minimum phase. We, in particular, deal with infinite Blaschke products and general singular inner functions induced by singular measures. A number of results known for rational functions are generalized to general inner functions. Both the discrete and continuous signals cases are rigorously treated.**

***Index Terms*—Hilbert transform, analytic signal, Hardy space, Hardy-Sobolev space, all-pass filter, inner function, outer function, Blaschke product, amplitude-phase representation of signal, instantaneous frequency, minimum phase signal,**

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I. Introduction

In signal processing, it is well known that any rational system function F may be decomposed into a minimum phase system function H and an all-pass system function G as $F = HG$ ([17]). There is an ample amount of studies of minimum phase systems and all-pass systems that are based on the above mentioned decomposition ([2], [3], [9], [13], [14], [15]). In [17] a minimum phase system is restricted to a system of finite order, and is defined in terms of locations of poles and zeros of the system. There are equivalent definitions through *phase-lag* function or *group delay*. The group delay is defined to be $-\phi'_F(\omega)$, where $F(e^{i\omega}) = |F(e^{i\omega})|e^{i\phi_F(\omega)}$. The definition via group delay amounts to say that a minimum phase system has a minimum group delay. The phenomenon of minimum group delay, however, is based on the fact that all-pass systems have positive group delay, i.e. $-\phi'_G \geq 0$. By definition positivity of group delay means negativity of the phase derivative. Thus the whole theory rests on the fact that suitably defined phase derivatives of all-pass systems are negative. To our knowledge, before the study in [23], the proofs of negativity of phase derivatives of all-pass systems were valid only for finite Blaschke products ([17], [5]). In [23] it is proved that all inner functions have positive phase derivatives (using variable z to replace $1/z$), where inner functions are identical with all-pass system functions in which finite Blaschke products are particular cases. Those particular cases are what practical signal analysis mainly concerns. Based on the positivity of phase derivatives of all-pass system functions, we can obtain that any causal and stable system function (a function in the Hardy space) can be factorized into a product of an all-pass system and a minimum phase system. In the present paper, we call signals corresponding to all-pass system functions *all-pass filters*, and signals corresponding to minimum-phase system functions *minimum phase signals*. The purpose of the paper is to give a systematical study on the all-pass filters and signals of minimum phase in both the discrete and continuous signals cases based on the mentioned phase derivative results. The theory for the discrete signals case may be said to have been partially formulated, while that for the continuous signals case is essentially new. For both cases we provide full details.

The theory of the functions being boundary limits of functions in the Hardy spaces has important applications to all-pass filters and signals of minimum phase (Section III). In the complex and harmonic analysis terminology, the $Z-$transforms of all-pass filters in discrete case are inner functions, and the $Z-$transforms of minimum phase signals are outer functions; in the continuous case, all-pass filters are distributional Fourier transforms of the boundary limits of inner functions, and signals of minimum phase are L^2-Fourier transforms of the boundary limits of outer functions. The relevant results in all-pass and minimum phase analysis are interplays between the two types of analytic functions. A series of papers by

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Kumaresan et al, including [13], [14] and [15] study minimum/maximum/all-pass decompositions in the time and frequency domains mainly for periodic analytic signal of finite bandwidth.

Comprehensive views of analytic signals and instantaneous frequencies with applications are contained in [2], [6], [5], [19]. The literature [27] contains a fundamental study on analytic signals with non-negative analytic instantaneous frequencies.

Instantaneous frequency is a fundamental concept in signal analysis. Its rigorous mathematical definition, however, has not been well agreed by signal analysts. The divergent understandings, as a matter of fact, have created many of the controversies ([6]). It is accepted that instantaneous frequency of a signal is its phase derivative with respect to the time variable (see [4], [20]). The question is how to uniquely determine a phase, and, once it is determined, in case the phase function is non-smooth, how to define a "phase derivative". For $n > 0$, it would be ambiguous to decide that both the signals $\sin nt$ and $\cos nt$ have the same phase function nt, and thus have the same frequency n. Signal analysts would write $\sin nt = \cos(nt - \pi/2)$ and then determine that $\sin nt$ has the same instantaneous frequency n. The background idea of transferring $\sin$ to $\cos$ would be traced back to Gabor. Until 1946, Gabor raised the concept *analytic signal* that is composed by a real-valued signal s and its Hilbert transform Hs in the pattern $As = s + iHs$ (see below), called the *analytic signal associated with* s. The *analytic instantaneous frequency* of a real-valued signal is then defined to be the *phase derivative* of the associated (complex-valued) analytic signal in its natural amplitude-phase, or quadrature, representation $As(t) = \rho(t)e^{i\varphi(t)}$ (see [10]), viz. $\varphi'(t)$, provided that the latter exists as a measurable function. It should then be called the *analytic phase derivative* or *analytic instantaneous frequency*.

On the other hand, any complex-valued signal s, including real-valued signals, or analytic, or non-analytic signals, has its natural quadrature representation, $s(t) = \rho(t)e^{i\varphi(t)}$, called *quadrature amplitude-phase representation*, or simply *amplitude-phase representation*, and the *phase derivative* $\varphi'(t)$ is defined to be the *quadrature instantaneous frequency*, or simply the *instantaneous frequency*, if exists. Therefore, the analytic instantaneous frequency of a real-valued signal s is the quadrature instantaneous frequency of the (complex-valued) analytic signal As. In this terminology, n is the analytic instantaneous frequency of both $\cos nt$ and $\sin nt$, obtained, respectively, as the quadrature instantaneous frequency of e^{int} and that of $e^{i(nt-\pi/2)}$.

The above introduced method of determining phase via the associated analytic signal is applicable to both real-valued and complex-valued signals. In general, a signal s of finite energy is decomposed into a sum of two signals of which one is boundary limit of a holomorphic function in the Hardy space of the upper-half plane, and the other is boundary limit of one in the Hardy space of the lower-half plane, viz.

$$\begin{aligned} s &= s^+ + s^- = \frac{1}{2}(s + iHs) + \frac{1}{2}(s - iHs) \\ &= \frac{1}{2}A^+ s + \frac{1}{2}A^- s, \end{aligned} \qquad (1.1)$$

where $A^{\pm}s = s \pm iHs$ are the *upper-* and *lower- analytic signals* associated with s (See Section II). The Hilbert transformation H on the line is defined through

$$Hs(t) \triangleq \frac{1}{\pi} \lim_{\epsilon \to 0} \int_{|t-u|>\epsilon} \frac{s(u)}{t-u} du. \qquad (1.2)$$

Under the extra condition $\omega\hat{s}(\omega) \in L^2(\mathbb{R})$, meaning that s is in the Sobolev space, s^+ and s^- both have certain smoothness and thus possess the *non-tangential analytic boundary derivatives* ${s^+}'$ and ${s^-}'$ ([8]). These allow to further formulate the so called *Hardy-Sobolev phase derivative* (see [7]):

$$\varphi^*(t) = \begin{cases} \mathrm{Im}\left[\frac{{s^+}'(t) + {s^-}'(t)}{s^+(t) + s^-(t)}\right] & \text{if } s^+(t) + s^-(t) \neq 0, \\ 0 & \text{if } s^+(t) + s^-(t) = 0. \end{cases} \qquad (1.3)$$

We similarly define Hardy-Sobolev derivatives of s and ρ, denoted by s^* and ρ^*, respectively. We prove in [7] that when the derivatives $\varphi'(t), s'(t)$ and $\rho'(t)$ exist in the classical sense, then the Hardy-Sobolev derivatives $\varphi^*(t), s^*(t)$ and $\rho^*(t)$ coincide with them. A large number of relations for smooth signals are extendable to general functions in the Sobolev space by using Hardy-Sobolev derivatives ([7]). The frequency spectra or inverse Fourier transforms, respectively for the discrete or continuous all-pass filters and signals of minimum phase, are themselves analytic signals. Their Hardy spaces decompositions then satisfy $s^- = 0$. This will be regarded as the "one-sided" case. In the case the Hardy-Sobolev derivatives are reduced to one-sided, too, namely

$$\varphi'(t) = \mathrm{Im}\left[\frac{{s^+}'(t)}{s^+(t)}\right],$$

called *non-tangential analytic phase derivative* or, in brief, *analytic phase derivative*. Although inner functions do not belong to the Sobolev space, their properties still guarantee the existence of the phase derivative. For signals of minimum phase the Sobolev space assumption should be added to ensure existence of such phase derivative. In Section 2 we provide more details for non-tangential analytic phase derivative.

The writing plan of the paper is as follows. In §2 we discuss non-tangential analytic phase derivatives for inner and outer functions in the unit disc and in the upper-half complex plan. In §3 we study the properties of all-pass filters and signals of minimum phase for both discrete and continuous signals.

II. Generalized Amplitude and Phase Derivatives

The quadrature amplitude and phase derivatives of a given real- or complex-valued signal $s(t) = \rho(t)e^{i\varphi(t)}$, $-\infty < t < \infty$, at the time moment t are defined to be the classical derivatives $\rho'(t)$ and $\varphi'(t)$, when exist, where $\rho(t)$ and $\varphi(t)$ are defined through

$$\rho(t) = |s(t)| \quad \text{and} \quad \varphi(t) = \arg \frac{s(t)}{|s(t)|}.$$

By definition, for a complex number z, $\arg z$ is any real number that satisfies the relation

$$z = |z|e^{i \arg z},$$

while $\mathrm{Arg}z$ is the branch of $\arg z$ in $(-\pi, \pi]$, called *the principal value* of $\arg z$. This shows that the mapping $s \to \varphi$ is not unique, and φ is, in fact, "set-valued." The function $\mathrm{Arg}s$ is uniquely defined, but it may not be differentiable. For a general square-integrable function one does not expect $\rho(t)$ and $\varphi(t)$ to be differentiable.

We note that if $s(t) = \rho(t)e^{i\varphi(t)}$, and $s(t)$, $\rho(t)$ and $\varphi(t)$ are differentiable and $\rho(t) \neq 0$, then differentiation of composed function gives

$$\rho'(t) = \rho(t)\mathrm{Re}\frac{s'(t)}{s(t)}, \qquad \varphi'(t) = \mathrm{Im}\frac{s'(t)}{s(t)}. \tag{2.4}$$

Our task is to define the phase derivatives for general square-integral signals.

The Fourier transform of $s \in L^1(\mathbb{R})$ is

$$\hat{s}(\omega) \triangleq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-it\omega} s(t)dt. \tag{2.5}$$

If $\hat{s}$ is also in $L^1(\mathbb{R})$, then the inversion formula holds, that is

$$s(t) = \hat{s}^\vee(t) \triangleq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{it\omega} \hat{s}(\omega)d\omega, \quad \text{a.e.} \tag{2.6}$$

Due to the Plancherel Theorem

$$\|\hat{s}\|_2^2 = \|s\|_2^2, \qquad s \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}),$$

both the Fourier transformation and its inverse can be extended, through a density argument, to $L^2(\mathbb{R})$, in which the Plancherel Theorem and the inversion formula remain to hold. The formulas (2.5) and (2.6) are valid for $s \in L^2(\mathbb{R})$, keeping in mind that convergence of the integrals are taken to be of the $L^2(\mathbb{R})$-norm sense.

The Hardy spaces decomposition refers to the decomposition of a signal $s \in L^2(\mathbb{R})$ into $s = s^+ + s^-$, where $s^\pm = (1/2)(s \pm iHs) \in H^2(\mathbb{C}^\pm)$. Note that the mappings from $H^2(\mathbb{C}^\pm)$ to their boundary values $H_b^2(\mathbb{R}) \subset L^2(\mathbb{R})$ are isometric isomorphisms. The Hardy-functions $s^\pm$ are given by

$$\begin{aligned} s^\pm(z) &= \frac{\pm 1}{2\pi i} \int_{-\infty}^{\infty} \frac{s(u)}{u - z} du \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{it\omega} \chi_\pm(\omega) e^{-y\omega} \hat{s}(\omega) d\omega, \end{aligned} \tag{2.7}$$

where $z = t + iy \in \mathbb{C}^\pm$. The last equal relation is a consequence of the Plancherel Theorem and the relation

$$\begin{aligned} [\frac{1}{(\cdot) - z}]^\vee(\omega) &= \pm\sqrt{2\pi} i \chi_\pm(\omega) e^{ix\omega} e^{-y\omega}, \\ & \qquad z = x + iy, \ \pm y > 0, \end{aligned} \tag{2.8}$$

where $\chi_\pm = \chi_{\mathbb{R}^\pm}$, $\mathbb{R}^+ = (0, +\infty)$ and $\mathbb{R}^- = (-\infty, 0)$. In general, χ_E denotes the characteristic function of the Lebesgue measurable set E that takes value 1 on E; and 0 otherwise. If either s^+ or s^- is the zero function, then s belongs to the one-sided category; and otherwise two-sided. From now on we concentrate in the one-sided case. The task of the rest part of this section is to define phase and amplitude derivatives of the non-tangential boundary values of some functions in the Hardy H^p spaces ([11], [12]). We will treat both the unit disc and a half complex plane cases.

(i) The unit disc context If $s \in H^p(\mathbb{D})$, then $s(z)$ has a non-tangential boundary value at almost all points on the boundary. We denote the limit function by s, that is

$$\lim_{r \to 1-} s(re^{it}) = s(e^{it}). \tag{2.9}$$

The Hardy space theory asserts that the limit does exist for almost all t, and the limit function s belongs to $L^p(\partial\mathbb{D})$. The boundary values of $H^p(\mathbb{D})$ form a closed subspace of the $L^p(\mathbb{D})$. For $p = 2$ the mapping is isometrically isomorphic (see [11]). The Nevanlinna Factorization theorem plays an important role in the phase-amplitude derivative theory.

Theorem 2.1: ([11]) If $s \in H^p(\mathbb{D})$, $p > 0$, then, apart from unimodular constants in the factors, $s(z)$ has a unique factorization representation

$$s(z) = cO(z)B(z)S(z)$$

where c is a constant and $|c| = 1$, $B(z)$ is a Blaschke product, $S(z)$ is a singular inner function, and $O(z)$ is an outer function in $H^p(\mathbb{D})$. They have the representations

$$B(z) = z^m \prod_{|z_k|\neq 0} \frac{-\bar{z}_k}{|z_k|} \frac{z - z_k}{1 - \bar{z}_k z}, \tag{2.10}$$

where m is a positive integer, z_k's are the zeros of $s(z)$ in the unit disk $\mathbb{D}$ that satisfy $\sum(1 - |z_k|) < \infty$;

$$S(z) = \exp\{-\int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\psi(\theta)\}, \tag{2.11}$$

where $d\psi(\theta)$ is a positive Borel measure singular to $d\theta$; and

$$O(z) = \exp\{\frac{1}{2\pi} \int_{-\pi}^{\pi} \ln|s(e^{i\theta})| \frac{e^{i\theta} + z}{e^{i\theta} - z} d\theta\}, \tag{2.12}$$

where for $1 \leq p \leq \infty$, there holds $\ln|s(e^{i\theta})| \in L^1(\partial\mathbb{D})$. For an analytic function $s : \mathbb{D} \to \mathbb{C}$, writing it in the form $s(re^{it}) = \rho_r(t)e^{i\varphi_r(t)}, r < 1$, and taking partial derivative with respect to t and then dividing the both sides by $s(re^{it})$, we obtain

$$\frac{\partial}{\partial t}\varphi_r(t) = \mathrm{Re}\left(z\frac{s'(z)}{s(z)}\right), \tag{2.13}$$

and

$$\frac{\partial}{\partial t}\rho_r(t) = -\rho_r(t)\mathrm{Im}\left(z\frac{s'(z)}{s(z)}\right). \tag{2.14}$$

Note that these relations may be extended to e^{it_0}, if s is analytic at the point e^{it_0}. In the case $s(e^{it}) = \rho_1(t)e^{i\varphi_1(t)}$, where φ_1 is the continuous continuation of $\varphi_r, r < 1$. Then $\varphi_1{}'(t_0)$ is the phase derivative of s at e^{it_0}. For a general point $z = e^{it}$, at which the function may not have an analytic continuation, a generalized phase derivative is defined through the *non-tangential limit* of the quantity given by (2.13). Throughout the paper whenever we concern boundary limit we always mean non-tangential boundary limit from inside of the domain (see [11]).

Definition 2.2: Let $s : \mathbb{D} \to \mathbb{C}$ be analytic, and $\zeta \in \partial\mathbb{D}$. If the limit $\lim_{z\to\zeta} \frac{zs'(z)}{s(z)}$ exists, then we denote

$$D_p s(\zeta) = \lim_{z\to\zeta} \mathrm{Re}\frac{zs'(z)}{s(z)},$$

and

$$D_a s(\zeta) = -\lim_{z\to\zeta} |s(z)|\mathrm{Im}\frac{zs'(z)}{s(z)},$$

and call them, respectively, the *phase derivative* and *amplitude derivative* of s at ζ. Note that for a given ζ, $D_p s(\zeta)$ or $D_a s(\zeta)$ or both of them may not exist, and when exist, may happen to be $\pm\infty$, or ∞. For $\zeta = e^{it}, s = \rho e^{i\varphi}$, the notation $D_p s(\zeta)$ and $D_a s(\zeta)$ are also denoted $\varphi'(t)$ and $\rho'(t)$. If s has an analytic continuation to ζ, then $D_p s$ and $D_a s$ coincide with the classical derivatives of the phase and amplitude.

As a direct consequence of Definition 2.2 we have

Theorem 2.3: If $s, s' \in \cup_{r>0} H^r(\mathbb{D})$, then both $D_p s$ and $D_a s$ are well defined measurable functions, and their function values are a.e. finite and non-zero.

Proof Since each of s and s' is in some Hardy space, their boundary values are a.e. finite and non-zero (Page 65, Corollary 4.2, [11]). The result follows.

[23] proves the positivity of phase derivatives of inner functions through the Julia-Wolff-Carathéodory's Theorem ([21] or [16]), stated as follows.

Theorem 2.4: If s is a non-trivial inner function, then $D_p(s)(t) > 0$ and $D_a(s)(t) = 0$, a.e. Moreover, if s has analytic continuation across an open interval containing $\zeta = e^{it_\zeta} \in \partial\mathbb{D}$, then with the angular parametrization $s(e^{it}) = e^{i\theta(t)}$ the phase function $\theta(t)$ is differentiable at $t = t_\zeta$, and $0 < \theta'(t_\zeta) = D_p(s)(e^{it_\zeta}) < +\infty$.

The preceding proofs of positivity of the phase derivatives of the boundary values of inner functions are only available for finite Blaschke products and for singular inner functions induced by finite linear combinations of shifted Dirac point measures ([5]).

It may be observed that for outer functions its boundary phase derivatives are sometimes positive and sometimes negative. As example, we consider a fractional linear transform that maps the unit disc to a disc that does not contain the origin. It is an outer function mapping the unit circle to the boundary of the disc in the same orientation. It follows that the phase is increasing in an open interval of the t variable, and then decreasing in an adjacent interval. The fact that the wending number being zero implies that the phase derivative has zero mean property. The following theorem concerns outer functions in a more general context.

Theorem 2.5: ([23]) Let s be an outer function in some H^p space for $0 < p \leq \infty$, and the analytic function s'/s belong to the Hardy $H^1(\mathbb{D})$ space. Then the boundary limits s' and s'/s both exist and are finite a.e., and s'/s is integrable with

$$\int_0^{2\pi} e^{it}\frac{s'(e^{it})}{s(e^{it})}dt = 0. \tag{2.15}$$

As consequence,

$$\int_0^{2\pi} D_p s(e^{it})dt = 0. \tag{2.16}$$

Theorem 2.5 addresses the fact that in general an outer functions has positive phase derivatives in a measurable set of positive measure, and has negative phase derivatives in a measurable set of positive measure as well.

(ii) The Upper- and Lower half complex planes context

The theory for a half complex plane is analogous with that for the unit disc. We only deal with the upper-half complex plane case.

Denote by κ the Cayley transformation that maps the upper-half complex plane conformally onto the disc $\mathbb{D}$ and the mapping continuously and in one to one manner extends to their boundaries:

$$\kappa : \mathbb{C}^+ \ \rightarrow \mathbb{D}, \quad \omega = \kappa(z) = \frac{i-z}{i+z},$$

$$z = \kappa^{-1}(\omega) = i\frac{1-\omega}{1+\omega}, \quad \kappa(i) = 0, \quad \kappa(\infty) = -1.$$

On the boundaries,

$$e^{it} = \frac{i-s}{i+s}, \ s = i\frac{1-e^{it}}{1+e^{it}},$$

$$\kappa((-\infty, \infty]) = \{e^{it} \ : \ -\pi < t < \pi\},$$

and

$$s = \tan\frac{t}{2}, \ \ t = 2\arctan s, \ \ \frac{dt}{ds} = \frac{2}{1+s^2}, \ \ \frac{ds}{dt} = \frac{1}{2}\sec^2\frac{t}{2}.$$

In the references [11] and [12] it is pointed out that $g(\omega) \in H^p(\mathbb{D})$ if and only if $F(z) = \frac{\pi^{-1/p}}{(z+i)^{2/p}} g(\omega(z)) \in H^p(\mathbb{C}^+)$. There is also a factorization theorem in $H^p(\mathbb{C}^+)$.

Theorem 2.6: ([12]) If $F \in H^r(\mathbb{C}^+), \ r > 0$, then $F(z)$ has a unique decomposition

$$F(z) = CB(z)S(z)O(z)$$

where $|C| = 1$, $B(z)$ is a Blaschke product, $S(z)$ is a singular function, and $O(z)$ is an outer function in $H^r(\mathbb{C}^+)$. They can be represented as following:

$$B(z) = \prod_{|z_k|\neq 0} e^{i\alpha_k}\frac{z-z_k}{z-\bar{z}_k}, \tag{2.17}$$

where $\{z_k\}$ are zeros of $F(z)$ in the $\mathbb{C}^+$ satisfying $\sum \frac{y_k}{1+|z_k|^2} < \infty$, $z_k = x_k + iy_k$, and $\{\alpha_k\}$ are selected so that $e^{i\alpha_k}\frac{i-z_k}{i-\bar{z}_k} \geq 0$;

$$S(z) = \exp\{-\frac{i}{\pi}\int_{-\infty}^{\infty}\frac{1+zt}{(z-t)(1+t^2)}d\sigma(t)\}, \tag{2.18}$$

where the measure $d\sigma(t)$ is positive and singular to dt, and satisfies $\int_{-\infty}^{\infty}\frac{d\sigma_F(t)}{1+t^2} < \infty$;

$$O(z) = \exp\{\frac{i}{\pi}\int_{-\infty}^{\infty}\ln|F(t)|\frac{1+zt}{(z-t)(1+t^2)}dt\}. \tag{2.19}$$

Similar to the unit disc case, we also have

Theorem 2.7: If $F, \ F' \in \cup_{r>0}H^r(\mathbb{C}^+)$, then both D_pF and D_aF are well defined measurable functions and finite a.e., where

$$\begin{aligned} D_pF(s) &\triangleq \lim_{z\to s}\text{Im}\left(\frac{F'(z)}{F(z)}\right) \\ &= \text{Im}\left(\frac{F'(s)}{F(s)}\right) \end{aligned} \tag{2.20}$$

$$\begin{aligned} D_aF(s) &\triangleq \lim_{z\to s}|F(z)|\text{Re}\left(\frac{F'(z)}{F(z)}\right) \\ &= |F(s)|\text{Re}\left(\frac{F'(s)}{F(s)}\right), \quad a.e. \end{aligned} \tag{2.21}$$

There is an alternative way to define the phase derivative of the non-tangential boundary value $F(s)$: We proceed by converting it to the unit disc. That is to map everything in the upper-half complex plane, through the Cayley transformation, to the unit disc. Note that Cayley transformation preserves complex analyticity and is of monotonicity when restricted to the boundaries. The phase derivative of F may be defined by $D(F\circ\kappa^{-1})(\kappa(s))\frac{dt}{ds}$. We now prove that the phase derivatives defined by the two methods are identical.

Theorem 2.8: Let F be an analytic function in $\mathbb{C}^+$. Denote by $D_{\mathbb{D}}$ the phase derivative defined by Definition 2.2 for the unit circle. There follows

$$\frac{dt}{ds}D_{\mathbb{D}}(F\circ\kappa^{-1})(\kappa(s)) = \lim_{z\to s}\text{Im}\frac{F'(z)}{F(z)}. \tag{2.22}$$

Proof of Theorem Let $F(z) = F(s+iy), \ y > 0$, we have

$$\begin{aligned} & \frac{dt}{ds}\cdot D_{\mathbb{D}}(F\circ\kappa^{-1})(\kappa(s)) \\ = \ & \frac{2}{1+s^2}\cdot\lim_{y\to 0^+}\text{Re}\frac{\kappa(s+iy)(F\circ\kappa^{-1})'(\kappa(s+iy))}{(F\circ\kappa^{-1})(\kappa(s+iy))} \\ = \ & \frac{2}{1+s^2}\lim_{y\to 0^+}\text{Re}\frac{\kappa(s+iy)F'(s+iy)(\kappa^{-1})'(\kappa(s+iy))}{F(s+iy)} \\ = \ & \lim_{y\to 0^+}\text{Re}\frac{-iF'(s+iy)}{F(s+iy)} \\ = \ & \lim_{z\to s}\text{Im}\frac{F'(z)}{F(z)}. \quad \Box \end{aligned}$$

Note that $\frac{dt}{ds}$ is always finite and positive, and $D_{\mathbb{D}}(F\circ\kappa^{-1})(\kappa(s)) = D_{\mathbb{D}}f(e^{it})$. Taking into account that the Caylay transformation maps inner functions in the upper-half complex plane to inner functions of the same type (the Blaschke product type or the singular inner function type) in the unit disc, and vise versa, we obtain

Theorem 2.9: If F is an inner function in the upper-half complex plane, then $D_pF > 0$ a.e. Moreover, if F has an analytic extension across an open interval containing s, then with the angular parametrization $F(s) = e^{i\phi(s)}$ the phase function $\phi(s)$ is differentiable at s, and $0 < \phi'(s) < +\infty$.

Theorem 2.10: Let F be an outer function in the upper-half complex plane, and the analytic function $\frac{F'}{F}$ belong to the Hardy $H^1(\mathbb{C}^+)$ space. Then the boundary limits F' and that of $\frac{F'}{F}$ both exist and are finite a.e., and the function $\frac{F'(s)}{F(s)}$ is integrable on $\mathbb{R}$ with

$$\int_{\mathbb{R}} \frac{F'(s)}{F(s)} ds = 0.$$

In particular,

$$\int_{\mathbb{R}} D_p F(s) ds = 0.$$

III. All-Pass Filters and Signals of Minimum Phase

In this part we discuss all-pass filters and signals of minimum phase and energy in the two contexts: the discrete and the continuous signals. The discrete signals $\{x(n)\}$ or continuous signals $f(s)$ we deal with are of finite energy, that is,

$$\sum_{n=-\infty}^{+\infty} |x(n)|^2 < \infty \quad \text{or} \quad \int_{-\infty}^{+\infty} |f(s)|^2 ds < +\infty. \tag{3.23}$$

A. Discrete signals

Definition 3.1: (i) A discrete signal $\{h(n)\}$ is said to be *physically realizable* if $\{h(n)\} \in l^2$ and

$$h(n) = 0, \quad if\ n < 0.$$

(ii) The Z-transform of a physically realizable signal $\{h(n)\}$ is

$$H(z) = \sum_{n=0}^{\infty} h(n) z^n.$$

Since $\{h(n)\} \in l^2$, it is easy to verify that $H(z)$ is well defined as an analytic function, and, in fact, an H^2-function in the unit disc. As consequence, it has non-tangential boundary values on $\partial\mathbb{D}$, denoted by $H(e^{i\omega})$, called the *frequency spectrum* of $\{h(n)\}$. In both the L^2- and the pointwise- convergence sense (Carleson's Theorem on pointwise convergence of Fourier series), we have

$$H(e^{i\omega}) = \sum_{n=0}^{+\infty} h(n) e^{in\omega}. \tag{3.24}$$

(iii) A discrete signal is said to be a *pure phase signal* or *all-pass filter* if it is physically realizable, and

$$|G(e^{i\omega})| = |\Sigma_{n=0}^{+\infty} g(n) e^{in\omega}| = 1, \quad \text{a.e.}$$

(iv) The phase derivative D_pH of the frequency spectrum $H(e^{i\omega})$ of a physically realizable signal $\{h(n)\}$ is also called the phase derivative of $\{h(n)\}$.

(v) Let $\{h_1(n)\}$ be a fixed physically realizable signal. If

$$D_pH \geq D_pH_1,$$

whenever

$$|H(e^{i\omega})| = |H_1(e^{i\omega})|,$$

where H and H_1, are, respectively, the Z-transforms of $\{h(n)\}$ and $\{h_1(n)\}$, then $\{h_1(n)\}$ is said to be a *minimum-phase signal*.

In accordance with the Nevanlinna Factorization Theorem, in the non-tangential boundary limit sense, for $\zeta = e^{i\omega}$,

$$\begin{aligned} H(\zeta) &= cO(\zeta)B(\zeta)S(\zeta) \\ &= c\rho_O(\omega)e^{i\theta_O(\omega)}e^{i\theta_B(\omega)}e^{i\theta_S(\omega)}, \end{aligned}$$

where O, B and S are the corresponding outer, Blaschke product and the singular inner functions (unique up to unimodular constants), and $\rho_O = |O|$. In the following proposition we discuss the existence of the boundary phase derivatives of $H(\zeta),\ O(\zeta),\ B(\zeta)$ and $S(\zeta)$.

Proposition 3.2: Let $H \in \cup_{p>0} H^p(\mathbb{D}),\ \zeta \in \partial\mathbb{D}$.

(i) If $H' \in \cup_{r>0} H^r(\mathbb{D})$, and $D_pB(\zeta),\ D_pS(\zeta)$ are a.e. finite, then $D_pH(\zeta),\ D_pO(\zeta)$ exist and are finite a.e.

(ii) If $O(\zeta)$ satisfies the conditions of Theorem 2.5, then $D_pO(\zeta)$ exists and is finite a.e.; and $D_pH(\zeta)$ exists a.e.

We omit the proof of the proposition and refer it to the results of §II. Under the condition of (i) or (ii) of Proposition 3.2, by invoking Theorem 2.4, we have

$$\begin{aligned} D_pH(\zeta) &= \mathrm{Re}\frac{\zeta H'(\zeta)}{H(\zeta)} \\ &= \mathrm{Re}\frac{\zeta O'(\zeta)}{O(\zeta)} + \mathrm{Re}\frac{\zeta B'(\zeta)}{B(\zeta)} + \mathrm{Re}\frac{\zeta S'(\zeta)}{S(\zeta)} \\ &= D_pO(\zeta) + D_pB(\zeta) + D_pS(\zeta) \\ &\geq D_pO(\zeta). \end{aligned} \tag{3.25}$$

We thus conclude that outer functions are minimum-phase signals. It amounts to say that of all physically realizable signals with the same amplitude spectrum $|H(e^{i\omega})|$, outer functions determined by the formula (2.12) are of minimum phase.

If $f(z)$ is in the Hardy $H^2(\mathbb{D})$, and its boundary value $f(e^{it})$ is in $L^r(\partial\mathbb{D})$, then $f(z)$ is in $H^r(\mathbb{D})$ (Chapter II, Corollary 4.3, [11]). Using this assertion to $r = \infty$ and $f(z) = H(z)$, where $H(z)$ is the Z-transform of a physically realizable signal, we conclude that $H(z)$ is an inner function if and only if the physically realizable signal associated with $H(z)$ is an all-pass filter. To summarize, we have the following

Theorem 3.3: (i) All-pass filters have a.e. positive phase derivatives. (ii) The discrete signals corresponding to outer functions are minimum-phase signals.

Now we prove a sufficient and necessary condition for minimum-phase signal.

Theorem 3.4: Let $\{h_1(n)\}$ be a physically realizable signal of finite energy, and its amplitude spectrum is $|H_1(e^{i\omega})|$. Then $\{h_1(n)\}$ is a minimum-phase signal if and only if, for any h and the corresponding $H(e^{i\omega})$, whenever $|H_1(e^{i\omega})| = |H(e^{i\omega})|$ we have

$$h_1(n) = e^{i\beta} h_O(n), \tag{3.26}$$

where β is a real constant, $\{h_O(n)\}$ is identical with an outer function $O_H(Z)$ generated from the amplitude spectrum $|H(e^{i\omega})|$ by using formula (2.12).

Proof of Theorem By definition, $\{h_1(n)\}$ is a minimum-phase signal if and only if for any physically realizable signal $\{h(n)\}$ with amplitude spectrum $|H_1(e^{i\omega})|$ the phase derivative $D_pH \geq D_pH_1$. We need to show $D_pH_1 = D_pO$, where O is the outer function in the Nevanlinna decomposition $H = cOBS$. On one hand, $O(z)$ itself is such a $H(z)$, therefore, $D_pO \geq D_pH_1$. On the other hand, since $|H_1| = |O|$, and the phase derivatives of B and S are a.e. positive (Theorem 2.4), we have $D_pH_1 \geq D_pO$. Therefore, $D_pO = D_pH_1$. The inner function part of H_1, therefore, is a constant. The latter is equivalent to $H_1(e^{i\omega}) = e^{i\beta}O_H(e^{i\omega})$, or $h_1(n) = e^{i\beta}h_O(n)$. Hence, $\{h_1(n)\}$ is a minimum-phase signal if and only if $h_1(n) = e^{i\beta}h_O(n)$. □

All-pass filters play a significant role in signal analysis also because of the total energy invariant and the energy delay properties. Those properties are not directly related to the positive phase derivative issue. Nevertheless, we provide the statements of the results for the completeness. The theory of discrete all-pass filters and discrete signals of minimum energy may be said to have been partially formulated (Ch. 9, §4, [5]). We decide to provide all the proofs of the propositions in the rest part of this section for two reasons. The principal reason is that some proofs are not valid in the literature (Theorem 3.5). The next reason is to make the existing proofs (of Theorem 3.6, 3.7) to be available in English (Ch. 9, §4, [5]) and thus the paper becomes self-containing.

Theorem 3.5: A physically realizable filter $\{g_n\}$ is an all-pass filter if and only if the total energy is invariant when any signal passes the filter $\{g_n\}$, that is, for any physically realizable input signal $\{x_n\}$, and the output signal $\{y_n\}: \ y_n = g_n * x_n = \sum_{k=0}^{+\infty} g_k x_{n-k}$, there holds

$$\sum_{n=0}^{+\infty} |x_n|^2 = \sum_{n=0}^{+\infty} |y_n|^2. \tag{3.27}$$

Proof Let X and G denote, respectively, the frequency spectrum of $\{x_n\}$ and $\{g_n\}$. The relation (3.27) is equivalent with

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{i\omega})|^2 d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} |G(e^{i\omega})|^2 |X(e^{i\omega})|^2 d\omega,$$

or

$$\int_{-\pi}^{\pi} \left(1 - |G(e^{i\omega})|^2\right) |X(e^{i\omega})|^2 d\omega = 0,$$

where X is any function in the Hardy space $H^2(\mathbb{D})$. The "only if" part is trivial. We now prove the "if" part from some particular choices of the test function $X(e^{i\omega})$. Substituting $X(e^{i\omega})$ with $X_r(e^{i\omega}) = \frac{\sqrt{1-r^2}}{1-re^{-i(t-\omega)}}, 0 < r < 1, t \in [-\pi, \pi]$, we have

$$\int_{-\pi}^{\pi} \left(1 - |G(e^{i\omega})|^2\right) \frac{1-r^2}{1-2r\cos(t-\omega)+r^2} d\omega = 0.$$

Letting $r \to 1$, since Poisson kernel is an approximation to identity ([25]), we have, pointwisely,

$$\lim_{r\to 1-} \left(1 - |G|^2\right) * P_r(t) = 1 - |G(e^{it})|^2 = 0, \quad \text{a.e.}$$

We thus conclude that $|G| = 1$ a.e. and so $\{g_n\}$ is an all-pass filter. □

We define the N-*partial energy* of physically realizable signal $\{x_n\}$ to be $\sum_{n=0}^{N} |x_n|^2$. For an all-pass filter $\{g_n\}$, we have

$$\sum_{n=0}^{N} |x_n|^2 = \sum_{n=0}^{+\infty} |\tilde{y}_n|^2, \quad \text{where } \tilde{y}_n = \sum_{\tau=0}^{N} x_\tau g_{n-\tau}. \tag{3.28}$$

Since

$$y_n = \sum_{\tau=0}^{+\infty} g_\tau x_{n-\tau} = \sum_{\tau=0}^{n} g_\tau x_{n-\tau} = \sum_{\tau=0}^{n} x_\tau g_{n-\tau}, \tag{3.29}$$

we obtain

$$\tilde{y}_n = y_n, \quad \text{when } n \leq N. \tag{3.30}$$

Theorem 3.6: Let $\{g_n\}$ be a physically realizable filter with the unit energy. Then $\{g_n\}$ is an all-pass filter if and only if for any physically realizable input signal $\{x_n\}$, the N-partial energy of the output $\{y_n\}: \ y_n = g_n * x_n$, is delayed, that is, for any $N > 0$,

$$\sum_{n=0}^{N} |x_n|^2 \geq \sum_{n=0}^{N} |y_n|^2. \tag{3.31}$$

Proof We assume that $\{g_n\}$ is an all-pass filter. For any physically realizable input signal $\{x_n\}$, we have

$$\sum_{n=0}^{N}|x_n|^2=\sum_{n=0}^{+\infty}|\tilde{y}_n|^2\geq\sum_{n=0}^{N}|\tilde{y}_n|^2=\sum_{n=0}^{N}|y_n|^2,\qquad N\geq 0$$

where $\tilde{y}_n$ and y_n are defined by (3.28) and (3.29).

We will prove the converse result by contradiction. Assume $G(e^{i\omega})\neq 1$. Take the input signal with the frequency spectrum $X(e^{i\omega})$ satisfying

$$|X(e^{i\omega})|^2=\begin{cases}2 & \text{if } |G(e^{i\omega})|^2-1>0,\\ 1 & \text{if } |G(e^{i\omega})|^2-1\leq 0.\end{cases}\tag{3.32}$$

Since

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|G(e^{i\omega})|^2d\omega=1,$$

then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|G(e^{i\omega})|^2|X(e^{i\omega})|^2d\omega>\frac{1}{2\pi}\int_{-\pi}^{\pi}|X(e^{i\omega})|^2d\omega,$$

and

$$\sum_{n=0}^{+\infty}|y_n|^2>\sum_{n=0}^{+\infty}|x_n|^2,$$

therefore there must exist some $N_0>0$ such that $\sum_{n=0}^{N_0}|y_n|^2>\sum_{n=0}^{N_0}|x_n|^2$, contradictory with (3.31). □

Theorem 3.7: Assume that the input $\{x_n\}$ is physically realizable, and the filter $\{g_n\}$ is an all-pass filter. Denote the output by $\{y_n\}$. Then for some N the equality

$$\sum_{n=0}^{N}|x_n|^2=\sum_{n=0}^{N}|y_n|^2\tag{3.33}$$

holds if and only if the Z-transform of the all-pass filter $\{g_n\}$ is of the form

$$G(Z)=\frac{y_0+y_1Z+...+y_NZ^N}{x_0+x_1Z+...+x_NZ^N},\tag{3.34}$$

where the denominator of $G(Z)$ has no zero in the closure of the unit disc.

Proof We first assume (3.33) and will show (3.34). For the signal $\{x_n\}'=\{x_1,...,x_N,0,0,...\}$, by (3.30), there holds

$$\sum_{n=0}^{N}|x_n|^2=\sum_{n=0}^{N}|\tilde{y}_n|^2.$$

Comparing it with (3.28), we conclude that $\tilde{y}_n=0$ when $n\geq N+1$. Then the Z-transform of $\{\tilde{y}_n\}$ is

$$Y_N(Z)=\tilde{y}_0+\tilde{y}_1Z+\ldots+\tilde{y}_NZ^N=y_0+y_1Z+\ldots+y_NZ^N,$$

while the Z-transform of $\{x_n\}'$ is

$$X_N(Z)=x_0+x_1Z+\ldots+x_NZ^N.$$

By $Y_N(Z)=G(Z)X_N(Z)$, we conclude (3.34). Since $\{g_n\}$ is an all-pass filter, $G(Z)$ is a H^∞-function in $\mathbb{D}$, and thus there is no singularity on the closed unit disc $\overline{\mathbb{D}}$.

Conversely, assume that (3.34) holds. Take the input signal to be $\{x_n\}$ whose Z-transform is $X(Z)=\sum_{n=0}^{N}x_nZ^n$. Then the output under the filter $\{g_n\}$ is $\{y_n\}$ whose Z-transform is $Y(Z)=\sum_{n=0}^{N}y_nZ^n$. Invoking Proposition 3.5, we get the equality (3.33). □

In digital signal processing, although minimum energy delay signal and minimum phase signal are two different concepts, they, in fact, are the same. Minimum energy delay signal is from the view of energy delay, and minimum phase signal is from the view of time delay.

B. Continuous Signal

Definition 3.8: (i) A signal of finite energy $f(t)$ is said to be *physically realizable* if it satisfies $f(t)=0,\ t<0$.

(ii) The *frequency spectrum* of a physically realizable signal $f(t)$ is defined to be the inverse Fourier transformation $\check{f}(\omega)$ of $f(t)$.

In fact, the frequency spectrum $\check{f}(\omega)$ of a physically realizable signal $f(t)$ is the non-tangential boundary value function of the $H^2(\mathbb{C}^+)$-function

$$\check{f}(z)=P_y*\check{f}(\omega)=\int P_y(t)\check{f}(\omega-t)dt,\quad z=\omega+iy,$$

where $P_y(t)=\frac{1}{\pi}\frac{y}{t^2+y^2}$ is the *Poisson kernel* of the upper-half complex plane (Chapter II, Corollary 3.2, [11]). In the sequel, we regard $\check{f}(z)$ as the $H^2(\mathbb{C}^+)$-function associated with $f(t)$. In view of the formulas in (2.7) we also have

$$\begin{aligned}\check{f}(z)&=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\check{f}(u)}{u-z}du\\&=\frac{1}{\sqrt{2\pi}}\int_{0}^{\infty}e^{i\omega t}e^{-yt}f(t)dt.\end{aligned}\tag{3.35}$$

Definition 3.9: Let $f_1(t)$ be a physically realizable signal. If for all physically realizable signals $f(t)$ such that $|\check{f}(\omega)|=|\check{f}_1(\omega)|$ we have $D_{\mathbb{R}}\check{f}\geq D_{\mathbb{R}}\check{f}_1$, then we say that $f_1(t)$ is a *minimum-phase signal*.

The Fourier transformation of a temperate distribution T is defined by the relation

$$\langle\hat{T},\phi\rangle=\langle T,\hat{\phi}\rangle,\qquad\phi\in\mathbb{S},\quad\text{the Schwartz class},$$

which coincides with the traditional definitions of Fourier transformations for functions in $L^p(\mathbb{R}),\ 1\leq p\leq 2$. Denote by $\mathbb{S}'$ the set of temperate distributions.

Set

$$\Phi^+\triangleq\{\phi\in\ C_0^\infty(\mathbb{R})\ :\ \text{supp }\phi\ \subset[0,\infty)\}$$

and

$$\Phi^- \triangleq \{\phi \in \ C_0^\infty(\mathbb{R}) \ : \ \text{supp}\ \phi \ \subset (-\infty, 0]\}.$$

The following theorem characterizes, in terms of Fourier spectrum, the boundary values of functions in the complex Hardy spaces $H^p(\mathbb{C}^\pm),\ 1 \le p \le \infty$. The case $p = \infty$ plays an important role in the all-pass filter theory for continuous signals.

Theorem 3.10: (see [22]) f is the non-tangential boundary value of a function in $H^p(\mathbb{C}^+),\ 1 \le p \le \infty$, if and only if $f \in L^p(\mathbb{R})$ *and* $\langle \hat{f}, \phi\rangle = 0$ for all $\phi \in \Phi^-$.

Theorem 3.11: (see p23, [26]) For $u \in \mathbb{S}'$ and $\varphi \in \mathbb{S}$, the convolution $u * \varphi$ is the function f, whose value at $x \in \mathbb{R}$ is $f(x) = u(\tau_x \tilde{\varphi})$, where τ_x denotes the variable translation operator by x, and $\tilde{\varphi}(x) = \varphi(-x)$. Moreover, f belongs to the class C^∞ and it, as well as all its derivatives, are slowly increasing.

Definition 3.12: Let g be a temperate distribution with supp $g \subset [0, \infty)$. We call g an *all-pass filter* if its inverse Fourier transformation $\check{g}$ is the non-tangential boundary value of an inner function in the $\mathbb{C}^+$.

¿From Theorem 3.10 and Definition 3.12, we conclude that a temperate distribution g with supp $g \subset [0, \infty)$ is an all-pass filter if and only if $\check{g}$ is a bounded analytic function satisfying

$$|\check{g}(\omega)| = 1,\ a.e.$$

Below for an all-pass filter g, the input signals are restricted to be physically realizable $f(t) \in \mathbb{S}$, and the output signals are $h(t) = (f * g)(t)$, owing to Theorem 3.11, are physically realizable signals belonging to C^∞.

According to the Factorization Theorem 2.6, any H^2-function $\check{f}(z)$ has the representation:

$$\begin{aligned}\check{f}(z) &= cO(z)B(z)S(z)\\ &= c\eta_{O,y}(\omega)e^{i\phi_{O,y}(\omega)}e^{i\phi_{B,y}(\omega)}e^{i\phi_{S,y}(\omega)},\\ &\qquad z = \omega + iy,\ y > 0,\end{aligned}$$

where O, B, S are the corresponding outer function, Blaschke product and singular inner function parts of $\check{f}$, where $O(z) = \eta_{O,y}(\omega)e^{i\phi_{O,y}(\omega)}, B(z) = e^{i\phi_{B,y}(\omega)}$ and $S(z) = e^{i\phi_{S,y}(\omega)}$.

Similarly to the discrete case, we study boundary phase derivatives of $\check{f}(z),\ B(z),\ S(z)$ and $O(z)$.

Proposition 3.13: Let $\check{f}(z) \in \cup_{r>0}H^r(\mathbb{C}^+)$.

(i) If $\check{f}'(z) \in \cup_{r>0}H^r(\mathbb{C}^+)$, and $D_{\mathbb{R}}B(\omega),\ D_{\mathbb{R}}S(\omega)$ are not equal to $+\infty$ on $\mathbb{R}$ a.e., then $D_{\mathbb{R}}\check{f}(\omega),\ D_{\mathbb{R}}O(\omega)$ exist, and are finite a.e.

(ii) If $O(z)$ satisfies the conditions of Theorem 2.10, then $D_{\mathbb{R}}O(\omega)$ exists and is finite a.e.; and $D_{\mathbb{R}}\check{f}(\omega)$ exists a.e.

Under the conditions of (i) or (ii) of Proposition 3.13, there follows

$$\begin{aligned}D_{\mathbb{R}}\check{f}(\omega) &= \text{Im}\frac{\check{f}'(\omega)}{\check{f}(\omega)}\\ &= \text{Im}\frac{O'(\omega)}{O(\omega)} + \text{Im}\frac{B'(\omega)}{B(\omega)} + \text{Im}\frac{S'(\omega)}{S(\omega)}\\ &= D_{\mathbb{R}}O(\omega) + D_{\mathbb{R}}B(\omega) + D_{\mathbb{R}}S(\omega)\\ &\ge D_{\mathbb{R}}O(\omega). \qquad (3.36)\end{aligned}$$

Thus $\hat{O}(t)$, corresponding to the outer function $O(z)$, is the minimum phase signal. It is to say that of all physically realizable signals with the same amplitude function, the one being the outer function is the minimum phase signal.

Theorem 3.14: Let $f_1(t)$ be a physically realizable signal. Then $f_1(t)$ is a minimum-phase signal if and only if

$$\check{f}_1(\omega) = e^{i\beta}O(\omega), \qquad (3.37)$$

where β is a real constant, O is an outer function formed from the boundary value $|\check{f}_1|$ by using (2.19).

In the continuous case, all-pass filters also have the same characteristic energy preserving property as for the discrete case.

Theorem 3.15: Assume that g is a temperate distribution with supp $g \subset [0, \infty)$ whose inverse Fourier transform $\check{g}(\omega)$ is a bounded measurable function. Then g is an all-pass filter if and only if

$$\int_0^\infty |f(t)|^2 dt = \int_0^\infty |h(t)|^2 dt. \qquad (3.38)$$

where $f(s)$ is any physically realizable input signal in $\mathbb{S}$, and $h(t) = (g * f)(t)$ is the output signal.

Proof of Theorem By the Plancherel Theorem, (3.38) holds if and only if

$$\int_{-\infty}^\infty |\check{f}(\omega)|^2 d\omega = \int_{-\infty}^\infty |\check{f}(\omega)\check{g}(\omega)|^2 d\omega.$$

That is,

$$\int_{-\infty}^\infty |\check{f}(\omega)|^2(1 - |\check{g}(\omega)|^2) d\omega = 0. \qquad (3.39)$$

If g is an all-pass filter, that is, $|\check{g}(\omega)| = 1$, a.e., then

$$\int_0^\infty |f(t)|^2 dt = \int_0^\infty |(g * f)(t)|^2 dt. \qquad (3.40)$$

We now show that the converse also holds. Assume that (3.40) holds. Let $F(t)$ be a function in the Schwartz class that has compact support in $[0, +\infty)$. Let $F_0(t) = F(-ty)e^{-itu},\ F_1(t) = \overline{F(-t)},\ F_2(t) = \frac{1}{\sqrt{2\pi}}(F_1 * F)(t)$, and $F_3(t) = e^{-itu}F_2(-ty)$, where $y > 0,\ u \in$

$(-\infty, +\infty)$. Since $F_0(t)$ is also in the Schwartz class and has compact support in $[0, +\infty)$, we have

$$\int_{-\infty}^{\infty} |\check{F}_0(\omega)|^2 (1 - |\check{g}(\omega)|^2) d\omega = 0.$$

We note that

$$\check{F}_3(\omega) = \frac{1}{y}\check{F}_2(\frac{u-\omega}{y}) = \frac{1}{y}|\check{F}(\frac{u-\omega}{y})|^2 = y|\check{F}_0(\omega)|^2.$$

Then

$$\int_{-\infty}^{\infty} \frac{1}{y}\check{F}_2(\frac{u-\omega}{y})(1 - |\check{g}(\omega)|^2) d\omega = 0.$$

In using the approximation to identity result (see p13, [26]) of the dilated convolution to the test function $1 - |\check{g}(\omega)|^2 \in L^\infty(\mathbb{R})$, we have

$$\begin{aligned} 0 &= \lim_{y\to 0} \int_{-\infty}^{\infty} \frac{1}{y}\check{F}_2(\frac{u-\omega}{y})(1 - |\check{g}(\omega)|^2) d\omega \\ &= (1 - |\check{g}(u)|^2) \int_{-\infty}^{\infty} \check{F}_2(\omega) d\omega, \qquad \text{a.e.} \end{aligned}$$

Since

$$\int_{-\infty}^{\infty} \check{F}_2(\omega) d\omega = \int_{-\infty}^{\infty} |\check{F}(\omega)|^2 d\omega \neq 0,$$

we have

$$1 - |\check{g}(u)|^2 = 0, \quad \text{a.e.} \quad \square$$

We regard $\int_0^N |f(t)|^2 dt$ as the *N-partial energy* of the physically realizable signal $f(t)$. Denoting $f_N(t) = \chi_{[0,N]}(t) f(t)$, for any all-pass filter g, from Theorem 3.15, there holds

$$\int_0^N |f(t)|^2 dt = \int_0^\infty |h_N(t)|^2 dt, \tag{3.41}$$

where

$$h_N(t) = (f_N * g)(t) = \langle g(\cdot), f_N(t-\cdot)\rangle = \langle g(t-\cdot), f_N(\cdot)\rangle.$$

On the other hand,

$$h(t) = (f * g)(t) = \langle g(\cdot), f(t-\cdot)\rangle = \langle g(t-\cdot), f(\cdot)\rangle.$$

Thus

$$h_N(t) = h(t), \quad t \le N. \tag{3.42}$$

The following theorem tells us that the partial energy delay property is also the characteristic property of all-pass filters in the continuous signals case.

Theorem 3.16: (i) If g is an all-pass filter, then for any $N > 0$, there holds

$$\int_0^N |f(t)|^2 dt \ge \int_0^N |h(t)|^2 dt,$$

where $f(t)$ is any physically realizable input signal in $\mathbb{S}$ and $h(t) = (g * f)(t)$ is the corresponding output signal.

(ii) Let g be a temperate distribution with $\operatorname{supp} g \subset [0, \infty)$ and the inverse Fourier transform of g be a measurable function, satisfying $\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|\check{g}(\omega)|^2}{1+\omega^2} d\omega = 1$. If for any $N > 0$,

$$\int_0^N |f(t)|^2 dt \ge \int_0^N |h(t)|^2 dt, \tag{3.43}$$

where $f(t)$ is any physically realizable input signal in $\mathbb{S}$ and $h(t) = (g * f)(t)$ is the corresponding output, then g is an all-pass filter.

Proof of Theorem 3.16 (1) From (3.41) and (3.42) we have

$$\begin{aligned} \int_0^N |f(t)|^2 dt &= \int_0^\infty |h_N(t)|^2 dt \\ &\ge \int_0^N |h_N(t)|^2 dt = \int_0^N |h(t)|^2 dt. \end{aligned}$$

(2) We will prove the assertion by introducing a contradiction. Assume $|\check{g}(\omega)| \neq 1, a.e.$

Let

$$|F(\omega)|^2 = \begin{cases} 2, & \text{if } |\check{g}(\omega)|^2 > 1, \\ 1, & \text{if } |\check{g}(\omega)|^2 < 1. \end{cases} \tag{3.44}$$

Since

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|\check{g}(\omega)|^2}{1+\omega^2} d\omega = 1 = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{1+\omega^2} d\omega,$$

we have

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|\check{g}(\omega)|^2 - 1}{1+\omega^2} d\omega = 1$$

and

$$\frac{1}{\pi}\int_{-\infty}^{\infty} (|\check{g}(\omega)|^2 - 1)\frac{|F(\omega)|^2}{1+\omega^2} d\omega > 0.$$

Let

$$|\check{f}(\omega)| = \frac{F(\omega)}{\sqrt{1+\omega^2}},$$

then

$$\int_{-\infty}^{\infty} (|\check{g}(\omega)|^2 - 1)|\check{f}(\omega)|^2 d\omega > 0,$$

and

$$\int_{-\infty}^{\infty} |\check{h}(\omega)|^2 d\omega = \int_{-\infty}^{\infty} |\check{g}(\omega)\check{f}(\omega)|^2 d\omega > \int_{-\infty}^{\infty} |\check{f}(\omega)|^2 d\omega.$$

Thus

$$\int_0^{+\infty} |h(t)|^2 dt > \int_0^{+\infty} |f(t)|^2 dt.$$

So we can select N_0 such that

$$\int_0^{N_0} |h(t)|^2 dt > \int_0^{N_0} |f(t)|^2 dt$$

which is contradictory with (3.43). Hence, $|\check{g}(\omega)| = 1$ a.e. $\square$

The following theorem tells that if there is some N that makes the equality holds in the inequality (3.43), then the all-pass filter will have to be a Blaschke product of a special structure.

We need the following definition.

Definition 3.17: If the zeros $z_k, k = 1, 2, ...,$ of a Blaschke product B defined in (2.17) do not have a finite accumulation point on the line, then B is said to be sparse.

It can be easily shown that if B is sparse on the line, then $\lim_{k\to\infty} |z_k| = \infty$.

We recall the Paley-Wiener class of analytic functions in the complex plane:

$$\begin{aligned} PW(N) \ &\triangleq \ \{f \ : \ f \text{ is an entirefunction}, \\ &\qquad f|_{\mathbb{R}} \in L^2(\mathbb{R}), \ \text{supp}\hat{f} \subset [-N, N]\}. \end{aligned}$$

The Paley-Wiener Theorem asserts that

$$\begin{aligned} PW(N) \ &= \ \{f \ : \ f \text{ is an entire function}, \\ &\qquad f|_{\mathbb{R}} \in L^2(\mathbb{R}), \ |f(z)| \le Ce^{N|z|}\} \\ &= \ \{f \ : \ f \text{ is an entire function}, \\ &\qquad f|_{\mathbb{R}} \in L^2(\mathbb{R}), \ |f(z)| \le C_\epsilon e^{(N+\epsilon)|z|}, \ \epsilon > 0\}. \end{aligned}$$

For a proof of the theorem see [24] or [26].

Theorem 3.18: Let g be an all-pass filter. Then there exist some $f \in \mathbb{S}$ and $N > 0$ such that for $h = g * f$,

$$\int_0^N |f(t)|^2 dt = \int_0^N |h(t)|^2 dt \tag{3.45}$$

if and only if $\check{g}(z)$ is a Blaschke product whose zeros, including the multiplicities, are zeros of an entire function belonging to the Paley-Wiener class $PW(N)$. In the case the Blaschke product is sparse.

Proof of Theorem 3.18 We first prove the "only if" part. We assume that (3.45) holds for some $f \in \mathbb{S}$ and some $N > 0$, and we are to show that the inner function $\check{g}(z)$ is a Blaschke product, and

$$\check{f}_N(z)\check{g}(z) = \check{h}_N,$$

where $\check{h}_N \in PW(N)$, as desired. Using the same notation as in the proof of Theorem 3.16, we have

$$\int_N^\infty |h_N(t)|^2 dt = 0,$$

and thus conclude that h_N has compact support in $[0, N]$, and therefore $\check{h}_N \in PW(N)$. We are to show that $\check{g}$ is a Blaschke product. Note that both $\check{f}_N$ and $\check{h}_N$ are H^2-functions in the upper-half complex plane. Owing to the Paley-Wiener Theorem, both can be analytically extended to become entire functions in the whole complex plane. By the Nevanlinna factorization Theorem, in the self-explanatory notation, we have

$$cO_{\check{f}_N} B_{\check{f}_N} S_{\check{f}_N} \cdot B_{\check{g}} S_{\check{g}} = c' O_{\check{h}_N} B_{\check{h}_N} S_{\check{h}_N},$$

where $|c| = |c'| = 1$. Now we show that $d\mu_{\check{h}_N} = 0$. Temporarily accepting this, we have $d\mu_{\check{f}_N} = d\mu_{\check{g}} = 0$, and then

$$S_{\check{f}_N}, S_{\check{g}}, S_{\check{h}_N}$$

are all unimodular constants, and thus

$$\check{g} = B_{\check{g}}.$$

Now, since $\check{h}_N$ has analytic continuation across any finite interval on the real line, both its inner and outer factors have analytic continuation across finite intervals. This fact is proved in Theorem 6.3, Chapter II of [11], through proving the sparseness of the Blaschke product part and the triviality of the singular inner function part, where the latter stands for $d\mu_{\check{h}_N} = 0$.

Next we prove the "if" part. We assume that $\check{g}(z)$ is a Blaschke product whose zeros, including their multiplicities, form part or all zeros of some entire function $\check{h}_N$ in $PW(N)$. Let

$$\check{g}(z) = \prod_{k=1}^\infty e^{i\alpha_k} \frac{z - a_k}{z - \overline{a}_k}.$$

Consider the expression

$$\check{f}_N(z) = \prod_{k=1}^\infty e^{-i\alpha_k} \frac{z - \overline{a}_k}{z - a_k} \check{h}_N(z).$$

we show that $f_N \in L^2(\mathbb{R})$ and $\text{supp} f_N \subset [0, N]$. To this end, as in [1], define

$$\sigma_n(t) = \begin{cases} -2y_n e^{-i\alpha_n} e^{-ia_n t} & \text{if } \ t \le 0, \\ 0 & \text{if } \ t > 0, \end{cases} \tag{3.46}$$

where $a_n = x_n + iy_n$. Then

$$e^{-i\alpha_n} \frac{\omega - \overline{a}_n}{\omega - a_n} \cdot \check{h}_N(\omega) = (T_n h_N)\check{}(\omega), \tag{3.47}$$

where

$$T_n h_N = \sigma_n * h_N + e^{-i\alpha_n} h_N. \tag{3.48}$$

Now we show that $T_n h_N \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, and $\text{supp} T_n h_N \subset [0, N]$.

Since σ_n is in L^1 and h_N is in $L^1 \cap L^2$, we have $T_n h_N \in L^1 \cap L^2$. Because

$$e^{-i\alpha_n} \frac{\omega - \overline{a}_n}{\omega - a_n} \cdot \check{h}_N(\omega)$$

is the boundary value of a function in the Hardy H^2 space, its Fourier transform, viz. $T_n h_N$, is supported in $[0, \infty)$. Now, a direct estimate gives

$$|e^{-i\alpha_n} \frac{z - \overline{a}_n}{z - a_n} \cdot \check{h}_N(z)| \le C_\epsilon e^{(N+\epsilon)|z|}, \quad \epsilon > 0,$$

that implies $(T_n h_N)\check{} \in PW(N)$, we conclude that $T_n h_N$ is supported in $[0, N]$.

Repeating this process we obtain

$$F_n = T_n \circ T_{n-1} \circ \cdots \circ T_1 h_N$$

that belongs to $L^1 \cap L^2$ and vanishes outside $[0, N]$. Note that Blaschke products are unimodular. By Lebesgue's dominated convergence theorem,

$$\lim_{n\to\infty} F_n = f_N$$

in the L^1-norm. By taking a subsequence we get that f_N vanishes almost everywhere outside $[0, N]$. The Paley-Wiener Theorem then guarantees that $\check{f}_N \in PW(N)$. Since the transformation from h_N to f_N is norm preserving, we finally have

$$\int_0^N |f(t)|^2 dt = \int_0^N |h(t)|^2 dt,$$

as desired. The proof is complete.

Example The following is an adaptation of the argument in the proof of Theorem XXIX in [18], combined with the argument on page 114 (30.51) of the same book . Take $a_0 = 0, a_{-n} = -a_n$, and

$$\lim_{n\to+\infty} \frac{a_n}{n} = N.$$

Then the function

$$\check{h}_N(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{a_n^2}\right)$$

belongs to L^2, if restricted to $\mathbb{R}$, with the property $\check{h}_N(a_n) = 0$, and supp$h_N \subset [0, N]$.

References

[1] Edwin J. Akutowicz, " On the determination of the phase of a Fourier integral, I," *Transactions of the American Mathematical Society,* vol. 83, no. 1, pp. 179-192, 1956.

[2] B. Boashashe, "Estimating and interpreting the instantaneous frequency of a signal-Part 1: Fundamentals," *Proc. IEEE*, vol. 80, no. 4, pp. 520-538, 1992.

[3] H. Boche and V. Pohl, "Robustness of the Inner-Outer Factorization and of the Spectral Factorization for FIR Data," *IEEE Transaction of Signal Processing*, vol. 56, no. 1, pp. 274-283, 2008.

[4] J.R. Carson and T.C. Fry, "Variable frequency electric circuit theory with application to the theory of frequency modulation," *Bell System Tech. Jour.,* vol. 16, pp. 513-540, 1937.

[5] Q.S. Cheng, *Digital Signal Processing,* Peking University Press, 2003, in Chinese.

[6] L. Cohen, *Time-Frequency Analysis: Theory and Applications.* Upper Saddle River, NJ: Prentice Hall, 1995.

[7] P. Dang and T. Qian, "Hardy-Sobolev Derivatives of Phase and Amplitude, and Their Applications," preprint.

[8] P. Dang, T. Qian and Z. You, "Hardy-Sobolev Spaces Decomposition in Signal Analysis ," *Journal of Fourier Analysis and Applications,* vol. 17, no. 1, pp. 36-64, 2011.

[9] M.F. Flanagan, M. McLaughlin and A.D. Fagan, "Gradient-adaptive algorithms for minimumphase-all-pass decomposition of a finite impulse response system," *Signal Processing, IET*, vol. 4, no. 1, pp. 12-21, 2010.

[10] D. Gabor, "Theory of communication," *J. IEE.* 93(III), pp. 429-457, 1946.

[11] J.B. Garnett, *Bounded Analytic Functions.* New York: Academic Press, 1981.

[12] P. Koosis, *Introduction to H_p Spaces.* New York: Cambridge University Press, 1980.

[13] R. Kumaresan and A. Rao, "On minimum/maximum/all-pass decompositions in time and frequency domains," *IEEE Transactions on Signal Processing,* vol. 48, no. 10, pp. 2973-2976, 2000.

[14] R. Kumaresan and A. Rao, "An algorithm for decomposing an analytic signal into AM and positive FM components," *Proc. IEEE Int. Conf. Acoust., Speech, Signal Process.*, vol. 3, pp. 1561 -1564, 1998.

[15] R. Kumaresan and Yadong Wang, "Covert zero-crossings represent envelope and phase of band-pass signals," Conference Record of the Thirty-Third Asilomar Conference on Signals, Systems, and Computers, vol. 2, pp. 1043-1046, 1999.

[16] S. Migliorini and F. Vlacci, "A new rigidity result for holomorphic maps," *Indag. Mathem., N.S.,* vol. 13, no. 4, pp. 537-549, 2002.

[17] A.V. Oppenheim and R.W. Schafer, *Discrete-Time Signal Processing*. Englewood Cliffs, N.J.: Prentice Hall, 1989.

[18] R. E. A. C. Paley and N. Wiener, *Fourier Transforms in the Complex Domain.* New York: American Mathematical Society, 1934.

[19] B. Picinbono, "On instantaneous amplitude and phase of signals," IEEE Transactions on Signal Processing, vol. 45, no. 3, pp. 552-560, 1997.

[20] Van der Pol, "The Fundamental Principles of Frequency Modulation," *B. Jour. IEE,* vol. 93, no. 23, pp. 153-158, 1946.

[21] Ch. Pommerrenke, *Boundary Behaviour of Conformal Maps.* Grundlehren der Mathematischen Wissenschaften. 299. Berlin: Springer- Verlag, 1992.

[22] T. Qian, Y.S. Xu, D.Y. Yan, L.X. Yan and B, Yu, "Fourier spectrum characterization of Hardy spaces and applications," *Proceedings of the American Mathematical Society,* vol. 137, pp. 971-980, 2009.

[23] T. Qian, "Boundary derivatives of the phases of inner and outer functions and applications ," *Math. Meth. Appl. Sci.,* vol. 32, no. 3, pp. 253-263, 2009.

[24] W. Rudin, *Real and Complex Analysis.* New York: McGraw Hill, 1966.

[25] E.M. Stein, *Singular Integrals and Differentiability Properties of Functions.* Princeton, New Jersey: Princeton University Press, 1970.

[26] E.M. Stein and G. Weiss, *Fourier Analysis on Euclidean Spaces.* Princeton, New Jersey: Princeton University Press, 1971.

[27] X.G. Xia and L. Cohen, "On analytic signals with nonnegative instantaneous frequencies," In: *Proceeding of the ICASSAP-99*, vol. 3, Washington: IEEE, pp. 1329-1332, 1999.

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